THE STATUS OF THE INTEGRABILITY PROBLEM IN MATHEMATICAL ECONOMICS

BY

ABDULKHALIQ SAIFALNASR

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Thesis Committee

Dr. Nicolas Hadjisavvas (Adviser)

Dr. Qamrul Hasan Ansari (Member)

Dr. Mohammed Alshahrani (Member)

Dr. Husain S. Al-Attas
Department Chairman

Dr. Salam A. Zummo
Dean of Graduate Studies

17/5/16
Date

Dedication

To my family

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NOTATION

- $v = (v_1, v_2, \ldots, v_n), u = (u_1, u_2, \ldots, u_n) \in \mathbb{R}^n, \langle v, u \rangle = \sum_{i=1}^n v_i u_i$
- $\mathbf{B}(x^*, \varepsilon) = \{y \in \mathbb{R}^n : \|y - x^*\| < \varepsilon\}$
- $\bar{\mathbf{B}}(x^*, \varepsilon) = \{y \in \mathbb{R}^n : \|y - x^*\| \leq \varepsilon\}$
- $\mathbb{R}^n_+ = [0, \infty)^n$
- $\mathbb{R}^n_{++} = (0, \infty)^n$
- For $x,\ y \in \mathbb{R}^n,\ y \ll x \Leftrightarrow y_i < x_i$ for all $i \in \{1, 2, \ldots, n\}$
- For $x,\ y \in \mathbb{R}^n,\ y \leq x \Leftrightarrow y_i \leq x_i$ for all $i \in \{1, 2, \ldots, n\}$
- $[x, y] = \{w : w = (1 - \lambda)x + \lambda y,\ \lambda \in [0, 1]\}$
- $\nabla f = (\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})$
- $JF = (\frac{\partial f_i}{\partial x_j})\ \ i, j = 1, \ldots, n$
- $\nabla^2 f = (\frac{\partial^2 f}{\partial x_i \partial x_j})\ \ i, j = 1, \ldots, n$

THESIS ABSTRACT

NAME: AbdulKhaliq SaifAlNasr

TITLE OF STUDY: The Status of the Integrability Problem in Mathematical Economics

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The integrability problem is one of the oldest problems in mathematical economics, dating from the end of the 19th century. Starting by the deeply influential work of Hurwicz and Uzawa in 1971, various proofs showed that the problem has a solution. In this thesis, we study the approach of Hurwicz and Uzawa in detail. We then highlight a shortcoming of their result, and we present an amendment proposed by Jackson. We conclude by presenting a recent approach by Hadjisavvas and Penot.

ملخص

تعد مسألة التكامل احدى أقدم المسائل في الاقتصاد الرياضي، حيث تعود نشأتها لنهاية القرن التاسع عشر للميلاد. يعتبر البحث العظيم لعالمي الرياضيات هورويكز وأوزاوا في عام ١٩٧١ من أوائل البراهين للمسألة. منذ ذلك الحين ظهرت عدة أبحاث لبرهنة المسألة، تحت افتراضات مختلفة. في هذه الرسالة، سنقوم بدراسة عمل هورويكز وأوزاوا بالتفصيل.

بعد ذلك نناقش نقطة ضعف في عمل المؤلفين، ونطرح تعديل لذلك العيب من قبل جاكسون. نختتم الرسالة بطرح عمل حديث من قبل حاجيسافاز وبينو.

CHAPTER 1

INTRODUCTION

The famous "integrability problem" was first mentioned by G. B. Antonelli back in 1886 [1], but it was formulated in a more precise way in 1892 by I. Fisher [7], in what Paul Samuelson called "perhaps the best of all doctoral dissertations in economics". Actually, Fisher thought initially that the problem did not have a solution. But the first deep study of the problem was made by P. A. Samuelson, in a series of seminal papers in 1938, 1948, and 1950 [20, 21, 23]. Samuelson gave an affirmative answer to the problem, with a complete proof, but only in the case of an economy where there are just two commodities. He also gave a heuristic proof of the general case of n commodities, but his proof was far from rigorous. The general case of n commodities remained unsolved for many years, until 1971 when Hurwicz and Uzawa gave a complete proof based on Nikliborc's theorem and on Thomas' theorem of the theory of PDEs [11]. Since then several other proofs appeared. For example, Chiappori and Ekeland in their book [3] make use of the Darboux theorem on differential forms. In [6], the proof is quite technical, but it is based only on the theory of ODEs. In [14, 8, 2], one makes use of the Frobenius theorem for PDEs. Besides using various methods, the assumptions of the existing proofs also differ. Some use additional technical assumptions, others do not. But the most important difference between results related to the integrability problem, concerns what is exactly proved. In many papers, actually only the local existence of a solution of a partial differential equation is deduced; this is in fact the easiest part of the problem. In others, like [11], one constructs a utility function that is not defined on the whole positive cone $\mathbb{R}^n_+$ (in economic terms, on the whole set of commodity bundles), but on a certain surface. In some others like [6], one constructs a solution defined on a compact "box" in $\mathbb{R}^n_+$, and then it is only affirmed that a similar proof can be made to construct the solution on the whole positive cone. A recent preprint [2] contains a simplified and amended version of the very complicated proof by Hurwicz and Uzawa.

Demand functions play a key role in developing the integrability problem, and historically there have been two main ways to study them in mathematical economics. (1) The finite approach. This involves finite sets of points. For example, the "revealed preference" approach, pioneered by P. A. Samuelson and developed by H. Houthaker. (2) The infinitesimal approach. This involves derivatives of demand functions. This was pioneered by Slutsky, in his development of substitution functions, as well as Hicks, Allen, and Samuelson. This is mainly the approach of the paper by Hurwicz and Uzawa [11]. In Hick's "Value and Capital" [10], and Samuelson's "Foundations of Economic Analysis" [22], it is established that the maximization of a consumer's "utility function" u, generates a demand function ξ. It also implies the symmetry as well as negative semidefiniteness of the so-called Slutsky matrix σ, a construction fundamental within the latter approach. The "Integrability Problem" concerns the converse question; given a function ξ that satisfies some conditions, such as the symmetry and negative semidefiniteness of the associated Slutsky matrix, can we construct a utility function u such that ξ is the corresponding demand function? In economic terms, if we know the commodities consumers are purchasing, given their budget and the prices of the commodities, can we infer their preferences, or 'utility', based on their choices?

In this thesis, we present several works dealing with the above question. In Chapter 2, we present preliminary notions essential for the development of the problem. Mainly, we recall definitions and notions from mathematical economics and convex analysis, which will be play an important role in the development of the problem. In Chapter 3, we talk about the Utility Maximization Problem and Expenditure Minimization Problem, two optimization problems of interest to our discussion, and conclude the chapter by defining the Slutsky matrix, a key notion in establishing a solution to the problem. In Chapter 4, we actually study, in detail, the integrability problem through the influential work by Hurwicz and Uzawa [11]. We then highlight a shortcoming of the authors' approach in Chapter 5, and present an amendment proposed by Jackson. We conclude in Chapter 6 by presenting a modern approach by Hadjisavvas and Penot, which, although might appear similar in some respects to the approach in [11], is in fact quite different.

CHAPTER 2

PRELIMINARIES

In this chapter, we present preliminary notions that will facilitate our discussion about consumer theory, and ultimately enable us to study and understand the integrability problem in mathematical economics. We also recall some classical definitions and results from convex analysis that are essential to the development of the main problem.

The model on which we base our discussion of consumer preferences is that of a *pure exchange economy.* In this model, the economic agents are only the consumers that can exchange goods; there are no producers of goods involved.

2.1 Commodities

Definition 2.1 *A commodity is essentially a good or service.*

That commodity could be tangible, e.g., a house or a car, or intangible, such as a service.

Definition 2.2 *A commodity bundle, or market basket, is a vector* $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n_+$*, where* x_i *represents the quantity of the ith commodity.*

We shall use $\mathbb{R}^n_+$ as the set of commodity bundles from which consumers can choose. It is worth mentioning that time, and even location, can be incorporated into the definition of a commodity. For example, water in Paris in the spring season is not as valuable as water in the arid Sahara desert in the summer. This perspective gives the two commodities different values, and as such makes them fundamentally different items.

2.2 Preference Relations

We assume that for each consumer there is a *preference relation* $\preceq$ on $\mathbb{R}^n_+$.

Definition 2.3 *The preference relation* $\preceq$ *is a binary relation that describes the consumer's preferences on* $\mathbb{R}^n_+$*.*

Given two commodity bundles x, $y \in \mathbb{R}^n_+$, we read $x \preceq y$ as "commodity bundle y is at least as desirable as commodity bundle x". We further assume that this relation is a *complete preorder relation.*

Definition 2.4 *We say that a binary relation* $\preceq$ *is complete (or total) if, for every* x, $y \in \mathbb{R}^n_+$, $x \preceq y$ *or* $y \preceq x$ *(or both).*

That is, the consumer can always compare any two commodity bundles. The strength of this assumption should not to be taken for granted; it is obvious how tricky it could be to compare alternatives that are radically different from one another.

Definition 2.5 *We say that a binary relation* $\preceq$ *is reflexive if* $x \preceq x$*, for all* $x \in \mathbb{R}^n_+$.

Definition 2.6 *We say that a binary relation* $\preceq$ *is transitive if, given* x, y, *and* $z \in \mathbb{R}^n_+$*, if* $x \preceq y$ *and* $y \preceq z$*, then* $x \preceq z$.

Definition 2.7 *A preorder is a binary relation that is both reflexive and transitive.*

Note that the assumption of reflexivity is implied by the completeness of $\preceq$. Once again, the significance of the assumption of transitivity should not be underestimated. It eliminates situations where the preferences of the consumer seem to cycle. For instance, say that a consumer prefers a BMW car model to a Mercedes, and prefers a Porsche to the BMW, then, had we not imposed transitivity, the consumer might prefer the Mercedes to the Porsche, leading to the cycle: Mercedes $\preceq$ BMW $\preceq$ Porsche $\preceq$ Mercedes. For a consistent and smooth study of economic theory, we impose transitivity in the sequel.

Additionally, note that we do not impose the assumption that $\preceq$ is antisymmetric; that is, $x \preceq y$ and $y \preceq x$ imply that $x = y$. In other words, a consumer could very well like two different commodity bundles as much as each other. In this respect, the preference relation $\preceq$ is not an order relation.

Definition 2.8 *When a consumer desires two different baskets* $x, y \in \mathbb{R}^n_+$ *as much as each other, i.e., when* $x \preceq y$ *and* $y \preceq x$*, we say that "*x *is indifferent to* y*", and we denote that by* $x \sim y$.

Proposition 2.1 *The relation* $\sim$ *has the following properties:*

(i) Reflexivity: $x \sim x$ *for all* $x \in \mathbb{R}^n_+$

(ii) Transitivity: if $x \sim y$ *and* $y \sim z$*, then* $x \sim z$

(iii) Symmetry: if $x \sim y$*, then* $y \sim x$

In other words, $\sim$ is an equivalence relation.

Definition 2.9 *Given a basket* $x \in X$*, and the preference relation* $\preceq$ *of a consumer, the indifference set (or curve) containing* x *is the set of all baskets indifferent to* x*, i.e.,* $\left\{y \in \mathbb{R}^n_+ : y \sim x\right\}$.

Definition 2.10 *We say that* x *is strictly preferred to* y*, and write* $x \prec y$*, if and only if* $x \preceq y$ *holds but* $y \preceq x$ *does not.*

Proposition 2.2 *The strict preference relation* $\prec$ *has the following properties:*

(i) Irreflexivity: $x \prec x$ *never holds*

(ii) Transitivity: if $x \prec y$ *and* $y \prec z$*, then* $x \prec z$

Definition 2.11 *We say that the preference relation* $\preceq$ *on* $\mathbb{R}^n_+$ *is monotone if* $x \in \mathbb{R}^n_+$*, and* $y \ll x$ *imply* $y \prec x$*. It is strongly monotone if* $y \leq x$ *and* $x \neq y$ *imply that* $y \leq x$.

Definition 2.12 *The preference relation* $\preceq$ *on* $\mathbb{R}^n_+$ *is locally non-satiated if, for every* $x \in \mathbb{R}^n_+$*, and every* $\varepsilon > 0$*, there exists a* $y \in \mathbb{R}^n_+$ *such that* $||y - x|| \leq \varepsilon$*, and* $x \prec y$*.*

2.3 Utility Functions

It is often convenient in economics to describe the preferences of consumers in terms of *utility functions* (or *indicators*). A utility function u compares different commodity bundles in $\mathbb{R}^n_+$ by assigning numerical values to each bundle, in accordance with the consumer's preferences. Formally, we define u as follows.

Definition 2.13 *A function* $u : \mathbb{R}^n_+ \to \mathbb{R}$ *is a utility function representing preference relation* $\preceq$ *if, for all* x*,* $y \in \mathbb{R}^n_+$*,*

$$x \preceq y \Leftrightarrow u(x) \leq u(y).$$

It is worth mentioning that the utility function u representing the relation $\preceq$ is not unique.

Proposition 2.3 *Consider a utility function* u *representing the preference relation* $\preceq$*. For any strictly increasing function* $f : \mathbb{R} \to \mathbb{R}$*, the composition function* $h = f \circ u$ *represents* $\preceq$ *as well.*

From the definition of a utility function u representing $\preceq$, we can infer the connection between the indifference curve containing x and the level set of u at x: $\left\{y \in \mathbb{R}^n_+ : y \sim x\right\} = \left\{y \in \mathbb{R}^n_+ : u(y) = u(x)\right\}$.

Definition 2.14 *We say that a preference relation $\preceq$ is continuous if for every $x \in \mathbb{R}^n_+$, the two sets*

$$\left\{y \in \mathbb{R}^n_+ : x \preceq y\right\} \text{ and } \left\{y \in \mathbb{R}^n_+ : y \preceq x\right\} \tag{2.1}$$

are closed.

This of course is equivalent to saying that their complements, i.e., the sets

$$\left\{y \in \mathbb{R}^n_+ : y \prec x\right\} \text{ and } \left\{y \in \mathbb{R}^n_+ : x \prec y\right\}$$

are open. From an economic point of view, this is a reasonable assumption: it means that if the consumer strictly prefers x to y, then by changing y very slightly, his preference will not change. The same is for the other set.

From a mathematical point of view it is also very convenient. In fact, if $\preceq$ may be represented by a continuous utility function, then the sets in (2.1) are the sets

$$\left\{y \in \mathbb{R}^n_+ : u(x) \le u(y)\right\} \text{ and } \left\{y \in \mathbb{R}^n_+ : u(y) \le u(x)\right\}$$

which are of course closed.

Definition 2.15 *A function $f : \mathbb{R}^n \to \mathbb{R}$ is lower semicontinuous at $\bar{x} \in \mathbb{R}^n$ if, for all $\alpha < f(\bar{x})$, there exists $\delta > 0$ such that, for all x with $\|x - \bar{x}\| < \delta$, $\alpha < f(x)$.*

Definition 2.16 *A function $f : \mathbb{R}^n \to \mathbb{R}$ is upper semicontinuous at $\bar{x} \in \mathbb{R}^n$ if $-f$ is lower semicontinuous at $\bar{x}$.*

In fact, the set $\{y \in \mathbb{R}^n_+ : u(y) \leq u(x)\}$ is closed if and only if u is lower semi-continuous. Similarly, the set $\{y \in \mathbb{R}^n_+ : u(y) \geq u(x)\}$ is closed if and only if u is upper semi-continuous.

Even more impressively, the following statement is true.

Theorem 2.1 *For every continuous preference relation, there exists a continuous utility map that represents it.*

The theorem is spectacular because it asserts the existence of a utility map that represents the preference relation, under the weak assumption of continuity.

Definition 2.17 *A utility function $u : \mathbb{R}^n_+ \to \mathbb{R}$ is called monotone if $x \in \mathbb{R}^n_+$ and $y \ll x$ imply $u(y) < u(x)$. It is strongly monotone if $y \leq x$ and $x \neq y$ imply $u(y) < u(x)$.*

In economics terms, the utility function u is monotone if all commodities are desirable by the consumer.

Proposition 2.4 *Let the preference relation $\preceq$ be represented by the utility function u. Then $\preceq$ is monotone if and only if u is monotone.*

Definition 2.18 *We say that a utility function u is locally nonsatiated if, given $x^* \in \mathbb{R}^n_+$, for all $\varepsilon > 0$, there exists a vector $x \in \mathbb{R}^n_+ \cap \mathbf{B}(x^*, \varepsilon)$ such that $u(x) > u(x^*)$.*

Proposition 2.5 *If u is a utility function representing a locally nonsatiated relation $\preceq$, then u is locally nonsatiated.*

2.4 Convexity and Concavity

Definition 2.19 *A set $A \subseteq \mathbb{R}^n$ is convex if, for all $x, y \in A$, $(1-\lambda)x + \lambda y \in A$, $\lambda \in [0,1]$.*

Definition 2.20 *Let $f : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}^n$ is convex. We say that f is convex if, for all $x, y \in A$, $f((1-\lambda)x + \lambda y) \leq (1-\lambda)f(x) + \lambda f(y)$, $\lambda \in [0,1]$.*

Definition 2.21 *A preference relation $\preceq$ is called convex if, for all x, y, and $z \in \mathbb{R}^n_+$ and all $\lambda \in [0,1]$, the relations $x \preceq y$ and $x \preceq z$ imply $x \preceq (1-\lambda)\ y + \lambda z$.*

Definition 2.22 *A preference relation is called strictly convex, if for all x, $y \in \mathbb{R}^n_+$ and all $\lambda \in (0,1)$, the relations $x \preceq y$ and $x \neq y$ imply $x \prec (1-\lambda)\ x + \lambda y$.*

Convexity (or strict convexity) of the preference relation means that the consumer prefers to diversify his acquisitions; for example, instead of a basket containing only eggs or a basket containing only oil, he would prefer a basket containing smaller quantities of both, because it is the only way to make an omelet.

Definition 2.23 *A function $u : \mathbb{R}^n_+ \to \mathbb{R}$ is called quasiconcave if for all x, $y \in \mathbb{R}^n_+$ and $z \in [x, y]$, the following relation holds:*

$$u(z) \geq \min\{u(x), u(y)\}.$$

Proposition 2.6 *u is quasiconcave if for all $\alpha \in \mathbb{R}$ the set $[u \geq \alpha] = \{x : u(x) \geq \alpha\}$ is convex.*

Definition 2.24 *u is strictly quasiconcave if for all x, $y \in \mathbb{R}^n_+$, $x \neq y$, and all $\lambda \in (0,1)$, $u((1-\lambda)x + \lambda y) > \min\{u(x), u(y)\}$.*

Proposition 2.7 *A function u is strictly quasiconcave if, for all x, $y \in \mathbb{R}^n_+$, $x \neq y$, and all $\lambda \in (0,1)$, one has*

$$[u(x) \leq u(y) \text{ and } x \neq y] \Rightarrow u(x) < u((1-\lambda)x + \lambda y).$$

Proposition 2.8 *Let the preference relation $\preceq$ be represented by the utility function u. Then:*

(i) $\preceq$ is convex if and only if u is quasiconcave;

(ii) $\preceq$ is strictly convex if and only if u is strictly quasiconcave

Due to the convenience they provide us, we will mostly deal with utility functions rather than preference relations.

2.5 Prices

Definition 2.25 *We denote by $p = (p_1, p_2, \ldots, p_n) \in \mathbb{R}^n$ a price vector, whose ith component p_i represents the price per unit of the ith commodity.*

Note that it is not obligatory for the vector p to be non-negative. However, for the purposes of our discussion, *we assume that prices are strictly positive*, i.e., $p \in \mathbb{R}^n_{++}$.

The reason is that if a price of a commodity is zero, then probably all consumers would like to acquire the whole available quantity of that commodity.

Given a price vector p and a commodity bundle x, the total worth of x is obviously $\sum_{i=1}^{n} p_i x_i = \langle p, x \rangle$.

We let $m \geq 0$ be a number representing the wealth of the consumer.

Definition 2.26 *The Walrasian budget set* $W_m(p) = \{y \in \mathbb{R}^n_+ : \langle p, y \rangle \leq m\}$ *is the set of all feasible commodity bundles that the consumer can afford, given his wealth* m *and the prices* p.

Definition 2.27 *We call the set* $\{y \in \mathbb{R}^n_+ : \langle p, y \rangle = m\}$ *the budget hyperplane.*

In the case $n = 2$, the budget hyperplane is called the *budget line.*

Proposition 2.9 *Let* $p \in \mathbb{R}^n_{++}$. *Then for any* $m \geq 0$, *the budget set* $W_m(p)$ *is nonempty, convex and compact.*

Proof. It is clear that $W_m(p)$ is nonempty (it contains 0), closed and convex. To see that it is bounded, just note that for all $y = (y_1,\ y_2, \ldots, y_n) \in W_m(p)$, one has $\sum_{i=1}^{n} p_i y_i \leq m$, so (since everything is non-negative), $p_i y_i \leq m$, i.e., $0 \leq y_i \leq m/p_i$. Since each coordinate of y is bounded by a constant, $W_m(p)$ is bounded. ∎

We assume that each consumer seeks to maximize his satisfaction by choosing the best affordable commodity bundle, i.e., that bundle in $W_m(p)$ that maximizes his utility. The following proposition shows that, if all prices are positive, then such a bundle exists.

Proposition 2.10 *Assume that* $p \in \mathbb{R}^n_{++}$*, and that the utility function* u *is upper semi-continuous. Then for any* $m \geq 0$*, there exists a commodity bundle* $x \in W_m(p)$ *such that* $u(y) \leq u(x)$ *for all* $y \in W_m(p)$*. If, in addition,* u *is strictly quasiconcave, then* x *is uniquely defined.*

Proof. Since $W_m(p)$ is nonempty, compact and the utility function u is upper semicontinuous, u has a maximum on $W_m(p)$, so there exists at least one maximizer of u on $W_m(p)$. Assume that there are two points x, $x' \in W_m(p)$ such that $u(x) = u(x') = \max_{y \in W_m(p)} u(y)$. By strict quasiconcavity, the point $x'' = (x + x')/2$ satisfies $u(x'') > u(x)$. In addition, $x'' \in W_m(p)$, since is convex. This contradicts the fact that x maximizes u in $W_m(p)$. ∎

Proposition 2.11 *The budget set* $W_m(p)$ *does not change upon multiplying all prices and budget by a number* $\lambda > 0$*.*

Proof. If we multiply all prices and budget by $\lambda > 0$, then the budget set becomes

$$W_{\lambda m}(\lambda p) = \{x \in \mathbb{R}^n_+ : \langle \lambda p, x \rangle \leq \lambda m\} = \{x \in \mathbb{R}^n_+ : \langle p, x \rangle \leq m\} = W_m(p)$$

i.e., it does not change. ∎

2.6 Demand Functions

Definition 2.28 *We call the function* $\xi : \mathbb{R}^n_{++} \times \mathbb{R}_+ \to \mathbb{R}^n_+$ *a demand function, if, for every* $(p, m) \in \mathbb{R}^n_{++} \times \mathbb{R}_+$*,* $\xi(p, m)$ *is the commodity bundle that yields* *the highest satisfaction for the consumer over his budget set. In other words,* $u(\xi(p,m)) \geq u(x)$*, for all* $x \in W_m(p)$*.*

Note that in the above definition we tacitly assume that $\xi(p,m)$ exists and is unique. For example, this happens if u is upper semicontiuous and strictly quasiconcave.

From Proposition (2.11), we immediately obtain the following assertion.

Proposition 2.12 *The demand function* $\xi(p,m)$ *is homogeneous of degree zero with respect to* (p,m)*. That is,* $\xi(\alpha p, \alpha m) = \xi(p,m)$*, for all* $(p,m) \in \mathbb{R}^n_{++} \times \mathbb{R}_+$*, and* $\alpha \in \mathbb{R}$*.*

Definition 2.29 *We say that the utility function* u *is nonsatiated if, for a given price vector* $p \in \mathbb{R}^n_{++}$*, a budget* $m \geq 0$*, and a basket* $x \in \mathbb{R}^n_+$ *such that* $\langle p, x \rangle < m$*, there exists another basket* $y \in \mathbb{R}^n_+$ *such that* $u(y) > u(x)$ *and* $\langle p, y \rangle \leq m$*.*

Note that this is equivalent to saying that $\langle p, \xi(p,m) \rangle = m$. This means that for each p, the value of the best bundle $\xi(p,m)$ exhausts the whole wealth of the consumer.

Proposition 2.13 *For a utility function* u*, the following series of implications hold:*

$$u \text{ is monotone } \Rightarrow u \text{ is locally nonsatiated } \Rightarrow u \text{ is nonsatiated}$$

Theorem 2.2 *Assume that the utility function* u *is continuous and locally nonsatiated. Then the demand function* $\xi : \mathbb{R}^n_{++} \times \mathbb{R}_+ \to \mathbb{R}^n_+$ *is continuous.*

Proof. Let $\{(p_k, m_k)\}$ be a sequence in $\mathbb{R}^n_{++} \times \mathbb{R}_+$ converging to $(p^*, m^*) \in \mathbb{R}^n_{++} \times \mathbb{R}_+$. By nonsatiation, $\langle p_k, \xi(p_k, m_k)\rangle = m_k$. Since $m_k \to m^*$, the sequence $\{m_k\}$ is bounded. Let $p_{k,i}$, p^*_i, $\xi_i(p_k, m_k)$ be, respectively, the *ith* coordinates of p_k, p^*, and $\xi(p_k, m_k)$. Then $p_{k,i}\xi_i(p_k, m_k) \leq \langle p_k, \xi(p_k, m_k)\rangle = m_k$. On the other hand, $p_{k,i} \to p^*_i > 0$, so for k large enough, $p_{k,i} > p^*_i/2$. This implies that

$$0 \leq \xi_i(p_k, m_k) \leq \frac{m_k}{p_{k,i}} \leq \frac{2m_k}{p^*_i} \tag{2.2}$$

We have two cases:

- If $m^* = 0$, then $W_0(p) = 0$, so $\xi(p, 0) = 0$. On the other hand, (2.2) together with $m_k \to 0$ imply that $\xi_i(p_k, m_k) \to 0$, i.e., $\xi(p_k, m_k) \to \xi(p, m^*)$
- If $m^* > 0$, then since the sequence m_k is bounded, (2.2) shows that for each i, $\langle \xi_i(p_k, m_k)\rangle$ is bounded.

Hence $\langle \xi(p_k, m_k)\rangle$ is bounded. By taking a subsequence, if necessary, we may assume that it converges to a vector y. We will show that $y = \xi(p^*, m^*)$. Since $\langle p_k, \xi(p_k, m_k)\rangle = m_k$, by taking limits we get $\langle p^*, y\rangle = m$. This implies that $y \in W_{m^*}(p^*)$. Let us show that y is the preferred bundle in $W_{m^*}(p^*)$. So let us first take any x such that $\langle p^*, x\rangle < m$. Then for k large enough, $\langle p_k, x\rangle < m_k$. Hence, $x \in W_{m_k}(p_k)$. By definition of ξ, we have $u(x) \leq u(\xi(p_k, m_k))$. By taking the limit and using continuity of u we obtain $u(x) \leq u(y)$, valid for all x with $\langle p^*, x\rangle < m$. Now assume that x is such that $\langle p^*, x\rangle = m$. Obviously, there exists a sequence $\{x_k\}$ such that $x_k \to x$ and $\langle p^*, x\rangle < m$. Since by the preceding proof $u(x_k) \le u(y)$, we obtain by continuity of u that $u(x) \le u(y)$. Thus, y is the preferred bundle in $W_{m^*}(p^*)$, i.e., $\xi(p^*, m^*) = y = \lim \xi(p_k, m_k)$. Thus ξ is continuous. ∎

CHAPTER 3

THE UMP AND EMP

Two optimization problems of particular interest to our discussion are the *Utility Maximization Problem (UMP)*, and the *Expenditure Minimization Problem (EMP).* As we shall see in Section 3.3, the two problems are closely related to each other. In fact, they are in a sense duals whose solutions coincide.

3.1 The Utility Maximization Problem (UMP)

Given a price vector $p \in \mathbb{R}^n_{++}$, and a budget $m \geq 0$ of a consumer, we formulate the *utility maximization problem* as

$$\max_{x \geq 0} \; u(x)$$

$$\text{s.t.} \; \langle p, x \rangle \leq m.$$

Basically, the consumer chooses the basket that maximizes his satisfaction from the feasible set of baskets. It is obvious that the feasible set for the UMP is the Walrasian budget set $W_m(p) = \{x \in \mathbb{R}^n_+ : \langle p, x\rangle \leq m\}$.

As we saw in Proposition 2.10, if u is upper semicontinuous, then the UMP has a solution.

Definition 3.1 *Given a price-budget pair $(p, m) \in \mathbb{R}^n_{++} \times \mathbb{R}_+$, the Walrasian demand correspondence $x^*(p, m)$ is the set of bundles that solves the UMP.*

Note that, in general, $x^*(p, m)$ is a set of bundles that satisfy the UMP. However, when u is strictly quasiconcave, Proposition 2.10 shows that $x^*(p, m)$ is a unique bundle. In this case, we say that $x^*(p, m) = \xi(p, m)$ is the *Walrasian demand function*. Here, $x^*(p, m) = \xi(p, m) = \arg\max\{u(x) : x \in \mathbb{R}^n_+, \langle p, x\rangle \leq m\}$

Proposition 3.1 *Suppose u is a continuous, locally nonsatiated utility function on $\mathbb{R}^n_+$. Then the Walrasian demand correspondence $x^*(p, m)$ has the following properties:*

1. *$x^*(p, m)$ is homogeneous of degree zero in (p, m): $x^*(\alpha p, \alpha m) = x^*(p, m)$ for any p, m and scalar $\alpha > 0$.*

2. *The budget exhaustion $\langle p, x\rangle = m$ holds for all $x \in x^*(p, m)$*

3. *If $\preceq$ is convex, so that u is quasiconcave, then $x^*(p, m)$ is a convex set. Moreover, if $\preceq$ is strictly convex, so that u is strictly quasiconcave, then $x^*(p, m)$ is the singleton $\xi(p, m)$.*

The proof is relatively straightforward and follows from the definition of x^*, therefore we omit it.

Definition 3.2 *Given a price-budget pair* (p, m), *we call the function* $v(p, m) = u(\xi(p, m))$ *the indirect utility function.*

Obviously, $v(p, m)$ is the maximum utility level that is achieved given the pair (p, m), i.e., it is the result of the UMP: $v(p, m) = \max\{u(x) : x \in \mathbb{R}^n_+, \langle p, x\rangle \leq m\}$.

Proposition 3.2 *Let* u *be a continuous, locally nonsatiated utility function on* $\mathbb{R}^n_+$. *The indirect utility function* $v(p, m)$ *has the following properties:*

1. *Homogeneity of degree zero;* $v(\alpha p, \alpha m) = v(p, m)$, $\alpha > 0$

2. *Strictly increasing in* m, *and nonincreasing in* p_i *for any* $i \in \{1, 2, \ldots, n\}$

3. *Quasiconvex; i.e., the set* $\{(p, m) : v(p, m) \leq \bar{v}\}$ *is convex for any* $\bar{v}$

4. *Continuous in* p *and* m

We omit the proof for this proposition since the properties follow from the definition of v.

We now turn our attention to the Expenditure Minimization Problem (EMP).

3.2 The Expenditure Minimization Problem (EMP)

Starting with the price vector $p \in \mathbb{R}^n_{++}$, and the fixed utility level $v \in \mathbb{R}$, we formulate the *expenditure minimization problem* as

$$\min_{x \geq 0} \langle p, x \rangle$$

$$\text{s.t. } u(x) \geq v$$

In the EMP, the objective is to minimize the amount of money spent in order to achieve a level of satisfaction that is at least v. Simply put, we aim to achieve at least utility level v as cheaply as we could.

We shall call the feasible set of the EMP $H(v) = \{x \in \mathbb{R}^n_+ : u(x) \geq v\}$ the *Hicksian budget set.*

Definition 3.3 *Given a price vector $p \in \mathbb{R}^n_{++}$, and utility level v, the Hicksian demand correspondence $h(p, v)$ is the set of bundles that solves the EMP given (p, v).*

The following proposition about the properties of the Hicksian demand correspondence parallels Proposition 3.1 for Walrasian demand.

Proposition 3.3 *Suppose u is a continuous, locally nonsatiated utility function on $\mathbb{R}^n_+$. Then the Hicksian demand correspondance $h(p, v)$ has the following properties:*

1. *Homogeneous of degree zero in* $p : h(\alpha p, v) = h(p, v)$ *for any* $p, v,$ *and* $\alpha > 0$.

2. *Minimum utility: For any* $x \in h(p, v)$, $u(x) = v$.

3. *Convexity/uniqueness: If* $\preceq$ *is convex, then* $h(p, v)$ *is a convex set. If, moreover,* $\preceq$ *is strictly convex, so that* u *is strictly quasiconcave, then* $h(p, v)$ *is a singleton.*

In the case when h is single-valued, $h(p, v)$ is called the Hicksian demand function. Here, $h(p, v) = \arg\min\{\langle p, x\rangle : x \in \mathbb{R}^n_+, u(x) \geq v\}$.

In parallel to the indirect utility function $v(p, m)$, we define the *Hicksian expenditure function* as follows.

Definition 3.4 *Given a price-utility pair* (p, v), *we call the function* $e(p, v) = \langle p, h(p, v)\rangle$ *the Hicksian expenditure function.*

It is clear that the Hicksian expenditure function is the result of the EMP given (p, v); i.e., $e(p, v) = \min\{\langle p, x\rangle : x \in \mathbb{R}^n_+, u(x) \geq v\}$.

In the same spirit as in Proposition 3.2, Proposition 3.4 gives a characterization of the properties of the expenditure function.

Proposition 3.4 *Let* u *be a continuous, locally nonsatiated utility function on* $\mathbb{R}^n_+$. *The expenditure function* $e(p, v)$ *has the following properties:*

1. *Homogeneous of degree one in* $p : e(\alpha p, v) = \alpha e(p, v)$

2. *Strictly increasing in* v *and nondecreasing in* p_i *for all* $i \in \{1, 2, \ldots, n\}$

3. *Concave in p*

4. *Continuous in p and v*

3.3 The Duality of the UMP and EMP

Before delving into the duality between the Utility Maximization Problem and the Expenditure Minimization Problem, we summarize the main constituents of each problem in the following table, where we consider a continuous, locally-nonsatiated, strictly quasiconcave utility function u, a price vector $p \in \mathbb{R}^n_+$, a budget $m \geq 0$, and a utility level $v \in \mathbb{R}$.

UMP	EMP
$\max_{x \geq 0} u(x)$ s.t. $\langle p, x \rangle \leq m$	$\min_{x \geq 0} \langle p, x \rangle$ s.t. $u(x) \geq v$
Walrasian budget set: $W_m(p) = \{x \in \mathbb{R}^n_+ : \langle p, x \rangle \leq m\}$	Hicksian budget set: $H(v) = \{x \in \mathbb{R}^n_+ : u(x) \geq v\}$
Indirect utility function: $v(p, m) = \max\{u(x) : x \in \mathbb{R}^n_+, \langle p, x \rangle \leq m\}$	Hicksian expenditure function: $e(p, v) = \min\{\langle p, x \rangle : x \in \mathbb{R}^n_+, u(x) \geq v\}$
Walrasian demand funcion: $\xi(p, m) = \arg\max\{u(x) : x \in \mathbb{R}^n_+, \langle p, x \rangle \leq m\}$	Hicksian demand function: $h(p, v) = \arg\min\{\langle p, x \rangle : x \in \mathbb{R}^n_+, u(x) \geq v\}$
$v(p, m) = u(\xi(p, m))$	$e(p, v) = \langle p, h(p, v) \rangle$

Table 3.1: Summary of EMP and UMP

We can understand the "duality" between the two problems in the following sense. Say we begin with the UMP. Given the price vector p, and the budget m, we calculate the Walrasian demand funcion $\xi(p,m)$ (the optimum bundle) and its corresponding utility $u(\xi(p,m)) = v(p,m)$.

Next, we set $v = v(p,m)$, and, along with the price vector p, use these quantities as inputs to the EMP. Doing so, we will find that the optimum bundle $h(p,v)$ is exactly $\xi(p,m)$, and the expenditure, i.e., the money necessary to buy $h(p,v)$, is exactly equal to m. That is, we have

$$h(p,v) = \xi(p,m)$$

$$e(p,v) = m$$

$$\text{where } v = v(p,m).$$

Dually, let us start from the EMP with a price vector p and a desired utility value v. We calculate the Hicksian demand function $h(p,v)$, the optimum bundle, and the expenditure $\langle p, h(p,v)\rangle = e(p,v)$. Then, using this expenditure as the budget in the UMP, i.e., taking $m = e(p,v)$, we will find that the optimum bundle $\xi(p,m)$ is exactly $h(p,v)$, and the corresponding utility $u(\xi(p,m)) = v(p,m)$ is exactly v. That is

$$\xi(p,m) = h(p,v)$$

$$v(p,m) = v$$

where $m = e(p, v)$.

Note that from the above it follows:

$$v = v(p, m) \Leftrightarrow e(p, v) = m$$

Let us prove these assertions.

Proposition 3.5 *Let u be a continuous, locally nonsatiated, and strictly quasi-concave utility function on $\mathbb{R}^n_+$. Then*

$$h(p, v) = \xi(p, e(p, v)) \tag{3.1}$$

$$\xi(p, m) = h(p, v(p, m)) \tag{3.2}$$

$$e(p, v(p, m)) = m \tag{3.3}$$

$$v(p, e(p, v)) = v \tag{3.4}$$

Proof. The single-valuedness of the Walrasian and Hicksian demand functions ξ and h is established in Proposition 3.1 and Proposition 3.3, respectively. To show (3.1), one needs to show that $h(p, v)$ belongs to the budget set $W_{e(p,v)}(p)$, and it maximizes the utility there. $h(p, v) \in W_{e(p,v)}(p)$ is true because $\langle p, h(p, v)\rangle = e(p, v)$, from the EMP. Then we have to show that $\langle p, x\rangle \le e(p, v)$ implies $u(x) \le u(h(p, v))$. Indeed, assume to the contrary that $u(x) > u(h(p, v)) \ge v$, but $\langle p, x\rangle \le e(p, v)$. Given that $e(p, v)$ is the minimal value of a basket with utility at least v, we would have that this minimum is achieved also by x. But this contradicts the uniqueness of $h(p, v)$. Hence (3.1) holds.

(3.2) can be proved similarly: we need to show that $\xi(p, m)$ has utility at least $v(p, m)$ (which is obviously true), and that it minimizes the value of all baskets with utility at least $v(p, m)$. Indeed, assume to the contrary that there is a basket y such that $u(y) \geq v(p, m)$ and $\langle p, y \rangle < \langle p, \xi(p, m) \rangle \leq m$.

Then $y \in W_m(p)$ so y maximizes the utility inside $W_m(p)$. But this contradicts the uniqueness of $\xi(p, m)$.

Since u is nonsatiated, we know that $m = \langle p, \xi(p, m) \rangle$. We deduce from (3.2) and $\langle p, h(p, v) \rangle = e(p, v)$ that $m = \langle p, h(p, v(p, m)) \rangle = e(p, v(p, m))$.

Finally, since we know that $u(h(p, v)) = v$, we deduce from (3.1) and $v(p, m) = u(\xi(p, m))$ that $v = u(\xi(p, e(p, v)) = v(p, e(p, v))$. ▮

3.4 The Slutsky Term

We now develop an important definition, which is that of the Slutsky matrix. In order to do so, however, we first recall a simple version of the Envelope Theorem.

Theorem 3.1 *(Envelope Theorem) Let $f : X \times T \to \mathbb{R}$ be a function of two variables, where X is any set and T an open interval. Define $V(t) = \sup\{f(x, t) : x \in X\}$. If $t_0 \in T$, $x_0 \in X$ are such that $f(x_0, t_0) = V(t_0)$, and V and $f(x, \cdot)$ are differentiable at t_0, then*

$$\frac{\partial V(t_0)}{\partial t} = \frac{\partial f(x_0, t_0)}{\partial t}.$$

That is, the derivative of V is the partial derivative of f, at any point x that achieves the maximum.

Proof. For every $s \in T$, $g(s) := f(x_0, s) - V(s) \leq 0$ by definition of V. Also, $g(t_0) = 0$. Thus, g has a maximum at t_0. Since g is differentiable by assumption, $g'(t_0) = 0$. This gives (3.1). ▮

A consequence is the so-called Shephard's lemma:

Lemma 3.1 *(Shephard) Assume that the function $e(p, v)$ is differentiable with respect to p (or at least that the partial derivatives exist). Then*

$$h_i(p, v) = \frac{\partial e(p, v)}{\partial p_i}.$$

Proof. The proof is a direct consequence of the Envelope Theorem: Since

$$e(p, v) = \min\{p_1 x_1 + p_2 x_2 + \ldots + p_n x_n : u(x_1, x_2, \ldots, x_n) \geq v\}$$

by the envelope theorem, $\frac{\partial e(p,v)}{\partial p_i} = \frac{\partial(p_1 x_1 + p_2 x_2 + \ldots + p_n x_n)}{\partial p_i} = x_{i,}$ where $(x_1, x_2, \ldots, x_n)$ is a solution of the minimization problem, i.e., $h(p, v)$. ▮

Now let us calculate $\frac{\partial h_i(p,v)}{\partial p_j}$.

Using (3.1), Shephard's lemma, as well as the chain rule, we get

$$\frac{\partial h_i(p, v)}{\partial p_j} = \frac{\partial \xi_i(p, e(p, v))}{\partial p_j} = \frac{\partial \xi_i(p, m)}{\partial p_j}|_{m=e(p,v)} + \frac{\partial \xi_i(p, m)}{\partial m}|_{m=e(p,v)} \; \frac{\partial e(p, v)}{\partial p_j}$$

$$= \frac{\partial \xi_i(p, m)}{\partial p_j}|_{m=e(p,v)} + \frac{\partial \xi_i(p, m)}{\partial m}|_{m=e(p,v)} \; \xi_j(p, e(p, v))$$

The last is true because $\frac{\partial e(p,v)}{\partial p_i} = h_i(p,v) = \xi_i(p, e(p,v))$. Thus, we proved:

$$\frac{\partial h_i(p,v)}{\partial p_j} = \frac{\partial \xi_i(p,m)}{\partial p_j} + \frac{\partial \xi_i(p,m)}{\partial m}\, \xi_j(p,m),$$

where $m = e(p,v)$ (or equivalently, $v = v(p,m)$).

Definition 3.5 *We call the term* $\sigma_{ij}(p,m) = \frac{\partial \xi_i(p,m)}{\partial p_j} + \frac{\partial \xi_i(p,m)}{\partial m}\, \xi_j(p,m)$ *the Slutsky substitution function. Additionally, the Slutsky matrix (sometimes the Slutsky-Hicks) is given by* $\sigma(p,m) = (\sigma_{ij}(p,m))_{1\le i,i\le n}$.

According to the derivation in this section, given m one has

$$\sigma_{ij}(p,m) = \frac{\partial h_i(p,v)}{\partial p_j}$$

with $v = v(p,m)$. Thus,

$$\sigma_{ij}(p,m) = \frac{\partial^2 e(p,v)}{\partial p_j \partial p_i}$$

so it is symmetric. Also, it is negative semidefinite since e is concave with respect to p.

CHAPTER 4

THE INTEGRABILITY PROBLEM

We are now ready to study the Integrability Problem, following the seminal paper of Hurwicz and Uzawa [11].

4.1 The Assumptions

The assumptions that we are going to impose on the demand function ξ are the following:

(A) Single-valuedness: $\xi(p, m)$ is a single-valued, n-vector function on $\mathbb{R}^n_{++} \times \mathbb{R}_+$

(B) The budget exhaustion (i.e., nonsatiation): $\langle p, \xi(p, m)\rangle = m$, for all $(p, m) \in \mathbb{R}^n_{++} \times \mathbb{R}_+$

(D) Differentiability: $\xi_i(p, m)$ are differentiable on $\mathbb{R}^n_{++} \times \mathbb{R}_+$, $i = 1, 2, \cdots, n$

(E) Boundedness: For any $0 \ll a' \ll a'' \in \mathbb{R}^n_{++}$, there exists a number $M_{a',a''} > 0$ such that, for all $m \geq 0$,

$$a' \leq p \leq a'' \Rightarrow \left| \frac{\partial \xi_i(p,m)}{\partial m} \right| \leq M_{a',a''}$$

We will also impose two additional assumptions that concern the Slutsky matrix:

(S) Symmetry: The Slutsky matrix $\sigma(p,m)$ is symmetric

(NSD) Negative semidefiniteness: The Slutsky matrix $\sigma(p,m)$ is negative semidefinite.

We already discussed conditions (A), (B), and (D). Condition (E) is imposed for mathematical convenience. As for conditions (S) and (NSD), we saw in Section 3.4 that they can be proved using some assumptions on the utility function u. In [11], the same is proved under slightly more general assumptions:

Theorem 4.1 *Let u be a utility function represented by a complete preorder $\preceq$ on a set $D \subseteq \mathbb{R}^n_+$. Let $\xi : \mathbb{R}^n_{++} \times \mathbb{R}_+ \to D$ be a single-valued demand function such that $\xi(p,m)$ uniquely maximizes $u(x)$, for all $x \in W_m(p)$. In addition to assumption (A), assume (B), and (D). Then the matrix $\sigma(p,m)$ is defined and is symmetric and negative semidefinite for all $(p,m) \in \mathbb{R}^n_{++} \times \mathbb{R}_+$.*

We omit the proof of Theorem 4.1. Let $X \subseteq \mathbb{R}^n_+$ denote the range of the demand function ξ, i.e., $X = \{\xi(p,m) : (p,m) \in \mathbb{R}^n_{++} \times \mathbb{R}_+\}$. The integrability problem is the following.

Theorem 4.2 ***(Main Theorem)*** *Let $\xi(p,m)$ satisfy (A), (B), (D), and (E). In addition, let the Slutsky matrix σ associated with ξ satisfy (S) and (NSD). Then there exists a utility function $u : X \to \mathbb{R}$ such that $\xi(p,m)$ is the unique maximizer of u over $W_m(p)$.*

We will not present the proof in full detail of Theorem 4.2; rather, we present, to a large extent, the general approach to recovering u from ξ. The construction of the utility function u involves the so-called income compensation function μ.

Definition 4.1 *The income compensation function is given by $\mu(p; p^0, m^0) = e(p, v(p^0, m^0))$.*

Clearly, $\mu(p^0; p^0, m^0) = e(p^0, v(p^0, m^0)) = m^0$.

We shall often deal with $\mu(p; p^0, m^0)$ as a function of the variable p only, i.e., $\mu(p; p^0, m^0) = \mu(p)$ with the understanding that the initial condition is (p^0, m^0).

It follows from the definition of μ and the envelope theorem that

$$\frac{\partial \mu(p; p^0, m^0)}{\partial p_i} = \frac{\partial e(p, v(p^0, m^0))}{\partial p_i} = h_i(p, v(p^0, m^0)) = \xi_i(p, e(p, v(p^0, m^0))) \quad (4.1)$$

$$= \xi_i(p, \mu(p; p^0, m^0)), \; i = 1, 2, \ldots, n.$$

4.2 Outline of the Argument

Our aim will be to first solve the so-called total differential equation (4.1). That is, given an initial condition (p^0, m^0), we aim to find a function μ such that

$$\frac{\partial \mu(p)}{\partial p_i} = \xi_i(p, \mu(p)),\ i = 1, 2, \ldots, n.$$

Once we have found such a function μ, we express it explicitly in terms of the initial condition $\mu(p^0) = m^0$ as $\mu(p; p^0, m^0)$.

We then interchange the roles of the vector p and the initial condition (p^0, m^0), by fixing p as p^*, and treating the initial data (p^0, m^0) as a variable $(p, m) \in \mathbb{R}^n_{++} \times \mathbb{R}_+$, from which we define the function

$$w(p, m) = \mu(p^*; p, m) = e(p^*, v(p, m)).$$

Observe that this function w is another utility function, in the sense that

$$w(p, m) \geq w(p', m') \Leftrightarrow v(p, m) \geq v(p', m')$$

This is true since the expenditure function e is increasing in v,

$$w(p, m) = e(p^*, v(p, m)) \geq e(p^*, v(p', m')) = w(p', m') \Leftrightarrow v(p, m) \geq v(p', m')$$

We then express the price-budget pair (p, m) in terms of the demand ξ.

Finally, we define the utility function u on the range of the demand function ξ as

$$u(x) = U_{p^*}(x) = w(p, m) = \mu(p^*; p, m)$$

where $x = \xi(p, m)$.

Before proceeding with the proof, we present two examples to illustrate the procedure outlined above.

4.3 Two Examples

Example 4.3 *Consider the Cobb-Douglas utility function* $u(x_1, x_2) = x_1^\alpha x_2^\beta$, *where* $\alpha + \beta = 1$. *Consider also a price vector* $\bar{p} = (p_1, p_2) \in \mathbb{R}^2_{++}$, *and a budget* $\bar{m} \geq 0$. *Recall that the demand function* $\xi(\bar{p}, \bar{m}) = (x_1^*(\bar{p}, \bar{m}), x_2^*(\bar{p}, \bar{m}))$ *is homogeneous of degree zero, thus we may divide the arguments by* p_2, *and take the price vector to be* $(p, 1)$, *and the budget as* m. *In order to find the demand components* x_1^* *and* x_2^*, *we solve the UMP*

$$\max u(x_1, x_2)$$

$$s.t. \;\; \langle (p, 1), (x_1, x_2) \rangle \leq m$$

We get

$$x_1^*(p, m) = \frac{\alpha m}{p}.$$

From the constraint of the problem $px_1 + x_2 = m$, we have

$$x_2^* = \beta m.$$

Substituting these values into the utility function u, we find the indirect utility function to be

$$v(p, m) = m\beta^\beta (\frac{\alpha}{p})^\alpha.$$

In order to find the expenditure function e, we solve the EMP

$$\min \langle (p, 1), (x_1, x_2) \rangle$$

$$s.t.\ u(x_1, x_2) \geq v$$

We find the expenditure function to be

$$e(p, v) = v\beta^{-\beta} (\frac{p}{\alpha})^\alpha.$$

Now choose (p^0, m^0), and define

$$\mu(p; p^0, m^0) = e(p, v(p^0, m^0)) = m^0 (\frac{p}{p^0})^\alpha$$

Clearly, $\mu(p^0; p^0, m^0) = m^0$.

Finally, note that this function μ satisfies the differential equation

$$\frac{\partial \mu(p)}{\partial p} = \alpha[m^0(p^0)^{-\alpha}]p^{\alpha-1} = \frac{\alpha\mu(p)}{p} = x_1^*(p, \mu(p)).$$

Example 4.4 *In this example, we concern ourselves with the reverse process to Example (4.3). Here, we illustrate the procedure that we outlined in the previous section to recover a utility function from a demand function. Given the price vector $(p, 1) \in \mathbb{R}^2_{++}$, the budget m, and the demand component $x_1^*(p, m) = \frac{\alpha m}{p}$, we aim to recover a utility function u such that $\xi(p, m) = (x_1^*(p, m), x_2^*(p, m))$ maximizes u subject to the constraint $\langle (p, 1), (x_1, x_2) \rangle \leq m$. From the budget constraint of the UMP $\langle (p, 1), (x_1, x_2) \rangle = m$, we get that $x_2^*(p, m) = (1 - \alpha)m$. Next, we solve the differential equation*

$$\frac{\partial \mu(p)}{\partial p} = x_1^*(p, \mu(p)) = \frac{\alpha\mu(p)}{p}$$
$$\Leftrightarrow \frac{\mu'}{\mu} = \frac{\alpha}{p}$$
$$\Leftrightarrow \mu(p) = cp^\alpha$$

Given the initial condition (p^0, m^0), we determine the value of the constant c by satisfying $\mu(p^0) = m^0$:

$$\mu(p^0) = c(p^0)^\alpha = m^0$$
$$\Leftrightarrow c = \frac{m^0}{(p^0)^\alpha}$$

Thus, the income compensation function μ *is*

$$\mu(p; p^0, m^0) = \frac{m^0}{(p^0)^\alpha} p^\alpha$$

We now interchange the roles of the variable p, *and the initial condition* (p^0, m^0), *by fixing the former to be* $p^* = 1$, *and letting the initial condition be the variable* (p, m), *to define the utility function* U_{p^*} :

$$U_{p^*} = w(p, m) = \mu(p^*; p, m) = \frac{m}{p^\alpha} \tag{4.2}$$

We are almost there, but we need to express the variables p *and* m *in terms of the demand components* x_1^* *and* x_2^*. *For simplicity, we call the latter* x_1 *and* x_2, *respectively, and we solve the equations* $x_1 = \frac{\alpha m}{p}$ *and* $x_2 = (1-\alpha)m$ *for* p *and* m. *Doing so, we obtain*

$$p = \frac{\alpha}{1-\alpha} \frac{x_2}{x_1}$$
$$m = \frac{x_2}{1-\alpha}$$

Finally, we substitute these values into (4.2) to obtain the utility function

$$U_{p^*}(x_1, x_2) = K x_1^\alpha x_2^{1-\alpha}$$

where K *is a constant. Hence, we have recovered the well-known Cobb-Douglas utility function from the given demand function.*

4.4 Solutions of Total Differential Equations

In order to proceed with the proof of the theorem, we first discuss solutions of total differential equations similar to (4.1).

Consider $\Omega = \Pi \times \Theta$, where $\Pi \subseteq \mathbb{R}^n$, $\Theta \subseteq \mathbb{R}$.

Let $f : \Pi \times \Theta \to \mathbb{R}^n$, with $f(x, z) = (f_1(x, z), f_2(x, z), \ldots, f_n(x, z))$

We are interested in solving the total differential equation

$$\frac{\partial z}{\partial x_i} = f_i(x_1, x_2, \ldots, x_n, z),\ i = 1, 2, \ldots, n. \tag{4.3}$$

Definition 4.2 *A real-valued function $z = w(x)$ defined on $\Pi^* \subseteq \Pi$ is a solution to (4.3), with initial condition $(x^0, z^0) \in \Omega$, if $x^0 \in \Pi^*$, and*

$$\frac{\partial w}{\partial x_i}(x) = f_i(x, w(x)),\ i = 1, 2, \ldots, n,\ for\ all\ x \in \Pi^*$$

and

$$w(x^0) = z^0$$

Now we consider the following assumptions:

(CD) Continuous differentiability: $\frac{\partial f_i}{\partial x_j}(x, z)$ and $\frac{\partial f_i}{\partial z}(x, z)$ exist and are continuous on Ω;

(DD) Differentiability: $f_i(x, z)$ are differentiable on Ω for $i = 1, 2, \ldots, n$;

(UD) Uniform boundedness of derivative: $\frac{\partial f_j}{\partial z}(x, z)$ are uniformly bounded on Ω; i.e., there is a $K \in \mathbb{R}$ such that

$$\left|\frac{\partial f_j}{\partial z}(x, z)\right| \leq K,$$

for all $(x, z) \in \Omega$;

(SS) Symmetry: $\frac{\partial f_i}{\partial x_j}(x, z) + \frac{\partial f_i}{\partial z}(x, z) \cdot f_j(x, z) = \frac{\partial f_j}{\partial x_i}(x, z) + \frac{\partial f_j}{\partial z}(x, z) \cdot f_i(x, z)$, $i = 1, 2, \ldots, n$, for all $(x, z) \in \Omega$.

The following theorem was proved by Thomas [24]. It is a version of the Frobenius theorem of PDEs.

Theorem 4.5 *(Thomas's) Let Let $f : \Pi \times \Theta \to \mathbb{R}^n$, where, for $a' < a''$,*

$$\Pi = \{x : a' < x_i < a'', i = 1, 2, \ldots, n\}, \ \Theta = \mathbb{R}$$

Assume (CD), (UD), (SS).

Then there exists a unique continuous solution $w(x) = w(x; x^0, z^0)$ of (4.3) with initial condition (x^0, z^0), for which the domain of definition is Π.

The following theorem, in which (CD) is relaxed into (DD), was proved by Hurwicz and Uzawa [11].

Theorem 4.6 *(Existence Theorem I) Let $f : \Pi \times \Theta \to \mathbb{R}^n$, where, for $a' < a''$,*

$$\Pi = \{x : a' < x_i < a'', i = 1, 2, \ldots, n\}, \ \Theta = \mathbb{R}$$

Assume (DD), (UD), (SS).

Then there exists a unique continuous solution $w(x) = w(x; x^0, z^0)$ *of (4.3) with initial condition* (x^0, z^0)*, for which the domain of definition is* Π*. Furthermore,* $w(x; x^0, z^0)$ *is continuous with respect to* x*, and continuous with respect to* (x^0, z^0)*.*

According to Hartman [9], $w(x; x^0, z^0)$ is in fact continuous with respect to the vector $(x; x^0, z^0)$ altogether.

The next theorem, also due to Hurwicz and Uzawa [11], refines the second component of the domain Θ into $\mathbb{R}^n_+$.

Theorem 4.7 *(Existence Theorem II) Let* $f : \Pi \times \Theta \to \mathbb{R}^n$*, where, for* $a' < a''$*,*

$$\Pi = \{x : a' \leq x_i \leq a'', i = 1, 2, \ldots, n\},\ \Theta = \mathbb{R}_+$$

Assume (DD), (UD), (SS). Assume, further, that

$$f_i(x, 0) = 0,\ \text{for all } x \in \Pi \tag{4.4}$$

Then there exists a unique continuous solution $w(x) = w(x; x^0, z^0)$ *of (4.3) with initial condition* (x^0, z^0)*, for which the domain of definition is* Π*.*

Furthermore, $w(x; x^0, z^0)$ *is continuous with respect to* x*, and continuous with respect to* (x^0, z^0)*.*

We are now ready to prove the theorem that is needed for our purpose.

Theorem 4.8 *(Existence Theorem III) Let* $f : \Pi \times \Theta \to \mathbb{R}^n$, *where*

$$\Pi = \mathbb{R}^n_{++},\ \Theta = \mathbb{R}_+$$

Assume (DD), (SS), (4.4). Assume, further,

(UD)': For any $0 \ll a' \ll a'' \in \mathbb{R}^n_{++}$, *there exists a number* $M_{a',a''} \in \mathbb{R}$ *such that, for all* $m \geq 0$,

$$a' \leq x \leq a'' \Rightarrow |\frac{\partial f_i(x,z)}{\partial z}| \leq M_{a',a''}$$

Then there exists a unique continuous solution $w(x) = w(x; x^0, z^0)$ *of (4.3) with initial condition* (x^0, z^0), *for which the domain of definition is* Π.

Furthermore, $w(x; x^0, z^0)$ *is continuous with respect to* x, *and continuous with respect to* (x^0, z^0).

Proof. We begin with an initial condition $(x^0, z^0) \in \Pi \times \Theta$, where, for some $0 < a' < a''$, $\Pi = \{x : a' \leq x_i \leq a''\}$, $\Theta = \mathbb{R}^n_+$. By Theorem (4.7), there exists a unique continuous solution $w(x) = w(x; x^0, z^0)$ of (4.3) with initial condition (x^0, z^0), for which the domain of definition is Π. Now let us start with the same initial condition (x^0, z^0). Take $a' = \frac{1}{n}$, and $a'' = n$, where $n \in \mathbb{N}$. For a sufficiently large n, the domain $\Pi = \{x : \frac{1}{n} \leq x_i \leq n\}$ contains x_0. Once again, by Existence Theorem III, there is a unique solution w to (4.3) passing through the point (x^0, z^0). Finally, we let $n \to \infty$ to get $\Pi = \mathbb{R}^n_+$, and the conclusion of the theorem follows. ∎

4.5 Construction of the Utility u from ξ

We now return to the total differential equation

$$\frac{\partial \mu(p)}{\partial p_i} = \xi_i(p, \mu(p)), \; i = 1, 2, \ldots, n \tag{4.5}$$

Or

$$\frac{\partial \mu(p)}{\partial p} = \xi(p, \mu(p)).$$

We are half-way through with the proof of Theorem 4.2. We now introduce a series of lemmas to complete the proof.

Lemma 4.1 *Let the demand function $\xi(p, m)$ satisfy (A), (D), (E), and (S). Then (4.5) is uniquely integrable; that is, for any $(p^*, m^*) \in \Omega = \mathbb{R}^n_{++} \times \mathbb{R}_+$, there exists a unique function $\mu(p; p^*, m^*)$ defined for all $p \in \mathbb{R}^n_{++}$ such that*

$$\mu(p^*; p^*, m^*) = m^* \tag{4.6}$$

$$\frac{\partial \mu(p; p^*, m^*)}{\partial p_i} = \xi_i(p, \mu(p; p^*, m^*)), \; i = 1, 2, \ldots, n, \text{ for all } p \in \mathbb{R}^n_{++}$$

Furthermore, $\mu(p^*; \cdot, \cdot)$ is continuous for every fixed p^*.

Lemma 4.2 *Consider (p', m') and (p'', m''), with $(p', m') \neq (p'', m'')$. If $\mu(p^0; p', m') = \mu(p^0; p'', m'')$, then*

$$\mu(p; p', m') = \mu(p; p'', m'') \text{ for all } p \in \Pi.$$

Proof. Assume $\mu(p^0;p',m') = \mu(p^0;p'',m'') = m^0$, for some $p^0 \in \mathbb{R}^n_{++}$, $m^0 \in \mathbb{R}_+$. By the definition of μ, the function $\mu(p;p^0,m^0)$ passes through the point (p^0,m^0), i.e., $\mu(p^0;p^0,m^0) = m^0$. But by the uniqueness of the solution of (4.5) from Theorem 4.8, $\mu(p;p^0,m^0) = \mu(p;p',m') = \mu(p;p'',m'')$, for all $p \in \Pi$. ▌

Lemma 4.3 *Consider (p',m') and (p'',m''), with $(p',m') \neq (p'',m'')$. If $\mu(p^0;p',m') < \mu(p^0;p'',m'')$, then*

$$\mu(p;p',m') < \mu(p;p'',m'').$$

Proof. Suppose there exist p^0, $p^1 \in \mathbb{R}^n_{++}$ such that $\mu(p^0;p',m') < \mu(p^0;p'',m'')$, but $\mu(p^1;p',m') > \mu(p^1;p'',m'')$. By Theorem 4.8, $\mu(\cdot;p^*,m^*)$ is continuous. Consider now the function $g(p) = \mu(p;p',m') - \mu(p;p'',m'')$. g is continuous with respect to p, since μ is continuous. Observe that $g(p^0) < 0$, while $g(p^1) > 0$. By the Intermediate Value theorem, there exists an element $\bar{p} = (1-\lambda)p^0 + \lambda p^1$, $\lambda \in (0,1)$, such that $g(\bar{p}) = 0$, i.e., $\mu(\bar{p};p',m') = \mu(\bar{p};p'',m'')$. But by Theorem 4.2, this would imply that $\mu(p;p',m') = \mu(p;p'',m'')$ for all $p \in \Pi$, a contradiction. ▌

Now we use the income compensation function $\mu(p^*;p,m)$ to construct, on the range X of the demand function ξ, the utility function $u(x)$ in Theorem 4.2.

We shall define u as

$$u(x) = U_{p^*}(x) = \mu(p^*;p,m), \text{ where } p^* \text{ is fixed, and } x = \xi(p,m)$$

Our objective now is to prove that this construction of $u(x)$ is indeed a utility function. First, however, we show that, for a given p^*, $U_{p^*}(x)$ is uniquely determined, irrespective of the choice of (p, m), such that $x = \xi(p, m)$. We now give further properties of μ.

Lemma 4.4 *Consider* (p^0, m^0), (p^1, m^1), *such that* $x^0 = \xi(p^0, m^0)$, $x^1 = \xi(p^1, m^1)$, *and* $x^0 \neq x^1$. *Assume that* $m^1 \geq \mu(p^1; p^0, m^0)$. *Then*

$$\langle p^0, x^1 \rangle > \langle p^0, x^0 \rangle$$

Proof. Define

$$p(t) = (1-t)p^0 + tp^1,$$

$$m(t) = \mu(p(t); p^0, m^0),$$

and

$$x(t) = \xi(p(t), m(t)).$$

Clearly, $p(0) = p^0$, $p(1) = p^1$. Thus, $x(0) = \xi(p^0, \mu(p^0; p^0, m^0)) = \xi(p^0, m^0) = x^0$, and $x(1) = \xi(p^1, \mu(p^1; p^0, m^0))$.

Let

$$\phi(t) = \langle p^0, x(t) \rangle = \sum_{i=1}^{n} p_i^0 \xi_i(p(t), \mu(p(t); p^0, m^0)) \tag{4.7}$$

We differentiate with respect to t and use the chain rule to get:

$$\begin{aligned}\phi'(t) &= \sum_{i=1}^{n} p_i^0 \frac{d}{dt}\xi_i(p(t), \mu(p(t); p^0, m^0)) \\ &= \sum_{i=1}^{n} p_i^0 \sum_{j=1}^{n} (\frac{\partial \xi_i}{\partial p_j} + \frac{\partial \xi_i}{\partial m}\frac{\partial \mu}{\partial p_j})(p_j^1 - p_j^0) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} (\frac{\partial \xi_i}{\partial p_j} + \frac{\partial \xi_i}{\partial m}\xi_j)(p_j^1 - p_j^0)p_i^0\end{aligned} \tag{4.8}$$

By the budget exhaustion condition (B), we get

$$m(t) = \langle p(t), x(t)\rangle = \sum_{i=1}^{n} p_i(t)\xi_i(p(t), \mu(p(t); p^0, m^0))$$

Differentiating m on the left-hand side with respect to t, we get

$$\frac{d}{dt}m(t) = \sum_{i=1}^{n} \frac{\partial \mu}{\partial p_i}\frac{d}{dt}p(t) = \sum_{i=1}^{n} \xi_i(p(t), m(t))(p_i^1 - p_i^0)$$

Differentiating the right-hand side, we get

$$\sum_{i=1}^{n}\{(p_i^1 - p_i^0)\xi_i(p(t), \mu(p(t); p^0, m^0)) + p_i(t)\sum_{i=1}^{n}(\frac{\partial \xi_i}{\partial p_j} + \frac{\partial \xi_i}{\partial m}\xi_j)(p_j^1 - p_j^0)\}.$$

From the two expressions above, we obtain,

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(\frac{\partial \xi_i}{\partial p_j} + \frac{\partial \xi_i}{\partial m}\xi_j)(p_j^1 - p_j^0)p_i(t) = 0.$$

We subtract this equation from (4.8) to get

$$\begin{aligned}
\phi'(t) &= \sum_{i=1}^{n}\sum_{j=1}^{n}(\frac{\partial \xi_i}{\partial p_j} + \frac{\partial \xi_i}{\partial m}\xi_j)(p_j^1 - p_j^0)(p_i^0 - p_i(t)) \\
&= -t\sum_{i=1}^{n}\sum_{j=1}^{n}(\frac{\partial \xi_i}{\partial p_j} + \frac{\partial \xi_i}{\partial m}\xi_j)(p_i^1 - p_i^0)(p_j^1 - p_j^0) \\
&= -t\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{i,j}(p(t), m(t))(p_i^1 - p_i^0)(p_j^1 - p_j^0)
\end{aligned}$$

By the (NSD), we get $\phi'(t) \geq 0$, and thus $\phi(1) \geq \phi(0)$. In other words,

$$\left\langle p^0, x^1 \right\rangle \geq \left\langle p^0, x^0 \right\rangle = m^0.$$

∎

Lemma 4.5 *(Weak Axiom of Revealed Preference) Let* $x^0 = \xi(p^0, m^0)$, $x^1 = \xi(p^1, m^1)$. *If*

$$\left\langle p^0, x^0 \right\rangle \geq \left\langle p^0, x^1 \right\rangle, \ x^0 \neq x^1$$

then

$$\left\langle p^1, x^0 \right\rangle > \left\langle p^1, x^1 \right\rangle$$

Proof. Assume the two bundles x^0, x^1 are different, and that, given the price system p^0, $\langle p^0, x^0 \rangle \geq \langle p^0, x^1 \rangle$, i.e., it is "revealed" that x^0 is preferable to x^1 under this system. Applying the contrapositive of Lemma 4.4, we obtain

$$\mu(p^1; p^1, m^1) = m^1 < \mu(p^1; p^0, m^0)$$

By Lemma 4.3, we obtain

$$\mu(p^0; p^1, m^1) < \mu(p^0; p^0, m^0) = m^0.$$

We apply Lemma 4.4 again to this inequality, and interchange the roles of 0 and 1, to conclude

$$\left\langle p^1, x^0 \right\rangle > \left\langle p^1, x^1 \right\rangle$$

∎

Lemma 4.6 *For any $x \in X$, the set $\{(p, m) : \xi(p, m) = x\}$ is convex.*

Proof. We make the following definitions:

$$\bar{x} = \xi(p^0, m^0) = \xi(p^1, m^1), \tag{4.9}$$

$$p(t) = p^0 + t(p^1 - p^0) \tag{4.10}$$

$$m(t) = m^0 + t(m^1 - m^0) \tag{4.11}$$

$$x(t) = \xi(p(t), m(t))$$

Thus

$$m(t) = \langle p(t), x(t) \rangle$$

We get from (4.9) that $m^0 = \langle p^0, \bar{x} \rangle$, and $m^1 = \langle p^1, \bar{x} \rangle$. We can also see that

$$\langle p(t), \bar{x} \rangle = \left\langle p^0 + t(p^1 - p^0), \bar{x} \right\rangle = m^0 + t(m^1 - m^0) = m(t) = \langle p(t), x(t) \rangle \tag{4.12}$$

Our objective now is to show that $x(t) = \bar{x}$, for all $t \in [0,1]$. We proceed by contradiction. Suppose there is a $t \in (0,1)$ for which $x(t) \neq \bar{x}$. From (4.12) and the Weak Axiom of Revealed Preference 4.5, we have

$$\left\langle p^0, \bar{x} \right\rangle < \left\langle p^0, x(t) \right\rangle \tag{4.13}$$

and

$$\left\langle p^1, \bar{x} \right\rangle < \left\langle p^1, x(t) \right\rangle \tag{4.14}$$

We now perform $(1-t)(4.13) + t(4.14)$:

$$\langle p(t), \bar{x} \rangle < \langle p(t), x(t) \rangle$$

contradicting (4.12). We conclude that $x(t) = \bar{x}$ is true for all $t \in [0,1]$, hence the set $\{(p,m) : \xi(p,m) = \bar{x}\}$ is convex. ∎

Lemma 4.7 *If* $\xi(p^0, m^0) = \xi(p^1, m^1)$, *then*

$$\mu(p; p^0, m^0) = \mu(p; p^1, m^1) \text{ for all } p$$

Proof. We use the definitions (4.10) and (4.11) for $p(t)$ and $m(t)$. We know from Lemma 4.6 that

$$\xi(p(t), m(t)) = \xi(p^0, m^0) = \bar{x}, \;\; t \in [0,1]$$

From (4.12), we have

$$m(t) = \langle \bar{x}, p(t) \rangle$$

Differentiating with respect to t, we get

$$\frac{dm(t)}{dt} = \bar{x}\frac{dp(t)}{dt} = \xi(p(t), m(t))\frac{dp(t)}{dt},\ t \in [0,1]. \tag{4.15}$$

On the other hand,

$$\frac{d}{dt}(\mu(p(t); p^0, m^0)) = \sum_{i=1}^{n} \frac{\partial \mu}{\partial p_i}\frac{dp_i}{dt} \tag{4.16}$$

For μ solution of (4.6),

$$\frac{\partial \mu(p)}{\partial p_i} = \xi_i(p, \mu(p)) \tag{4.17}$$

Combining (4.16) and (4.17), we obtain

$$\frac{dm}{dt} = \frac{d}{dt}\mu(p(t), p^0, m^0) \tag{4.18}$$

So μ and m differ by a constant. For $t = 0$, we get that the constant is zero. Thus, we have

$$\mu(p(t); p^0, m^0) = m(t),\ t \in [0,1]$$

which yields, when $t = 1$,

$$\mu(p^1; p^0, m^0) = m^1.$$

Applying Lemma 4.2 one more time, we obtain

$$\mu(p;p^0,m^0) = \mu(p;p^1,m^1), \text{ for all } p.$$

∎

Lemma 4.8 *Let the demand function ξ satisfy (A), (B), (D), (E), (S), and (NSD). For any p^*, $U_{p^*}(x) = \mu(p^*;p,m)$ is a single-valued function defined on the range X of the demand function ξ, and for any p and m,*

$$U_{p^*}(\xi(p,m)) > U_{p^*}(x)$$

for all $x \in X$, subject to $\langle p,x \rangle \leq m$, $x \neq \xi(p,m)$.

Proof. The utility function U_{p^*} is single-valued from Lemmas 4.1, 4.2, and 4.7. Now let

$$x^0 = \xi(p^0,m^0),\ \left\langle p^0,x^0 \right\rangle = m^0$$

$$x^1 = \xi(p^1,m^1),\ \left\langle p^1,x^1 \right\rangle = m^1$$

and

$$\left\langle p^0,x^1 \right\rangle \leq m^0,\ x^0 \neq x^1.$$

Using the contrapositive of Lemma 4.4,

$$m^1 < \mu(p^1;p^0,m^0).$$

But $m^1 = \mu(p^1; p^1, m^1)$ from Lemma 4.1, thus,

$$\mu(p^1; p^1, m^1) < \mu(p^1; p^0, m^0)$$

By Lemma 4.3,

$$\mu(p; p^1, m^1) < \mu(p; p^0, m^0), \text{ for all } p.$$

In particular, for $p = p^*$, we obtain

$$U_{p^*}(x^0) > U_{p^*}(x^1).$$

∎

Lemma 4.8 concludes the proof of Theorem 4.2.

4.6 Some Properties of the Utility Function u

According to the discussion of the previous sections, the integrability problem has a solution. That is, given a demand function ξ that satisfies some properties, there exists a utility function u such that ξ maximizes u on the budget set. It is clear that u is not uniquely defined since it depends on the choice of p^*. We will now show, however, that the corresponding preference relation does not depend on the choice of p^*.

Definition 4.3 *Two functions $f, g : A \to \mathbb{R}$, are said to induce the same ordering* *on* A *if: for all* $a', a'' \in A$,

$$f(a') > f(a'') \Leftrightarrow g(a') > g(a''.)$$

Note that whenever f, g induce the same ordering, one has

$$f(a') = f(a'') \Leftrightarrow g(a') = g(a'') \tag{4.19}$$

Proposition 4.1 *If two real-valued functions* f, g *induce the same ordering on a set* A, *then one of them is the composition of a strictly increasing function with the other.*

Proof. If the functions f, g, induce the same ordering, we would like to determine

if, say, $g = h \circ f$, where h is a strictly increasing function. Let $f(A)$ and $g(A)$ denote the ranges of f and g, respectively. Define a function $h : f(A) \to g(A)$, as follows: For every $y \in f(A)$, there exists at least one element $x \in A$ such that $y = f(x)$. We set $h(y) = g(x)$. Then obviously $h(f(x)) = g(x)$.

- h is well-defined. Consider $x_1,\ x_2 \in A$. If $f(x_1) = f(x_2) = y$, then, from (4.19), we obtain $g(x_1) = h(f(x_1)) = h(f(x_2)) = g(x_2)$. $h(y)$ is thus well-defined.

- h is strictly increasing. Let $y_1,\ y_2 \in f(A)$, with $y_1 < y_2$. Clearly, $y_1 = f(x_1)$, $y_2 = f(x_2)$, for some $x_1,\ x_2 \in A$. Now, $y_1 < y_2 \Rightarrow f(x_1) < f(x_2)$, and since f, g induce the same ordering on A, we get $g(x_1) < g(x_2)$. But by definition,

$g(x_1) = h(y_1) < h(y_2) = g(x_2)$. In other words, h is strictly increasing.

∎

Theorem 4.9 *For any price vectors p^*, $p^{**} \in \mathbb{R}^n_{++}$, the functions U_{p^*} and $U_{p^{**}}$ induce the same ordering on the range X of $\xi(p, m)$.*

Proof. The result follows from the definitions of U_{p^*} and $U_{p^{**}}$ and 4.3. ∎

We will now examine other properties of the constructed utility, such as monotonicity, quasiconcavity, and upper semicontinuity.

Definition 4.4 *The indifference sets of $f : A \to \mathbb{R}$ are said to be strictly convex toward the origin if, for any $a^0 \in A$, there exists a $q \in \mathbb{R}^n_{++}$ such that $\langle q, a^0 \rangle < \langle q, a \rangle$ for all $a \in A$ satisfying $f(a) = f(a^0)$, $a \neq a^0$.*

Theorem 4.10 *For any $p^* \in \mathbb{R}^n_{++}$, $U_{p^*}(x)$ is monotone increasing with respect to the vectorial ordering of X, and the indifference sets of U_{p^*} are strictly convex toward the origin.*

Proof. Lemma 4.8 guarantees the monotonicity of U_{p^*}. As for the strict convexity toward the origin of the indifference sets of U_{p^*}, it suffices to prove that for any $(p^0, m^0) \in \mathbb{R}^n_{++} \times \mathbb{R}_+$,

$$\left\langle p^0, x \right\rangle > \left\langle p^0, x^0 \right\rangle,$$

for all $x \in X$ satisfying $U_{p^*}(x) = U_{p^*}(x^0)$. Now let $x^0 = \xi(p^0, m^0)$, and let $x^1 = \xi(p^1, m^1)$ be a bundle satisfying

$$U_{p^*}(x^1) = U_{p^*}(x^0)$$

Consider $\phi(t)$ defined by (4.7). We know from the proof of Lemma 4.4 that $\phi(0) < \phi(1)$ whenever $x^0 \neq x^1$, i.e., $\langle p^0, x \rangle > \langle p^0, x^0 \rangle$. ∎

Theorem 4.11 *Under the assumptions of Theorem 4.2, $U_{p^*}(x)$ is upper semicontinuous in x, for every p^*.*

Proof. We prove that the lower level sets $\{x : x \in X, U_{p^*}(x) < \alpha\}$ are open for every $\alpha \in \mathbb{R}$. In other words, we prove that for every $x^1 \in X$ such that $U_{p^*}(x^1) < \alpha$, there exists a δ-neighborhood around x^1, $\|x - x^1\| < \delta$, such that for every vector x of which we have

$$U_{p^*}(x) < \alpha.$$

Let $x^1 = \xi(p^1, m^1)$. Now, $U_{p^*}(x^1) = \mu(p^*; p^1, m^1) < \alpha$. Due to the continuity of $\mu(p^*; p^1, m)$ with respect to m, there exists an $\varepsilon > 0$ such that

$$\mu(p^*; p^1, m^1 + \varepsilon) < \alpha. \tag{4.20}$$

Additionally, $m^1 = \langle p^1, x^1 \rangle$, and the function $\langle p^1, x \rangle$ is continuous with respect to x. Therefore, there exists a $\delta > 0$ such that

$$\left\langle p^1, x \right\rangle < m^1 + \varepsilon \tag{4.21}$$

for all x satisfying $\|x - x^1\| < \delta$. Now let $x^\varepsilon = \xi(p^1, m^1 + \varepsilon)$. Applying Lemma 4.8

and (4.21) above, we obtain

$$U_{p^*}(x) < U_{p^*}(x^\varepsilon),$$

for all x satisfying $\|x - x^1\| < \delta$. By the way we have constructed x^ε, (4.20) may be written as

$$U_{p^*}(x^\varepsilon) < \alpha.$$

This implies that for all x satisfying $\|x - x^1\| < \delta$, $U_{p^*}(x) < \alpha$, and thus the sets $\{x : x \in X, U_{p^*}(x) < \alpha\}$ are open for all $\alpha \in \mathbb{R}$. ▮

The last theorem, whose proof we omit, examines the continuity of the constructed utility.

Theorem 4.12 *The utility function $U_{p^*}(x)$ is lower semicontinuous in $x \in X$ for every choice of p^* if, in addition to the assumptions (A), (B), (D), and (E) of Theorem 4.2, we also have any one of the following three conditions*

(a) If the sequence $\{p^k\}$, $k \in \mathbb{N}$, converges to some p^0 such that $p^0 \neq 0$, $p^0 \ngtr 0$, then the sequence $\{x^k\}$, where $x^k = \xi\left(p^k, \mu(p^k; p, m)\right)$, is unbounded for every choice of (p, m), $p > 0$, $m \geq 0$;

(b) There exists a single-valued inverse demand function; that is

$$\xi(p', m') = \xi(p'', m'')$$

implies $m' = m'' = 0$, or $\frac{p'}{m'} = \frac{p''}{m''}$;

(c) $\xi(p, m)$ *is Lipschitzian with respect to the boundary; that is, for every* p^0 *such that* $p^0 \geqq 0$, $p^0 \neq 0$, $p^0 \not> 0$, *there exist positive numbers* ε *and* $K = K_{\varepsilon, p^0}$ *such that*

$$\|\xi(p, m') - \xi(p, m'')\| < K\,|m' - m''|$$

for all $m',\ m'' \geq 0$ *and all* $p > 0$, $\|p - p^0\| < \varepsilon$.

CHAPTER 5

EXTENSION OF THE DOMAIN OF THE UTILITY FUNCTION

A drawback of the work of Hurwicz and Uzawa that we presented in the previous chapter was that the constructed utility was not defined on the whole cone of commodity bundles, $\mathbb{R}^n_+$, but only on the range X of the demand function ξ. This drawback was remedied later by the work of Jackson [12], where it was shown that, under the same assumptions in [11], one can extend the domain of definition of u to the whole cone $\mathbb{R}^n_+$. In this chapter, we will present this approach.

The main theorem of this chapter is the following.

Theorem 5.1 *Assume that a function* $\xi : \mathbb{R}^n_{++} \times \mathbb{R}_+ \to \mathbb{R}^n_+$ *satisfies conditions (A), (B), (D), and (E), and that its associated Slutsky matrix* σ *satisfies the conditions (S) and (NSD). Then there exists an upper semicontinuous, quasi-concave, and increasing utility function* U *defined on* $\mathbb{R}^n_+$ *such that* $\xi(p, m)$ *is the unique maximizer of* $U(x)$ *over the set* $\{x \in \mathbb{R}^n_+ : \langle p, x\rangle \leq m\}$.

5.1 Procedure for Extending the Domain of the Utility Function

In order to prove Theorem 5.1, we consider expenditure functions and utility functions having specific properties. We then establish two theorems showing the existence of bijections between some classes of the above expenditure and utility functions. Afterwards, we utilize earlier results to define an expenditure function in terms of the income compensation function given by Definition 4.1. Finally, we use a bijection to map this expenditure function onto a utility function defined over the entire space of $\mathbb{R}^n_+$.

Consider the set of expenditure functions $\{\bar{e} : \mathbb{R}^n_{++} \times \mathbb{R}_+ \to \mathbb{R}_+\}$, each of which having at least one of the following properties:

(E1) $\bar{e}(p, \cdot)$ is continuous

(E2) $\bar{e}(p, u) = 0 \Leftrightarrow u = 0$

(E3) $\bar{e}(p, \cdot)$ is unbounded above

(E4) $\bar{e}(\cdot, u)$ is concave

(E5) $\bar{e}(\cdot, u)$ is positive homogeneous degree one

(E6) For each $x^0 \in \mathbb{R}^n_+, u \geq 0$, and $\varepsilon > 0$ such that $\langle p, x^0 \rangle \geq \bar{e}(p, u + \varepsilon)$ for all $p \in \mathbb{R}^n_{++}$, there exists a number $\delta \in (0, 1)$ such that $\langle p, \delta x^0 \rangle > \bar{e}(p, u)$ for all $p \in \mathbb{R}^n_{++}$.

(E7) $\bar{e}(p, \cdot)$ is non-decreasing

(E8) $\bar{e}(\cdot, u)$ is non-decreasing

Now consider the set of expenditure functions $\{\bar{U} : \mathbb{R}^n_+ \to \mathbb{R}_+\}$, each of which having at least one of the following properties:

(U1) $\bar{U}$ is increasing

(U2) $\bar{U}$ is continuous

(U3) $\bar{U}(0) = 0$

(U4) $\bar{U}$ is unbounded above

(U5) $\bar{U}$ is quasi-concave

(U6) $\bar{U}$ is upper semi-continuous

Additionally, define the following two functions as in [12]:

$$[\psi(\bar{U})](p, u) = \min\{\langle p, x\rangle : \bar{U}(x) \geq u\} \tag{5.1}$$

$$[\psi^{-1}(\bar{e})](x) = \max\{u : \langle p, x\rangle \geq \bar{e}(p, u),\ \ for\ all\ p\} \tag{5.2}$$

Theorem 5.2 *The function ψ, defined by (5.1), is a bijection, with inverse ψ^{-1}, defined by (5.2), between the class of expenditure functions having properties (E1)→(E5), (E7), and (E8), and the class of utility functions having properties (U1), and (U3)→(U6).*

Theorem 5.3 *The function ψ, defined by (5.1), is a bijection, with inverse ψ^{-1}, defined by (5.2), between the class of expenditure functions having properties (E1)→(E6), and the class of utility functions having properties (U1)→(U5).*

Now choose any $(p^0, m^0) \in \mathbb{R}^n_{++} \times \mathbb{R}_+$. Define an expenditure function

$$\bar{e}(p, u) = \mu(p; p^0, um^0), \tag{5.3}$$

Lemma 5.1 *Pick any $(p', m') \in \mathbb{R}^n_{++} \times \mathbb{R}_+$, and $p'' \in \mathbb{R}^n_{++}$. Then $\langle p, \xi(p'', \mu(p''; p', m'))\rangle \geq \mu(p; p', m')$ for all p.*

Proof. Choose any p^*, and define $x'' = \xi(p'', \mu(p''; p', m'))$, $x^* = \xi(p^*, \mu(p^*; p', m'))$. By Lemma 4.1, $\mu(p^*; p', m') = \mu(p^*; p^*, \mu(p^*; p', m'))$. By Lemma 4.2, $\mu(p''; p', m') = \mu(p''; p^*, \mu(p^*; p', m'))$. If $x'' \neq x^*$, then, by Lemma 4.4, $\langle p^*, x''\rangle > \langle p^*, x^*\rangle$. If $x'' = x^*$, then $\langle p^*, x''\rangle = \langle p^*, x^*\rangle$. Thus, $\langle p^*, x''\rangle \geq \langle p^*, x^*\rangle$. By the budget exhaustion condition (B), $\langle p^*, \xi(p'', \mu(p''; p', m'))\rangle \geq \mu(p^*; p', m')$. But the choice of p^* was arbitrary, thus the conclusion is true for all p. ∎

Using Lemmas 4.1, 4.2, 4.3, 4.4, 4.7, and 5.1, we can verify that the expenditure function $\bar{e}$ defined by (5.3) is increasing, and possesses properties (E1)→(E5), (E7), and (E8).

We now use Theorem 5.2 to define the utility function

$$\bar{U}(x) = \max\{u : \langle p, x\rangle \geq \mu(p; p^0, um^0) \ for\ all\ p\} \tag{5.4}$$

We can verify that this utility function possesses properties (U1), and (U3)→(U6). Finally, we show that the maximization of the utility function $\bar{U}$ gives rise to the demand function $\xi(p, m)$, according to the following lemma.

Lemma 5.2 *Suppose that $\xi(p, m)$ satisfies conditions (A), (B), (D), (E), (S), and (NSD). Then for any $(p^0, m^0) \in \mathbb{R}^n_{++} \times \mathbb{R}_+$, $\bar{U}(x)$, given by (5.4), is single-valued, defined on $\mathbb{R}^n_+$, and for any $(p, m) \in \mathbb{R}^n_{++} \times \mathbb{R}_+$, $\bar{U}(\xi(p, m)) > \bar{U}(x)$ for all $x \in \mathbb{R}^n_+$ subject to $\langle p, x\rangle \leq m$, and $x \neq \xi(p, m)$.*

Proof. The definition of $\bar{U}$ and Theorem 5.2 yield its single-valuedness.

Pick $(p', m') \in \mathbb{R}^n_{++} \times \mathbb{R}_+$, and let $x' = \xi(p', m')$. Choose $x^2 \neq x'$ such that $\langle p, x^2\rangle \leq m'$. Since $\bar{e}(p, \cdot)$ is continuous, unbounded, increasing, and $\bar{e}(p, 0) = 0$, we can find a utility level u' such that $\bar{e}(p', u') = m'$.

By the budget exhaustion condition (B), $\langle p', x'\rangle = \bar{e}(p', u')$. By Lemma 5.1, $\bar{U}(x') = u'$. Since $\langle p', x^2\rangle \leq \bar{e}(p', u')$, we have $\bar{U}(x^2) \leq u' = \bar{U}(x')$. Suppose the equality holds, i.e., $\bar{U}(x^2) = \bar{U}(x')$.

Now since $x' \neq x^2$, $\bar{U}(x^2) = u'$, and $\langle p', x^2\rangle \geq \bar{e}(p', u')$, then $\langle p', x^2\rangle = \bar{e}(p', u') = \langle p', x'\rangle$. It follows that there is an $i \in \{1, 2, \ldots, n\}$ such that $x^2_i < x'_i$.

Let $p^\gamma = p' + \gamma e_i$, where e_i is the *ith* component of the unit basis vector. Then $\langle p^\gamma, x'\rangle > \langle p^\gamma, x^2\rangle \geq \bar{e}(p^\gamma, u')$ for all $\gamma > 0$.

Let $x^\gamma = \xi(p^\gamma, e(p^\gamma, u'))$. By the condition (D), ξ is differentiable and continuous in (p, m), and since $\bar{e}(\cdot, u)$ is concave and thus continuous, it follows that $\xi(p, \bar{e}(p, u'))$ is continuous in p. Let $\varepsilon = \langle e_i, x' - x^2\rangle$. We can then find δ such that if $\|p^\gamma - p'\| < \delta$, then $\|x^\gamma - x'\| < \varepsilon$.

Therefore, if $\|p^\gamma - p'\| < \delta$, then $\langle e_i, x' - x^\gamma \rangle < \langle e_i, x' - x^2 \rangle$ and so $\langle e_i, x^\gamma \rangle > \langle e_i, x^2 \rangle$. By Lemma 5.1, $\langle p', x^\gamma \rangle \geq \langle p', x^2 \rangle = \bar{e}(p', u')$, from which we get $\langle p^\gamma, x^\gamma \rangle \geq \langle p^\gamma, x^2 \rangle$, which contradicts the fact that $\langle p^\gamma, x^2 \rangle \geq \bar{e}(p^\gamma, u') = \langle p^\gamma, x^\gamma \rangle$. Hence, $\bar{U}(x') > \bar{U}(x^2)$. ∎

Definition 5.1 *We say that a function ξ is strongly non-inferior if*

- $\xi(p, m^1) \geq \xi(p, m^2)$ *for all* $p \in \mathbb{R}^n_{++}$ *and* $m^1 > m^2$, *and*
- *whenever* $\xi_i(p, m^2) > 0$ *and* $\xi(p, m^2) \not> 0$, *we have* $\xi_i(p, m^1) \geq \xi_i(p, m^2)$.

Theorem 5.4 *Let $\xi(p, m)$ satisfy conditions (A), (B), (D), (E), (S), and (NSD). In addition, let ξ be strongly non-inferior. If the range of $\xi(p, m)$ is $\mathbb{R}^n_+$, then there exists a continuous, quasi-concave, and increasing utility function U defined on $\mathbb{R}^n_+$ such that $\xi(p, m)$ is the unique maximizer of U over the set $\{x \in \mathbb{R}^n_+ : \langle p, x \rangle \leq m\}$.*

We have shown in Theorem 4.12 that if a function ξ satisfies the conditions (A), (B), (D), (E), (S), and (NSD), in addition to any one of the conditions (a), (b), or (c), then the utility function u is continuous and defined on the range X of the demand function ξ. In the next theorem, we establish that either condition (a) or (c) suffices to guarantee that the utility function u is defined on all of the closed positive orthant $\mathbb{R}^n_+$.

Theorem 5.5 *Let $\xi(p, m)$ satisfy conditions (A), (B), (D), (E), (S), and (NSD). In addition, let ξ satisfy either condition (a) or (c). Then there exists a continuous, quasi-concave, and increasing utility function U defined on $\mathbb{R}^n_+$ such that $\xi(p, m)$ is the unique maximizer of U over the set $\{x \in \mathbb{R}^n_+ : \langle p, x \rangle \leq m\}$.*

Note that this theorem rules out demand functions where boundary points are demanded for some price-budget pairs (p, m).

Furthermore, condition (a) requires knowledge about the income compensation function μ.

We now define another condition that enables us to deal with the demand function ξ directly, without involving μ.

(C) For any sequence $\{(p_k, m_k)\} \subseteq \mathbb{R}^{n+1}_{++}$, with $(p_k, m_k) \to (p', m')$, such that $p' \neq 0$, $p' \not> 0$, and $\lim_{k\to\infty} \xi(p_k, m_k) = x'$. Either

- There is a set $P \subseteq \mathbb{R}^{n+1}_{++}$ and $\delta \in (0,1)$ such that P is closed, and for any $t \in [\delta, 1]$,

$$tx' = \xi(p, \langle p, tx' \rangle)$$

  for some $p \in P$, or

- given $\varepsilon > 0$, there exists a number K such that $\|\xi(p, m') - \xi(p, m'')\| < K\,|m' - m''|$ for $m',\ m'' \geq 0$, $m' \neq m''$, and p satisfying $\|p - p'\| < \varepsilon$.

Theorem 5.6 *Let $\xi(p, m)$ satisfy conditions (A), (B), (D), (E), (S), and (NSD). In addition, let ξ satisfy condition (C). Then there exists a continuous, quasi-concave, and increasing utility function U defined on $\mathbb{R}^n_+$ such that $\xi(p, m)$ is the unique maximizer of U over the set $\{x \in \mathbb{R}^n_+ : \langle p, x \rangle \leq m\}$.*

CHAPTER 6

ALTERNATIVE APPROACH

In this chapter, we take a different approach to the integrability problem. We present the approach by Hadjisavvas and Penot in [8]. The authors deal with Banach spaces in their paper. However, in order to maintain the uniformity of our treatment, we deal with the Euclidean space $\mathbb{R}^n$ only. Many of the proofs in this chapter are quite technical, therefore we will omit them.

6.1 An Interlude: The Theorem of Frobenius

The theorem of Frobenius asserts the existence of solutions for a system of PDEs. It is a generalization of the following theorem, known from advanced calculus.

Theorem 6.1 *Let $F : A \to \mathbb{R}^n$, $F = (f_1, \ldots, f_n)$, be a C^1 vector field, defined on an open, simply connected[1] subset $A \subseteq \mathbb{R}^n$. Then there exists a function $s : A \to \mathbb{R}$ such that $\nabla s(x) = F(x), x \in A$, if and only if the Jacobian JF of F is symmetric.*

[1] A set $A \subseteq \mathbb{R}^n$ is called simply connected if every simple closed curve can be continuously mapped into a point in the set. Loosely speaking, if A has no "holes".

That is,

$$\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i},$$

$i, j = 1, \ldots, n.$

In other words, this theorem gives a condition for the system of PDEs $\nabla s(x) = F(x), x \in A$ to have solution. The theorem of Frobenius treats the more general case of a system of PDEs of the form $\nabla s(x) = (F(x), s(x))$. We will need the theorem in the following form.

Theorem 6.2 *(Frobenius) Let $K \subseteq \mathbb{R}^n \times \mathbb{R}$ be open, and $F : K \to \mathbb{R}^n$ be a C^1 function. Suppose that, for all $(x, t) \in K$, the following condition is satisfied: the matrix $J_x F(x,t) + \frac{\partial F(x,t)}{\partial t}^T F(x,t)$ is symmetric. Then, for all $(\bar{x}, \bar{t}) \in K$, there exist some neighborhoods U and V of $\bar{x}$ and $\bar{t}$, respectively, and a unique C^1 function $s : U \times U \times V \to \mathbb{R}$ such that*

$$s(x, x, t) = t$$

and

$$\nabla_x s(x, x', t') = F(x, s(x, x', t'))$$

for all $(x, x', t') \in U \times U \times V$.

In the above theorem, $J_x F$ is the Jacobian of F with respect to x, and $\nabla_x s$ is the gradient of s with respect to x. The function $s(x, x', t')$ is a solution of the system $\nabla s(x) = F(x, s(x))$, depending on x and the initial condition: If we write $s(x, x', t') = s_{x', t'}(x)$, then $s_{x', t'}$ is a solution of the system that satisfies the initial condition $s_{x',t'}(x') = t'$. We also have a global version of the theorem, for which we need the notion of a starshaped set.

Definition 6.1 *We say that a subset $U \subseteq \mathbb{R}^n$ is starshaped with respect to $u \in U$ if, for all $u' \in U$, and $t \in [0,1]$, $(1-t)u + tu' \in U$.*

Clearly, U is convex if and only if it is starshaped with respect to all its elements.

Theorem 6.3 *Let K and F be as in the Frobenius theorem. Then, for every given $(\bar{x}, \bar{t}) \in K$, there exists a greatest open set $U_{\bar{x},\bar{t}}$ which is starshaped with respect to $\bar{x}$ and a unique C^1 map $s_{\bar{x},\bar{t}} : U_{\bar{x},\bar{t}} \to \mathbb{R}$ such that*

$$s_{\bar{x},\bar{t}}(\bar{x}) = \bar{t}$$

and

$$\nabla_x s_{\bar{x},\bar{t}}(x) = F(x, s_{\bar{x},\bar{t}}(x))$$

for all $x \in U_{\bar{x},\bar{t}}$.

6.2 Normalizations and Important Tools

Recall the Walrasian budget set: $W_m(p) = \{x : \langle p, x \rangle \leq m\}$. By Proposition 2.10, we know that the budget set remains the same upon division of both p and m by a constant. We use this fact and normalize $W_m(p)$ by dividing the price vectors and budget by m, thus obtaining the *normalized budget set*:

$$W(p) = \{x : \langle p, x \rangle \leq 1\}$$

In the context of the above normalization, we define the following notions:

- The *normalized demand correspondence*

$$X(p) = \{x \in W(p) : u(x) \geq u(y), \forall y \in W(p)\}$$

- The budget exhaustion condition (i.e., nonsatiation): for all $x \in X(p)$,

$$\langle p, x \rangle = 1$$

- The *normalized indirect utility function*

$$v(p) = \max\{u(x) : x \in \mathbb{R}^n_+, \langle p, x \rangle \leq 1\}$$

As we saw in Proposition 3.2, v is nonincreasing in p, as well as quasiconvex.

Definition 6.2 *We say that a set $K \subset \mathbb{R}^n$ is evenly convex if it can be written as an intersection of open half spaces.*

It is a consequence of the Hahn-Banach separation theorem that every open convex set, as well as every closed convex set, is evenly convex. However, a set such as $A = ([-1, 1] \times [-1, 1]) \backslash (\{1\} \times [0, 1])$ is convex, but not evenly convex.

Definition 6.3 *We say that a function* $f : \mathbb{R}^n \to \mathbb{R}$ *is evenly quasiconvex if its lower level sets are evenly convex. That is, for any* $\alpha \in \mathbb{R}$*, the lower level sets* $\{x \in \mathbb{R}^n : f(x) \leq \alpha\}$ *are intersections of open half spaces in* $\mathbb{R}^n$*.*

The following result is classic, and establishes a duality between the utility function u and the indirect utility function v.

Proposition 6.1 *Let* $v : \mathbb{R}^n_{++} \to \mathbb{R}$ *be evenly quasiconvex and nonincreasing. Then* v *is the indirect utility function of the quasiconcave utility function* $u : \mathbb{R}^n_{+} \to \mathbb{R} \cup \{-\infty\}$ *given by*

$$u(x) = \min\{v(p) : p \in \mathbb{R}^n_{++}, \langle p, x\rangle \leq 1\}$$

Moreover, if v *has no local minimizers, then the nonsatiation condition holds.*

Let $x \in X(p)$ be given. As a consequence of the definitions of v and $X(p)$, we have the following result.

Proposition 6.2 $v(p) = \min\{v(p') : p' \in \mathbb{R}^n_{++}, \langle p, x\rangle \leq 1\}$.

Proof. For $p' \in \mathbb{R}^n_{++}$, $\langle p, x\rangle \leq 1$, we have $x \in W(p)$, hence $v(p') \geq u(x) = v(p)$. ∎

Using the Karush-Kuhn Tucker conditions, we obtain that there exists a number $\mu \geq 0$ such that

$$\nabla v(p) = -\mu x. \tag{6.1}$$

Furthermore, if $\nabla v(p) \neq 0$, and the nonsatiation condition holds, then, multiplying both sides of (6.1) by p, we obtain

$$\mu = -\langle \nabla v(p), p \rangle \neq 0 \tag{6.2}$$

and therefore x is uniquely determined in $X(p)$ by

$$x = \frac{\nabla v(p)}{-\mu} = \frac{\nabla v(p)}{\langle \nabla v(p), p \rangle}$$

When we have the above situation, it is clear that μ and x could be treated as functions of p, i.e., $\mu(p) = -\langle \nabla v(p), p \rangle$, and $x(p) = \frac{\nabla v(p)}{\langle \nabla v(p), p \rangle}$. We therefore obtain

$$\nabla v(p) = -\mu(p)x(p) \tag{6.3}$$

If v is twice differentiable, it follows that $\mu(p)$ is differentiable.

From (6.2), we know that $\mu(p) \neq 0$, from which we see that $x(p) = \frac{\nabla v(p)}{-\mu(p)}$ is orthogonal at p to the *indifference price surface*

$$S(p) = \{p' \in \mathbb{R}^n : v(p') = v(p)\}.$$

From (6.3), we get

$$\nabla^2 v(p) = -x(p)^T \nabla \mu(p) - \mu(p) J x(p).$$

Thus, or every p_1, $p_2 \in x(p)^\perp = \{p' \in \mathbb{R}^n : \langle p', x(p) \rangle = 0\}$, we obtain

$$\left\langle p_2, \nabla^2 v(p) p_1 \right\rangle = -\mu(p) \left\langle p_2, Jx(p) p_1 \right\rangle = -\mu(p) \left\langle p_2, Jx(p) p_1 \right\rangle$$

Similarly, $\langle p_1, \nabla^2 v(p) p_2 \rangle = -\mu(p) \langle p_1, Jx(p) p_2 \rangle$.

We know that $\nabla^2 v(p)$ is symmetric, therefore $\langle p_2, \nabla^2 v(p) p_1 \rangle = \langle p_1, \nabla^2 v(p) p_2 \rangle$. We deduce that $Jx(p)$ is symmetric on the subspace $x(p)^\perp$, provided that $\nabla v(p) \neq 0$. Since $v(p)$ is quasiconvex, for all $p \in \mathbb{R}^n_{++}$, $\nabla^2 v(p)$ is positive semidefinite on $x(p)^\perp$. So $\langle q, \nabla^2 v(p) q \rangle = -\mu(p) \langle q, Jx(p) q \rangle$ for all $q \in x(p)^\perp$. When $\nabla v(p) \neq 0$, this yields that $Jx(p)$ is negative semidefinite on $x(p)^\perp$.

6.3 Assumptions

We will impose the following assumptions on the function $x : \mathbb{R}^n_{++} \to \mathbb{R}^n_+$:

(DN) Differentiability: x is C^1

(BN) Budget exhaustion (nonsatiation): $\langle p, x(p) \rangle = 1$

(SNSD) Symmetry and negative semidefiniteness: The restriction of $Jx(p)$ to the subspace $x(p)^\perp$ is symmetric and negative semidefinite

Basically, our objective is to establish that the function x, having the above properties, is a demand function maximizing some utility function $u : \mathbb{R}^n_+ \to \mathbb{R} \cup \{-\infty\}$. We are familiar with conditions (DN) and (BN), for they are the normalized versions of conditions (D) and (B), respectively. Further, it is not difficult to link condition (SNSD) to the symmetry and negative semidefiniteness of the Slutsky matrix. In order to see this, however, we recall John's [13] definition of the Slutsky matrix:

$$s(p) = Jx(p) - Jx(p)p^T x(p) \tag{6.4}$$

for all $p' \in \mathbb{R}^n$.

When conditions (DN), (BN), and (SNSD) hold, $s(p)$ possesses the following properties:

- $s(p)$ is symmetric if and only if the restriction of $Jx(p)$ to $x(p)^\perp$ is symmetric
- $s(p)$ is negative semidefinite if and only if the restriction of $Jx(p)$ to $x(p)^\perp$ is negative semidefinite

We now present the main theorem.

Theorem 6.4 *Let $x : \mathbb{R}^n_{++} \to \mathbb{R}^n_+$ be a function satisfying conditions (BN), (DN), and (SNSD). Then there exists a quasiconcave and upper semicontinuous utility function $u : \mathbb{R}^n_+ \to \mathbb{R}$ such that the nonsatiation condition holds, and, for all $p \in \mathbb{R}^n_{++}$, the normalized demand set $X(p)$ associated to u is $\{x(p)\}$. Moreover, the associated indirect utility function v is real-valued, differentiable, decreasing along rays, nonincreasing, and pseudoconvex.*

To prove the theorem, we present some notions and decompose the commodity and price spaces in a specific manner.

6.4 A Decomposition of $\mathbb{R}^n$

Let $e \in \mathbb{R}^n_{++}$ be the unit vector

$$e = \frac{1}{\sqrt{n}}(1, \ldots, 1)$$

and Q be the subspace

$$Q = e^{\perp} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=1}^{n} x_i = 0\}.$$

Then $\mathbb{R}^n$ can be written as

$$\mathbb{R}^n = Q \oplus \mathbb{R}e.$$

Using this decomposition, we can express every vector $x \in \mathbb{R}^n$ as $x = y + te$, with $y \in Q$ and $t \in \mathbb{R}$. For simplicity, we write $x = (y, t)$. In this way, if $x(p)$ is the demand function, we express p as $p = (q, r)$ and $x(p)$ as

$$x(p) = x(q, r) = (a(q, r), b(q, r)) = (a(p), b(p)) \in Q \times \mathbb{R}.$$

Observe that the component $b(p) > 0$, since $b(p) = \langle e, x(p) \rangle > 0$ due to the budget exhaustion condition (BN).

6.5 Recovering the Utility Function u

Our method for recovering the utility function u will involve the indirect utility function v. Essentially, we will first construct v, and then apply Proposition 6.1 to recover u. The procedure for doing so is as follows. Say we start with the price vector $p_0 = (q_0, r_0) \in \mathbb{R}^n_{++}$. We first determine the indifference price surface $S(p_0)$ as the graph of a C^1 function $s : Q_0 \to \mathbb{R}$, where $Q_0 \subset Q$ is open. We show that $0 \in Q_0$, so that the ray $(0, \infty)e$ intersects the graph of s. By prescribing a given value to v on $(0, \infty)e$, it is thus possible to define v on the graph of s, and ultimately on the whole space $\mathbb{R}^n_{++}$. The uniqueness of v will follow from the uniqueness of the solution s.

We show that this function v has good properties, and defines a utility function u for which $x(\cdot)$ is the demand function.

6.5.1 Construction of v

If v exists and is C^1, then $\nabla v(p) = -\mu(p)x(p)$ should hold, and $x(p)$ should be orthogonal to the graph of s at the point $p = (q, s(q))$, $q \in Q_0$. But $x(p) = x(q, s(q)) = (a(q, s(q)), b(q, s(q)))$, thus it is then natural to seek s as a solution of the total differential equation

$$\nabla s(q) = -\frac{a(q, s(q))}{b(q, s(q))}, \tag{6.5}$$

with $s(q_0) = r_0$.

In order to proceed with the proof, we define

$$r(q) = \inf\{t \in \mathbb{R} : q + te \in \mathbb{R}^n_{++}\}.$$

We also define the open convex cone K by

$$K = \{(q,r) \in Q \times \mathbb{R} : q + re \in \mathbb{R}^n_{++}\} = \{(q,r) \in Q \times \mathbb{R} : r > r(q)\}$$

and the map $F : K \to Q$ by

$$F(q,r) = -\frac{a(q,r)}{b(q,r)}.$$

A calculation shows that the symmetry condition (SNSD) implies the symmetry requirement of the Frobenius Theorem; consequently, (6.5) is locally solvable around any point $(q_0, r_0) \in \mathbb{R}^n_{++}$, and we obtain the following result.

Proposition 6.3 *For every $p_0 = q_0 + r_0 e \in \mathbb{R}^n_{++}$, (6.5) has a unique solution in a neighborhood of p_0.*

From now on, we mean by a solution of (6.5) the uniquely defined solution s with greatest starshaped domain with respect to q_0, according to Theorem 6.3.

Proposition 6.4 *If $s : U \to \mathbb{R}$ is the solution to (6.5) with initial condition $(q_0, r_0) \in K$, with largest starshaped domain with respect to q_0, then $0 \in U$.*

Given $(q,r) \in \mathbb{R}^n_{++}$, we denote by $s_{q,r}(\cdot)$ the solution to (6.5) issued from $(q_0, r_0) :=$

(q,r). Since F is C^1, it follows that s is C^1. By Proposition (6.4), $s_{q,r}(0)$ is defined for all $(q,r) \in \mathbb{R}^n_{++}$.

Given a smooth decreasing function $h : (0,\infty) \to \mathbb{R}$ satisfying $h'(t) < 0$ for all $t \in (0,\infty)$, and given $p = q + re \in \mathbb{R}^n_{++}$, we set

$$v(p) = h(s_{q,r}(0)). \tag{6.6}$$

Since, $s_{0,r}(0) = r$, for all $r > 0$ by construction, we get $v(re) = h(r)$ for all $r > 0$. The C^1 function v thus defined is constant on the graph of $s_{q,r}(\cdot)$, since for all q' in U, setting $s_{q,r}(q') = r'$ yields $s_{q',r'}(q') = r' = s_{q,r}(q')$. Thus, by uniqueness, $s_{q',r'}(0) = s_{q,r}(0)$.

Obviously, because h was arbitrarily chosen, v is not uniquely defined, even though s is. However, once the values of v along $(0,\infty)e$ are assigned, v is uniquely determined.

6.5.2 Some Properties of v

Because v is constant on the graph of s, and because $s_{q,r}(q) = r$ for every $(q,r) \in K$, we have

$$\nabla_q v(q,r) + \frac{\partial v(q,r)}{\partial r} \nabla_q s_{q,r}(q) = 0$$

or

$$\nabla_q v(q,r) - \frac{\partial v(q,r)}{\partial r} \frac{a(q,r)}{b(q,r)} = 0.$$

Setting

$$\mu(q,r) = -\frac{\frac{\partial v(q,r)}{\partial r}}{b(q,r)},$$

it follows that $(\nabla_q v(q,r), \frac{\partial v(q,r)}{\partial r}) = -\mu(q,r)(a(q,r), b(q,r))$ or

$$\nabla v(p) = -\mu(p)x(p). \tag{6.7}$$

Since the nonsatiation condition holds $\langle p, x(p)\rangle = 1$, we obtain

$$\mu(p) = -\langle p, \nabla v(p)\rangle .$$

One can show the following result.

Lemma 6.1 *For all $p \in \mathbb{R}^n_{++}$, one has $\nabla v(p) \neq 0, \mu(p) > 0$, and in fact $\langle e, \nabla v(p)\rangle < 0$.*

We now recall some notions of generalized convexity.

Definition 6.4 *A C^1 function $f : U \to \mathbb{R}$, where $U \subseteq \mathbb{R}^n$ is open and convex, is said to be pseudoconvex if, for all $u, v \in U$, one has $f(v) \geq f(u)$ whenever $\langle \nabla f(u), (v-u)\rangle \geq 0$.*

Definition 6.5 *An operator $T : U \to \mathbb{R}^n$ is said to be pseudomonotone if, for every $p', p'' \in U$, one has $\langle Tp', p' - p\rangle \geq 0$ whenever $\langle Tp, p' - p\rangle \geq 0$.*

The following characterization is well-known.

Proposition 6.5 *A function f is pseudoconvex if and only if ∇f is pseudomonotone.*

The following result is due to John [13].

Proposition 6.6 *If conditions (BN), (DN), and (SNSD) hold, then $-x(\cdot)$ is pseudomonotone.*

We are now ready to prove Theorem 6.4.

Proof. We define v by the construction (6.6). v in this case is real-valued, (6.7) holds, $\mu(p) > 0$ for all $p \in \mathbb{R}^n_{++}$, and $-x(\cdot)$ is pseudomonotone. The operator $\nabla v(\cdot) = -\mu(\cdot)x(\cdot)$ is therefore pseudomonotone. By Proposition 6.5, v is pseudoconvex.

Now, for all $p', p'' \in \mathbb{R}^n_{++}$,

$$\langle p', \nabla v(p)\rangle = -\mu(p)\,\langle p', x(p)\rangle < 0,$$

since $-\mu(p) < 0$ and $\langle p', x(p)\rangle > 0$. The map $t \longmapsto v(p + tp')$ is decreasing and $v(p + p') < v(p)$. In particular, v is decreasing along rays. If, moreover, $p \in \mathbb{R}^n_{++}$, $p' \in \mathbb{R}^n_{+}$, taking a sequence $\{p'_n\} \subseteq \mathbb{R}^n_{++}$, with $p'_n \to p'$, we get $v(p + p') \leq v(p)$ since v is continuous and is of class C^1.

We now define $u : \mathbb{R}^n_+ \to \mathbb{R} \cup \{-\infty\}$ as

$$u(x) = \inf\{v(p) : p \in \mathbb{R}^n_{++}, \langle p, x\rangle \leq 1\}. \tag{6.8}$$

By Proposition 6.1, v is the indirect utility function associated with u, i.e., for all

$p \in \mathbb{R}^n_{++}$,

$$v(p) = \max\{u(x) : x \in \mathbb{R}^n_+, \langle p, x \rangle \leq 1\}. \tag{6.9}$$

We now show that, given $p \in \mathbb{R}^n_{++}$, $u(x(p)) = v(p)$. Since $\langle p, x(p) \rangle = 1$, we get that $x(p)$ is a maximizer of u on $W(p)$, i.e., $x(p) \in X(p)$.

Further, for all $p' \in \mathbb{R}^n_{++}$ satisfying $\langle p', x(p) \rangle \leq 1$, we have $\langle p' - p, x(p) \rangle \leq 0$. Thus,

$$\langle p' - p, \nabla v(p) \rangle = -\mu(p) \langle p' - p, x(p) \rangle \geq 0.$$

Now, v is pseudoconvex, so we get $v(p') \geq v(p)$. (6.8) then shows $u(x(p)) \geq v(p)$, and, since the reverse inequality holds because v satisfies (6.9) and $x(p) \in W(p)$, we have

$$u(x(p)) = v(p).$$

Since v is pseudoconvex, it is also quasiconvex. Moreover, it is continuous and decreasing along rays, thus Proposition 6.1 shows that the nonsatiation condition holds. It follows by ([19], Lemma 1) that, since v is continuous, then u is uppersemicontinuous.

Now if $x' \in X(p)$, then $v(p) = u(x')$, thus by (6.8),

$$v(p) = \min\{v(p') : p' \in \mathbb{R}^n_{++}, \langle p', x' \rangle \leq 1\},$$

i.e., p is a minimizer of v subject to $\langle p, x' \rangle \leq 1$. Since v is smooth, there exists a number $\lambda \geq 0$ such that $\nabla v(p) = -\lambda x'$. Here, we use the fact that $x' \neq 0$ by the nonsatiation. Finally, using the facts $\nabla v(p) = -\mu(p)x(p)$, $\nabla v(p) = -\lambda x'$, and $\langle p, x'\rangle = \langle p, x(p)\rangle = 1$, we deduce that $x' = x(p)$, i.e., that $X(p) = \{x(p)\}$. ∎

Example 6.5 *Consider the commodity bundles space* $\mathbb{R}^n_+$, *and the price space* $\mathbb{R}^n_{++}$. *Take* $e = \frac{1}{\sqrt{n}}(1, 1, \ldots, 1) \in \mathbb{R}^n$. *It is easy to check that*

$$x(p) = \frac{e}{\langle e, p\rangle}$$

$p \in \mathbb{R}^n_{++}$, *is a demand map satisfying conditions (DN), (BN), and (SNSD). We also have*

$$x(p) = x(q, r) = (a(q, r), b(q, r)) = (0, \frac{1}{\langle p, e\rangle}).$$

Thus, the solution to (6.5) is $s(\cdot, q, r) = r$. *Taking* $h(r) = \frac{1}{r}$ *for* $r \in (0, \infty)$, *we get* $v(p) = \frac{1}{\langle p,e\rangle}$. *It can be shown, moreover, that the utility function* u *recovered from* v *is given by* $u(x) = \max\{t \in \mathbb{R}_+ : te \leq x\}$, $x \in \mathbb{R}^n_+$. *This yields:* $u(x_1, x_2, \ldots, x_n) = \sqrt{n}\min(x_1, x_2, \ldots, x_n)$, *which is a nonsmooth utility function. In fact, for all* $p \in \mathbb{R}^n_{++}$, *we have* $u(x(p)) = \frac{1}{\langle p,e\rangle}$, *and if* $u(x) > \frac{1}{\langle p,e\rangle}$, *we have* $x_i > \frac{c_i}{\langle p,e\rangle}$ *for all* $i = 1, 2, \ldots, n$, *thus* $\langle p, x\rangle > (p_1 + \ldots + p_n)/\langle p, e\rangle = 1$ *and* $x \notin W(p)$.

So for all $x \in W(p)$, *we have* $u(x) \leq \frac{1}{\langle p,e\rangle}$ *and* $v(p) = \frac{1}{\langle p,e\rangle}$.

This is the famous Leontieff function.

CONCLUSION

There are various approaches to study and understand the integrability problem, from which we presented two. We discussed conditions under which the problem has a solution, as was proposed by Hurwicz and Uzawa. But as it turned out, their constrcution of the utility function contained a shortcoming, which Jackson later remedied. We then explored the approach by Hadjisavvas and Penot. The authors, albeit utilized theorems similar to those used by Hurwicz and Uzawa, took a radically different approach to the problem and succeeded in recovering a utility function in a Banach space setting. To maintain the uniformity of the discussion, however, we restricted the treatment to the Euclidean space $\mathbb{R}^n$ only. The significance of the integrability problem in mathematical economics stems from the fact that knowing the preferences and tastes of consumers enables producers to allocate their resources efficiently and therefore to maximize their profit. This, in addition to maximizing the utility of consumers, yield an overall improved economy.

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Vitae

- Name: AbdulKhaliq SaifAlNasr
- Nationality: Egyptian
- Date of Birth: 20/June/1991
- Email: abdulkaliq_alnasr@live.com
- Permenant Address: Riyadh, Saudi Arabia
- Bachelor's of Science in Electrical Engineering with a minor in Mathematics from KFUPM
- Master's of Science in Mathematics from KFUPM