ON THE NONEXISTENCE OF STRONGLY REGULAR GRAPHS WITH PARAMETERS SRG(76, 21, 2, 7)

BY

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عمادة الدراسات العليا
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To my parents and my wife

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Since it is said that as it was in the beginning so shall it be in the end, I want to conclude my acknowledgement and appreciations by giving all Thanks and Praise to Almighty Allah, The Most Exalted, The Most High.

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G, H	graphs
I	incidence relation
Ω	adjacency algebra
$\mathcal{L}$	set of lines
$D_i(u)$	vertices at distance i from u in G
$G_i(u)$	induced subgraph by $D_i(u)$
$V(G)$	vertex set of G
$E(G)$	edge set of G
$G(u)$	neighborhood of u in G
$L(G)$	line graph of G
k	valency of a regular graph
d	diameter of G
$d(u, v)$	distance between u and v
g	girth of G
$\overline{G}$	complement of G
G/P	quotient graph of G (with respect to P)
K_n	complete graph with n vertices
$K_{n,n}$	complete bipartite graph with n vertices in each part
C_n	cycle with length n
$\iota(G)$	intersection array of a distance-regular graph$G$
π	partition of $V(G)$
a_i, b_i, c_i	intersection numbers
DRG	distance-regular graph
SRG	strongly-regular graph
$srg(n, k, \lambda, \mu)$	parameters of an SRG

ABSTRACT

In this thesis, we worked on strongly regular graphs. Our aim in this work is to apply the unit vector representation of distance regular graphs to the nonexistence of $G = srg(76, 21, 2, 7)$. We gave some introduction to the distance regular graph paradigm and also presented very important results on the feasibility of intersection arrays of distance regular graphs. We also discuss the particular subclass of distance regular graphs which is of interest to us, namely the strongly regular graphs. We discuss the feasibility conditions for the parameters of strongly regular graphs and also give the formulas used to obtain the spectrum of strongly regular graphs. We then discuss the vector representation of distance regular graphs. This vector representation has the interesting property that the angle between any two vector images depend solely on the distance in G between the vertices preimage. We then construct unit vector representation for G, via the sequence of cosines and examined the nonexistence of G.

ملخص البحث

يكمن الموضوع الرئيس لهذه الرسالة في دراسة الرسوم قوية الانتظام. هدفنا في هذا البحث هو تطبيق تمثيل متجه الوحدة للرسوم منتظمة المسافات على الرسم قوي الانتظام $G = srg(76, 21, 2, 7)$. اعطينا مقدمة عن الرسوم المنتظمة المسافات، كما قدمنا نتائج مهمة جدا عن القوائم العددية الممكنة لتمثيل هذه الرسوم. كما قدمنا بتقديم الرسوم قوية الانتظام والتي هي من صميم اهتمامنا في هذه الرسالة بشكل فيه نوع من التفصيل، وهي بالأساس نوع فرعي من الرسوم المنتظمة المسافات. بعدها بينّا الشروط اللازمة للقوائم العددية لكي تمثل الرسوم قوية الانتظام. كما أعطينا الصيغ اللازمة للحصول على طيف هذه الرسوم. تمت بعدها مناقشة تمثيل متجه الوحدة للرسوم منتظمة المسافات بعمومها. هذا المتجه يملك خاصية مهمة ألا وهي أن الزاوية بين المتجهات تعتمد فقط علي المسافة بين الرؤوس الممثلة لهذه المتجهات في الرسم قوي الانتظام G .بعدها قمنا ببناء متجهات الوحدة الممثلة للرسم ج عبر سلسلة من جيوب التمام ومن ثم فحصنا عدم الوجود للرسم G .

1

Introduction

"The theory of graphs is one of the few fields of mathematics with a definite birth date." [...] Early geometry concerned distance, lengths, angles, areas and volumes, which were used for surveying, construction and astronomy. While geometry dealt with magnitudes, the German mathematician Gottfried Leibniz introduced another branch of geometry called the geometry of position. This branch of geometry did not deal with measurements and calculations, but rather with the determination of position and its properties. The famous mathematician Leonhard Euler said it had not been determined what kinds of problems could be studied with the aid of the geometry of position but in 1736 he believed that he had found one, which would lead to the origin of graph theory. It is this event to which Oystein Ore was referring in his quote above.

– G. Chartrand, L. Lesnak, P. Zhang, *Graphs and Digraphs*

1.1 Organization of this thesis

In this thesis, we worked on the non-existence of a strongly regular graph $G = srg(76, 21, 2, 7)$ using the unit vector representation of distance regular graphs.

In the first chapter, we introduce the notions of graphs and adjacency matrix. Starting with basic definitions and some examples, then discuss the adjacency matrix and its spectrum. The eigenvectors of a graph are very important in its vector representation, so we shall also discuss them. We shall also discuss equitable partitions as a preparation towards the distance regular graphs. And we end the chapter with a discussion of incidence graphs which are obtained from incidence structures. This topic is of particular interest in this thesis because the $srg(76, 21, 2, 7)$ which is of concern to us is a point graph of the partial geometry $pg(3, 6, 1)$.

In chapter 2, the first section is on distance regular graphs while the second part discusses the notion of strongly regular graphs. The bases of existence or nonexistence of distance regular graphs is the feasibility of intersection arrays. Thus, we would basically be establishing the feasibility conditions of intersection arrays of distance regular graphs. We shall prove that the number $s_{hi}(u, v)$ depends only on the distance $d(u, v)$ in G between the vertices u and v. Then we would move on to discuss the strongly regular graphs and give some important results. We shall deduce the conditions for feasibility of parameters of strongly regular graphs, and obtain their eigenvalues together with their multiplicities from the parameters of the graph and briefly discuss the status of strongly regular graphs.

Chapter 3 encapsulates the main concept of vector representation. There, we shall discuss the vector representation of distance regular graphs. And we would also give very important results about the inner product between vectors in a vector representation of distance regular graphs and obtain the formula for the cosine sequence of a distance regular graph. And in chapter 4, we shall present the result we obtained via the unit vector representation of

distance regular graphs on the nonexistence of $srg(76, 21, 2, 7)$.

1.2 Graphs

Informally, the study of graphs can be viewed as a study of the modeling of pairwise relations between object(s).

Definition 1.2.1. ***Graph:*** *A graph G is a nonempty set V of objects called* ***vertices*** *(**vertex** in singular) together with a possibly empty set E of 2-element subsets of V called* ***edges****. In this thesis we shall only consider finite graphs i.e., the case where $V(G)$ and $E(G)$ are both finite.*

We shall take note of the following notions as they come up very often in our work:

- We say two vertices u and v are **adjacent** in G if uv is an edge in G i.e., $uv \in E(G)$. We denote adjacency of u and v by $u \sim v$. If we add direction to an edge uv, then it is called an **arc** and denoted (u, v). A **loop** in a graph is an edge from a vertex u back to itself.

- The **degree** of v is the number of vertices adjacent to v. A graph G is **regular** if every vertex of G has same degree.

- If u and v are adjacent, we say that u is a **neighbour** of v. The set of all neighbours of v is called the **neighbourhood** of v and is denoted by $N(v)$.

- For any two vertices u and v in G, a $u - v$ **walk** W in G is a sequence of vertices in G, beginning with u and ending at v such that consecutive vertices in W are adjacent in G. A **path** in G is a walk in G, in which no vertex is repeated.

- A **cycle** in G is a sequence of vertices in which consecutive vertices are connected by an edge, such that the first vertex is same as the last.

- The length of the shortest cycle in a graph G is called the **girth** of G and is denoted by g.

- The **distance** between any two vertices u and v in G denoted $d(u, v)$ is the 1 plus the number of intermediary vertices between u and v. The **diameter** $d(G)$ or simply d of a graph G is defined by $d = \max_{u,v \in G}\{d(u, v)\}$.

- The **order** of a graph is the number of its vertices while the **size** is its number of edges. A graph G of order n such that any vertex is adjacent to every other vertex is called a **complete graph** denoted by K_n.

- The **complement** of a graph G of order n is another graph $\overline{G}$ having vertex set $V(G)$ and edge set $E(\overline{G}) = E(K_n) \setminus E(G)$.

Definition 1.2.2. ***Subgraph:*** *A graph H is a subgraph of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.*

Definition 1.2.3. ***Directed graph:*** *We say a graph G is directed if the edges have directions.*

Remark 1.2.1. *A graph is called* ***simple*** *if it is not directed and contains no loop. Throughout this thesis, by the word graph, we shall be referring to simple graphs unless otherwise stated.*

Definition 1.2.4. ***Line Graph:*** *The line graph of a graph G is a graph denoted by $L(G)$ having $E(G)$ as its vertex set and two edges e_1 and e_2 of G are adjacent in $L(G)$ if and only if they have a common endpoint in G.*

Definition 1.2.5. ***Bipartite Graph:*** *A graph G is bipartite if $V(G)$ can be partitioned into two sets U and W such that any edge of G is only between a vertex in U and another vertex in W.*

Definition 1.2.6. *A* ***complete bipartite graph****, denoted $K_{m,n}$ has each vertex in one partition of m vertices adjacent to all the vertices in the other partition of n vertices.*

Definition 1.2.7. ***Isomorphic Graphs:*** *Two graphs G_1 and G_2 are said to be* ***isomorphic*** *if there exists a bijection $\alpha : V(G_1) \to V(G_2)$ such that any two vertices $u, v \in G_1$ are adjacent if and only if $\alpha(u)$ and $\alpha(v)$ are adjacent in G_2.*

Example 1.2.1. 1. K_4 is an example of a complete graph. It is 3-regular with girth $g = 3$.

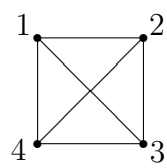

Figure 1.1: The complete graph K_4

2. $K_{2,3}$ is an example of a complete bipartite graph. The bipartition is $\{4, 5\}$ and $\{1, 2, 3\}$.

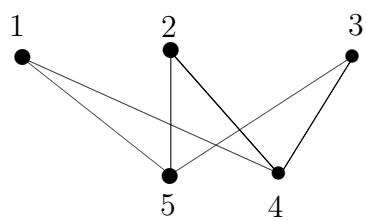

Figure 1.2: complete bipartite graph $K_{2,3}$

3. The Mckay graph is an example of a non-regular graph.

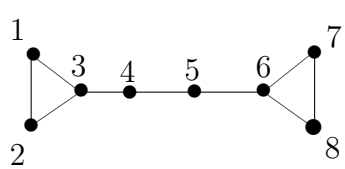

Figure 1.3: Mckay graph

4. The Petersen graph is an example of a 3-regular graph, having diameter 2 and girth $g = 5$.

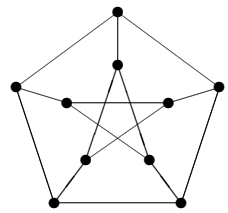

Figure 1.4: Petersen graph

Definition 1.2.8. ***The odd graphs*** $O_k,\ k \geq 2$***:*** *The odd graph O_k has one vertex for each of the $(k-1)$-element subsets of a $(2k-1)$-element set. Two vertices are connected by an edge if and only if the corresponding subsets are disjoint. O_2 is a triangle, while O_3 is the familiar Petersen graph.*

1.3 Adjacency Matrix and Adjacency Algebra

Definition 1.3.1. ***Adjacency Matrix:*** *The adjacency matrix of a graph G, usually denoted by $A(G)$, is a 01-matrix whose rows and columns are indexed by the vertices of G and the entries are given by*

$$a_{ij} = \begin{cases} 1 & if \ \ i \sim j \\ 0 & otherwise. \end{cases}$$

It follows directly from the definition that $A(G)$ is a real symmetric matrix.

Example 1.3.1. The adjacency matrix of a complete graph of order n is an $n \times n$ matrix having zeros on its diagonal and ones everywhere else.

Definition 1.3.2. *The* ***Spectrum*** *of a graph G is the set of eigenvalues of $A(G)$, together with their multiplicities. If $A(G)$ has s distinct eigenvalues λ_i such that $\lambda_0 > \lambda_1 > \lambda_2 > \ldots > \lambda_{s-1}$ with multiplicities $m(\lambda_0), m(\lambda_1), \ldots, m(\lambda_{s-1})$, then we shall write*

$$Spec(G) = \begin{pmatrix} \lambda_0 & \lambda_1 & \ldots & \lambda_{s-1} \\ m(\lambda_0) & m(\lambda_1) & \ldots & m(\lambda_{s-1}) \end{pmatrix}.$$

If λ is an eigenvalue of $A(G)$ we simply say it is an eigenvalue of G.

Example 1.3.2. (The spectrum of the Petersen graph O_3)

$$Spec(O_3) = \begin{pmatrix} 3 & 1 & -2 \\ 1 & 5 & 4 \end{pmatrix}.$$

Generally, regular graphs have some interesting spectral properties. There has been a lot of study dedicated to the spectrum of regular graphs. We shall see that the adjacency matrix of the strongly regular graphs (SRGs) have very interesting properties with respect to their spectrum.

We end this subsection with the definition of adjacency algebra, which is needed in our work. Suppose A is the adjacency matrix of a graph G. Then the set of polynomials in A, with complex coefficients, forms an algebra over the complex numbers, under the usual matrix operations. This algebra has finite dimension as a complex vector space.

Definition 1.3.3. *The* ***Adjacency Algebra*** *of a graph G denoted $\Omega(G)$ is the algebra of polynomials in the adjacency matrix $A(G)$.*

1.3.1 Eigenvectors

The eigenvectors of a graph forms another set of objects which play a very important role in the vector representation of DRGs, which is actually the theme of this thesis.

Let G be a graph on n vertices, $A := A(G)$ and $x = (x_1, x_2, \ldots, x_n)$ be an eigenvector of A corresponding to an eigenvalue λ. As A is a 01-matrix , it is easy to see that

$$Ax = \lambda x. \tag{1.1}$$

is equivalent to

$$\lambda x_i = \sum_{j \sim i} x_j \quad i = 1, 2, \ldots, n. \tag{1.2}$$

If we view x as a function from $V(G)$ to the set of real numbers, then equation (1.2) implies that λ multiplied by the value of this function on vertex i is equal to the sum of the values of x on the neighbors of i.

Conversely, any real-valued function on $V(G)$ which satisfies this condition can be seen to be an eigenvector of A. If λ is an eigenvalue of A with multiplicity m, we can go further and let U be an $n \times m$ matrix whose columns form a basis for the eigenspace corresponding to λ. It is not difficult to see that

$$AU = \lambda U. \tag{1.3}$$

Hence the rows of U give rise to a vector-valued function, u say on $V(G)$ such that

$$\lambda u(i) = \sum_{j \sim i} u_j \quad i = 1, 2, \ldots, n. \tag{1.4}$$

We again find that any vector-valued function satisfying this condition determines an eigenspace of A. Any such vector-valued function will be called a *representation* of the graph G.

Example 1.3.3. Let G be the complete graph K_4 with $V(K_4) = \{1, 2, 3, 4\}$.

The spectrum of K_4 is given by

$$Spec(K_4) = \begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix}.$$

We consider the eigenvalue $\lambda = -1$ having multiplicity 3. Applying equation (1.2), we

get the following system:

$$-x_1 = x_2 + x_3 + x_4$$

$$-x_2 = x_1 + x_3 + x_4$$

$$-x_3 = x_1 + x_2 + x_4$$

$$-x_4 = x_1 + x_2 + x_3.$$

These equations are equivalent to $x_1+x_2+x_3+x_4 = 0$, which has three linearly independent solution vectors. So we set x_2, x_3, x_4 as free variables. Choosing the respective variables we get

$[u_1, u_2, u_3] = \{[1, -1, 0, 0]^T, [-1, 0, 0, 1]^T, [-1, 0, 1, 0]^T\}$, which span the 3 dimension eigenspace for $\lambda = -1$.

Now, we have

$$U = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \text{ and } AU = \begin{pmatrix} -1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}.$$

We can easily verify that $AU = -U$. We shall see, in chapter 3, that this defines a vector representation of K_4 where $\lambda = -1$ is the corresponding eigenvalue.

Example 1.3.4. We recall that the spectrum of the Petersen graph is given by

$$Spec(O_3) = \begin{pmatrix} 3 & 1 & -2 \\ 1 & 5 & 4 \end{pmatrix}.$$

Let $u = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10})$ be an eigenvector of O_3 corresponding to $\lambda = 1$, then applying equation (1.2), we get ten linear equations that have solutions of the

form

$$(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}) =$$

$$(x_1, x_2, x_2 - x_7 - x_1, x_4, x_5, x_1 - x_2 - x_5, x_7, -x_7 - x_1 - x_4, x_4 + x_7 + x_1 - x_2 - x_5, x_5 - x_1 - x_4).$$

We can now construct five linearly independent vectors for O_3 corresponding to $\lambda = 1$ which span the eigenspace by choosing x_1, x_2, x_4, x_5, x_7 as free variables. In chapter 3 we shall revisit these eigenvectors when we discuss the vector representation of the Petersen Graph.

1.4 Quotient Graphs

We turn our attention to a combinatorial concept which is very useful in the study of DRGs, particularly SRGs. We use equitable partitions to obtain information about the eigenvalues and eigenvectors of a graph from a smaller 'quotient'. Let us start this section with definitions and examples that illustrate such a concept.

Definition 1.4.1. *Let G be a graph, a partition π of the vertex set $V(G)$ is a disjoint collection $\pi = \{C_1, \ldots, C_k\}$ of subsets of $V(G)$ such that $V(G) = \bigcup_{i=1}^{k} C_i$.*

Definition 1.4.2. *A partition is equitable if for all i and j, the number of neighbours which a vertex in C_i has in the cell C_j is independent of the choice of a vertex in C_i.*

Example 1.4.1. Consider the Mckay graph in example 1.1.1 (Figure 1.3), the partition $\pi = \{C_1 = \{3, 6\},\ C_2 = \{1, 2, 4, 5, 7, 8\}\}$ is an equitable partition.

Definition 1.4.3. *A distance partition $\pi(u)$ of a graph G is a collection of sets $D_i(u)$ of vertices at distance i from u in G.*

It is easy to check that $\pi(u)$ partitions $V(G)$ for any choice of a vertex u.

Example 1.4.2. Consider the Petersen graph in example 1.1.1 (Figure 1.4), fix the vertex 1 and construct the distance partition with respect to 1, we have $D_0(1) = \{1\},\ D_1(1) =$

$\{2, 5, 6\}$ and $D_2(1) = \{3, 4, 7, 8, 9, 10\}$. One can easily check that $\pi = \{D_0(1), D_1(1), D_2(1)\}$ forms an equitable partition.

Remark 1.4.1. *The induced subgraph of each cell C_i of an equitable partition is necessarily a regular graph, since each vertex in the cell C_i has the same number of neighbours in C_i.*

Let $\pi = \{C_1, \ldots, C_k\}$ be an equitable partition and c_{ij} be the number of edges between any fixed vertex in C_i and the vertices in C_j. The quotient graph of G with respect to π is denoted G/π.

Definition 1.4.4. *The **quotient graph** G/π is a directed graph having the cells of π as its vertices and c_{ij} arcs going from C_i to C_j.*

Remark 1.4.2. *It can be observed immediately that G/π in general can contain loops.*

The adjacency matrix of a quotient graph which we shall introduce shortly, is a very important matrix in the study of SRGs. This matrix would be revisited again in chapter 3 but purely from the linear algebraic perspective. It makes sense for us to reiterate that we actually introduced this section in order to give a feel of some blend between our work and the combinatorial structure. In other words, to see how we can simplify some concepts of SRGs using combinatorics.

We denote the adjacency matrix of the quotient graph of G with respect to π by $A(G/\pi)$ and $[A(G/\pi)]_{ij} = c_{ij}$.

Example 1.4.3. The quotient graph of the Petersen graph with respect to distance partition π, denoted O_3/π, is given below together with its adjacency matrix.

$$A(O_3/\pi) = \begin{pmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{pmatrix}.$$

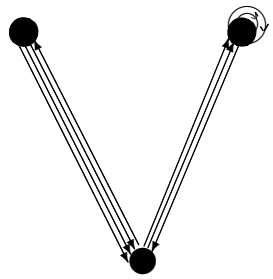

Figure 1.5: Quotient of Petersen Graph

Example 1.4.4. The quotient of the Mckay graph with respect to the equitable partition in example 1.3.1 is given below together with its adjacency matrix.

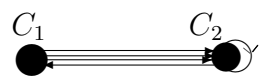

Figure 1.6: Quotient of Mckay Graph

$$A(G/\pi) = \begin{pmatrix} 1 & 1 \\ 3 & 0 \end{pmatrix}.$$

Definition 1.4.5. *The characteristic matrix $P = P(\pi)$ of a partition $\pi = \{C_1, ldots, C_k\}$ of a set of n elements $\{u_1, \ldots, u_n\}$ is an $n \times k$ matrix defined by*

$$[P(\pi)]_{ij} = \begin{cases} 1 & \text{if } u_i \in C_j \\ 0 & otherwise \end{cases}.$$

The following lemma provides information about the eigenvalues and eigenvectors of the adjacency matrix $A(G)$ from the quotient matrix $A(G/\pi)$, where π is an equitable partition of the vertex set of G.

Lemma 1.4.1 ([4], p. 78)**.** *Let π be an equitable partition of a graph G with c cells. Assume $P = P(\pi)$, $A = A(G)$ and $B = A(G/\pi)$. We have:*

(a) If $Bx = \lambda x$ then $APx = \lambda Px$.

(b) If $Ay = \lambda y$ then $y^T PB = \lambda y^T P$.

(c) The characteristic polynomial of B divides the characteristic polynomial of A.

Remark 1.4.3.

- *From (a), we see that the eigenvectors of the quotient graph can be 'lifted' to an eigenvector of G. In other words, if x is an eigenvector of the quotient, then Px is an eigenvector of G*

- *From (b) if y is an eigenvector of A, then $y^T P$ is a left eigenvector of B if and only if it is not zero. While the statement of (c) is self explanatory.*

Example 1.4.5. Consider the distance partition of example 1.3.2 $\pi = \{D_0(1), D_1(1), D_2(1)\}$, then

$$P(\pi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

and $x = [3, 1, -1]^T$ is an eigenvector for $A(O_3/\pi)$ with eigenvalue $\lambda = 1$. Moreover, the vector $Px = [3, 1, -1, -1, 1, 1, -1, -1, -1, -1]^T$ is an eigenvector for the Petersen graph corresponding to the eigenvalue $\lambda = 1$, which satisfies the vector form obtained in example 1.2.4.

1.5 Incidence Graphs

We shall introduce the notion of incidence structures and their corresponding incidence graphs. These structures are useful in the study of DRGs and in particular of SRGs. When considering some feasible parameters of an SRG, we sometimes prove that they represent an incidence graph and then use the theory of incidence structure to determine the existence or nonexistence.

Definition 1.5.1. *An **Incidence Structure** is a triple $S = (P, L, I)$ where P is a set of "Points", L is a set of "Lines" and $I \subseteq P \times L$ is the incidence relation. The elements $(p, l) \in I$ are called **flags** and the elements $(p, l) \in (P \times L) \setminus I$ are called **antiflags**. If $(p, l) \in I$ we say the point p "lies on" l denoted pI_l. If $S = (P, L, I)$ is an incident structure then its **dual** is given by $S^* = (L, P, I^*)$, where $I^* = \{(l, p) : (p, l) \in I\}$*

Definition 1.5.2. ***Incidence Graph:*** *The incidence graph $G(S)$ of an incidence structure S is the graph with vertex set $P \cup L$, where two vertices are adjacent if and only if they are incident.*

Definition 1.5.3. ***Partial Linear Space:*** *This is an incidence structure in which any two points are incident with at most one line. This implies that any two lines are incident with at most one point.*

Lemma 1.5.1. *The incidence graph G of a partial linear space has girth at least six.*

Proof. First note that an incidence graph is bipartite. Hence the smallest possible value of a girth g of any partial linear space P_l is four. So let P_l be a partial linear space with girth $g = 4$. This implies the existence of points p_1, p_2 and lines l_1, l_2 such that $p_1 I_{l_1} I p_2 I_{l_2} I p_1$. Hence both p_1 and p_2 are incident to l_1 and l_2, which yields a contradiction. □

Definition 1.5.4. ***Order of an incidence structure***

An incidence structure is of order (s,t) if:

(i) all vertices of the incidence graph corresponding to the elements of L have the same degree s + 1 for some natural number s; in other words, every line contains exactly s + 1 points,

(ii) all vertices of the incidence graph corresponding to the elements of P have the same degree t + 1 for some natural number t; in other words, every point lies on exactly t + 1 lines.

If s=t then we simply say the incidence structure is of order s.

Definition 1.5.5. ***Partial Geometry:*** *A partial geometry $pg(s,t,\alpha)$ is a partial linear space $pls(s,t)$ such that if there exists a point $p \in P$ that does not lie on a line $l \in L$, then there are exactly α lines which pass through p and are concurrent with l.*

One of the most interesting classes of incidence structures is that of projective planes. In chapter 3, we shall see that all incidence graphs of the projective plane are distance regular.

Definition 1.5.6. *A* ***projective plane*** *is a partial linear space satisfying the following three conditions:*

(i) Any two lines meet in a unique point.

(ii) Any two points lie in a unique line.

(iii) There are three points which are pairwise collinear but all three points are not simultaneously colinear (a triangle)

The first two conditions are duals of each other, while the third is self dual, so the dual of a projective plane is again a projective plane. The first two conditions are the important conditions, with the third serving to eliminate uninteresting " 1-dimensional" cases, such as partial linear spaces where all the points lie on a single line or all the lines on a single point [5]. We now give an example for constructing the Fano plane by considering a vector space

V over a finite field F of characteristic q, in this case $q = 2$. For the general method for constructing the projective geometry $PG(2, q)$, we refer the reader to [5].

Example 1.5.1. ***Fano Plane:*** Let $F = \{0, 1\}$ so that $q = 2$ and V be a subspace of R^3 over F. The lines are the two dimensional subspaces of V while the points are the one dimensional subspaces of V. We have $q^2 + q + 1 = 7$ points and lines respectively. Also each line passes through $q + 1 = 3$ points and each point is incident to $q + 1 = 3$ lines. The points are given by $p_1 = \{(0, 0, 0), (1, 0, 0)\}$, $p_2 = \{(0, 0, 0), (0, 1, 0)\}$, $p_3 = \{(0, 0, 0), (0, 0, 1)\}$, $p_4 = \{(0, 0, 0), (1, 1, 0)\}$, $p_5 = \{(0, 0, 0), (1, 0, 1)\}$, $p_6 = \{(0, 0, 0), (0, 1, 1)\}$, $p_4 = \{(0, 0, 0), (1, 1, 1)\}$ and the lines are given by $l_1 = \{(0, 0, 0), (1, 0, 0), (0, 1, 0), (1, 1, 0)\}$,
$l_2 = \{(0, 0, 0), (1, 0, 0), (0, 0, 1), (1, 0, 1)\}$
$l_3 = \{(0, 0, 0), (0, 1, 0), (0, 0, 1), (0, 1, 1)\}$, $l_4 = \{(0, 0, 0), (1, 0, 0), (0, 1, 1), (1, 1, 1)\}$
$l_5 = \{(0, 0, 0), (0, 1, 0), (1, 0, 1), (1, 1, 1)\}$, $l_6 = \{(0, 0, 0), (0, 0, 1), (1, 1, 0), (1, 1, 1)\}$
$l_7 = \{(0, 0, 0), (1, 1, 0), (0, 1, 1), (1, 0, 1)\}$, with
$I = \{(p_1, l_1), (p_1, l_2), (p_1, l_4), (p_2, l_3), (p_2, l_5), (p_3, l_2), (p_3, l_3), (p_3, l_6), (p_4, l_1), (p_4, l_6), (p_4, l_7), (p_5, l_2), (p_5, l_5), (p_5, l_7), (p_6, l_3), (p_6, l_4), (p_6, l_7), (p_7, l_4), (p_7, l_5), (p_7, l_6), \}$. The resulting structure is the Fano plane given below.

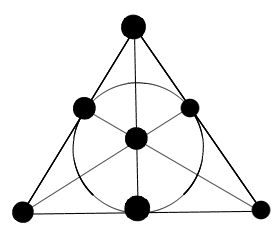

Figure 1.7: The Fano Plane

Definition 1.5.7. *A generalized quadrangle is a partial linear space satisfying the following two conditions:*

(i) Given any line L and a point p not on L there is a unique point p' on L such that p

and p' are collinear. In other words, let (p, l_1) *be an antiflag then there exists a unique flag* (q, l_2) *such that* $pI_{l_2}IqI_{l_1}$

(ii) There are noncollinear points and nonconcurrent lines.

Definition 1.5.8. *The* ***point graph*** *of a generalized quadrangle is the graph with the points of the quadrangle as its vertices, with two points adjacent if and only if they are collinear.*

2

Distance-Regular Graphs

In this chapter we shall introduce the notions of distance regular graphs and then consider the particular cases of diameter 2 which yields strongly regular graphs. We would discuss some concepts, interesting results and illustrate these concepts with example.

2.1 Distance-Regular Graphs

If u and v are vertices in the graph G, we shall denote the distance between u and v in G by $d(u, v)$ and the diameter of G by d. The set of vertices at distance r from u will be denoted by $D_r(u)$. In other words, $D_r(v) = \{u \in V(G) : d(u, v) = r,\ 0 \leq r \leq d\}$. In chapter 1, we gave an example of equitable partition obtained from the distance partition of the Petersen graph and we saw that if we fix any vertex of the Petersen graph O_3, then the distance partition corresponding to the fixed vertex forms an equitable partition. This phenomenon is not random but in fact it constitutes the concept we shall study in the first half of this chapter.

Definition 2.1.1. *For any connected graph G with diameter d, and any vertices u, v of G*

let

$$s_{hi}(u,v) = |\{w \in V(G) : d(u,w) = h \text{ and } d(v,w) = i\}|;$$

i.e., $s_{hi}(u,v)$ *is the number of vertices of* G *whose distance from* u *is* h *and at distance* i *from* v*, where distance between* u *and* v *is* $0 \leq d(u,v) \leq d$.

Definition 2.1.2. ***Distance Regular Graphs (DRGs):*** *We say a connected regular graph* G *is distance regular, if the number* $s_{hi}(u,v)$ *does not depend on the individual pair* (u,v) *but only on the distance* $d(u,v)$*. If* $d(u,v) = j$ *we write* $s_{hi}(u,v) = s_{hij}$.

Let u be a fixed vertex in a DRG G, we shall consider the case when $h = 1$, so that s_{1ij} is read as the number of vertices which are adjacent to u in G, but at distance i from v where $d(u,v) = j$. Now let $w \sim u$ and $d(u,v) = j$, then clearly $d(v,w) = j-1$ or $j+1$ or j; in other words $s_{1ij} = 0$ if $i \neq j-1$ or $j+1$ or j, where $1 \leq j \leq d-1$.

We now introduce the following notations: $c_j = s_{1,j-1,j}$, $a_j = s_{1,j,j}$ $b_j = s_{1,j+1,j}$ where $0 \leq j \leq d$. Except that c_0 and b_d are undefined.

These numbers have the following simple interpretation in terms of the diagrammatic representation of a DRG G.

Pick an arbitrary vertex u and a vertex v in $D_j(u)$. Then v is adjacent to c_j vertices in $D_{j-1}(u)$, a_j vertices in $D_j(u)$ and b_j vertices in $D_{j+1}(u)$. These numbers are independent of u and v, provided that $d(u,v) = j$.

Given $c_j = s_{1,j-1,j}$, $\quad a_j = s_{1,j,j}$, $\quad b_j = s_{1,j+1,j}$, take $d(u,v) = j$ then:

$c_j = |D_1(u) \cap \{w : d(v,w) = j-1\}| = |D_1(u) \cap D_{j-1}(v)|$,

$a_j = |D_1(u) \cap \{w : d(v,w) = j\}| = |D_1(u) \cap D_j(v)|$,

$b_j = |D_1(u) \cap \{w : d(v,w) = j+1\}| = |D_1(u) \cap D_{j+1}(v)|$, with $0 \leq j \leq d$.

We shall denote $|D_i(u)|$ by k_i, for any vertex u, where $0 \leq i \leq d$. Clearly $k_0 = 1$, $k_1 = k = b_0$ and $c_1 = 1$.

Definition 2.1.3. ***Intersection arrays:*** *The intersection array of a distance regular graph*

G is given by

$$\iota(G) = \begin{pmatrix} - & c_1 & \dots & c_j & \dots & c_d \\ a_0 & a_1 & \dots & a_j & \dots & a_d \\ b_0 & b_1 & \dots & b_j & \dots & - \end{pmatrix}.$$

Moreover, $k = b_i + a_i + c_i$ *where* $1 \leq i \leq d-1$ *and* $a_0 = 0,\ a_d = k - c_d$.

Each column of the intersection array sums to k, if we are given the first and the third rows, we can calculate the middle row. Thus it suffices to use the alternate notation

$$\iota(G) = \{k, b_1, \dots, b_{d-1}; 1, c_2, \dots, c_d\}.$$

Example 2.1.1. The intersection array of $K_{3,3}$ is $\iota(K_{3,3}) = \{3, 2; 1, 3\}$.

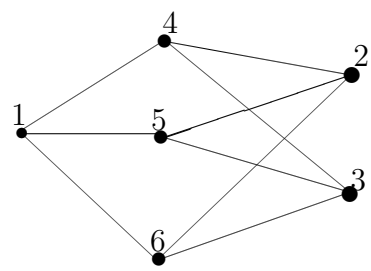

Figure 2.1: The complete bipartite $K_{3,3}$

2.1.1 Parameter matrix

We recall that when we discussed the equitable partitions, we introduced the quotient matrix of a graph G with respect to a partition π, denoted $B = A(G/\pi)$. As for any distance regular graph G, the distance partition with respect to any vertex of G forms an equitable partition,

we have the $(d+1)\times(d+1)$ parameter matrix B defined by

$$B = \begin{pmatrix} 0 & b_0 & & & & & & \\ c_1 & a_1 & b_1 & & & & & \\ & c_2 & a_2 & b_2 & & & & \\ & & \cdot & \cdot & \cdot & & & \\ & & & \cdot & \cdot & \cdot & & \\ & & & & \cdot & \cdot & \cdot & \\ & & & & & \cdot & a_{d-1} & b_{d-1} \\ & & & & & & c_d & a_d \end{pmatrix}.$$

Since the numbers a_i, b_i and c_i are independent of u, the matrix B is also obviously independent of u.

The following is a quite useful result for distance regular graphs.

Lemma 2.1.1. *[[12], p. 135] Let G be a distance-regular graph whose intersection array is $\{k, b_1, \ldots, b_{d-1}; 1, c_2, \ldots, c_d\}$. Then we have the following equations and inequalities.*

- $k_{i-1}b_{i-1} = k_i c_i$ $(1 \leq i \leq d)$.
- $1 \leq c_2 \leq c_3 \leq \ldots \leq c_d$.
- $k \geq b_1 \geq b_2 \geq \ldots \geq b_{d-1}$.

We end this section by giving the intersection arrays of some well known classes of DRGs. For more details about such graphs, we refer the reader to [1]

Intersection arrays of complete graphs K_k and **complete bipartite graphs** $K_{k,k}$ For $k \geq 2$, K_k and $K_{k,k}$ are DRGs with diameters 1 and 2 respectively. Their respective intersection arrays are:

$$\iota(K_k) = \{k-1\ ;1\}, \quad \iota(K_{k,k}) = \{k, k-1\ ;1, k\}.$$

Intersection array of the Petersen graph O_3: The Petersen graph O_3 (figure 1.4) is a DRG with diameter 2 and valency 3. Its intersection array is given by

$$\iota(Q_3) = \{3, 2, 2\ ; 1, 1, 1\}.$$

Definition 2.1.4. ***The Cube Graphs:*** *The k-cube Q_k is defined as follows: The vertices of Q_k are the 2^k symbols $(\epsilon_1, \epsilon_2, \ldots, \epsilon_k)$, where $\epsilon_i = 0$ or 1, $1 \leq i \leq k$ and two vertices are adjacent when their symbols differ in exactly one coordinate.*

Intersection array of k-cube $k \geq 2$: The k-cubes are DRGs with diameter k, valency k and their intersection array is given by

$$\iota(Q_k) = \{k, k-1, \ldots, 1; 1, 2, \ldots, k\}.$$

Example 2.1.2. The intersection array of Q_3 is given by $\iota(Q_3) = \{3, 2, 1\ ; 1, 2, 3\}$.

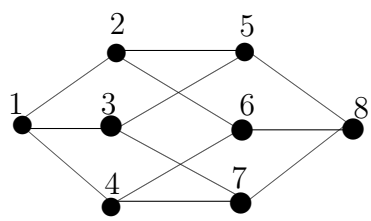

Figure 2.2: The 3-cube Q_3

Definition 2.1.5. ***The Triangle Graphs (Δ_t):*** *This is the line graph of the complete graph K_t, $(L(K_t))$. Δ_t can be described by saying that its vertices correspond to the $\frac{1}{2}t(t-1)$ pairs of numbers from the set $\{1, 2, \ldots, t\}$ where two vertices are adjacent whenever the corresponding pairs have just one common member.*

Intersection array of triangle graphs: The triangle graphs are DRGs with diameter 2 and for $t \geq 4$, the intersection array of Δ_t is given by: $\iota(\Delta_t) = \{2t-4, t-3\ ; 1, 4\}$.

Intersection array of odd graphs O_k, $k \geq 2$: The odd graphs O_k are distance regular graphs with diameter $k-1$. The intersection arrays when k is odd or even are given respectively as

$$\iota(Q_k) = \begin{cases} \{2l-1, 2l-2, 2l-2, \ldots, l+1, l+1, l\ ; 1, 1, 2, 2, \ldots, l-1, l-1\}, & k = 2l-1; \\ \{2l, 2l-1, 2l-1, \ldots, l+1, l+1\ ; 1, 1, 2, 2, \ldots, l-1, l-1, l\}, & k = 2l. \end{cases}$$

We now introduce the distance matrices of a graph which shall be used to construct a basis for the adjacency algebra of a distance-regular graph.

2.1.2 Distance Matrices

Definition 2.1.6. *Let G be a graph with diameter d and vertex set $\{v_1, \ldots, v_n\}$, the* ***distance matrices*** *$\{A_0, \ldots, A_d\}$ of G are the $n \times n$ matrices with entries*

$$(A_h)_{rs} = \begin{cases} 1 & \text{if } d(v_r, v_s) = h \\ 0 & \text{otherwise.} \end{cases}$$

Remark 2.1.1. *One can easily see from the definition of distance matrices that:*

- *$A_0 = I$, A_1 is the adjacency matrix $A(G)$.*
- *$A_0 + A_1 + \cdots + A_d = J$, where J is the all-one matrix.*

We now give some results on the feasibility of intersection arrays. The proof of such results can been found in [1].

Lemma 2.1.2 ([12], p. 136)**.** *Let G be a distance-regular graph with adjacency matrix A and intersection array $\{k, b_1, \ldots, b_{d-1}; 1, c_2, \ldots, c_d\}$. For $1 \leq i \leq d-1$, put $a_i = k - b_i - c_i$ then*

$$AA_i = b_{i-1}A_{i-1} + a_iA_i + c_{i+1}A_{i+1} \quad (1 \leq i \leq d-1). \tag{2.1}$$

Proof. We recall here that A is the adjacency matrix of G while A_i is a distance matrix of G. Then by definitions of A and A_i we see that the $(AA_i)_{rs}$ entry is the number of vertices $u \in G$ such that $d(v_r, u) = 1$ and $d(u, v_s) = i$ i.e $u \in D_1(v_r) \cap D_i(v_s)$. Whenever such vertices u exist then $d(v_r, v_s)$ must be one of $i-1, i,$ *and* $i+1$. Since G is distance-regular, it follows that the number of vertices u is one of b_{i-1}, a_i or c_{i+1} respectively. Hence the entry $(AA_i)_{rs}$ is equal to the (r, s) entry of the right hand side of equation (1.1). □

Theorem 2.1.1 ([12], p. 136)**.** *Let G be a distance-regular graph with diameter d and adjacency algebra $\Omega(G)$. Then $\{A_0, A_1, A_2, \ldots, A_d\}$ is a basis for $\Omega(G)$ and consequently $\Omega(G)$ is of dimension $d+1$.*

Lemma 2.1.3 is a very important result, which shows the very important property that the numbers $s_{hi}(u, v)$ only depend on $d(u, v)$ in G.

Lemma 2.1.3 ([12], p. 137)**.** *Let G be a distance-regular graph with diameter d.*

- *The numbers $s_{hi}(u, v)$ depend only on $d(u, v)$, for $h, i \in \{0, 1, 2, \ldots, d\}$.*
- *if $s_{hi}(u, v) = s_{hij}$ when $d(u, v) = j$, then*

$$A_h A_i = \sum_{j=0}^{d} s_{hij} A_j.$$

Proof. From theorem 2.1.1 $\{A_0, A_1, A_2, \ldots, A_d\}$ is a basis for the adjacency algebra $\Omega(G)$, it follows that $A_h A_i$ is a linear combination of the form

$$A_h A_i = \sum t_{hij} A_j \quad \text{for some scalars } t_{hij},\ 1 \leq h, i, j, \leq d. \tag{2.2}$$

and as we saw in the proof of lemma 2.1.2, $(A_h A_i)_{rs}$ is the number of vertices u such that $d(v_r, u) = h$ and $d(v_s, u) = i$. So $(A_h A_i)_{rs} = s_{hi}(u, v)$, but there is just one member of

the basis whose entry (r, s) equals 1 while others have zero, which is in fact that A_j with $d(v_r, v_s) = j$. Hence $s_{hi}(v_r, v_s) = t_{hij}$ is the intersection number s_{hij}. $\square$

We shall end this section with additional inequalities on the intersection numbers of DRGs.

Lemma 2.1.4 ([12], p. 139)**.** *If G is a DRG with intersection array $\{k, b_1, \ldots, b_{d-1}\,; 1, c_2, \ldots, c_d\}$, then we have the following inequalities:*

- *If $1 \leq i \leq \frac{1}{2}d$, then $b_i \geq c_i$.*
- *If $1 \leq i \leq d-1$, then $b_1 \geq c_i$.*
- $c_2 \geq k - 2b_1$.

2.1.3 Feasibility of Intersection Arrays

Our aim in this subsection is to answer the question: Given an arbitrary array when does this array represent the intersection numbers of a DRG? This is a very difficult question and has been a major area of focus in the study of DRGs. In this section, we are going to obtain some very restrictive necessary conditions for the feasibility of an intersection array of a DRG. It should be noted that the conditions obtained are only necessary conditions but not sufficient. The inequalities given in lemmas 2.1.1 and 2.1.4 also represent some necessary conditions on an intersection array of a DRG. But these conditions are not very restrictive, as there are many arrays which satisfy these inequalities but still are not feasible. For example $\{3, 2, 1\,; 1, 1, 3\}$ passes all the tests of both lemmas 2.1.1 and 2.1.4. Thus special arguments are needed in these cases to eliminate such arrays. The conditions obtained at the end of this section are more restrictive and rule out many of the arrays which are not feasible.

Let $A_h \in \Omega(G)$, $0 \leq h \leq d$. By theorem 2.1.1, the set $B = \{A_0, A_1, A_2, \ldots, A_d\}$ is a basis for $\Omega(G)$. Let $f : \Omega(G) \to \Omega(G)$ be a linear transformation defined on the basis B by

$f(A_i) = A_h A_i$ $(0 \leq i \leq d)$. By lemma 2.1.3, $A_h A_i = \sum_{j=0}^{d} s_{hij} A_j$ $(0 \leq i \leq d)$ and hence a matrix representation of f in B is the $(d+1) \times (d+1)$ matrix T_h given by $(T_h)_{ij} = s_{hij}$, $0 \leq h, i, j \leq d$. We have

Lemma 2.1.5 ([12], p. 141)**.** *The adjacency algebra $\Omega(G)$ of a distance regular graph G with diameter d can be faithfully represented by an algebra of $(d+1) \times (d+1)$ matrices. A basis for this representation is the set $\{T_0, T_1, T_2, \ldots, T_d\}$ where $(T_h)_{ij}$ is the intersection number s_{hij} for $0 \leq i, j, h \leq d$.*

We can now regard the members of $\Omega(G)$ as square matrices of size $d+1$ (instead of n). Since s_{hij} only depends on j, it suffices to choose $h = 1$, thus we only consider the matrix T_1. To see this, we notice first that, since $(T_1)_{ij} = T_{1ij}$, T_1 is tridiagonal:

$$T_1 = \begin{bmatrix} 0 & c_1 & & & & & & \\ k & a_1 & c_2 & & & & & \\ & b_1 & a_2 & c_3 & & & & \\ & & \cdot & \cdot & \cdot & & & \\ & & & \cdot & \cdot & \cdot & & \\ & & & & \cdot & \cdot & \cdot & \\ & & & & & \cdot & a_{d-1} & c_d \\ & & & & & & b_{d-1} & a_d \end{bmatrix}.$$

We shall simply denote $T_1 = T$ and refer to it as the intersection matrix of G. Now, since the matrices T_h are images of the matrices A_h under a faithful representation, by lemma 2.1.2:

$$TT_i = b_{i-1}T_{i-1} + a_i T_i + c_{i+1}T_{i+1} \quad (1 \leq i \leq d-1).$$

Consequently, each T_i $(i \geq 2)$ is a polynomial in T with coefficients which depend only on the entries of T.

Lemma 2.1.6 ([12], p. 141)**.** *Let G be a DRG with valency k and diameter d. Then G has $d+1$ distinct eigenvalues $k = \lambda_0, \lambda_1, \ldots, \lambda_d$ which are the eigenvalues of the intersection matrix T.*

Remark 2.1.2. *Every eigenvalue λ of A is also an eigenvalue of T with multiplicity one. (see [12] and [4]).*

We now discuss briefly how the multiplicity $m(\lambda)$ of an eigenvalue λ of A is obtained from T. For a detailed study of this process, we refer to [12] pages 142 - 144.

We regard λ as an indeterminate and define the following recursions:

$$x_0(\lambda) = 1, \;\; x_1(\lambda) = \lambda,$$

$$c_{i+1}x_{i+1}(\lambda) + (a_i - \lambda)x_i(\lambda) + b_{i-1}x_{i-1}(\lambda) = 0 \;\; (1 \leq i \leq d-1). \tag{2.3}$$

Remark 2.1.3. *The polynomial $x_i(\lambda)$ has degree i in λ. Moreover, $A_i = x_i(A) \quad (0 \leq i \leq d)$, by lemma 2.1.2.*

Define the column vector $x(\lambda) = [x_0(\lambda), x_1(\lambda), \ldots, x_d(\lambda)]^t$, the right eigenvector x_i of T corresponding to λ_i has components $(x_i)_j = x_j(\lambda_i)$ and we define the left eigenvector y_i of T corresponding to λ_i as the row vector satisfying $y_iT = \lambda_i y_i$. The vector x is called standard if $x_0 = 1$.

Lemma 2.1.7 ([12], p. 142)**.** *Suppose x_i and y_i are standard right and left eigenvectors of T respectively corresponding to an eigenvalue λ_i of T. Then $(x_i)_j = k_j(y_i)_j$, for all $i, j \in \{0, 1, \ldots, d\}$ where $k_j = |D_j|$ (number of vertices in D_j) .*

Observe that the standard inner product $(y_i, x_l) = 0$ if $i \neq l$, since

$$\lambda_i(y_i, x_l) = y_iTx_l = \lambda_l(y_i, x_l).$$

The vectors x_i and y_i are standard right and left eigenvectors of T respectively corresponding to an eigenvalue λ_i of T.

Theorem 2.1.2 ([12], p. 142)**.** *The multiplicity $m(\lambda_i)$ of a DRG on n vertices is given by*

$$m(\lambda_i) = \frac{n}{(y_i, x_i)}.$$

Remark 2.1.4. *All our computations till now have been completely determined by the intersection array of the given DRG. Which is a quite interesting property of DRGs.*

Until now, we have been able to impose many conditions on the intersection array of a DRG which in turn yields a very powerful definition/result.

Definition 2.1.7. *The array $\{k, b_1, \ldots, b_{d-1}\ ; 1, c_2, \ldots, c_d\}$ is **feasible** if the following conditions are satisfied.*

- *The numbers $k_i = (kb_1 \ldots b_{i-1})/(c_2c_3 \ldots c_i)$ are integers $(2 \leq i \leq d)$.*
- $k \geq b_1 \geq \ldots \geq b_{d-1}; 1 \leq c_2 \leq \ldots \leq c_d$.
- *If $n = 1 + k + k_2 + \ldots + k_d$ and $a_i = k - b_i - c_i$ $(1 \leq i \leq d-1), a_d = k - c_d$ then nk is even and $k_i a_i$ is even.*
- *The numbers $n/(y_i, x_i) \qquad (0 \leq i \leq d)$ are positive integers.*

Remark 2.1.5. *Notice that:*

- *The conditions in definition 2.1.7 are only necessary conditions.*
- *Despite not being sufficient conditions, these conditions are so stringent that most of the feasible arrays (in the sense of definition 2.1.7) have corresponding distance regular graphs.*

A more convenient way to compute $m(\lambda_i)$ is to first note that

$$(y_i, x_i) = \sum \frac{(x_i)_j^2}{k_j}.$$

So we have

$$m(\lambda_i) = \frac{n}{\sum \frac{(x_i)_j^2}{k_j}}. \tag{2.4}$$

Example 2.1.3. Is the array $\{3, 2, 1; 1, 2, 3\}$ feasible? If yes is it realizable?

Recall that $k = b_0 = 3$. We now compute a_i, $0 \leq i \leq 3$, and form the 4×4 matrix T. $a_i = k - b_i - c_i$ $(1 \leq i \leq d-1), a_d = k - c_d$, so we obtain $a_0 = a_1 = a_2 = a_3 = 0$,

$$T = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3 & 0 & 2 & 0 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

and $k_0 = 1, \quad k_1 = 3, \quad k_2 = 3, \quad k_3 = 1$. Conditions 1 to 3 of definition 2.1.7 have been verified, we now compute the multiplicities of these eigenvalues. $x(\lambda_j) = x_j$ and $x_j = [(x_j)_0, (x_j)_1, \ldots, (x_j)_d]^t$. We shall now apply the recurrence (2.3) to obtain the eigenvectors.

$\lambda_1 = 1,\ x(\lambda_1) = x_1.\ (x_1)_0 = 1,\ (x_1)_1 = 1,\ (x_1)_2 = -1,\ (x_1)_3 = -1$, thus

$$x_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}.$$

Hence $m(\lambda_1) = \frac{n}{(y_1, x_1)} = \frac{n}{\sum_{j=0}^{d} \frac{(x_1)_j^2}{k_j}} = 3$. In a similar way, we get $m(\lambda_2 = 1) = 3$ and $m(\lambda_3 = -3) = 1$. Thus we have shown that $\frac{n}{(y_i, x_i)}$, $0 \leq i \leq 3$ are all integers as such the

last condition of definition 2.1.7. Hence the array $\{3, 2, 1; 1, 2, 3\}$ is feasible and it is in fact realizable. See example 2.1.2.

Example 2.1.4. Is the array $\{3, 2, 1; 1, 1, 3\}$ feasible?

We check the feasibility conditions of definition 2.1.7 as we did in last example. $k_0 = 1,\ k_1 = 3,\ k_2 = 6,\ k_3 = 2$, $n = 12$ and $a_0 = 0,\ a_1 = 0,\ a_2 = 1,\ a_3 = 0$. Therefore, conditions 1 to 3 are easily verified. We now check the fourth condition.

$$T = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 0 & 2 & 1 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

The eigenvalues of T are $\lambda_0 = 3,\ \lambda_1 = 469/360,\ \lambda_2 = -829/360,\ \lambda_3 = -1$. We now obtain the corresponding eigenvectors using the recurrences (2.3).

$$x_1 = \begin{bmatrix} 1 \\ 469/360 \\ 1.30277 \\ 1 \end{bmatrix}.$$

Its multiplicity is obtained by $m(\lambda_1) = n/(y_1, x_1) = 5.109$, which is not an integer. Hence $\{3, 2, 1; 1, 1, 3\}$ is not feasible and thus not realizable as a DRG.

Remark 2.1.6. *There are some feasible arrays which are not realizable.*

- *The arrays* $\{52, 35, 16; 1, 4, 28\}$, $\{69, 48, 24; 1, 4, 46\}$ *and* $\{45, 30, 7; 1, 2, 27\}$ *are all feasible but not realizable. See [2] and [3] respectively.*

- *Graphs with intersection arrays* $\{3\alpha + 3, 2\alpha + 2, \alpha + 2 - \beta;\ 1, 2, 3\beta\}$ *for all* $\beta > 1$ *do not exist. See [3].*

2.2 Strongly Regular Graphs

In this section, we shall study a particular class of DRGs known as the strongly regular graphs. The SRGs are DRGs of diameter 2 and as such inherit all the properties of a DRG. The SRGs though have some other special properties which are not common to all classes of DRGs. We shall be studying these properties which are specific to SRGs in addition to those common to all DRGs.

In addition to being regular, a strongly regular graph has the property that the number of common neighbours of two distinct vertices depends only on whether they are adjacent or nonadjacent. The strongly regular graph represents the most important type of graphs as far as this thesis is concerned.

By way of example, the Petersen graph O_3 (figure 1.4) has "diameter 2 and girth 5". We can read the properties in quotes as: having girth 5 implies O_3 contains no triangle and thus any two adjacent vertices have no neighbours in common. Combining both conditions i.e diameter 2 and girth 5, it follows that two nonadjacent vertices have exactly one common neighbour. So we can deduce that the Petersen graph has 10 vertices, it is 3-regular, adjacent vertices have 0 neighbours in common and non-adjacent vertices have exactly 1 common neighbour. These numbers $10, 3, 0, 1$ are not random as we will see.

2.2.1 Definition and Parameters

Definition 2.2.1. *Let $G = (V, E)$ be a regular graph of degree k and order n different from the complete graph and empty graph. G is said to be* ***strongly regular*** *if there exist $\lambda, \mu \in \mathbb{N} \cup \{0\}$ such that:*

- *Every two adjacent vertices have λ common neighbours.*
- *Every two nonadjacent vertices have μ common neighbours.*

A graph of this kind is denoted $srg(n, k, \lambda, \mu)$. These numbers n, k, λ, μ are called parameters of G.

Example 2.2.1. (i) The Petersen graph is $srg(10, 3, 0, 1)$

(ii) The cycle of length 5 is 2-regular, any two adjacent vertices have no common neighbour while two nonadjacent vertices have exactly one common neighbour. Thus C_5 is an $srg(5, 2, 0, 1)$ and C_4 is an $srg(4, 2, 0, 2)$. Moreover for any bipartite graph $\lambda = 0$ but the converse is not true. C_5 provides a counterexample.

(iii) The triangle graphs Δ_t (line graphs of K_t) are $srg(\frac{t(t-1)}{2}, 2t-4, t-2, 4)$.

(iv) The line graphs of $K_{k,k}$ are $srg(k^2, 2k-2, k-2, 4)$.

Now we give some fundamental results on SRGs which are in fact very useful.

Lemma 2.2.1. *Let G be an $srg(n, k, \lambda, \mu)$, then the complement of G, $\overline{G}$, is an $srg(n, n-k-1, n-2-2k+\mu, n-2k+\lambda)$ denoted by $srg(n, \bar{k}, \bar{\lambda}, \bar{\mu})$.*

Proof. Obviously if G is k regular then $\overline{G}$ is $n-k-1$ regular, so $\bar{k} = n-k-1$. We now consider two vertices u and v which are adjacent in $\overline{G}$, it follows that they are nonadjacent in G and as such they have μ neighbours in common in G. Thus they have $k-\mu$ non-common neighbours in G. This implies there are $n-\mu-(k-\mu)-(k-\mu)-2$ vertices in $G \setminus \{u, v\}$ that are nonadjacent to both u and v thus $\bar{\lambda} = n-2k+\mu-2$.

Now suppose $u \not\sim v$ in $\overline{G}$, then $u \sim v$ in G. Thus u and v have λ common neighbours and $k-\lambda$ non-common neighbours in G, it follows that there are $n-\lambda-(k-\lambda)-(k-\lambda)$ vertices in $G \setminus \{u, v\}$ that are nonadjacent to both u and v. Hence $\bar{\mu} = n-2k+\lambda$. $\square$

Definition 2.2.2. *Let G be an SRG, we say G is **primitive** if both G and $\overline{G}$ are connected, otherwise G is **imprimitive**.*

Remark 2.2.1. *A primitive SRG is a connected DRG of diameter 2 having a connected complement.*

Lemma 2.2.2 ([5], p. 218)**.** *Let G be an srg(n,k,λ, μ). Then the following are equivalent:*

- *G is not connected,*
- *μ=0,*
- $\lambda = k - 1$,
- *G is isomorphic to mK_{k+1} for some $m > 1$.*

Remark 2.2.2. *mK_{k+1} denotes m disjoint copies of K_{k+1}.*

Corollary 2.2.1. Suppose G is an imprimitive SRG, then either $G \cong mK_{k+1}$ or $G \cong \overline{mK_{k+1}}$ for some $m > 1$.

Relationship between parameters: We shall now discuss the relations between the parameters (n, k, λ, μ) of an SRG and intersection numbers as a DRG. Since an SRG G is a DRG of diameter 2, its intersection array is of the form $\iota(G) = \{b_0, b_1; c_1, c_2\}$, where
$a_0 = 0,\ b_0 = k,\ a_1 = \lambda,\ b_1 = k - \lambda - 1,\ c_1 = 1,\ a_2 = k - \mu,\ c_2 = \mu.$
And as such, the parameter matrix of an SRG is given by

$$B = \begin{bmatrix} 0 & k & 0 \\ 1 & \lambda & k - \lambda - 1 \\ 0 & \mu & k - \mu \end{bmatrix}.$$

Feasibility of Parameters: At this stage, it makes sense to ask whether a given set (n, k, λ, μ) with $k < n$ and $\lambda,\ \mu \leq k$ represents the parameters of an SRG. In fact the four parameters in an $srg(n, k, \lambda, \mu)$ are not independent and must obey the following relation.

$$(n - k - 1)\mu = k(k - \lambda - 1). \tag{2.5}$$

Equation (2.5) can be obtained by applying some counting mechanisms. Since diameter of any SRG is 2, if we fix a vertex say $u \in G$ and count the number of edges between $D_1(u)$ and $D_2(u)$ in both directions, we obtain the desired relationship. Equation (2.5) may be viewed as a feasibility condition for any given parameter set of an SRG, in addition to those feasibility conditions of the DRG we have seen earlier in definition 2.1.7. We shall see more restrictive and interesting conditions on the parameter set.

2.2.2 Eigenvalues

The computation of the eigenvalues of an SRG is perhaps the most interesting thing about its spectrum. How we compute the eigenvalues of an SRG together with their multiplicities is truly a good achievement.

Let J denote the matrix whose entries are all one. If A is the adjacency matrix of a regular graph of valency k, we have $AJ = JA = kJ$. Now given an $srg(n, k, \lambda, \mu)$, we want to discuss how we obtain the eigenvalues of the adjacency matrix A from the parameters of G.

The uv-entry of the matrix A^2 is the number of walks of length two from the vertex u to the vertex v. In a strongly regular graph this number is determined only by whether u and v are equal, adjacent or non-adjacent. Therefore, we get the equation:

$$A^2 = kI + \lambda A + \mu(J - I - A),$$

which can be rewritten as

$$A^2 - (\lambda - \mu)A - (k - \mu)I = \mu J.$$

We can use this equation to find the eigenvalues of A.

Recall from linear algebra: Let A be a real symmetric matrix. If u and v are eigenvectors of A corresponding to different eigenvalues, then u and v are orthogonal. Since G is regular, k is an eigenvalue of A with eigenvector $\mathbf{1}$ (the column vector with all n entries equal to 1). Thus any other eigenvector of A is orthogonal to $\mathbf{1}$.

Now let z be an eigenvector for A with eigenvalue $\theta \neq k$. Then

$$A^2z - (\lambda - \mu)Az - (k - \mu)Iz = \mu Jz = 0.$$

so,

$$\theta^2 - (\lambda - \mu)\theta - (k - \mu) = 0.$$

Therefore, the eigenvalues of A different from k must be zeros of the quadratic $x^2 - (\lambda - \mu)x - (k - \mu) = 0$. If we set $\Delta = (\lambda - \mu)^2 + 4(k - \mu)$ (the discriminant of the quadratic) and denote the two zeros of this polynomial by θ and τ, we get

$$\theta = \frac{(\lambda - \mu) + \sqrt{\Delta}}{2}. \tag{2.6}$$

$$\tau = \frac{(\lambda - \mu) - \sqrt{\Delta}}{2}. \tag{2.7}$$

We see that $\theta\tau = \mu - k$ and so provided $\mu < k$, we get that θ and τ are nonzero with opposite signs. We shall usually assume that $\theta > 0$. Though eigenvalues of a strongly regular graph are determined by its parameters, SRGs with the same parameters need not be isomorphic. The multiplicities of the eigenvalues of A can also be computed from the parameters of the SRG having A as its adjacency matrix. Now let m_θ and m_τ be the multiplicities of θ and τ, respectively, then $m_\theta + m_\tau = n - 1,$ and $m_\theta\theta + m_\tau\tau = -k$. Hence after a couple of calculations, we get

$$m_\theta = \frac{1}{2}\left((n-1) - \frac{2k + (n-1)(\lambda - \mu)}{\sqrt{\Delta}}\right). \tag{2.8}$$

$$m_\tau = \frac{1}{2}\left((n-1) + \frac{2k + (n-1)(\lambda - \mu)}{\sqrt{\Delta}}\right). \tag{2.9}$$

This argument yields a powerful feasibility condition. Given a parameter set we can compute m_θ and m_τ using these equations. If the results are not positive integers, then there cannot be a strongly regular graph with these parameters. In practice this is a very useful condition because for (n, k, λ, μ) to be feasible, both m_θ and m_τ must be integers. A classical application of this idea is to determine the possible valencies for a Moore graph (see [11]) with diameter two.

Lemma 2.2.3 ([5], p. 220)**.** *A connected regular graph with exactly three distinct eigenvalues is strongly regular.*

Example 2.2.2. Determine the feasibility of $srg(486, 165, 36, 66)$. $n = 486$, $k = 165$, $\lambda = 36$, $\mu = 66$. First we verify equation (2.5), $(n-k-1)\mu = (486-165-1)66 = 21120 = k(k-\lambda-1) = 165(165-36-1)$. Thus (2.5) is satisfied. We now proceed to compute the eigenvalues. $\Delta = (\lambda-\mu)^2 + 4(k-\mu) = 900 + 4(165-66) = 1296$, so $\sqrt{\Delta} = 36$. $\theta = \frac{(\lambda-\mu)+\sqrt{\Delta}}{2} = 3$ *and* $\tau = -33$. Their respective multiplicities are $m_\theta = 440$ and $m_\tau = 45$. Thus $(486, 165, 36, 66)$ is feasible. We even have more, as we can verify that $m_\tau + m_\theta + 1 = 486 = n$, which is the dimension of the adjacency matrix. But [8] has shown that this SRG does not exist.

Definition 2.2.3. *Let G be an SRG with $m_\theta = m_\tau$. Then G is said to be a conference graph.*

The name conference graph arises from the fact that these graphs are associated to so called symmetric conference matrices.

Definition 2.2.4. *A* ***conference matrix*** C *of order* n *is an* $n \times n$, $(0, 1, -1)$ *matrix whose diagonal entries are all zeros and* $CC^T = (n-1)I$.

If C is a symmetric conference matrix, we may assume the first row and column of C are all ones (if not we can permute the rows and columns) except $a_{11} = 0$. We now delete the first row and column, and switch all $+1's$ to 0 and all $-1's$ to 1 to obtain a new $(n-1) \times (n-1)$ matrix A_C. And we let G_C be the graph whose adjacency matrix is A_C. One can check that G_C is a conference graph. It is a fact that if $n \equiv 2$ $(mod\ 4)$ then a conference matrix of order n exists if and only if $n-1$ is a sum of two squares (see [14]).

Corollary 2.2.2. A conference graph with n vertices exists if and only if n is the sum of two squares.

Example 2.2.3. Determine the existence of $srg(21, 10, 4, 5)$.
First, one can easily see that equation (2.5) is satisfied. Now, $\Delta = 21$, so the eigenvalues are $\theta = \frac{-1+\sqrt{21}}{2}$, and $\tau = \frac{-1-\sqrt{21}}{2}$. Their multiplicities are $m_\theta = m_\tau = 10$. Thus the parameters $(21, 10, 4, 5)$ are feasible. Since $m_\theta = m_\tau$, if $G = srg(21, 10, 4, 5)$ is to exist, then G must be a conference graph. But 21 is not a sum of two squares. Hence by corollary 2.2.2 an $srg(21, 10, 4, 5)$ does not exist.

We shall round up this subsection with some results on SRGs.

Lemma 2.2.4 ([5], p. 222)**.** *Let* G *be an srg(n,k,*λ, μ*) with eigenvalues* k, θ *and* τ. *Then either*

- G *is a conference graph, or*
- $(\theta - \tau)^2$ *is a perfect square and* θ, τ *are integers.*

We thus have the following consequence.

Lemma 2.2.5 ([5], p. 222)**.** *Let* G *be a strongly regular graph with* p *vertices, where* p *is prime. Then* G *is a conference graph.*

2.2.3 Status Of Existence of SRGs

One of the most important problems about strongly regular graphs is their existence. While many classes of strongly regular graphs have construction mechanisms, and there are some non-existence results, for many parameter sets the existence of a strongly regular graph is still unknown. The existence problem is also the theme of this thesis and the methods for determining the existence or nonexistence of an SRG with a given parameter set are non-trivial.

The first table below which is extracted from [5] shows a list of small SRGs whose existence or nonexistence are known, together with their parameters. While the second table shows a list of parameters with ($n \leq 100$) whose existence are yet to be determined.

Table 2.1: Parameters of SRGs with $n \leq 25$

n	k	λ	μ	θ	τ	m_θ	m_τ
5	2	0	1	$(-1+\sqrt{5})/2$	$(-1-\sqrt{5})/2$	2	2
9	4	1	2	1	-2	4	4
10	3	0	1	1	-2	5	4
13	6	2	3	$(-1+\sqrt{13})/2$	$(-1-\sqrt{13})/2$	6	6
15	6	1	3	1	-3	9	5
16	5	0	2	1	-3	10	5
16	6	2	2	2	-2	6	9
17	8	3	4	$(-1+\sqrt{17})/2$	$(-1-\sqrt{17})/2$	8	8
21	10	3	6	2	-2	14	6
21	10	4	5	$(-1+\sqrt{21})/2$	$(-1-\sqrt{21})/2$	10	10
21	10	5	4	3	-2	6	14
25	8	3	2	3	-2	8	16
25	12	5	6	2	-3	12	12

Table 2.1 shows us that the conference graphs form a very important class of strongly regular graphs as it can be read that in many cases $m_\theta = m_\tau$. Also two non-isomorphic graphs might have the same parameters as an SRG. In other words the existence of an SRG with given parameters does not necessarily imply uniqueness, for example there are two Latin squares (see [5]) of order 5, yielding two $srg(25, 12, 5, 6)$. An exhaustive computer search

has shown that there are 10 strongly regular graphs with these parameters. Such exhaustive computer searches have also been performed for a very limited number of other parameter sets. The smallest parameter set for which the exact number of strongly regular graphs is unknown is $(37, 18, 8, 9)$. Of all the parameters given in table 2.1, only $srg(21, 10, 4, 5)$ does not exist, see example 2.2.3.

Table 2.2: Parameters of SRGs with unknown existence having $n \leq 100$

n	k	λ	μ
65	32	15	16
69	20	7	5
75	32	10	16
76	30	8	14
76	35	18	14
85	14	3	2
85	30	11	10
85	42	20	21
88	27	6	9
95	40	12	20
96	35	10	14
96	38	10	18
96	45	24	18
99	14	1	2
99	42	21	15
100	33	8	12

In most of the classical works on SRGs, the partial geometries are also discussed because of their close relations with SRGs. In chapter one we introduced the incidence structures and incidence graphs, we shall now end this section with a discussion of point graphs.

The next result shows that any point graph of a nontrivial generalized quadrangle is strongly regular.

Lemma 2.2.6 ([5], p. 235)**.** *Let G be the point graph of a generalized quadrangle of order (s, t). Then G is strongly regular with parameters*

$$((s+1)(st+1), s(t+1), s-1, t-1).$$

Example 2.2.4. The point graph of the generalized quadrangle on the edges and one-factor of K_6 is $\overline{L(K_6)}$ which is a strongly regular graph.

3

Vector Representations

Our proof methodology in this thesis is based on the unit vector representation of DRGs. In this chapter we shall discuss the concept of vector representation of graphs in the first section, and also the special cases of DRGs. Then we round off the section with the idempotent matrices of SRGs. Subsequently, in the next section we discuss the proof methodology applied by Haemers in his proof of the nonexistence of an $srg(76, 21, 2, 7)$. And in the last section we discuss the motivation behind our unit vector representation approach to prove the nonexistence of an $srg(76, 21, 2, 7)$.

3.1 Vector Representation of DRGs.

A vector representation of a graph G is a mapping of its vertices into a real vector space, constructed from an eigenspace of its adjacency matrix $A(G)$. If G is a DRG, then this mapping has the property that the distance between the images of any two vertices is determined by the distance between the preimages in G.

Definition 3.1.1. *A pair of functions $\rho_v : V(G) \to V$ and $\rho_\epsilon : E(G) \to P(V)$ is called a V-representation of a graph G if for any edge $e = uv \in E(G)$, we have $\{\rho_v(u), \rho_v(v)\} \subseteq \rho_\epsilon(uv)$.*

Let G be a graph with $V(G) = \{1, 2, \ldots, n\}$ having $A(G)$ as its adjacency matrix. Suppose θ is an eigenvalue of $A(G)$ having multiplicity $m(\theta) = m$. We shall let U_θ be an $n \times m$ matrix whose columns form an orthonormal basis for the eigenspace corresponding to θ. Then $AU_\theta = \theta U_\theta$. If we let $u_\theta(i)$ be the ith row of U_θ, then

$$\sum_{j \sim i} u_\theta(j) = \theta u_\theta(i) \tag{3.1}$$

Definition 3.1.2. ***Principal idempotent*** *of $\Omega(G)$ is an idempotent e such that $fef = fe$ for all idempotents $f \in \Omega(G)$.*

Remark 3.1.1. *The Gram matrix of the vectors $u_\theta(i)$ is the principal idempotent of $\Omega(G)$ corresponding to θ and obtained as $U_\theta U_\theta^T$.*

Definition 3.1.3. *Given the matrix U satisfying* (3.1)*, the mapping $\rho : V(G) \to R^m$ defined by $\rho(i) = u_i(\theta)$ is called a* ***vector representation*** *of G in the m dimensional eigenspace of θ.*

Definition 3.1.4. ***Characteristic vector:*** *Let G be a graph on n vertices, having $V(G) = \{v_1, v_2, ..., v_n\}$. The characteristic vector of a vertex i is a vector $x_i = (x_1, x_2, ..., x_n)$ such that*

$$(x_i)_j = \begin{cases} 1, & \text{if } i = j \text{ or } v_i \sim v_j; \\ 0, & \text{otherwise.} \end{cases}$$

We shall now give some examples of the vector representation of some DRGs

Example 3.1.1. Let $G = J(5, 2, 0)$, the Johnson's graph, whose vertices are the two elements subsets of $N = \{1, 2, 3, 4, 5\}$ where two vertices are adjacent if and only if the intersections of the subsets representing them are empty. Thus,
$V(G) = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{2, 3\}, \{2, 4\}, \{2, 5\}, \{3, 4\}, \{3, 5\}, \{4, 5\}\}$. For simplicity we can assume $V(G) = \{a, b, c, d, e, f, g, h, i, j\}$ respectively. The characteristic vector of

vertex a is $x_a = (1,0,0,0,0,0,0,1,1,1)$. We can easily see that for any vertex $s \in G$, its standard norm $|x_s| = 2$. Now define a new vector corresponding to a vertex s by $w_s = x_s - \frac{2}{5}j$, where $j = (1,1,1,1,1,1,1,1,1,1) \in R^{10}$. One can check that each of these w_s is perpendicular to j as such $w_s \in R^9$. The mapping $\rho : V(G) \to R^9$ defined by $\rho(s) = w_s$ is a vector representation of G. For instance $a \mapsto (3/5, -2/5, -2/5, -2/5, -2/5, -2/5, -2/5, 3/5, 3/5, 3/5)$.

Example 3.1.1 shows the vector representation of G into a vector space constructed from the characteristic vector of its vertices. We shall now give more examples which show a representation of G into a subspace constructed from an eigenspace corresponding to one of its eigenvalues.

Example 3.1.2. Consider the complete graph K_4. Let $V(K_4) = \{1,2,3,4\}$. In example 1.2.3, we saw that the vectors $[u_1, u_2, u_3] = \{[1,-1,0,0]^T, [-1,0,0,1]^T, [-1,0,1,0]^T\}$ span the 3 dimensional eigenspace corresponding to $\lambda = -1$. Let U be given by $U = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, then $AU = \begin{pmatrix} -1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$, where A is the adjacency matrix of K_4.

We see that for $\lambda = -1$, the matrix U satisfies $AU = \lambda U = -U$. And as such it immediately satisfies equation (3.1). If we let $u(i)$ be the ith row of U, then the mapping $\rho : V(K_4) \to R^3$ with $i \mapsto u(i)$ for each $i \in V(G)$ is a vector representation of K_4 in the eigenspace of $\lambda = -1$.

Example 3.1.3. Consider the Petersen graph O_3. Let $V(O_3) = \{1,2,3,4,5,6,7,8,9,10\}$. In example 1.2.4 we saw that the eigenspace of the Petersen graph corresponding to $\lambda = 1$ is spanned by 5 linearly independent vectors of the form $(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}) = (x_1, x_2, x_2 - x_7 - x_1, x_4, x_5, x_1 - x_2 - x_5, x_7, -x_7 - x_1 - x_4, x_4 + x_7 + x_1 - x_2 - x_5, x_5 - x_1 - x_4)$, where x_1, x_2, x_4, x_5, x_7 are free variables. We thus have the following eigenvectors $v_1 = [0,1,1,0,-1,0,0,0,0,-1]^T$, $v_2 = [0,0,-1,-1,-1,1,1,0,1,0]^T$, $v_3 = [0,0,-1,1,0,0,1,-2,2,-1]^T$,

$v_4 = [1, 1, 0, -1, 0, 0, 0, 0, -1, 0]^T$ and $v_5 = [3, 1, -1, -1, 1, 1, -1, -1, -1, -1]^T$. The matrix U whose columns are the eigenvectors of O_3 is given by

$$U = \begin{pmatrix} 0 & 0 & 0 & 1 & 3 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & -1 & -1 & 0 & -1 \\ 0 & -1 & 1 & -1 & -1 \\ -1 & -1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & -2 & 0 & -1 \\ 0 & 1 & 2 & -1 & -1 \\ -1 & 0 & -1 & 0 & -1 \end{pmatrix}.$$

One can now check that $AU = \lambda U = U$, where A is the adjacency matrix of O_3 and $\lambda = 1$. Hence, equation (3.1) is thus satisfied. Thus the function $\rho : V(O_3) \to R^5$, such that $i \mapsto u(i)$ for each vertex $i \in V(O_3)$ where $u(i)$ is the ith row of the matrix U is a vector representation of O_3 to the eigenspace corresponding to $\lambda = 1$.

Remark 3.1.2. *We should note that the vector representation of a graph G is not unique. In example* 3.1.1*, we gave a vector representation of $J(5, 2, 0)$ which is the Petersen graph, by mapping vertices of G into vectors constructed from their characteristic vectors. While in example* 3.1.3*, we gave a vector representation of the same Petersen graph O_3 in one of its eigenspaces.*

Though most of our examples so far are distance regular graphs, the methods we have used can be applied for any graph. We now turn our attention to the cases where G is necessarily distance regular. We shall state some results and develop the concept which serves as our method of proof in this thesis.

Given that $E_\theta = U_\theta U_\theta^t$ is a principal idempotent of a graph G, we recall that any polynomial $p(A)$ can be given by: $p(A) = \sum_{\theta \in ev(A)} p(\theta) E_\theta$.

We then have the following results.

Lemma 3.1.1 ([4], p. 262)**.** *Let G be a graph with adjacency matrix A, and s, t be any two vertices of G and r be a non-negative integer, then*

$$(A^r)_{st} = \sum_{\theta \in ev(A)} \theta^r (u_\theta(s), u_\theta(t)). \tag{3.2}$$

Lemma 3.1.2 ([4], p. 262)**.** *Let G be a DRG having θ as one of its eigenvalues. Suppose s and t are two vertices of G then the dot product $(u_\theta(s), u_\theta(t))$ is determined only by the distance in G between s and t.*

Remark 3.1.3. *From lemma 3.1.2, if $d(s,t) = 0$ i.e., $s = t$, then $(u_\theta(s), u_\theta(t))$ is just the length of the vector $u_\theta(s)$. Since this dot product depends only on the distance between s and t, it follows that all the vectors $u_\theta(j)$ have the same length for all $j \in V(G)$. Thus $u_\theta(j)$ maps $V(G)$ into a sphere in R^m*

3.1.1 The Sequence of Cosines

Suppose G is a DRG with adjacency matrix $A(G)$ having θ as one of its eigenvalues. Let s and t be two vertices at distance r. Then we define the rth cosine (with respect to θ) as:

$$w_r(\theta) = \frac{(u_\theta(s), u_\theta(t))}{(u_\theta(s), u_\theta(s))}. \tag{3.3}$$

We recall that $(u_\theta(s), u_\theta(t))$ is only determined by $d(s,t)$ i.e., $(u_\theta(s), u_\theta(s)) = (u_\theta(t), u_\theta(t))$. Thus (3.3) is symmetric in s and t. Moreover, if we scale our vector representations to be unit vectors, then $w_r(\theta) = (u_\theta(s), u_\theta(t))$ which just determines the dot products of vertices s and t at distance r from each other. We again recall that any vector representation of G

must satisfy equation (3.1). Taking the inner product of both sides of (3.1) with $u_\theta(l)$ where $d(l,s) = r$, we have the recurrence:

$$\theta w_r = c_r w_{r-1} + a_r w_r + b_r w_{r+1}. \tag{3.4}$$

This implies that

$$w_{r+1} = \frac{1}{b_r}\theta w_r - (c_r w_{r-1} + a_r w_r).$$

It is obvious from (3.3) that $w_0(\theta) = 1$. While others can be computed from the recurrence (3.4) as $w_1(\theta) = \frac{\theta}{k}$, $w_2(\theta) = \frac{1}{kb_1}(\theta^2 - a_1\theta - k)$. The sequence $\{w_i(\theta)\}_{i=1}^d$ is called the cosine sequence of G with respect to θ

Remark 3.1.4.

- *We observe that the sequence cosines $\{w_r\}_{r=0}^d$ is determined just from the parameters of the DRG and its eigenvalues.*
- *From the first point, it follows that if the DRG is in particular an SRG then this cosine sequence can be completely determined without having the adjacency matrix of G. In other words, when G is an SRG, the sequence of cosines can be completely determined from its parameters (n, k, λ, μ).*
- *The inner product between the vectors in our graph representation of a DRG can be completely determined from the cosine sequences whenever we assume that the representation is a unit vector representation.*

We now discuss the idempotent matrices of a graph, in a special way which is exclusive to the SRGs in their vector representation or spectral analysis. This method can only be used when the graph is known to exist and in fact requires that the adjacency matrix of the graph is known.

3.1.2 Idempotent Matrices of SRGs

In this subsection, we discuss the idempotent matrices of SRGs, which plays an important role in their study. We shall study the idempotent matrices via the spectral decomposition theorem.

Definition 3.1.5. *Let $\{V_i\}_{i=1}^m$ be subspaces of a vector space V. The sum $\sum_{i=1}^{m} V_i$ is the set of all vectors v of the form $v = \sum_{i=1}^{m} v_i$ where $v_i \in V_i$ for $1 \leq i \leq m$.*

Definition 3.1.6. *Subspaces $\{U_i\}_{i=1}^n$ of a vector space V are said to be **linearly independent** from one another if $\sum_{i=1}^{n} \alpha_i = 0$, $\alpha_i \in U_i$ implies that each α_i is 0.*

Definition 3.1.7. *We say a vector space V is a **direct sum** of $\{V_i\}_{i=1}^m$, written $V = V_1 \oplus V_2 \oplus \ldots \oplus V_m$, if and only if $\{V_i\}_{i=1}^m$ are linearly independent and V is a sum of V_i for $1 \leq i \leq m$.*

Definition 3.1.8. *A linear transformation E is called a projection or idempotent if $E^2 = E$.*

We now give the spectral theorem which we shall apply to the adjacency matrix $A(G)$ of $G = srg(n, k, \lambda, \mu)$.

Theorem 3.1.1. ***(Spectral Theorem)*** *Let V be a finite dimensional inner product space over R and $T : V \to V$ be a self-adjoint linear operator. Suppose T has k distinct eigenvalues $\{\lambda_i\}_{i=1}^k$ with associated eigenspaces $\{V_i\}_{i=1}^k$ respectively. Let E_i be a projection of V on V_i. Then*

(i) $V = V_1 \oplus V_2 \oplus \ldots \oplus V_k$,

(ii) $E_1 + E_2 + \ldots + E_k = I$,

(iii) $T = \lambda_1 E_1 + \ldots + \lambda_k E_k$.

We now apply the spectral theorem 3.1.1 to the adjacency matrix $A(G)$ of our $srg(n, k, \lambda, \mu)$ to find the idempotent matrices E_0, E_1, E_2. Let $V = R^n$ and V_0, V_1 and V_2 be the eigenspaces of A corresponding to eigenvalues k, θ, τ respectively. Let E_i be the projection of V onto V_i for $0 \leq i \leq 2$. From (ii) and (iii) of the spectral theorem, it follows that

$$I = E_0 + E_1 + E_2 \tag{3.5}$$

$$A = kE_0 + \theta E_1 + \tau E_2 \tag{3.6}$$

$$\overline{A} = J - A - I = (n - k - 1)E_0 - (\theta + 1)E_1 - (\tau + 1)E_2. \tag{3.7}$$

where $\overline{A}$ is the adjacency matrix of $\overline{G}$. Solving equations (3.5), (3.6) and (3.7) we get:

$$E_0 = \frac{1}{n}J \tag{3.8}$$

$$E_1 = \frac{1}{\theta - \tau}(A - \tau I + \frac{\tau - k}{n}J) \tag{3.9}$$

$$E_2 = \frac{1}{\tau - \theta}(A - \theta I + \frac{\theta - k}{n}J). \tag{3.10}$$

Remark 3.1.5. *The column spaces of E_0, E_1 and E_2 are the eigenspaces of k, θ and τ respectively. In other words, a basis for the column spaces span these eigenspaces and can in fact be chosen as eigenvectors with respect to the corresponding eigenvalues.*

3.2 Haemers' Approach to the Nonexistence $srg(76, 21, 2, 7)$

In this section, we shall briefly discuss the work by W.H Haemers [10] on his nonexistence proof of an $srg(76, 21, 2, 7)$. Its main idea is to figure out that an $srg(76, 21, 2, 7)$ exists if and only if it is a point graph of some partial geometry. And he then used the richness of

the existing results in geometry to arrive at his conclusion of nonexistence. In the sequel we denote the subgraph of G induced by vertices of $D_i(u)$ by $G_i(u)$, for any $u \in V(G)$.

Haemers applied a sophisticated counting mechanism in his proof. First, since G is an SRG and thus a DRG, he fixed a vertex say $u \in G$ and then considered the distance partitions $D_1(u)$ and $D_2(u)$. He then used the property $\lambda = 2$ which implies that the induced subgraph $G_1(u)$ is a disjoint union of cycles and then employed a counting approach based on the assumption that any vertex $i \in D_2(u)$ connects to exactly c_i vertices on a cycle in $D_1(u)$. This led him to the assertion that every cycle in $G_1(u)$ has length k which is always a multiple of 3.

Subsequently, Haemers also figured out that the graph G cannot contain a subgraph isomorphic to $K_4 \setminus \{e\}$ where e is any edge on K_4. Then based on these results, he discovered that every edge of G must lie on a unique 4-clique. Thus he proved that if an $srg(76, 21, 2, 7)$ exists, then it must be a point graph of the partial geometry $pg(3, 6, 1)$.

But S. Dixmier and F.Zara already proved that such an incidence structure does not exist. Hence he deduced that an $srg(76, 21, 2, 7)$ does not exist.

3.3 Our Unit Vector Representation Approach

We shall now discuss the motivation behind our unit vector approach to the proof of nonexistence of an SRG and then we give an overview of such method.

The motivation behind this approach can be deduced from remark 3.1.4. The nice algebraic properties of the SRG allow us to compute the sequence of cosines from its parameters. Now, if we choose a unit vector representation, then the sequence of cosines ω_r are just the inner products of vectors representing vertices at distance r from each other. Thus we can explore the distance regularity property of the SRG and choose to fix a vertex say $u \in G$ so that ω_0 is the length of any vector in the representation, ω_1 is the inner product of any

two vectors representing adjacent vertices and ω_2 is the inner product of vectors representing nonadjacent vertices.

Hence if we choose such a unit vector representation in any eigenspace of G, the sequence of cosines enables us to endow an inner product on this eigenspace and provides us with many sophisticated manipulations.

It is believed that via the available inner products with respect to some manipulations of the vectors and the known parameters of the SRG, we can sufficiently determine many properties of the SRG, discover some impossible configurations and figure out some forbidden structures in this graph which contradicts its being a strongly regular graph. This so called forbidden configurations would be completely deduced based on the linear algebraic properties of the vectors in this representation.

4

On Nonexistence of $srg(76, 21, 2, 7)$

We shall now present our own work and result in the proof of nonexistence of a strongly regular graph $G = srg(76, 21, 2, 7)$ inspired by the work of Ivanov and Shpectorov [6].

4.1 The Vector Representation Method

Firstly, we compute the spectrum of G and its sequence of cosines. The largest eigenvalue of G is $k = 21$ having multiplicity $m_k = m_0 = 1$. The other eigenvalues of G are obtained from equations (2.5) and (2.6) and thus $\theta = 2$ and $\tau = -7$. By equations (2.7) and (2.8), we have $m_\theta = m_1 = 56$ and $m_\tau = m_2 = 19$. Hence

$$Spec(G) = \begin{pmatrix} 21 & 2 & -7 \\ 1 & 56 & 19 \end{pmatrix}.$$

As $\tau = -7$ has smaller and then more realizable eigenspace than $\theta = 2$, we study the representation of G in the eigenspace of $\tau = -7$.

We now compute the sequence of cosines of G for $\tau = -7$. From the relationship between the parameters of SRGs and the intersection arrays of DRGs, we recall that $a_1 = \lambda = 2$,

$b_1 = k - \lambda - 1 = 18$. So $\omega_0(-7) = 1$, $\omega_1(-7) = \frac{\tau}{k} = -\frac{1}{3}$, and $\omega_2(-7) = \frac{1}{kb_1}(\theta^2 - a_1\theta - k) = \frac{1}{9}$. Hence the sequence of cosines corresponding to $\tau = -7$ is $(1, -\frac{1}{3}, \frac{1}{9})$.

Construction of the Vector Representation.

In this construction, when the vertices which are preimages of the vectors u and v are adjacent, we simply say u is adjacent to v, denoted $u \sim v$ and if they are at distance 2 from each other we say that u is at distance 2 from v or vice versa.

Let u and v be the vector images of any two vertices in G. Since we have chosen a unit vector representation of G into the eigenspace of $\tau = -7$, coupled with the fact that an SRG has diameter 2 and from the sequence of cosines corresponding to $\tau = -7$, it follows that

- if $u = v$, then $(u, v) = 1$.
- if $u \sim v$, then $(u, v) = -\frac{1}{3}$.
- if $d(u, v) = 2$, then $(u, v) = \frac{1}{9}$.

Now let a vector u be the image of a fixed vertex in G. Then there are 21 vectors at distance 1 from u, denoted as $D_1(u) = \{v_1, v_2, \ldots, v_{21}\}$. We construct an 18 dimensional subspace for this representation by defining vectors $w_i \in u^\perp$ as follows. For each v_i, define $w_i = v_i + \frac{1}{3}u$, for $1 \leq i \leq 21$. One can easily verify that $(w_i, u) = 0$, for $1 \leq i \leq 21$. Moreover, we have

$$(w_i, w_j) = \begin{cases} 8/9 & \text{if } i = j, \\ -4/9 & \text{if } i \neq j \text{ and } v_i \sim v_j, \\ 0 & \text{if } i \neq j \text{ and } v_i \text{ not adjacent to } v_j. \end{cases}$$

Then we scale w_i to have length 2 by defining $\tilde{w}_i = \frac{3}{2}w_i$. So, we have

$$(\tilde{w}_i, \tilde{w}_j) = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } i \neq j \text{ and } v_i \text{ adjacent to } v_j, \\ 0 & \text{if } i \neq j \text{ and } v_i \text{ not adjacent to } v_j. \end{cases}$$

Thus for each v_i, we have a corresponding $\tilde{w}_i$ in the 18 dimensional subspace $u^{\perp}$. So, invariably, the fixed vertex in G having image u has its neighbours mapped into the vectors $\{\tilde{w}_1, \tilde{w}_2, \ldots, \tilde{w}_{21}\}$.

We have the following results.

Lemma 4.1.1. *Let $C = \{v_1, v_2, \ldots, v_t\}$ be a cycle of length t in $G_1(u)$. Then we have:*

(i) $\tilde{w}_1 + \tilde{w}_2 + \cdots + \tilde{w}_t = 0$

(ii) The set $\Phi = \{\tilde{w}_1, \tilde{w}_2, \ldots, \tilde{w}_{t-1}\}$ is linearly independent.

Proof. (i) Let $\tilde{w} = \tilde{w}_1 + \tilde{w}_2 + \cdots + \tilde{w}_t$. Then

$$(\tilde{w}, \tilde{w}) = (\tilde{w}_1, \tilde{w}) + (\tilde{w}_2, \tilde{w}) + \cdots + (\tilde{w}_t, \tilde{w}). \tag{4.1}$$

Let $v_{t+1} := v_1$, then v_i, $2 \leq i \leq t$, is adjacent to both v_{i-1} and v_{i+1} and non-adjacent to all others, while v_1 is adjacent to both v_2 and v_{t+1}. Therefore,

$$(w_i, w_j) = \begin{cases} 2 & \text{if } i = j,\ 1 \leq i, j \leq t, \\ -1 & \text{if } i = j - 1 \text{ or } i = j + 1,\ 2 \leq i, j \leq t \\ 0 & \text{otherwise.} \end{cases}$$

Thus

$$(\tilde{w}_i, \tilde{w}) = (\tilde{w}_i, \tilde{w}_{i-1}) + (\tilde{w}_i, \tilde{w}_i) + (\tilde{w}_i, \tilde{w}_{i+1}),\ 2 \leq i \leq t.$$

$$\Rightarrow (\tilde{w}_i, \tilde{w}) = -1 + 2 - 1 = 0,\ 2 \leq i \leq t.$$

And

$$(\tilde{w}_1, \tilde{w}) = (\tilde{w}_1, \tilde{w}_1) + (\tilde{w}_1, \tilde{w}_2) + (\tilde{w}_1, \tilde{w}_t) = -1 + 2 - 1 = 0.$$

Thus by (4.2) we have $(\tilde{w}, \tilde{w}) = 0$ which implies that $\tilde{w} = 0$. As desired.

(ii) Let A_n $(n \geq 3)$ be the $n \times n$ tridiagonal Gram matrix $G(\tilde{w}_1, \tilde{w}_2, \ldots, \tilde{w}_{t-1})$ of the set Φ,

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 & . & 0 \\ -1 & 2 & -1 & 0 & . & 0 \\ 0 & -1 & 2 & -1 & . & 0 \\ . & . & . & . & . & . \\ 0 & . & . & -1 & 2 & -1 \\ 0 & . & . & . & -1 & 2 \end{pmatrix},$$

and let $d_r = det(A_r)$. By cofactor expansion along the first row, we have

$$d_n = 2d_{n-1} + \begin{vmatrix} -1 & 2 & -1 & 0 & . & 0 \\ 0 & -1 & 2 & -1 & . & 0 \\ . & . & . & . & . & . \\ 0 & . & . & -1 & 2 & -1 \\ 0 & . & . & . & -1 & 2 \end{vmatrix}.$$

By similar expansion along the first column of the above $(n-1) \times (n-1)$ determinant, we obtain

$$d_n = 2d_{n-1} - d_{n-2}. \tag{4.2}$$

Claim

$$d_n = n + 1, \tag{4.3}$$

we shall prove (4.3) by induction, first we observe that $d_1 = 2$ and $d_2 = 3$, then from (4.2), $d_3 = 4$. If we assume that $d_i = i + 1$ for all $i < n$, then from (4.2) $d_n = 2d_{n-1} - d_{n-2} = 2(n)-(n-1)$ thus $d_n = n+1$ as desired. Hence the set of vectors Φ is linearly independent. □

Corollary 4.1.1. The set of vectors $\{\tilde{w}_1, \tilde{w}_2, \ldots, \tilde{w}_{t-1}, \tilde{w}_t\}$ spans a $(t-1)$-dimensional subspace.

Proof. This result follows immediately since the whole set is linearly dependent and contains $t-1$ linearly independent vectors. □

Lemma 4.1.2. *The neighbourhood $G_1(u)$ of a vertex u of G consists of a disjoint union of cycles with lengths divisible by 3. Moreover, if t is the girth of any such cycle, then each vertex $x \in V(G)$ with $d(u, x) = 2$ connects to exactly $\frac{t}{3}$ vertices of C_t.*

Proof. Since $\lambda = 2$, the induced subgraph $G_1(u)$ is 2-regular. Hence it is either a cycle by itself or a disjoint union of cycles.

We now prove that the length t of our cycle is a multiple of 3. Let $C_t = \{v_1, v_2, \ldots, v_t\}$ be a circle of girth t in $G_1(u)$. Consider the corresponding vectors $\tilde{w}_i$, $1 \leq i \leq t$. Notice that $\tilde{w}_i = \frac{3}{2}(v_i + \frac{1}{3}u) = \frac{3}{2}v_i + \frac{1}{2}u$, $1 \leq i \leq t$. Now let x be a vector-vertex at distance 2 from u. Suppose that x connects to s vertices of C_t. This implies that it does not connect to exactly $t - s$ vertices of C_t. As $x \in G_2(u)$ we know that $(u, x) = \frac{1}{9}$. Now for

$$\tilde{w} = \tilde{w}_1 + \tilde{w}_2 + \cdots + \tilde{w}_t,$$

we have

$$(\tilde{w}, x) = (\tilde{w}_1, x) + (\tilde{w}_2, x) + \cdots + (\tilde{w}_t, x) = 0.$$

This implies that

$$(\frac{3}{2}v_1 + \frac{1}{2}u, x) + (\frac{3}{2}v_2 + \frac{1}{2}u, x) + \cdots + (\frac{3}{2}v_t + \frac{1}{2}u, x) = 0.$$

Hence

$$\frac{3}{2}(v_1, x) + \frac{1}{2}(u, x) + \frac{3}{2}(v_2, x) + \frac{1}{2}(u, x) + \cdots + \frac{3}{2}(v_t, x) + \frac{1}{2}(u, x) = 0.$$

Since x connects exactly to s of the vectors v_i, s of the above inner products (v_i, x) give $\frac{-1}{3}$ while $t - s$ of them give $\frac{1}{9}$. Thus we have $\frac{t}{2}\frac{1}{9} + \frac{3}{2}s(\frac{-1}{3}) + \frac{3}{2}(t-s)\frac{1}{9} = 0$. Hence $\frac{t}{3} = s$. But as s is necessarily an integer, t must be divisible by 3. □

Remark 4.1.1.

- *The representation of the* 21*-Cycle would require* 20 *linearly independent vectors which is impossible in our* 18 *dimensional subspace.*

- *If we have an* 18*-Cycle, then the other disjoint vertices must form a* 3*-Cycle. Thus we require* 19 *linearly independent vectors for such a representation which is also impossible in our* 18 *dimensional subspace.*

- *In the same way, by lemma 4.1.1, if we let C_t be a t-Cycle in $G_1(U)$, and define its complement $\overline{C_t}$ to be a $(21-t)$-Cycle in $G_1(U)$, then $G_1(U)$ cannot contain both C_t and $\overline{C_t}$.*

We now construct the vector representation for the vertices in $G_2(u)$. We denote the vectors at distance 2 from u by x_i, so that $D_2(u) = \{x_1, x_2, \ldots, x_{54}\}$. Again we map the vertices at distance 2 from u to the 18 dimensional subspace $u^\perp$ constructed as follows. We construct vector $z_i \in u^\perp$ as $z_i = 3x_i - \frac{1}{3}u$. Hence for each x_i $\exists$ $z_i \in u^\perp$ such that $(z_i, u) = 0$. We have the following inner products

$$(z_i, z_j) = \begin{cases} 80/9 & \text{if } i = j, \\ -28/9 & \text{if } i \neq j \text{ and } x_i \text{ adjacent to } x_j, \\ 8/9 & \text{if } i \neq j \text{ and } x_i \text{ not adjacent to } x_j. \end{cases}$$

Furthermore, we construct vectors $\tilde{z}_i$ which would have integer norms by taking $\tilde{z}_i = \frac{3}{4} z_i$. So we have

$$(\tilde{z}_i, \tilde{z}_j) = \begin{cases} 5 & \text{if } i = j, \\ -7/4 & \text{if } i \neq j \text{ and } x_i \text{ adjacent to } x_j, \\ 1/2 & \text{if } i \neq j \text{ and } x_i \text{ not adjacent to } x_j. \end{cases}$$

Thus for each x_i, we have a corresponding $\tilde{z}_i$ in the 18 dimensional subspace $u^\perp$. So, invariably, the fixed vertex in G having image u has the vertices at distance two mapped into the vectors $\{\tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_{54}\}$.

Keeping in mind the definitions of $\tilde{z}_i$ and $\tilde{w}_j$, we get

$$(\tilde{z}_i, \tilde{w}_j) = \begin{cases} -1 & \text{if } x_i \text{ adjacent to } v_j, \\ 1/2 & \text{if } x_i \text{ not adjacent to } v_j. \end{cases}$$

Given all the results we have had till now and the various constructions of inner products, we have enough tools to begin our search for the forbidden configurations in G, or at least to restrict the edge connectivity as much as possible so that some counting techniques can be used to conclude the nonexistence.

Our next approach is to consider the projection of vectors in $G_2(u)$ onto the subspace spanned by the neighbours of u. To do this, we pick a vector $\tilde{z}_i$ corresponding to a vector x_i in $G_2(u)$, then we take a projection of $\tilde{z}_i$ onto the subspace spanned by the vectors on a cycle C_t in $G_1(u)$.

We know that any vector x_i in $G_2(u)$ connects to exactly $t/3$ vectors of C_t in $G_1(u)$. We shall also see that these projections depend on the edge connectivity between the vector x_i

and its neighbours on C_t. Such dependence of the projections provides us with a very useful information.

We now recall the basic definition of a projection.

Definition 4.1.1. *The* ***Projection*** *of a vector $\vec{x}$ to the subspace spanned by vectors $V = \{v_1, v_2, \ldots, v_n\}$ is a unique vector say $\vec{x}^p \in span(V)$ such that*

$$\vec{x} = \vec{x}^p + y, \tag{4.4}$$

where y is a unique vector in $span(V)^\perp$. And as such, $\vec{x}^p$ is of the form

$$\vec{x}^p = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n,$$

where $\{\alpha_i\}_{i=1}^n$ are all real numbers.

4.1.1 The 3-Cycle

Let $C_3 = \{v_1, v_2, v_3\}$ be a 3-cycle in $G_1(u)$ and $\{\tilde{w}_1, \tilde{w}_2, \tilde{w}_3\}$ be its corresponding vectors. Define a projection of $\tilde{z}_i$, $1 \leq i \leq 3$, to the subspace spanned by $\{\tilde{w}_1, \tilde{w}_2, \tilde{w}_3\}$. Let $A_3 = \{\tilde{w}_1, \tilde{w}_2, \tilde{w}_3\}$ and $A_2 = \{\tilde{w}_1, \tilde{w}_2\}$, then clearly $SpanA_2 \subseteq SpanA_3$. But by lemma 4.1.1, we know that $dim(A_2) = dim(A_3)$. Hence $SpanA_3 = SpanA_2$.

Now for a vector $x_i \in G_2(u)$, we first compute its corresponding vector $\tilde{z}_i \in u^\perp$, then seek its projection to $S = SpanA_2$. By definition 4.1.2, the *projection* of $\tilde{z}_i$ to S is the unique vector say $\tilde{w} \in A_2$ such that

$$\tilde{z}_i = \tilde{w} + y,$$

where $y \in S^\perp$ is also unique. Thus $\tilde{w}$ is of the form

$$\tilde{w} = \alpha_1 \tilde{w}_1 + \alpha_2 \tilde{w}_2,$$

for some $\alpha_1, \alpha_2 \in R$. It follows that $(\tilde{w}, y) = 0$. WLOG, we may assume that the vector $x_i \in G_2(u)$ is adjacent to $v_1 \in G_1(u)$. So $(\tilde{z}_i, \tilde{w}_1) = -1$ and $(\tilde{z}_i, \tilde{w}_2) = (\tilde{z}_i, \tilde{w}_3) = \frac{1}{2}$. By computing the inner products of both $(\tilde{z}_i, \tilde{w}_1)$ and $(\tilde{z}_i, \tilde{w}_2)$, we get the following equations

$$-1 = 2\alpha_1 - \alpha_2.$$
$$\frac{1}{2} = -\alpha_1 + 2\alpha_2.$$

Hence $\alpha_1 = -\frac{1}{2}$ and $\alpha_2 = 0$. So the projection of $\tilde{z}_i$ to the S is $\tilde{w} = -\frac{1}{2}\tilde{w}_1$. Generally, if x_i connects to v_j on a 3-cycle, then the projection of $\tilde{z}_i$ to the subspace spanned by $\{\tilde{w}_1, \tilde{w}_2, \tilde{w}_3\}$ is $\tilde{w} = -\frac{1}{2}\tilde{w}_j$ with length $(\tilde{w}, \tilde{w}) = \frac{1}{4}(\tilde{w}_j, \tilde{w}_j) = \frac{1}{2}$. This shows that the 3-Cycle might not be a forbidden structure which is consistent with the work of Haemers [10]. We should note that the conditions we are looking for are only necessary conditions.

4.1.2 The 6-Cycle

In the case of t-Cycle with $t \geq 6$, where we have the vector $x_i \in G_2(u)$ connecting to more than one vector on C_t, we use different method of connecting x_i to its neighbours on C_t based on the distance between them on the cycle C_t itself. Also, we shall keep in mind the symmetry of the cycle. So it suffices to assume that the vector x_i connects to $\tilde{w}_1$ and $\frac{t}{3} - 1$ other vertices on C_t. Such idea described above would come into fruition as we consider the various cases of the 6-Cycle.

Let $C_6 = \{v_1, v_2, v_3, v_4, v_5, v_6\}$, its corresponding vectors $A_6 = \{\tilde{w}_1, \tilde{w}_2, \tilde{w}_3, \tilde{w}_4, \tilde{w}_5, \tilde{w}_6\}$ and $A_5 = \{\tilde{w}_1, \tilde{w}_2, \tilde{w}_3, \tilde{w}_4, \tilde{w}_5\}$. Now we seek the projection of $\tilde{z}_i$ to the subspace spanned by A_6 using the same argument as in the case of A_3. Lemma 4.1.1 gives $SpanA_6 = SpanA_5$. So we simply seek the projection of $\tilde{z}_i$ onto $S = SpanA_5$. By definition 4.1.2, $proj_S\tilde{z}_i$ is the unique vector $\tilde{w} \in S$ such that

$$\tilde{z}_i = \tilde{w} + y,$$

where $y \in S^{\perp}$ is also unique. Thus again, $\tilde{w}$ is of the form

$$\tilde{w} = \alpha_1\tilde{w}_1 + \alpha_2\tilde{w}_2 + \alpha_3\tilde{w}_3 + \alpha_4\tilde{w}_4 + \alpha_5\tilde{w}_5,$$

for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \in R$. It follows that $(\tilde{w}, y) = 0$. Now we consider three different cases.

Case 1: x_i is connected to two adjacent vertices $(v_k \sim v_j)$. WLOG, we may assume x_i is adjacent to v_1 and v_2 on C_6. Hence for the vector $\tilde{z}_i$ corresponding to $x_i \in G_2(u)$, we have $(\tilde{z}_i, \tilde{w}_1) = (\tilde{z}_i, \tilde{w}_2) = -1$ and $(\tilde{z}_i, \tilde{w}_3) = (\tilde{z}_i, \tilde{w}_4) = (\tilde{z}_i, \tilde{w}_5) = \frac{1}{2}$. By computing the inner products $(\tilde{z}_i, \tilde{w}_i)$, $1 \leq i \leq 5$, we get the following equations

$$\begin{aligned} -1 &= 2\alpha_1 - \alpha_2. \\ -1 &= -\alpha_1 + 2\alpha_2 - \alpha_3. \\ \frac{1}{2} &= -\alpha_2 + 2\alpha_3 - \alpha_4. \\ \frac{1}{2} &= -\alpha_3 + 2\alpha_4 - \alpha_5. \\ \frac{1}{2} &= -\alpha_4 + 2\alpha_5. \end{aligned}$$

Thus $\alpha_1 = \alpha_2 = -1$ and $\alpha_3 = 0$ $\alpha_4 = \alpha_5 = \frac{1}{2}$. Therefore,

$$proj_S\tilde{z}_i = \tilde{w} = -\tilde{w}_1 - \tilde{w}_2 + \frac{1}{2}\tilde{w}_4 + \frac{1}{2}\tilde{w}_5.$$

As $(\tilde{w}_1, \tilde{w}_4) = (\tilde{w}_1, \tilde{w}_5) = (\tilde{w}_2, \tilde{w}_4) = (\tilde{w}_2, \tilde{w}_5) = 0$, then the length of our projection is: $(\tilde{w}, \tilde{w}) = (\tilde{w}_1, \tilde{w}_1) + (\tilde{w}_1, \tilde{w}_2) + (\tilde{w}_2, \tilde{w}_1) + (\tilde{w}_2, \tilde{w}_2) + \frac{1}{4}(\tilde{w}_4, \tilde{w}_4) + \frac{1}{4}(\tilde{w}_4, \tilde{w}_5) + \frac{1}{4}(\tilde{w}_5, \tilde{w}_4) + \frac{1}{4}(\tilde{w}_5, \tilde{w}_5) = \frac{5}{2}$.

Case 2: x_i is connected to two vertices v_k, v_j with $d(v_k, v_j) = 2$. WLOG, we may

assume x_i is adjacent to v_1 and v_3 on C_6. Hence for the vector $\tilde{z}_i$ corresponding to $x_i \in G_2(u)$, we have $(\tilde{z}_i, \tilde{w}_1) = (\tilde{z}_i, \tilde{w}_3) = -1$ and $(\tilde{z}_i, \tilde{w}_2) = (\tilde{z}_i, \tilde{w}_4) = (\tilde{z}_i, \tilde{w}_5) = \frac{1}{2}$. By computing the inner products $(\tilde{z}_i, \tilde{w}_i)$, $1 \leq i \leq 5$, we get the following equations

$$\begin{aligned} -1 &= 2\alpha_1 - \alpha_2. \\ \frac{1}{2} &= -\alpha_1 + 2\alpha_2 - \alpha_3. \\ -1 &= -\alpha_2 + 2\alpha_3 - \alpha_4. \\ \frac{1}{2} &= -\alpha_3 + 2\alpha_4 - \alpha_5. \\ \frac{1}{2} &= -\alpha_4 + 2\alpha_5. \end{aligned}$$

Thus $\alpha_1 = \alpha_3 = -\frac{3}{4}$, $\alpha_4 = 0$, $\alpha_2 = -\frac{1}{2}$ and $\alpha_5 = \frac{1}{4}$. Therefore,

$$proj_S \tilde{z}_i = \tilde{w} = -\frac{3}{4}\tilde{w}_1 - \frac{1}{2}\tilde{w}_2 - \frac{3}{4}\tilde{w}_3 + \frac{1}{4}\tilde{w}_5.$$

As $(\tilde{w}_1, \tilde{w}_3) = (\tilde{w}_1, \tilde{w}_5) = (\tilde{w}_2, \tilde{w}_5) = (\tilde{w}_3, \tilde{w}_5) = 0$, then the length of our projection is $(\tilde{w}, \tilde{w}) = \frac{9}{16}(\tilde{w}_1, \tilde{w}_1) + \frac{3}{8}(\tilde{w}_1, \tilde{w}_2) + \frac{3}{8}(\tilde{w}_2, \tilde{w}_1) + \frac{1}{4}(\tilde{w}_2, \tilde{w}_2) + \frac{3}{8}(\tilde{w}_2, \tilde{w}_3) + \frac{3}{8}(\tilde{w}_3, \tilde{w}_2) + \frac{9}{16}(\tilde{w}_3, \tilde{w}_3) + \frac{1}{16}(\tilde{w}_5, \tilde{w}_5) = \frac{11}{8}$.

Case 3: x_i is connected to two vertices v_k, v_j with $d(v_k, v_j) = 3$. WLOG, we may assume x_i is adjacent to v_1 and v_4 on C_6. Hence for the vector $\tilde{z}_i$ corresponding to $x_i \in G_2(u)$, we have $(\tilde{z}_i, \tilde{w}_1) = (\tilde{z}_i, \tilde{w}_4) = -1$ and $(\tilde{z}_i, \tilde{w}_2) = (\tilde{z}_i, \tilde{w}_3) = (\tilde{z}_i, \tilde{w}_5) = \frac{1}{2}$. By computing the

inner products $(\tilde{z}_i, \tilde{w}_i)$, $1 \leq i \leq 5$, we get the following equations

$$\begin{aligned} -1 &= 2\alpha_1 - \alpha_2. \\ \frac{1}{2} &= -\alpha_1 + 2\alpha_2 - \alpha_3. \\ \frac{1}{2} &= -\alpha_2 + 2\alpha_3 - \alpha_4. \\ -1 &= -\alpha_3 + 2\alpha_4 - \alpha_5. \\ \frac{1}{2} &= -\alpha_4 + 2\alpha_5. \end{aligned}$$

Thus $\alpha_1 = \alpha_4 = -\frac{1}{2}$ and $\alpha_2 = \alpha_3 = \alpha_5 = 0$. Therefore,

$$proj_S \tilde{z}_i = \tilde{w} = -\frac{1}{2}\tilde{w}_1 - \frac{1}{2}\tilde{w}_4.$$

And hence the length of our projection is $(\tilde{w}, \tilde{w}) = \frac{1}{4}(\tilde{w}_1, \tilde{w}_1) + \frac{1}{4}(\tilde{w}_1, \tilde{w}_4) + \frac{1}{4}(\tilde{w}_4, \tilde{w}_1) + \frac{1}{4}(\tilde{w}_4, \tilde{w}_4) = 1$.

So far we have fixed the vector v_1 on C_6 to be a neighbour of $x \in G_2(u)$ on the 6-cycle along with one other vertex. So we shall denote the neighbours of x on the 6-cycle by $N(x) = \{v_1, v_i\}$. Then we let $\tilde{z}^p_{1i}$ be the projection of $\tilde{z}$ onto the subspace $S = span(A_6)$. We get the following summary table

Table 4.1: 6-Cycle: projections onto A_6 with their Lengths

Projection	Length
$\tilde{z}^p_{12} = -\tilde{w}_1 - \tilde{w}_2 + \frac{1}{2}\tilde{w}_4 + \frac{1}{2}\tilde{w}_5.$	$\frac{5}{2}$
$\tilde{z}^p_{13} = -\frac{3}{4}\tilde{w}_1 - \frac{1}{2}\tilde{w}_2 - \frac{3}{4}\tilde{w}_3 + \frac{1}{4}\tilde{w}_5.$	$\frac{11}{8}$
$\tilde{z}^p_{14} = -\frac{1}{2}\tilde{w}_1 - \frac{1}{2}\tilde{w}_4.$	1

We observe that any two vertices on the 6-Cycle neighbourhood of u have no other common neighbour on the 6-Cycle. Since $\lambda = 2$, it follows that there must be a vector say x_i adjacent to two adjacent vectors on C_6 and for each such vector x_i, there is a corresponding vector $\tilde{z}_i$. Hence there are six such vectors (one for each edge on C_6). We find the projections

of these 6 vectors say $\{\tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_6\}$ onto $A_6 = \{\tilde{w}_1, \tilde{w}_2, \ldots, \tilde{w}_6\}$. Then we form their Gram matrix. So, let us find such vectors.

Case 1: $N(x_1) = \{v_1, v_2\}$**.** We seek the projection of $\tilde{z}_1$ onto A_6. Going as before, we obtain the following system of equations

$$
\begin{aligned}
-1 &= 2\alpha_1 - \alpha_2 - \alpha_6. \\
-1 &= -\alpha_1 + 2\alpha_2 - \alpha_3. \\
\frac{1}{2} &= -\alpha_2 + 2\alpha_3 - \alpha_4. \\
\frac{1}{2} &= -\alpha_3 + 2\alpha_4 - \alpha_5. \\
\frac{1}{2} &= -\alpha_4 + 2\alpha_5 - \alpha_6. \\
\frac{1}{2} &= -\alpha_1 - \alpha_5 + 2\alpha_6.
\end{aligned}
$$

Thus $\alpha_1 = -5/2, \alpha_2 = -5/2, \alpha_3 = -3/2, \alpha_4 = -1, \alpha_5 = -1, \alpha_6 = -3/2$. Therefore,

$$\tilde{z}_1^p = -\frac{5}{2}\tilde{w}_1 - \frac{5}{2}\tilde{w}_2 - \frac{3}{2}\tilde{w}_3 - \tilde{w}_4 - \tilde{w}_5 - \frac{3}{2}\tilde{w}_6,$$

with length 2.5.

Case 2: $N(x_2) = \{v_2, v_3\}$**.** We seek the projection of $\tilde{z}_2$ onto A_6. Going as before, we obtain the following system of equations

$$\frac{1}{2} = 2\alpha_1 - \alpha_2 - \alpha_6.$$
$$-1 = -\alpha_1 + 2\alpha_2 - \alpha_3.$$
$$-1 = -\alpha_2 + 2\alpha_3 - \alpha_4.$$
$$\frac{1}{2} = -\alpha_3 + 2\alpha_4 - \alpha_5.$$
$$\frac{1}{2} = -\alpha_4 + 2\alpha_5 - \alpha_6.$$
$$\frac{1}{2} = -\alpha_1 - \alpha_5 + 2\alpha_6.$$

Thus $\alpha_1 = -3/4, \alpha_2 = -7/4, \alpha_3 = -7/4, \alpha_4 = -3/4, \alpha_5 = -1/4, \alpha_6 = -1/4$. Therefore,

$$\tilde{z}_2^p = -\frac{3}{4}\tilde{w}_1 - \frac{7}{4}\tilde{w}_2 - \frac{7}{4}\tilde{w}_3 - \frac{3}{4}\tilde{w}_4 - \frac{1}{4}\tilde{w}_8 - \frac{1}{4}\tilde{w}_9,$$

with length 2.5.

Case 3: $N(x_3) = \{v_3, v_4\}$ We seek the projection of $\tilde{z}_3$ onto A_6. We have the following system of equations

$$\frac{1}{2} = 2\alpha_1 - \alpha_2 - \alpha_6.$$
$$\frac{1}{2} = -\alpha_1 + 2\alpha_2 - \alpha_3.$$
$$-1 = -\alpha_2 + 2\alpha_3 - \alpha_4.$$
$$-1 = -\alpha_3 + 2\alpha_4 - \alpha_5.$$
$$\frac{1}{2} = -\alpha_4 + 2\alpha_5 - \alpha_6.$$
$$\frac{1}{2} = -\alpha_1 - \alpha_5 + 2\alpha_6.$$

Thus $\alpha_1 = 3/2, \alpha_2 = 1, \alpha_3 = 0, \alpha_4 = 0, \alpha_5 = 1, \alpha_6 = 3/2$. Therefore,

$$\tilde{z}_3^p = \frac{3}{2}\tilde{w}_1 + \tilde{w}_2 + \tilde{w}_5 + \frac{3}{2}\tilde{w}_6,$$

with length 2.5.

Case 4: $N(x_4) = \{v_4, v_5\}$ We seek the projection of $\tilde{z}_4$ onto A_6. We have the following equations

$$\begin{aligned}
\frac{1}{2} &= 2\alpha_1 - \alpha_2 - \alpha_6. \\
\frac{1}{2} &= -\alpha_1 + 2\alpha_2 - \alpha_3. \\
\frac{1}{2} &= -\alpha_2 + 2\alpha_3 - \alpha_4. \\
-1 &= -\alpha_3 + 2\alpha_4 - \alpha_5. \\
-1 &= -\alpha_4 + 2\alpha_5 - \alpha_6. \\
\frac{1}{2} &= -\alpha_1 - \alpha_5 + 2\alpha_6.
\end{aligned}$$

Thus $\alpha_1 = 3/2,\ \alpha_2 = 3/2,\ \alpha_3 = 1,\ \alpha_4 = 0,\ \alpha_5 = 0,\ \alpha_6 = 1$. Therefore,

$$\tilde{z}_4^p = \frac{3}{2}\tilde{w}_1 + \frac{3}{2}\tilde{w}_2 + \tilde{w}_3 + \tilde{w}_6,$$

with length 2.5.

Case 5: $N(x_5) = \{v_5, v_6\}$ We seek the projection of $\tilde{z}_5$ onto A_6.

$$
\begin{aligned}
-1 &= 2\alpha_1 - \alpha_2 - \alpha_9. \\
\frac{1}{2} &= -\alpha_1 + 2\alpha_2 - \alpha_3. \\
\frac{1}{2} &= -\alpha_2 + 2\alpha_3 - \alpha_4. \\
\frac{1}{2} &= -\alpha_3 + 2\alpha_4 - \alpha_5. \\
-1 &= -\alpha_4 + 2\alpha_5 - \alpha_6. \\
-1 &= -\alpha_1 - \alpha_5 + 2\alpha_6.
\end{aligned}
$$

Thus $\alpha_1 = 1,\ \alpha_2 = 3/2,\ \alpha_3 = 3/2,\ \alpha_4 = 1,\ \alpha_5 = 0,\ \alpha_6 = 0$. Therefore,

$$
\tilde{z}_5^p = \frac{1}{4}\tilde{w}_1 + \frac{3}{2}\tilde{w}_2 + \frac{3}{2}\tilde{w}_3 + \tilde{w}_4,
$$

with length 2.5.

Case 6: $N(x_6) = \{v_1, v_6\}$ We seek the projection of $\tilde{z}_6$ onto A_6. We have the following equations

$$
\begin{aligned}
-1 &= 2\alpha_1 - \alpha_2 - \alpha_6. \\
\frac{1}{2} &= -\alpha_1 + 2\alpha_2 - \alpha_3. \\
\frac{1}{2} &= -\alpha_2 + 2\alpha_3 - \alpha_4. \\
\frac{1}{2} &= -\alpha_3 + 2\alpha_4 - \alpha_5. \\
\frac{1}{2} &= -\alpha_4 + 2\alpha_5 - \alpha_6. \\
-1 &= -\alpha_7 - \alpha_5 + 2\alpha_6.
\end{aligned}
$$

Thus $\alpha_1 = 0,\ \alpha_2 = 1,\ \alpha_3 = 3/2,\ \alpha_4 = 3/2,\ \alpha_5 = 1,\ \alpha_6 = 0$. Therefore,

$$\tilde{z}_6^p = \tilde{w}_2 + \frac{3}{2}\tilde{w}_3 + \frac{3}{2}\tilde{w}_4 - \tilde{w}_5,$$

with length 2.5.

The inner products among all these projection vectors form the Gram matrix, see Table 4.2 below. From such matrix and its eigenvalues, we cannot quite discover any forbidden configuration with respect to the 6-Cycle. But these projections have given us insight on how the edges from vertices in $G_2(u)$ can connect to vertices on the cycle.

Table 4.2: The Gram matrix of projections onto the 6-cycle

	$\tilde{z}_1^p$	$\tilde{z}_2^p$	$\tilde{z}_3^p$	$\tilde{z}_4^p$	$\tilde{z}_5^p$	$\tilde{z}_6^p$
$\tilde{z}_1^p$	2.5	1	-1.25	-2	-1.25	1
$\tilde{z}_2^p$	1	2.5	-1	-1.25	-2	-1.25
$\tilde{z}_3^p$	-1.25	1	2.5	1	-1.25	-2
$\tilde{z}_4^p$	-2	-1.25	1	2.5	1	-1.25
$\tilde{z}_5^p$	-1.25	-2	-1.25	1	2.5	1
$\tilde{z}_6^p$	1	-1.25	-2	-1.25	1	2.5

4.1.3 The 9-Cycle

Let $C_9 = \{v_1, v_2, v_3, \ldots, v_9\}$, v_k, v_l and v_j be the neighbors of $x_i \in G_2(u)$ on C_9 and $\tilde{z}_i$ be the vector in $u^\perp$ corresponding to x_i. Let $A_9 = \{\tilde{w}_1, \tilde{w}_2, \ldots, \tilde{w}_9\}$ and $A_8 = \{\tilde{w}_1, \tilde{w}_2, \ldots, \tilde{w}_8\}$. As before, by lemma 4.1.1, we have $SpanA_9 = SpanA_8 = S$. We now seek the projection $\tilde{w} = Proj_S \tilde{z}_i$. By definition 4.1.2,

$$\tilde{z}_i = \tilde{w} + y,$$

where $y \in S^\perp$ and $\tilde{w}$ is of the form

$$\tilde{w} = \alpha_1\tilde{w}_1 + \alpha_2\tilde{w}_2 + \cdots + \alpha_8\tilde{w}_8.$$

Case 1: x_i is connected to v_k, v_j, v_l with $v_k \sim v_j$, $v_k \sim v_l$ and $d(v_j, v_l) = 2$. WLOG, we may assume x_i is adjacent to v_1, v_2 and v_3 on C_9. Hence for the vector $\tilde{z}_i$ corresponding to $x_i \in G_2(u)$, we have $(\tilde{z}_i, \tilde{w}_1) = (\tilde{z}_i, \tilde{w}_2) = (\tilde{z}_i, \tilde{w}_3) = -1$ and $(\tilde{z}_i, \tilde{w}_3) = (\tilde{z}_i, \tilde{w}_4) = (\tilde{z}_i, \tilde{w}_5) = \frac{1}{2}$. By computing the inner products $(\tilde{z}_i, \tilde{w}_i)$ we get the following equations

$$
\begin{aligned}
-1 &= 2\alpha_1 - \alpha_2. \\
-1 &= -\alpha_1 + 2\alpha_2 - \alpha_3. \\
-1 &= -\alpha_2 + 2\alpha_3 - \alpha_4. \\
\frac{1}{2} &= -\alpha_3 + 2\alpha_4 - \alpha_5. \\
\frac{1}{2} &= -\alpha_4 + 2\alpha_5 - \alpha_6. \\
\frac{1}{2} &= -\alpha_5 + 2\alpha_6 - \alpha_7. \\
\frac{1}{2} &= -\alpha_6 + 2\alpha_7 - \alpha_8. \\
\frac{1}{2} &= -\alpha_7 + 2\alpha_8.
\end{aligned}
$$

Thus $\alpha_1 = \alpha_3 = -\frac{3}{2}$ and $\alpha_2 = 2$ $\alpha_4 = 0$ $\alpha_6 = \alpha_7 = \frac{3}{2}$ $\alpha_5 = \alpha_8 = 1$. Therefore,

$$proj_S \tilde{z}_i = \tilde{w} = -\frac{3}{2}\tilde{w}_1 - 2\tilde{w}_2 - \frac{3}{2}\tilde{w}_3 + \tilde{w}_5 + \frac{3}{2}\tilde{w}_6 + \frac{3}{2}\tilde{w}_7 + \tilde{w}_8.$$

As $(\tilde{w}_1, \tilde{w}_3) = (\tilde{w}_1, \tilde{w}_5) = (\tilde{w}_1, \tilde{w}_6) = (\tilde{w}_1, \tilde{w}_7) = (\tilde{w}_1, \tilde{w}_8) = (\tilde{w}_2, \tilde{w}_5) = (\tilde{w}_2, \tilde{w}_6) = (\tilde{w}_2, \tilde{w}_7) = (\tilde{w}_2, \tilde{w}_8) = (\tilde{w}_3, \tilde{w}_5) = (\tilde{w}_3, \tilde{w}_6) = (\tilde{w}_3, \tilde{w}_7) = (\tilde{w}_3, \tilde{w}_8) = (\tilde{w}_5, \tilde{w}_7) = (\tilde{w}_5, \tilde{w}_8) = (\tilde{w}_6, \tilde{w}_8) = 0$, the length of our projection is $(\tilde{w}, \tilde{w}) = \frac{9}{4}(\tilde{w}_1, \tilde{w}_1) + 3(\tilde{w}_1, \tilde{w}_2) + 3(\tilde{w}_2, \tilde{w}_1) + 4(\tilde{w}_2, \tilde{w}_2) + 3(\tilde{w}_2, \tilde{w}_3) + 3(\tilde{w}_3, \tilde{w}_2) + + \frac{9}{4}(\tilde{w}_3, \tilde{w}_3) + (\tilde{w}_5, \tilde{w}_5) + \frac{3}{2}(\tilde{w}_5, \tilde{w}_6) + \frac{3}{2}(\tilde{w}_6, \tilde{w}_5) + \frac{9}{4}(\tilde{w}_6, \tilde{w}_6) + \frac{9}{4}(\tilde{w}_6, \tilde{w}_7) + \frac{9}{4}(\tilde{w}_7, \tilde{w}_6) + \frac{9}{4}(\tilde{w}_7, \tilde{w}_7) + \frac{3}{2}(\tilde{w}_7, \tilde{w}_8) + \frac{3}{2}(\tilde{w}_8, \tilde{w}_7) + (\tilde{w}_8, \tilde{w}_8) = \frac{15}{2}$.

Case 2: x_i is connected to v_k, v_j, v_l with $d(v_k, v_j) = d(v_k, v_l) = 2$ and $d(v_j, v_l) = 4$. WLOG, we may assume x_i is adjacent to say v_1, v_3, v_5 on C_9. Hence for the vector $\tilde{z}_i$

corresponding to $x_i \in G_2(u)$, we have $(\tilde{z}_i, \tilde{w}_1) = (\tilde{z}_i, \tilde{w}_3) = (\tilde{z}_i, \tilde{w}_5) = -1$ and $(\tilde{z}_i, \tilde{w}_2) = (\tilde{z}_i, \tilde{w}_4) = (\tilde{z}_i, \tilde{w}_6) = (\tilde{z}_i, \tilde{w}_7) = (\tilde{z}_i, \tilde{w}_8) = \frac{1}{2}$. By computing the inner products $(\tilde{z}_i, \tilde{w}_i)$, we get the following equations

$$
\begin{aligned}
-1 &= 2\alpha_1 - \alpha_2. \\
\frac{1}{2} &= -\alpha_1 + 2\alpha_2 - \alpha_3. \\
-1 &= -\alpha_2 + 2\alpha_3 - \alpha_4. \\
\frac{1}{2} &= -\alpha_3 + 2\alpha_4 - \alpha_5. \\
-1 &= -\alpha_4 + 2\alpha_5 - \alpha_6. \\
\frac{1}{2} &= -\alpha_5 + 2\alpha_6 - \alpha_7. \\
\frac{1}{2} &= -\alpha_6 + 2\alpha_7 - \alpha_8. \\
\frac{1}{2} &= -\alpha_7 + 2\alpha_8.
\end{aligned}
$$

Thus $\alpha_1 = \alpha_2 = \alpha_4 = \alpha_5 = -1$ and $\alpha_3 = \frac{3}{2}$ $\alpha_6 = 0$ $\alpha_7 = \alpha_8 = \frac{1}{2}$. Therefore,

$$proj_S \tilde{z}_i = \tilde{w} = -\tilde{w}_1 - \tilde{w}_2 - \frac{3}{2}\tilde{w}_3 - \tilde{w}_4 - \tilde{w}_5 + \frac{1}{2}\tilde{w}_7 + \frac{1}{2}\tilde{w}_8,$$

with length $(\tilde{w}, \tilde{w}) = (\tilde{w}_1, \tilde{w}_1) + (\tilde{w}_1, \tilde{w}_2) + (\tilde{w}_2, \tilde{w}_1) + (\tilde{w}_2, \tilde{w}_2) + \frac{3}{2}(\tilde{w}_2, \tilde{w}_3) + \frac{3}{2}(\tilde{w}_3, \tilde{w}_2) + \frac{9}{4}(\tilde{w}_3, \tilde{w}_3) + \frac{3}{2}(\tilde{w}_3, \tilde{w}_4) + \frac{3}{2}(\tilde{w}_4, \tilde{w}_3) + (\tilde{w}_4, \tilde{w}_4) + (\tilde{w}_4, \tilde{w}_5) + (\tilde{w}_5, \tilde{w}_4) + (\tilde{w}_5, \tilde{w}_5) + \frac{1}{4}(\tilde{w}_7, \tilde{w}_7) + \frac{1}{4}(\tilde{w}_7, \tilde{w}_8) + \frac{1}{4}(\tilde{w}_8, \tilde{w}_7) + \frac{1}{4}(\tilde{w}_8, \tilde{w}_8) = 3$.

Case 3: x_i is connected to v_k, v_l, v_j with $d(v_k, v_l) = d(v_k, v_j) = d(v_l, v_j) = 3$. WLOG, we may assume x_i is adjacent to v_1, v_4 and v_7 on C_9. Hence for the vector $\tilde{z}_i$ corresponding to $x_i \in G_2(u)$, we have $(\tilde{z}_i, \tilde{w}_1) = (\tilde{z}_i, \tilde{w}_4) = (\tilde{z}_i, \tilde{w}_7) = -1$ and $(\tilde{z}_i, \tilde{w}_2) = (\tilde{z}_i, \tilde{w}_3) = (\tilde{z}_i, \tilde{w}_5) = (\tilde{z}_i, \tilde{w}_6) = (\tilde{z}_i, \tilde{w}_8) = \frac{1}{2}$. By computing the inner products $(\tilde{z}_i, \tilde{w}_i)$, we get the following

equations

$$
\begin{aligned}
-1 &= 2\alpha_1 - \alpha_2. \\
\frac{1}{2} &= -\alpha_1 + 2\alpha_2 - \alpha_3. \\
\frac{1}{2} &= -\alpha_2 + 2\alpha_3 - \alpha_4. \\
-1 &= -\alpha_3 + 2\alpha_4 - \alpha_5. \\
\frac{1}{2} &= -\alpha_4 + 2\alpha_5 - \alpha_6. \\
\frac{1}{2} &= -\alpha_5 + 2\alpha_6 - \alpha_7. \\
-1 &= -\alpha_6 + 2\alpha_7 - \alpha_8. \\
\frac{1}{2} &= -\alpha_7 + 2\alpha_8.
\end{aligned}
$$

Thus $\alpha_1 = \alpha_4 = \alpha_7 = -\frac{1}{2}$ and $\alpha_2 = \alpha_3 = \alpha_5 = \alpha_6 = \alpha_8 = 0$. Therefore,

$$proj_S \tilde{z}_i = \tilde{w} = -\frac{1}{2}\tilde{w}_1 - \frac{1}{2}\tilde{w}_4 - \frac{1}{2}\tilde{w}_7,$$

with length $(\tilde{w}, \tilde{w}) = \frac{1}{4}(\tilde{w}_1, \tilde{w}_1) + \frac{1}{4}(\tilde{w}_4, \tilde{w}_4) + \frac{1}{4}(\tilde{w}_7, \tilde{w}_7) = \frac{3}{2}$.

Case 4: x_i is adjacent to v_k, v_l, v_j with $v_k \sim v_l$ and $d(v_k, v_j) = 2$. WLOG, we may assume that x_i is adjacent to v_1, v_2, v_4(We note that the case where x_i is adjacent to v_1, v_2, v_3 has already been considered as case 1). By similar procedure as the previous three cases we get

$$proj_S \tilde{z}_i = \tilde{w} = -\frac{4}{3}\tilde{w}_1 - \frac{5}{3}\tilde{w}_2 - \tilde{w}_3 - \frac{5}{6}\tilde{w}_4 + \frac{1}{3}\tilde{w}_5 + \tilde{w}_6 + \frac{7}{6}\tilde{w}_7 + \frac{5}{6}\tilde{w}_8,$$

with length equal to 5.

Case 5: x_i is connected to v_k, v_l, v_j with $v_k \sim v_l$ and $d(v_k, v_j) = 3$. WLOG, we may

assume x_i is adjacent to v_1, v_2, v_5. By similar procedure as the previous cases we get

$$proj_S \tilde{z}_i = \tilde{w} = -\frac{7}{6}\tilde{w}_1 - \frac{4}{3}\tilde{w}_2 - \frac{1}{2}\tilde{w}_3 - \frac{1}{6}\tilde{w}_4 - \frac{1}{3}\tilde{w}_5 + \frac{1}{2}\tilde{w}_6 + \frac{5}{6}\tilde{w}_7 + \frac{2}{3}\tilde{w}_8,$$

with length equal to 5.5.

Case 6: x_i is connected to v_k, v_l, v_j with $v_k \sim v_l$ and $d(v_k, v_j) = 4$ WLOG, we may assume x_i is adjacent to v_1, v_2, v_6. By similar procedure as the previous cases we get

$$proj_S \tilde{z}_i = \tilde{w} = -\tilde{w}_1 - \tilde{w}_2 + \frac{1}{2}\tilde{w}_4 + \frac{1}{2}\tilde{w}_5 + \frac{1}{2}\tilde{w}_7 + \frac{1}{2}\tilde{w}_8,$$

with length equal to 3.

Case 7: x_i is connected to v_k, v_l, v_j with $d(v_k, v_l) = 2$, $d(v_k, v_j) = 3$ and $d(v_l, v_j) = 4$. WLOG, we may assume x_i is adjacent to v_1, v_3, v_6. By similar procedure as the previous cases we get

$$proj_S \tilde{z}_i = \tilde{w} = -\frac{5}{6}\tilde{w}_1 - \frac{2}{3}\tilde{w}_2 - \tilde{w}_3 - \frac{1}{3}\tilde{w}_4 - \frac{1}{6}\tilde{w}_5 - \frac{1}{2}\tilde{w}_6 + \frac{1}{6}\tilde{w}_7 + \frac{1}{3}\tilde{w}_8,$$

with length equal to 2.

So far we have fixed the vector v_1 on C_9 to be a neighbour of $x_i \in G_2(u)$ on the 9-cycle along with two other vertices. So we shall denote the neighbours of x_i on the 9-cycle by $N(x) = \{v_1, v_i, v_j\}$. We let $\tilde{z}^p_{ij}$ be the projection of $\tilde{z}$ onto $span(A_9)$. We get the following summary table

Remark 4.1.2. *From the table it can be read immediately that cases 1 and 4 are not possible since they give projections of length greater or equal to 5, whereas the vectors $\tilde{z}_i$ themselves are of length 5.*

Table 4.3: 9-Cycle: projections onto A_9 with their Lengths

Projection	Length
$\tilde{z}^p_{23} = -\frac{3}{2}\tilde{w}_1 - 2\tilde{w}_2 - \frac{3}{2}\tilde{w}_3 + \tilde{w}_5 + \frac{3}{2}\tilde{w}_6 + \frac{3}{2}\tilde{w}_7 + \tilde{w}_8$	$\frac{15}{2}$
$\tilde{z}^p_{35} = -\tilde{w}_1 - \tilde{w}_2 - \frac{3}{2}\tilde{w}_3 - \tilde{w}_4 - \tilde{w}_5 + \frac{1}{2}\tilde{w}_7 + \frac{1}{2}\tilde{w}_8$	3
$\tilde{z}^p_{47} = -\frac{1}{2}\tilde{w}_1 - \frac{1}{2}\tilde{w}_4 - \frac{1}{2}\tilde{w}_7$	$\frac{3}{2}$
$\tilde{z}^p_{24} = -\frac{4}{3}\tilde{w}_1 - \frac{5}{3}\tilde{w}_2 - \tilde{w}_3 - \frac{5}{6}\tilde{w}_4 + \frac{1}{3}\tilde{w}_5 + \tilde{w}_6 + \frac{7}{6}\tilde{w}_7 + \frac{5}{6}\tilde{w}_8$	5
$\tilde{z}^p_{25} = -\frac{7}{6}\tilde{w}_1 - \frac{4}{3}\tilde{w}_2 - \frac{1}{2}\tilde{w}_3 - \frac{1}{6}\tilde{w}_4 - \frac{1}{3}\tilde{w}_5 + \frac{1}{2}\tilde{w}_6 + \frac{5}{6}\tilde{w}_7 + \frac{2}{3}\tilde{w}_8$	3.5
$\tilde{z}^p_{26} = -\tilde{w}_1 - \tilde{w}_2 + \frac{1}{2}\tilde{w}_4 + \frac{1}{2}\tilde{w}_5 + \frac{1}{2}\tilde{w}_7 + \frac{1}{2}\tilde{w}_8$	3
$\tilde{z}^p_{36} = -\frac{5}{6}\tilde{w}_1 - \frac{2}{3}\tilde{w}_2 - \tilde{w}_3 - \frac{1}{3}\tilde{w}_4 - \frac{1}{6}\tilde{w}_5 - \frac{1}{2}\tilde{w}_6 + \frac{1}{6}\tilde{w}_7 + \frac{1}{3}\tilde{w}_8$	2

4.1.4 The 12-Cycle

Let $A_{12} = \{\tilde{w}_1, \tilde{w}_2, \ldots, \tilde{w}_{12}\}$ and the neighbours of $x \in G_2(u)$ on the 12-cycle be $N(x) = \{v_1, v_i, v_j, v_k\}$. Let $\tilde{z}^p_{ijk}$ be the projection of $\tilde{z}$ onto $span(A_{12})$. As exactly we did for the 9-cycle, we consider all the different cases of how the vector $x_i \in G_2(u)$ connects to vectors on the 12-Cycle. We have the following cases:

Case 1: $N(x) = \{v_1, v_2, v_3, v_4\}$**.** The projection is

$$\tilde{z}^p_{234} = -2\tilde{w}_1 - 3\tilde{w}_2 - 3\tilde{w}_3 - 2\tilde{w}_4 + \frac{3}{2}\tilde{w}_6 + \frac{5}{2}\tilde{w}_7 + 3\tilde{w}_8 + 3\tilde{w}_9 + \frac{5}{2}\tilde{w}_{10} + \frac{3}{2}\tilde{w}_{11},$$

with length equal to 17.

Case 2: $N(x) = \{v_1, v_2, v_3, v_5\}$**.** The projection is

$$\tilde{z}^p_{235} = -\frac{15}{8}\tilde{w}_1 - \frac{11}{4}\tilde{w}_2 - \frac{21}{8}\tilde{w}_3 - \frac{3}{2}\tilde{w}_4 - \frac{7}{8}\tilde{w}_5 + \frac{3}{4}\tilde{w}_6 + \frac{15}{8}\tilde{w}_7 + \frac{5}{2}\tilde{w}_8 + \frac{21}{8}\tilde{w}_9 + \frac{9}{4}\tilde{w}_{10} + \frac{11}{8}\tilde{w}_{11},$$

with length equal to 13.0625.

Case 3: $N(x) = \{v_1, v_2, v_3, v_6\}$**.** The projection is

$$\tilde{z}^p_{236} = -\frac{7}{4}\tilde{w}_1 - \frac{5}{2}\tilde{w}_2 - \frac{9}{4}\tilde{w}_3 - \tilde{w}_4 - \frac{1}{4}\tilde{w}_5 + \frac{5}{4}\tilde{w}_7 + 2\tilde{w}_8 + \frac{9}{4}\tilde{w}_9 + 2\tilde{w}_{10} + \frac{5}{4}\tilde{w}_{11},$$

with length equal to 10.25.

Case 4: $N(x) = \{v_1, v_2, v_3, v_7\}$**.** The projection is

$$\tilde{z}^p_{237} = -\frac{13}{8}\tilde{w}_1 - \frac{9}{4}\tilde{w}_2 - \frac{15}{8}\tilde{w}_3 - \frac{1}{2}\tilde{w}_4 + \frac{3}{8}\tilde{w}_5 + \frac{3}{4}\tilde{w}_6 + \frac{5}{8}\tilde{w}_7 + \frac{3}{2}\tilde{w}_8 + \frac{15}{8}\tilde{w}_9 + \frac{7}{4}\tilde{w}_{10} + \frac{9}{8}\tilde{w}_{11},$$

with length equal to 8.5625.

Case 5: $N(x) = \{v_1, v_2, v_4, v_5\}$**.** The projection is

$$\tilde{z}^p_{245} = -\frac{7}{4}\tilde{w}_1 - \frac{5}{2}\tilde{w}_2 - \frac{9}{4}\tilde{w}_3 - \frac{5}{2}\tilde{w}_4 - \frac{7}{4}\tilde{w}_5 + \frac{5}{4}\tilde{w}_7 + 2\tilde{w}_8 + \frac{9}{4}\tilde{w}_9 + 2\tilde{w}_{10} + \frac{5}{4}\tilde{w}_{11},$$

with length equal to 11.75.

Case 6: $N(x) = \{v_1, v_2, v_4, v_6\}$**.** The projection is

$$\tilde{z}^p_{246} = -\frac{13}{8}\tilde{w}_1 - \frac{9}{4}\tilde{w}_2 - \frac{15}{8}\tilde{w}_3 - 2\tilde{w}_4 - \frac{9}{8}\tilde{w}_5 - \frac{3}{4}\tilde{w}_6 + \frac{5}{8}\tilde{w}_7 + \frac{3}{2}\tilde{w}_8 + \frac{15}{8}\tilde{w}_9 + \frac{7}{4}\tilde{w}_{10} + \frac{9}{8}\tilde{w}_{11},$$

with length equal to 8.5625.

Case 7: $N(x) = \{v_1, v_2, v_4, v_7\}$**.** The projection is

$$\tilde{z}^p_{247} = -\frac{3}{2}\tilde{w}_1 - 2\tilde{w}_2 - \frac{3}{2}\tilde{w}_3 - \frac{3}{2}\tilde{w}_4 - \frac{1}{2}\tilde{w}_5 + \tilde{w}_8 + \frac{3}{2}\tilde{w}_9 + \frac{3}{2}\tilde{w}_{10} + \tilde{w}_{11},$$

with length equal to 6.5.

Case 8: $N(x) = \{v_1, v_2, v_4, v_8\}$**.** The projection is

$$\tilde{z}^p_{248} = -\frac{11}{8}\tilde{w}_1 - \frac{7}{4}\tilde{w}_2 - \frac{9}{8}\tilde{w}_3 - \tilde{w}_4 + \frac{1}{8}\tilde{w}_5 + \frac{3}{4}\tilde{w}_6 + \frac{7}{8}\tilde{w}_7 + \frac{1}{2}\tilde{w}_8 + \frac{9}{8}\tilde{w}_9 + \frac{5}{4}\tilde{w}_{10} + \frac{7}{8}\tilde{w}_{11},$$

with length equal to 5.5625.

Case 9: $N(x) = \{v_1, v_2, v_4, v_9\}$**.** The projection is

$$\tilde{z}^p_{249} = -\frac{5}{4}\tilde{w}_1 - \frac{3}{2}\tilde{w}_2 - \frac{3}{4}\tilde{w}_3 - \frac{1}{2}\tilde{w}_4 + \frac{3}{4}\tilde{w}_5 + \frac{3}{2}\tilde{w}_6 + \frac{7}{4}\tilde{w}_7 + \frac{3}{2}\tilde{w}_8 + \frac{3}{4}\tilde{w}_9 + \tilde{w}_{10} + \frac{3}{4}\tilde{w}_{11},$$

with length equal to 5.75.

Case 10: $N(x) = \{v_1, v_2, v_5, v_6\}$**.** The projection is

$$\tilde{z}^p_{256} = -\frac{3}{2}\tilde{w}_1 - 2\tilde{w}_2 - \frac{3}{2}\tilde{w}_3 - \frac{3}{2}\tilde{w}_4 - 2\tilde{w}_5 - \frac{3}{2}\tilde{w}_6 + \tilde{w}_8 + \frac{3}{2}\tilde{w}_9 + \frac{3}{2}\tilde{w}_{10} + \tilde{w}_{11},$$

with length equal to 8.

Case 11: $N(x) = \{v_1, v_2, v_5, v_7\}$**.** The projection is

$$\tilde{z}^p_{257} = -\frac{11}{8}\tilde{w}_1 - \frac{7}{4}\tilde{w}_2 - \frac{9}{8}\tilde{w}_3 - \tilde{w}_4 - \frac{11}{8}\tilde{w}_5 - \frac{3}{4}\tilde{w}_6 - \frac{5}{8}\tilde{w}_7 + \frac{1}{2}\tilde{w}_8 + \frac{9}{8}\tilde{w}_9 + \frac{5}{4}\tilde{w}_{10} + \frac{7}{8}\tilde{w}_{11},$$

with length equal to 5.5625.

Case 12: $N(x) = \{v_1, v_2, v_5, v_8\}$**.** The projection is

$$\tilde{z}^p_{258} = -\frac{5}{4}\tilde{w}_1 - \frac{3}{2}\tilde{w}_2 - \frac{3}{4}\tilde{w}_3 - \frac{1}{2}\tilde{w}_4 - \frac{3}{4}\tilde{w}_5 + \frac{1}{4}\tilde{w}_7 + \frac{3}{4}\tilde{w}_9 + \tilde{w}_{10} + \frac{3}{4}\tilde{w}_{11},$$

with length equal to 4.25.

Case 13: $N(x) = \{v_1, v_2, v_5, v_9\}$**.** The projection is

$$\tilde{z}^p_{259} = -\frac{9}{8}\tilde{w}_1 - \frac{5}{4}\tilde{w}_2 - \frac{3}{8}\tilde{w}_3 - \frac{1}{8}\tilde{w}_5 + \frac{3}{4}\tilde{w}_6 + \frac{9}{8}\tilde{w}_7 + \tilde{w}_8 + \frac{3}{8}\tilde{w}_9 + \frac{3}{4}\tilde{w}_{10} + \frac{5}{8}\tilde{w}_{11},$$

with length equal to 4.0625.

Case 14: $N(x) = \{v_1, v_2, v_6, v_7\}$**.** The projection is

$$\tilde{z}^p_{267} = -\frac{5}{4}\tilde{w}_1 - \frac{3}{2}\tilde{w}_2 - \frac{3}{4}\tilde{w}_3 - \frac{1}{2}\tilde{w}_4 - \frac{3}{4}\tilde{w}_5 - \frac{3}{2}\tilde{w}_6 - \frac{5}{4}\tilde{w}_7 + \frac{3}{4}\tilde{w}_9 + \tilde{w}_{10} + \frac{3}{4}\tilde{w}_{11},$$

with length equal to 5.75.

Case 15: $N(x) = \{v_1, v_2, v_6, v_8\}$**.** The projection is

$$\tilde{z}^p_{268} = -\frac{9}{8}\tilde{w}_1 - \frac{5}{4}\tilde{w}_2 - \frac{3}{8}\tilde{w}_3 - \frac{1}{8}\tilde{w}_5 - \frac{3}{4}\tilde{w}_6 - \frac{3}{8}\tilde{w}_7 - \frac{1}{2}\tilde{w}_8 + \frac{3}{8}\tilde{w}_9 + \frac{3}{4}\tilde{w}_{10} + \frac{5}{8}\tilde{w}_{11},$$

with length equal to 4.0625.

Case 16: $N(x) = \{v_1, v_2, v_6, v_9\}$**.** The projection is

$$\tilde{z}^p_{269} = -\tilde{w}_1 - \tilde{w}_2 + \frac{1}{2}\tilde{w}_4 + \frac{1}{2}\tilde{w}_5 + \frac{1}{2}\tilde{w}_7 + \frac{1}{2}\tilde{w}_8 + \frac{1}{2}\tilde{w}_{10} + \frac{1}{2}\tilde{w}_{11},$$

with length equal to 3.5.

Case 17: The neighbours of x_i on C_{12} are at distance 3 from each other. The projection is

$$-\frac{2}{8}\tilde{w}_1 - \frac{1}{2}\tilde{w}_4 - \frac{1}{2}\tilde{w}_7 - \frac{1}{2}\tilde{w}_{10},$$

with length equal to 2.

We get the following summary table

Table 4.4: 12-Cycle: projections onto A_{12} with their Lengths

Projection	Length	Projection	Length
$\tilde{z}^p_{234}$	17	$\tilde{z}^p_{235}$	13.0625
$\tilde{z}^p_{236}$	10.25	$\tilde{z}^p_{237}$	8.5265
$\tilde{z}^p_{245}$	11.75	$\tilde{z}^p_{246}$	8.5265
$\tilde{z}^p_{247}$	6.5	$\tilde{z}^p_{248}$	5.5625
$\tilde{z}^p_{249}$	5.75	$\tilde{z}^p_{256}$	8
$\tilde{z}^p_{257}$	5.5625	$\tilde{z}^p_{258}$	4.25
$\tilde{z}^p_{259}$	4.0625	$\tilde{z}^p_{267}$	5.75
$\tilde{z}^p_{268}$	4.0625	$\tilde{z}^p_{269}$	3.5
$\tilde{z}^p_{375}$	5.75	$\tilde{z}^p_{4710}$	2
$\tilde{z}^p_{238}$	18	$\tilde{z}^p_{2410}$	7.0625
$\tilde{z}^p_{2411}$	9.5	$\tilde{z}^p_{2510}$	5
$\tilde{z}^p_{278}$	5	$\tilde{z}^p_{358}$	4.0625
$\tilde{z}^p_{359}$	3.5	$\tilde{z}^p_{368}$	3.5
$\tilde{z}^p_{369}$	2.5625	$\tilde{z}^p_{3610}$	2.75
$\tilde{z}^p_{379}$	2.75		

All the 29 possible cases were considered up to the symmetry of the 12-cycle. It can be read from this table that it is impossible to have the 12-cycle. In fact, the number of projections having length less than 5 do not provide us with sufficient edge connectivity to the 12-Cycle.

4.1.5 The 15-Cycle

Here we shall simply present the lengths of the projections considered. It should be noted that as with previous cases, we considered the projections up to symmetry of the 15-Cycle. Let $A_{15} = \{\tilde{w}_1, \tilde{w}_2, \ldots, \tilde{w}_{15}\}$ and the neighbours of $x \in G_2(u)$ on the 15-cycle be $N(x) = \{v_1, v_i, v_j, v_k, v_l\}$. Let $\tilde{z}^p_{ijkl}$ be the projection of $\tilde{z}$ onto $span(A_{15})$. The following table provide such projections with their corresponding length.

It can be read from table 4.5 that it is impossible to have the 15-cycle. In fact, the number of projections having length less than 5 do not provide us with sufficient edge connectivity to the 15-Cycle.

4.1.6 Integration with counting methods

We shall integrate our vector representation of $G = srg(76, 21, 2, 7)$ with some combinatorial properties in order to figure out some forbidden configurations inside G, and subsequently give our conclusions.

Let S denotes the graph $K_4 \setminus e$, where $e \in E(K_4)$ is any edge. If we can show that S cannot be an induced subgraph of G via our unit vector representation, then we ruled out all t-cycle cases, where $t > 3$. Hence we only left with the 3-cycle case in the neighborhood of any fixed vertex $u \in G$. However, this implies that such neighborhood of u including u itself consists of disjoint union of K_4 cliques. Hence by S. Dixmier and F.Zara [10] G is actually the nonexistence partial geometry $pg(3, 6, 1)$.

Table 4.5: 15-Cycle: projections onto A_{15} with their Lengths

Projection	Length	Projection	Length	Projection	Length
z^p_{2345}	32.5	z^p_{2346}	27.1	z^p_{2347}	22.9
z^p_{2348}	19.9	z^p_{2349}	18.1	z^p_{23410}	17.5
z^p_{2356}	24.4	z^p_{2357}	19.9	z^p_{2359}	14.5
z^p_{23510}	13.6	z^p_{23511}	13.9	z^p_{23512}	15.4
z^p_{23513}	18.1	z^p_{23514}	22	z^p_{2367}	18.1
z^p_{2368}	14.5	z^p_{2369}	12.1	z^p_{23610}	10.9
z^p_{23611}	10.9	z^p_{23612}	12.1	z^p_{23613}	14.5
z^p_{2378}	13.6	z^p_{2379}	10.9	z^p_{23710}	9.4
z^p_{23711}	9.1	z^p_{23712}	10	z^p_{2389}	10.9
z^p_{23810}	9.1	z^p_{23811}	8.5	z^p_{23910}	10
z^p_{2456}	24.4	z^p_{2457}	19.6	z^p_{2457}	19.6
z^p_{2458}	16	z^p_{2459}	13.6	z^p_{24510}	12.4
z^p_{24511}	12.4	z^p_{2467}	17.5	z^p_{2468}	13.6
z^p_{2469}	10.9	z^p_{24610}	9.4	z^p_{24611}	9.1
z^p_{24612}	10	z^p_{24613}	12.1	z^p_{24614}	15.4
z^p_{2478}	12.4	z^p_{2479}	9.4	z^p_{24710}	7.6
z^p_{24711}	7	z^p_{24712}	7.6	z^p_{24713}	9.4
z^p_{24714}	12.4	z^p_{2489}	9.1	z^p_{24810}	7
z^p_{24811}	6.1	z^p_{24812}	6.4	z^p_{24813}	7.9
z^p_{24814}	10.6	z^p_{24910}	7.1	z^p_{24911}	6.4
z^p_{24912}	6.4	z^p_{24913}	7.6	z^p_{24914}	10
z^p_{241011}	7.9	z^p_{241012}	7.6	z^p_{241013}	8.5
z^p_{2567}	18.6	z^p_{2569}	10.9	z^p_{25610}	9.1
z^p_{25611}	8.5	z^p_{2579}	10.9	z^p_{25710}	9.1
z^p_{25711}	6.1	z^p_{25712}	6.4	z^p_{25713}	7.9
z^p_{2589}	8.5	z^p_{25810}	6.1	z^p_{25811}	4.9
z^p_{25812}	4.9	z^p_{25813}	6.1	z^p_{25910}	6.4
z^p_{25911}	4.9	z^p_{25912}	4.6	z^p_{25913}	5.5
z^p_{251011}	6.1	z^p_{251012}	5.5	z^p_{251112}	7.6
z^p_{26711}	6.4	z^p_{26810}	6.4	z^p_{26811}	4.9
z^p_{26812}	4.6	z^p_{26911}	4.6	z^p_{26912}	4.0
z^p_{261011}	5.5	z^p_{27911}	5.5	z^p_{3579}	10
z^p_{35710}	7.6	z^p_{35711}	6.4	z^p_{35810}	6.4
z^p_{35811}	4.9	z^p_{35812}	4.6	z^p_{35813}	5.5
z^p_{35911}	4.6	z^p_{35912}	4.0	z^p_{36811}	4.6
z^p_{36812}	4.0	z^p_{36911}	4.0	z^p_{36912}	3.1
z^p_{36913}	3.4	z^p_{361012}	3.4	z^p_{471013}	2.5

Now suppose by contradiction S is an induced subgraph of G. We simply fix any vertex in G say u and then pick any of its 3 neighbours. But these neighbours should form P_3 (an open path of length 3 on any t-Cycle, $t > 3$, in $G_1(u)$). Hence we have the following structure:

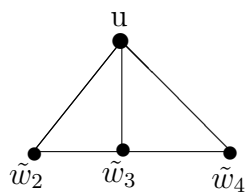

Figure 4.1: $K_4 \setminus e$

Let $S_i = \{v \in G \mid v \notin S \ \& \ v$ adjacent to exactly i vertices in $S\}$, where $0 \leq i \leq 4$. It is clear that $S_i \cap S_j = \phi$, where $0 \leq i \neq j \leq 4$. Since $\lambda = 2$, it follows that $S_4 = \phi$. Again since $\lambda = 2$, $\tilde{w}_i \notin S_3$. Now in $G_2(u)$, x_i is adjacent to 3 or more vertices on C_t if and only if $t \geq 9$. But if such exists then the lengths of their projections $|\tilde{z}_i^p| > 5$. Thus $S_3 = \phi$. Now, $S_2 = \{5$ vertices in $G_2(u)$ adjacent to $\tilde{w}_2$ and $\tilde{w}_4$, 1 vertex in $G_2(u)$ adjacent to $\tilde{w}_2$ and $\tilde{w}_3$, 1 vertex in $G_2(u)$ adjacent to $\tilde{w}_3$ and $\tilde{w}_4$, 1 vertex in $G_1(u)$ adjacent to u and $\tilde{w}_2$ *and* 1 vertex in $G_1(u)$ adjacent to u and $\tilde{w}_4$ $\}$. Hence S_2 contains exactly 9 vertices where these numbers were obtained based on $\lambda = 2$ and $\mu = 7$.

The degrees sum of all 4 vertices in S is $4k = 84$. Hence as $|S_1| + 2|S_2| + 10 = 4k$, we have $|S_1| = 56$. Since all the S_i together with S partition $V(G)$, we have

$$n = |S| + \sum_{i=0}^{4} |S_i|. \tag{4.5}$$

Hence $|S_0| = 7$. We now investigate this $|S_0|$ via our vector representation.

For the 6-cycle=$\{\tilde{w}_1, \tilde{w}_2, \tilde{w}_3, \tilde{w}_4, \tilde{w}_5, \tilde{w}_6\}$, we have the following graph

By Lemma 4.1.2, we know that any vector (vertex) $x_i \in G_2(u)$ connects to exactly 2 vertices on C_6. Hence as in the figure 4.2, the vertices of S_0 are actually the 5 vertices in $G_2(u)$ adjacent to both $\tilde{w}_1$, $\tilde{w}_5$ and two other vertices in $G_2(u)$, one of them adjacent to both $\tilde{w}_1$

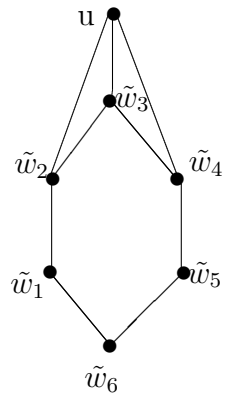

Figure 4.2: Neighbours of u on C_6

and $\tilde{w}_6$ while the other one is adjacent to both $\tilde{w}_5$ and $\tilde{w}_6$. Hence for C_6, $|S_0| = 7$ which coincides with what we have before. Hence we have no conclusion in this case.

For the 9-Cycle, by lemma 4.1.2, we know that any vertex $x_i \in G_2(u)$ connects to exactly 3 vertices in C_9. By using the same argument as in the previous case together with the facts we have about the lengths of our projections and the permissible edge connectivity, we get $|S_0| = 12$. This contradicts equation (4.6). Hence we can not have 9-Cycles in G.

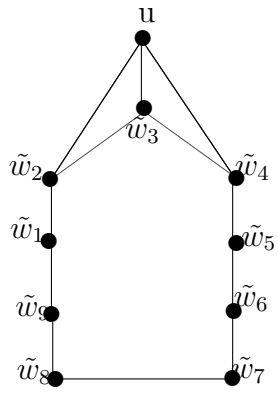

Figure 4.3: Neighbours of u on C_9

For $C_9 = \{\tilde{w}_1, \tilde{w}_2, \ldots, \tilde{w}_9\}$ where $\tilde{w}_2$, $\tilde{w}_3$ and $\tilde{w}_4$ are in S as shown in figure 4.3, we can find exactly 5 vertices in $G_2(u)$ which are adjacent to $\tilde{w}_6$, $\tilde{w}_8$ and $\tilde{w}_1$, and another 1 vertex in $G_2(u)$ adjacent to $\tilde{w}_6$, $\tilde{w}_7$ and $\tilde{w}_1$. Also we get 5 vertices in $G_2(u)$ which are adjacent to $\tilde{w}_5$, $\tilde{w}_7$ and $\tilde{w}_9$, and another 1 vertex in $G_2(u)$ adjacent to $\tilde{w}_5$, $\tilde{w}_6$ and $\tilde{w}_9$. All these edge

connectivities are assumed based on what is permissible from our projection onto the 9-Cycle and we observe these 12 vertices in $G_2(u)$ do not connect to vertices in S and they are the only possibilities. Hence we get $|S_0| = 12$, which is a contradiction.

4.2 Conclusion and Recommendation

First we draw our conclusions on the vector representation approach towards the nonexistence proof of SRGs and then we give some recommendations for future work.
In conclusion:

(1) We have been able to successfully use the vector representation approach to determine some necessary structures of components in the graph.

(2) The cases of 18 and 21 cycles have been immediately shown to be impossible as established in remark 4.1.1.

(3) In the cases of possible existence of the 6 and 9 cycles which are actually supposed to be forbidden structures, via our constructions of projections, we have been able to eliminate some edge connectivity from neighbours in $G_2(u)$ to the cycles in $G_1(u)$. This helps to have a better insight on such a subgraph component if it were to exist.

(4) After computations of all the projections, the lengths of the projections indicate immediately the impossibilities of having the 12-cycle and the 15-cycle as a subgraph in $G_1(u)$.

(5) We were able to integrate the counting method with our unit vector representation approach, which was quite successful in the elimination of the 9-Cycle.

(6) It is still an open problem if the forbidden structure in a subgraph can be figured out using the Gram matrix.

We now give the following recommendations:

(1) There are so many ways in which this work can be extended or improved on. Firstly we observe that if a vector representation proof can be given to show that C_6 does not exist in $G_1(u)$, then we are done from the vector representation proof of the nonexistence of G.

(2) It can also be seen that for any feasible parameter set having $\lambda = 2$, the method used in this thesis would be very useful.

(3) Even more, if $k - \mu = 2$, then this method could also be interesting as it could yield similar outcome.

(4) From the last point of our conclusion, we recommend that in addition to our method, forbidden structures in an SRG should be studied via Gram matrix. In this case we take the gram matrix of a subgraph component in the SRG and try to figure out a forbidden configuration by constructing the Gram matrix of the vectors representing this subgraph.

(5) It remains an interesting problem to generalize this method to other cases rather than just the $srg(76, 21, 2, 7)$.

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