ON THE LONG-TIME BEHAVIOR OF SOME INFINITE-HISTORY VISCOELASTIC PROBLEMS

by

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THESIS ABSTRACT

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In this dissertation, we study the well-posedness as well as the asymptotic behavior of some viscoelastic problems in the presence of infinite history. In this regard, we prove several general decay results under appropriate assumptions on the kernels and the structural parameters of the equations. We use the multiplier method, the well depth method and/or convexity argument to establish the desired stability results of the problems. Our results generalize many results existing in the literature.

CHAPTER 1

INTRODUCTION

1.1 Viscoelasticity

Elasticity is the material deformation behavior described by Hooke's law which states that displacement is linearly proportional to the applied load. An elastic material returns to the undeformed state once the loads are removed and the effects of multiple load systems can be computed by simple linear superposition. Moreover, the work done by the forces is calculated by multiplying the loads by the displacements. On the other hand, viscosity is an internal property of a fluid that offers resistance to flow. Viscous liquid has no definite shape and it flows irreversibly under the action of external forces. However, there are materials with properties that are intermediate between elasticity and viscosity.

Viscoelasticity, as its name suggests, incorporates aspects of both time dependent fluid behavior (viscous) and time independent solid behavior (elastic). Viscoelastic materials share some properties with elastic solids and some others with Newtonian viscous fluid. They exhibit an instantaneous elasticity effect and creep characteristics at the same time. In fact they can display all the intermediate range of properties. For instance, at low temperatures, or high frequencies of measurement, a polymer may be glass-like and it will break or flow at great strains. On the other hand, at high temperatures, permanent deformation occurs under load, and polymer behaves like a highly viscous liquid. However, in an intermediate temperature or frequency range, commonly called the glass transition range, the polymer is neither glassy nor rubberlike. Hence, polymers are usually described as viscoelastic materials and may dissipate a considerable amount of energy on being strained. In the rubber-like state, a polymer may be subjected to large deformation and still shows a complete recovery. To a good approximation, this is an elastic behavior at large strain [34], [52].

The importance of the viscoelastic properties of materials has been realized because of the rapid developments in rubber and plastics industry. Many advances in the studies of constitutive relations, failure theories and life prediction of viscoelastic materials and structures were reported and reviewed in the last two decades [24]. Time dependence of mechanical behavior of viscoelastic materials reveals the existence of inner clock or intrinsic time, which can be influenced by many factors such as temperature [3], physical [15], [83], [84], damage, pressure and solvent concentration [47], [55], strain and stress level [11], [50], [75].

Depending on the change of strain rate versus stress inside a material the viscosity can be categorized as having a linear, non-linear, or plastic response. When a material exhibits a linear response it is categorized as a Newtonian material. In this case the stress is linearly proportional to the strain rate. If the material exhibits a non-linear response to the strain rate, it is categorized as Non-Newtonian fluid. There is also an interesting case where the viscosity decreases as the shear/strain rate remains constant. A material which exhibits this type of behavior is known as thixotropic. In addition, when the stress is independent of this strain rate, the material exhibits plastic deformation [53].

Boltzmann (1844-1906) first proposed to use superposition to compute the stress-strain response of a viscoelastic solid subjected to an arbitrary loading history. He assumed that creep at any time is a function of the entire prior loading history and that each loading step makes an independent contribution to the deformation. Hence for an applied stress $\sigma(t)$, the strain is

$$\epsilon(t) = \int_{-\infty}^{t} J(t-s)d\sigma(s),$$

where J is time dependent creep compliance.

Likewise, if a strain $\epsilon(t)$ is applied

$$\sigma(t) = \int_{-\infty}^{t} G(t-s)d\epsilon(s),$$

where G is time dependent stress relaxation modulus [53].

We consider viscoelasticity in the isothermal approximation, which means that the temperature does not enter the model (state and constitutive relation). So the state involves the deformation gradient only while the constitutive equation is in fact a stress-strain relation. We obtain [14], [53]

$$\sigma(t) = G(t)\epsilon(0) + \int_0^t G(t-s)\frac{\partial \epsilon(s)}{\partial s}ds.$$

The integrating functions $G(t)$ are mechanical properties of the material and are called relaxation functions. It can be considered to be the formulation of Boltzmann's superposition principle such that the current stress is determined by the superposition of the responses to the complete spectrum of increments of strain. The relaxation function G brings about damping effect of the solutions to the problem. This viscous damping ensures global existence of smooth solutions decaying uniformly under constant density as time goes to infinity. This is true for sufficiently smooth and/or small data and history. We shall mainly be concerned with this phenomenon in our study.

1.2 Result Description

The aim of this dissertation is to investigate the well-posedness as well as the asymptotic behavior of solutions of some viscoelastic problems in the presence of infinite history. In this regard, we study several problems and establish several general decay results under some suitable assumptions. This study improves and generalizes several eariler results mentioned in Section 4 in this chapter. In particular, we extended results of general decay by the new approach introduced by

Guesmia and Messaoudi [41] to some nonlinear problems and systems. Also, we extended, using the convexity arguments used in [38] and [51], the results of some nonlinear viscoelastic problems with finite history to infinite history.

Our contributions start from Chapter two, where we investigate the asymptotic stability of solutions of the following nonlinear wave equation with infinite history

$$\begin{cases} |u_t(x,t)|^\rho u_{tt}(x,t) - \Delta u - \Delta u_{tt}(x,t) + \int_0^{+\infty} g(s)\Delta u(x,t-s)ds = 0, & \text{in } \Omega \times \mathbb{R}^+, \\ u(x,t) = 0, & \text{on } \Gamma \times \mathbb{R}^+, \\ u(x,-t) = u_0(x,t), \quad u_t(x,0) = u_1(x), & \text{in } \Omega \times \mathbb{R}^+, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^N (N \geq 1)$ with a smooth boundary Γ, u is the transverse displacement of waves, the relaxation function g is positive and decreasing, the exponent ρ is a positive real number satisfying some conditions to be specified later. We consider (1.1) and establish a general decay result for the associated energy functional.

In Chapter three, we study the asymptotic behavior of the following nonlinear system of wave equations with infinite memories

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) + \displaystyle\int_0^{+\infty} g(s)\Delta u(x,t-s)ds + \lambda |u_t(x,t)|^{m-1}u_t(x,t) \\ \qquad\qquad\qquad\qquad\qquad\qquad = f_1(u,v), \text{ in } \Omega \times \mathbb{R}^+, \\ v_{tt}(x,t) - \Delta v(x,t) + \displaystyle\int_0^{+\infty} h(s)\Delta v(x,t-s)ds + \mu |v_t(x,t)|^{r-1}v_t(x,t) \\ \qquad\qquad\qquad\qquad\qquad\qquad = f_2(u,v), \text{ in } \Omega \times \mathbb{R}^+, \\ u(x,t) = v(x,t) = 0, \qquad\qquad \text{in } \partial\Omega \times \mathbb{R}^+, \\ u(x,-t) = u_0(x,t), u_t(x,0) = u_1(x), v(x,-t) = v_0(x,t), v_t(x,0) = v_1(x), \quad \text{in } \Omega \times \mathbb{R}^+, \end{cases} \tag{1.2}$$

$$\begin{cases} f_1(u,v) = a|u+v|^{2(\rho+1)}(u+v) + b|u|^{\rho}u|v|^{\rho+2}, \\ f_2(u,v) = a|u+v|^{2(\rho+1)}(u+v) + b|v|^{\rho}v|u|^{\rho+2}, \end{cases} \tag{1.3}$$

where u and v denote the transverse displacements of waves, Ω is a bounded domain of $\mathbb{R}^N (N = 1, 2, 3)$ with a smooth boundary $\partial\Omega$, ρ, m, r, λ, μ are positive constants, the kernels g and h are satisfying some conditions to be specified later and the nonlinear coupling functions, f_1 and f_2, describe the interaction between the two waves.

We establish, for a class of initial data, a general decay result using the well depth and the multiplier methods. This result extends the result obtained by [79]. Furthemore; in contrast to [38], we allow a wider class of relaxation functions and we do not use the convexity argument

In Chapter four, we study the asymptotic stability of the following viscoelastic equation with infinite history and boundary feedback

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) + \int_0^{+\infty} g(s)\Delta u(x,t-s)ds = 0, & \text{in } \Omega \times \mathbb{R}^+, \\ \frac{\partial u}{\partial \nu}(- \int_0^{+\infty} g(s)\frac{\partial u}{\partial \nu}(x,t-s)ds + h(u_t) = 0, & \text{on } \Gamma_1 \times \mathbb{R}^+, \\ u(t) = 0, & \text{on } \Gamma_0 \times \mathbb{R}^+, \\ u(x,-t) = u_0(t), u_t(x,0) = u_1(x), & \text{in } \Omega \times \mathbb{R}^+, \end{cases} \tag{1.4}$$

where u denotes the transverse displacement of waves, Ω is a bounded domain of $\mathbb{R}^N (N \geq 1)$ with a smooth boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1$ such that Γ_0 and Γ_1 are closed and disjoint, with $|\Gamma_0| > 0$, ν is the unit outer normal to $\partial\Omega$ and g, h are specific functions.

We establish, without imposing any growth condition on h and for a wide class of relaxation functions g, an explicit decay result of the solution of the problem.

Chapter five is devoted to the study of the well-posedness as well as the asymptotic behavior of a viscoelastic equation with infinite history and nonlinear localized damping. Namely,

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) + \int_0^{+\infty} g(s)\Delta u(x,t-s)ds + a(x)|u_t(x,t)|^{m-2}u_t(x,t) = 0, & \text{in } \Omega \times \mathbb{R}^+, \\ u(x,t) = 0, & \text{on } \partial\Omega \times \mathbb{R}^+, \\ u(x,-t) = u_0(x,t),\ u_t(x,0) = u_1(x), & \text{in } \Omega \times \mathbb{R}^+, \end{cases} \tag{1.5}$$

where u denotes the transverse displacement of waves, Ω is a bounded domain of $\mathbb{R}^N$ $(N \geq 1)$ with a smooth boundary $\partial\Omega$, g is positive and decreasing function,

a is nonnegative bounded measurable function defined on Ω and $m > 1$.

We establish the well-posedness when $m \geq 2$ using the standard Faedo-Galerkin method and obtain an explicit decay result for the associated energy functional under some suitable assumptions.

1.3 Methodology

We use the multiplier method, the well depth method and/or a convexity argument to establish the desired stability results of the systems. The multiplier method relies mostly on the construction of an appropriate Lyapunov functional $\mathcal{L}$ equivalent to the energy of the solution E. By equivalence $\mathcal{L} \sim E$, we mean

$$\alpha_1 E(t) \leq \mathcal{L}(t) \leq \alpha_2 E(t), \quad \forall t \in \mathbb{R}^+, \tag{1.6}$$

for two positive constants α_1 and α_2. To prove the exponential stability, we show that $\mathcal{L}$ satisfies

$$\mathcal{L}'(t) \leq -c_1 \mathcal{L}(t), \quad \forall t \in \mathbb{R}^+, \tag{1.7}$$

for some $c_1 > 0$. A simple integration of (1.7) over $(0, t)$ together with (1.6) gives the desired exponential stability result.

In the case of a general decay result, the obtained decay rate depends on the relaxation function g, which is assumed to satisfy the following two conditions

(A1) $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a C^1 decreasing function satisfying

$$g(0) > 0, \quad 1 - \int_0^{+\infty} g(s)ds = l > 0,$$

(A2) There exists a positive nonincreasing differentiable function $\xi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$g'(t) \leq -\xi(t)g(t), \quad t \in \mathbb{R}^+.$$

Then, we show that

$$(\xi(t)\mathcal{L}(t) + \beta_1 E(t))' \leq -c_1\xi(t)E(t) + \beta_2 r(t), \ \forall t\mathbb{R}^+,$$

where $r(t) = \xi(t) \int_t^{+\infty} g(s)ds$.

After that, we exploit (1.6) to prove that

$$\mathcal{F}(t) = \xi(t)\mathcal{L}(t) + \beta_1 E(t) \sim E(t). \tag{1.8}$$

Direct integtation on $(0, T)$ gives

$$\mathcal{F}(T) \leq e^{-\delta_0 \int_0^T \xi(s)ds} \left(F(0) + \beta_2 \int_0^T e^{\delta_0 \int_0^t \xi(s)ds} r(t)dt \right). \tag{1.9}$$

Use of integration by parts, properties of ξ and g and (1.8) lead to the general decay result. In addition, we use the well depth method, in Chapter three, to guarantee the non-negativeness of the energy functional. In Chapter four and five, we use a convexity argument to obtain the desired decay result. For the well-posedness, we employ the standard Galerkin method.

1.4 Literature Review

1.4.1 Viscoelastic Problems with Finite Memory

We start with the pioneer work of Dafermos [28], [29], where he considered a one-dimensional viscoelastic problem of the form

$$\rho u_{tt} = c u_{xx} - \int_{-\infty}^{t} g(t-s) u_{xx}(s) ds$$

and established various existence results and then proved, for smooth monotone decreasing relaxation functions, that the solutions go to zero as t goes to infinity. However, no rate of decay has been specified. Hrusa [45] considered a one-dimensional nonlinear viscoelastic equation of the form

$$u_{tt} - c u_{xx} + \int_{0}^{t} m(t-s)(\psi(u_x(s)))_x ds = f(x,t)$$

and proved several global existence results for large data. He also proved an exponential decay result for strong solutions when $m(s) = e^{-s}$ and ψ satisfies certain conditions. In [31], Dassios and Zafiropoulos considered a viscoelastic problem in $\mathbb{R}^3$ and proved a polynomial decay result for exponentially decaying kernels. In their book, Fabrizio and Morro [33] established a uniform stability of some problems in linear viscoelasticity. After that, Rivera [69] considered equations for linear isotropic homogeneous viscoelastic solids of integral type which occupy bounded domains or the whole space $\mathbb{R}^n$. In the bounded-domain case and for exponentially decaying memory kernels and regular solutions, he showed that the sum of the first and the second energy decays exponentially. For the whole-space case and for exponentially decaying memory kernels, he showed that the rate of decay of energy is of algebraic type and depends on the regularity of the solution. This result was later generalized to a situation, where the kernel is decaying algebraically but not exponentially by Cabanillas and Rivera [16]. In their paper, the authors considered the case of bounded domains as well as the case when the material is occupying the entire space and showed that the decay of solutions is also algebraic, at a rate which can be determined by the rate of the decay of the relaxation function. This latter result was later improved by Baretto et al. [10], where equations related to linear viscoelastic plates were treated. Precisely, they showed that the solution energy decays at the same decay rate of the relaxation function. For partially viscoelastic materials, Rivera and Salvatierra [80] showed that the energy decays exponentially, provided the relaxation function decays in a similar fashion and the dissipation is acting on a part of the domain near to the boundary. Also, Rivera et al. [73], [74] established the same result as in [80] regardless to the size of the viscoelastic part of the material. Fabrizio and Polidoro

[34] studied the following problem

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau + u_t = 0, & \text{in } \Omega \times \mathbb{R}^+, \\ u = 0, & \text{on } \partial\Omega \times \mathbb{R}^+ \end{cases}$$

and showed that the exponential decay of the relaxation function is a necessary condition for the exponential decay of the solution energy. In [72], a class of abstract viscoelastic equations of the form

$$u_{tt} + Au + \beta u - (g * A^{\alpha}u)(t) = 0, \tag{1.10}$$

for $0 \leq \alpha \leq 1$ and $\beta \geq 0$, was investigated. The main focus was on the case when $0 < \alpha < 1$ and the main result was that solutions for (1.10) decay polynomially even if the kernel g decay exponentially. This result has been generalized by Rivera et al. [71], where the authors studied a more general abstract problem than (1.10) and established a necessary and sufficient condition to obtain an exponential decay. For quasilinear problems, Cavalcanti et al. [18] studied, in a bounded domain, the following equation

$$|u_t|^{\rho}u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-\tau)\Delta u(\tau)d\tau - \gamma\Delta u_t = 0, \tag{1.11}$$

for $\rho > 0$. A global existence result for $\gamma \geq 0$, as well as an exponential decay result for $\gamma > 0$, have been established. This latter result was then extended to a situation, where $\gamma = 0$, by Messaoudi and Tatar [65, 66], and exponential and polynomial decay results have been established in the absence, as well as in the presence, of a source term. In all the above mentioned works, the rates of decay in relaxation functions were either of exponential or polynomial type. In [21], Cavalcanti et al. considered

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds + a(x)u_t + |u|^{p-1}u = 0, \quad \text{in } \Omega \times \mathbb{R}^+,$$

where $a : \Omega \to \mathbb{R}^+$ is a function which may vanish on a part of the domain Ω but satisfies $a(x) \geq a_0$ on $\omega \subset \Omega$ and g satisfies, for two positive constants ξ_1 and ξ_2,

$$-\xi_1 g(t) \leq g'(t) \leq -\xi_2 g(t), \quad t \in \mathbb{R}^+$$

and established an exponential decay result under some restrictions on ω. Berrimi and Messaoudi [12] established the result of [21], under weaker conditions on both a and g, to a problem where a source term is competing with the damping term. Cavalcanti and Oquendo [23] considered the following problem

$$u_{tt} - k_0\Delta u + \int_0^t \text{div}[a(x)g(t-s)\Delta u(s)]ds + b(x)h(u_t) + f(u) = 0 \tag{1.12}$$

and established, for $a(x) + b(x) \geq \rho > 0$, an exponential stability result for g decaying exponentially and h linear, and a polynomial stability result for g decaying polynomially and h nonlinear. Li et al. [54] treated (1.12) with $b(x) = 0$ and $f(u) = -|u|^{\gamma}u$, $\gamma > 0$. They showed the global existence and uniqueness of global solution of problem (1.12) and established uniform decay rate of the energy under suitable conditions on the initial data and the relaxation function g. Messaoudi [57] investigated the equation

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds + au_t|u_t|^m = b|u|^\gamma u, \ \text{ in } \Omega \times \mathbb{R}^+$$

and showed, under suitable conditions on g, that solutions with negatve energy blow up in finite time if $\gamma > m$, and continue to exist if $m \geq \gamma$. In the absence of the viscoelastic term ($g = 0$), the problem has been extensively studied and many results concerning global existence and nonexistence have been proved. For instance, for the problem

$$u_{tt} - \Delta u + \alpha u_t|u_t|^m = b|u|^\gamma u, \ \text{ in } \Omega \times \mathbb{R}^+,$$

with m, $\gamma \geq 0$, it is well known that, for $a = 0$, the source term $b|u|^\gamma u$ ($\gamma > 0$) causes finite time blow up of solutions with negative initial energy [9], [46] and for $b = 0$, the damping term $au_t|u_t|^m$ assures global existence for arbitrary initial data [42], [49]. For more general decaying relaxation functions, Messaoudi [59, 60] considered

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau = b|u|^{p-2}u, \tag{1.13}$$

for $p \geq 2$, $b \in \{0, 1\}$ and g satisfying ($A1$), ($A2$), and established a more general decay result, from which the usual exponential and polynomial decay rates are only special cases. Alabau-Boussouira and Cannarsa [4] considered (1.13) with

$b = 0$ and relaxation functions satisfying

$$g'(t) \leq -\chi(g(t)),$$

where χ is a non-negative function, with $\chi(0) = \chi'(0) = 0$, and χ is strictly increasing and stictly convex on $(0, k_0]$, for some $k_0 > 0$. They also required that

$$\int_0^{k_0} \frac{dx}{\chi(x)} = +\infty, \quad \int_0^{k_0} \frac{xdx}{\chi(x)} < 1, \quad \liminf_{s \to 0^+} \frac{\chi(s)/s}{\chi'(s)} > \frac{1}{2}$$

and claimed a decay result for the energy of (1.13). In addition to these assumptions, if

$$\limsup_{s \to 0^+} \frac{\chi(s)/s}{\chi'(s)} < 1 \text{ and } g'(t) = -\chi\left(g(t)\right),$$

then, in this case, an explicit rate of decay is given. Very recently, Messaoudi and Mustafa [62] considered (1.13), with $b = 0$, and for relaxation functions satisfying

$$g'(t) \leq -H(g(t)),$$

for some positive convex function H. They used the properties of the convex functions together with the generalized Young inequality and established a general decay result depending on g and H.

For Frictional dissipative boundary condition, Lasiecka and Tataru [51] investigated the following problem

$$\begin{cases} u_{tt} = \Delta u - f_0(u), & \text{in } \Omega \times \mathbb{R}^+, \\ \frac{\partial u}{\partial \nu} = -g(u_t|_{\Gamma_1}) - f_1(u|_{\Gamma_1}), & \text{on } \Gamma_1 \times \mathbb{R}^+, \\ u = 0, & \text{on } \Gamma_0 \times \mathbb{R}^+, \\ u(0) = u_0 \in H_0^1(\Omega), \quad u_t(0) = u_1 \in L^2(\Omega) \end{cases}$$

and proved, without imposing any growth condition on h, that the energy decays as fast as the solution of an associated differential equation whose coefficients depend on the damping term. A nonlinear wave equation with viscoelastic boundary condition was studied by Rivera and Andrade [70] and existence and uniform decay results, under some restriction on the initial data, were established. Santos [82] considered a one-dimensional wave equation with viscoelastic boundary feedback and showed, under some assumptions on both g' and g'', that the dissipation is strong enough to produce exponential (polynomial) decay of the solution, provided the relaxation function also decays exponentially (polynomially) respectively. Messaoudi and Soufyane considered a semilinear wave equation of the form

$$\begin{cases} u_{tt} - \Delta u + f(u) = 0, & \text{in } \Omega \times \mathbb{R}^+, \\ u = 0, & \text{on } \Gamma_0 \times \mathbb{R}^+, \\ u = -\int_0^t g(t-s) \frac{\partial u}{\partial \nu}(s) ds, & \text{on } \Gamma_1 \times \mathbb{R}^+, \end{cases}$$

where $f \in C^1(\mathbb{R})$ is a function satisfying $uf(u) \geq bF(u) \geq 0$, for $b > 2$,

$$F(u) = \int_0^u f(\xi)d\xi,$$

with

$$F(u) \leq d|u|^p, \ \forall u \in \mathbb{R},$$

for some constant $d > 0$ and $p \geq 1$ such that $(n-2)p \leq n$, and established a general decay result. Cavalcanti et al. [20] studied the following problem

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds = 0, & \text{in } \Omega \times \mathbb{R}^+, \\ u = 0, & \text{on } \Gamma_1 \times \mathbb{R}^+, \\ \frac{\partial u}{\partial \nu} - \int_0^t g(t-s)\frac{\partial u}{\partial \nu}(s)ds + h(u_t) = 0, & \text{on } \Gamma_0 \times \mathbb{R}^+, \end{cases} \tag{1.14}$$

where $h : \mathbb{R} \to \mathbb{R}$ is a nondecreasing C^1 function such that

$$h(s)s > 0, \ \text{for all}\, s \neq 0$$

and there exist $C_i > 0$, $i = 1, 2, 3, 4$, such that

$$\begin{cases} C_1|s|^p \leq |h(s)| \leq C_2|s|^{\frac{1}{p}}, & \text{if } |s| \leq 1, \\ C_3|s| \leq |h(s)| \leq C_4|s|, & \text{if } |s| > 1, \end{cases}$$

where $p \geq 1$. He established a global existence of strong and weak solutions and some uniform decay rate results under restrictive assumptions on both the damping function h and the kernel g. After that, Cavalcanti et al. [19] weakened the conditions on both h and g and established a uniform stability depending on the behavior of h and g. Messaoudi and Mustafa [61] considered (1.14), for more general type of the relaxation functions and, without imposing any growth condition on h, they established a general deacy result depending on g and h. Cavalcanti and Guesmia [22] studied the following hyperbolic problem

$$\begin{cases} u_{tt} - \Delta u + F(x,t,\nabla u) = 0, & \text{in } \Omega \times \mathbb{R}^+, \\ u = 0, & \text{on } \Gamma_0, \\ u + \int_0^t g(t-s)\frac{\partial u}{\partial \mu}(s)ds = 0, & \text{on } \Gamma_1 \times \mathbb{R}^+ \end{cases} \tag{1.15}$$

and proved, under some conditions, that the dissipation given by the memory term is strong enough to assure stability of our system. Precisely, they showed that if the relaxation function decays exponentially (or polynomially), then the solution also decays exponentially (or polynomially) and with the same decay rate.

1.4.2 Viscolestic Systems with Finite Memory

Andrade and Mognon [6] treated the following problem

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g_1(t-s)\Delta u(s)ds + f_1(u,v) = 0, & \text{in } [0,T] \times \Omega, \\ v_{tt} - \Delta v + \int_0^t g_2(t-s)\Delta v(s)ds + f_2(u,v) = 0, & \text{in } [0,T] \times \Omega, \end{cases} \tag{1.16}$$

with

$$f_1(u,v) = |u|^{p-2}u|v|^p \text{ and } f_2(u,v) = |v|^{p-2}v(t)|u|^p,$$

where $p > 1$ if $N = 1, 2$ and $1 < p \leq \frac{N-1}{N-2}$ if $N = 3$. They proved the well posedness under the following assumptions on the relaxation functions

$$1 - \int_0^{+\infty} g_i(s)ds > 0, \; g_i'' \in L^1(0,\infty) \text{ for } i = 1, 2$$

and for some positive constants α and β

$$-\alpha g_i(t) \leq g_i'(t) \leq -\beta g_i(t).$$

In [81], Santos considered (1.16) with

$$f_1(u,v) = a(u-v) \text{ and } f_2(u,v) = -a(u-v),$$

where a is a positive constant and the relaxation functions satisfy

$$\begin{cases} -a_1 g_i^p(t) \leq g_i'(t) \leq -a_2 g_i^p(t) \\ 0 \leq g_i''(t) \leq \gamma g_i^p(t), \quad i = 1, 2, \end{cases}$$

for $1 \leq p < 2$. He proved an exponential decay result for the kernels decaying exponentially ($p = 1$) and a polynomial decay result for the kernels decaying polynomially ($p > 1$). Messaoudi and Tatar [67] considered the following problem

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds + f(u,v) = 0, & x \in \Omega, t \in \mathbb{R}^+, \\ v_{tt} - \Delta v + \int_0^t h(t-s)\Delta v(s)ds + k(u,v) = 0, & x \in \Omega, t \in \mathbb{R}^+, \end{cases} \tag{1.17}$$

where the functions f and k satisfy, for all $(u, v) \in \mathbb{R}^2$, the following assumptions

$$\begin{cases} |f(u,v)| \le d(|u|^{\beta_1} + |v|^{\beta_2}), \\ |k(u,v)| \le d(|u|^{\beta_3} + |v|^{\beta_4}), \end{cases}$$

for some constant $d > 0$ and

$$\beta_i \ge 1, \quad (N-2)\beta_i \le N, \quad i = 1, 2, 3, 4.$$

They proved an exponential decay result if both g and h are decaying exponentially and a polynomial decay result otherwise. Recently, Mustafa [68] investagiated the well-posedness and the asymptotic behavior of the system (1.17) for relaxation functions satisfy

$$g'(t) \le -\xi_1(t)g(t), \quad h'(t) \le -\xi_2(t)h(t),$$

and for more general forms of f and k and established a general decay result depending on g and h. Han and Wang [44] studied the following problem

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds + |u_t|^{m-1}u_t = f(u,v), & x \in \Omega, t \in \mathbb{R}^+, \\ v_{tt} - \Delta v + \int_0^t h(t-s)\Delta v(s)ds + |v_t|^{r-1}v_t = k(u,v), & x \in \Omega, t \in \mathbb{R}^+, \end{cases} \tag{1.18}$$

and established global existence and blow-up results. However, the decay issue was not discussed. In [64], Messaoudi and Said-Houari studied (1.18) for f and k given in (3.2), and proved, under specific assumptions on the kernels and the initial data, a global nonexistence theorem with positive initial energy. Also, Said-Houari et al. [79] studied (1.18) and proved, for certain class of relaxation functions and for some restrictions on the initial data, that the rate of decay of the total energy depends on those of the relaxation functions.

1.4.3 Viscoelastic Problems with Infinite History

Giorgi et al. [36] considered the following semilinear hyperbolic equation, in a bounded domain $\Omega \subset \mathbb{R}^3$,

$$u_{tt} - K(0)\Delta u - \int_0^{+\infty} K'(s)\Delta u(t-s)ds + g(u) = f,$$

with $K(0)$, $K(\infty) > 0$ and $K' \leq 0$ and gave the existence of global attractors for the problem. Conti and Pata [25] considered the following semilinear hyperbolic equation with linear memory in a bounded domain $\Omega \subset \mathbb{R}^n$,

$$u_{tt} + \alpha u_t - K(0)\Delta u - \int_0^{+\infty} K'(s)\Delta u(t-s)ds + g(u) = f, \quad \text{in } \Omega \times \mathbb{R}^+, \quad (1.19)$$

where the memory kernel is a convex decreasing smooth function such that $K(0) > K(\infty) > 0$ and $g : \mathbb{R} \to \mathbb{R}$ is a nonlinear function of at most cubic growth satisfying some conditions and proved the existence of a regular global attractor. In [7], Appleby et al. studied the linear integro-differential equation

$$u_{tt} + Au + \int_{-\infty}^{t} K(t-s)Au(s)ds = 0, \quad t \in \mathbb{R}^+$$

and established results of exponential decay of strong solutions in a Hilbert space. Pata [76] discussed the decay properties of the semigroup generated by the following equation

$$u_{tt} + \alpha Au + \beta u_t - \int_0^{+\infty} \mu(s)Au(t-s)ds = 0,$$

where A is a strictly positive self-adjoint linear operator and $\alpha > 0$, $\beta \geq 0$ and the memory kernel μ is a decreasing function satisfying some specific conditions. He established the necessary as well as the sufficient conditions for the exponential stability. In [38], Guesmia considered the following abstract equation

$$u_{tt} + Au - \int_0^{+\infty} g(s)Bu(t-s)ds = 0, \quad (1.20)$$

where A and B are two positive self-adjoint operator satisfy some conditions, and introduced a new ingenuous approach based on the properties of convex functions and the use of the generalized Young inequality which allows a larger class of infinite history kernels than the one considered in the literature thereby established a more general decay result for a class of hyperbolic problems. Using this approach, Guesmia and Messaoudi [40] later looked into

$$u_{tt} - \Delta u + \int_0^t g_1(t-s) div(a_1(x)\nabla u(s))ds + \int_0^{+\infty} g_2(s) div(a_2(x)\nabla u(t-s))ds = 0,$$

in a bounded domain and under suitable conditions on a_1, a_2 and for a wide class of relaxation functions g_1 and g_2 which are not necessarily decaying polynomially or exponentially, and established a general decay result from which the usual exponential and polynomial decay rates are only special cases. In [41], Guesmia and Messaoudi considered (1.20) with $B = A$. They adopted the method introduced in [60], for finite history, with some modifications imposed by the nature of their problem, to establish a general decay result which depends only on the behavior of the relaxation function. Very recently, Guesmia [39] considered the following coupled abstract evolution equations

$$\begin{cases} u_{tt} + Au - \int_0^{+\infty} g(s)Bu(t-s)ds + \tilde{B}v = 0, \ \forall t \in \mathbb{R}^+, \\ v_{tt} + \tilde{A}v + \tilde{B}u = 0, \qquad \forall t \in \mathbb{R}^+, \end{cases} \tag{1.21}$$

where A, $\tilde{A}$ and B are self-adjoint linear positive definite operators, whereas $\tilde{B}$ is a self-adjoint linear bounded operator. Under a boundedness condition on the past history data, he proved that the stability of (1.21) holds for convolution kernels having much weaker decay rates than the exponential one. Also, he showed that the general and precise decay estimate of solution he obtained depends on the growth of the convolution kernel at infinity, the regularity of the initial data, and the connection between the operators describing the considered equations. Conti et al [27] studied the asymptotic properties of the semigroup $S(t)$ arising from the nonlinear viscoelastic equation with hereditary memory on a bounded three-dimensional domain

$$|u_t|^\rho u_{tt} - \Delta u_{tt} - \Delta u_t - \left(1 + \int_0^{+\infty} \mu(s) ds\right) \Delta u + \int_0^{+\infty} \mu(s) \Delta u(t-s) ds + f(u) = h,$$

written in the past history framework of Dafermos [28]. They established the existence of the global attractor of optimal regularity for $S(t)$ when $\rho \in [0, 4)$ and f has polynomial growth of (at most) critical order 5. Very recently, Conti et al. [26] considered the following nonlinear viscoelastic equation

$$|u_t|^\rho u_{tt} - \Delta u_{tt} + \gamma(-\Delta)^\theta u_t - \alpha \Delta u + \int_0^{+\infty} \mu(s) \Delta u(t-s) ds + f(u) = h, \quad (1.22)$$

with hereditary memory on a bounded three-dimensional domain, the parameter ρ belongs to the interval $[0, 4]$, while f can reach the critical polynomial order 5. They established an existence, uniqueness and continuous dependence result for the weak solutions to (1.22).

1.5 Notation and some Useful Inequalities

Throughout this dissertation, we use the standard $L^2(\Omega)$ and $H^1(\Omega)$ spaces. The space $H^1(\Omega)$ is defined to be

$$H^1(\Omega) = \Big\{ u \in L^2(\Omega); \exists g_1, g_2, \cdots g_n \in L^2(\Omega) : \\ \int_\Omega u \frac{\partial \varphi}{\partial x_i} = -\int_\Omega g_i \varphi \quad \forall \varphi \in C_0^\infty(\Omega) \Big\},$$

where $g_i = \dfrac{\partial u}{\partial x_i}$ in weak sense.

The space $H^1(\Omega)$ is equipped with the norm

$$\|u\|^2_{H^1(\Omega)} = \|u\|_2 + \|\nabla u\|_2,$$

where$\|u\|_2 = \|u\|^2_{L^2(\Omega)}$.

For $H_0^1(\Omega)$

$$H_0^1(\Omega) = \{u \in H^1(\Omega) \colon u = 0 \text{ on } \partial\Omega\},$$

where $\partial\Omega$ is the boundary of Ω, which is equipped with the norm

$$\|u\|^2_{H_0^1(\Omega)} = \|\nabla u\|_2^2,$$

provided that Ω is a bounded domain. The following notations are used in the dissertation

- $\Delta = \partial^2_{x_1} + \partial^2_{x_2} + ... + \partial^2_{x_n}$

- $\nabla = (\partial_{x_1},\ \partial_{x_2}, ..., \partial_{x_n})$

- $u_t = \dfrac{\partial u}{\partial t}, \quad u_{tt} = \dfrac{\partial^2 u}{\partial t^2},$

- $C^1(\Omega)$ denotes the space of all continuously differentiable functions on Ω,

- $C_0^1(\Omega)$ denotes the space of all continuously differentiable functions with compact support in Ω. The support of a continuous function f defined on Ω is the closure of the set of point where $f(x)$ is nonzero. That is

$$\text{supp}(f) :\ = \overline{\{x \in \Omega \mid f(x) \neq 0\}}.$$

Furthermore, we sometimes use c to denote a generic positive constant. The following inequalities are repeatedly used in the dissertation

1. **Hölder's inequality**. Let $1 \leq p,\, q \leq \infty$ such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$. If $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$, then $uv \in L^1(\Omega)$ and

$$\int_\Omega |uv| \leq \|u\|_p \|v\|_q.$$

   By taking $p = q = 2$, we have the **Cauchy-Schwarz inequality**.

2. **Young's inequality**. Let $1 < p,\, q < \infty$ such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Then for any $\varepsilon > 0$, we have

$$ab \leq \varepsilon a^p + C_\varepsilon b^q, \quad \forall a,\, b \geq 0,$$

where $C_\varepsilon = \dfrac{1}{q(\varepsilon p)^{\frac{q}{p}}}$. For $p = q = 2$, we have

$$ab \leq \varepsilon a^2 + \frac{b^2}{4\varepsilon}.$$

3. **Green's formula**. Let Ω be a bounded domain of $\mathbb{R}^n$ with smooth boundary. Then

$$\int_\Omega u\Delta v dx = -\int_\Omega \nabla u \cdot \nabla v dx + \int_{\partial\Omega} u\nabla v \cdot \nu ds, \qquad \forall u \in H^1(\Omega) \text{ and } v \in H^2(\Omega).$$

where ν is the outer unit normal to $\partial\Omega$. If $u \in H_0^1$, the Green's formula becomes

$$\int_\Omega u\Delta v dx = -\int_\Omega \nabla u \cdot \nabla v dx.$$

4. **Poincaré's inequality**. Let $1 \leq p < \infty$ and Ω be a bounded domain of $\mathbb{R}^n$. Then there exists a constant C (depending on Ω and p only) such that

$$\|u\|_{L^p(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)}, \quad \forall u \in W_0^{1,p}(\Omega).$$

If $p = 2$, then we set $H_0^1(\Omega) = W_0^{1,2}(\Omega)$.

Publications

The following results were Published/Submitted from our research

(1) Salim A. Messaoudi and Mohammad M. Al-Gharabli, *A general stability result for a nonlinear wave equation with infinite memory*, Appl. Math. Letters (26), 1082-1086 (2013)

(2) Salim A. Messaoudi and Mohammad M. Al-Gharabli, *A general decay result of a viscoelastic equation with past history and boundary* feedback, ZAMP, DoI 10.1007/s00033-014-0476-8 (2014)

(3) Salim A. Messaoudi and Mohammad M. Al-Gharabli, *A general decay result of a nonlinear system of wave equations with infinite memories* Appl. Math. and Computations (under 2nd Review)

(4) Salim A. Messaoudi and Mohammad M. Al-Gharabli, *A general decay result of a viscoelastic equation with infinite history and nonlinear damping* (submitted).

CHAPTER 2

A GENERAL STABILITY RESULT FOR A NONLINEAR WAVE EQUATION WITH INFINITE HISTORY

In this chapter, we consider the following problem

$$\begin{cases} |u_t(x,t)|^\rho u_{tt}(x,t) - \Delta u(x,t) - \Delta u_{tt}(x,t) + \int_0^{+\infty} g(s)\Delta u(x,t-s)ds = 0, & \text{in } \Omega\times\mathbb{R}^+, \\ u(x,t) = 0, & \text{on } \Gamma\times\mathbb{R}^+, \\ u(x,-t) = u_0(x,t), \quad u_t(x,0) = u_1(x), & \text{in } \Omega\times\mathbb{R}^+, \end{cases} \tag{2.1}$$

where Ω is a bounded domain of $\mathbb{R}^N (N \geq 1)$ with a smooth boundary Γ, u is the transverse displacement of waves, the relaxation function g is positive and decreasing, the exponent ρ is a positive real number satisfying some conditions to be specified later. This is a nonlinear wave equation with the presence of a viscoelastic damping with infinite memory supplemented by a history function u_0 and initial data u_1. In section 2.1, we give some assumptions needed in this chapter. Some technical lemmas and the statement with proof of the main results are given in section 2.2 and section 2.3, respectively. Finally, we give some examples to illustrate our result.

2.1 Assumptions

In this section, we present some material needed in the proof of our result. We use the standard Lebesgue space $L^2(\Omega)$ and the Sobolev space $H_0^1(\Omega)$ with their usual scalar products and norms. For the relaxation function g, we assume the following

$(A1)$ $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a C^1 decreasing function satisfying

$$g(0) > 0, \;\; 1 - \int_0^{+\infty} g(s)ds = \ell > 0. \tag{2.2}$$

$(A2)$ There exists a nonincreasing differentiable function $\xi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$g'(t) \leq -\xi(t)g(t), \;\; \forall t \in \mathbb{R}^+. \tag{2.3}$$

$(A3)$ For the nonlinearity, we assume that

$$0 < \rho \leq \frac{2}{N-2}, \; N \geq 3 \; \textit{and} \; 0 < \rho, \; N = 1, 2. \tag{2.4}$$

$(A4)$ There exists a positive constant m_0, such that

$$||u_0(s)||_2 \leq m_0, \; \forall s \geq 0. \tag{2.5}$$

The energy associated with problem (2.1) is

$$E(t) = \frac{1}{\rho+2}\int_\Omega |u_t|^{\rho+2}dx + \frac{\ell}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{2}\int_\Omega |\nabla u_t|^2 dx + \frac{1}{2}(go\nabla u)(t), \tag{2.6}$$

where

$$(go\nabla u)(t) = \int_\Omega \int_0^{+\infty} g(s)|\nabla u(t-s) - \nabla u(t)|^2 ds dx.$$

For completeness we state, without proof, the existence result of [26].

Proposition 2.1 *Let* $(u_0(.,0), u_1) \in H_0^1(\Omega) \times H_0^1(\Omega)$ *be given. Assume that* $(A1) - (A4)$ *are satisfied; then problem* (2.1) *has a unique global (weak) solution*

$$u \in C^1(\mathbb{R}^+; H_0^1(\Omega)).$$

2.2 Technical Lemmas

In this section, we establish several lemmas needed for the proof of our main result.

Lemma 2.1 *For $u \in H_0^1(\Omega)$, we have*

$$\int_\Omega \left(\int_0^{+\infty} g(s)(u(t) - u(t-s))ds \right)^2 dx \leq (1-\ell) C_p^2 (go\nabla u)(t),$$

where C_p is the Poincaré constant.

Proof.

$$\int_\Omega \left(\int_0^{+\infty} g(s)(u(t) - u(t-s))ds \right)^2 dx = \int_\Omega \left(\int_0^{+\infty} \sqrt{g(s)}\sqrt{g(s)}(u(t) - u(t-s))ds \right)^2 dx.$$

By applying Cauchy-Schwarz' and Poincaré's inequalities, we can show that

$$\begin{aligned} &\int_\Omega \left(\int_0^{+\infty} g(s)(u(t) - u(t-s))ds \right)^2 dx \\ &\quad \leq \int_\Omega \left(\int_0^{+\infty} g(s)ds \right) \left(\int_0^{+\infty} g(s)(u(t) - u(t-s))^2 ds \right) dx \\ &\quad \leq (1-\ell) C_p^2 (go\nabla u)(t). \end{aligned}$$

∎

Lemma 2.2 *Suppose that $(A1) - (A3)$ hold. Let u be the solution of the system* (2.1)*, then the energy functional satisfies*

$$E'(t) = \frac{1}{2}(g' o \nabla u)(t) \leq 0. \tag{2.7}$$

Proof. By multiplying equation in (2.1) by u_t, integrating over Ω, using integration by parts, and hypotheses (A1) and (A2) we obtain

$$\begin{aligned}
&\frac{d}{dt}\left[\frac{1}{\rho+2}\int_\Omega |u_t|^{\rho+2}dx + \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{2}\int_\Omega |\nabla u_t|^2 dx\right] \\
&\quad - \int_\Omega \nabla u_t(t).\int_0^{+\infty} g(s)(\nabla u(t-s) - \nabla u(t))ds \\
&\quad - (1-\ell)\int_\Omega \nabla u \nabla u_t dx = 0,
\end{aligned} \tag{2.8}$$

for any regular solution. This result remains valid for weak solutions by a simple density argument. For the last term on the left side of (2.8) we have

$$\begin{aligned}
&\int_\Omega \nabla u_t(t)\int_0^{+\infty} g(s)(\nabla u(t-s) - \nabla u(t))dsdx \\
&\quad = -\frac{1}{2}\int_\Omega\int_0^{+\infty} g(s)\frac{\partial}{\partial s}|\nabla u(t-s) - \nabla u(t)|^2 dsdx \\
&\quad\quad - \frac{1}{2}\int_\Omega\int_0^{+\infty} g(s)\frac{\partial}{\partial t}|\nabla u(t-s) - \nabla u(t)|^2 dsdx.
\end{aligned}$$

Using integration by parts, we get

$$\int_\Omega \nabla u_t(t)\int_0^{+\infty} g(s)(\nabla u(t-s) - \nabla u(t))dsdx = -\frac{1}{2}(g' o \nabla u)(t) + \frac{1}{2}\frac{d}{dt}(go\nabla u)(t). \tag{2.9}$$

Inserting (2.9) into (2.8), we obtain

$$\frac{d}{dt}\left[\frac{1}{\rho+2}\int_\Omega |u_t|^{\rho+2}dx + \frac{\ell}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{2}\int_\Omega |\nabla u_t|^2 dx + \frac{1}{2}(go\nabla u)(t)\right] = \frac{1}{2}(g' o\nabla u)(t). \tag{2.10}$$

Which completes the proof. ∎

Lemma 2.3 *Under the assumptions* $(A1)-(A3)$, *the functional*

$$\psi(t) := \frac{1}{\rho+1}\int_\Omega |u_t|^\rho u_t u dx + \int_\Omega \nabla u.\nabla u_t dx$$

satisfies, along the solution of (2.1), *the estimate*

$$\psi'(t) \leq -\frac{\ell}{2}\int_\Omega |\nabla u|^2 dx + \int_\Omega |\nabla u_t|^2 dx + \frac{1}{\rho+1}\int_\Omega |u_t|^{\rho+2} dx + \frac{1-\ell}{2\ell}(go\nabla u)(t). \tag{2.11}$$

Proof. Direct differentiation of ψ, using (2.1), yields

$$\begin{aligned}\psi'(t) = &-\int_\Omega |\nabla u|^2 dx + \int_\Omega \nabla u(t).\int_0^{+\infty} g(s)\nabla u(t-s) ds dx \\ &+ \int_\Omega |\nabla u_t|^2 dx + \frac{1}{\rho+1}\int_\Omega |u_t|^{\rho+2} dx.\end{aligned} \tag{2.12}$$

We then estimate the second term on the right side of (2.12), using Young's inequality, as follows

$$\begin{aligned}&\int_\Omega \nabla u(t).\int_0^{+\infty} g(s)\nabla u(t-s) ds dx \\ &\quad \leq \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{2}\int_\Omega \left(\int_0^{+\infty} g(s)(|\nabla u(t-s) - \nabla u(t)| + |\nabla u(t)|) ds\right)^2 dx.\end{aligned}$$

By exploiting (2.2), Lemma 2.1 and

$$(a+b)^2 \leq (1+\eta)a^2 + \left(1+\frac{1}{\eta}\right) b^2, \quad \forall \eta > 0,$$

we arrive at

$$\begin{aligned}
&\int_\Omega \nabla u(t). \int_0^{+\infty} g(s)\nabla u(t-s)dsdx \\
&\quad\le \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{2}\left(1+\frac{1}{\eta}\right)\int_\Omega \left(\int_0^{+\infty} g(s)|\nabla u(t-s)-\nabla u(t)|ds\right)^2 dx \\
&\qquad\qquad + \frac{1}{2}(1+\eta)(1-\ell)^2 \int_\Omega |\nabla u|^2 dx \\
&\quad\le \frac{1}{2}\left[1+(1+\eta)(1-\ell)^2\right]\int_\Omega |\nabla u|^2 dx + \frac{1}{2}\left(1+\frac{1}{\eta}\right)(1-\ell)(go\nabla u)(t).
\end{aligned}$$

By taking $\eta = \frac{\ell}{1-\ell}$, we find

$$\int_\Omega \nabla u(t). \int_0^{+\infty} g(s)\nabla u(t-s)dsdx \le \frac{2-\ell}{2}\int_\Omega |\nabla u|^2 dx + \frac{1-\ell}{2\ell}(go\nabla u)(t). \tag{2.13}$$

Inserting (2.13) in (2.12), estimate (2.11) is established. ▮

Lemma 2.4 *Under the assumptions* $(A1)-(A4)$, *the functional*

$$\chi(t) := \int_\Omega \left(\Delta u_t - \frac{|u_t|^\rho u_t}{\rho+1}\right)\int_0^{+\infty} g(s)(u(t)-u(t-s))dsdx$$

satisfies, along the solution of (2.1) *and for any* $\delta_1, \delta_2 > 0$, *the estimate*

$$\begin{aligned}
\chi'(t) &\le (1+2(1-\ell)^2)\delta_1 \int_\Omega |\nabla u|^2 dx - \frac{1-\ell}{\rho+1}\int_\Omega |u_t|^{\rho+2} dx \\
&\quad + (1-\ell)\left(2\delta_1 + \frac{1}{2}\delta_1\right)(go\nabla u)(t) - \frac{g(0)}{4\delta_2}\left(1+\frac{C_p}{\rho+1}\right)(g'o\nabla u)(t) \\
&\quad + \left[\delta_2 + c\frac{\delta_2}{\rho+1}(2E(0))^\rho - (1-\ell)\right]\int_\Omega |\nabla u_t|^2 dx,
\end{aligned} \tag{2.14}$$

where c *is a positive constant and* C_p *is the Poincaré constant.*

Proof. Differentiating χ with respect to t and making use of (2.1), we find

$$\begin{aligned}
\chi'(t) = &\int_\Omega \nabla u(t). \int_0^{+\infty} g(s)(\nabla u(t-s) - \nabla u(t))dsdx \\
&- \int_\Omega \left(\int_0^{+\infty} g(s)\nabla u(t-s)ds\right).\left(\int_0^{+\infty} g(s)(\nabla u(t-s) - \nabla u(t))ds\right)dx \\
&- (1-\ell)\int_\Omega |\nabla u_t|^2 dx - \int_\Omega \nabla u_t(t). \int_0^{+\infty} g'(s)(\nabla u(t-s) - \nabla u(t))dsdx \\
&- \frac{1}{\rho+1}\int_\Omega |u_t|^\rho u_t \int_0^{+\infty} g'(s)(u(t-s) - u(t))dsdx \\
&- \frac{1}{\rho+1}(1-\ell)\int_\Omega |u_t|^{\rho+2}dx.
\end{aligned} \tag{2.15}$$

Now we proceed, using repeatedly Cauchy-Schwarz' inequality, Young's inequality and Lemma 2.1 to estimate each term in the right-hand side of (2.15). The first term may be estimated as follows

$$\begin{aligned}
&\int_\Omega \nabla u(t). \int_0^{+\infty} g(s)(\nabla u(t-s) - \nabla u(t))dsdx \\
&\qquad \le \delta_1 \int_\Omega |\nabla u|^2 dx + \frac{1-\ell}{4\delta_1}(go\nabla u)(t), \ \forall \delta_1 > 0.
\end{aligned} \tag{2.16}$$

For the second term, we recall (2.2) and the fact that $(a+b)^2 \le 2(a^2+b^2)$ to get

$$\begin{aligned}
&\int_\Omega \left(\int_0^{+\infty} g(s)\nabla u(t-s)ds\right).\left(\int_0^{+\infty} g(s)(\nabla u(t-s) - \nabla u(t))ds\right)dx \\
&\le \delta_1 \int_\Omega \left| \int_0^{+\infty} g(s)\nabla u(t-s)ds \right|^2 dx + \frac{1}{4\delta_1}\int_\Omega \left| \int_0^{+\infty} g(s)(\nabla u(t-s) - \nabla u(t))ds \right|^2 dx \\
&\le \delta_1 \int_\Omega \left(\int_0^{+\infty} g(s)\left(\left| \nabla u(t-s) - \nabla u(t) \right| + |\nabla u(t)|\right)ds\right)^2 dx + \frac{1-\ell}{4\delta_1}(go\nabla u)(t) \\
&\le \left(2\delta_1 + \frac{1}{4\delta_1}\right)(1-\ell)(go\nabla u)(t) + 2\delta_1(1-\ell)^2\int_\Omega |\nabla u|^2 dx.
\end{aligned} \tag{2.17}$$

For the fourth term, it is easy to see that, $\forall \delta_2 > 0$,

$$\begin{aligned}
&\int_\Omega \nabla u_t . \int_0^{+\infty} g'(s)(\nabla u(t-s) - \nabla u(t)) ds dx \\
&\quad \leq \delta_2 \int_\Omega |\nabla u_t|^2 dx + \frac{g(0)}{4\delta_2} \int_\Omega \int_0^{+\infty} -g'(s) |\nabla u(t-s) - \nabla u(t)|^2 ds dx.
\end{aligned} \tag{2.18}$$

The fifth term may be handled similarly

$$\begin{aligned}
&\frac{1}{\rho+1} \int_\Omega |u_t|^\rho u_t \int_0^{+\infty} g'(s)(u(t-s) - u(t)) ds dx \\
&\quad \leq \frac{1}{\rho+1} \left[\delta_2 \int_\Omega |u_t|^{2(\rho+1)} dx + \frac{g(0)}{4\delta_2} C_p \int_\Omega \int_0^{+\infty} -g'(s) |\nabla u(t-s) - \nabla u(t)|^2 ds dx \right],
\end{aligned} \tag{2.19}$$

where C_p is the Poincaré constant. By exploiting the Sobolev embedding

$$H_0^1(\Omega) \hookrightarrow L^{2(\rho+1)}(\Omega) \text{ for } 0 < \rho \leq \frac{2}{N-2} \text{ if } N \geq 3 \text{ and } \rho > 0 \text{ if } n = 1, 2. \tag{2.20}$$

and the fact that $E(t) \leq E(0)$, $\forall t \in \mathbb{R}^+$, we obtain

$$\int_\Omega |u_t|^{2(\rho+1)} dx \leq c(2E(0))^\rho \int_\Omega |\nabla u_t|^2 dx. \tag{2.21}$$

Therefore (2.19) takes the form

$$\begin{aligned}
&\frac{1}{\rho+1} \int_\Omega |u_t|^\rho u_t \int_0^{+\infty} g'(s)(u(t-s) - u(t)) ds dx \\
&\quad \leq c\delta_2 (2E(0))^\rho \int_\Omega |\nabla u_t|^2 dx - \frac{g(0) C_p}{4\delta_2(\rho+1)} (g' o \nabla u)(t).
\end{aligned} \tag{2.22}$$

Combining (2.15)-(2.18) and (2.22), estimate (2.14) is established. ∎

2.3 General Decay Result

In this section we state and prove our main result. For this purpose we introduce the following lemmas

Lemma 2.5 *Assume that* $(A1)-(A4)$ *hold. Then there exist constants* $\varepsilon, \alpha_1, \alpha_2, M > 0$ *such that the functional*

$$L = ME + \varepsilon\psi + \chi$$

satisfies, for all $t \in \mathbb{R}^+$,

$$L \sim E \tag{2.23}$$

and

$$L'(t) \leq -\alpha_1 E(t) + \alpha_2 \int_0^{+\infty} g(s)||\nabla u(t-s) - \nabla u(t)||_2^2 ds. \tag{2.24}$$

Proof. To prove (2.23), we use Young's inequality and the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{\rho+2}(\Omega)$ to obtain

$$\begin{aligned}
|L(t) - ME(t)| \leq & \frac{\varepsilon}{\rho+2}||u_t||_{\rho+2}^{\rho+2} + \frac{\varepsilon}{(\rho+1)(\rho+2)}||u||_{\rho+2}^{\rho+2} + \frac{\varepsilon}{2}||\nabla u_t||_2^2 \\
& + \frac{\varepsilon}{2}||\nabla u||_2^2 + \frac{1}{2(\rho+1)}||u_t||_{2(\rho+1)}^{2(\rho+1)} + \frac{1-\ell}{2(\rho+1)}C_p(go\nabla u)(t) \\
& + \frac{1}{2}||\nabla u_t||_2^2 + \frac{1-\ell}{2}(go\nabla u)(t) \\
\leq & \varepsilon E(t) + \varepsilon\frac{C^{\rho+2}}{(\rho+1)(\rho+2)}2^{\frac{\rho+2}{2}}\ell^{-\frac{\rho+2}{2}}(E(0))^{\frac{\rho}{2}}E(t) + \varepsilon(1+\ell^{-1})E(t) \\
& + \frac{C_p^{2(\rho+1)}}{\rho+1}2^{\rho}(E(0))^{\rho}E(t) + \frac{(1-\ell)}{2(\rho+1)}C_pE(t) + E(t) + (1-\ell)E(t).
\end{aligned} \tag{2.25}$$

By fixing M large enough, we obtain $|L(t) - ME(t)| \leq C_1 E(t)$ which gives the desired result.

For the proof of (2.24), using (2.7), (2.11) and (2.14), we can show that

$$\begin{aligned} L'(t) \leq & \left[\frac{M}{2} - \frac{g(0)}{4\delta_2} \left(1 + \frac{C_p}{\rho+1} \right) \right] (g' o \nabla u)(t) - \frac{1-\ell-\varepsilon}{\rho+1} \int_\Omega |u_t|^{\rho+2} dx \\ & - \left[\varepsilon \frac{\ell}{2} - (1 + 2(1-\ell)^2)\delta_1 \right] \int_\Omega |\nabla u|^2 dx \\ & - [1 - \ell - \varepsilon - \delta_2 - c\delta_2 (2E(0))^\rho] \int_\Omega |\nabla u_t|^2 dx \\ & + (1-\ell) \left(\frac{\varepsilon}{2\ell} + 2\delta_1 + \frac{1}{2\delta_1} \right) (g o \nabla u)(t). \end{aligned} \tag{2.26}$$

We now choose our constant carefully. First, we pick $\varepsilon < 1 - \ell$, and then δ_1 and δ_2 small enough that

$$\varepsilon \frac{\ell}{2} - (1 + 2(1-\ell)^2)\delta_1 > 0, \qquad 1 - \ell - \varepsilon - \delta_2 - c\delta_2 (2E(0))^\rho > 0.$$

Then we take M sufficiently large that

$$\frac{M}{2} - \frac{g(0)}{4\delta_2} \left(1 + \frac{C_p}{\rho+1} \right) \geq 0.$$

Therefore, (2.26) reduces to (2.24) for two positive constants α_1 and α_2. ∎

Lemma 2.6 *Assume that (A1) − (A4). Then there exist positive constants β_1 and β_2 such that for all $t \in \mathbb{R}^+$,*

$$\xi(t) L'(t) + \beta_1 E'(t) \leq -\alpha_1 \xi(t) E(t) + \beta_2 \xi(t) \int_t^{+\infty} g(s) ds. \tag{2.27}$$

Proof. Using (2.3), (2.7) and the fact that ξ and g are noninceasing, we get

$$\begin{aligned} \xi(t)\int_0^t g(s)||\nabla u(t-s)-\nabla u(t)||_2^2 ds &\leq -\int_0^t g'(s)||\nabla u(t-s)-\nabla u(t)||_2^2 \\ &\leq -2E'(t), \quad \forall t\in\mathbb{R}^+. \end{aligned} \tag{2.28}$$

On the other hand, (2.6) and the fact that E is nonincreasing imply that

$$\begin{aligned} ||\nabla u(t-s)-\nabla u(t)||_2^2 &\leq 2||\nabla u(t-s)||_2^2+2||\nabla u(t)||_2^2 \\ &\leq 4\sup_{s>0}||\nabla u(s)||_2^2+2\sup_{\tau<0}||\nabla u(\tau)||_2^2 \\ &\leq \frac{8}{\ell}E(0)+2\sup_{\tau>0}||\nabla u_0(\tau)||_2^2, \quad \forall t,\, s\in\mathbb{R}^+. \end{aligned}$$

Hence, combination with (2.5) yields, for all $t\in\mathbb{R}^+$,

$$\xi(t)\int_t^{+\infty} g(s)||\nabla u(t-s)-\nabla u(t)||_2^2 ds \leq \left(\frac{8}{\ell}E(0)+2m_0^2\right)\xi(t)\int_t^{+\infty} g(s)ds. \tag{2.29}$$

Finally, multiplying (2.24) by $\xi(t)$ and combining with (2.28) and (2.29), we get (2.27). ∎

Theorem 2.1 *Let $(u_0(.,0),u_1)\in H_0^1(\Omega)\times H_0^1(\Omega)$ be given. Assume that $(A1)-(A4)$ hold. Then, there exist constants $\gamma_0\in(0,1)$ and $\delta_1>0$ such that, for all $t\in\mathbb{R}^+$ and for all $\delta_0\in(0,\gamma_0]$,*

$$E(t)\leq \delta_1\left(1+\int_0^t (g(s))^{1-\delta_0}ds\right)e^{-\delta_0\int_0^t \xi(s)ds}+\delta_1\int_t^{+\infty} g(s)ds. \tag{2.30}$$

Proof. We will use the approach of [41] to prove this theorem. Consider the functionals

$$F(t) = \xi(t)L(t) + \beta_1 E(t) \ \text{ and } \ h(t) = \xi(t)\int_t^{+\infty} g(s)ds. \qquad (2.31)$$

Using the facts that $F \sim E$ and ξ is nonincreasing, (2.27) gives

$$F'(t) \leq -\gamma_0\xi(t)F(t) + \beta_2 h(t), \quad \forall t \in \mathbb{R}^+,$$

for some $\gamma_0 > 0$. This last inequality remains true for any $\delta_0 \in (0, \gamma_0]$; that is

$$F'(t) \leq -\delta_0\xi(t)F(t) + \beta_2 h(t), \quad \forall t \in \mathbb{R}^+.$$

Therefore, direct integration leads to

$$F(T) \leq e^{-\delta_0\int_0^T \xi(s)ds}\left(F(0) + \beta_2\int_0^T e^{\delta_0\int_0^t \xi(s)ds}h(t)dt\right)$$

and the fact that $F \sim E$ gives

$$E(T) \leq \frac{1}{\beta_1}e^{-\delta_0\int_0^T \xi(s)ds}\left(F(0) + \beta_2\int_0^T e^{\delta_0\int_0^t \xi(s)ds}h(t)dt\right). \qquad (2.32)$$

Noting that

$$e^{\delta_0\int_0^t \xi(s)ds}h(t) = \frac{1}{\delta_0}\left(e^{\delta_0\int_0^t \xi(s)ds}\right)'\int_t^{+\infty} g(s)ds, \quad \forall t \in \mathbb{R}^+,$$

integration by parts then leads to

$$\int_0^T e^{\delta_0 \int_0^t \xi(s)ds} h(t)dt$$
$$= \frac{1}{\delta_0}\left(e^{\delta_0 \int_0^T \xi(s)ds} \int_T^{+\infty} g(s)ds - \int_0^{+\infty} g(s)ds + \int_0^T e^{\delta_0 \int_0^t \xi(s)ds} g(t)dt \right).$$

Consequently, combining with (2.32), we have

$$E(T) \leq \frac{1}{\beta_1}\left(F(0) + \frac{\beta_2}{\delta_0} \int_0^T e^{\delta_0 \int_0^t \xi(s)ds} g(t)dt \right) e^{-\delta_0 \int_0^T \xi(s)ds} + \frac{\beta_2}{\beta_1 \delta_0} \int_T^{+\infty} g(s)ds. \tag{2.33}$$

Also, (2.3) implies that

$$\left(e^{\int_0^t \xi(s)ds} g(t) \right)' \leq 0, \quad \forall t \in \mathbb{R}^+.$$

Consequently, we have $e^{\int_0^t \xi(s)ds} g(t) \leq g(0)$ and

$$\int_0^T e^{\delta_0 \int_0^t \xi(s)ds} g(t)dt \leq (g(0))^{\delta_0} \int_0^T (g(t))^{1-\delta_0}\, dt. \tag{2.34}$$

Finally, we obtain (2.30) by combining (2.6), (2.33) and (2.34). ∎

Remark 2.1 *If there exists* $\varepsilon_0 \in (0,1)$, *for which*

$$\int_0^{+\infty} (g(s))^{1-\varepsilon_0} ds < +\infty, \tag{2.35}$$

then we can choose $\delta_0 \in (0, \gamma_1], \gamma_1 = \min\{\varepsilon_0, \gamma_0\}$ *such that*

$$\int_0^{+\infty} (g(s))^{1-\delta_0}\, ds < +\infty,$$

and consequently, (2.30) *takes the form*

$$E(t) \le \delta_2 \left(e^{-\delta_0 \int_0^t \xi(s)ds} + \int_t^{+\infty} g(s)ds \right). \tag{2.36}$$

Now, we give some examples to illustrate the energy decay rates obtained by Theorem 2.1.

1. Let $g(t) = de^{-(1+t)^q}$ with $0 < q \le 1$ and $d > 0$ small enough that (2.2) and (2.3), with $\xi(t) = q(1+t)^{q-1}$, hold. Then, (2.35) is satisfied and consequently, (2.36) gives, for two positive constants c_1 and c_2,

$$E(t) \le c_1 e^{-c_2(1+t)^q}, \quad \forall t \in \mathbb{R}^+.$$

2. Let $g(t) = \frac{d}{(1+t)^q}$ with $q > 1$ and $d > 0$ small enough that (2.2) and (2.3), with $\xi(t) = q(1+t)^{-1}$, hold. Then (2.35) is satisfied and consequently, (2.36) gives, for two positive constants c_1 and c_2,

$$E(t) \le \frac{c_1}{(t+1)^{c_2}}, \quad \forall t \in \mathbb{R}^+.$$

CHAPTER 3

A GENERAL DECAY RESULT OF A NONLINEAR SYSTEM OF WAVE EQUATIONS WITH INFINITE HISTORIES

In this chapter, we investigate the asymptotic behavior of a system of viscoelastic wave equations with infinite histories. We use the multiplier method and the well depth method to establish a general stability result of the system. To this end, we consider the following viscoelastic problem

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) + \int_0^{+\infty} g(s)\Delta u(x,t-s)ds + \lambda |u_t|^{m-1}u_t = f_1(u,v), & \text{in } \Omega \times \mathbb{R}^+, \\ v_{tt}(x,t) - \Delta v(x,t) + \int_0^{+\infty} h(s)\Delta v(x,t-s)ds + \mu |v_t|^{r-1}v_t = f_2(u,v), & \text{in } \Omega \times \mathbb{R}^+, \\ u(x,t) = v(x,t) = 0, & \text{in } \partial\Omega \times \mathbb{R}^+, \\ u(x,-t) = u_0(x,t), u_t(x,0) = u_1(x), v(x,-t) = v_0(x,t), v_t(x,0) = v_1(x), & \text{in } \Omega \times \mathbb{R}^+, \end{cases} \tag{3.1}$$

with

$$\begin{cases} f_1(u,v) = a|u+v|^{2(\rho+1)}(u+v) + b|u|^\rho u|v|^{\rho+2}, \\ f_2(u,v) = a|u+v|^{2(\rho+1)}(u+v) + b|v|^\rho v|u|^{\rho+2}, \end{cases} \tag{3.2}$$

where u and v denote the transverse displacements of waves, Ω is a bounded domain of $\mathbb{R}^N (N = 1,2,3)$ with a smooth boundary $\partial\Omega, \rho, m, r,\ \lambda, \mu$ are positive constants, the kernels g and h are satisfying some conditions to be specified later and the nonlinear coupling functions, $f_1,\ f_2$, describe the interaction between the two waves.

The rest of the chapter is organized as follows. In section 3.1, we introduce some assumptions needed in this chapter. Some technical lemmas and the statement with proof of our main result will be given in section 3.2 and section 3.3, respectively. Finally, we give some examples to illustrate our result.

3.1 Assumptions

In this section, we present some materials needed in the proof of our result. We use the standard Lebesgue space $L^2(\Omega)$ and the Sobolev space $H_0^1(\Omega)$ with their usual scalar products and norms and we denote by H the following space $H = H_0^1(\Omega) \times H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega)$. We assume the following

$(A1)$ $g, h : \mathbb{R}^+ \to \mathbb{R}^+$ are C^1 functions satisfying

$$\begin{cases} g(0) > 0, \ 1 - \int_0^{+\infty} g(s)ds = \ell > 0. \\ h(0) > 0, \ 1 - \int_0^{+\infty} h(s)ds = \kappa > 0. \end{cases} \tag{3.3}$$

$(A2)$ There exist two positive nonincreasing differentiable functions ξ_1 and ξ_2 such that

$$\begin{cases} g'(t) \leq -\xi_1(t)g(t), \, t \in \mathbb{R}^+. \\ h'(t) \leq -\xi_2(t)h(t), t \in \mathbb{R}^+. \end{cases} \tag{3.4}$$

For the nonlinearity, we suppose that

$(A3)$

$$\begin{cases} -1 < \rho \ \text{ if } N = 1, 2 \ \text{ and } \ -1 < \rho \leq \frac{3-N}{N-2} \ \text{ if } N = 3, \\ 1 < r, m \ \text{ if } N = 1, 2 \ \text{ and } \ 1 < r, m \leq \frac{N+2}{N-2} \ \text{ if } N = 3. \end{cases} \tag{3.5}$$

$(A4)$ There exists $m_0 > 0$ such that

$$||\nabla u_0(s)||_2 \leq m_0 \ \text{ and } \ ||\nabla v_0(s)||_2 \leq m_0, \ \ \forall s \geq 0. \tag{3.6}$$

Remark 3.1 *Concerning the functions f_1 and f_2 , we note that*

$$uf_1(u,v) + vf_2(u,v) = 2(\rho+2)F(u,v), \quad \forall (u,v) \in \mathbb{R}^2,$$

where

$$F(u,v) = \frac{1}{2(\rho+2)}\left[a|u+v|^{2(\rho+2)} + 2b|uv|^{\rho+2}\right].$$

3.2 Technical Lemmas

In this section we establish several lemmas needed for the proof of our main result.

First, we introduce the following functionals

$$J(t) = \frac{\ell}{2}|| \nabla u||_2^2 + \frac{\kappa}{2}|| \nabla v||_2^2 + \frac{1}{2}[(go\nabla u)(t) + (ho\nabla v)(t)] - \int_\Omega F(u,v)dx, \quad (3.7)$$

where

$$(go\nabla u)(t) = \int_\Omega \int_0^{+\infty} g(s)|\nabla u(t-s) - \nabla u(t)|^2 ds dx$$

and

$$\begin{aligned} I(t) = \ell|| \nabla u||_2^2 + \kappa|| \nabla v||_2^2 + (go\nabla u)(t) + (ho\nabla v)(t) \\ - 2(\rho+2)\int_\Omega F(u,v)dx. \end{aligned} \quad (3.8)$$

The "modified" energy functional E associated to system (3.1) is

$$\begin{aligned} E(t) = &\frac{1}{2}(||u_t||_2^2 + ||v_t||_2^2) + \frac{\ell}{2}||\ \nabla u||_2^2 + \frac{\kappa}{2}||\ \nabla v||_2^2 \\ &+\frac{1}{2}[(go\nabla u)(t) + (ho\nabla v)(t)] - \int_\Omega F(u,v)dx. \end{aligned} \tag{3.9}$$

Lemma 3.1 *Suppose that* $(A1)-(A3)$ *hold. Let* (u,v) *be the solution of the system* (3.1), *then the energy functional is a non-increasing function and*

$$\frac{dE(t)}{dt} = -\left[\lambda||u_t||_{m+1}^{m+1} + \mu||v_t||_{r+1}^{r+1} - \frac{1}{2}(g'o\nabla u)(t) - \frac{1}{2}(h'o\nabla v)(t)\right] \leq 0. \tag{3.10}$$

Proof. By multiplying the first equation in (3.1) by u_t, integrating over Ω, using integration by parts, and hypotheses $(A1)$ and $(A2)$ we obtain

$$\begin{aligned} &\frac{d}{dt}\left[\frac{1}{2}\int_\Omega u_t^2 dx - \frac{1}{2}\int_\Omega |\nabla u|^2 dx\right] \\ &\quad - \int_\Omega \nabla u_t(t).\int_0^{+\infty} g(s)(\nabla u(t-s) - \nabla u(t))ds \\ &\quad - (1-\ell)\int_\Omega \nabla u(t)\nabla u_t(t)ds + \lambda\int_\Omega |\ u_t|^{m+1} = \int_\Omega u_t f_1(u,v)dx, \end{aligned} \tag{3.11}$$

for any regular solution. This result remains valid for weak solutions by a simple density argument. For the second term on the left side of (3.11) we have

$$\begin{aligned} &\int_\Omega \nabla u_t(t)\int_0^{+\infty} g(s)(\nabla u(t-s) - \nabla u(t))dsdx \\ &\quad = \frac{1}{2}\int_\Omega\int_0^{+\infty} g(s)\frac{\partial}{\partial s}|\nabla u(t-s) - \nabla u(t)|^2 dsdx \\ &\quad\quad - \frac{1}{2}\int_\Omega\int_0^{+\infty} g(s)\frac{\partial}{\partial t}|\nabla u(t-s) - \nabla u(t)|^2 dsdx. \end{aligned}$$

Using integration by parts, we get

$$\int_\Omega \nabla u_t(t) \int_0^{+\infty} g(s)(\nabla u(t-s) - \nabla u(t))dsdx = -\frac{1}{2}(g'o\nabla u)(t) + \frac{1}{2}\frac{d}{dt}(go\nabla u)(t). \tag{3.12}$$

Inserting (3.12) into (3.11), we obtain

$$\frac{1}{2}\frac{d}{dt}\left[||u_t||_2^2 + \ell||\nabla u||_2^2 + (go\nabla u)(t)\right] = \frac{1}{2}(g'o\nabla u)(t) - \lambda\int_\Omega |u_t|^{m+1}dx + \int_\Omega u_t f_1(u,v)dx. \tag{3.13}$$

Similarly, if we multiply the second equation by v_t and integrate over Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}\left[||v_t||_2^2 + \kappa||\nabla v||_2^2 + (ho\nabla v)(t)\right] = \frac{1}{2}(h'o\nabla v)(t) - \mu\int_\Omega |v_t|^{r+1}dx + \int_\Omega v_t f_2(u,v)dx. \tag{3.14}$$

Hence, after adding (3.13) to (3.14), (5.7) is established. ▌

Lemma 3.2 *[79] There exist two positive constants c_0 and c_1 such that*

$$\frac{c_0}{2(\rho+2)}\left(|u|^{2(\rho+2)} + |v|^{2(\rho+2)}\right) \le F(u,v) \le \frac{c_1}{2(\rho+2)}\left(|u|^{2(\rho+2)} + |v|^{2(\rho+2)}\right). \tag{3.15}$$

Proof. The right-hand side of inequality (3.15) is trivial. For the left-hand side, the result is also trivial if $u = v = 0$.

If, without loss of generality, $v \neq 0$, then either $|u| \le |v|$ or $|u| > |v|$.

For $|u| \leq |v|$, we get

$$F(u,v) = \frac{1}{2(\rho+2)}|v|^{2(\rho+2)} \left[a\left| 1 + \frac{u}{v} \right|^{2(\rho+2)} + 2b\left| \frac{u}{v} \right|^{\rho+2} \right].$$

Consider the continuous function

$$j(s) = a|1+s|^{2(\rho+2)} + 2b|s|^{\rho+2}, \quad \text{over } [-1,1].$$

So $\min j(s) \geq 0$. If $\min j(s) = 0$ then, for some $s_0 \in [-1,1]$, we have

$$j(s_0) = a|1+s_0|^{2(\rho+2)} + 2b|s_0|^{\rho+2} = 0.$$

This implies that $|1+s_0| = |s_0| = 0$, which is impossible. Thus $\min j(s) = 2c_0 > 0$. Therefore

$$F(u,v) \geq \frac{c_0}{\rho+2}|v|^{2(\rho+2)} \geq \frac{c_0}{\rho+2}|u|^{2(\rho+2)}.$$

Consequently,

$$2F(u,v) \geq \frac{c_0}{\rho+2}(|v|^{2(\rho+2)} + |u|^{2(\rho+2)})$$

and then

$$\frac{c_0}{2(\rho+2)}(|v|^{2(\rho+2)} + |u|^{2(\rho+2)}) \leq F(u,v).$$

If $|u| \geq |v|$, then

$$
\begin{aligned}
F(u,v) &= \frac{1}{2(\rho+2)}|u|^{2(\rho+2)}\left[a\left|\,1+\frac{v}{u}\,\right|^{2(\rho+2)}+2b\left|\,\frac{v}{u}\,\right|^{\rho+2}\right] \\
&\geq \frac{c_0}{\rho+2}|u|^{2(\rho+2)} \\
&\geq \frac{c_0}{\rho+2}|v|^{2(\rho+2)}.
\end{aligned}
$$

This leads to the desired result and completes the proof of Lemma 3.2. ∎

Lemma 3.3 *[79] There exist two positive constants* Λ_1 *and* Λ_2 *such that*

$$
\int_\Omega |f_i(u,v)|^2 dx \leq \Lambda_i\big(\ell\|\,\nabla u\|_2^2+\kappa\|\,\nabla v\|_2^2\big)^{2\rho+3}, \quad i=1,2. \tag{3.16}
$$

Proof. we prove inequality (3.16) for f_1 and the same result holds for f_2. It's clear that

$$
\begin{aligned}
|f_1(u,v)| &\leq C\left(|u+v|^{2\rho+3}+|u|^{\rho+1}|v|^{\rho+2}\right) \\
&\leq C\left(|u|^{2\rho+3}+|v|^{2\rho+3}+|u|^{\rho+1}|v|^{\rho+2}\right).
\end{aligned} \tag{3.17}
$$

From (3.17) and Young's inequality, with

$$
q=\frac{2\rho+3}{\rho+1}, \quad q'=\frac{2\rho+3}{\rho+2},
$$

we get

$$
|u|^{\rho+1}|v|^{\rho+2} \leq c_1|u|^{2\rho+3}+c_2|v|^{2\rho+3},
$$

hence

$$|f_1(u,v)| \leq C\left[|u|^{2\rho+3} + |v|^{2\rho+3}\right].$$

Consequently, by using Poincaré's inequality and (3.5), we obtain

$$\int_\Omega |f_1(u,v)|^2 dx \leq C\left(||\ \nabla u||_2^{2(2\rho+3)} + ||\ \nabla v||_2^{2(2\rho+3)}\right)$$
$$\leq \Lambda_1(\ell||\ \nabla u||_2^2 + \kappa||\ \nabla v||_2^2)^{2\rho+3}.$$

This completes the proof of Lemma 3.3 ∎

Lemma 3.4 *[79] Suppose that* (3.5) *holds. Then there exists* $\eta > 0$ *such that for any* $(u,v) \in H_0^1(\Omega) \times H_0^1(\Omega)$*, we have*

$$||u+v||_{2(\rho+2)}^{2(\rho+2)} + 2||uv||_{\rho+2}^{\rho+2} \leq \eta\big(\ell||\ \nabla u||_2^2 + \kappa||\ \nabla v||_2^2\big)^{\rho+2}. \tag{3.18}$$

Proof. It is clear that, by using the Minkowski inequality we get

$$||u+v||_{2(\rho+2)}^2 \leq 2(||u||_{2(\rho+2)}^2 + ||v||_{2(\rho+2)}^2).$$

Also, Hölder's and Young's inequalities give us

$$||uv||_{\rho+2} \leq ||u||_{2(\rho+2)}||v||_{2(\rho+2)} \leq c(\ell||\nabla u||_2^2 + \kappa||\nabla v||_2^2).$$

A combination of the two last inequalities and the embedding $H_0^1(\Omega) \hookrightarrow L^{2(\rho+2)}(\Omega)$ yields (3.18).

∎

Lemma 3.5 *[79] Suppose that (A1) − (A4) hold. Then for any $(u_0(0), v_0(0), u_1, v_1) \in H$ satisfying*

$$\begin{cases} \beta = \eta\left[\frac{2(\rho+2)}{\rho+1}E(0)\right]^{\rho+1} < 1, \\ I(0) = I(u_0(0), v_0(0)) > 0, \end{cases} \tag{3.19}$$

we have

$$I(t) = I(u(t), v(t)) > 0, \quad \forall t \in \mathbb{R}^+. \tag{3.20}$$

Proof. Since $I(0) > 0$, then by continuity,

$$I(t) \geq 0, \quad \text{on } (0, \delta), \ \delta > 0.$$

Let T_m be such that

$$\left\{ I(T_m) = 0 \ \ and \ \ I(t) > 0, \ \ \forall \, 0 \leq t < T_m \right\}, \tag{3.21}$$

which implies that, for all $t \in [0, T_m]$,

$$\begin{aligned} J(t) &= \frac{1}{2(\rho+2)}I(t) + \frac{\rho+1}{2(\rho+2)}\left\{\ell||\nabla u||_2^2 + \kappa||\nabla v||_2^2 + (go\nabla u)(t) + (ho\nabla v)(t)\right\}, \\ &\geq \frac{\rho+1}{2(\rho+2)}\left\{\ell||\nabla u||_2^2 + \kappa||\nabla u||_2^2 + (go\nabla u)(t) + (ho\nabla v)(t)\right\}. \end{aligned} \tag{3.22}$$

By using (3.7) , (3.10) and (3.22), we easily get

$$
\begin{aligned}
\ell||\nabla u||_2^2 + \kappa||\nabla v||_2^2 &\leq \frac{2(\rho+2)}{\rho+1} J(t) \\
&\leq \frac{2(\rho+2)}{\rho+1} E(t) \qquad (3.23)\\
&\leq \frac{2(\rho+2)}{\rho+1} E(0), \quad \forall t \in [0, T_m].
\end{aligned}
$$

By exploiting (3.18), (3.19) and (3.23), we obtain

$$
\begin{aligned}
2(\rho+2)\int_\Omega F(u(T_m), v(T_m))dx &\leq \eta\big(\ell||\nabla u(T_m)||_2^2 + \kappa||\nabla v(T_m)||_2^2\big)^{\rho+2} \\
&= \eta\big(\ell||\nabla u(T_m)||_2^2 + \kappa||\nabla v(T_m)||_2^2\big)^{\rho+1} \\
&\quad \times \big(\ell||\nabla u(T_m)||_2^2 + \kappa||\nabla v(T_m)||_2^2\big) \\
&\leq \eta\left[\frac{2(\rho+2)}{\rho+1} E(0)\right]^{\rho+1} \times \big(\ell||\nabla u(T_m)||_2^2 + \kappa||\nabla v(T_m)||_2^2\big).
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
2(\rho+2)\int_\Omega F(u(T_m), v(T_m))dx &\leq \beta\left(\ell||\nabla u(T_m)||_2^2 + \kappa||\nabla v(T_m)||_2^2\right) \\
&< \left(\ell||\nabla u(T_m)||_2^2 + \kappa||\nabla v(T_m)||_2^2\right).
\end{aligned}
$$

Therefore, by using (3.8), we conclude that

$$
I(T_m) > 0.
$$

This contradicts (3.21). So $I(t) > 0, \;\; \forall t \in \mathbb{R}^+$.

∎

Remark 3.2 *We can easily deduce from Lemma 3.5 that*

$$\ell||\nabla u||_2^2 + \kappa||\nabla v||_2^2 \leq \frac{2(\rho+2)}{\rho+1}E(0), \quad \forall t \in \mathbb{R}^+. \tag{3.24}$$

Remark 3.3 *The restriction* (3.19) *on the initial data will guarantee the nonnegativeness of* $E(t)$.

Proposition 3.1 *Let* $N = 1, 2, 3$. *Assume that* $(A1) - (A4)$ *and* (3.19) *hold. Then for any initial data* $u_0(.,0), v_0(.,0) \in H_0^1(\Omega)$ *and* $u_1, v_1 \in L^2(\Omega)$, *there exists a unique global weak solution* (u, v) *of* (3.1).

This proposition can be established by a standard Galerkin method similar to [44] and [68].

Lemma 3.6 *Under the assumptions* $(A1) - (A4)$, *the functional*

$$\psi(t) := \int_\Omega u_t u dx + \int_\Omega v_t v dx$$

satisfies, along the solution of system (3.1), *the estimate*

$$\begin{aligned}
\psi'(t) \leq & \left(1 + \frac{C_*}{\ell}\right)||u_t||_2^2 + \left(1 + \frac{C_*}{\kappa}\right)||v_t||_2^2 - \frac{\ell}{4}||\ \nabla u||_2^2 - \frac{\kappa}{4}||\ \nabla v||_2^2 \\
& + \frac{1-\ell}{2\ell}(go\nabla u)(t) + \frac{1-\kappa}{2\kappa}(ho\nabla v)(t) \\
& + \left[\lambda\frac{m}{m+1}\beta_1^{-\frac{m+1}{m}}\right]||u_t||_{m+1}^{m+1} + \left[\mu\frac{r}{r+1}\beta_2^{-\frac{r+1}{r}}\right]||v_t||_{r+1}^{r+1} \\
& + 2(\rho+2)\int_\Omega F(u,v)dx,
\end{aligned} \tag{3.25}$$

for some constants β_1, β_2 *and* C_* *is the Poincaré constant.*

Proof. Direct computations, using (3.1), yield

$$\psi'(t) = ||u_t||_2^2 + ||v_t||_2^2 - ||\ \nabla u||_2^2 - ||\ \nabla v||_2^2 + \int_\Omega u f_1(u,v)dx + \int_\Omega v f_2(u,v)dx$$
$$+ \int_\Omega \nabla u. \left(\int_0^{+\infty} g(s).\nabla u(t-s)ds \right) dx + \int_\Omega \nabla v. \left(\int_0^{+\infty} h(s).\nabla v(t-s)ds \right) dx$$
$$- \int_\Omega \lambda u|u_t|^{m-1}u_t dx - \int_\Omega \mu v|v_t|^{r-1}v_t dx. \tag{3.26}$$

Young's inequality, Poincaré's inequality and (3.19) imply, for some $\beta_1,\ \beta_2 > 0$,

$$\left| \int_\Omega \lambda u|u_t|^{m-1}u_t dx \right| \le \lambda \frac{\beta_1^{m+1}}{m+1} ||u||_{m+1}^{m+1} + \lambda \frac{m}{m+1} {\beta_1}^{-\frac{m+1}{m}} ||u_t||_{m+1}^{m+1}$$
$$\le \lambda \frac{{\beta_1}^{m+1} C_*^{m+1}}{m+1} \left(\frac{2(\rho+2)}{\ell\,(\rho+1)} E(0) \right)^{\frac{m-1}{2}} ||\nabla u||_2^2$$
$$+ \lambda \frac{m}{m+1} {\beta_1}^{-\frac{m+1}{m}} ||u_t||_{m+1}^{m+1} \tag{3.27}$$

and

$$\left| \int_\Omega \mu v|v_t|^{r-1}v_t dx \right| \le \mu \frac{{\beta_2}^{r+1} C_*^{r+1}}{r+1} \left(\frac{2(\rho+2)}{\kappa\,(\rho+1)} E(0) \right)^{\frac{r-1}{2}} ||\nabla v||_2^2$$
$$+ \mu \frac{r}{r+1} {\beta_2}^{-\frac{r+1}{r}} ||v_t||_{r+1}^{r+1}. \tag{3.28}$$

Also, using Cauchy$-$Schwarz' and Young's inequalities as in [60], we obtain for all $\mu_1,\ \mu_2 > 0$.

$$
\begin{aligned}
&\int_\Omega \nabla u(t). \left(\int_0^{+\infty} g(s)\nabla u(t-s)ds\right)dx \\
&\quad \leq \frac{1}{2}\left\{||\nabla u||_2^2 + \left(1+\frac{1}{\mu_1}\right)(1-\ell)(go\nabla u) + (1+\mu_1)(1-\ell)^2||\nabla u||_2^2\right\}
\end{aligned}
\tag{3.29}
$$

and

$$
\begin{aligned}
&\int_\Omega \nabla v(t). \left(\int_0^{+\infty} g(s)\nabla v(t-s)ds\right)dx \\
&\quad \leq \frac{1}{2}\left\{||\nabla v||_2^2 + \left(1+\frac{1}{\mu_2}\right)(1-\kappa)(ho\nabla v) + (1+\mu_2)(1-k)^2||\nabla v||_2^2\right\}.
\end{aligned}
\tag{3.30}
$$

Inserting (3.27), (3.28), (3.29) and (3.30) into (3.26), we arrive at

$$
\begin{aligned}
\psi'(t) \leq & \left(1+\frac{C_*}{\ell}\right)||u_t||_2^2 + \left(1+\frac{C_*}{\kappa}\right)||v_t||_2^2 - \kappa_1||\ \nabla u||_2^2 - \kappa_2||\ \nabla v||_2^2 \\
& + \frac{1}{2}\left(1+\frac{1}{\mu_1}\right)(1-\ell)\,(go\nabla u) + \frac{1}{2}\left(1+\frac{1}{\mu_2}\right)(1-\kappa)\,(ho\nabla v) \\
& + \left[\lambda\frac{m}{m+1}\beta_1^{-\frac{m+1}{m}}\right]||u_t||_{m+1}^{m+1} + \left[\mu\frac{r}{r+1}\beta_2^{-\frac{r+1}{r}}\right]||v_t||_{r+1}^{r+1} \\
& + 2(\rho+2)\int_\Omega F(u,v)dx,
\end{aligned}
\tag{3.31}
$$

where

$$
\kappa_1 = \frac{1}{2}\left(1-(1+\mu_1)(1-\ell)^2\right) - \frac{\lambda\beta_1^{m+1}C_*^{m+1}}{m+1}\left(\frac{2(\rho+2)}{\ell(\rho+1)}E(0)\right)^{\frac{m-1}{2}}
$$

and

$$\kappa_2 = \frac{1}{2}\left(1 - (1+\mu_2)(1-\kappa)^2\right) - \frac{\mu\beta_2^{r+1}C_*^{r+1}}{r+1}\left(\frac{2(\rho+2)}{\kappa(\rho+1)}E(0)\right)^{\frac{r-1}{2}}.$$

Choosing $\mu_1 = \frac{\ell}{1-\ell}$, $\mu_2 = \frac{\kappa}{1-\kappa}$ and piking β_1 and β_2 small enough such that

$$\frac{\lambda\beta_1^{m+1}C_*^{m+1}}{m+1}\left(\frac{2(\rho+2)}{\ell(\rho+1)}E(0)\right)^{\frac{m-1}{2}} \leq \frac{\ell}{4}$$

and

$$\frac{\mu\beta_2^{r+1}C_*^{r+1}}{r+1}\left(\frac{2(\rho+2)}{\kappa(\rho+1)}E(0)\right)^{\frac{r-1}{2}} \leq \frac{\kappa}{4}$$

estimate (3.25) follows.

∎

Lemma 3.7 *Assume that* $(A1)-(A4)$ *hold. Then the functional*

$$\chi_1(t) := -\int_\Omega u_t \int_0^{+\infty} g(s)(u(t) - u(t-s))dsdx. \tag{3.32}$$

satisfies, along the solution of system (3.1) *and for all* $\delta > 0$, *the estimate*

$$\begin{aligned}\chi_1'(t) \leq &\left[\left(2\delta + \frac{1}{4\delta}\right)(1-\ell) + \frac{1-\ell}{4\delta} + \lambda C_*^{m+1}N_1^{\frac{m-1}{2}}\frac{\delta^{m+1}}{m+1}(1-\ell)^m + \frac{(1-\ell)C_*^2}{4\delta}\right](go\nabla u)(t)\\ &+ (2\delta(1-\ell)^2 + \delta + \ell\delta\Lambda_3)||\nabla u||_2^2 + \delta\kappa\Lambda_3||\nabla v||_2^2 - ((1-\ell)-\delta)||u_t||_2^2\\ &- \frac{g(0)}{4\delta}C_*^2(g'o\nabla u)(t) + \lambda\frac{m}{m+1}\delta^{-\frac{m+1}{m}}||u_t||_{m+1}^{m+1},\end{aligned} \tag{3.33}$$

where

$$N_1 = \frac{8(\rho+2)}{\ell(\rho+1)}E(0) + 2m_0^2$$

and

$$\Lambda_3 = \Lambda_1\left(\frac{2(\rho+2)}{\rho+1}E(0)\right)^{2(\rho+1)}.$$

Proof. Differentiate (3.32) and use the first equation in system (3.1) to get

$$\begin{aligned}
\chi_1'(t) = &\int_\Omega \nabla u(t).\left(\int_0^{+\infty} g(s)(\nabla u(t) - \nabla u(t-s))ds\right)dx \\
&- \int_\Omega \left(\int_0^{+\infty} g(s)\nabla u(t-s)ds\right)\left(\int_0^{+\infty} g(s)(\nabla u(t) - \nabla u(t-s))ds\right)dx \\
&- \int_\Omega u_t \int_0^{+\infty} g'(s)(u(t)-u(t-s))dsdx - (1-\ell)||u_t||_2^2 \\
&+ \lambda\int_\Omega |u_t|^{m-1}u_t\left(\int_0^{+\infty} g(s)(u(t)-u(t-s))ds\right)dx \\
&- \int_\Omega f_1(u,v)\left(\int_0^{+\infty} g(s)(u(t)-u(t-s))ds\right)dx.
\end{aligned} \tag{3.34}$$

Similarly, as in the previous Lemma, we estimate the right-hand side terms of (3.34) as follows. By using Young's and Lemma 2.1, we obtain for any $\delta > 0$,

$$\begin{aligned}
&\int_\Omega \nabla u(t).\left(\int_0^{+\infty} g(s)(\nabla u(t) - \nabla u(t-s))ds\right)dx \\
&\quad \leq \delta||\ \nabla u||_2^2 + \frac{1-\ell}{4\delta}(go\nabla u)(t).
\end{aligned} \tag{3.35}$$

Also, the second term can be estimated as follows

$$\begin{aligned}
&\int_\Omega \left(\int_0^{+\infty} g(s)\nabla u(t-s)ds\right)\left(\int_0^{+\infty} g(s)(\nabla u(t) - \nabla u(t-s)ds\right)dx \\
&\quad \leq \left(2\delta + \frac{1}{4\delta}\right)(1-\ell)(go\nabla u)(t) + 2\delta(1-\ell)^2||\ \nabla u||_2^2.
\end{aligned} \tag{3.36}$$

To estimate the third term, we use Young's, Poincaré's and Cauchy−Schwarz' inequalities to get

$$\int_\Omega u_t \int_0^{+\infty} g'(s)(u(t)-u(t-s))dsdx \leq \delta||u_t||_2^2 - \frac{g(0)}{4\delta}C_*^2(g'o\nabla u)(t). \qquad (3.37)$$

By exploiting Young's, Hölder's and Poincaré's inequalities, we get

$$\begin{aligned}
&\lambda\int_\Omega |u_t|^{m-1}u_t\left(\int_0^{+\infty} g(s)(u(t)-u(t-s))ds\right)dx\\
&\qquad \leq \lambda\frac{m}{m+1}\delta^{-\frac{m+1}{m}}||u_t||_{m+1}^{m+1} + \lambda\frac{\delta^{m+1}}{m+1}\int_\Omega\left[\int_0^{+\infty} g(s)|u(t)-u(t-s)|ds\right]^{m+1}dx\\
&\qquad \leq \lambda\frac{m}{m+1}\delta^{-\frac{m+1}{m}}||u_t||_{m+1}^{m+1}\\
&\qquad\qquad + \lambda(1-\ell)^m C_*^{m+1}\frac{\delta^{m+1}}{m+1}\int_0^{+\infty} g(s)||\nabla u(t)-\nabla u(t-s)||_2^{m+1}ds.
\end{aligned} \qquad (3.38)$$

Now, using (3.24) and (3.6) we obtain

$$\begin{aligned}
||\nabla u(t)-\nabla u(t-s)||_2^2 &\leq 2||\nabla u(t)||_2{}^2 + 2||\nabla u(t-s)||_2^2\\
&\leq 4\sup_{s>0}||\nabla u(s)||_2{}^2 + 2\sup_{\tau<0}||\nabla u(\tau)||_2{}^2\\
&\leq 4\sup_{s>0}||\nabla u(s)||_2{}^2 + 2\sup_{\tau>0}||\nabla u_0(\tau)||_2{}^2\\
&\leq \frac{8(\rho+2)}{\ell(\rho+1)}E(0) + 2m_0^2 := N_1.
\end{aligned} \qquad (3.39)$$

Therefore, we arrive at

$$\lambda\int_\Omega |u_t|^{m-1}u_t\left(\int_0^{+\infty} g(s)(u(t)-u(t-s))ds\right)dx$$
$$\le \lambda\frac{m}{m+1}\delta^{-\frac{m+1}{m}}||u_t||_{m+1}^{m+1}+\lambda(1-\ell)^m C_*^{m+1}N_1^{\frac{m-1}{2}}\frac{\delta^{m+1}}{m+1}(go\nabla u)(t). \tag{3.40}$$

To estimate the last term, we use Young's inequality, (3.16), (3.24) and Lemma 2.1 to obtain

$$\int_\Omega f_1(u,v)\left(\int_0^{+\infty} g(s)(u(t)-u(t-s))ds\right)dx$$
$$\le \delta\left(\int_\Omega |f_1(u,v)|^2dx\right)+\frac{1}{4\delta}\int_\Omega\left(\int_0^{+\infty} g(s)(u(t)-u(t-s))ds\right)^2dx$$
$$\le \Lambda_1\delta(\ell||\nabla u||_2^2+\kappa||\nabla v||_2^2)^{2\rho+3}+\frac{(1-\ell)C_*^2}{4\delta}(go\nabla u)(t)$$
$$\le \Lambda_1\delta\left(\frac{2(\rho+2)}{\rho+1}E(0)\right)^{2\rho+1}(\ell||\nabla u||_2^2+\kappa||\nabla v||_2^2)+\frac{(1-\ell)C_*^2}{4\delta}(go\nabla u)(t). \tag{3.41}$$

By combining (3.35) - (3.37), (3.40) and (3.41), the asseration of Lemma 3.7 is established.

∎

By repeating the same steps as in Lemma 3.7, we have the following

Lemma 3.8 *Assume that* (A1) − (A4) *hold, then the functional*

$$\chi_2(t):=-\int_\Omega v_t\int_0^{+\infty} h(s)(v(t)-v(t-s))dsdx$$

satisfies, along the solution of system (3.1) *and for all* $\delta>0$, *the estimate*

$$\chi_2'(t) \leq \left[\left(2\delta + \frac{1}{4\delta}\right)(1-\kappa) + \frac{1-\kappa}{4\delta} + \mu C_*^{r+1} N_2^{\frac{r-1}{2}} \frac{\delta^{r+1}}{r+1}(1-\kappa)^r + \frac{(1-\kappa)C_*^2}{4\delta}\right](ho\nabla v)(t)$$
$$+ (2\delta(1-\kappa)^2 + \delta + \kappa\delta\Lambda_4)||\nabla v||_2^2 + \delta\ell\Lambda_4||\nabla u||_2^2 - ((1-\kappa)-\delta)||v_t||_2^2$$
$$- \frac{h(0)}{4\delta} C_*^2 (h'o\nabla v)(t) + \mu\frac{r}{r+1}\delta^{-\frac{r+1}{r}}||v_t||_{r+1}^{r+1}, \tag{3.42}$$

where $N_2 = \frac{8(\rho+2)}{\kappa(\rho+1)}E(0) + 2m_0^2$ and $\Lambda_4 = \Lambda_2\left(\frac{2(\rho+2)}{\rho+1}E(0)\right)^{2(\rho+1)}$.

Lemma 3.9 *Assume that (A1) − (A4). Then there exist constants* ε_1, ε_2, α_1, $\alpha_2 > 0$ *such that the functional*

$$L(t) = E(t) + \varepsilon_1\psi(t) + \varepsilon_2(\chi_1(t) + \chi_2(t))$$

satisfies, for all $t \in \mathbb{R}^+$,

$$L \sim E \tag{3.43}$$

and

$$L'(t) \leq -\alpha_1 E(t) + \alpha_2\left[(go\nabla u)(t) + (ho\nabla v)(t)\right]. \tag{3.44}$$

Proof. To prove (3.43), we follow the same techniques used in [59]. Therefore, using Young's inequality, (3.9) and Lemma 2.1, we get

$$\begin{aligned}
L(t) &\le E(t) + \frac{\varepsilon_1}{2}\left(||u_t||_2^2 + ||v_t||_2^2\right) + \frac{\varepsilon_1}{2}\left(||u||_2^2 + ||v||_2^2\right) + \frac{\varepsilon_2}{2}\left(||u_t||_2^2 + ||v_t||_2^2\right) \\
&\quad + \frac{\varepsilon_2}{2}\int_\Omega \left(\int_0^{+\infty} g(s)(u(t) - u(t-s))ds\right)^2 dx \\
&\quad + \frac{\varepsilon_2}{2}\int_\Omega \left(\int_0^{+\infty} h(s)(u(t) - u(t-s))ds\right)^2 dx \\
&\le \frac{1}{2}(1+\varepsilon_1+\varepsilon_2)\left(||u_t||_2^2 + ||v_t||_2^2\right) + \frac{1}{2}(\ell + \varepsilon_1 C_*^2)||\nabla u||_2^2 + \frac{1}{2}(\kappa + \varepsilon_1 C_*^2)||\nabla v||_2^2 \\
&\quad + \frac{1}{2}\left(1 + \varepsilon_2 C_*^2(1-\kappa)\right)(ho\nabla v)(t) + + \frac{1}{2}\left(1 + \varepsilon_2 C_*^2(1-\ell)\right)(go\nabla u)(t) \\
&\quad - \int_\Omega F(u,v)dx.
\end{aligned} \tag{3.45}$$

Consequently, by using (3.8), we conclude that

$$\begin{aligned}
2E(t) - L(t) &\ge \frac{1}{2}\left[1 - (\varepsilon_1 + \varepsilon_2)\right]||u_t||_2^2 + \frac{1}{2}\left[1 - (\varepsilon_1 + \varepsilon_2)\right]||v_t||_2^2 + \frac{1}{2(\rho+2)}I(t) \\
&\quad + \left[\frac{\rho+1}{2(\rho+2)} - \frac{\varepsilon_2}{2}C_*^2(1-\ell)\right](go\nabla u)(t) + \left[\frac{(\rho+1)\ell}{2(\rho+2)} - \frac{\varepsilon_1}{2}C_*^2\right]||\nabla u||_2^2 \\
&\quad + \left[\frac{\rho+1}{2(\rho+2)} - \frac{\varepsilon_2}{2}C_*^2(1-\kappa)\right](ho\nabla v)(t) + \left[\frac{(\rho+1)\kappa}{2(\rho+2)} - \frac{\varepsilon_1}{2}C_*^2\right]||\nabla v||_2^2.
\end{aligned} \tag{3.46}$$

By fixing ε_1 and ε_2 small enough, we obtain $2E(t) - L(t) \ge 0$. Also, Young's inequality implies that

$$\begin{aligned}
L(t) - \frac{1}{2}E(t) &\ge \frac{1}{2}\left(\frac{1}{2} - \varepsilon_1 - \varepsilon_2\right)\left(||u_t||_2^2 + ||v_t||_2^2\right) + \frac{1}{4(\rho+2)}I(t) \\
&\quad + \left[\frac{\rho+1}{4(\rho+2)} - \frac{\varepsilon_2}{2}C_*^2(1-\ell)\right](go\nabla u)(t) + \left[\frac{(\rho+1)\ell}{4(\rho+2)} - \frac{\varepsilon_1}{2}C_*^2\right]||\nabla u||_2^2 \\
&\quad + \left[\frac{\rho+1}{4(\rho+2)} - \frac{\varepsilon_2}{2}C_*^2(1-\kappa)\right](ho\nabla v)(t) + \left[\frac{(\rho+1)\kappa}{4(\rho+2)} - \frac{\varepsilon_1}{2}C_*^2\right]||\nabla v||_2^2.
\end{aligned} \tag{3.47}$$

By fixing ε_1 and ε_2 small enough, we get $L(t) - \frac{1}{2}E(t) \geq 0$. This completes the proof of (3.43).

For the proof of (3.44), we can deduce from (3.10), (3.25), (3.33) and (3.42) we have

$$
\begin{aligned}
L'(t) \leq & - \left[\varepsilon_2\{(1-\ell)-\delta\} - \varepsilon_1\left(1+\frac{C_*}{\ell}\right)\right] ||u_t||_2^2 \\
& - \left[\varepsilon_2\{(1-\kappa)-\delta\} - \varepsilon_1\left(1+\frac{C_*}{\kappa}\right)\right] ||v_t||_2^2 \\
& + \left(\frac{1}{2} - \varepsilon_2\frac{g(0)}{4\delta}C_*^2\right)(g'o\nabla u)(t) + \left(\frac{1}{2} - \varepsilon_2\frac{h(0)}{4\delta}C_*^2\right)(h'o\nabla v)(t) \\
& - \left[\varepsilon_1\frac{\ell}{4} - \varepsilon_2\delta\{2(1-\ell)^2 + 1 + \Lambda_3\ell + \Lambda_4\ell\}\right] ||\ \nabla u||_2^2 \\
& - \left[\varepsilon_1\frac{\kappa}{4} - \varepsilon_2\delta\{2(1-\kappa)^2 + 1 + \Lambda_3\kappa + \Lambda_4\kappa\}\right] ||\ \nabla v||_2^2 \\
& + \left(\varepsilon_1\frac{1-\ell}{2\ell} + \varepsilon_2\left(\overline{\alpha}_1 + \frac{(1-\ell)C_*^2}{4\delta}\right)\right)(go\nabla u)(t) \\
& + \left(\varepsilon_1\frac{1-\kappa}{2\kappa} + \varepsilon_2\left(\overline{\alpha}_2 + \frac{(1-\kappa)C_*^2}{4\delta}\right)\right)(ho\nabla v)(t) \\
& + \overline{\beta}_1(\varepsilon_1,\varepsilon_2)||u_t||_{m+1}^{m+1} + \overline{\beta}_2(\varepsilon_1,\varepsilon_2)||v_t||_{r+1}^{r+1} + 2(\rho+2)\varepsilon_1\int_\Omega F(u,v)dx,
\end{aligned}
\tag{3.48}
$$

where

$$
\overline{\alpha}_1 = \left(2\delta + \frac{1}{4\delta}\right)(1-\ell) + \frac{1-\ell}{4\delta} + \lambda C_*^{m+1}N_1^{\frac{m-1}{2}}\frac{\delta^{m+1}}{m+1}(1-\ell)^m,
$$

$$
\overline{\alpha}_2 = \left(2\delta + \frac{1}{4\delta}\right)(1-\kappa) + \frac{1-\kappa}{4\delta} + \mu C_*^{r+1}N_2^{\frac{r-1}{2}}\frac{\delta^{r+1}}{r+1}(1-\kappa)^r,
$$

$$
\overline{\beta}_1(\varepsilon_1,\varepsilon_2) = \left(\varepsilon_1\lambda\frac{m}{m+1}\beta_1^{-\frac{m+1}{m}} + \varepsilon_2\lambda\frac{m}{m+1}\delta^{-\frac{m+1}{m}} - \lambda\right)
$$

and

$$\overline{\beta}_2(\varepsilon_1,\varepsilon_2) = \left(\varepsilon_1\mu\frac{r}{r+1}\beta_2^{-\frac{r+1}{r}} + \varepsilon_2\mu\frac{r}{r+1}\delta^{-\frac{r+1}{r}} - \mu\right).$$

At this point, we choose δ small enough so that

$$\delta \leq \frac{1}{2}\min\{1-\ell, 1-\kappa\}$$

and

$$\begin{cases} \frac{4}{\ell}\delta\left(2(1-\ell)^2+1+\Lambda_3\ell+\Lambda_4\ell\right) < \frac{1}{4\left(1+\frac{C_*}{\ell}\right)}(1-\ell), \\ \frac{4}{\kappa}\delta\left(2(1-\kappa)^2+1+\Lambda_3\kappa+\Lambda_4\kappa\right) < \frac{1}{4\left(1+\frac{C_*}{\kappa}\right)}(1-\kappa). \end{cases}$$

Once δ is fixed, the choice of any two positive constants ε_1 and ε_2 satisfying

$$\begin{cases} \frac{1}{4\left(1+\frac{C_*}{\ell}\right)}(1-\ell)\varepsilon_2 < \varepsilon_1 < \frac{1}{2\left(1+\frac{C_*}{\ell}\right)}(1-\ell)\varepsilon_2, \\ \frac{1}{4\left(1+\frac{C_*}{\kappa}\right)}(1-\kappa)\varepsilon_2 < \varepsilon_1 < \frac{1}{2\left(1+\frac{C_*}{\kappa}\right)}(1-\kappa)\varepsilon_2 \end{cases} \tag{3.49}$$

will make

$$\kappa_1 = \varepsilon_2\{(1-\ell)-\delta\} - \varepsilon_1\left(1+\frac{C_*}{\ell}\right) > 0,$$

$$\kappa_2 = \varepsilon_2\{(1-\kappa)-\delta\} - \varepsilon_1\left(1+\frac{C_*}{\kappa}\right) > 0,$$

$$\kappa_3 = \varepsilon_1 \frac{\ell}{4} - \varepsilon_2 \delta \{2(1-\ell)^2 + 1 + \Lambda_3 \ell + \Lambda_4 \ell\} > 0$$

and

$$\kappa_4 = \varepsilon_1 \frac{\kappa}{4} - \varepsilon_2 \delta \{2(1-\kappa)^2 + 1 + \Lambda_3 \kappa + \Lambda_4 \kappa\} > 0.$$

We then pick ε_1 and ε_2 so small that (3.49) and (3.43) remain valid and further

$$\kappa_5 = \frac{1}{2} - \varepsilon_2 \frac{g(0)}{4\delta} C_*^2 > 0, \quad \kappa_6 = \frac{1}{2} - \varepsilon_2 \frac{h(0)}{4\delta} C_*^2 > 0, \quad k_0 - (\rho + 2)\varepsilon_1,$$

where $k_0 = \min_{1 \leq i \leq 4} k_i$, and

$$\overline{\beta}_1(\varepsilon_1, \varepsilon_2) < 0, \quad \overline{\beta}_2(\varepsilon_1, \varepsilon_2) < 0$$

Therefore, (3.48) reduces to (3.44) for two positive constants α_1 and α_2. ▮

Lemma 3.10 *Assume that* $(A1) - (A4)$, *then there exist two positive constants,* β_1, β_2, *such that,* $\forall t \in \mathbb{R}^+$

$$\xi(t) L'(t) + \beta_1 E'(t) \leq -\alpha_1 \xi(t) E(t) + \beta_2 \xi(t) \int_t^{+\infty} f(s) ds, \tag{3.50}$$

where $\xi(t) = \min\{\xi_1(t), \xi_2(t)\}$ *and* $f(t) = \max\{g(t), h(t)\}, \ \forall t \in \mathbb{R}^+$.

Proof. By using the fact that ξ_1 and ξ_2 are nonincreasing, we obtain

$$
\begin{aligned}
\xi(t)\int_0^t g(s)||\nabla u(t)-\nabla u(t-s)||_2^2 ds &= \int_0^t \xi(t)g(s)||\nabla u(t)-\nabla u(t-s)||_2^2 ds \\
&\leq \int_0^t \xi_1(t)g(s)||\nabla u(t)-\nabla u(t-s)||_2^2 ds \\
&\leq \int_0^t \xi_1(s)g(s)||\nabla u(t)-\nabla u(t-s)||_2^2 ds \\
&\leq \int_0^t -g'(s)||\nabla u(t)-\nabla u(t-s)||_2^2 ds \\
&\leq \int_0^{+\infty} -g'(s)||\nabla u(t)-\nabla u(t-s)||_2^2 ds \\
&\leq -2E'(t)
\end{aligned}
\tag{3.51}
$$

and similarly,

$$
\xi(t)\int_0^t h(s)||\nabla v(t)-\nabla v(t-s)||_2^2 ds \leq -2E'(t). \tag{3.52}
$$

A combination of (3.39), (3.51) and (3.52) leads to

$$
\begin{aligned}
\xi(t)\left[(go\nabla u)(t)+(ho\nabla v)(t)\right] &\leq -4E'(t)+\xi(t)\left(N_1\int_t^{+\infty} g(s)ds + N_2\int_t^{+\infty} h(s)ds\right) \\
&\leq -4E'(t)+C\xi(t)\int_t^{+\infty} f(s)ds.
\end{aligned}
\tag{3.53}
$$

Thus, estimate (3.50) is obtained from (3.44) and (3.53). ∎

3.3 The Main Result

In this section we state and prove our main result.

Theorem 3.1 *Let $(u_0(.,0), v_0(.,0), u_1, v_1) \in H$ be given. Assume that $(A1)-(A4)$ and (3.19) hold. Then, there exist constants $\gamma_0 \in (0,1)$ and δ_1 such that, for all $t \in \mathbb{R}^+$ and for all $\delta_0 \in (0, \gamma_0]$,*

$$E(t) \leq \delta_1 \left(1 + \int_0^t f(s)^{1-\delta_0} ds\right) e^{-\delta_0 \int_0^t \xi(s)ds} + \delta_1 \int_t^{+\infty} f(s)ds, \tag{3.54}$$

where $\xi(t) = \min\{\xi_1(t), \xi_2(t)\}$ and $f(t) = \max\{g(t), h(t)\}$.

Proof. We will use the approach of [41] to prove this lemma. Let

$$L_1(t) = \xi(t)L(t) + \beta_1 E(t) \sim E(t). \tag{3.55}$$

By using (3.55), the fact that $\xi'(t) \leq 0$ for a.e. $t \in \mathbb{R}^+$ and $\xi(t) \geq 0$ and letting

$$r(t) = \xi(t) \int_t^{+\infty} f(s)ds$$

we obtain, for some $\alpha_0 > 0$,

$$L_1'(t) \leq -\alpha_0 \xi(t) L_1(t) + \beta_2 r(t) \;\; a.e. \; t \in \mathbb{R}^+. \tag{3.56}$$

This last inequality remains true for any $\delta_0 \in (0, \alpha_0]$; that is

$$L_1'(t) \leq -\delta_0 \xi(t) L_1(t) + \beta_2 r(t) \;\; a.e. \;\; t \in \mathbb{R}^+.$$

Direct integration then leads to

$$L_1(T) \le e^{-\delta_0 \int_0^T \xi(s)ds} \left(L_1(0) + \beta_2 \int_0^T e^{\delta_0 \int_0^T \xi(s)ds} r(t)dt \right)$$

and the fact that $L \sim E$ gives

$$E(T) \le \frac{1}{\beta_1} e^{-\delta_0 \int_0^T \xi(s)ds} \left(L_1(0) + \beta_2 \int_0^T e^{\delta_0 \int_0^T \xi(s)ds} r(t)dt \right). \tag{3.57}$$

Noting that

$$e^{\delta_0 \int_0^t \xi(s)ds} r(t) = \frac{1}{\delta_0} \left(\int_t^{+\infty} f(s)ds \right) \left(e^{\delta_0 \int_0^t \xi(s)ds} \right)'.$$

Integration by parts then yields

$$\int_0^T e^{\delta_0 \int_0^T \xi(s)ds} r(t)dt = \frac{1}{\delta_0} \left(e^{\delta_0 \int_0^T \xi(s)ds} \int_T^{+\infty} f(s)ds - \int_0^{+\infty} f(s)ds + \int_0^T e^{\delta_0 \int_0^t \xi(s)ds} f(t)dt \right).$$

Consequently, combining with (3.57), we have

$$E(T) \le \frac{1}{\beta_1} \left(L_1(0) + \frac{\beta_2}{\delta_0} \int_0^T e^{\delta_0 \int_0^t \xi(s)ds} f(t)dt \right) e^{-\delta_0 \int_0^T \xi(s)ds} + \frac{\beta_2}{\beta_1 \delta_0} \int_T^{+\infty} f(s)ds. \tag{3.58}$$

Also, using (A2), we have for a.e. $t \in \mathbb{R}^+$

$$\begin{aligned}
\left(e^{\int_0^t \xi(s)ds} \left(g(t) + h(t) \right) \right)' &= \left(g'(t) + h(t)' \right) e^{\int_0^t \xi(s)ds} + \left(g(t) + h(t) \right) \xi(t) e^{\int_0^t \xi(s)ds} \\
&\le \left(-\xi_1(t) g(t) - \xi_2(t) h(t) \right) e^{\int_0^t \xi(s)ds} + \xi(t) \left(g(t) + h(t) \right) e^{\int_0^t \xi(s)ds} \\
&\le \left[(\xi(t) - \xi_1(t)) g(t) + (\xi(t) - \xi_2(t)) h(t) \right] e^{\int_0^t \xi(s)ds} \le 0.
\end{aligned}$$

Consequently, we have

$$e^{\int_0^t \xi(s)ds} f(t) \le e^{\int_0^t \xi(s)ds} (g(t) + h(t)) \le (g(0) + h(0)) \le 2f(0)$$

and

$$\int_0^T e^{\delta_0 \int_0^t \xi(s)ds} f(t)dt \le (2f(0))^{\delta_0} \int_0^T (f(t))^{1-\delta_0} dt. \tag{3.59}$$

Finally, we obtain (3.54) by combining (3.9), (3.58) and (3.59). ∎

Remark 3.4 *For the exponential decay case (ξ_1, ξ_2 are positive constants), the general decay result* (3.54) *remains true without imposing condition* (3.6)*. Precisely, we have*

Theorem 3.2 *Let $(u_0(.,0), v_0(.,0), u_1, v_1) \in H$ be given. Assume that $(A1)-(A3)$ and* (3.19) *hold. Then, there exist strictly positive constants c_1 and c_2 such that*

$$E(t) \le c_1 e^{-c_2 t}.$$

Remark 3.5 *If there exists $\varepsilon_0 \in (0, 1)$, for which*

$$\int_0^{+\infty} (f(s))^{1-\varepsilon_0} ds < +\infty, \tag{3.60}$$

then we can choose $\delta_0 \in (0, \gamma_1], \gamma_1 = \min\{\varepsilon_0, \gamma_0\}$ such that

$$\int_0^{+\infty} (f(s))^{1-\delta_0} ds < +\infty$$

and consequently, (3.54) *takes the form*

$$E(t) \leq \delta_2 \left(e^{-\delta_0 \int_0^t \xi(s)ds} + \int_t^{+\infty} f(s)ds \right). \tag{3.61}$$

Examples

(1) Let $g(t) = a_1 e^{-b_1(t)},\ h(t) = a_2 e^{-b_2(t)}$, with $b_i > 0$ and $a_i > 0$ small enough so that (3.3) and (3.4), with $\xi_i(t) = b_i$, hold. In this case $\xi(t) = \min\{b_1, b_2\} = b_0$ and $f(t) = A_0 e^{-b_0(t)}$, where $A_0 = \max\{a_1, a_2\}$. Then, (3.60) is satisfied and consequently, (3.61) gives, for two positive constants c_1, c_2,

$$E(t) \leq c_1 e^{-c_2 t}, \quad \forall t \in \mathbb{R}^+.$$

(2) Let $g(t) = \frac{a_1}{(1+t)^{b_1}},\ h(t) = \frac{a_2}{(1+t)^{b_2}}$, with $b_i > 1$ and $a_i > 0$ small enough so that (3.3) and (3.4), with $\xi_i(t) = \frac{b_i}{1+t}$, hold. In this case $\xi(t) = \frac{b_0}{1+t}$ and $f(t) = \frac{A_0}{(1+t)^{b_0}}$, where $A_0 = \max\{a_1, a_2\}$ and $b_0 = \min\{b_1, b_2\}$. Then, (3.60) is satisfied and hence, (3.54) yields,

$$E(t) \leq \frac{c_1}{(1+t)^{c_2}}, \quad \forall t \in \mathbb{R}^+.$$

(3) Let $g(t) = a_1 e^{-b_1(1+t)^{p_1}},\ h(t) = a_1 e^{-b_1(1+t)^{p_2}}$, with a_i, $b_i > 0$ and $0 < p_i < 1$ (i=1,2). It is clear that (3.3) and (3.4) hold for $\xi_i(t) = b_i p_i (1+t)^{(p_i - 1)}$. Consequently, applying (3.61), we obtain for some constant c the following subexponential decay

$$E(t) \leq \delta_1 e^{-c(1+t)^{\min\{p_1, p_2\}}}.$$

CHAPTER 4

A GENERAL DECAY RESULT OF A VISCOELASTIC EQUATION WITH INFINITE HISTORY AND BOUNDARY FEEDBACK

In this chapter, we consider the following problem

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) + \int_0^{+\infty} g(s)\Delta u(x,t-s)ds = 0, & \text{in } \Omega \times \mathbb{R}^+ \\ \frac{\partial u}{\partial \nu}(x,t) - \int_0^{+\infty} g(s)\frac{\partial u}{\partial \nu}(x,t-s)ds + h(u_t) = 0, & \text{on } \Gamma_1 \times \mathbb{R}^+, \\ u(x,t) = 0, & \text{on } \Gamma_0 \times \mathbb{R}^+, \\ u(x,-t) = u_0(x,t),\ u_t(x,0) = u_1(x), & \text{in } \Omega \times \mathbb{R}^+, \end{cases} \tag{4.1}$$

where u denotes the transverse displacement of waves, Ω is a bounded domain of $\mathbb{R}^N (N \geq 1)$ with a smooth boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1$ such that Γ_0 and Γ_1 are closed and disjoint, with $|\Gamma_0| > 0$, ν is the unit outer normal to $\partial\Omega$, and g, h are specific functions. In section 4.1, we introduce some assumptions needed throughout the chapter. In section 4.2 we give technical lemmas. The statement with proof of the main results are given in section 4.3 and 4.4. Finally, we give an example to illustrate our result.

4.1 Assumptions

In this section, we present some material needed in the proof of our result. We use the standard Lebesgue space $L^2(\Omega)$ and the Sobolev space $H_0^1(\Omega)$ with their usual scalar products and norms and denote by V the following space

$$V = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_0\}.$$

We consider the following hypotheses on g and h

$(A1)$ $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a C^1 nonincreasing function satisfying

$$g(0) > 0, \; 1 - \int_0^{+\infty} g(s)ds = \ell > 0, \tag{4.2}$$

and there exists an increasing strictly convex function $G : \mathbb{R}^+ \to \mathbb{R}^+$ of class

$C^1(\mathbb{R}^+) \cap C^2(0,\infty)$ satisfying

$$G(0) = G'(0) = 0 \text{ and } \lim_{t\to\infty} G'(t) = +\infty \tag{4.3}$$

such that

$$\int_0^{+\infty} \frac{g(s)}{G^{-1}(-g'(s))} ds + \sup_{s\in\mathbb{R}^+} \frac{g(s)}{G^{-1}(-g'(s))} < +\infty. \tag{4.4}$$

$(A2)$ $h : \mathbb{R} \to \mathbb{R}$ is a nondecreasing C^0 function such that there exist a strictly increasing function $h_0 \in C^1(\mathbb{R}^+)$, with $h_0(0) = 0$, and positive constants c_1, c_2, ε such that

$$\begin{aligned} h_0(|s|) \le |h(s)| \le h_0^{-1}(|s|) \quad & \text{for all } |s| \le \varepsilon, \\ c_1|s| \le |h(s)| \le c_2|s| \qquad & \text{for all } |s| \ge \varepsilon. \end{aligned} \tag{4.5}$$

In addition, we assume that the function H, defined by $H(s) = \sqrt{s}h_0(\sqrt{s})$, is a strictly convex C^2 function on $(0, r^2]$, for some $r > 0$, when h_0 is nonlinear.

$(A3)$ There exists a positive constant m_0, such that

$$||\nabla u_0(s)||_2 \le m_0, \quad \forall s \ge 0. \tag{4.6}$$

Remark 4.1 *It is worth noting that kernels satisfying* (4.2)-(4.3) *were first introduced in [38]. Whereas, condition* (4.5) *was first introduced in [51].*

Remark 4.2 *We can deduce using* $(A1)$ *that if* $g'(s_0) = 0$ *for some* $s_0 \ge 0$*, then* $g(s) = 0$ *for all* $s \ge s_0$*.*

Remark 4.3 *Hypothesis (A2) implies that $sh(s) > 0$, for all $s \neq 0$.*

Proposition 4.1 *Let $(u_0(.,0), u_1) \in V \times L^2(\Omega)$ be given. Assume that $(A1)-(A3)$ are satisfied, then the problem (4.1) has a unique global weak solution*

$$u \in C(\mathbb{R}^+; V) \cap C^1(\mathbb{R}^+; L^2(\Omega)).$$

This proposition can be established by a standard Galerkin method, similar to [19], (see also [17] [51].

Now, we introduce the "modified" energy associated to problem (4.1)

$$E(t) = \frac{1}{2}||u_t(t)||_2^2 + \frac{1-\ell}{2}||\ \nabla u(t)||_2^2 + \frac{1}{2}(go\nabla u)(t), \tag{4.7}$$

where

$$(go\nabla u)(t) = \int_0^{+\infty} g(s)||\nabla u(t) - \nabla u(t-s)||_2^2 ds.$$

Direct differentiation, using (4.1), leads to

$$E'(t) = \frac{1}{2}(g'o\nabla u)(t) - \int_{\Gamma_1} u_t h(u_t) d\Gamma \leq 0. \tag{4.8}$$

4.2 Technical Lemmas

In this section, we establish several lemmas needed for the proof of our main result.

Lemma 4.1 *Under the assumptions (A1) and (A2), the functional*

$$\psi(t) := \int_\Omega u u_t dx$$

satisfies, along the solution, the estimate

$$\psi'(t) \leq -\frac{\ell}{2}||\ \nabla u||_2^2 + ||u_t||_2^2 + c(go\nabla u)(t) + c\int_{\Gamma_1} h^2(u_t)d\Gamma, \quad \forall t \in \mathbb{R}^+, \tag{4.9}$$

where c is a positive generic constant.

Proof. Direct computations, using (4.1), yield

$$\begin{aligned}\psi'(t) &= \int_\Omega u_t^2 dx + \int_\Omega u\Delta u dx - \int_\Omega u\int_0^{+\infty} g(s)\Delta u(t-s)dsdx \\ &= \int_\Omega u_t^2 dx - \ell\int_\Omega |\nabla u|^2 dx - \int_{\Gamma_1} uh(u_t)d\Gamma \\ &\quad + \int_\Omega \nabla u.\int_0^{+\infty} g(s)(\nabla u(t-s) - \nabla u(t))dsdx.\end{aligned} \tag{4.10}$$

Using Young's inequality and Lemma 2.1, we obtain

$$\begin{aligned}&\int_\Omega \nabla u.\int_0^{+\infty} g(s)(\nabla u(t-s) - \nabla u(t))dsdx \\ &\quad \leq \delta\int_\Omega |\nabla u|^2 dx + \frac{1}{4\delta}\int_\Omega\left(\int_0^{+\infty} g(s)|\nabla u(t-s) - \nabla u(t)|ds\right)^2 dx \\ &\quad \leq \delta\int_\Omega |\nabla u|^2 dx + \frac{c}{\delta}(go\nabla u)(t).\end{aligned} \tag{4.11}$$

Also, use of Young's and Poincaré's inequalities gives

$$
\begin{aligned}
-\int_{\Omega} uh(u_t)d\Gamma &\leq \delta \int_{\Gamma_1} u^2 d\Gamma + \frac{1}{4\delta}\int_{\Gamma_1} h^2(u_t)d\Gamma \\
&\leq c\delta \int_{\Omega} |\nabla u|^2 d\Gamma + \frac{1}{4\delta}\int_{\Gamma_1} h^2(u_t)d\Gamma.
\end{aligned}
\tag{4.12}
$$

Combining (4.10)-(4.12) and choosing δ small enough give (4.9).

∎

Lemma 4.2 *Under the assumptions* $(A1)-(A3)$, *the functional*

$$
\chi(t) := -\int_{\Omega} u_t \int_0^{+\infty} g(s)(u(t)-u(t-s))dsdx
$$

satisfies, along the solution, the estimate

$$
\begin{aligned}
\chi'(t) \leq -\delta || \nabla u||_2^2 - [(1-\ell)-\delta]\,||u_t||_2^2 + \frac{c}{\delta}(go\nabla u)(t) \\
-\frac{c}{\delta}(g'o\nabla u)(t) + c\int_{\Gamma_1} h^2(u_t)d\Gamma, \quad \forall t \in \mathbb{R}^+ \textit{ and } \forall \delta > 0.
\end{aligned}
\tag{4.13}
$$

Proof. By exploiting Eqs. (4.1) and performing integration by parts, we arrive at

$$\begin{aligned}
\chi'(t) = &\int_\Omega \nabla u. \int_0^{+\infty} g(s)(\nabla u(t-s) - \nabla u(t))dsdx \\
&- \int_\Omega \left(\int_0^{+\infty} g(s)\nabla u(t-s)ds\right). \left(\int_0^{+\infty} g(s)(\nabla u(t-s) - \nabla u(t))ds\right) dx \\
&+ \int_{\Gamma_1} \left(\int_0^{+\infty} g(s)(u(t-s) - u(t))ds\right) h(u_t)d\Gamma \\
&- \int_\Omega u_t \int_0^{+\infty} g'(s)(u(t-s) - u(t))dsdx - (1-\ell)\int_\Omega {u_t}^2 dx \\
= &\ \ell \int_\Omega \nabla u. \int_0^{+\infty} g(s)(\nabla u(t-s) - \nabla u(t))dsdx \\
&+ \int_\Omega \left| \int_0^{+\infty} g(s)(\nabla u(t-s) - \nabla u(t))ds \right|^2 dx \\
&+ \int_{\Gamma_1} \left(\int_0^{+\infty} g(s)(u(t-s) - u(t))ds\right) h(u_t)d\Gamma \\
&- \int_\Omega u_t \int_0^{+\infty} g'(s)(u(t-s) - u(t))dsdx - (1-\ell)\int_\Omega u_t^2 dx.
\end{aligned}$$

Using Young's inequality and Lemma 2.1, we obtain

$$\ell \int_\Omega \nabla u. \int_0^{+\infty} g(s)(\nabla u(t-s) - \nabla u(t))dsdx \leq \delta \int_\Omega |\nabla u|^2 dx + \frac{c}{\delta}(go\nabla u)(t)$$

$$\int_{\Gamma_1} \left(\int_0^{+\infty} g(s)(u(t-s) - u(t))ds\right) h(u_t)d\Gamma \leq c(go\nabla u)(t) + c\int_{\Gamma_1} h^2(u_t)d\Gamma$$

and

$$-\int_\Omega u_t \int_0^{+\infty} g'(s)(u(t-s) - u(t))dsdx \leq \delta \int_\Omega u_t^2 dx - \frac{c}{\delta}(g'o\nabla u)(t).$$

Combining all the above estimates, (4.13) is established. ∎

Lemma 4.3 *Assume that* $(A1)-(A3)$ *hold. Then there exist constants* M_1, M_2, m, $c>0$ *such that the functional*

$$L(t)=M_1E(t)+M_2\chi(t)+\psi(t)$$

satisfies, for all $t\in\mathbb{R}^+$,

$$L'(t)\le -mE(t)+c(go\nabla u)(t)+c\int_{\Gamma_1}h^2(u_t)d\Gamma. \tag{4.14}$$

Proof. By using (4.8), (4.9) and (4.13), we easily see that

$$\begin{aligned}L'(t)\le& -\frac{\ell}{4}||\ \nabla u||_2^2-\left(M_2(1-\ell)-\frac{\ell}{4}-1\right)||u_t||_2^2+\left(\frac{4c}{\ell}M_2^2+c\right)(go\nabla u)(t)\\&+\left(\frac{1}{2}M_1-\frac{4c}{\ell}M_2^2\right)(g'o\nabla u)(t)+(cM_2+c)\int_{\Gamma_1}h^2(u_t)d\Gamma.\end{aligned}$$

At this point, we choose M_2 large enough so that

$$\alpha:=M_2(1-\ell)-\frac{\ell}{4}-1>0$$

and then M_1 large enough so that

$$\frac{1}{2}M_1-\frac{4c}{\ell}M_2^2>0.$$

So, we arrive at

$$L'(t) \leq -\frac{\ell}{4}|| \nabla u||_2^2 - \alpha||u_t||_2^2 + c(go\nabla u)(t) + c\int_{\Gamma_1} h^2(u_t)d\Gamma \tag{4.15}$$

Therefore, (4.15) reduces to (4.14) for two positive constants m and c. On the other hand (see [12]), we can choose M_1 even larger (if needed) so that

$$L \sim E. \tag{4.16}$$

∎

Lemma 4.4 *[38] Assume that* $(A1)$ *and* $(A2)$ *are satisfied. Then there exists* $\beta_1 > 0$ *such that for all* $\delta_0 > 0$ *and* $t \in \mathbb{R}^+$,

$$G'(\delta_0 E(t))(go\nabla u)(t) \leq -\beta_1 E'(t) + \beta_1 \delta_0 E(t) G'(\delta_0 E(t)). \tag{4.17}$$

Proof. We will use the approach of [38] to prove this lemma. First, using Remark 4.2, we can assume without loss of generality that $g' < 0$. Let G^* be the convex conjugate of G in the sense of Young (see [8], p.61-64); then

$$\begin{aligned} G^*(t) &= t(G')^{-1}(t) - G\left((G')^{-1}(t)\right) \\ &\leq t(G')^{-1}(t), \qquad\qquad \forall t \in \mathbb{R}^+. \end{aligned} \tag{4.18}$$

Using the general Young inequality: $t_1 t_2 \leq G(t_1) + G^*(t_2)$, with

$$t_1 = G^{-1}\left(-\tau_2 g'(s)||\nabla u(t) - \nabla u(t-s)||_2^2\right), \; t_2 = \frac{\tau_1 G'(\delta_0 E(t)) g(s)||\nabla u(t) - \nabla u(t-s)||_2^2}{G^{-1}\left(-\tau_2 g'(s)||\nabla u(t) - \nabla u(t-s)||_2^2\right)},$$

where δ_0, τ_1, $\tau_2 > 0$, we get for all $t \in \mathbb{R}^+$,

$$\begin{aligned}
(go\nabla u)(t) &= \int_0^{+\infty} g(s)||\nabla u(t) - \nabla u(t-s)||_2^2 ds \\
&= \frac{1}{\tau_1 G'(\delta_0 E(t))} \int_0^{+\infty} \Bigg\{ G^{-1}\left(-\tau_2 g'(s)||\nabla u(t) - \nabla u(t-s)||_2^2\right) \\
&\qquad \times \frac{\tau_1 G'(\delta_0 E(t)) g(s)||\nabla u(t) - \nabla u(t-s)||_2^2}{G^{-1}\left(-\tau_2 g'(s)||\nabla u(t) - \nabla u(t-s)||_2^2\right)} \Bigg\} ds \\
&\leq -\frac{\tau_2}{\tau_1 G'(\delta_0 E(t))} (g' o \nabla u)(t) \\
&\quad + \frac{1}{\tau_1 G'(\delta_0 E(t))} \int_0^{+\infty} G^* \left(\frac{\tau_1 G'(\delta_0 E(t)) g(s)||\nabla u(t) - \nabla u(t-s)||_2^2}{G^{-1}\left(-\tau_2 g'(s)||\nabla u(t) - \nabla u(t-s)||_2^2\right)} \right) ds.
\end{aligned}$$

Combing (4.8) and (4.18), we have for all $t \in \mathbb{R}^+$,

$$\begin{aligned}
(go\nabla u)(t) &\leq -\frac{2\tau_2}{\tau_1 G'(\delta_0 E(t))} E'(t) \\
&\quad + \int_0^{+\infty} \Bigg\{ \frac{g(s)||\nabla u(t) - \nabla u(t-s)||_2^2}{G^{-1}\left(-\tau_2 g'(s)||\nabla u(t) - \nabla u(t-s)||_2^2\right)} \\
&\qquad \times (G')^{-1} \left(\frac{\tau_1 G'(\delta_0 E(t)) g(s)||\nabla u(t) - \nabla u(t-s)||_2^2}{G^{-1}\left(-\tau_2 g'(s)||\nabla u(t) - \nabla u(t-s)||_2^2\right)} \right) \Bigg\} ds.
\end{aligned} \tag{4.19}$$

Exploiting (4.7), (4.8), (4.6), we obtain $\forall t,\ s \in \mathbb{R}^+$,

$$
\begin{aligned}
||\nabla u(t) - \nabla u(t-s)||_2^2 \le 2||\nabla u(t)||_2{}^2 + 2||\nabla u(t-s)||_2^2 \\
&\le 4 \sup_{s>0} ||\nabla u(s)||_2{}^2 + 2 \sup_{\tau<0} ||\nabla u(\tau)||_2{}^2 \\
&\le 4 \sup_{s>0} ||\nabla u(s)||_2^2 + 2 \sup_{\tau>0} ||\nabla u_0(\tau)||_2{}^2 \\
&\le \frac{8}{1-\ell} E(0) + 2m_0^2 := N_1.
\end{aligned}
\tag{4.20}
$$

On the other hand, using the fact that G^{-1} is concave and $G^{-1}(0) = 0$, the function $K(s) = \frac{s}{G^{-1}(s)}$ satisfies, for any $0 \le s_1 < s_2$,

$$
K(s_1) = \frac{s_1}{G^{-1}\left(\frac{s_1}{s_2} s_2 + \left(1 - \frac{s_1}{s_2}\right) 0\right)} \le \frac{s_1}{\frac{s_1}{s_2} G^{-1}(s_2) + \left(1 - \frac{s_1}{s_2}\right) G^{-1}(0)} = \frac{s_2}{G^{-1}(s_2)} = K(s_2).
$$

Therefore, using (4.20) and the fact that K is nondecreasing, we get

$$
\begin{aligned}
&(G')^{-1} \left(\frac{\tau_1 G'(\delta_0 E(t)) g(s) ||\nabla u(t) - \nabla u(t-s)||_2^2}{G^{-1}\left(-\tau_2 g'(s) ||\nabla u(t) - \nabla u(t-s)||_2^2\right)} \right) \\
&\quad = (G')^{-1} \left(\frac{\tau_1 G'(\delta_0 E(t)) g(s)}{-\tau_2 g'(s)} K\left(-\tau_2 g'(s) ||\nabla u(t) - \nabla u(t-s)||_2^2\right) \right) \\
&\quad \le (G')^{-1} \left(\frac{\tau_1 G'(\delta_0 E(t)) g(s)}{-\tau_2 g'(s)} K\left(-\tau_2 N_1 g'(s)\right) \right) \\
&\quad \le (G')^{-1} \left(\frac{\tau_1 N_1 G'(\delta_0 E(t)) g(s)}{G^{-1}(-\tau_2 N_1 g'(s))} \right).
\end{aligned}
$$

Thus, we obtain from (4.19) and (4.20) that, for all $t \in \mathbb{R}^+$,

$$
(go\nabla u)(t) \le -\frac{2\tau_2}{\tau_1 G'(\delta_0 E(t))} E'(t) + N_1 \int_0^{+\infty} \frac{g(s)}{G^{-1}(-\tau_2 N_1 g'(s))} (G')^{-1} \left(\frac{\tau_1 N_1 G'(\delta_0 E(t)) g(s)}{G^{-1}(-\tau_2 N_1 g'(s))} \right) ds.
$$

Condition (4.4) implies that

$$\sup_{s\in\mathbb{R}^+} \frac{g(s)}{G^{-1}(-g'(s))} = N_2 < +\infty$$

and

$$\int_0^{+\infty} \frac{g(s)}{G^{-1}(-g'(s))} ds = N_3 < +\infty.$$

Then choosing $\tau_1 = \frac{1}{N_1 N_2}$, $\tau_2 = \frac{1}{N_1}$ and using the fact that $(G')^{-1}$ is nondecreasing, we obtain for all $t \in \mathbb{R}^+$,

$$\begin{aligned}(go\nabla u)(t) &\leq -\frac{2N_2}{G'(\delta_0 E(t))} E'(t) + N_1 (G')^{-1} \left(G'(\delta_0 E(t))\right) \int_0^{+\infty} \frac{g(s)}{G^{-1}(-g'(s))} ds \\ &\leq -\frac{2N_2}{G'(\delta_0 E(t))} E'(t) + N_1 N_3 \delta_0 E(t), \quad \forall t \in \mathbb{R}^+,\end{aligned}$$

which gives (4.17) with $\beta_1 = max\{2N_2, N_1 N_3\}$. ∎

4.3 The Main Result

In this section we state and prove the main result in this chapter.

Theorem 4.1 *Let $(u_0(.,0), u_1) \in V \times L^2(\Omega)$ be given. Assume that $(A1) - (A3)$ hold. Then, there exist positive constants k_2 , k_3, k_4, δ_1, ε_0 such that*

$$E(t) \leq k_4 W_1^{-1} \left(k_2 t + k_3\right), \quad \forall t \in \mathbb{R}^+, \tag{4.21}$$

where

$$W_1(\tau) = \int_{\tau}^{1} \frac{1}{W_2(s)} ds \quad and \quad W_2(t) = tG'\left(\delta_1 t\right) H'\left(\varepsilon_0 t\right).$$

Proof. **Case 1**. h_0 is linear. Then, using $(A2)$ we have

$$c_1'|s| \leq |h(s)| \leq c_2'|s|, \quad \forall s \in \mathbb{R}$$

and hence

$$h^2(s) \leq c_2' s h(s), \quad \forall s \in \mathbb{R}. \tag{4.22}$$

Therefore using (4.22) and (4.8), estimate (4.14) becomes

$$L'(t) \leq -mE(t) + c(go\nabla u)(t) - cE'(t), \qquad \forall t \in \mathbb{R}^+.$$

Consequently, $F_0(t) := L(t) + cE(t)$ satisfies

$$F_0'(t) \leq -mE(t) + c(go\nabla u)(t) \tag{4.23}$$

and, recalling (4.16), we have $F_0 \sim E$. Now, we multiply (4.23) by $G'(\delta_0 E(t))$ and use (4.17) to obtain:

$$\begin{aligned} G'(\delta_0 E(t))F_0'(t) &\leq -mG'(\delta_0 E(t))E(t) - c\beta_1 E'(t) + c\beta_1 \delta_0 E(t) G'(\delta_0 E(t)) \\ &= -(m - c\beta_1 \delta_0) E(t) G'(\delta_0 E(t)) - c\beta_1 E'(t). \end{aligned}$$

Choosing δ_0 small enough so that $\beta_2 := m - c\beta_1\delta_0 > 0$ and putting

$$F_1(t) := G'(\delta_0 E(t))F_0(t) + c\beta_1 E(t),$$

we deduce (note that $G'(\delta_0 E(t))$ is nonincerasing) that

$$F_1 \sim E \quad \text{ and } F_1'(t) \leq -k_1 F_1(t) G'(\delta_1 F_1(t)).$$

The last inequality implies that $(W_1(F_1))' \geq k_1$, where

$$W_1(\tau) = \int_\tau^1 \frac{1}{csG'(\delta_1 s)} ds,$$

for $0 < \tau \leq 1$. Then, by integrating over $[0,t]$, we get,

$$F_1(t) \leq W_1^{-1}(k_1 t + k_2), \quad \forall t \in \mathbb{R}^+.$$

The equivalence $F_1 \sim E$ and the definition (4.7) give the desired result. Note that in this case, $H(s) = \sqrt{s}h_0(\sqrt{s}) = cs$ and $W_2(s) = csG'(\delta_1 s)$

Case 2. h_0 is nonlinear on $[0, \varepsilon]$.

First, we assume that $\max\{r, h_0(r)\} < \varepsilon$; otherwise we take r smaller. Let $\varepsilon_1 = \min\{r, h_0(r)\}$. Using $(A2)$, we have, for $\varepsilon_1 \leq |s| \leq \varepsilon$,

$$|h(s)| \leq \frac{h_0^{-1}(|s|)}{|s|}|s| \leq \frac{h_0^{-1}(|\varepsilon|)}{|\varepsilon_1|}|s|$$

and

$$|h(s)| \geq \frac{h_0(|s|)}{|s|}|s| \geq \frac{h_0(|\varepsilon_1|)}{|\varepsilon|}|s|.$$

So, we deduce that

$$\begin{cases} h_0(|s|) \leq |h(s)| \leq h_0^{-1}(|s|) \ \text{ for all } \ |s| < \varepsilon_1 \\ c_1'|s| \leq |h(s)| \leq c_2'|s| \ \text{ for all } \ |s| \geq \varepsilon_1. \end{cases} \tag{4.24}$$

Then (4.24) yields, for all $|s| \leq \varepsilon_1$,

$$H(h^2(s)) = |h(s)|h_0(|h(s)|) \leq sh(s),$$

which gives

$$h^2(s) \leq H^{-1}(sh(s)) \ \text{ for all } \ |s| \leq \varepsilon_1. \tag{4.25}$$

To estimate the last integral in (4.14), we define the following partition which is based on an idea of Komornik (see [48], p.142)

$$\Gamma_{11} = \{x \in \Gamma_1 : |u_t| > \varepsilon_1\}, \quad \Gamma_{12} = \{x \in \Gamma_1 : |u_t| \leq \varepsilon_1\}.$$

Using (4.24) and recalling the definition of ε_1, we get on Γ_{12}

$$u_t h(u_t) \leq \varepsilon_1 h_0^{-1}(\varepsilon_1) \leq h_0(r)r = H(r^2). \tag{4.26}$$

Then, with $J(t)$ defined by

$$J(t) := \frac{1}{|\Gamma_{12}|} \int_{\Gamma_{12}} u_t h(u_t) d\Gamma,$$

Jensen's inequality, with the fact that H^{-1} is concave, gives

$$H^{-1}\left(J(t)\right) \geq c \int_{\Gamma_{12}} H^{-1}(u_t h(u_t)) d\Gamma. \tag{4.27}$$

Thus, combining (4.25) and (4.27), we arrive at

$$\begin{aligned} \int_{\Gamma_1} h^2(u_t) d\Gamma &= \int_{\Gamma_{12}} h^2(u_t) d\Gamma + \int_{\Gamma_{11}} h^2(u_t) d\Gamma \\ &\leq \int_{\Gamma_{12}} H^{-1}\left(u_t h(u_t)\right) d\Gamma + c \int_{\Gamma_{11}} u_t h(u_t) d\Gamma \\ &\leq c H^{-1}(J(t)) - cE'(t). \end{aligned} \tag{4.28}$$

Therefore, (4.14) becomes

$$L_0'(t) \leq -mE(t) + c(go\nabla u)(t) + cH^{-1}(J(t)), \quad \forall t \in \mathbb{R}^+, \tag{4.29}$$

where $L_0 = L + cE$, which is clearly equivalent to E.

Now, for $\varepsilon_0 < r^2$ and $c_0 > 0$, let

$$L_1(t) := H'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) L_0(t) + c_0 E(t),$$

provided that $E(0) > 0$; otherwise $E(t) = 0$, $\forall t \in \mathbb{R}^+$, hence theorem is verified.

By exploiting the properties of E and H, we conclude that L_1 satisifies

$$\begin{aligned} L_1'(t) \leq -mE(t)H'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) + cH^{-1}\left(J(t)\right)H'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) \\ + c_0 E'(t) + cH'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)(go\nabla u)(t). \end{aligned} \tag{4.30}$$

Let H^* be the convex conjugate of H in the sense of Young(see [8], p.61-64), then

$$\begin{aligned} H^*(s) &= s(H')^{-1}(s) - H[(H')^{-1}(s)] \quad \text{if } s \in (0, H'(r^2)] \\ &\leq s(H')^{-1}(s). \end{aligned} \tag{4.31}$$

Using the general Young inequality

$$AB \leq H^*(A) + H(B), \quad \text{if } A \in (0, H'(r^2)],\ B \in (0, r^2],$$

with

$$A = H'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) \quad \text{and} \quad B = H^{-1}\left(J(t)\right),$$

we get

$$\begin{aligned} L_1'(t) &\leq -mE(t)H'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) + c\varepsilon_0 \frac{E(t)}{E(0)} H'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) - cE'(t) \\ &\quad + c_0 E'(t) + cH'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)(go\nabla u)(t) \\ &= -(mE(0) - c\varepsilon_0) \frac{E(t)}{E(0)} H'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) - (c - c_0)E'(t) \\ &\quad + cH'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)(go\nabla u)(t). \end{aligned}$$

Consequently, by picking ε_0 small enough so that $mE(0) - c\varepsilon_0 > 0$ and c_0 large such that $c - c_0 < 0$, we obtain, for all $t \in \mathbb{R}^+$,

$$L_1'(t) \leq -k\frac{E(t)}{E(0)}H'\left(\varepsilon_0\frac{E(t)}{E(0)}\right) + cH'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)(go\nabla u)(t). \tag{4.32}$$

Multiplying (4.32) by $G'(\delta_0 E(t))$ and taking in account (4.17), we get

$$\begin{aligned} G'(\delta_0 E(t))\, L_1'(t) &\leq -k\frac{E(t)}{E(0)}G'(\delta_0 E(t))\, H'\left(\varepsilon_0\frac{E(t)}{E(0)}\right) - \beta_2 E'(t)H'\left(\varepsilon_0\frac{E(t)}{E(0)}\right) \\ &\quad + \beta_2\delta_0 E(t)H'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)G'(\delta_0 E(t)) \\ &\leq -k\frac{E(t)}{E(0)}G'(\delta_0 E(t))\, H'\left(\varepsilon_0\frac{E(t)}{E(0)}\right) - CE'(t) \\ &\quad + \beta_2\delta_0 E(t)H'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)G'(\delta_0 E(t)). \end{aligned} \tag{4.33}$$

Now, using (4.16), (4.33) and the fact that $E' \leq 0$ and $G'' > 0$, we find that the functional L_2 defined by

$$L_2(t) := G'(\delta_0 E(t))\, L_1(t) + CE(t)$$

satisfies, for some $\alpha_1,\ \alpha_2 > 0$

$$\alpha_1 L_2(t) \leq E(t) \leq \alpha_2 L_2(t) \tag{4.34}$$

and

$$L_2'(t) \leq -k\frac{E(t)}{E(0)}G'\left(\delta_0 E(t)\right)H'\left(\varepsilon_0\frac{E(t)}{E(0)}\right) + \beta_2\delta_0 E(t)H'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)G'\left(\delta_0 E(t)\right)$$
$$= -\left(k-\beta_2\delta_1\right)\frac{E(t)}{E(0)}G'\left(\delta_1\frac{E(t)}{E(0)}\right)H'\left(\varepsilon_0\frac{E(t)}{E(0)}\right), \tag{4.35}$$

where $\delta_1 = E(0)\delta_0$. After choosing δ_0 small enough so that $k_1 := k - \beta_2\delta_1 > 0$, (4.35) becomes

$$L_2'(t) \leq -k_1\frac{E(t)}{E(0)}G'\left(\delta_1 E(t)\right)H'\left(\varepsilon_0\frac{E(t)}{E(0)}\right) = -k_1 W_2\left(\frac{E(t)}{E(0)}\right), \tag{4.36}$$

where $W_2(\tau) = \tau H'(\varepsilon_0\tau)G'(\delta_1\tau)$. Thus, (4.34) and (4.36) yield that $R(t) = \frac{\alpha_1 L_2(t)}{E(0)}$ satisfies

$$R(t) \sim E(t) \tag{4.37}$$

and, for some $k_2 > 0$,

$$R'(t) \leq -k_2 W_2(R(t)). \tag{4.38}$$

Inequality (4.38) implies that $(W_1(R))' \geq k_2$, where

$$W_1(\tau) = \int_\tau^1 \frac{1}{W_2(s)}ds \ \text{ for } \tau \in (0,1].$$

Thus, by integrating over $[0,t]$, we get

$$R(t) \leq W_1^{-1}\left(k_2 t + k_3\right), \quad \forall t \in \mathbb{R}^+. \tag{4.39}$$

Finally, we obtain (4.21) by combing (4.7), (4.37) and (4.39). ∎

4.4 Kernels with Exponential Decay

In this section, we discuss the case of exponentially decaying kernels and the following decay result can be obtained without condition (4.6).

Theorem 4.2 *Assume that* (*A*2) *holds and the kernel* g *satisfies* (4.2) *and, for some positive constant* ξ,

$$g'(t) \leq -\xi g(t). \tag{4.40}$$

Then, there exist positive constants c_1, c_2, c_3 *and* ε_0 *such that*

$$E(t) \leq c_3 H_1^{-1}\left(c_1 t + c_2\right),\ \forall t \in \mathbb{R}^+, \tag{4.41}$$

where

$$H_1(\tau) = \int_\tau^1 \frac{1}{H_2(s)} ds$$

and

$$H_2(t) = tH'(\varepsilon_0(t)).$$

Proof. We multiply (4.14) by ξ and use (4.40) and (4.8), to get

$$\begin{aligned} \xi L'(t) &\leq -\xi m E(t) - c(g' o \nabla u)(t) + c\xi \int_{\Gamma_1} h^2(u_t) d\Gamma \\ &\leq -m\xi E(t) + c\xi \int_{\Gamma_1} h^2(u_t) d\Gamma - cE'(t). \end{aligned}$$

Using (4.16), the functional $F := \xi L + cE$ satisfies

$$F \sim E$$

and

$$F'(t) \leq -m\xi E(t) + c\xi \int_{\Gamma_1} h^2(u_t) d\Gamma. \tag{4.42}$$

Case 1. h_0 is linear. Then recalling (4.22), estimate (4.42) becomes

$$F'(t) \leq -m\xi E(t) - cE'(t),$$

which gives

$$(F + cE)'(t) \leq -m\xi E(t).$$

Hence, using the fact that $F + cE \sim E$, we easily obtain

$$E(t) \leq c_1 e^{-ct}$$

and since in this case $H(s) = \sqrt{s} h_0(\sqrt{s}) = cs$, then

$$E(t) \leq c_1 H_1^{-1}(t).$$

Case 2. h_0 is nonlinear on $[0, \varepsilon]$. Using (4.40) and repeating the same analysis as in the proof of Theorem 4.1, we obtain the desired result. ∎

Now, we give an example to illustrate the energy decay rate obtained by Theorem 4.1.

Let $g(t) = \frac{a}{(1+t)^r}$ with $r > 1$, and $a > 0$ small enough so that (4.2) holds. Hypothesis $(A1)$ is satisfied with $G(t) = t^{\frac{1}{p}}$ for any $p \in (0, \frac{r-1}{2})$. Now, if h satisfies

$$c_1 \min\{|s|, |s|^r\} \leq |h(s)| \leq c_2 \max\{|s|, |s|^{\frac{1}{r}}\},$$

for some c_1, c_2 and $r > 1$. Hypothesis $(A2)$ is satisfied with $h_0(s) = cs^r$ and $H(s) = cs^{r+1}$. Then (4.21) gives

$$E(t) \leq \frac{1}{(1+t)^{\frac{1}{p}+\frac{r+1}{2}-1}}.$$

CHAPTER 5

A GENERAL DECAY RESULT OF A VISCOELASTIC EQUATION WITH INFINITE HISTORY AND NONLINEAR DAMPING

In this chapter, we consider the well-posedness and the asymptotic behavior of a viscoelastic equation with infinite memory and nonlinear damping. That is,

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) + \int_0^{+\infty} g(s)\Delta u(x,t-s)ds + a(x)|u_t(x,t)|^{m-2}u_t(x,t) = 0, & \text{in } \Omega \times \mathbb{R}^+, \\ u(x,t) = 0, & \text{on } \partial\Omega \times \mathbb{R}^+, \\ u(x,-t) = u_0(x,t),\ u_t(x,0) = u_1(x), & \text{in } \Omega \times \mathbb{R}^+, \end{cases} \tag{5.1}$$

where u denotes the transverse displacement of waves and Ω is a bounded domain of $\mathbb{R}^N (N \geq 1)$ with a smooth boundary $\partial\Omega$, g is positive and decreasing function, a is a bounded measurable nonnegative function defined on Ω and $m > 1$. In section 5.1, we introduce some assumptions needed in this chapter. In section 5.2 we use the Galerkin method to prove the well-posedness of the problem. Some technical lemmas are given in section 5.3. The statement with proof of our main results will be given in section 5.4 and section 5.5.

5.1 Assumptions

In this section, we present some materials needed in the proof of our results and establish the well-posedness of the problem. We use the standard Lebesgue space $L^2(\Omega)$ and the Sobolev space $H_0^1(\Omega)$ with their usual scalar products and norms and assume the following hypotheses

$(A1)$ $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a C^1 nonincreasing function satisfying

$$g(0) > 0,\ 1 - \int_0^{+\infty} g(s)ds = \ell > 0 \tag{5.2}$$

and there exists an increasing strictly convex function $G : \mathbb{R}^+ \to \mathbb{R}^+$ of class $C^1(\mathbb{R}^+) \cap C^2(0, +\infty)$ satisfying

$$G(0) = G'(0) = 0 \text{ and } \lim_{t \to +\infty} G'(t) = +\infty \tag{5.3}$$

such that

$$\int_0^{+\infty} \frac{g(s)}{G^{-1}(-g'(s))} ds + \sup_{s \in \mathbb{R}^+} \frac{g(s)}{G^{-1}(-g'(s))} < +\infty. \tag{5.4}$$

$(A2)$ $a : \Omega \to \mathbb{R}^+$ is a bounded and measurable function.

$(A3)$ For the nonlinearity in the damping, we assume that

$$1 < m \leq \frac{2n}{n-2}, \text{ if } n > 2 \text{ and } m > 1, \text{ if } n = 1, 2. \tag{5.5}$$

$(A4)$ There exists $m_0 \in \mathbb{R}^+$, such that

$$||\nabla u_0(s)||_2 \leq m_0, \quad \forall s \geq 0. \tag{5.6}$$

We introduce the "modified" energy associated to problem (5.1)

$$E(t) = \frac{1}{2}||u_t||_2^2 + \frac{1-\ell}{2}||\ \nabla u||_2^2 + \frac{1}{2}(go\nabla u)(t), \tag{5.7}$$

where

$$(go\nabla u)(t) = \int_0^{+\infty} g(s)||\nabla u(t) - \nabla u(t-s)||_2^2 ds.$$

Direct differentiation, using (5.1), leads to

$$E'(t) = \frac{1}{2}(g'o\nabla u)(t) - \int_\Omega a(x)|u_t|^m dx \leq 0. \tag{5.8}$$

5.2 The Well-posedness of the Problem

In this section, we give the existence and uniqueness result for problem (5.1), for $m \geq 2$, using the Galerkin method.

Proposition 5.1 *Let $(u_0(.,0), u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ be given. Under the assumptions $(A1)-(A4)$ and $m \geq 2$, the problem (5.1) has a unique weak global solution.*

Proof. *We use the standard Faedo-Galerkin method to prove our result. Let $\{w_j\}_{j=1}^{\infty}$ be the eigenfunctions of the Laplacian operator subject to Dirichlet boundary conditions. Then $\{w_j\}_{j=1}^{\infty}$ is orthogonal basis of $H_0^1(\Omega)$ as well as for $L^2(\Omega)$. Let $V_k = span\{w_1, w_2, ..., w_k\}$ and the projection of the history and initial data on the finite-dimensional subspace V_k is given by*

$$u_0^k(t) = \sum_{j=1}^{k} a_j(t) w_j, \qquad u_1^k = \sum_{j=1}^{k} b_j w_j,$$

where,

$$\begin{cases} u_0^k(t) \to u_0(t) \text{ in } H_0^1(\Omega) \text{ for each } t \in \mathbb{R}^+ \\ \text{and} \\ u_1^k \to u_1 \qquad \text{in } L^2(\Omega). \end{cases} \tag{5.9}$$

We search solutions of the form

$$u^k(x,t) = \sum_{j=1}^{k} h^{j,k}(t) w_j(x),$$

for the approximate problem in V_k

$$\begin{cases} \displaystyle\int_\Omega u_{tt}^k w dx + \int_\Omega \nabla u^k . \nabla w dx - \int_\Omega \int_0^{+\infty} g(s) \nabla u^k (t-s) . \nabla w ds dx \\ \qquad\qquad + \displaystyle\int_\Omega a(x) |u_t^k|^{m-2} u_t^k w dx = 0, \; \forall w \in V_k \\ u^k(-t) = u_0^k(t), \; u_t^k(0) = u_1^k. \end{cases} \tag{5.10}$$

This leads to a system of ODE's for unknown functions $h^{j,m}(t)$. Based on standard existence theory for ODE, the system (5.10) *admits a solution u^k on a maximal time interval $[0, t_k)$, $0 < t_k < T$ for each $k \in \mathbb{N}$. In fact $t_k = T = +\infty$ and to show this, replace w by u_t^k in* (5.10) *and integrate by parts to obtain*

$$\begin{aligned} &\frac{d}{dt}\left(\frac{1}{2}||u_t^k||_2^2 + \frac{\ell}{2}||\nabla u^k||_2^2 + \frac{1}{2}(go\nabla u^k)(t)\right) \\ &\qquad = \frac{1}{2}(g' o \nabla u^k)(t) - \int_\Omega a(x) |u_t^k(t)|^m dx \le 0. \end{aligned} \tag{5.11}$$

Integrate (5.11) *over* $(0, t)$ *to obtain*

$$\begin{aligned} &\frac{1}{2}\left(||u_t^k||_2^2 + \ell||\nabla u^k||_2^2 + (go\nabla u^k)(t)\right) + \int_0^t \int_\Omega a(x)|u_t^k(s)|^m dx ds \\ &\qquad = \frac{1}{2}\left(||\nabla u_0^k(t)||_2^2 + ||u_1^k||_2^2 + (go\nabla u^k)(0)\right) + \frac{1}{2}\int_0^t (g' o \nabla u^k)(s) ds. \end{aligned} \tag{5.12}$$

This means, using $(A1)$ *and* $(A4)$*, that, for some positive constant C independent of t and k,*

$$E^k(t) \le E^k(0) \le C.$$

Thus, we can extend t_k to infinity and, in addition, we have

$$\begin{cases} (u^k) \text{ is a bounded sequence in } L^\infty(0,T;H_0^1(\Omega)). \\ (u_t^k) \text{ is a bounded sequence in } L^\infty(0,T;L^2(\Omega)) \cap L_a^m(\Omega \times (0,T)), \end{cases}$$

where $L_a^m(\Omega \times (0,T))$ is the a-weighted L^m space which is defined as follows

$$v \in L_a^m(\Omega \times (0,T)) \ \text{ if and only if } \ \int_0^T \int_\Omega a(x)|u(x,t)|^m dxdt < +\infty.$$

Therefore, there exists a subsequence (u^k), still denoted by (u^k), such that

$$\begin{cases} u^k \rightharpoonup^* u \ \text{ in } L^\infty(0,T;H_0^1(\Omega)), \\ u_t^k \rightharpoonup^* u_t \ \text{ in } L^\infty(0,T;L^2(\Omega)), \\ u_t^k \rightharpoonup u_t \ \text{ in } L_a^m(\Omega \times (0,T)). \end{cases} \tag{5.13}$$

Since (u_t^k) is bounded in $L_a^m(\Omega \times (0,T))$, then $(|u_t^k|^{m-2}u_t^k)$ is bounded in $L_a^{\frac{m}{m-1}}(\Omega \times (0,T))$. Hence, up to a subsequence,

$$|u_t^k|^{m-2}u_t^k \rightharpoonup \psi \ \text{ in } L_a^{\frac{m}{m-1}}(\Omega \times (0,T)). \tag{5.14}$$

Now, our task to show that $\psi = |u_t|^{m-2}u_t$. For this purpose, integrate (5.10) over

$(0,t)$ *to obtain*

$$
\begin{aligned}
&\int_\Omega u_t^k(t) w dx - \int_\Omega u_1^k w dx + \int_0^t \int_\Omega \nabla u^k(s).\nabla w dx ds \\
&\quad - \int_\Omega \int_0^t \left(\int_0^{+\infty} g(\tau) \nabla u^k(s-\tau) d\tau \right) \nabla w ds dx \\
&\qquad + \int_\Omega a(x) \int_0^t |u_s^k(s)|^{m-2} u_s^k(s) w ds dx = 0, \quad \forall w \in V_j,\ \forall j = 1, 2, ..., k.
\end{aligned}
\tag{5.15}
$$

Convergences (5.9), (5.13) *and* (5.14) *are sufficient to pass to the limit in* (5.15), *as* $k \to +\infty$, *and get*

$$
\begin{aligned}
&\int_\Omega u_t(t) w dx - \int_\Omega u_1 w dx + \int_0^t \int_\Omega \nabla u(s).\nabla w dx ds \\
&\quad - \int_\Omega \int_0^t \left(\int_0^{+\infty} g(\tau) \nabla u(s-\tau) d\tau \right) \nabla w ds dx \\
&\qquad + \int_\Omega a(x) \int_0^t \psi(s) w ds dx = 0, \quad \forall w \in V_k,\ \forall k \geq 1,
\end{aligned}
\tag{5.16}
$$

which implies that (5.16) *is valid for any* $w \in H_0^1(\Omega)$. *Using the fact that the left hand side of* (5.16) *is an absolutely continuous function, hence it is differentiable for a.e* $t \in \mathbb{R}^+$, *we get*

$$
\begin{aligned}
&\int_\Omega u_{tt}(x,t) w(x) dx + \int_\Omega \nabla u(x,t).\nabla w(x) dx \\
&\quad - \int_\Omega \left(\int_0^{+\infty} g(s) \nabla u(x, t-s) ds \right) \nabla w(x) dx \\
&\qquad + \int_\Omega a(x) \psi(t) w(x) dx = 0, \quad \forall w \in H_0^1(\Omega).
\end{aligned}
\tag{5.17}
$$

Now, define

$$X^k = \int_0^T \int_\Omega \left(|u_t^k|^{m-2} u_t^k - |v_t|^{m-2} v \right) (u_t^k - v) dxdt \geq 0, \quad \forall v \in L^m((0,T), H_0^1(\Omega)). \tag{5.18}$$

This is true by the following elementary inequality

$$(|a|^{q-1}a - |b|^{q-1}b)(a-b) \geq C|a-b|^{q+1}, \quad \textit{for } a,\, b \in \mathbb{R},\ q \geq 1. \tag{5.19}$$

So, by using (5.12)*, we get*

$$\begin{aligned} X^k &= \frac{1}{2}\left(||\nabla u_0^k(t)||_2^2 + ||u_1^k||_2^2 + (go\nabla u^k)(0) \right) - \frac{1}{2}\int_0^t (g' o \nabla u^k)(s) ds \\ &\quad - \frac{1}{2}\left(||u_t^k||_2^2 + \ell ||\nabla u^k||_2^2 + (go\nabla u^k)(t) \right) - \int_0^T \int_\Omega a(x)|u_t^k|^{m-2}|u_t^k| v dxdt \\ &\quad - \int_0^T \int_\Omega a(x)|v|^{m-2} v(u_t - v) dxdt. \end{aligned}$$

Taking $k \to +\infty$*, we obtain*

$$\begin{aligned} 0 \leq \limsup X^k &= \frac{1}{2}\left(||\nabla u_0(t)||_2^2 + ||u_1||_2^2 + (go\nabla u)(0) \right) - \frac{1}{2}\int_0^t (g' o \nabla u)(s) ds \\ &\quad - \frac{1}{2}\left(||u_t||_2^2 + \ell ||\nabla u||_2^2 + (go\nabla u)(t) \right) \\ &\quad - \int_0^T \int_\Omega a(x)\psi(t) v dxdt - \int_0^T \int_\Omega a(x)|v|^{m-2} v(u_t - v) dxdt. \end{aligned} \tag{5.20}$$

Replacing w *by* u_t *in* (5.17) *and integrating over* $(0,T)$*, to obtain*

$$\begin{aligned} &\frac{1}{2}\left(||\nabla u_0(t)||_2^2 + ||u_1||_2^2 + (go\nabla u)(0) \right) - \frac{1}{2}\int_0^t (g' o \nabla u)(s) ds \\ &\quad - \frac{1}{2}\left(||u_t||_2^2 + \ell ||\nabla u||_2^2 + (go\nabla u)(t) \right) + \int_0^T \int_\Omega a(x)\psi u_t dxdt = 0. \end{aligned} \tag{5.21}$$

Compining (5.20) *and* (5.21), *we arrive at*

$$\begin{aligned} 0 \leq \limsup X^k &= \int_0^T \int_\Omega \psi u_t dxdt - \int_0^T \int_\Omega \psi v dxdt \\ &\quad - \int_0^T \int_\Omega a(x)|v|^{m-2} v (u_t - v) dxdt \\ &\leq \int_0^T \int_\Omega a(x)(\psi - |v|^{m-2} v)(u_t - v) dxdt. \end{aligned}$$

Hence,

$$\int_0^T \int_\Omega a(x)(\psi - |v|^{m-2} v)(u_t - v) dxdt \geq 0,, \quad \forall v \in L_a^m(\Omega \times (0,T)),$$

by density of $H_0^1(\Omega)$ *in* $L^m(\Omega)$.

Let $v = \lambda z + u_t$, $z \in L_a^m(\Omega \times (0,T))$. *So, we get,* $\forall \lambda \neq 0$,

$$-\lambda \int_0^T \int_\Omega a(x) \left(\psi - |\lambda z + u_t|^{m-2} (\lambda w + u_t) \right) z dxdt \geq 0, \quad z \in L_a^m(\Omega \times (0,T)).$$

Let $\lambda > 0$. *So we have*

$$\int_0^T \int_\Omega a(x) \left(\psi - |\lambda z + u_t|^{m-2} (\lambda z + u_t) \right) z dxdt \leq 0, \quad z \in L_a^m(\Omega \times (0,T)).$$

As $\lambda \to 0$, *we get*

$$\int_0^T \int_\Omega a(x) \left(\psi - |u_t|^{m-2} u_t \right) z dxdt \leq 0, \quad z \in L_a^m(\Omega \times (0,T)) \tag{5.22}$$

Similarly, for $\lambda < 0$*, we get*

$$\int_0^T \int_\Omega a(x) \left(\psi - |u_t|^{m-2} u_t\right) z dx dt \le 0, \quad z \in L_a^m(\Omega \times (0,T)). \tag{5.23}$$

Thus, (5.22) *and* (5.23) *imply that* $\psi = |u_t|^{m-2}u_t$*. Hence* (5.16) *becomes*

$$\begin{aligned} &\int_\Omega u_{tt}(x,t)w(x)dx + \int_\Omega \nabla u(x,t).\nabla w(x)dx \\ &\quad - \int_\Omega \left(\int_0^{+\infty} g(s)\nabla u(x,t-s)ds\right) \nabla w(x)dx \\ &\quad\quad + \int_\Omega a(x)|u_t|^{m-2}u_t w(x)dx = 0, \quad \forall w \in H_0^1(\Omega), \end{aligned}$$

which gives

$$u_{tt} - \Delta u + \int_0^{+\infty} g(s)\Delta(t-s)ds + a(x)|u_t|^{m-2}u_t = 0, \quad \text{in } D'(\Omega \times (0,T)).$$

For uniqueness, let us assume that problem (5.1) *has two solutions* u *and* v*. Then,* $w = u - v$ *satisfies*

$$\begin{cases} w_{tt} - \Delta w + \int_0^{+\infty} g(s)\Delta w(t-s)ds + a(x)\left(|u_t|^{m-2}u_t - |v_t|^{m-2}v_t\right) = 0, & \text{in } \Omega \times (0,T), \\ w = 0, & \text{on } \partial\Omega \times (0,T), \\ w(x,-t) = w_t(x,0) = 0, & \text{in } \Omega \times (0,T), \end{cases} \tag{5.24}$$

Now, multiply (5.24) *by* w_t *and integrate over* $\Omega \times (0,t)$ *to obtain*

$$||w_t||_2^2 + ||\nabla w||_2^2 + (go\nabla u)(t) - \int_0^t (g'o\nabla u)(s)ds$$
$$+ 2\int_0^t \int_\Omega a(x)\left(|u_t|^{m-2}u_t - |v_t|^{m-2}v_t\right)(u_t - v_t)dxds = 0.$$

Hence, by using inequality (5.19), *we have*

$$||w_t||_2^2 + ||\nabla w||_2^2 \leq 0,$$

which implies that $w = C$. *In fact,* $C = 0$ *since* $w = 0$ *on* $\partial\Omega$. *Which completes the proof.* ▮

5.3 Technical Lemmas

In this section, we establish several lemmas needed for the proof of our main result.

Lemma 5.1 *Under the assumptions* $(A1) - (A4)$, *the functional*

$$\psi(t) := \int_\Omega uu_t dx$$

satisfies, along the solution, the estimate

$$\begin{aligned}\psi'(t) \leq -\frac{\ell}{4}||\ \nabla u||_2^2 + ||u_t||_2^2 + \frac{1-\ell}{2\ell}(go\nabla u)(t) \\ + c(\delta)\int_\Omega a(x)|u_t|^m dx, \qquad \textit{if}\ m \geq 2\end{aligned} \tag{5.25}$$

and

$$\begin{aligned}\psi'(t) \leq &-\frac{\ell}{4}||\ \nabla u||_2^2 + ||u_t||_2^2 + \frac{1-\ell}{2\ell}(go\nabla u)(t)\\ &+ c(\delta, ||a||_\infty, \Omega)\left(\int_\Omega a(x)|u_t|^m dx\right)^{\frac{2m-2}{m}}, \qquad \textit{if}\ m < 2.\end{aligned} \tag{5.26}$$

Proof. Case $m \geq 2$.

By using (5.1), we easily see that

$$\begin{aligned}\psi'(t) = &||u_t||_2^2 - ||\ \nabla u||_2^2 - \int_\Omega a(x)|u_t|^{m-2}u_t u dx\\ &+ \int_\Omega \nabla u(t). \int_0^{+\infty} g(s)\nabla u(t-s)dsdx.\end{aligned} \tag{5.27}$$

We now estimate the third term of the RHS of (5.27), using Young's inequality and (5.5),

$$\begin{aligned}\int_\Omega a(x)|u_t|^{m-2}u_t u dx &\leq \delta \int_\Omega a(x)|u|^m dx + c(\delta)\int_\Omega a(x)|u_t|^m dx\\ &\leq c(\delta)\int_\Omega a(x)|u_t|^m dx + \delta C||\nabla u||_2^{m-2}||\nabla u||_2^2\\ &\leq c(\delta)\int_\Omega a(x)|u_t|^m dx + \delta C E^{\frac{m-2}{2}}(0)||\nabla u||_2^2.\end{aligned} \tag{5.28}$$

The fourth term in the RHS of (5.27) can be handled as follows

$$\begin{aligned}&\int_\Omega \nabla u(t). \int_0^{+\infty} g(s)\nabla u(t-s)dsdx\\ &\quad\leq \frac{1}{2}||\nabla u||_2^2 + \frac{1}{2}\int_\Omega \left(\int_0^{+\infty} g(s)|\nabla u(t-s)|ds\right)^2 dx\\ &\quad\leq \frac{1}{2}||\nabla u||_2^2 + \frac{1}{2}\int_\Omega \left(\int_0^{+\infty} g(s)|\nabla u(t-s) - \nabla u(t)| + |\nabla u(t)|ds\right)^2 dx.\end{aligned} \tag{5.29}$$

We then use Cauchy−Schwarz inequality, Young's inequality to obtain, for any $\eta > 0$,

$$\begin{aligned}
&\int_\Omega \left(\int_0^{+\infty} g(s)|\nabla u(t-s) - \nabla u(t)| + |\nabla u(t)| ds\right)^2 dx \\
&\quad \le \int_\Omega \left(\int_0^{+\infty} g(s)|\nabla u(t-s) - \nabla u(t)| ds\right)^2 dx + \int_\Omega \left(\int_0^{+\infty} g(s)|\nabla u(t)| ds\right)^2 dx \\
&\qquad + 2\int_\Omega \left(\int_0^{+\infty} g(s)|\nabla u(t-s) - \nabla u(t)| ds\right)\left(\int_0^{+\infty} g(s)|\nabla u(t)| ds\right) dx \\
&\quad \le \left(1+\frac{1}{\eta}\right)\int_\Omega \left(\int_0^{+\infty} g(s)|\nabla u(t-s) - \nabla u(t)| ds\right)^2 dx \\
&\qquad + (1+\eta)\int_\Omega \left(\int_0^{+\infty} g(s)|\nabla u(t)| ds\right)^2 dx \\
&\quad \le \left(1+\frac{1}{\eta}\right)(1-\ell)(go\nabla u)(t) + (1+\eta)(1-\ell)^2 ||\nabla u||^2.
\end{aligned} \tag{5.30}$$

By combining (5.27)-(5.30), we have

$$\begin{aligned}
\psi'(t) \le ||u_t||_2^2 - \frac{1}{2}\left[1 - (1+\eta)(1-\ell)^2 - 2\delta C E^{\frac{m-2}{2}}(0)\right] ||\ \nabla u||_2^2 \\
+ \frac{1}{2}\left(1+\frac{1}{\eta}\right)(1-\ell)(go\nabla u)(t) + c(\delta)\int_\Omega a(x)|u_t|^m dx.
\end{aligned} \tag{5.31}$$

By choosing $\eta = \frac{\ell}{1-\ell}$ and $\delta = \frac{\ell}{4CE^{\frac{m-2}{2}}(0)}$, (5.25) is established.

For the case of $m < 2$, we re-estimate (5.28), as follows

$$\begin{aligned}
&\int_\Omega a(x)|u_t|^{m-2} u_t u dx \\
&\quad \le \delta ||u||^2 + c(\delta)\int_\Omega a^{2m-2}(x)|u_t|^{2m-2} dx \\
&\quad \le \delta C||\nabla u||^2 + C(\delta, ||a||_\infty, \Omega)\left(\int_\Omega a(x)|u_t|^m dx\right)^{\frac{2m-2}{m}}.
\end{aligned} \tag{5.32}$$

By combining (5.27), (5.29), (5.30), (5.32) and choosing the same values of η and δ, (5.26) is obtained. ∎

Lemma 5.2 *Under the assumptions* $(A1)-(A4)$, *the functional*

$$\chi(t) := -\int_\Omega u_t \int_0^{+\infty} g(s)(u(t)-u(t-s))dsdx$$

satisfies, along the solution, the estimate

$$\begin{aligned}\chi'(t) \leq & -\delta[1+2(1-\ell)^2]||\ \nabla u||_2^2 - ((1-\ell)-\delta)||u_t||_2^2 + C(\delta)(go\nabla u)(t)\\ & + \frac{g(0)}{4\delta}(-(g'o\nabla u))(t) + C(\delta)\int_\Omega a(x)|u_t|^m dx, \qquad \textit{if } m \geq 2\end{aligned} \tag{5.33}$$

and

$$\begin{aligned}\chi'(t) \leq & -\delta[1+2(1-\ell)^2]||\ \nabla u||_2^2 - ((1-\ell)-\delta)||u_t||_2^2 + C(\delta)(go\nabla u)(t)\\ & + \frac{g(0)}{4\delta}(-(g'o\nabla u))(t) + c(\delta, ||a||_\infty, \Omega)\left(\int_\Omega a(x)|u_t|^m dx\right)^{\frac{2m-2}{m}}, \quad \textit{if } m < 2.\end{aligned} \tag{5.34}$$

Proof. By using (5.1), we easily see that

$$\begin{aligned}\chi'(t) = & \int_\Omega \nabla u(t). \left(\int_0^{+\infty} g(s)(\nabla u(t) - \nabla u(t-s))ds\right) dx \\ & - \int_\Omega \left(\int_0^{+\infty} g(s)\nabla u(t-s)ds\right). \left(\int_0^{+\infty} g(s)(u(t) - u(t-s))ds\right) dx \\ & + \int_\Omega a(x)|u_t|^{m-2}u_t \int_0^{+\infty} g(s)(u(t) - u(t-s))dsdx \\ & - \int_\Omega u_t \int_0^{+\infty} g'(s)(u(t) - u(t-s))dsdx - (1-\ell)\int_\Omega u_t^2 dx.\end{aligned} \tag{5.35}$$

Similarly to (5.27), we estimate the RHS terms of (5.35). So for any $\delta > 0$ we have (see [60]).

$$\int_\Omega \nabla u(t). \left(\int_0^{+\infty} g(s)(\nabla u(t) - \nabla u(t-s))ds\right) dx \leq \delta||\ \nabla u||_2^2 + \frac{1-\ell}{4\delta}(go\nabla u)(t). \tag{5.36}$$

The second term can be estimated as (5.30). Thus, we have

$$\begin{aligned}& \int_\Omega \left(\int_0^{+\infty} g(s)\nabla u(t-s)ds\right). \left(\int_0^{+\infty} g(s)(u(t) - u(t-s))ds\right) dx \\ & \quad \leq \left(2\delta + \frac{1}{4\delta}\right)(1-\ell)(go\nabla u)(t) + 2\delta(1-\ell)^2||\ \nabla u||_2^2.\end{aligned} \tag{5.37}$$

Exploiting (5.6), (5.7) and (5.8) and we obtain $\forall t, s \in \mathbb{R}^+$,

$$\begin{aligned}||\nabla u(t) - \nabla u(t-s)||_2^2 & \leq 2||\nabla u(t)||_2{}^2 + 2||\nabla u(t-s)||_2^2 \\ & \leq 4\sup_{s>0}||\nabla u(s)||_2^2 + 2\sup_{\tau<0}||\nabla u(\tau)||_2^2 \\ & \leq 4\sup_{s>0}||\nabla u(s)||_2^2 + 2\sup_{\tau>0}||\nabla u_0(\tau)||_2^2 \\ & \leq \frac{8}{1-\ell}E(0) + 2m_0^2 := N_1.\end{aligned} \tag{5.38}$$

Using (5.38), the third term of (5.35) can be estimated, for $m \geq 2$, as follows

$$\begin{aligned}
&\int_\Omega a(x)|u_t|^{m-2}u_t \int_0^{+\infty} g(s)(u(t)-u(t-s))dsdx \\
&\quad \leq \delta ||a||_\infty \int_\Omega \left| \int_0^{+\infty} g(s)(u(t)-u(t-s))ds \right|^m dx + C(\delta)\int_\Omega a(x)|u_t|^m dx \\
&\quad \leq \delta ||a||_\infty (1-\ell)^{m-1} \int_\Omega \int_0^{+\infty} g(s)|u(t)-u(t-s)|^m dsdx + C(\delta)\int_\Omega a(x)|u_t|^m dx \\
&\quad \leq \delta ||a||_\infty (1-\ell)^{m-1} C \int_0^{+\infty} g(s)||\nabla u(t)-\nabla u(t-s)||_2^m ds + C(\delta)\int_\Omega a(x)|u_t|^m dx \\
&\quad \leq \delta ||a||_\infty (1-\ell)^{m-1} C N_1^{\frac{m-2}{2}} (go\nabla u)(t) + C(\delta)\int_\Omega a(x)|u_t|^m dx.
\end{aligned} \tag{5.39}$$

The forth term

$$-\int_\Omega u_t \int_0^{+\infty} g'(s)(u(t)-u(t-s))dsdx \leq \delta ||u_t||_2^2 + \frac{g(0)}{4\delta} C \int_\Omega (-g' o\nabla u)(t). \tag{5.40}$$

A combination of (5.35)-(5.40) then yields (5.33). To establish (5.34) we re-estimate (5.39), as follows

$$\begin{aligned}
&\int_\Omega a(x)|u_t|^{m-2}u_t \int_0^{+\infty} g(s)(u(t)-u(t-s))dsdx \\
&\quad \leq \delta C(go\nabla u)(t) + C(\delta, ||a||_\infty, \Omega)\left(\int_\Omega a(x)|u_t|dx\right)^{\frac{2m-2}{m}}.
\end{aligned} \tag{5.41}$$

Hence, a combination of (5.35)-(5.37), (5.40) and (5.41) then gives (5.34). ∎

Lemma 5.3 *Assume that* $(A1)-(A4)$*. Then there exist strictly positive constants* ε_1, ε_2, λ, c *such that the functional*

$$L(t) = E(t) + \varepsilon_1 \psi(t) + \varepsilon_2 \chi(t)$$

satisfies, for all $t \in \mathbb{R}^+$,

$$L \sim E, \tag{5.42}$$

$$L'(t) \leq -\lambda E(t) + c(go\nabla u)(t), \qquad \textit{if } m \geq 2 \tag{5.43}$$

and

$$L'(t) \leq -\lambda E(t) + c(go\nabla u)(t) + c\left(\int_\Omega a(x)|u_t|^m dx\right)^{\frac{2m-2}{m}}, \quad \textit{if } m < 2. \tag{5.44}$$

Proof. For the proof of (5.42), it is straightforward to see, using Young's inequality and Lemma 2.1, that

$$\begin{aligned} L(t) &\leq E(t) + \frac{\varepsilon_1}{2}||u_t||_2^2 + \frac{\varepsilon_1}{2}||u||_2^2 + \frac{\varepsilon_2}{2}||u_t||_2^2 \\ &\quad + \frac{\varepsilon_2}{2}\int_\Omega \left(\int_0^{+\infty} g(s)(u(t) - u(t-s))ds\right)^2 dx \\ &\leq E(t) + \frac{\varepsilon_1}{2}||u_t||_2^2 + \frac{\varepsilon_1}{2}C_p||\nabla u||_2^2 \\ &\quad + \frac{\varepsilon_2}{2}||u_t||_2^2 + \frac{\varepsilon_2}{2}C_p(1-\ell)(go\nabla u)(t) \\ &\leq C_2 E(t) \end{aligned} \tag{5.45}$$

and

$$\begin{aligned} L(t) \geq & \left(\frac{1-\ell}{2} - \frac{\varepsilon_2}{2}C_p\right)||\nabla u||_2^2 + \left(\frac{1}{2} - \frac{\varepsilon_1 + \varepsilon_2}{2}\right)||u_t||_2^2 \\ & + \left(\frac{1}{2} - \frac{\varepsilon_2}{2}C_p(1-\ell)\right)(go\nabla u)(t). \end{aligned} \tag{5.46}$$

By fixing ε_1 and ε_2 small enough, we obtain $L(t) \geq C_1E(t)$. Now, we prove inequality (5.43). By using (5.8), (5.25) and (5.33), we obtain

$$\begin{aligned} L'(t) \leq & -[1-(\varepsilon_1+\varepsilon_2)C(\delta)]\int_\Omega a(x)|u_t|^m dx \\ & -[\varepsilon_2((1-\ell)-\delta)-\varepsilon_1]\,||u_t||_2^2 - \left[\frac{\varepsilon_1\ell}{4}-\varepsilon_2\delta(1+2(1-\ell)^2)\right]||\,\nabla u||_2^2 \\ & + \left[\frac{1}{2}-\varepsilon_2\frac{g(0)}{4\delta}C\right](g'o\nabla u)(t) + \left[\varepsilon_1\frac{1-\ell}{2\ell}+\varepsilon_2C(\delta)\right](go\nabla u)(t). \end{aligned} \tag{5.47}$$

At this point we choose δ so small that

$$(1-\ell)-\delta > \frac{1}{2}(1-\ell)$$

and

$$\frac{4}{\ell}\delta\left(1+2(1-\ell)^2\right) < \frac{1}{4}(1-\ell).$$

Whence δ is fixed, the choice of any two positive constants ε_1 and ε_2 satisfying

$$\frac{1}{4}(1-\ell)\varepsilon_2 < \varepsilon_1 < \frac{1}{2}(1-\ell)\varepsilon_2 \tag{5.48}$$

will make

$$k_1 = \varepsilon_2\left((1-\ell)-\delta\right)-\varepsilon_1 > 0$$

and

$$k_2 = \frac{\varepsilon_1\ell}{4}-\varepsilon_2\delta\left(1+2(1-\ell)^2\right) > 0.$$

We then select ε_1 and ε_2 so small that (5.42) and (5.48) remain valid and, further,

$$1 - (\varepsilon_1 + \varepsilon_2)C(\delta) > 0$$

and

$$\frac{1}{2} - \varepsilon_2 \frac{g(0)}{4\delta} C > 0$$

Therefore (5.43) is established for two positive constants $c, \lambda > 0$. The same calculations, for $m < 2$, using (5.8), (5.26) and (5.34), give (5.44). ∎

5.4 The Main Result

In this section we state and prove our main result. Our main result is

Theorem 5.1 *Let $(u_0(.,0), u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ be given. Assume that $(A1) - (A4)$ hold. Then, there exist positive constants k_i, λ_j, δ_1, for $i, j = 1, 2, 3$, such that*

$$E(t) \leq k_1 G_1^{-1}\left(k_2 t + k_3\right), \quad \forall t \in \mathbb{R}^+, \quad \text{if } m \geq 2, \tag{5.49}$$

where

$$G_1(\tau) = \int_\tau^1 \frac{1}{G_2(s)} ds \quad \text{and} \quad G_2(t) = tG'\left(\delta_1 t\right)$$

and

$$E(t) \leq \lambda_1 W_1^{-1}\left(\lambda_2 t + \lambda_3\right), \quad \forall t \in \mathbb{R}^+, \quad \text{if } m < 2, \tag{5.50}$$

where

$$W_1(\tau) = \int_\tau^1 \frac{1}{W_2(s)} ds \quad and \quad W_2(t) = t^{\frac{m}{2m-2}} G'\left(\delta_1 t\right).$$

Proof. We distinguish two cases

Case $m \geq 2$.

Multiplying (5.43) by $G'(\delta_0 E(t))$ and using (4.17), we get

$$\begin{aligned} G'(\delta_0 E(t)) L'(t) &\leq -\lambda E(t) G'(\delta_0 E(t)) - c\beta_1 E'(t) + c\beta_1 \delta_0 E(t) G'(\delta_0 E(t)) \\ &= -(\lambda - c\beta_1\delta_0) E(t) G'(\delta_0 E(t)) - c\beta_1 E'(t). \end{aligned}$$

Choosing δ_0 small enough so that $\beta_2 := \lambda - c\beta_1\delta_0 > 0$ and puting

$$F(t) := G'(\delta_0 E(t)) L(t) + c\beta_1 E(t)$$

we deduce that $F \sim E$ and

$$F'(t) \leq -k_2 F(t) G'(\delta_1 F(t)).$$

The last inequality implies that $(G_1(F))' \geq k_2$, where $G_1(\tau) = \int_\tau^1 \frac{1}{sG'(\delta_0 s)} ds$ for $0 < \tau \leq 1$. Then, by integrating over $[0, t]$, we get

$$F(t) \leq G_1^{-1}(k_2 t + k_3), \quad \forall t \in \mathbb{R}^+.$$

The equivalence $F \sim E$ and the definition (5.7) give (5.49).

Case $1 < m < 2$.

Multiplying (5.44) by $G'(\delta_0 E(t))$ and using (4.17), we find that the functional F_1 defined by

$$F_1(t) := G'(\delta_0 E(t))L(t) + c\beta_1 E(t)$$

satisfies

$$F_1'(t) \leq -\lambda G'(\delta_0 E(t))E(t) + cG'(\delta_0 E(t))\left(\int_\Omega a(x)|u_t|^m dx\right)^{\frac{2m-2}{m}}. \tag{5.51}$$

By multiplying (5.51) by $E^q(t)$, $q > 0$, and using (5.8), and using Young's inequality, we get

$$\begin{aligned} E^q(t)F_1'(t) &\leq -\lambda G'(\delta_0 E(t))E^{q+1}(t) + cE^q(t)G'(\delta_0 E(t))\left(-E'(t)\right)^{\frac{2m-2}{m}} \\ &\leq -\lambda G'(\delta_0 E(t))E^{q+1}(t) + \varepsilon G'(\delta_0 E(t))E^{\frac{qm}{2-m}} + C(\varepsilon)G'(\delta_0 E(0))(-E'(t)). \end{aligned} \tag{5.52}$$

By choosing $q = \frac{2-m}{2m-2}$, (5.52) yields

$$\begin{aligned} E^q(t)F_1'(t) &\leq -\lambda G'(\delta_0 E(t))E^{q+1}(t) + \varepsilon G'(\delta_0 E(t))E^{q+1} + C(\varepsilon)(-E'(t)) \\ &= -(\lambda - \varepsilon)G'(\delta_0 E(t))E^{q+1}(t) - C(\varepsilon)E'(t). \end{aligned} \tag{5.53}$$

Let $F_2(t) := E^q(t)F_1(t) + C(\varepsilon)E(t)$ then using (5.8), (5.53), (5.42) and choosing ε small enough, we get

$$F_2'(t) \leq -\lambda_2 G'(\delta_1 F_2(t))F_2^{q+1}(t) = -\lambda_2 W_2(F_2(t)), \tag{5.54}$$

where $W_2(\tau) = \tau^{q+1} G'(\delta_1 \tau)$. Inequality (5.54) implies that

$$(W_1(F_2))' \geq \lambda_2,$$

where

$$W_1(\tau) = \int_\tau^1 \frac{1}{W_2(s)} ds \quad \text{for } \tau \in (0, 1].$$

Then integrating over $[0, t]$, we obtain

$$W_1(F_2(t)) \geq \lambda_2 t + W_1(F_2(0))$$

since W_1^{-1} is nonincreasing,

$$F_2(t) \leq W_1^{-1}(\lambda_2 t + \lambda_3).$$

The equivalence $F_2 \sim E$ and the definition (5.7) give (5.50). ▮

5.5 Kernels with Exponential Decay

In this section, we discuss the case of exponentially decaying kernels. Though this type of kernels is a special case of our general class of kernels we considered above, the decaying result here can be obtained without condition (5.6).

Theorem 5.2 *Assume that* $(A2)-(A3)$ *hold and the kernel* g *satisfies* (5.2) *and,* *for some positive constant* ξ,

$$g'(t) \leq -\xi g(t). \tag{5.55}$$

Then, there exist positive constants c_1, c_2 *and* c_3 *such that*

$$E(t) \leq c_2 e^{-c_1 t}, \quad \forall t \in \mathbb{R}^+, \quad \textit{if}\, m \geq 2 \tag{5.56}$$

and

$$E(t) \leq c_3 (1+t)^{-\frac{2m-2}{2-m}}, \quad \forall t \in \mathbb{R}^+, \quad \textit{if}\, m < 2. \tag{5.57}$$

Proof. **Case** $m \geq 2$.

We multiply (5.43) by ξ and use (5.55) and (5.8), to get

$$\begin{aligned} \xi L'(t) &\leq -\lambda \xi E(t) + c\xi (go\nabla u)(t) \\ &\leq -\lambda \xi E(t) - c(g' o\nabla u)(t) \\ &\leq -\lambda \xi E(t) - cE'(t). \end{aligned} \tag{5.58}$$

Let $L_1(t) := \xi L(t) + cE(t)$, then

$$L_1 \sim E \tag{5.59}$$

and, by using (5.58),

$$L_1'(t) \leq -c_1 L_1(t). \tag{5.60}$$

A simple integration of (5.60) leads to

$$L_1(t) \leq L(0)e^{-c_1 t}. \tag{5.61}$$

Thus (5.59), (5.60) give the desired result.

Case $1 < m < 2$.

We multiply (5.44) by ξ and use (5.55) and (5.8), to find that the functional defined by

$$R(t) = \xi L(t) + cE(t),$$

satisfies

$$R \sim E \tag{5.62}$$

and

$$R'(t) \leq -\xi\lambda E(t) + c\xi \left(E'(t)\right)^{\frac{2m-2}{m}}. \tag{5.63}$$

By multiplying (5.63) by $E^q(t)$, $q > 0$, and using Young's inequality, we get

$$E^q(t)R'(t) \leq -\xi\lambda E^{q+1}(t) + \varepsilon E^{\frac{qm}{2-m}} + C(\varepsilon)(-E'(t)). \tag{5.64}$$

By selecting $q = \frac{2-m}{2m-2}$, defining $R_1(t) = E^q(t)R(t) + C(\varepsilon)E(t)$ and choosing ε small enough, we easily see that

$$R_1 \sim E \tag{5.65}$$

and

$$R_1'(t) \leq -c' R_1^{q+1}(t). \tag{5.66}$$

A simple integration of (5.66) and using (5.65) gives (5.57). ∎

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Publications

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