LOCAL AND NON-LOCAL LINEARIZATION OF SCALAR SECOND ORDER ODES IN THE NORMAL FORM

BY

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This thesis, written by **RAED ALI MARA'BEH** under the direction of his thesis adviser and approved by his thesis committee, has been presented to and accepted by the Dean of Graduate Studies, in partial fulfillment of the requirements for the degree of **MASTER OF SCIENCE IN MATHEMATICS AND STATISTICS**.

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To my father Ali,

To my mother Aida,

To my brothers Mohammad, Saed, Baha' and Ala'.

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All praise is to Allah Who is One, and His blessings and peace be upon him whom there is no messenger after him.

First and the foremost acknowledgments are due to my beloved parents and brothers for their supporting, encouraging, and prayers.

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THESIS ABSTRACT

NAME: Raed Ali Mara'Beh

TITLE OF STUDY: Local and Non-local Linearization of Scalar Second Order ODEs in the Normal Form

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The mathematical modeling of most of the natural and physical processes leads to nonlinear differential equations and no general theory exists for finding exact solutions of such nonlinear differential equations. Because of the difficulties associated with solving nonlinear problems, simplifying assumptions are usually made to consider linear approximations of such problems. However, such assumptions may lead to significant error in the solution. Therefore, in general it is required to solve nonlinear differential equations in the form that models the nonlinear real life problem. The linearization process of transforming nonlinear differential equations to linear form by means of transformations of the independent and dependent variables provides a powerful technique of determining exact solutions of the nonlinear problem. The aim of this work is to investigate questions regarding linearization of second order ODEs via local or *non-local transformations. The first problem considered is concerned with the subject of linearization of nonlinear second order ODEs via local transformations. An alternative proof of Lie's approach for linearization of second order ODEs is derived using the relationship between λ-symmetries and the first integrals. This relation further leads to a new λ-symmetry linearization criteria for second order ODEs which provides a new approach for constructing the linearization transformations with lower complexity. The effectiveness of the approach is illustrated by obtaining the general local linearization transformations for the nonlinear ODEs of the form* $y'' + F_1(x, y)y' + F(x, y) = 0$. *Examples of linearizing nonlinear ODEs which are quadratic or cubic in the first derivative are also presented. The second problem investigated is the linearization problem for nonlinear second order ODEs to the Laguerre form by means of generalized Sundman transformations (S-transformations). A characterization of these S-linearizable equations in terms of first integral and procedure for construction of linearizing S-transformations has been given recently by Muriel-Romero [35, 37]. Here we give a new characterization of S-linearizable equations in terms of the coefficients and one auxiliary function. This new criterion is used to obtain the general solutions for the first integral explicitly, providing a direct alternative procedure for constructing the first integrals and Sundman transformations. The effectiveness of this approach is demonstrated by applying it to find the general solution for geodesics on surfaces of revolution of constant curvature in a unified manner.*

ملخص الرسالة

الاســــــم: رائد علي عايش مراعبه
عنوان الرسالـة: التحويل الخطي للمعادلات التفاضلية العادية من الدرجة الثانية باستخدام التحويلات الخطية المحلية وغيرالمحلية
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إن المحاكاة الرياضية لمعظم العمليات الطبيعية و الفيزيائية تؤدي إلى معادلات تفاضلية غير خطية ولا توجد نظرية عامة لإيجاد حلول لهذه المعادلات الغير خطية.

ونتيجة لهذه الصعوبات المصاحبة لحل المعادلات الغير خطية فإن فرضيات التقريب الخطي تعتبر وسيلة لتبسيط هذه المعادلات . ومع ذلك، فإن هذه الفرضيات قد تؤدي الى خطأ كبير في الحل. لذلك ، بشكل عام أصبح لزاما حل المعادلات التفاضليه الغير خطية التي تحاكي المسائل الحياتية بدون اللجوء إلى التقريب الخطي. إن عمليات التحويل الخطي تقدم تقنيات قوية لإيجاد حلول للمعادلات الغير خطية. هذه العمليات تقوم على تحويل المعادلات التفاضلية الغير خطية إلى معادلات خطية من خلال تعريف متغيرات مستقله ومتغيرات تابعة جديدة. إن الهدف من هذه الرسالة هو البحث في المسائل المتعلقة بالتحويل الخطي للمعادلات التفاضلية العادية من الدرجة الثانية باستخدام التحويلات الخطية المحلية وغير المحلية.

فيما يتعلق بالمسألة الأولى في هذا البحث والتي تركز على موضوع التحويل الخطي للمعادلات التفاضلية العادية من الدرجة الثانية عبر التحويلات الخطية المحلية، لقد تم مراجعة طريقة التحويل الخطي الخاصة بلي (Sophus_Lie) . كما تم تقديم برهان بديل باستخدام العلاقة بين محاور تماثل لامدا والتكاملات الأولى للمعادلات التفاضلية. إضافة الى ذلك، هذه العلاقة أنتجت معايير جديدة لإمكانية التحويل الخطي كما أوجدت طريقة أقل تعقيدا لإيجاد التحويلات الخطية .

لقد تم توضيح فعالية هذه الطريقة من خلال إيجاد التحويلات الخطية المحلية للمعادلات التفاضلية العادية الغير خطية من الدرجة الثانية على شكل $y'' + F_1(x,y)y' + F(x,y) = 0$. كما تم تطبيق هذه الطريقة على المعادلات التفاضلية العادية الغير خطية من الدرجة الثانية ذات المشتقة الأولى المربعة أو المكعبة.

فيما يتعلق بالمسألة الثانية في هذا البحث والتي تركز على موضوع التحويل الخطي للمعادلات التفاضلية العادية من الدرجة الثانية الى صيغة لاجيرعبر تحويلات سندمان الغيرمحلية. فقد قدم موريل و روميرو [37,35] وصفا دقيقا للمعادلات الغير خطية التي يمكن تحويلها الى خطية بواسطة تحويلات سندمان بدلالة التكامل الأول، كما عرضوا طريقة مفصلة لايجاد تحويلات سندمان.

في هذا البحث، قدمنا وصفا جديدا للمعادلات الغير خطية التي يمكن تحويلها الى خطية بواسطة تحويلات سندمان بدلالة المعاملات واقتران مساعد. ومن ثم قدمنا صيغة صريحة عامة للتكامل الأول للمعادلات التي تحقق المعيار الجديد. وبذلك فإن المعيار الجديد يقدم طريقة جديدة لايجاد التكاملات الأولى و تحويلات سندمان.

لقد تم توضيح فعالية هذه الطريقة من خلال إيجاد الحل العام للمعادلات الجيوديسية على الاسطح الدورانية ذات الانحناء الثابت بطريقة موحدة.

CHAPTER 1

LIE SYMMETRY METHOD FOR ORDINARY DIFFERENTIAL EQUATIONS

This chapter provides an introduction to Lie symmetries of ODEs and sets up the basic framework for the research work carried out in this thesis. Sections (1.1) to (1.3) introduce the concepts of one parameter groups, infinitesimal generators and the relation between these. The key concept of prolongations is discussed in section (1.4) which is utilized in the next two sections to present the notion of symmetries of ODEs and the procedure for finding symmetries of ODEs. One of the main applications of Lie symmetries of ODEs is to reduce of order of ODEs. Section (1.7) provides details about the reduction of order of ODEs using symmetries. The set of symmetries of differential equations has a special algebraic structure, namely the symmetries of differential equations form a Lie algebra. The chapter closes with a brief introduction to Lie algebras and their fundamental properties.

1.1 One parameter group of transformations

The study of Lie symmetries of ODEs involves one parameter group of transformations

Definition 1.1 *A point transformation in the* $xy-$ *plane is a function*

$$T:\ R^2 \rightarrow\ R^2 \tag{1.1}$$

given by

$$T(x,y)=(\bar{x},\bar{y}) \tag{1.2}$$

where

$$\begin{aligned}\bar{x}&=\varphi(x,y)\\ \bar{y}&=\psi(x,y)\end{aligned} \tag{1.3}$$

A transformation in the xy-plane transforms a point (x,y) *to another point* $(\bar{x},\bar{y})$ *.*

Some examples of transformations in xy - plane are given below

- Translations (shifts)

$$\begin{aligned}\bar{x}&=x+a\\ \bar{y}&=y+b\end{aligned} \tag{1.4}$$

- Rotation

$$\begin{aligned}\bar{x}&=x\cos\theta-y\sin\theta\\ \bar{y}&=x\sin\theta+y\cos\theta\end{aligned} \tag{1.5}$$

- Re-scaling

$$\begin{aligned}\bar{x}&=e^{a}x\\ \bar{y}&=\ e^{b}y\end{aligned} \tag{1.6}$$

Definition 1.2 *A one parameter transformation is a transformation that depends on one parameter only. If the parameters is ε, it is denoted as $T_\varepsilon : \Re^2 \to \Re^2$ and is of the form*

$$\begin{aligned} \bar{x} &= \varphi(x, y, \varepsilon) \\ \bar{y} &= \psi(x, y, \varepsilon) \end{aligned} \tag{1.7}$$

Consider a parameter ε belonging to an interval $I \subset \Re$. The set of transformations $\{T_\varepsilon\}$ is a one parameter group of transformations if the following properties hold:

- T_0 is the identity transformation , i.e. $T_0(x, y) = (x, y)$.
- The set $\{T_\varepsilon\}$ contains the inverse transformation ${T_\varepsilon}^{-1} = T_{-\varepsilon}$, i.e. $T_\varepsilon T_{-\varepsilon} = T_{-\varepsilon} T_\varepsilon = T_0$.
- Composition law, i.e. $T_{\varepsilon_1} \circ T_{\varepsilon_2} = T_{\varepsilon_1 + \varepsilon_2}$.

Example 1.1 *Consider the set of vertical transformations $T_\varepsilon : \Re^2 \to \Re^2$ given by*

$$\begin{aligned} \bar{x} &= x \\ \bar{y} &= y + \varepsilon \end{aligned} \tag{1.8}$$

Show that this set of transformations is a one parameter group.

Solution

- *Clearly $T_0(x, y) = (x, y)$.*
- *The inverse of T_ε is $T_\varepsilon^{-1} = T_{-\varepsilon}$.*

  Because

$$T_\varepsilon \circ T_{-\varepsilon} = T_\varepsilon(T_{-\varepsilon}) = T_\varepsilon(x, y - \varepsilon) = (x, y)$$

and

$$T_{-\varepsilon} \circ T_{\varepsilon} = T_{-\varepsilon}(T_{\varepsilon}) = T_{-\varepsilon}(x, y + \varepsilon) = (x, y)$$

- $T_{\varepsilon_1} \circ T_{\varepsilon_2} = T_{\varepsilon_1}(T_{\varepsilon_2}) = T_{\varepsilon_1}(x, y + \varepsilon_2) = (x, y + \varepsilon_1 + \varepsilon_2) = T_{\varepsilon_1 + \varepsilon_2}$.

So the properties of one parameter group of transformations are holds.

Example 1.2 *The set of transformation* $T_{\varepsilon} : \Re^2 \to \Re^2$ *is defined by*

$$\begin{aligned} \bar{x} &= e^{k_1 \varepsilon} x \\ \bar{y} &= e^{k_2 \varepsilon} y, \end{aligned} \tag{1.9}$$

is a one parameter group.

Solution*:*

- *It is clear that* $T_0(x, y) = (x, y)$. *So* T_0 *is the identity transformation.*

- $T_{\varepsilon} T_{-\varepsilon}(x, y) = T_{-\varepsilon} T_{\varepsilon}(x, y) = (x, y)$, *which implies that the transformation* $T_{-\varepsilon}$ *is inverse of the transformation* T_{ε} .

- $T_{\varepsilon_1} \circ T_{\varepsilon_2} = (e^{k_1 \varepsilon_1}(e^{k_1 \varepsilon_2} x), e^{k_2 \varepsilon_1}(e^{k_2 \varepsilon_2} y)) = (e^{k_1 \varepsilon_1} e^{k_1 \varepsilon_2} x, e^{k_2 \varepsilon_1} e^{k_2 \varepsilon_2} y) = (e^{k_1(\varepsilon_1 + \varepsilon_2)} x, e^{k_2(\varepsilon_1 + \varepsilon_2)} y) = T_{\varepsilon_1 + \varepsilon_2}(x, y)$.

1.2 Infinitesimal generators of one parameter group of transformations

Given a one parameter group of transformations

$$\begin{aligned}\bar{x} &= \varphi(x, y, \varepsilon)\\ \bar{y} &= \psi(x, y, \varepsilon)\end{aligned} \tag{1.10}$$

and let

$$\tau(x, y) = \left.\frac{\partial \varphi}{\partial \varepsilon}\right|_{\varepsilon=0}, \qquad \eta(x, y) = \left.\frac{\partial \psi}{\partial \varepsilon}\right|_{\varepsilon=0} \tag{1.11}$$

then the vector field

$$X = \tau(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y} \tag{1.12}$$

is known as infinitesimal generator of the one parameter group of transformation.

In fact we can represent the functions φ and ψ via their Taylor series expansion with respect the parameter ε in the neighborhood of $\varepsilon = 0$ by

$$\begin{aligned}\bar{x} &= x + \varepsilon\tau(x, y)\\ \bar{y} &= y + \varepsilon\eta(x, y)\end{aligned} \tag{1.13}$$

The form (1.13) is called infinitesimal form of transformations (1.10).

Example 1.3 *Find the infinitesimal generator of the group of* *i) non-uniform scaling*

$$\begin{aligned} \bar{x} &= e^{k_1 \varepsilon} x \\ \bar{y} &= e^{k_2 \varepsilon} \mathrm{y} \end{aligned} \tag{1.14}$$

Solution

We have

$$\varphi(x, y, \varepsilon) = e^{k_1 \varepsilon} x, \quad \psi(x, y, \varepsilon) = e^{k_2 \varepsilon} y \tag{1.15}$$

Therefore

$$\tau(x, y) = \left. \frac{\partial \varphi}{\partial \varepsilon} \right|_{\varepsilon=0} = k_1 x \tag{1.16}$$

$$\eta(x, y) = \left. \frac{\partial \psi}{\partial \varepsilon} \right|_{\varepsilon=0} = k_2 y \tag{1.17}$$

So the infinitesimal generator is

$$X = k_1 x \frac{\partial}{\partial x} + k_2 y \frac{\partial}{\partial y} \tag{1.18}$$

ii) the transformation

$$\bar{x} = \tfrac{x}{1 - \varepsilon y}, \;\; \overline{y} = \tfrac{y}{1 - \varepsilon y} \tag{1.19}$$

Solution

$$\begin{aligned} \tau(x, y) &= \tfrac{\partial \bar{x}}{\partial \varepsilon}\Big|_{\varepsilon=0} = xy \\ \eta(x, y) &= \tfrac{\partial \overline{y}}{\partial \varepsilon}\Big|_{\varepsilon=0} = y^2 \end{aligned} \tag{1.20}$$

Therefore, the associated infinitesimal generator is

$$X = xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y} \tag{1.21}$$

1.3 Determining one parameter group corresponding to infinitesimal generator

Given an infinitesimal generator

$$X = \tau(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}, \tag{1.22}$$

the associated one parameter group of transformations

$$\bar{x} = \varphi(x,y,\varepsilon), \quad \bar{y} = \psi(x,y,\varepsilon) \tag{1.23}$$

with

$$\bar{\varphi}(x,y,0) = x, \quad \psi(x,y,0) = y \tag{1.24}$$

can be found by solving the system of ODEs given by

$$\frac{\partial\,\bar{x}}{\partial\varepsilon} = \tau(x,y) \tag{1.25}$$

$$\frac{\partial\,\bar{y}}{\partial\varepsilon} = \eta(x,y) \tag{1.26}$$

subject to the initial conditions

$$\begin{aligned} \bar{x}|_{\varepsilon=0} &= x \\ \bar{y}|_{\varepsilon=0} &= y. \end{aligned} \tag{1.27}$$

Example 1.4 *Find the one parameter group corresponding to the generator:*

$$X = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} \tag{1.28}$$

Solution

we need to solve the system

$$\begin{aligned} \tfrac{\partial\,\bar{x}}{\partial\,\varepsilon} &= a\bar{x} \\ \tfrac{\partial\,\bar{y}}{\partial\varepsilon} &= b\bar{y} \end{aligned} \tag{1.29}$$

with the initial conditions

$$\begin{aligned} \bar{x}(0) &= x \\ \bar{y}(0) &= y. \end{aligned} \tag{1.30}$$

The solution of the system is given by

$$\begin{aligned} \bar{x} &= c_1\, e^{a\varepsilon} \\ \bar{y} &= c_2\, ye^{b\varepsilon} \end{aligned} \tag{1.31}$$

Applying the initial conditions (1.30) implies

$$\begin{aligned} \bar{x} &= xe^{a\varepsilon} \\ \bar{y} &= ye^{b\varepsilon} \end{aligned} \tag{1.32}$$

1.4 Prolongations of infinitesimal generators

This section presents the notion of prolongation of infinitesimal generator which plays an important role in symmetry analysis of ODEs. We begin with the definition of total differentiation operator.

Definition 1.3 *The total differentiation operator is defined by*

$$\begin{aligned} D_x &= \tfrac{\partial}{\partial x} + \sum_{i=1}^{\infty} y^{(i)} \tfrac{\partial}{\partial y^{(i-1)}} \\ &= \tfrac{\partial}{\partial x} + y^{(1)} \tfrac{\partial}{\partial y} + y^{(2)} \tfrac{\partial}{\partial y^{(1)}} \cdots + y^{(n)} \tfrac{\partial}{\partial y^{(n-1)}} + \cdots \end{aligned} \tag{1.33}$$

For a differentiable function $F(x, y, y^{(1)}, \dots, y^{(n)})$*, its total derivative is given by*

$$D_x F(x, y, y^{(1)}, \dots, y^{(n)}) = \frac{\partial F}{\partial x} + y^{(1)} \frac{\partial F}{\partial y} + y^{(2)} \frac{\partial F}{\partial y^{(1)}} + \cdots + y^{(n)} \frac{\partial F}{\partial y^{(n-1)}}\,. \tag{1.34}$$

The operator

$$X = \tau(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y} \tag{1.35}$$

provides information on how the variables are transformed. i.e. using Taylor's series, the infinitesimal transformation of the corresponding group is given by

$$\begin{aligned} \bar{x} &= x + \varepsilon \tau(x, y) + o(\varepsilon^2) \\ \bar{y} &= y + \varepsilon \eta(x, y) + o(\varepsilon^2). \end{aligned} \tag{1.36}$$

However this information is not enough to apply these operators to study symmetries of ODEs. We also need to know how the derivatives are transformed via the transformation group (1.36) or equivalently the operator (1.35).

1.4.1 First prolongation

Consider the operator

$$X = \tau(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} \tag{1.37}$$

As already mentioned this is equivalently to the infinitesimal transformation

$$\begin{aligned} \bar{x} &= x + \varepsilon\tau(x,y) + o(\varepsilon^2) \\ \bar{y} &= y + \varepsilon\eta(x,y) + o(\varepsilon^2). \end{aligned} \tag{1.38}$$

We want to find the transformation of the first derivative for which we need to extend the symmetry operator to the operator

$$X^{[1]} = \tau(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} + \eta^{[1]}(x,y)\frac{\partial}{\partial y'} \tag{1.39}$$

where $y' = \frac{dy}{dx}$ is transformed to $\bar{y}' = \frac{d\bar{y}}{dx}$ Hence we want to obtain the function $\eta^{[1]}(x,y,y')$ such that

$$\bar{y}' = y' + \varepsilon\eta^{[1]}(x,y,y') + o(\varepsilon^2). \tag{1.40}$$

To find it we proceed as follows:

$$\begin{aligned}\bar{y}' = \tfrac{d\,\bar{y}}{d\,\bar{x}} = \tfrac{dy+\varepsilon d\eta+o(\varepsilon^2)}{dx+\varepsilon d\tau+o(\varepsilon^2)} &= \tfrac{\frac{dy}{dx}+\varepsilon\frac{d\eta}{dx}}{1+\varepsilon\frac{d\tau}{dx}}\\ &= \left(y'+\varepsilon\tfrac{d\eta}{dx}\right)\left(1+\varepsilon\tfrac{d\tau}{dx}\right)^{-1}\\ &= \left(y'+\varepsilon\tfrac{d\eta}{dx}\right)\left(1-\varepsilon\tfrac{d\tau}{dx}+o(\varepsilon^2)\right)\\ &= y'+\varepsilon\left(\tfrac{d\eta}{dx}-y'\tfrac{d\tau}{dx}\right)+o(\varepsilon^2).\end{aligned} \tag{1.41}$$

This implies

$$\begin{aligned}\eta^{[1]}(x,y,y') &= \tfrac{d\eta}{dx}-y'\tfrac{d\tau}{dx}\\ &= \eta_x+y'\left(\eta_y-\tau_x\right)-{y'}^2\tau_y.\end{aligned} \tag{1.42}$$

Thus, the first prolongation is given by

$$X^{[1]} = \tau(x,y)\frac{\partial}{\partial x}+\eta(x,y)\frac{\partial}{\partial y}+\Big(\eta_x+y'\left(\eta_y-\tau_x\right)-{y'}^2\tau_y\Big)\frac{\partial}{\partial y'}. \tag{1.43}$$

1.4.2 Second prolongation

Let

$$X = \tau(x,y)\frac{\partial}{\partial x}+\eta(x,y)\frac{\partial}{\partial y} \tag{1.44}$$

be infinitesimal transformation and

$$X^{[1]} = \tau(x,y)\frac{\partial}{\partial x}+\eta(x,y)\frac{\partial}{\partial y}+\eta^{[1]}(x,y)\frac{\partial}{\partial y'} \tag{1.45}$$

be the its first prolongation. Then, the transformation of x , y and y' are given by

$$\begin{aligned}
\bar{x} &= x + \varepsilon\tau(x,y) + o(\varepsilon^2)\\
\bar{y} &= y + \varepsilon\eta(x,y) + o(\varepsilon^2)\\
\bar{y}' &= y' + \varepsilon\eta'(x,y) + o(\varepsilon^2).
\end{aligned} \tag{1.46}$$

We need to obtain the function $\eta^{[2]}(x,y,y')$ such that

$$\bar{y}'' = y'' + \varepsilon\eta^{[2]}(x,y,y',y'') + o(\varepsilon^2). \tag{1.47}$$

As above we have

$$\begin{aligned}
\bar{y}'' = \tfrac{d\,\bar{y}'}{d\,\bar{x}} = \tfrac{dy' + \varepsilon d\eta^{[1]} + o(\varepsilon^2)}{dx + \varepsilon d\tau + o(\varepsilon^2)} &= \tfrac{y'' + \varepsilon\frac{d\eta^{[1]}}{dx}}{1 + \varepsilon\frac{d\tau}{dx}}\\
&= \left(y'' + \varepsilon\tfrac{d\eta^{[1]}}{dx}\right)\left(1 + \varepsilon\tfrac{d\tau}{dx}\right)^{-1}\\
&= \left(y'' + \varepsilon\tfrac{d\eta^{[1]}}{dx}\right)\left(1 - \varepsilon\tfrac{d\tau}{dx} + o(\varepsilon^2)\right)\\
&= y'' + \left(\varepsilon\tfrac{d\eta^{[1]}}{dx} - y''\tfrac{d\tau}{dx}\right) + o(\varepsilon^2)
\end{aligned} \tag{1.48}$$

which implies

$$\begin{aligned}
\eta^{[2]}(x,y,y') &= \frac{d\eta^{[1]}}{dx} - y''\frac{d\tau}{dx}\\
&= \eta_{xx} + (2\eta_{xy} - \tau_{xx})\,y' + (\eta_{yy} - 2\tau_{xy})\,{y'}^2 - \tau_{yy}{y'}^3 + (\eta_y - 2\tau_x - 3y'\tau_y)\,y''.
\end{aligned}$$

Therefore, the second prolongation is given by

$$X^{[2]} = \tau(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} + \eta^{[1]}\frac{\partial}{\partial y'} + \eta^{[2]}\frac{\partial}{\partial y''} \tag{1.49}$$

1.4.3 Higher order prolongations

Generalizing to higher order, we have

Definition 1.4 *The n^{th} prolongation of the vector field*

$$X = \tau(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} \tag{1.50}$$

is given by

$$X^{[n]} = \tau(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} + \sum_{k=1}^{n} \eta^{(k)} \frac{\partial}{\partial y^{(k)}} \tag{1.51}$$

where

$$\eta^{[i]} = \eta^{[i]}(x, y, y^{(1)}, y^{(2)}, \ldots, y^{(i-1)}) = D_x \eta^{[i-1]} - y^{(i)} D_x \tau \tag{1.52}$$

and $\eta^{[0]} = \eta$.

indeed We can verify formula (1.52) by induction. Indeed if $\eta^{[i-1]}$ is known then relations

$$\begin{aligned} \bar{y}^{(i)} &= \tfrac{d\,\bar{y}^{(i-1)}}{d\,\bar{x}} = \tfrac{dy^{(i-1)} + \varepsilon d\eta^{(i-1)} + o(\varepsilon^2)}{dx + \varepsilon d\tau + o(\varepsilon^2)} \\ &= \left(y^{(i)} + \varepsilon \tfrac{d\eta^{(i-1)}}{dx}\right)\left(1 + \varepsilon \tfrac{d\tau}{dx}\right)^{-1} \\ &= \left(y^{(i)} + \varepsilon \tfrac{d\eta^{(i-1)}}{dx}\right)\left(1 - \varepsilon \tfrac{d\tau}{dx} + o(\varepsilon^2)\right) \\ &= y^{(i)} + \varepsilon \left(\tfrac{d\eta^{(i-1)}}{dx} - y^{(i)} \tfrac{d\tau}{dx}\right) + o(\varepsilon^2) \end{aligned} \tag{1.53}$$

Example 1.5 *Find the second prolongation of*

$$X = y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \tag{1.54}$$

Solution

we have

$$\tau(x,y) = y\,, \quad \eta(x,y) = x \tag{1.55}$$

so

$$\eta^{[1]} = \eta_x + y'\left(\eta_y - \tau_x\right) - {y'}^2\tau_y = 1 - {y'}^2$$

$$\eta^{[2]} = \eta_{xx} + \left(2\eta_{xy} - \tau_{xx}\right)y' - \left(\eta_{yy} - 2\tau_{xy}\right){y'}^2 - \tau_{yy}{y'}^3 + \left(\eta_y - 2\tau_x - 3y'\tau_y\right)y'' = -3y'y''$$

Therefore,

$$X^{[2]} = y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + (1 - y'^2)\frac{\partial}{\partial y'} - 3y'y''\frac{\partial}{\partial y''}$$

1.5 Symmetries of ordinary differential equations

Consider an ODE

$$F(x, y, y^{(1)}, y^{(2)}, \ldots, y^{(n)}) = 0 \tag{1.56}$$

A one parameter group of transformations

$$\bar{x} = \varphi(x, y, \varepsilon), \quad \bar{y} = \psi(x, y, \varepsilon) \tag{1.57}$$

is called a symmetry of ODE (1.56) if the form of the ODE remains unchanged under the transformation (1.57). i.e

$$F(\bar{x}, \bar{y}, \overline{y^{(1)}}, \ldots, \overline{y^{(n)}}) = 0 \tag{1.58}$$

whenever

$$F(x, y, y^{(1)}, \dots, y^{(n)}) = 0 \tag{1.59}$$

Example 1.6 *Show that the transformation*

$$\bar{x} = x + \varepsilon xy \tag{1.60}$$

$$\bar{y} = y + \varepsilon y^2 \tag{1.61}$$

is a symmetry of

$$\frac{dy}{dx} = 0 \tag{1.62}$$

Solution

From (1.60) we have:

$$d\,\bar{x} = \ (1 + \varepsilon y)\, dx + \varepsilon x\, dy. \tag{1.63}$$

and equation (1.61) implies

$$d\,\bar{y} = (1 + 2\varepsilon y)\, dy \tag{1.64}$$

Dividing equation (1.63) over equation (1.64) gives

$$\frac{d\,\bar{y}}{d\,\bar{x}} = \frac{(1 + 2\varepsilon y)\, dy}{(1 + \varepsilon y)\, dx + \varepsilon x\, dy} = \frac{(1 + 2\varepsilon y)\, {}^{dy}\!/_{dx}}{(1 + \varepsilon y)\ + \varepsilon x\, {}^{dy}\!/_{dx}} = 0. \tag{1.65}$$

Therefore transformation $\bar{x} = x + \varepsilon xy$ *and* $\bar{y} = y + \varepsilon y^2$ *is a symmetry of* $\frac{dy}{dx} = 0$.

The above symmetry invariance condition can be written in the operator form in the following manner.

Definition 1.5 *The infinitesimal transformation*

$$\begin{aligned}\bar{x} &= x + \varepsilon\tau(x,y)\\ \bar{y} &= y + \varepsilon\eta(x,y)\end{aligned} \tag{1.66}$$

or the equivalent generator

$$X = \tau(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} \tag{1.67}$$

is a symmetry of ODEs

$$F(x, y, y^{(1)}, \ldots, y^{(n)}) = 0 \tag{1.68}$$

if an only if

$$X^{[n]}F = 0 \tag{1.69}$$

when

$$F = 0. \tag{1.70}$$

where $X^{[n]}$ *is the* n^{th} *prolongation of* X*.*

Example 1.7 *Show that*

$$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \tag{1.71}$$

is a symmetry of the ODE

$$y'' - \frac{1}{x}y' - \frac{1}{x}y'^3 = 0. \tag{1.72}$$

Solution

The associated function for the ODE is

$$F(x,y,y',y'') = y'' - \frac{1}{x}y' - \frac{1}{x}{y'}^3. \tag{1.73}$$

The second prolongation of X is

$$X^{[2]} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - y''\frac{\partial}{\partial y''}. \tag{1.74}$$

Hence,

$$\begin{aligned} X^{[2]}F &= x\tfrac{\partial F}{\partial x} + y\tfrac{\partial F}{\partial y} - y''\tfrac{\partial F}{\partial y''} \\ &= x(\tfrac{y'+{y'}^3}{x^2}) - y'' \\ &= \tfrac{y'+{y'}^3}{x} - \tfrac{y'+{y'}^3}{x} \\ &= 0 \ \text{ when } \ F = 0. \end{aligned} \tag{1.75}$$

So

$$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}. \tag{1.76}$$

is a symmetry of the equation (1.72).

1.6 Procedure for finding symmetries of ODEs

Given an operator

$$X = \tau(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} \tag{1.77}$$

the criteria $X^{[n]}F = 0$ is called the invariance criteria for symmetries of ODE

$$F(x, y, y^{(1)}, \dots, y^{(n)}) = 0 \tag{1.78}$$

and serves as the main tool for finding symmetries of ODEs. This generates an over determined system of linear PDEs in τ and η called the determining system for τ and η. By solving this system One can obtain the symmetries of ODE (1.78).

In general, the steps of finding symmetries of ODE (1.78) are summarized as follows:

1. Find the n^{th} prolongation $X^{[n]}$, where n is the order of the ODE.
2. Apply the invariance criteria $X^{[n]}F|_{F=0} = 0$.
3. Obtain the determining system from step (2) by comparing coefficients of derivatives of y.
4. Simplify and solve the system of determining equations.

The symmetries of different equations form a lie algebra. A brief introduction to lie algebra is provided in the last section of this chapter.

Example 1.8 *Find all symmetries of the equation*

$$y'' = y^{-3}. \tag{1.79}$$

Solution

Assume

$$F(x, y, y', y'') = y'' - y^{-3} \tag{1.80}$$

and the symmetry generator of the

$$X = \tau(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} \tag{1.81}$$

- **Step 1**: *Writing the* 2^{nd} *prolongation*

$$\eta^{[1]} = \eta_x + y'\left(\eta_y - \tau_x\right) - y'^2\tau_y$$

$$\eta^{[2]}(x,y,y') = \eta_{xx} + \left(2\eta_{xy} - \tau_{xx}\right)y' + \left(\eta_{yy} - 2\tau_{xy}\right)y'^2 - \tau_{yy}y'^3 + \left(\eta_y - 2\tau_x - 3y'\tau_y\right)y''$$

So the 2^{nd} *prolongation of* X *is:*

$$X^{[2]}(F) = \tau(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} + \eta^{[1]}\frac{\partial}{\partial y'} + \eta^{[2]}\frac{\partial}{\partial y''}$$

- **Step 2**: *Applying the invariance criteria* $X^{[2]}(F)\big|_{F=0} = 0$ *implies*

$$3\eta y^{-4} + \eta_{xx} + (2\eta_{xy} - \tau_{xx})y' - (\eta_{yy} - 2\tau_{xy})y'^2 - \tau_{yy}y'^3 + (\eta_y - 2\tau_x - 3y'\tau_y)y^{-3} = 0 \tag{1.82}$$

- **Step 3**: *Finding the determining equations as follows:*

$$\text{coefficient of}(y')^3 \quad : \ \tau_{yy} = 0 \tag{1.83}$$

$$\text{coefficient of}(y')^2 \quad : \eta_{yy} - 2\tau_{xy} = 0 \tag{1.84}$$

$$\text{coefficient of}(y')^1 \quad : \ -3\tau_y + 2\eta_{xy}\, y^3 - \tau_{xx}\, y^3 = 0 \tag{1.85}$$

$$\text{coefficient of}(y')^0 \quad : \ 3\eta - 2y\tau_x \, + \, \eta_y y + \, \eta_{xx} y^4 = 0\,. \tag{1.86}$$

- **Step 4**: *Solving the over determined system of linear PDEs (1.83)-(1.86)*
  - *From equation (1.83),* τ *is a linear in* y*. So*

$$\tau = f_1(x)y + f_2(x). \tag{1.87}$$

  - *Substituting the expression (1.87) in equation, (1.84),*

$$\eta = {f_1}'(x)y^2 + f_3(x)y + f_4(x). \tag{1.88}$$

  - *Substituting equations (1.87) and (1.88) in equation (1.85) gives*

$$2{f_3}'(x) - {f_2}''(x) = 0 \tag{1.89}$$

and

$$f_1(x) = 0. \tag{1.90}$$

Hence the τ and η become respectively

$$\tau = f_2(x) \tag{1.91}$$

$$\eta = f_3(x)y + f_4(x). \tag{1.92}$$

- *Substituting equations (1.91) and (1.92) in equation (1.86) gives*

$$f_2(x) = c_1x^2 + 2c_2x + c_3 \tag{1.93}$$

$$f_3(x) = c_1x + c_2 \tag{1.94}$$

and

$$f_4(x) = 0. \tag{1.95}$$

- *Hence, the final formulas are*

$$\tau = c_1x^2 + 2c_2x + c_3 \tag{1.96}$$

$$\eta = (c_1x + c_2)\, y. \tag{1.97}$$

Therefore, we obtain 3 independent parameters which provide 3 dimensional Lie algebra generated by the symmetries found as follows:

1. *If* $c_1 = 1$ *and the other constants vanish implies then* $X_1 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}$.
2. *If* $c_2 = 1$ *and the other constants vanish implies then* $X_2 = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$.

3. If $c_3 = 1$ *and the other constants vanish implies then* $X_3 = \frac{\partial}{\partial x}$.

Example 1.9 *Find all symmetries of the Blausius equation*

$$y''' + yy'' = 0. \tag{1.98}$$

Solution

Let $F(x, y, y', y'', y''') = y''' + yy''$ *and let the symmetry be of the form*

$$X = \tau(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y}. \tag{1.99}$$

Then the invariance criteria

$$X^{[3]}(F)\big|_{F=0} = 0 \tag{1.100}$$

gives

$$\tau(x, y)\frac{\partial F}{\partial x} + \eta(x, y)\frac{\partial F}{\partial y} + \eta^{[1]}\frac{\partial F}{\partial y'} + \eta^{[2]}\frac{\partial F}{\partial y''} + \eta^{[3]}\frac{\partial F}{\partial y'''} = 0 \tag{1.101}$$

or equivalently

$$\tau(x, y).\,(0) + \eta(x, y)y'' + \eta^{[1]}.\,(0) + \eta^{[2]}y + \eta^{[3]} = 0. \tag{1.102}$$

Substituting the expressions for the second and the third prolongation into (1.102), *and gathering coefficients of derivatives of y gives the following determining equations*

$$\text{Coefficient of } (y'')^2 \ : \ \ \tau_y = 0 \tag{1.103}$$

$$\text{Coefficient of } ({y'}^2 y'') \ : \ \ \tau_{yy} = 0 \tag{1.104}$$

$$\text{Coefficient of } (y'y'') \ : \ \ \eta_{yy} = 0 \tag{1.105}$$

$$\text{Coefficient of } (y')^3 \ : \ \ \eta_{yyy} = 0 \tag{1.106}$$

$$\text{Coefficient of } (y')^0 \ : \ \ y\eta_{xx} + \eta_{xxx} = 0 \tag{1.107}$$

$$\text{Coefficient of } (y') \ : \ \ 2y\eta_{xy} - y\tau_{xx} + 3\eta_{xxy} - \tau_{xxx} = 0 \tag{1.108}$$

$$\text{Coefficient of } (y'') \ : \ \ 3\eta_{xy} - 3\tau_{xx} + \eta + y\tau_x = 0. \tag{1.109}$$

Now to derive the corresponding symmetry we solve the over determined system of linear PDEs (1.103)-(1.109).

- *From eq(1.103) we get :*

$$\tau = f_1(x) \tag{1.110}$$

- *From eq(1.105) we get :*

$$\eta = f_2(x)y + f_3(x). \tag{1.111}$$

- *Using equation (1.111) in equation (1.107) gives:*

$$f_2{}''(x)y^2 + f_3{}''(x)y + f_2{}'''(x) + f_3{}'''(x) = 0. \tag{1.112}$$

This implies

$$f_2(x) = c_1 x + c_2 \tag{1.113}$$

and

$$f_3(x) = c_3 x + c_4. \tag{1.114}$$

- *Substituting* $f_2(x)$ *and* $f_3(x)$ *in (1.108) gives*

$$\tau = c_1 x^2 + c_5 x + c_6. \tag{1.115}$$

- *By substituting equations (1.113)-(1.115) in (1.109), we conclude that*

$$c_1 = c_3 = c_4 = 0 \tag{1.116}$$

$$c_2 + c_5 = 0. \tag{1.117}$$

So we have

$$\tau = c_5 x + c_6 \tag{1.118}$$

$$\eta = -c_5 y. \tag{1.119}$$

Hence the associated symmetry algebra is generated by:

$$X_1 = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} \tag{1.120}$$

$$X_2 = \frac{\partial}{\partial x}. \tag{1.121}$$

Example 1.10 *Consider the equation*

$$y'' = \frac{y'^2}{y} + f'(x)y^{p+1} + pf(x)y'y^p \tag{1.122}$$

where $p \neq 0$ is a real constant and $f \neq 0$ is an arbitrary function of the independent variable x. If we apply the second prolongation for (1.122) we get the following determining system

$$\begin{aligned}
&y\tau_{yy} + \tau_y = 0\\
&-2\tau_y\, pf\,(x)\, y^{p+2} + (\eta_{yy} - 2\tau_{xy})\, y^2 + \eta - y\eta_y = 0\\
&-\eta p^2 f\,(x)\, y^p - (\tau_x pf\,(x) + \tau pf'\,(x))\, y^{p+1} + 2\eta_{xy}y - \tau_{xx}y - 2\eta_x - 3\tau_y f'\,(x)\, y^{p+2} = 0\\
&(-\eta f'\,(x)\, p - \eta f'\,(x))\, y^p + (-2\tau_x\,(x,y)\, f'\,(x) + \eta_y f'\,(x) - \tau f''\,(x))\, y^{p+1} - \eta_x pf\,(x)\, y^p + \eta_{xx} = 0
\end{aligned} \tag{1.123}$$

and the solution of this system is $\tau(x,y) = \eta(x,y) = 0$, This equation does not possess Lie point symmetries for general $f(x)$.

In the next chapter we introduce the concept of λ-symmetries and show that ODE (1.122) admit $\lambda-$symmetry that can be used to find its solution

1.7 Reduction of higher order ODEs using symmetry method

One of the main applications of Lie symmetry is to reduce the order of differential equations. The reduction of order of ODE

$$F(x, y, y', \ldots, y^{(n)}) = 0 \tag{1.124}$$

using the symmetry

$$X = \tau(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y} \tag{1.125}$$

is based on the a standard procedure of change of variables by obtaining the invariants through

$$X^{[n]}F = 0. \tag{1.126}$$

This method is illustrated in the example below.

Example 1.11 *Find the solution of ODE*

$$x^2y'' - xy' + y = 0 \tag{1.127}$$

by reducing the order via the symmetry

$$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \tag{1.128}$$

Solution

Let

$$F = x^2y'' - xy' + y = 0 \tag{1.129}$$

The 2^{nd} *prolongation of* X *is*

$$X^{[2]} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 0\frac{\partial}{\partial y'} - y''\frac{\partial}{\partial y''}. \tag{1.130}$$

Then the characteristic system of the invariants obtained through $X^{[2]}F = 0$ *is*

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dy'}{0} = \frac{dy''}{-y''} \tag{1.131}$$

- *Solving* $\frac{dx}{x} = \frac{dy}{y}$ *implies* $\frac{y}{x} = \text{constant}$,
- *Solving* $\frac{dy}{y} = \frac{dy'}{0}$ *implies* $y' = \text{constant}$.

So we get two invariants of $X^{[2]}$ *given by:*

$$u = \frac{y}{x} \tag{1.132}$$

$$v = y'. \tag{1.133}$$

Hence

$$x^2y'' = x^2\frac{dy'}{dx} = x^2\frac{dv}{dx} = x^2\frac{dv}{du}\frac{du}{dx} = x^2\frac{dv}{du}\frac{x\frac{dy}{dx} - y}{x^2} = (xv - y)\frac{dv}{du}. \tag{1.134}$$

By rewriting the equation 1.127 in terms of u and v we get

$$(xv - y)\frac{dv}{du} - xv + y = 0. \tag{1.135}$$

Dividing equation (1.135) by x gives the reduced first order ODE

$$(v - u)\frac{dv}{du} - v + u = 0. \tag{1.136}$$

Hence,

$$\frac{dv}{du} = \frac{v - u}{v - u} = 1 \tag{1.137}$$

which gives

$$v = u + c_1. \tag{1.138}$$

Going back to original variables we have

$$\frac{dy}{dx} = \frac{y}{x} + c_1 \tag{1.139}$$

which gives the solution

$$y(x) = x\,(c_1 \ln x + c_2.) \tag{1.140}$$

1.8 Lie algebra of vector fields

This section provides a brief introduction to lie algebras. It is a well known fact that if X and Y are vector fields and f, g, h are functions, then

- $(fX + gY)(h) = fX(h) + gY(h)$

- $X(af + bg) = aX(f) + bX(g)$

- $X(fg) = X(f)g + fX(g)$

Let us define a very important operation on the set of vector fields.

Definition 1.6 *The commutator Lie bracket or simply Lie bracket* $[X, Y]$*, of two vector fields* X *and* Y *is the vector field such that*

$$[X, Y](f) = X(Y(f)) - Y(X(f)). \tag{1.141}$$

The Lie bracket of vector fields $[X, Y] = XY - YX$ satisfies the following property:

- $[X, X] = 0$

- $[X, Y] = -[Y, X]$

- $[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0$

- for functions f and g : $[f\,X, g\,Y] = f\,g\,[X, Y] + f\,X(g)Y - gY(f)X$

Example 1.12 *Compute* $[X, Y]$ *for* $X = y\frac{\partial}{\partial y},\ Y = y\frac{\partial}{\partial x}$.

Solution

$$\left[y\frac{\partial}{\partial y},\ y\frac{\partial}{\partial x}\right] = y\frac{\partial}{\partial y}\left(y\frac{\partial}{\partial x}\right) - y\frac{\partial}{\partial x}\left(y\frac{\partial}{\partial y}\right) = y\frac{\partial}{\partial x} + y^2\frac{\partial^2}{\partial y\partial x} - y^2\frac{\partial^2}{\partial x\partial y} = y\frac{\partial}{\partial x} = Y.$$

Example 1.13 *Find all functions $\phi(x, y)$ such that*

$$\left[\phi\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right] = 0$$

Solution

$$\left[\phi\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right](f) = \phi\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) - \frac{\partial}{\partial y}\left(\phi\frac{\partial f}{\partial x}\right) = \phi\frac{\partial^2 f}{\partial x \partial y} - \phi\frac{\partial^2 f}{\partial y \partial x} - \frac{\partial \phi}{\partial y}\frac{\partial f}{\partial x} = -\frac{\partial \phi}{\partial y}\frac{\partial f}{\partial x} = -\left[\frac{\partial \phi}{\partial y}\frac{\partial}{\partial x}\right](f)$$

Hence, $\left[\phi\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right] = 0$ *if and only if* $\frac{\partial \phi}{\partial y} = 0$. *Therefore* $\phi = \phi(x)$.

Definition 1.7 *Let Ł be a finite dimensional vector space of vectors. The vector space Ł is called a Lie algebra if it is closed under the Lie bracket.*

i.e

$$[X, Y] \in L \quad \forall X, Y \in L. \tag{1.142}$$

Definition 1.8 *A subspace l of Ł is a subalgebra of Ł if it is closed under Lie bracket.*

i.e.

$$[X, Y] \in l \quad \forall X, Y \in l \tag{1.143}$$

If the vectors $X_1, X_2, \ldots, X_n$ are a basis of Ł (as a vector space), then Ł is called n-dimensional lie algebra generated by $X_1, X_2, \ldots, X_n$ and is denoted as $\langle X_1, X_2, \ldots, X_n \rangle = \text{Ł}^n$.

Definition 1.9 *A Lie algebra L is called abelian if*

$$[X, Y] = 0 \quad \forall\, X, Y \in L \tag{1.144}$$

Example 1.14 *Find the commutator table for* $L = \langle X_1, X_2, X_3, X_4 \rangle$ *where*

$$\begin{aligned} X_1 &= x^2 \tfrac{\partial}{\partial x} + xy \tfrac{\partial}{\partial y} \\ X_2 &= xy \tfrac{\partial}{\partial x} \ + y^2 \tfrac{\partial}{\partial y} \\ X_3 &= x \tfrac{\partial}{\partial x} \\ X_4 &= y \tfrac{\partial}{\partial x}. \end{aligned} \tag{1.145}$$

Solution

	X_1	X_2	X_3	X_4
X_1	0	0	$-X_1$	$-X_2$
X_2	0	0	0	0
X_3	X_1	0	0	$-X_4$
X_4	X_2	0	X_4	0

Table 1.1: Commutator table for the vector fields (1.145).

Example 1.15 *The commutator table for the infinitesimal generators of examples (1.8) and (1.9) are respectively formalized*

	X_1	X_2	X_3
X_1	0	$-2X_1$	$-X_2$
X_2	$2X_1$	0	$-2X_3$
X_3	X_2	$2X_3$	0

Table 1.2: Commutator table for the infinitesimal generators of example (1.8).

	X_1	X_2
X_1	0	$-X_2$
X_2	X_2	0

Table 1.3: Table for the Infinitesimal Generators of Example (1.9).

Definition 1.10 *A subalgebra* $\text{Ł}^r \subset \text{Ł}^n$, *where* $r < n$ *is called an ideal of* Ł^r *if* $[X, Y] \in \text{Ł}^r$ *for all* $X \in \text{Ł}^r$ *and* $Y \in \text{Ł}^n$, *.*

Definition 1.11 *The Lie algebra* Ł^r *is* $r-$*dimensional solvable Lie algebra if there exists a chain of subalgebras*

$$\text{Ł}^0 \subset \text{Ł}^1 \subset \text{Ł}^2 ... \subset \text{Ł}^{r-1} \subset \text{Ł}^r \tag{1.146}$$

such that Ł^k *is a k-dimensional Lie algebra and* Ł^{k-1} *is an ideal subalgebra of* Ł^k *for* $k = 1, 2, ..., r$. *Here* Ł^0 *is the null ideal consisting of only the zero vector.*

We close this section with two fundamental results for solvable lie algebra .

Theorem 1.16 *Every Abelian Lie algebra is solvable.*

Theorem 1.17 *Every two-dimensional Lie algebra is solvable.*

The reader is referred to [19] for the proofs of the results stated in this section as well as for comprehensive introduction to the subject of lie algebras.

CHAPTER 2

λ-SYMMETRIES AND FIRST INTEGRALS

The main aim of this chapter is to present the basic ideas of λ-symmetries required for the investigations carried out in chapter 3. Section (2.1) provides the notions of λ-prolongations and λ-symmetries, and outlines an algorithmic procedure for constructing λ-symmetries. Examples, demonstrating the implementation of this procedure, are also presented. section (2.2) consists of a discussion of the relation between λ-symmetries, first integral and integrating factors. The remaining sections show, with the aid of examples, how to find first integrals of second order ODEs with known λ-symmetry .

2.1 λ-Symmetries for ODEs

In this section we present the basic concept of λ-symmetries, the associated notation of prolongations of these vector fields as well as a procedure for finding λ-symmetries.

Muriel and Romero in [32] introduced the concept of λ-symmetries to justify the existence of some special cases of reduction for ODE's. Pucci and Saccomandsi proved in [44] that the λ-symmetries are the largest class of vector fields allowing to reduce the order of an ODE. In [36] it has been proved that second order differential equations with first integrals possess an equivalence condition between λ-symmetries depending on the Lie point symmetry coefficients. Nucci and Levi in [41] used the Jacobi last multiplier to obtain the function λ for λ-symmetries. A large amount of literature about the λ-symmetry theory and its applications a is available, e.g. [3,12,13,23,34,42].

Definition 2.1 *[32] Given the ODE*

$$y^{(n)} = f(x, y, y', y'', \dots, y^{(n-1)}). \tag{2.1}$$

Let $X = \tau(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}$ *be a vector field, and let* $\lambda(x, y, y', ...y^{(n-1)})$ *be an arbitrary function. The* λ*-prolongation of order* n *of* X*, denoted by* $X^{[n,\lambda]}$*, is the vector field defined by*

$$X^{[n,\lambda]} = \tau(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} + \sum_{k=1}^{n} \sigma^{[k]} \frac{\partial}{\partial y^{(k)}}, \tag{2.2}$$

where

$$\sigma^{[k+1]} = (D_x + \lambda)\sigma^{[k]} - y^{[k+1]}(D_x + \lambda)\tau \tag{2.3}$$

for $1 \le k \le n$*.*

Note that when $\lambda = 0$, the prolongation $X^{[0,\lambda]}$ is the usual Lie symmetry prolongation $X^{[n]}$.

Definition 2.2 *[32] We will say that a vector field X is a λ-symmetry of the ODE (2.1) if there exists a function λ such that*

$$X^{[n,\lambda]}(y^{(n)} - f)|_{y^{(n)}=f} = 0. \tag{2.4}$$

An efficient method for constructing is based on following theorem

Theorem 2.1 *[33] Let $y^{(n)} = f(x, y, y', ..., y^{(n-1)})$ be a n^{th} order differential equation, then there exist a function $\lambda(x, y, y', ..., y^{(k)})$, for some $k < n$ such that the vector field $X = \frac{\partial}{\partial y}$ is a λ-symmetry of the equation.*

The investigations of this thesis involve second order ODEs of the form

$$y'' = f(x, y, y') \tag{2.5}$$

Specializing to these second order ODEs, the λ prolongation of $X = \frac{\partial}{\partial y}$ for equation (2.5) can be derived as follows:

$$\sigma^{[0]} = 1$$

$$\sigma^{[1]} = (D_x + \lambda)1 = \lambda$$

$$\sigma^{[2]} = (D_x + \lambda)\lambda = \lambda_x + y'\lambda_y + y''\lambda_{y'} + \lambda^2$$

where $\lambda = \lambda(x, y, y'.)$ Now applying the invariance condition (2.4) for equation (2.5) gives

$$X^{[2,\lambda]}(y'' - f(x, y, y'))|_{y''=f} = \sigma^{[2]} - f_{y'}\sigma^{[1]} - f_y = -f_y - \lambda f_{y'} + \lambda_x + y'\lambda_y + f\lambda_{y'} + \lambda^2 = 0.$$

We can summarize the above discussion in the following corollary

Corollary 2.2 *The vector field $X = \frac{\partial}{\partial y}$ is a λ-symmetry of equation $y'' = f(x, y, y')$ if and only if λ is a solution of the equation*

$$f_y + \lambda f_{y'} = \lambda_x + y'\lambda_y + f\lambda_{y'} + \lambda^2. \tag{2.6}$$

Hence the λ-symmetry $X = \frac{\partial}{\partial y}$ of equation $y'' = f(x, y, y')$ for some function $\lambda(x, y, y')$ can be found by the following algorithmic procedure:
By theorem (2.1), equation $y'' = f(x, y, y')$ admits the vector field $X = \frac{\partial}{\partial y}$ as a λ-symmetry. By corollary (2.2) λ is any particular solution to the equation (2.6), and for simplicity we consider the form $\lambda(x, y) = \alpha(x, y)y' + \beta(x, y)$ to determine λ. This procedure is implemented in the examples below.

Example 2.3 *The second order ODE*

$$y'' - \frac{y'^2}{y} - \frac{x^2 + x}{y}y' + 2x + 1 = 0 \tag{2.7}$$

does not admit Lie point symmetries, but we show that it has λ-Symmetry. By Theorem (2.1) equation (2.7) admits the vector field $X = \frac{\partial}{\partial y}$ as a λ-symmetry. Moreover *by corollary (2.2) λ is any particular solution to the following equation*

$$\begin{aligned}&(-2xy^2 - y^2 + yy'^2 + x^2yy' + xyy')\lambda_{y'} + y^2y'\lambda_y + y^2\lambda_x\\&+\lambda^2y^2 - (x^2y + xy + 2yy')\lambda + x^2y' + xy' + y'^2 = 0\end{aligned} \tag{2.8}$$

For simplicity assume the solution λ of (2.8) of the form $\lambda(x, y, y') = \alpha(x, y)y' + \beta(x, y)$. This anasatz leads to the following system

$$\text{Coefficients of } \mathrm{y}'^2 : \quad \alpha^2y^2 + \alpha_yy^2 - \alpha y + 1 = 0 \tag{2.9}$$

$$\text{Coefficients of } \mathrm{y}' : \quad 2\alpha\beta y^2 + \beta_yy^2 + \alpha_xy^2 - 2\beta y + x^2 + x = 0 \tag{2.10}$$

$$\text{Coefficients of } (\mathrm{y}')^0 : \quad \beta^2y^2 - 2x\alpha y^2 - \alpha y^2 + \beta_xy^2 - x^2\beta y - x\beta y = 0 \tag{2.11}$$

It is clear that (2.9) has a particular solution $\alpha(x, y) = \frac{1}{y}$, so, by substituting α, the two remaining equations become:

$$\beta_yy^2 + x^2 + x = 0 \tag{2.12}$$

$$\beta^2y^2 + \beta_xy^2 - 2xy - x^2\beta y - x\beta y - y = 0 \tag{2.13}$$

The general solution of the equation (2.12) is given by

$$\beta(x, y) = \frac{x^2}{y} + \frac{x}{y} + h(x)$$

substituting β *in the equation (2.13) gives*

$$\left(h(x)^2 + h'(x)\right) y^2 + \left(h(x)x^2 + h(x)x\right) y = 0$$

So we can choose $h(x) = 0$. *Therefore* $X = \frac{\partial}{\partial y}$ *is a* λ*-symmetry for* $\lambda = \frac{y'+x^2+x}{y}$.

Example 2.4 *Consider the differential equation*

$$y'' = \frac{y'^2}{y} + f'(x)y^{p+1} + pf(x)y'y^p \tag{2.14}$$

where $p \neq 0$ *is a real constant and* $f \neq 0$ *is an arbitrary function of the independent variable* x. *By Theorem (2.1) equation (2.14) admits the vector field* $X = \frac{\partial}{\partial y}$ *as a* λ*-symmetry. Taking* λ *of the form* $\lambda(x, y, y') = \alpha(x, y)y' + \beta(x, y)$, *leads to the following system*

$$\alpha_y y^2 + \alpha^2 y^2 - \alpha y + 1 = 0 \tag{2.15}$$

$$p^2 f(x)y^p + 2\beta - \alpha_x y - \beta_y y - 2\alpha\beta y = 0 \tag{2.16}$$

$$pf'(x)y^p + f'(x)y^p + \beta pf(x)y^p - \beta_x + \alpha f'(x)y^{p+1} + \beta^2 = 0 \tag{2.17}$$

A particular solution of the equation (2.15) is given by $\alpha = \frac{1}{y}$. *Substituting* $\alpha = \frac{1}{y}$ *the equation (2.16) becomes:*

$$p^2 f(x)y^p - \beta_y y = 0 \tag{2.18}$$

which implies

$$\beta = pf(x)y^p + h(x) \tag{2.19}$$

Substituting α and β in the equation (2.17) provides

$$h(x) + pf(x)h(x)y^p + h^2(x)$$

So we can choose the $h(x) = 0$. Therefore $X = \frac{\partial}{\partial y}$ is a λ-symmetry for $\lambda = \frac{y' + pf(x)y^{p+1}}{y}$.

2.2 First Integrals and λ-symmetries of second order ODEs

Sometimes it might not be easy to completely integrate an ODE, however it could be possible to write the solution of differential equations explicitly in terms of elementary functions that are constant on every solution of differential equation, that is, first integral. In the other word, we have differential equation of order n, and if by integrability an equation of order $(n-1)$ involving an arbitrary constant is obtained, then the latter is known as the first integral of the given equation.

Definition 2.3 *[2] A first integral of an nth-order ODE*

$$y^{(n)} = f(x, y, y', \dots, y^{(n-1)}) \tag{2.20}$$

is a function $w(x, y, y', \dots, y^{(n-1)})$ with essential dependence on $y^{(n-1)}$ satisfying

$$D_x(w) = 0 \quad \textit{when } y^{(n)} = f$$

i.e, $w(x, y, y', \ldots, y^{(n-1)})$ *is constant for every solution* y *of ODE (2.20).*

In fact, finding a first integral of a given ODE is equivalent to obtaining a function called an integrating factor.

Definition 2.4 *[2] An integrating factor of an nth-order ODE (2.20) is a non zero function* $\Lambda(x, y, y', \ldots, y^{(l)})$, $0 \leq l \leq n-1$ *that satisfies*

$$\Lambda(x, y, y', \ldots, y^{(l)}) \left(y^{(n)} - f(x, y, y', \ldots, y^{(n-1)})\right) = D_x \left(w(x, y, y', \ldots, y^{(n-1)})\right)$$

for some function $w(x, y, y', \ldots, y^{(n-1)})$ *which has an essential dependence on* $y^{(n-1)}$*. The highest-order* l *of the derivatives of* y *in* $\Lambda(x, y, y', \ldots, y^{(l)})$ *is called the order of the integrating factor.*

Next theorem provides a well known relation between first integrals and integrating factors.

Theorem 2.5 *[34] If* $w(x, y, y')$ *is a first integral of equation*

$$y'' = f(x, y, y') \tag{2.21}$$

then $\Lambda = w_{y'}$ *is an integrating factor of (2.21). Conversely, if* $\Lambda(x, y, y')$ *is an integrating factor of (2.21), then there exist a first integral* $w(x, y, y')$ *of (2.21) such that* $\Lambda = w_{y'}$*.*

Proof. Assume $w(x, y, y')$ is a first integral of equation (2.21), then

$$D_x(w)|_{y''=f} = w_x + y' w_y + f(x, y, y') w_{y'} = 0$$

Therefore

$$w_x + y'w_y = -f(x, y, y')w_{y'}$$

But

$$D_x(w) = w_x + y'w_y + y''w_{y'} = -f(x, y, y')w_{y'} + y''w_{y'} = w_{y'}(y'' - f(x, y, y'))$$

So $\Lambda = w_{y'}$ is an integrating factor.

Conversely, assume that $\Lambda(x, y, y')$ is an integrating factor of (2.21), then for some function $w(x, y, y')$

$$\Lambda(y'' - f(x, y, y')) = D_x(w) \tag{2.22}$$

Hence

$$\Lambda(y'' - f(x, y, y')) = w_x + y'w_y + y''w_{y'}$$

then necessarily implies

$$\Lambda = w_{y'} \tag{2.23}$$

and

$$-\Lambda f(x, y, y') = w_x + y'w_y$$

This proves that

$$D_x(w) = w_x + y'w_y + f(x, y, y')w_{y'} = 0 \tag{2.24}$$

Therefore by (2.23) and (2.24), $w(x, y, y')$ is a first integral of equation (2.21), such that $\Lambda = w_{y'}$. ▮

The following theorem establishes the significant connection between λ-symmetries and integrating factors of second order ODEs. This relation will play an important role in obtaining results of chapter 3.

Theorem 2.6 *[34] If $I(x,y,y')$ is a first integral of (2.21), then the vector field $X = \frac{\partial}{\partial y}$ is a λ-symmetry of (2.21) for $\lambda = -\frac{I_y}{I_{y'}}$ and $X^{[\lambda,1]}(I) = 0$. Conversely, If $X = \frac{\partial}{\partial y}$ is a λ-symmetry of (2.21) for some function $\lambda(x,y,y')$, then there exists a first integral $I(x,y,y')$ of (2.21) such that $X^{[\lambda,1]}(I) = 0$.*

2.3 Reduction of second-order ODEs using λ symmetries

If we know that $\frac{\partial}{\partial y}$ is a λ-symmetry of the ODE

$$y'' = f(x,y,y') \tag{2.25}$$

for some known function $\lambda(x,y,y')$, then a procedure to find a first integral $I(x,y,y')$ of (2.25), and consequently an integrating factor of that equation is as follows:

1. Find invariant $w(x,y,y')$ of $X^{[\lambda,1]}$, i.e a particular solution of the equation

$$w_y + \lambda w_{y'} = 0. \tag{2.26}$$

2. Evaluate $D_x(w)|_{y''=f}$ and express it in terms of (x,w) as $D_x(w)|_{y''=f} = F(x,w)$.

3. Find invariant $G(x, w)$ of $\frac{\partial}{\partial x} + F(x, w)\frac{\partial}{\partial w}$.

4. Evaluate $I(x, y, y') = G(x, w(x, y, y'))$.

Then $I(x, y, y')$ is a first integral of (2.25)and $\mu(x, y, y') = I_{y'}$ is an integrating factor of (2.25).

Example 2.7 *For the ODE*

$$y'' - \frac{y'^2}{y} - \frac{x^2 + x}{y}y' + 2x + 1 = 0 \tag{2.27}$$

we have shown in the example(2.3) that the equation (2.27) has $X = \frac{\partial}{\partial y}$ *as a* λ*-symmetry with* $\lambda = \frac{y'+x^2+x}{y}$.

- *We want to find a first order invariant* $w(x, y, y')$ *of* $X^{[\lambda,1]}$. *Equation(2.26) gives*

$$w_y + \frac{y' + x^2 + x}{y}w_{y'} = 0 \tag{2.28}$$

  which has a solution

$$w(x, y, y') = \frac{y' + x^2 + x}{y}$$

- *Now*

$$\begin{aligned} D_x(w)|_{y''=f} &= w_x + y'w_y + y''w_{y'} \\ &= \tfrac{2x+1}{y} - y'(\tfrac{y'+x^2+x}{y^2}) + \tfrac{1}{y}(y'') = 0 \end{aligned}$$

  So $F(x, w) = 0$.

- *Invariant* G *of* $\frac{\partial}{\partial x}$ *is* $G(x, w) = G(w)$.

- *The first integral I is defined as*

$$I(x,y,y') = G\left(\frac{y'+x^2+x}{y}\right)$$

for simplicity we choose $I(x,y,y') = \frac{y'+x^2+x}{y}$, hence $\mu(x,y,y') = I_{y'} = \frac{1}{y}$ is an integrating factor of (2.27).

Moreover, from definition of first integral we conclude that $I(x,y,y') = c\,, c \in \Re$. Therefore equation(2.27) is reduced to the equation

$$y' - cy + x^2 + x = 0. \tag{2.29}$$

We observe that

- *If $c = 0$ then the equation (2.29) has the solution*

$$y = -\frac{x^3}{3} - \frac{x^2}{2} + c_1$$

- *For $c \neq 0$, the first order linear equation (2.29) can be solved as follows: Multiplying both sides by the integrating factor e^{-cx} gives*

$$(ye^{-cx})_x = -(x^2+x)e^{-cx} \tag{2.30}$$

Integrate both sides of equation (2.30)

$$\int (e^{-cx}y)_x dx = -\int x^2 e^{-cx} dx - \int x e^{-cx} dx \tag{2.31}$$

gives the general solution

$$y = c_1 e^{cx} + \frac{x^2}{c} + \frac{2x}{c^2} + \frac{2}{c^3} + \frac{x}{c} + \frac{1}{c^2}.$$

Example 2.8 *We have verified in example (2.4) that the ODE*

$$y'' = \frac{y'^2}{y} + f'(x)y^{p+1} + pf(x)y'y^p \tag{2.32}$$

admits the vector field $X = \frac{\partial}{\partial y}$ *as a* λ*-symmetry for* $\lambda = \frac{y'+pf(x)y^{p+1}}{y}$.

- *We want to find a first order invariant* $w(x, y, y')$ *of* $X^{[\lambda,1]}$. *Equation (2.26) gives*

  $$w_y + \frac{y' + pf(x)y^{p+1}}{y} w_{y'} = 0 \tag{2.33}$$

  It is clear that $w(x, y, y') = \frac{y'-f(x)y^{p+1}}{y}$ *is a solution of (2.33).*

- *From*

  $$\begin{aligned} D_x(w)|_{y''=f} &= w_x + y'w_y + y''w_{y'} \\ &= -f'(x)y^{(p+1)} - y'(\tfrac{y'}{y^2} + pf(x)y^{p-1}) + \tfrac{y''}{y} = 0 \end{aligned}$$

  we got $F(x, w) = 0$.

- *Invariant* G *of* $\frac{\partial}{\partial x}$ *is* $G(x, w) = G(w)$.

- *The first integral I is defined as*

$$I(x,y,y') = G\left(\frac{y' - f(x)y^{p+1}}{y}\right)$$

For simplicity we consider $I(x,y,y') = \frac{y'-f(x)y^{p+1}}{y}$, *then* $\mu(x,y,y') = I_{y'} = \frac{1}{y}$.

Since $I(x,y,y') = c\,, c \in \Re$, *the equation (2.32) is reduced to the Bernoulli equation*

$$y' - f(x)y^{p+1} - cy = 0 \tag{2.34}$$

To solve this equation set $w = y^{-p}$, *so that equation (2.34) can be written as*

$$w' + cpw = -pf(x) \tag{2.35}$$

The linear equation (2.35) has the solution

$$w = -e^{-cpx}\left(\int p\,f(x)e^{cpx}dx + d\right) \tag{2.36}$$

Therefore the general solution of (2.14) is given by

$$y(x) = \frac{e^{cx}}{\left(\int -pf(x)e^{cpx}dx + d\right)^{1/p}}.$$

2.4 An Application: Modified Emden Type Equation with additional linear forcing

Let us consider the modified Emden type equation with additional linear forcing [38]:

$$\ddot{x} + 3kx\dot{x} + k^2x^3 + c^2x = 0. \tag{2.37}$$

By Theorem (2.1) equation (2.37) admits the vector field $X = \frac{\partial}{\partial x}$ as a λ-symmetry. Also for $\lambda(t, x, \dot{x}) = \alpha(t, x)\dot{x} + \beta(t, x)$, Corollary (2.2) leads to the following system

$$\alpha_x + \alpha^2 = 0 \tag{2.38}$$

$$\alpha_t + \beta_x + 2\alpha\beta + 3k = 0 \tag{2.39}$$

$$3kx\beta + \beta_t - \alpha k^2x^3 - c^2\alpha x + \beta^2 + 3k^2x^2 + c^2 = 0 \tag{2.40}$$

Using the separation of variables equation (2.38) has the general solution

$$\alpha(t, x) = \frac{1}{x + h_1(t)}$$

Since $h_1(x)$ is an arbitrary function, we can choose $h_1(t) = 0$. So equation (2.39) becomes

$$\beta_x + 2\frac{\beta}{x} = -3k \tag{2.41}$$

Multiply both sides of equation (2.41) by the integrating factor x^2 provides

$$(x^2\beta)_x = -3kx^2 \tag{2.42}$$

integrating both sides with respect to x gives

$$\beta = -kx + \frac{h_2(t)}{x^2}$$

Hence equation (2.40) becomes

$$kx^3h_2(t) + h_2'(t)x^2 + h_2^2(t) = 0 \tag{2.43}$$

It is clear that $h_2(t) = 0$ satisfy equation (2.43). Therefore equation (2.37) admits the vector field $X = \frac{\partial}{\partial x}$ as a λ-symmetry for $\lambda(t, x, \dot{x}) = \frac{\dot{x}}{x} - kx$. We can obtain the corresponding first integral as follows:

- Equation (2.26) gives

  $$w_x + \left(\frac{\dot{x}}{x} - kx\right) w_{\dot{x}} = 0 \tag{2.44}$$

  which has a solution

  $$w(t, x, \dot{x}) = \frac{\dot{x}}{x} + kx$$

- Observe that

$$\begin{aligned} D_t(w) &= w_t + \dot{x}w_x + \ddot{x}w_{\dot{x}} \quad \text{provided equation (2.37)} \quad = 0 \\ &= -\left(\left(\tfrac{\dot{x}}{x}\right)^2 + 2k\tfrac{\dot{x}}{x} + k^2x^2\right) - c^2 \\ &= -w^2 - c^2 \end{aligned}$$

So $F(t,w) = -w^2 - c^2$.

- To find the invariant G of $\frac{\partial}{\partial t} + F(t,w)\frac{\partial}{\partial w}$ we solve the PDE

$$G_t + FG_w = 0 \tag{2.45}$$

which depends on the values of c.

i. $c = 0$:

In this case equation (2.45) has the characteristic equation

$$dt = -\frac{dw}{w^2}$$

Integrating both sides gives

$$G = t - \frac{1}{w}$$

Therefore the first integral I is defined as

$$I(t,x,\dot{x}) = t - \frac{x}{\dot{x} + kx^2}$$

From the definition of first integral $I(t, x, \dot{x}) = c_1, c_1 \in \Re$, therefore the equation (2.37) is reduced to the Bernoulli equation

$$\dot{x} + kx^2 = \frac{x}{t - c_1} \tag{2.46}$$

To solve this equation let $w = \frac{1}{x}$, So the equation (2.46) can be written as

$$\dot{w} + \frac{w}{t - c_1} = k \tag{2.47}$$

Multiply this equation by the integrating factor $t - c_1$

$$((t - c_1)w)_t = (t - c_1)k \tag{2.48}$$

integrate both sides

$$w = \frac{kt^2 - 2c_1kt + 2c_2}{2(t - c_1)} \tag{2.49}$$

rewrite the equation (2.49) in the term of x gives the general solution

$$x(t) = \frac{2(t - c_1)}{kt^2 - 2c_1kt + 2c_2} \tag{2.50}$$

ii. $c \neq 0$:

In this case equation (2.45) has the characteristic equation

$$dt = -\frac{dw}{w^2 + c^2}$$

Integrating both sides gives

$$G = t + \frac{1}{c}\tan^{-1}(\frac{w}{c})$$

So the first integral I is defined as

$$I(t, x, \dot{x}) = t + \frac{1}{c}\tan^{-1}\left(\frac{\dot{x} + kx^2}{cx}\right)$$

Since $I(t, x, \dot{x}) = c_1, c_1 \in \Re$ the equation (2.37) is reduced to the Bernoulli equation

$$\dot{x} - c\tan(c_1c - ct)x = -kx^2 \tag{2.51}$$

To solve this equation let $w = \frac{1}{x}$, So the equation (2.51) can be written as

$$\dot{w} + c\tan(c_1c - ct)w = k \tag{2.52}$$

Multiply this equation by the integrating factor $\cos(c_1c - ct)$

$$\left(c_1\cos(c - ct)w\right)_t = \cos(c_1c - ct)k \tag{2.53}$$

integrate both sides

$$w = \frac{c_2c - k\sin(c_1c - ct)}{c\cos(c_1c - ct)} \tag{2.54}$$

rewrite the equation (2.54) in the term of x gives the general solution

$$x(t) = \frac{-c\cos(c_1 c - ct)}{k\sin(c_1 c - ct) - c_2 c} \tag{2.55}$$

CHAPTER 3

λ-SYMMETRY AND LINEARIZATION OF SECOND-ORDER ODES VIA LOCAL TRANSFORMATIONS

An alternative proof of Lie's approach for linearization of scalar second order ODEs is derived using the relationship between λ-symmetries and first integrals. This relation further leads to a new λ-symmetry linearization criteria for second order ODEs which provides a new approach for constructing the linearization transformations with lower complexity. The effectiveness of the approach is illustrated by obtaining the local linearization transformations for the linearizable nonlinear ODEs of the form $y'' + F_1(x, y)y' + F(x, y) = 0$. Examples of linearizing nonlinear ODEs which are quadratic or cubic in the first derivative are also presented.

3.1 Introduction

The initial seminal studies of scalar second-order ordinary differential equations (ODEs) which are linearizable by means of point transformations are due to Lie [24] and Tressé [48]. In recent decades there have been a resurgence of interest in this topic (see the reviews [14, 29, 45]).

It was shown by Lie [24] that any second-order ODE

$$y'' = f(x, y, y') \tag{3.1}$$

which is linearizable via point transformations is at most cubic in the first derivative, i.e. it has the form

$$y'' + F_3(x, y)y'^3 + F_2(x, y)y'^2 + F_1(x, y)y' + F(x, y) = 0. \tag{3.2}$$

It is well known [15] that any second order linear ODE can be transformed via point transformations to the free particle equation

$$u_{tt} = 0. \tag{3.3}$$

Therefore, all linearizable second-order ODEs (3.1) are obtained from the free particle equation (3.3) via point transformations. Precisely, the free particle equation (3.3) can be transformed by an arbitrary change of variables

$$t=\phi\left(x,y\right),\ u=\psi\left(x,y\right),\quad J=\phi_x\psi_y-\phi_y\psi_x\neq 0, \tag{3.4}$$

where J is the Jacobian, to the family of ODEs (3.2) with the coefficients $F(x,y), F_1(x,y)$, $F_2(x,y)$ and $F_3(x,y)$ satisfying the following system of partial differential equations

$$F_3(x,y)=A,\quad F_2(x,y)=B+2w,\quad F_1(x,y)=P+2z,\quad F(x,y)=Q, \tag{3.5}$$

in which

$$\begin{aligned}
&A=\tfrac{\phi_y\psi_{yy}-\psi_y\phi_{yy}}{\phi_x\psi_y-\phi_y\psi_x},\quad B=\tfrac{\phi_x\psi_{yy}-\psi_x\phi_{yy}}{\phi_x\psi_y-\phi_y\psi_x},\quad w=\tfrac{\phi_y\psi_{xy}-\psi_y\phi_{xy}}{\phi_x\psi_y-\phi_y\psi_x},\\
&Q=\tfrac{\phi_x\psi_{xx}-\psi_x\phi_{xx}}{\phi_x\psi_y-\phi_y\psi_x},\quad z=\tfrac{\phi_x\psi_{xy}-\psi_x\phi_{xy}}{\phi_x\psi_y-\phi_y\psi_x},\quad P=\tfrac{\phi_y\psi_{xx}-\psi_y\phi_{xx}}{\phi_x\psi_y-\phi_y\psi_x}.
\end{aligned} \tag{3.6}$$

This can be shown as follows:

$$u_t=\frac{du}{dt}=\frac{\psi_x dx+\psi_y dy}{\phi_x dx+\phi_y dy}=\frac{\psi_x+y'\psi_y}{\phi_x+y'\phi_y}=\frac{D_x(\psi)}{D_x(\phi)}. \tag{3.7}$$

similarly

$$\begin{aligned}
u_{tt}\ &=\tfrac{du_t}{dt}=\tfrac{D_x(u_t)}{D_x(\phi)}=\tfrac{1}{D_x(\phi)}D_x\left(\tfrac{D_x(\psi)}{D_x(\phi)}\right)\\
&=\tfrac{D_x(\phi){D_x}^2(\psi)-D_x(\psi){D_x}^2(\phi)}{(D_x(\phi))^3}\\
&=\tfrac{(\phi_x+\phi_y y')D_x(\psi_x+\psi_y y')-(\psi_x+\psi_y y')D_x(\phi_x+\phi_y y')}{(\phi_x+\phi_y y')^3}\\
&=\tfrac{(\phi_x+\phi_y y')(\psi_{xx}+2\psi_{xy}y'+\psi_y y''+\psi_{yy}y'^2)-(\psi_x+\psi_y y')(\phi_{xx}+2\phi_{xy}y'+\phi_y y''+\phi_{yy}y'^2)}{(\phi_x+\phi_y y')^3}
\end{aligned}$$

Then, the free particle equation (3.3) can be transformed under the point transformation (3.4) to the form (3.2) where $F(x,y), F_1(x,y), F_2(x,y)$ and $F_3(x,y)$ satisfy the system of PDEs (3.5).

Lie [24] crucially also found the following over-determined system of four equations

$$\begin{aligned}
w_x &= zw - F\,F_3 - \tfrac{1}{3}\tfrac{\partial F_1}{\partial y} + \tfrac{2}{3}\tfrac{\partial F_2}{\partial x},\\
w_y &= -w^2 + F_2 w + F_3 z + \tfrac{\partial F_3}{\partial x} - F_1 F_3,\\
z_x &= z^2 - F_1 z - Fw + \tfrac{\partial F}{\partial y} + FF_2,\\
z_y &= -\,zw + F\,F_3 - \tfrac{1}{3}\tfrac{\partial F_2}{\partial x} + \tfrac{2}{3}\tfrac{\partial F_1}{\partial y},
\end{aligned} \tag{3.8}$$

which are called the Lie conditions. The compatibility of Lie's conditions gives the following well known Lie-Tressé linearization test for ODEs of the form (3.2), viz.

$$\begin{gathered}
\tfrac{\partial^2 F_1}{\partial y^2} - 2\,\tfrac{\partial^2 F_2}{\partial y \partial x} + 3\,\tfrac{\partial^2 F_3}{\partial x^2} - 3\,\tfrac{\partial F_1}{\partial x}F_3 - 3\,F_1\tfrac{\partial F_3}{\partial x} + 6\,\tfrac{\partial F}{\partial y}F_3 + 3\,F\tfrac{\partial F_3}{\partial y} - F_2\tfrac{\partial F_1}{\partial y} + 2\,F_2\tfrac{\partial F_2}{\partial x} = 0\\
\tfrac{\partial^2 F_2}{\partial x^2} - 2\,\tfrac{\partial^2 F_1}{\partial y \partial x} + 3\,\tfrac{\partial^2 F}{\partial y^2} + 3\,\tfrac{\partial F}{\partial y}F_2 + 3\,F\tfrac{\partial F_2}{\partial y} - 3\,\tfrac{\partial F}{\partial x}F_3 - 6\,F\tfrac{\partial F_3}{\partial x} + F_1\tfrac{\partial F_2}{\partial x} - 2\,F_1\tfrac{\partial F_1}{\partial y} = 0.
\end{gathered} \tag{3.9}$$

This can be shown as follows:

$w_{xy} = w_{yx}$ implies

$$\begin{aligned}
&z_y w + z w_y - F_3\tfrac{\partial F}{\partial y} - F\tfrac{\partial F_3}{\partial y} - \tfrac{1}{3}\tfrac{\partial^2 F_1}{\partial y^2} + \tfrac{2}{3}\tfrac{\partial^2 F_2}{\partial y \partial x} + 2ww_x - \tfrac{\partial F_2}{\partial x}w - F_2 w_x\\
&-\tfrac{\partial F_3}{\partial x}z - F_3 z_x - \tfrac{\partial^2 F_3}{\partial x^2} + \tfrac{\partial F_1}{\partial x}F_3 + F_1\tfrac{\partial F_3}{\partial x} = 0
\end{aligned} \tag{3.10}$$

Substituting the values of w_x, w_y, z_x and z_y given by the system (3.8) into the equation (3.10) provides the first equation in (3.9).

Similarly $z_{xy} = z_{yx}$ leads to

$$\begin{aligned} &2z\, z_y - \tfrac{\partial F}{\partial y} w - F w_y - \tfrac{\partial F_1}{\partial y} z - F_1 z_y + \tfrac{\partial^2 F}{\partial y^2} + \tfrac{\partial F}{\partial y} F_2 + F \tfrac{\partial F_2}{\partial y} + z_x w \\ &+ z w_x - \tfrac{\partial F}{\partial x} F_3 - F \tfrac{\partial F_3}{\partial x} + \tfrac{1}{3} \tfrac{\partial^2 F_2}{\partial x^2} - \tfrac{2}{3} \tfrac{\partial^2 F_1}{\partial y \partial x} = 0 \end{aligned} \tag{3.11}$$

Substituting the values of w_x, w_y, z_x and z_y given by the system (3.8) into the equation (3.11) provides the second equation in (3.9). It was Tressé [48] who first obtained the invariant criteria (3.9).

We point out that the conditions (3.9) can also be deduced via the Cartan equivalence approach [11]. More recently in two independent papers [15, 28], the authors present geometrical proofs for the determination of the invariant conditions (3.9). It should be noted that the projections utilized in [15, 28] were different to each other. In [1] Lie's linearizability criteria were utilized to determine a linearizable class of a system of two second order equations that are obtainable from complex scalar second order ODEs. Furthermore, Mahomed and Qadir [27] found criteria for conditional linearizability of third order ODEs via point transformation subject to a Lie-Tressé linearizable second order ODE.

One can mention here as well the results on the algebraic criteria for linearization by point transformations of scalar second order ODEs (3.1). It is well known from Lie's work that such linearizable ODEs possess eight point symmetries. The question arises if one can conclude linearization of (3.1) in case one has knowledge of fewer than eight symmetries of the ODE (3.1). The answer is affirmative. In fact Lie was the first to obtain algebraic criteria for linearization when the ODE (3.1) admits two symmetries which are connected. Some further studies [25, 46] provided input on the algebraic criteria for linearization when the ODE (3.1) has two unconnected symmetries.

Lie proved that the ODE (3.1) is linearizable by point transformations if and only if the over-determined system (3.8) is compatible [24]. In order to prove the sufficiency of the compatibility of (3.8) for linearization, Lie showed that the system (3.8) can be linearized and the resulting linear system belongs to a class of special type of linear systems which can be reduced to a linear third-order ODE. Thus the three solutions $(z, w), (z_1, w_1)$ and (z_2, w_2) of the system (3.8) can be found by solving this linear third-order ODE. Finally, these solutions can be used as a basis for constructing the linearizing transformations by solving the quadratures

$$\begin{aligned} \tfrac{\phi_x}{\phi} &= z - z_1, \quad \tfrac{\phi_y}{\phi} = w_1 - w, \\ \tfrac{\psi_x}{\psi} &= z - z_2, \quad \tfrac{\psi_y}{\psi} = w_2 - w. \end{aligned} \tag{3.12}$$

The aim of this chapter is to investigate the linearization of second order ODEs using λ-symmetries. The outline of this chapter is as follows. The alternative proof of Lie's approach for linearization of second order ODEs is provided in Section (3.2). This section also presents a new λ-symmetry linearization criteria which provides an alternative approach for constructing the linearization transformations. The relationship between λ-symmetries and the first integrals of ODEs play the key role in the proving the results of Section (3.2). A familiarity with standard results about the theory of λ-symmetries is assumed and the reader is referred to chapter two. In Section (3.3), we apply the new approach to linearize the ODEs of the form (3.2) with

$F_3 = F_2 = 0$. Section (3.5) consists of examples illustrating the application of the new approach to linearize ODEs of the form (3.2) with $F_3 \neq 0$ or $F_2 \neq 0$.

3.2 Alternative proof of Lie's linearization approach & new λ-symmetry linearization criteria

In this section, an alternative proof of Lie's approach to linearization of second order ODEs is presented. It is noticed that the relation between the λ-symmetries and the first integrals provides a direct method to derive both the Lie's conditions and the quadratures. In addition, a λ-symmetry criteria for linearization via a point transformation is stated. This criteria provides a new approach for constructing the linearizing transformations, utilizing λ-symmetries.

The two first integrals

$$I_1 = u_t, \quad I_2 = u - t\ u_t \tag{3.13}$$

of the free particle equation (3.3) take the form

$$I_1 = \tfrac{\psi_x + \psi_y y'}{\phi_x + \phi_y y'}, \quad I_2 = \psi - \phi\left(\tfrac{\psi_x + \psi_y y'}{\phi_x + \phi_y y'}\right), \tag{3.14}$$

when expressed in the new variables x and $y(x)$ defined by equation (3.4). It follows that all linearizable equations (3.2) should have the two first integrals (3.14). Hence, the relationship between the first integrals and the λ-symmetries [33, 34] implies that all linearizable equations (3.2) should have the two λ-symmetries equivalent to the canonical pairs $(\frac{\partial}{\partial y}, \lambda_1)$ and $(\frac{\partial}{\partial y}, \lambda_2)$ given by

$$\begin{aligned}\lambda_1 &= -\tfrac{I_{1y}}{I_{1y'}} = -A\ y'^2 - (B+w)\,y' - z\\ \lambda_2 &= -\tfrac{I_{2y}}{I_{2y'}} = -A\ y'^2 - \Big(B + w - \tfrac{\phi_y}{\phi}\Big)\,y' - z + \tfrac{\phi_x}{\phi},\end{aligned} \tag{3.15}$$

where A, B, w and z are given by equation (3.6).

The free particle equation (3.3) possesses the functionally dependent quotient first integral $I_3 = \frac{I_2}{I_1} = \frac{u - t\ u_t}{u_t}$ as well. It is worthwhile to mention here that the three triplets of Lie algebras of I_1, I_2 and I_3 which are isomorphic to each other generate the Lie algebra $sl(3, R)$ of the free particle equation [20]. Similar to the above, it can be seen that all linearizable equations (3.2) should have the third associated λ-symmetry equivalent to the canonical pair $(\frac{\partial}{\partial y}, \lambda_3)$, where

$$\lambda_3 = -\tfrac{I_{3y}}{I_{3y'}} = -A\ y'^2 - \Big(B + w - \tfrac{\psi_y}{\psi}\Big)\,y' - z + \tfrac{\psi_x}{\psi}. \tag{3.16}$$

The expressions for λ_1, λ_2 and λ_3 can be simplified in term of w and z, using the system (3.5), as

$$\begin{aligned}\lambda_1 &= -F_3\ y'^2 - (F_2 - w)\,y' - z\\ \lambda_2 &= -F_3\ y'^2 - \Big(F_2 - w - \tfrac{\phi_y}{\phi}\Big)\,y' - z + \tfrac{\phi_x}{\phi}\\ \lambda_3 &= -F_3\ y'^2 - \Big(F_2 - w - \tfrac{\psi_y}{\psi}\Big)\,y' - z + \tfrac{\psi_x}{\psi}.\end{aligned} \tag{3.17}$$

Employing the definition of λ-symmetry [32], the canonical pair $(\frac{\partial}{\partial y}, \lambda_1)$ leads to the following system for w and z

$$\begin{aligned} w_y &= -w^2 + F_2 w + F_3 z + \tfrac{\partial F_3}{\partial x} - F_1 F_3, \\ w_x - z_y &= 2\,wz - 2\,F_3 F + \tfrac{\partial F_2}{\partial x} - \tfrac{\partial F_1}{\partial y} \\ z_x &= z^2 - F_1 z - Fw + \tfrac{\partial F}{\partial y} + FF_2. \end{aligned} \tag{3.18}$$

Similarly, it follows by applying the definition of λ-symmetry for λ_2 and λ_3, along with using the system (3.18), that the transformations ϕ and ψ can be given by finding two solutions which non-zero Jacobian of the following over-determined system

$$\begin{aligned} &S_{yy} + (2\,w - F_2)\,S_y + F_3\,S_x = 0 \\ &S_{xy} + w\ S_x - z\ S_y = 0 \\ &S_{xx} + (F_1 - 2\,z)\,S_x - F\ S_y = 0 \end{aligned} \tag{3.19}$$

Based on the above discussion we have obtained the following two λ-symmetry criteria for linearization via point transformations.

Theorem 3.1 *A scalar second-order ODE (3.1) is linearizable via point transformations (3.4) if and only if it has the cubic in first derivative form (3.2) with the λ-symmetries equivalent to the canonical pair $(\frac{\partial}{\partial y}, \lambda_1)$ for $\lambda_1 = -F_3\ {y'}^2 - (F_2 - w)\,y' - z$ and the transformations ϕ and ψ satisfying equation (3.19), where w and z are auxiliary functions.*

Proof. The proof in one direction is given in the preceding discussion, so we prove the other way.

Assume that equation (3.2) admits the λ-symmetries equivalent to the canonical pair $(\frac{\partial}{\partial y}, \lambda_1)$ for $\lambda_1 = -F_3\, {y'}^2 - (F_2 - w)\, y' - z$. Then, the system (3.18) for w and z is obtained using the definition of λ-symmetry. Since the transformations ϕ and ψ satisfy equation (3.19), the compatibility of the system (3.19), i.e. $S_{xy} = S_{yx}$, $S_{xyx} = S_{xxy}$ and $S_{xyy} = S_{yyx}$, leads to the system

$$\begin{aligned}
&\phi_y \left(-z_x + z^2 - F_1 z - F w + \tfrac{\partial F}{\partial y} + F F_2,\right) - \phi_x \left(-2\, z_y - z w - w_x + \tfrac{\partial F_1}{\partial y} + F F_3\right) = 0 \\
&\phi_x \left(w_y + w^2 - F_2 w - F_3 z - \tfrac{\partial F_3}{\partial x} + F_1 F_3\right) - \phi_y \left(2\, w_x - z w + z_y - \tfrac{\partial F_2}{\partial x} + F F_3\right) = 0.
\end{aligned} \tag{3.20}$$

Substituting the system (3.18) into the system (3.20) and noting that the Jacobian $J = \phi_x \psi_y - \phi_y \psi_x \neq 0$, one finds

$$w_x + z_y = \frac{1}{3}\left(\frac{\partial F_2}{\partial x} + \frac{\partial F_1}{\partial y}\right). \tag{3.21}$$

Finally, system (3.18) and equation (3.21) are equivalent to the Lie's conditions (3.8) for linearizable equations and so the compatibility, $w_{xy} = w_{yx}$ and $z_{xy} = z_{yx}$, of the Lie's conditions (3.8) leads to the invariant equations (3.9). ▮

Corollary 3.2 *A scalar second-order ODE (3.1) is linearizable via point transformations if and only if it has the cubic in first derivative form (3.2) with the λ-symmetries equivalent to the canonical pair $(\frac{\partial}{\partial y}, \lambda_1)$ for $\lambda_1 = -F_3\, {y'}^2 - (F_2 - w)\, y' - z$, where w and z are auxiliary functions satisfying the equation $w_x + z_y = \frac{1}{3}\left(\frac{\partial F_2}{\partial x} + \frac{\partial F_1}{\partial y}\right)$.*

Remark 3.3 The compatibility of the system (3.19) is guaranteed by both the system (3.18) and equation (3.21), i.e. $S_{xy} = S_{yx}$, $S_{xyx} = S_{xxy}$ and $S_{xyy} = S_{yyx}$. Hence for each w and z given by solving (3.18) and (3.21), one can construct transformations by solving the over-determined system (3.19). This provides a new approach for constructing the linearizing transformations whose implementation requires finding only one solution of the system (3.8). In comparison, the standard implementation of Lie's linearization approach involves solving Lie's quadratures (3.12) which requires finding three solutions of the system (3.8).

Finally, in order to obtain an alternative proof of Lie's quadratures (3.12) using λ-symmetries, it is noticed that $\lambda_1,\ \lambda_2$ and λ_3 given by (3.17) can be written as

$$\begin{aligned}\lambda_1 &= -F_3\, {y'}^2 - (F_2 - w)\, y' - z,\\ \lambda_2 &= -F_3\, {y'}^2 - (F_2 - w_1)\, y' - z_1,\\ \lambda_3 &= -F_3\, {y'}^2 - (F_2 - w_2)\, y' - z_2,\end{aligned}$$

where $w_1 = w + \frac{\phi_y}{\phi}, z_1 = z - \frac{\phi_x}{\phi}, w_2 = w + \frac{\psi_y}{\psi}$ and $z_2 = z - \frac{\psi_x}{\psi}$. Therefore, using the definition of λ-symmetry shows that $(z, w), (z_1, w_1)$ and (z_2, w_2) are the three solutions for the system (3.8). This completes the alternative proof of Lie's approach.

3.3 Linearization of the ODE's of the form (3.2) with $F_3 = F_2 = 0$

The non-linear second order ODEs of the form

$$y'' + F_1(x, y)y' + F(x, y) = 0 \tag{3.22}$$

satisfy the Lie-Tressé linearization criteria (3.9)

$$\frac{\partial^2 F_1}{\partial y^2} = 0 \tag{3.23}$$

$$2F_1\frac{\partial F_1}{\partial y} - 3\frac{\partial^2 F}{\partial y^2} + 2\frac{\partial^2 F_1}{\partial y \partial x} = 0 \tag{3.24}$$

if and only if

$$F_1(x,y) = a(x)y+b(x), \quad F(x,y) = \frac{a^2(x)}{9}y^3+\frac{1}{3}\left(\frac{d\,a(x)}{dx} + a(x)b(x)\right)y^2+c(x)y+d(x)$$

Here we consider the class of nonlinear equations

$$\frac{d^2y}{dx^2}+\left(a\left(x\right)y+b\left(x\right)\right)\frac{dy}{dx}+\frac{1}{9}\left(a\left(x\right)\right)^2y^3+\frac{1}{3}\left(\frac{da(x)}{dx} + a\left(x\right)b\left(x\right)\right)y^2+c\left(x\right)y+d(x) = 0 \tag{3.25}$$

where $a(x) \neq 0$.

In this case, the system (3.8) becomes

$$w_x - zw + \frac{1}{3}\frac{\partial F_1}{\partial y} = 0 \tag{3.26}$$

$$w_y + w^2 = 0 \tag{3.27}$$

$$z_x - z^2 + F_1 z + Fw - \frac{\partial F}{\partial y} = 0 \tag{3.28}$$

$$z_y + zw - \frac{2}{3}\frac{\partial F_1}{\partial y} = 0 \tag{3.29}$$

and it can be solved as follows:

The solution of equation (3.27) can be given as

$$w = \frac{1}{y + h_1(x)} \tag{3.30}$$

Now, using the multiplier $y + h_1(x)$, the solution of equation (3.29) is

$$z = \frac{\frac{2}{3}\int \frac{\partial F_1}{\partial y}(y + h_1(x))\, dy + h_2(x)}{y + h_1(x)}, \tag{3.31}$$

Substituting equations (3.30) and (3.31) in equation (3.26) gives

$$h_2(x) = -\frac{2}{3}\int (y + h_1(x))\,\frac{\partial F_1}{\partial y}\, dy + \frac{1}{3}\,(y + h_1(x))^2\,\frac{\partial F_1}{\partial y} - h_1'(x) \tag{3.32}$$

Substituting equations (3.30),(3.31) and (3.32) in equation (3.28) gives

$$\begin{aligned} &9{h_1}'' - 9\tfrac{\partial F_1}{\partial y} h_1 {h_1}' + 9\left(F_1 - \tfrac{\partial F_1}{\partial y} y\right) {h_1}' - 3\left(F_1 \tfrac{\partial F_1}{\partial y} + \tfrac{\partial^2 F_1}{\partial x \partial y} - \left(\tfrac{\partial F_1}{\partial y}\right)^2 y\right) {h_1}^2 + \left(\tfrac{\partial F_1}{\partial y}\right)^2 {h_1}^3 \\ &+ \left(9\tfrac{\partial F}{\partial y} + 3\left(\tfrac{\partial F_1}{\partial y}\right)^2 y^2 - 6F_1 \tfrac{\partial F_1}{\partial y} y - 6\tfrac{\partial^2 F_1}{\partial x \partial y} y\right) h_1 - 9F + \left(\tfrac{\partial F_1}{\partial y}\right)^2 y^3 + 9\tfrac{\partial F}{\partial y} y - 3F_1 \tfrac{\partial F_1}{\partial y} y^2 - 3\tfrac{\partial^2 F_1}{\partial x \partial y} y^2 = 0 \end{aligned} \tag{3.33}$$

Equation (3.33) can be linearized by the ansatz (see [18])

$$h_1(x) = \alpha(x)\frac{Y'(x)}{Y(x)} \tag{3.34}$$

Substituting the ansatz (3.34) in equation (3.33) and equate the coefficients of non linear terms to zero gives the ansatz (3.34) with for $\alpha = -3/\frac{\partial F_1}{\partial y}$ transforms equation

(3.33) into the following linear third-order ODE:

$$Y'''(x) + \left(b(x) - 2\tfrac{a'(x)}{a(x)}\right) Y''(x) - \left(\tfrac{a''(x)}{a(x)} - 2\tfrac{a'(x)^2}{a(x)^2} + b(x)\tfrac{a'(x)}{a(x)} - c(x)\right) Y'(x) + \tfrac{d(x)a(x)}{3} Y(x) = 0. \tag{3.35}$$

Therefore solving the system (3.8) gives the three solutions (w, z), (w_1, z_1) and (w_2, z_2)

$$\begin{aligned}
w(x,y) &= \tfrac{a(x)g_1(x)}{ya(x)g_1(x)-3\,{g_1}'(x)}, & z(x,y) &= \tfrac{1}{3}\,\tfrac{9\,a(x){g_1}''(x)-\left(9\,a'(x)+6\,ya(x)^2\right){g_1}'(x)+y^2a(x)^3g_1(x)}{a(x)(ya(x)g_1(x)-3\,{g_1}'(x))},\\
w_1(x,y) &= \tfrac{a(x)g_2(x)}{ya(x)g_2(x)-3\,{g_2}'(x)}, & z_1(x,y) &= \tfrac{1}{3}\,\tfrac{9\,a(x){g_2}''(x)-\left(9\,a'(x)+6\,ya(x)^2\right){g_2}'(x)+y^2a(x)^3g_2(x)}{a(x)(ya(x)g_2(x)-3\,{g_2}'(x))},\\
w_2(x,y) &= \tfrac{a(x)g_3(x)}{ya(x)g_3(x)-3\,{g_3}'(x)}, & z_2(x,y) &= \tfrac{1}{3}\,\tfrac{9\,a(x){g_3}''(x)-\left(9\,a'(x)+6\,ya(x)^2\right){g_3}'(x)+y^2a(x)^3g_3(x)}{a(x)(ya(x)g_3(x)-3\,{g_3}'(x))},
\end{aligned} \tag{3.36}$$

where $g_i(x), i = 1, 2, 3$, are the three linearly independent solutions of the linear third order ODE (3.35).

Now, in order to construct the linearizing transformation using Lie's linearization approach, one should solve the quadratures (3.12) by considering the three solutions (w, z), (w_1, z_1) and (w_2, z_2). However, our approach requires utilizing any one the three solutions (w, z), (w_1, z_1), (w_2, z_2) in order to determine two non-constant solutions of the over-determined system (3.19) which yield the linearization transformations.

As an illustration, we consider ODEs (3.25) with $d(x) = 0$. For this case $g_1(x) = 1$ is a constant solution of the ODE (3.35). So equation (3.36) gives the solution $(w, z) = \left(\frac{1}{y}, \frac{1}{3}ya(x)\right)$ of the system (3.8). So the system (3.19) becomes

$$\begin{aligned}
&S_{yy} + \tfrac{2}{y}S_y = 0\\
&S_{xy} + \tfrac{1}{y}S_x - \tfrac{1}{3}y\,a(x)\,S_y = 0\\
&S_{xx} + \left(F_1 - \tfrac{2}{3}a(x)\,y\right)S_x - F\,S_y = 0
\end{aligned} \tag{3.37}$$

Solving the first equation in the system (3.37) gives

$$S(x,y) = -\frac{h(x)}{y} + k(x) \tag{3.38}$$

Substituting equation (3.38) in the second equation of the system (3.37) gives

$$k(x) = \frac{1}{3}\int h(x)\,a(x)\,dx + C_1 \tag{3.39}$$

Substituting equations (3.38) and (3.39) in the third equation of the system (3.37) gives

$$h''(x) + b(x)\,h'(x) + c(x)\,h(x) = 0 \tag{3.40}$$

Thus, the linearizing point transformations are

$$\phi(x,y) = \tfrac{1}{3}\textstyle\int a(x)h_1(x)dx - \frac{h_1(x)}{y}, \quad \psi(x,y) = \tfrac{1}{3}\textstyle\int a(x)h_2(x)dx - \frac{h_2(x)}{y}. \tag{3.41}$$

with $J = \frac{-1}{y^3}\,W(h_1(x),h_2(x))$ where $W(h_1(x),h_2(x))$ denotes to the Wronskian of the two linearly independent solutions of the linear second order ODE (3.40).

Finally, since $u(t) = c_1t + c_2$ is the general solution of the free particle equation $u_{tt} = 0$, the nontrivial solution of the ODE (3.25) can be given as

$$\frac{1}{3}\int a(x)h_2(x)dx - \frac{h_2(x)}{y} = c_1\left(\frac{1}{3}\int a(x)h_1(x)dx - \frac{h_1(x)}{y}\right) + c_2. \tag{3.42}$$

Example 3.4 *We consider the ODE (3.25) with* $a(x) = 3, b(x) = 0$ *and* $c(x) = 0$ *that* *gives the modified Emden equation*

$$y'' + 3yy' + y^3 = 0,$$

for which the ODE (3.40) reduces to

$$h''(x) = 0.$$

This implies two linearly independent solutions

$$h_1(x) = x\,, \quad h_2(x) = 1$$

Hence the linearizing point transformations can be written using (3.41) as

$$\phi(x,y) = x - \tfrac{1}{y}, \quad \psi(x,y) = \tfrac{x^2}{2} - \tfrac{x}{y}.$$

with $J = \frac{-1}{y^3}$*. Finally, the nontrivial solution can be written using (3.42) as*

$$y(x) = \frac{2x + c_1}{x^2 + c_1 x + c_2}.$$

Example 3.5 *For* $a(x) = 3k, b(x) = b$ *and* $c(x) = \frac{b^2}{4}$*, ODE (3.25) gives the* Liénard *type equation*

$$y'' + (b + 3ky)y' + k^2y^3 + bky^2 + \frac{b^2}{4}y = 0,$$

and the ODE (3.40) reduces to

$$h''(x) + bh'(x) + \frac{b^2}{4}h(x) = 0. \tag{3.43}$$

which gives the two linearly independent solutions

$$h_1(x) = e^{\frac{-bx}{2}}, \quad h_2(x) = xe^{\frac{-bx}{2}}$$

So the linearizing point transformations can be stated, using (3.41), as

$$\phi(x,y) = \tfrac{2ky+b}{bky}e^{-\frac{b}{2}x}, \quad \psi(x,y) = \tfrac{2bkxy+4ky+b^2x}{b^2ky}e^{-\frac{b}{2}x},$$

with $J = \frac{e^{-bx}}{k^2y^3}$ *which lead to the nontrivial solution via (3.42) as*

$$y(x) = \frac{b^2(c_1 - x)}{2bkx + 4k - 2c_1bk + c_2b^2ke^{\frac{b}{2}x}}.$$

3.4 An Application: Closed form solution for non-linear oscillation of a cantilever beam under different conditions

Cantilever beams are widely used in mechanical systems [21] and building constructions [22] because of their unique flexural characteristics. The cantilever beam with non-linear damping and stiffness parameters undergoes a non-linear flexural motion when the tension at the free end is released. Since the cantilever beam under consideration has considerably smaller thickness than its length, the beam can be considered to be a thin plate and the non-linear cantilever flexural motion problem reduces to one-dimensional form. This situation can also be represented as a mass with supported by non-linear damper and stiffness as shown in figures 3.1 and 3.2.

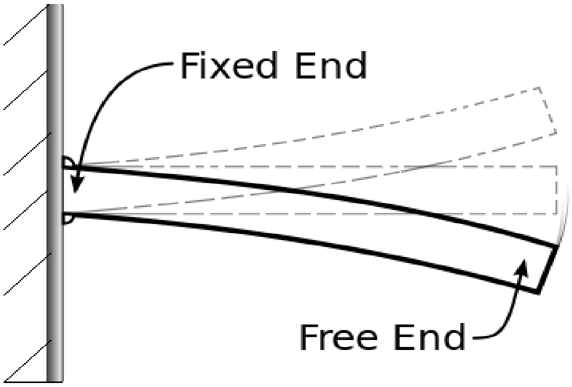

Figure 3.1: Schematic view of the cantilever beam considered in the analysis.

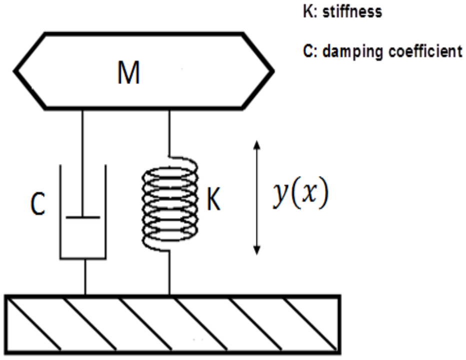

Figure 3.2: Representation of cantilever beam with a mechanical system.

This situation has been modeled by the nonlinear ODE

$$m\,y'' + c\,y' + k\,y = 0, \tag{3.44}$$

where $c = c_0 + c_1 y$ and $k = k_1 + k_2 y + k_3 y^2$.

In application, involving some metallic materials, the damping coefficient and stiffness can be simplified through a polynomial form of displacement. equation (3.44) satisfies the lie linearization test (3.9) if and only if $k_2 = \frac{c_0\, c_1}{3m}$, and $k_3 = \frac{c_1^2}{9m}$.

In this section, we will approximate the values of k_2 and k_3 to make the ODE (3.44) linearizable. So it can be written as

$$y'' + \frac{(c_0 + c_1 y)}{m} y' + \frac{k_1 y}{m} + \frac{c_1 c_0 y^2}{3m^2} + \frac{c_1^2 y^3}{9m^2} = 0 \tag{3.45}$$

The Lie condition system (3.8) has a particular solution $w(x, y) = \frac{1}{y}$, $z(x, y) = \frac{c_1 y}{3m}$. Then the system (3.19) has two solutions that provide the point transformations

$$\psi(x,y) = \frac{\mathrm{e}^{-\frac{c_0\, x}{2m}}}{6k_1 m y} \left(c_1 \sqrt{4k_1 m - c_0^2}\, y \cos\left(\frac{\sqrt{4k_1 m - c_0^2}}{2m} x \right) + (c_1 c_0 y + 6k_1 m) \sin\left(\frac{\sqrt{4k_1 m - c_0^2}}{2m} x \right) \right)$$

$$\phi(x,y) = \frac{\mathrm{e}^{-\frac{c_0\, x}{2m}}}{6k_1 m y} \left((c_0 c_1 y + 6k_1 m) \cos\left(\frac{\sqrt{4k_1 m - c_0^2}}{2m} x \right) - c_1 \sqrt{4k_1 m - c_0^2}\, y \sin\left(\frac{\sqrt{4k_1 m - c_0^2}}{2m} x \right) \right)$$

with the nonzero Jacobian for our considered parameters

$$J = \frac{\mathrm{e}^{-\frac{c_0\, x}{m}} \sqrt{4k_1 m - c_0^2}}{2my^3}$$

that transform the ODE (3.44) to the laguerre form $u_{tt} = 0$.
So, the general solution of ODE (3.44) can be given as

$$y(x) = -\frac{6k_1 m \left(s_1 \cos\left(\frac{\sqrt{4k_1 m - c_0^2}}{2m} x \right) - \sin\left(\frac{\sqrt{4k_1 m - c_0^2}}{2m} x \right) \right)}{\left(\left(s_1 c_0 c_1 - c_1 \sqrt{4k_1 m - c_0^2} \right) \cos\left(\frac{\sqrt{4k_1 m - c_0^2}}{2m} x \right) - \left(c_0 c_1 + s_1 c_1 \sqrt{4k_1 m - c_0^2} \right) \sin\left(\frac{\sqrt{4k_1 m - c_0^2}}{2m} x \right) - 6k_1 s_2 m \mathrm{e}^{\frac{c_0\, x}{2m}} \right)} \tag{3.46}$$

where s_1 and s_2 are arbitrary constants.

One can see that general solution of equation (3.44) in the linear case when $c_1 = 0$ can be given as follows:

$$y(x) = \mathrm{e}^{-\frac{c_0\,x}{2m}} \left(A\cos\left(\frac{\sqrt{4k_1 m - c_0^2}}{2m} x \right) + B\sin\left(\frac{\sqrt{4k_1 m - c_0^2}}{2m} x \right) \right) \tag{3.47}$$

Now, we will provide the values of s_1 and s_2 for the following cases:

- Case 1: Zero initial displacement and zero velocity for some non-zero instant For the conditions $y(0) = 0\,,\ y'(x_0) = 0,\ \ x_0 \neq 0$, the values of s_1 and s_2 can be given as

$$s_1 = 0 \tag{3.48}$$

$$s_2 = \frac{c_1 \mathrm{e}^{-\frac{c_0\,x_0}{2m}} \left(c_0^2 - 4k_1 m\right)}{6k_1 m \left(\sqrt{4k_1 m - c_0^2}\ \cos\left(\frac{\sqrt{4k_1 m - c_0^2}}{2m} x_0 \right) - c_0 \sin\left(\frac{\sqrt{4k_1 m - c_0^2}}{2m} x_0 \right) \right)} \tag{3.49}$$

  Note that in the linear case $y = 0$.

- Case 2: Zero initial displacement and the non-zero displacement for some non-zero instant For the conditions $y(0) = 0\,,\ y(x_0) = y_0,\ \ x_0 \neq 0, y_0 \neq 0$, the values of s_1 and s_2 can be given as

$$s_1 = 0 \tag{3.50}$$

$$\begin{split} s_2 = -\frac{\mathrm{e}^{-\frac{c_0\,x_0}{2m}}}{6y_0 k_1 m} \left(6mk_1 \sin\left(\frac{\sqrt{4k_1 m - c_0^2}}{2m} x_0 \right) + c_1 y_0 \sqrt{4k_1 m - c_0^2} \cos\left(\frac{\sqrt{4k_1 m - c_0^2}}{2m} x_0 \right) \right) \\ - \frac{\mathrm{e}^{-\frac{c_0\,x_0}{2m}}}{6y_0 k_1 m} c_0 c_1 y_0\ \sin\left(\frac{\sqrt{4k_1 m - c_0^2}}{2m} x_0 \right) \end{split} \tag{3.51}$$

  Moreover, in the linear case

$$s_1 = 0 \tag{3.52}$$

$$s_2 = -\frac{e^{-\frac{c_0 x_0}{2m}}}{y_0} \sin\left(\frac{\sqrt{4k_1 m - c_0^2}}{2m} x_0\right) \tag{3.53}$$

- Case 3: Initial displacement and initial velocity For the conditions $y(0) = y_0\,,\ y'(0) = y_1$, the values of s_1 and s_2 can be given as

$$s_1 = -\frac{3\sqrt{4k_1 m - c_0^2}\, y_0}{6y_1 m + 3c_0 y_0 + 2c_1 y_0^2} \tag{3.54}$$

$$s_2 = -\frac{\left(c_1^2 y_0^2 + (3y_1 m + 3c_0 y_0)\, c_1 + 9k_1 m\right)\sqrt{4k_1 m - c_0^2}}{3k_1 m \left(6y_1 m + 3c_0 y_0 + 2c_1 y_0^2\right)} \tag{3.55}$$

In the linear case:

$$s_1 = -\frac{\sqrt{4k_1 m - c_0^2}\, y_0}{2y_1 m + c_0 y_0} \tag{3.56}$$

$$s_2 = -\frac{\sqrt{4k_1 m - c_0^2}}{2y_1 m + c_0 y_0} \tag{3.57}$$

A computer code is developed to simulate the non-linear flexural characteristics of the cantilever beam for three cases considered. The physical properties used in the simulations are given in table 3.1.

Property	Numerical value
c_0	0.42 Ns/m
c_1	368 Ns/m^2
k_1	875 N/m
k_2	416.323 N/m^2
k_3	1.216×10^5 N/m^3
m	0.12375 kg
y_0	0.04 m
y_1	10 m/s
x_0	0.5 s

Table 3.1: The physical properties used in the simulations.

The following figures show displacement with time curves for non-linear and linear behavior of the cantilever beam for the three cases.

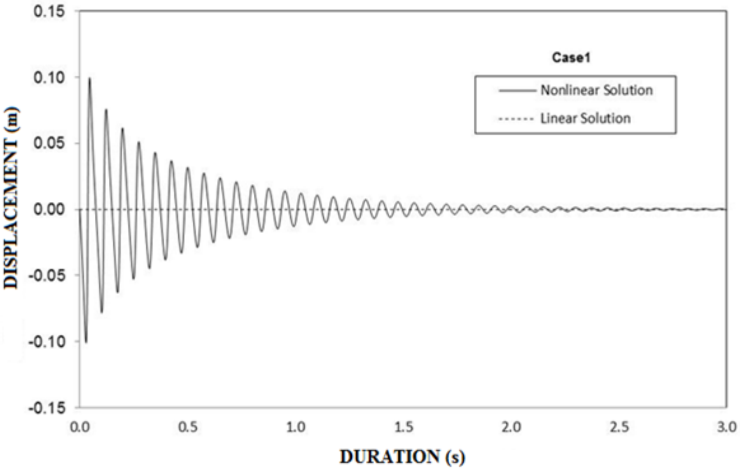

Figure 3.3: Displacement with duration for case 1; for the condition $y(0) = 0\,,\ y'(x_0) = 0,\ \ x_0 \neq 0$.

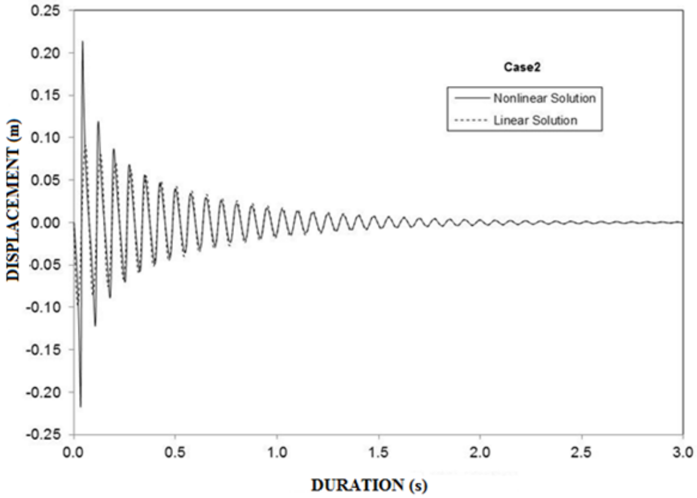

Figure 3.4: Displacement with duration for case 2; for the condition $y(0) = 0\,,\ y(x_0) = y_0,\ \ x_0 \neq 0, y_0 \neq 0$.

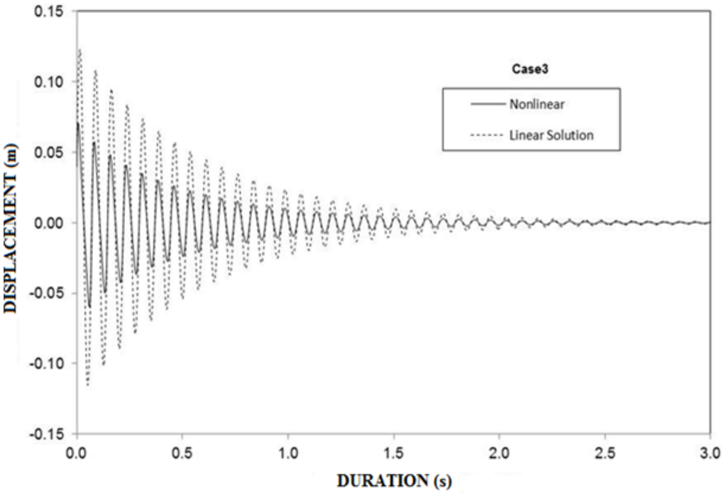

Figure 3.5: Displacement with duration for case 3; for the condition $y(0) = y_0\,,\ y'(0) = y_1$.

3.5 Examples of linearization of the ODEs of the form (3.2) with $F_3 \neq 0$ or $F_2 \neq 0$

In this section, we illustrate the application of our approach to obtain linearization transformations for ODEs that are cubic or quadratic in the first derivative.

Example 3.6 *As the first example, we consider the nonlinear ODE*

$$y'' - \frac{2}{x+y}y'^2 - \frac{1}{x+y}y' = 0 \tag{3.58}$$

which satisfies the Lie-Tressé linearization test (3.9). In this case, the system (3.8) *becomes*

$$zw + \frac{1}{(x+y)^2} - w_x = 0 \tag{3.59}$$

$$w_y + w^2 + \frac{2w}{x+y} = 0 \tag{3.60}$$

$$z^2 + \frac{z}{x+y} - z_x = 0 \tag{3.61}$$

$$z_y + zw = 0 \tag{3.62}$$

and it can be solved as follows:

The Bernoulli equation (3.60) has the solution

$$w = \frac{1}{h_1(x)(x+y)^2 - (x+y)} \tag{3.63}$$

Substituting equation (3.63) in equation (3.62) gives

$$z = \frac{(x+y)h_2(x)}{h_1(x)(x+y) - 1} \tag{3.64}$$

Substituting equations (3.63) and (3.64) in equations (3.59) and (3.61) provides respectively

$$h_1' + h_1^2 + h_2 = 0 \tag{3.65}$$

$$h_1'h_2\,(x+y) - h_1h_2'\,(x+y) + h_2^2\,(x+y) + h_1h_2 + h_2' = 0 \tag{3.66}$$

It is clear that $h_1 = h_2 = 0$ *satisfy the system (3.65-3.66). Therefore, the Lie's* *conditions (3.8) have a solution*

$$w(x,y) = -\frac{1}{x+y}, \quad z(x,y) = 0 \tag{3.67}$$

Hence the over-determined system (3.19) becomes

$$\begin{aligned} &S_{yy} = 0 \\ &S_{xy} - \tfrac{S_x}{x+y} = 0 \\ &S_{xx} - \tfrac{S_x}{x+y} = 0 \end{aligned} \tag{3.68}$$

This system can be solved as follows:

The solution of the first equation of system (3.68) is given by

$$S(x,y) = f_1(x)y + f_2(x) \tag{3.69}$$

Substitution equation (3.69) in the second equation of system (3.68) gives

$$f_2(x) = \int x{f_1}'(x)\,dx + c \tag{3.70}$$

Substituting equations (3.69) and (3.70) in the third equation of system (3.68) leads to

$$f_1(x) = Ax + B \tag{3.71}$$

Therefore

$$S(x,y) = Axy + By + \frac{Ax^2}{2} + c \tag{3.72}$$

Two solutions which results in the point transformations

$$\phi(x,y) = y, \quad \psi(x,y) = x(x+2y).$$

with $J = 2x + 2y$ *that linearize the ODE (3.58) to the free particle equation* $u_{tt} = 0$. *So the general solution of the ODE (3.58) is*

$$x(x+2y) = c_1 y + c_2.$$

Example 3.7 *The nonlinear ODE*

$$xy'' - y'^3 - y' = 0 \tag{3.73}$$

satisfies the Lie-Tressé linearization test (3.9). In this case, the system (3.8) becomes

$$w_x - zw = 0 \tag{3.74}$$

$$w_y + w^2 + \frac{z}{x} = 0 \tag{3.75}$$

$$z_x - z^2 - \frac{z}{x} = 0 \tag{3.76}$$

$$z_y + zw = 0 \tag{3.77}$$

It is clear that $w(x,y) = 0$ *and* $z(x,y) = 0$ *is a solution of the Lie's conditions. Now,* *the corresponding over-determined system (3.19) is*

$$\begin{aligned} &S_{yy} - \tfrac{S_x}{x} = 0 \\ &S_{xy} = 0 \\ &S_{xx} - \tfrac{S_x}{x} = 0 \end{aligned} \tag{3.78}$$

The system (3.78)can be solved as follows:

The third equation has the solution

$$S(x,y) = f_1(y)x^2 + f_2(y) \tag{3.79}$$

Substituting equation (3.79) in the second equation of system (3.78) leads to

$$f_1(y) = c_1 \tag{3.80}$$

Also substituting equations (3.79) and (3.80) in the first equation of system (3.78) provides

$$f_2(y) = c_1y^2 + c_2y + c_3 \tag{3.81}$$

Therefore

$$S(x,y) = c_1(x^2 + y^2) + c_2y + c_3$$

Hence we have the two non-constant solutions which gives rise to the point transformation

$$\phi(x,y) = y, \quad \psi(x,y) = x^2 + y^2$$

with $J = -2x$ *that linearize the ODE (3.73) to the free particle equation* $u_{tt} = 0$*. So the general solution of the ODE (3.73) is obtained as*

$$x^2 + y^2 = c_1 + c_2 y.$$

Example 3.8 *In this example, we find the general solution of the nonlinear ODE*

$$y'' - \frac{1}{x}y'^3 + \frac{2y}{y^2-1}y'^2 - \frac{1}{x}y' = 0 \tag{3.82}$$

via linearization transformations. ODE (3.82) satisfies the Lie-Tressé linearization test (3.9). In this case, the system (3.8) becomes

$$w_x - zw = 0 \tag{3.83}$$

$$w_y + w^2 - \frac{2yw}{y^2-1} + \frac{z}{x} = 0 \tag{3.84}$$

$$z_x - z^2 - \frac{z}{x} = 0 \tag{3.85}$$

$$z_y + zw = 0 \tag{3.86}$$

and a solution (w, z) *of the Lie's conditions (3.8) is*

$$w(x, y) = 0, \quad z(x, y) = 0.$$

Hence, the corresponding system (3.19) gives

$$\begin{aligned}
&S_{yy} - \tfrac{2y}{y^2-1} S_y - \tfrac{1}{x} S_x = 0 \\
&S_{xy} = 0 \\
&S_{xx} - \tfrac{1}{x} S_x = 0
\end{aligned}$$

Now, solving this system gives the two solutions which gives the point transformation

$$\phi(x,y) = (y^3 - 3y), \quad \psi(x,y) = (y^3 - 3y + 2)\ln|y-1| + (2 + 3y - y^3)\ln|y+1| + 3x^2 - y^2$$

with $J = -18x(y-1)(y+1)$ *that linearize the ODE (3.82) to the free particle equation* $u_{tt} = 0$. *Hence the general solution of the ODE (3.82) can be written as*

$$(y^3 - 3y + 2)\ln|y-1| + (2 + 3y - y^3)\ln|y+1| + 3x^2 - y^2 = c_1 + c_2 y(y^2 - 3).$$

3.6 Conclusion

The question of linearization of second order ODEs is investigated employing λ-symmetries. The relationship between λ-symmetries and the first integrals plays an important role and provides a direct method to derive both of Lie's conditions (3.8) and quadratures (3.12). The relationship further leads to a λ-symmetry criteria for linearization via point transformations. This criteria provides a new approach for constructing the linearizing transformations whose implementation requires finding only one solution of the Lie's conditions (3.8). In comparison, the standard implementation of Lie's linearization approach involves solving Lie's quadratures (3.12) which requires finding three solutions of the Lie's conditions (3.8).

CHAPTER 4

ON THE LINEARIZATION OF SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS TO THE LAGUERRE FORM VIA GENERALIZED SUNDMAN TRANSFORMATIONS

The linearization problem for nonlinear second order ODEs to the Laguerre form by means of generalized Sundman transformations (S-transformations) is considered, which has been investigated by Duarte et al earlier. A characterization of these S-linearizable equations in terms of first integral and procedure for construction of linearizing S-transformations has been given recently by Muriel-Romero. Here we give a new characterization of S-linearizable equations in terms of the coefficients of ODE and one auxiliary function. This new criterion is used to obtain the general solutions for the first integral explicitly, providing a direct alternative procedure for constructing the first integrals and Sundman transformations. The effectiveness of this approach is demonstrated by applying it to find the general solution for geodesics on surfaces of revolution of constant curvature in a unified manner.

4.1 Introduction

The mathematical modeling of many physical phenomena leads to such nonlinear ordinary differential equations (ODEs) whose analytical solutions are hard to find directly. Therefore, the approach of investigating nonlinear ODEs via transforming to simpler ODEs becomes important and has been quite fruitful in analysis of physical problems. This includes the classical linearization problem of finding transformations that linearize a given ODE. For the linearization problem of second order ODEs via point transformations, it is known that these must be at most cubic in the first order derivative and its coefficients should satisfy the Lie linearization test [15, 16, 26, 31]. The implementation of the Lie linearization method requires solving systems of partial differential equations (PDEs). It is also as mentioned above only second order ODEs admitting 8 dimensional Lie symmetry algebra pass the Lie linearization test, which makes it a restricted class of ODEs. In order to consider a larger class of ODEs, linearization problem via nonlocal transformations has been investigated in [4, 7, 10].

Many of these transformations are of the form

$$u(t) = \psi(x, y)\,,\ dt = \phi(x, y, y')\,dx, \qquad \psi_y\phi \neq 0, \tag{4.1}$$

and the linearization problem via transformations (4.1), in general, is an open problem. In case that $\phi = \phi(x, y)$, the transformations of type (4.1) are called generalized Sundman transformations [9] and equations that can be linearized by means of generalized Sundman transformations to the Laguerre form $u_{tt} = 0$ are called S-linearizable [37]. These transformations have also been utilized to define Sundman symmetries of ODEs [8–10]. It should be mentioned that another special classes of nonlocal transformations of type (4.1) with polynomials of first degree in y' for $\phi(x, y, y')$ have also been studied in [5, 39].

Duarte et al [7] showed that the S-linearizable second-order equations

$$y'' = f(x, y, y') \tag{4.2}$$

are at most quadratic in the first derivative, i.e. belong to the family of equations of the form

$$y'' + F_2(x, y){y'}^2 + F_1(x, y)y' + F(x, y) = 0. \tag{4.3}$$

Precisely, the free particle equation

$$u_{tt} = 0 \tag{4.4}$$

can be transformed by an arbitrary generalized Sundman transformation

$$u(t) = \psi\,(x, y)\,, \; dt = \phi\,(x, y)\,dx, \qquad \psi_y\phi \neq 0, \tag{4.5}$$

to the family of equations of the form (4.3) with the coefficients $F(x, y), F_1(x, y)$ and $F_2(x, y)$ satisfying the following system of partial differential equations

$$AF_2 = A_y, \quad AF_1 = B_y + A_x, \quad AF = B_x. \tag{4.6}$$

where

$$A = \tfrac{\psi_y}{\varphi}, \quad B = \tfrac{\psi_x}{\varphi}. \tag{4.7}$$

In fact the derivatives under the Sundman transformations (4.5) are transformed as follows:

$$u_t = \frac{du}{dt} = \frac{\psi_x dx + \psi_y dy}{\phi dx} = \frac{(\psi_x + y'\psi_y)}{\phi} = \frac{D_x(\psi)}{\phi} \tag{4.8}$$

and

$$u_{tt} = \frac{D_x(u_t)}{\phi} = \frac{1}{\phi}D_x\left(\frac{D_x(\psi)}{\phi}\right) = \frac{1}{\phi}D_x\left(\frac{\psi_x}{\phi} + \frac{\psi_y}{\phi}y'\right) \tag{4.9}$$

On the other hand, if we equate the second derivative transformation (4.9) to zero, we get

$$\left(\frac{\psi_x}{\phi}\right)_x + \left(\frac{\psi_x}{\phi}\right)_y y' + \left[\left(\frac{\psi_y}{\phi}\right)_x + \left(\frac{\psi_y}{\phi}\right)_y y'\right] y' + \frac{\psi_y}{\phi}y'' = 0$$

and by collecting the coefficients of $y'^2, y', (y')^0$ we get the ODE (4.3) where $F(x, y), F_1(x, y),$ and $F_2(x, y)$ are given by (4.6).

Duarte et al [7] also gave a characterization of these S-linearizable equations in terms of the coefficients. Muriel and Romero [37] further studied S-linearizable equations and proved that these must admit first integrals that are polynomials of first degree in the first-order derivative.

Theorem 4.1 *[37] The ODE (4.2) is S-linearizable if and only if it admits a first integral of the form* $w(x,y,y') = A(x,y)y' + B(x,y)$. *In this case ODE has the form (4.3). If a linearizing S-transformation (4.5) is known then a first integral* $w(x,y,y') = A(x,y)y' + B(x,y)$ *of (4.3) is defined by (4.7). Conversely, if a first integral* $w(x,y,y') = A(x,y)y' + B(x,y)$ *of (4.3) is known then a linearizing S-transformation can be determined by*

$$\begin{aligned} &\psi(x,y) = \eta(I(x,y)), \\ &\phi(x,y) = \tfrac{\psi_y}{A} \ or \ \phi(x,y) = \tfrac{\psi_x}{B} \ if \ B \neq 0, \end{aligned} \tag{4.10}$$

where $I(x,y)$ *is the first integral of*

$$y' = -\frac{B}{A}. \tag{4.11}$$

Proof. We suppose that A and B are two functions satisfying system (4.6), and define $w = A(x,y)y' + B(x,y)$, then

$$\begin{aligned} D_x(w) &= \tfrac{\partial w}{\partial x} + y'\tfrac{\partial w}{\partial y} + y''\tfrac{\partial w}{\partial y'} \\ &= Ay'' + A_y y'^2 + (B_y + A_x)y' + B_x \\ &= A\left(y'' + \tfrac{A_y}{A}y'^2 + (\tfrac{B_y+A_x}{A})y' + \tfrac{B_x}{A}\right) \end{aligned}$$

Therefore, w is a first integral of (4.2) and A is an integrating factor of (4.3).

Conversely, $w = A(x,y)y' + B(x,y)$ be a first integral of (4.2), then $D_x(w)|_{y''=f} = 0$; i.e.

$$D_x(w) = \frac{\partial w}{\partial x} + y'\frac{\partial w}{\partial y} + y''\frac{\partial w}{\partial y'} = 0.$$

Hence,

$$Ay'' + A_y y'^2 + (A_x + B_y)y' + B_x = 0$$

after dividing by A, we get the equation of the form (4.3) where the coefficients $F(x,y)$, $F_1(x,y)$, and $F_2(x,y)$ satisfying system (4.6), So it is S-linearizable.

To prove that in this case equation (4.3) is linearizable by an S-transformation (4.5), We first try to find a function ψ such that

$$B\psi_y - A\psi_x = 0$$

and the characteristic equation of this first order linear partial differential equations is given by

$$y' = -\frac{B}{A} \tag{4.12}$$

For any solution $I(x,y) = K$, $K \in \Re$ of equation (4.12) we can choose ψ to be any non-constant function of the form

$$\psi(x,y) = \eta(I(x,y))$$

Then ϕ is uniquely determined by

$$\phi(x,y) = \frac{\psi_y}{A} \ \ or \ \ \phi(x,y) = \frac{\psi_x}{B} \ \ if \ B \neq 0.$$

∎

Moreover, Muriel and Romero in [35,37] revisited Duarte results [7], presented the following equivalent characterization of S-linearizable ODE of the form (4.3), and also provided a constructive methods to derive the linearizing S-transformations.

Theorem 4.2 *[37] Let us consider an equation of the form (4.3) and let S_1 and S_2 be the functions defined by*

$$\begin{aligned} S_1(x,y) &= F_{1y} - 2F_{2x}, \\ S_2(x,y) &= (FF_2 + F_y)_y + (F_{2x} - F_{1y})_x + (F_{2x} - F_{1y})F_1. \end{aligned} \tag{4.13}$$

The following alternatives hold.

- *If $S_1 = 0$ then equation (4.3) is S-linearizable if and only if $S_2 = 0$*
- *If $S_1 \neq 0$, let S_3 and S_4 be the functions defined by*

$$\begin{aligned} S_3(x,y) &= \left(\tfrac{S_2}{S_1}\right)_y - (F_{2x} - F_{1y}), \\ S_4(x,y) &= \left(\tfrac{S_2}{S_1}\right)_x + \left(\tfrac{S_2}{S_1}\right)^2 + F_1\left(\tfrac{S_2}{S_1}\right) + FF_2 + F_y. \end{aligned} \tag{4.14}$$

  equation (4.3) is S-linearizable if and only if $S_3 = 0$ and $S_4 = 0$.

Theorem 4.3 *[37] Consider an equation of the form (4.3) and let S_1 and S_2 be the functions defined by (3.13). The following alternative hold:*

- *If* $S_1 = 0$ *then the equation has a first integral of the form* $w = A(x,y)y' + B(x,y)$ *if and only if* $S_2 = 0$. *In this case* A *and* B *can be given as* $A = q\ e^P$, $B = Q$, *where* P *is a solution of the system*

  $$P_x = \frac{1}{2}F_1, \qquad P_y = F_2, \tag{4.15}$$

  q *is a nonzero solution of*

  $$q''(x) + f(x)q(x) = 0 \tag{4.16}$$

  where

  $$f(x) = FF_2 + F_y - \frac{1}{2}F_{1x} - \frac{1}{4}{F_1}^2 \tag{4.17}$$

  and Q *is a solution of the system*

  $$Q_x = F\,q\,e^P\,, \qquad Q_y\ = \left(\frac{1}{2}F_1 - \frac{q'}{q}\right) q\,e^{\,P}. \tag{4.18}$$

- *If* $S_1 \neq 0$ *then the equation has a first integral has a first integral of the form* $w = A(x,y)y' + B(x,y)$ *if and only if* $S_3 = S_4 = 0$, *where* S_3 *and* S_4 *are the functions defined by (4.14). In this case* A *and* B *can be given as* $A = e^P$, $B = Q$, *where* P *is a solution of the system*

  $$P_x = F_1 + \frac{S_2}{S_1}\,, \qquad P_y = F_2, \tag{4.19}$$

and Q is a solution of the system

$$Q_x = F \ e^P , \quad Q_y = -\left(\frac{S_2}{S_1}\right) \ e^P . \tag{4.20}$$

The following examples illustratable the algorithms of construction the first integrals of the form $A(x,y)y' + B(x,y)$and the Sundman transformations.

Example 4.4 *Consider the second order nonlinear ODE*

$$y'' - \frac{2y'^2}{y} + \frac{2y}{x^2} = 0 \tag{4.21}$$

It can be shown that the coefficients of equation (4.21) satisfy $S_1 = S_2 = 0$

- *For equation (4.21) the corresponding system (4.15) is*

  $$P_x = 0 \ , \quad P_y = \frac{-2}{y}.$$

  a solution of this system is given by $P(x,y) = -2\ln|y|$

- *Equation (4.16) for this case is*

  $$q''(x) - \frac{2}{x^2}q(x) = 0 \ . \tag{4.22}$$

  To solve this Cauchy-Euler differential equation, assume $q = x^m$. Hence equation

(4.22) gives the characteristic equation

$$m^2 - m - 2 = 0$$

which leads to the following two linearly independent solutions of equation (4.22)

$$q_1(x) = x^2 \ , \quad q_2(x) = \frac{1}{x} \ .$$

- *Consider the solution $q_1(x) = x^2$, then the corresponding system (4.18) is*

$$Q_x = \frac{2}{y} \ , \qquad Q_y = \frac{-2x}{y^2}.$$

Hence a solution of this system is $Q_1 = \frac{2x}{y}$.

- *Therefore we get*

$$A = \frac{x^2}{y^2},$$

$$B = \frac{2x}{y}.$$

So a first integral of (4.21) is given by

$$w^1 = \frac{x^2}{y^2}y' + \frac{2x}{y}$$

- *Now using relation (4.11), we find the first integral of*

$$y' = -\frac{2y}{x}$$

as

$$I = x^2y.$$

- *The Sundman transformation is*

$$\psi(x, y) = \eta(x^2y)$$

$$\phi(x, y) = y^2\eta'(x^2y)$$

- *Now consider* $q_2(x) = \frac{1}{x}$, *so the system (4.18) becomes*

$$Q_x = \frac{2}{x^3y},$$

$$Q_y = \frac{1}{x^2y^2}.$$

and a solution of this system is

$$Q_2 = -\frac{1}{x^2y}.$$

- *Therefore*

$$A = \frac{1}{xy^2},$$

$$B = -\frac{1}{x^2y}.$$

Hence a second first integral of (4.21) is given by

$$w^2 = \frac{1}{xy^2}y' - \frac{1}{x^2y}.$$

- *Now using relation (4.11), we find the first integral of*

$$y' = \frac{1}{x^2y}$$

as

$$I = \frac{y}{x}.$$

- *Using (4.10), the Sundman transformation is defined as*

$$\psi(x,y) = \eta(\frac{y}{x})$$

$$\phi(x,y) = y^2\eta'(\frac{y}{x})$$

Example 4.5 *Consider the ODE*

$$y'' + \left(\frac{y-1}{y}\right)y'^2 - \frac{xe^{-y}}{y}y' + e^{-y} = 0 \qquad (4.23)$$

It can be shown that the coefficients of equation (4.23) satisfy $S_3 = S_4 = 0$

The corresponding system (4.19) is

$$P_x = 0\,,\;\; P_y = 1 - \frac{1}{y}$$

and a solution of this system is

$$P(x,y) = y - \ln|y|.$$

- *The corresponding system (4.20) is*

$$Q_x = \frac{1}{y}\,, \quad Q_y = \frac{-x}{y^2}$$

which has a solution

$$Q(x,y) = \frac{x}{y}.$$

- *Therefore*

$$A = \frac{e^y}{y},$$

$$B = \frac{x}{y}.$$

- *Hence, equation (4.23) has a first integral*

$$w = \frac{e^y}{y}y' + \frac{x}{y}.$$

- *The equation corresponding to (4.11) is given by*

$$y' = -xe^{-y}$$

which implies

$$I(x,y) = e^{y} + \frac{x^2}{2}.$$

- *Finally, the Sundman transformations is defined as*

$$\begin{aligned}\psi(x,y) &= \eta(e^{y} + \tfrac{x^2}{2}),\\ \phi(x,y) &= y\,\eta'(e^{y} + \tfrac{x^2}{2}).\end{aligned}$$

Example 4.6 *The nonlinear ODE*

$$y'' - \left(\tan(y) + \frac{1}{y}\right)y'^2 + \left(\frac{1}{x} - \frac{\tan(y)}{xy}\right)y' - \frac{\tan(y)}{x^2} = 0 \tag{4.24}$$

has coefficients satisfy $S_3 = S_4 = 0$

- *The corresponding system (4.19) is*

$$P_x = 0\ ,\quad P_y = -\tan(y) - \frac{1}{y},$$

which a the solution

$$P = \ln\left|\frac{\cos(y)}{y}\right|.$$

- *The corresponding system (4.20) is*

$$Q_x = -\frac{\sin(y)}{x^2 y}, \quad Q_y = \frac{\cos(y)\,(y - \tan(y))}{xy^2}$$

which has a solution

$$Q(x, y) = \frac{\sin(y)}{xy}.$$

- *Therefore*

$$A = \frac{\cos(y)}{y},$$

$$B = \frac{\sin(y)}{xy},$$

Hence

$$w = \frac{\cos(y)}{y} y' + \frac{\sin(y)}{xy}.$$

is a first integral of equation (4.24)

- *The corresponding equation of (4.11) is given by*

$$y' = -\frac{\tan(y)}{x}.$$

- *which implies*

$$I(x, y) = x\sin(y).$$

- *Therefore the Sundman transformations is given as*

$$\psi(x,y) = \eta(x\sin(y)),$$

$$\phi(x,y) = xy\,\eta'(x\sin(y)).$$

Remark 4.7 *The general solution of equation (4.3) which satisfy* $S_1 = S_2 = 0$ *can be found by eliminating* y' *from* $w_1 = C_1$ *and* $w_2 = C_2$*,* $C_1, C_2 \in \Re$*. For instance the ODE (4.21) has two functionally independent first integrals, namely:*

$$w_1 = \frac{x^2}{y^2}y' + \frac{2x}{y}$$

$$w_2 = \frac{1}{xy^2}y' - \frac{1}{x^2y}$$

Set $w_1 = C_1$ *and* $w_2 = C_2$ *to get*

$$\frac{x^2}{y^2}y' + \frac{2x}{y} = C_1\,, \quad \frac{1}{xy^2}y' - \frac{1}{x^2y} = C_2 \tag{4.25}$$

from the first equation in (4.25) we have

$$y' = C_1\frac{y^2}{x^2} - 2\frac{y}{x} \tag{4.26}$$

substitute the expression of y' *(4.26) in the second equation of (4.25) to get the general solution of the ODE (4.21)*

$$y(x) = \frac{3x}{C_1 - C_2x^3}.$$

In this chapter , a new characterization of S-linearizable equations in terms of the coefficients and one auxiliary function is given, and the equivalence with the old criteria is proved. This criterion is used to provide explicit general solutions for the auxiliary functions A and B given in (4.7) which can be directly utilized to obtain the first integral of (4.3). So, using Theorem 4.1, the linearizing generalized Sundman transformations can be constructed by solving the first order ODE (4.11). The method is illustrated in examples where we recover the Sundman transformations of Muriel and Romero in [37].

As an application, we express the system of geodesic equations for surfaces of revolution as a single second order ODE and use our method to find the general solution for geodesics on surfaces of revolution of constant curvature in a unified manner.

In this chapter, we have focused on S-linearization to the Laguerre form $u_{tt} = 0$. For an account of S-linearization to any linear second order ODE, the reader is referred to [40].

4.2 The method for constructing the first integrals and Sundman transformations

When the ODE (4.3) is S-linearizable, Theorem 4.2 does not give a method to construct the linearizing generalized Sundman transformations. In order to derive a method to obtain linearizing generalized Sundman transformations (4.5) of a given

S-linearizable equation (4.3), Muriel and Romero [37] found additional relationships between the functions ϕ and ψ in (4.5) and the functions $F(x,y), F_1(x,y)$ and $F_2(x,y)$, in (4.3), and used these to provide constructive methods to derive the linearizing S-transformations for the case $S_1 = S_2 = 0$ and the case $S_1 \neq 0$ but $S_3 = S_4 = 0$. This section provides an alternative procedure for constructing the first integrals and Sundman transformations for S-linearizable equations, which can be applied to both of the cases.

The key idea in this chapter is that instead of finding additional relationships between the functions ϕ and ψ, we find additional relationship between the functions A and B in (4.7) and the functions $F(x,y), F_1(x,y)$ and $F_2(x,y)$, in (4.3). Equations (4.6) implies

$$\begin{aligned}\left(\tfrac{B_y - A_x}{A}\right)_y &= \tfrac{B_{yy}A - A_{xy}A - A_yB_y + A_yA_x}{A^2}\\ &= \tfrac{A(B_{yy}+A_{xy}) - A_y(B_y+A_x)}{A^2} - 2\left(\tfrac{A_{xy}A - A_xA_y}{A^2}\right)\\ &= \left(\tfrac{B_y+A_x}{A}\right)_y - 2\left(\tfrac{A_y}{A}\right)_x\\ &= F_{1y} - 2F_{2x}\end{aligned}$$

The formula

$$\left(\frac{B_y - A_x}{A}\right)_y = F_{1y} - 2F_{2x}, \tag{4.27}$$

leads to the following missing relationship

$$B_y - A_x = A(F_1 - 2h_x) \tag{4.28}$$

where

$$h = \int F_2(x,y)dy + g(x), \tag{4.29}$$

and $g(x)$ can be determined using Theorem 4.8 in case that the ODE is S-linearizable.

This missing equation jointly with (4.6) give a new compact S-linearizability criterion for ODE of the form (4.3), given in Theorem 4.8. The S-linearizability criterion is used to provide explicit general solutions for the auxiliary functions A and B which can be directly utilized to obtain the first integral of (4.6), given in Theorem 4.10, and hence the Sundman transformations can be constructed using Theorem 4.1. Thus an alternative procedure for constructing the first integral and Sundman transformation is obtained.

Theorem 4.8 *Let us consider an equation of the form (4.3) and let* $h = \int F_2(x,y)dy + g(x)$*. Equation (4.3) is S-linearizable if and only if*

$$F_{1x} + F_1 h_x - {h_x}^2 - h_{xx} - F_y - F F_2 = 0, \tag{4.30}$$

for some auxiliary function $g(x)$*.*

Proof. Using the new relationship (4.28) with equations (4.6) one can get the following equations

$$\begin{aligned} A_x &= A h_x, \\ A_y &= A F_2, \\ B_x &= A F, \\ B_y &= A(F_1 - h_x). \end{aligned} \tag{4.31}$$

The compatibility of the system (4.31), i.e. $A_{xy} = A_{yx}$ and $B_{xy} = B_{yx}$, leads to the criteria (4.30). In order to show that the new criterion (4.30) is equivalent to the one given in Theorem 4.2, we note that the system consisting of equation (4.30) and the second derivatives of h given by (4.29)

$$\begin{aligned} h_{xx} &= F_{1x} + F_1 h_x - {h_x}^2 - F_y - FF_2, \\ h_{yx} &= F_{2x}, \\ h_{yy} &= F_{2y}. \end{aligned} \tag{4.32}$$

is compatible i.e. $h_{xy} = h_{yx}, h_{xxy} = h_{yxx}$ and $h_{yyx} = h_{yxy}$, when the following equation holds

$$h_x(F_{1y} - 2F_{2x}) + F_{2x}F_1 + F_{1xy} - F_{2xx} - F_{yy} - F_yF_2 - FF_{2y} = 0. \tag{4.33}$$

Now, using S_1 and S_2 defined by (4.13), equation (4.33) can be rewritten as

$$S_1 h_x = S_2 + F_1 S_1. \tag{4.34}$$

Then clearly if $S_1 = 0$, then $S_2 = 0$ and if $S_1 \neq 0$, then

$$h_x = \frac{S_2}{S_1} + F_1. \tag{4.35}$$

Finally, substituting (4.35) in (4.32), gives

$$
\begin{aligned}
S_3(x,y) &= \left(\tfrac{S_2}{S_1}\right)_y - (F_{2x} - F_{1y}) = 0,\\
S_4(x,y) &= \left(\tfrac{S_2}{S_1}\right)_x + \left(\tfrac{S_2}{S_1}\right)^2 + F_1\left(\tfrac{S_2}{S_1}\right) + FF_2 + F_y = 0.
\end{aligned}
$$

∎

Remark 4.9 *In the case $S_1 = S_2 = 0$, the criteria (4.30) can be transformed by the change of variable*

$$g(x) = \ln|q(x)| + k(x) \tag{4.36}$$

where $k'(x) = \frac{1}{2}F_1 - \int F_{2x}dy$ to the well-defined ODE, equation (4.16), in Theorem 4.3 and $P = h - \ln|q|$ verifies the system (4.15) in theorem 4.3.

Moreover, in the case $S_3 = S_4 = 0$, the criteria (4.30) implies equation (4.35) which shows that $P = h$ verifies the system (4.19) in Theorem 4.3 and hence solving this system provides a well-defined ODE

$$g'(x) = k(x) \tag{4.37}$$

where $k(x) = \frac{S_2}{S_1} + F_1 - \int F_{2x}dy$.

Hence when equation (4.3) is S-linearizable, one can solve the criteria (4.30) for a function $g(x)$ by considering both of x and y as independent variables. Or equivalently one can get the function $g(x)$ by equation (4.36) when $S_1 = S_2 = 0$ whereas when $S_3 = S_4 = 0$, the function $g(x)$ can be obtained from equation (4.37).

In the next theorem, the general solution of the first integral is given explicitly in terms of the function $h(x)$ where $h = \int F_2(x,y)dy + g(x)$. It can be verified that this solution coincides with the solution of the systems given in theorem 4.3. Hence, it provides an alternate direct procedure for constructing the first integrals and the S-transformations.

Theorem 4.10 *Let us assume that equation (4.3) is S-linearizable. Then (4.3) has the first integral* $w(x,y,y') = A(x,y)y' + B(x,y)$ *where* A *and* B *are given by*

$$\begin{aligned} &A(x,y) = \mathrm{e}^h, \\ &B(x,y) = \int F\,\mathrm{e}^h dx + \int\, \left(\mathrm{e}^h(F_1 - h_x) - \int \mathrm{e}^h\,(F_y + FF_2)\,dx\right) dy. \end{aligned} \tag{4.38}$$

and h *is given by (4.29).*

Proof. The functions A and B defined by (4.7) can be given explicitly by finding the general solution of the system (4.31) where the second and the third equations of the system (4.31) have general solution

$$\begin{aligned} &A = v\,(x)\,\mathrm{e}^h, \\ &B = \int FA\;dx + z\,(y)\,, \end{aligned} \tag{4.39}$$

for arbitrary functions $v\,(x)$ and $z\,(y)$.

Substituting (4.39) in the first and the fourth equations of the system (4.31) gives

$$\begin{aligned} &v\,(x) = c_1, \\ &z(y) = \int(A(F_1 - h_x) - \int(FA_y + F_yA)\;dx)dy + k(x), \end{aligned} \tag{4.40}$$

for arbitrary functions $k\,(x)$.

Now, differentiating (4.40) with respect to x and using the criterion (4.30) gives

$$k_x = -\int A(F_{1x} + F_1 h_x - {h_x}^2 - h_{xx} - F_y - FF_2)dy = 0. \tag{4.41}$$

So $k(x) = c_2$, and finally, from Theorem 4.1, equation (4.3) has the first integral $w(x, y, y') = A(x, y)y' + B(x, y)$ and without loss of generality, one can choose $c_1 = 1$ and $c_2 = 0$ by relabeling of $\frac{w(x,y,y')-c_2}{c_1}$. ▌

Remark 4.11 *An algorithmic implementation of our approach can be carried out as summarized below:*

Given a S-linearizable ODE of the form (4.3). Use Theorem 4.8 to determine an auxiliary function $g(x)$*. Find the first integrals using* $g(x)$ *and Theorem 4.10. Construct the Sundman transformations (4.1) using the first integral and Theorem 4.1. Since the free particle equation* $u_{tt} = 0$ *has the general solution* $u(t) = c_1 + c_2 t$*, finally using the Sundman transformations leads to the second integral of the ODE (4.3)*

$$\psi(x, y) = c_1 + c_2\ \mu(x), \tag{4.42}$$

where $t = \mu(x)$ *is a solution of the first order ODE*

$$\tfrac{dt}{dx} = \phi(x, \gamma(x, t)). \tag{4.43}$$

and $y = \gamma(x, t)$ *can be obtained by solving* $c_1 + c_2 t = \psi\,(x, y)$ *for* y*.*

But in case that $\phi = \phi(x)$*,* $\mu(x) = \int \phi(x)dx$ *and so the Sundman transformation* *is a point transformation and leads to the general solution [40].*

In the next two examples, we apply our approach to construct the first integrals and use these to recover the Sundman transformations of Muriel and Romero in [37]. In addition, we provide the two parameters family of solution in the first example.

Example 4.12 Consider the ODE for the variable frequency oscillator [30]

$$y'' + yy'^2 = 0, \tag{4.44}$$

Theorem 4.2 shows that the coefficients of the equation satisfy $S_1 = 0, S_2 = 0$. By Theorem (4.8), ODE (4.44) is S-linearizable if and only if

$$g'' + g'^2 = 0 \tag{4.45}$$

for some auxiliary function $g(x)$. The solution of ODE (4.45) can be given, by the change of variable $v = g'$ as

$$g(x) = \ln|x + c_1| + c_2$$

A particular solution of (4.45) is $g(x) = \ln|x|$ so by (4.29) we have $h = \frac{y^2}{2} + \ln|x|$ and hence using Theorem 4.10, we can get the first integral $w(x, y, y') = A(x, y)y' + B(x, y)$ where

$$\begin{aligned} A(x, y) &= x\ exp(\tfrac{y^2}{2}), \\ B(x, y) &= -\textstyle\int exp(\tfrac{y^2}{2})dy. \end{aligned} \tag{4.46}$$

Finally, the Sundman transformations can be constructed using Theorem 4.1 as follows:

$$\begin{aligned} \psi(x,y) &= \eta(I(x,y)), \\ \phi(x,y) &= \tfrac{1}{x^2}\eta'(I(x,y)), \end{aligned} \tag{4.47}$$

where $I(x,y) = \frac{\int exp(\frac{y^2}{2})dy}{x}$.

Now choosing $\eta(I) = I$ makes $\phi(x,y) = \phi(x)$ and so using Remark 4.11 gives the two parameters family solution of the ODE (4.44)

$$erfi(\tfrac{y}{\sqrt{2}}) = C_1 x + C_2, \tag{4.48}$$

where $erfi(y) = \frac{2}{\sqrt{\pi}}\int_0^y e^{t^2}dt$ is the imaginary error function.

Example 4.13 *Consider the equation*

$$y'' - \left(\tan y + \frac{1}{y}\right)y'^2 + \left(\frac{1}{x} - \frac{\tan y}{xy}\right)y' - \frac{\tan y}{x^2} = 0 \tag{4.49}$$

Theorem 4.2 shows that the coefficients of this equation satisfy $S_1 \neq 0$ but $S_3 = S_4 = 0$. In addition, it follows from Theorem 4.8 that ODE (4.49) is S-linearizable if and only if

$$g'' + g'^2 - (\tfrac{1}{x} - \tfrac{\tan y}{xy})g' = 0 \tag{4.50}$$

for some auxiliary function $g(x)$. The only solution of (4.50) is $g(x) = C$ so by (4.29) we have $h = \ln\left|\frac{\cos y}{y}\right|$ and hence using Theorem 4.10, we can get the first integral

$w(x, y, y') = A(x, y)y' + B(x, y)$ where

$$A(x, y) = \frac{\cos y}{y}$$

$$B(x, y) = \frac{\sin y}{xy}$$

Finally, the Sundman transformations can be constructed using Theorem 4.1 as follows:

$$\psi(x, y) = \eta(I(x, y)),$$

$$\phi(x, y) = xy\ \eta'(I(x, y)),$$

where $I(x, y) = x \sin y$.

One can show that there is no $\eta(I)$ which makes $\phi = \phi(x)$ and so using Remark 4.11 for $\eta(I) = I$ gives the two parameters family solution of the ODE (4.49)

$$x \sin y = c_1 + c_2\mu(x),$$

where the function $t = \mu(x)$ is a solution of the equation

$$\frac{dt}{dx} = x \sin^{-1}\left(\frac{c_1+c_2t}{x}\right).$$

For example, if $c_2 = 0$, then one obtains the solution of ODE (4.49) as

$$x \sin y = c_1.$$

As another application, we solve geodesic equations for surfaces of revolution of constant curvature in a unified manner. Consider a surface of revolution with parameterizations $(f(y)\cos x, f(y)\sin x, g(y))$ obtained by revolving the unit speed curve $(f(y), g(y))$. The geodesic equations are [43].

$$
\begin{aligned}
&\ddot{y} = f(y)f'(y)\dot{x}^2,\\
&\tfrac{d}{dt}\left(f(y)^2\dot{x}\right) = 0,
\end{aligned}
$$

where $\dot{y} = \frac{dy}{dt}$ and $\dot{x} = \frac{dx}{dt}$.

Using the formulas $\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}}$ and $\frac{d^2y}{dx^2} = \frac{\dot{x}\ddot{y}-\dot{y}\ddot{x}}{\dot{x}^3}$ gives

$$
y'' - 2\tfrac{f'(y)}{f(y)}y'^2 - f'(y)f(y) = 0, \tag{4.51}
$$

which as special case for $f(y) = \sin y$ includes the equation for geodesics on unit sphere given in [2, 47].

Example 4.14 We consider the nonlinear second order ODE (4.51) for $f(y) = y, f(y) = b + y, f(y) = \sin y$ and $f(y) = \sinh y$ describing the geodesics on cone, plane, sphere and surface of conic type respectively.

For the surfaces under consideration we have $f'^2(y) - f(y)f''(y) = 1$. It can be checked from Theorem 4.2 that the coefficients of the equation satisfy $S_1 = 0, S_2 = 0$. In addition, it follows from Theorem 4.8 that ODE (4.51) for each of $f(y) = y, f(y) = b + y, f(y) = \sin y$ and $f(y) = \sinh y$ is S-linearizable if and only if

$$
g'' + g'^2 + 1 = 0, \tag{4.52}
$$

The solution of ODE (4.52) can be given, by the change of variable $v = g'$ as

$$g(x) = \ln|\cos(c_1 - x)| = \ln|\cos(c_1)\cos(x) + \sin(c_1)\sin(x)| + c_2$$

therefore particular solution of (4.52) is $g(x) = \ln|\sin x|$, so by (4.29) we have $h = \ln\left|\frac{\sin x}{f^2(y)}\right|$ and hence using Theorem 4.10, we can get the first integral $w(x, y, y') = A(x, y)y' + B(x, y)$ where

$$\begin{aligned} A(x,y) &= \tfrac{\sin x}{f^2(y)}, \\ B(x,y) &= \tfrac{f'(y)}{f(y)} \cos x. \end{aligned} \tag{4.53}$$

Finally, the Sundman transformations can be constructed using Theorem 4.1 as follows:

$$\begin{aligned} \psi(x,y) &= \eta(I(x,y)), \\ \phi(x,y) &= \left(\tfrac{f(y)}{f'(y)}\right)^2 \eta'(I(x,y)), \end{aligned} \tag{4.54}$$

where $I(x, y) = \frac{f(y)}{f'(y)} \sin x$.

Now choosing $\eta(I) = \frac{1}{I}$ makes $\phi(x, y) = \phi(x)$. Hence (4.54) becomes

$$\begin{aligned} \psi(x,y) &= \tfrac{f'(y)}{f(y)sin(x)}, \\ \phi(x,y) &= -\csc^2(x), \end{aligned}$$

therefore $\mu(x) = cot(x)$. So using Remark 4.11 gives the two parameters family solution of the ODE (4.51)

$$c_1 f(y)\sin x + c_2 f(y)\cos x = f'(y). \tag{4.55}$$

Example 4.15 We consider the nonlinear second order ODE

$$y'' - 2\tanh y \; y'^2 - \cosh y \sinh y = 0, \tag{4.56}$$

that describes the geodesics on hyperboloid of one sheet.

Theorem 4.2 shows that the coefficients of the equation satisfy $S_1 = 0, S_2 = 0$. It follows from Theorem 4.8 that ODE (4.56) is S-linearizable if and only if

$$g'' + g'^2 - 1 = 0, \tag{4.57}$$

for some auxiliary function $g(x)$. The solution of ODE (4.57) can be given, by the change of variable $v = g'$ as

$$g(x) = \int \tanh(x + c_1)\,dx = \ln|c_2\cosh(x+c_1)| = \ln|c_2\sinh(x)\sinh(c_1)+c_2\cosh(x)\cosh(c_1)|$$

A particular solution of (4.57) is $g(x) = \ln|\sinh x|$, so by (4.29) we have $h = \ln\left|\frac{\sinh x}{\cosh^2 y}\right|$ and hence using Theorem 4.10, we can get the first integral $w(x, y, y') = A(x, y)y' + B(x, y)$ where

$$\begin{aligned} A(x,y) &= \tfrac{\sinh x}{\cosh^2 y}, \\ B(x,y) &= -\tfrac{\sinh y}{\cosh y}\cosh x. \end{aligned} \tag{4.58}$$

Finally, the Sundman transformations can be constructed using Theorem 4.1 as follows:

$$\begin{aligned} \psi(x,y) &= \eta(I(x,y)), \\ \phi(x,y) &= csch^2x\ \eta'(I(x,y)), \end{aligned} \tag{4.59}$$

where $I(x,y) = \frac{\tanh y}{\sinh x}$.

Now choosing $\eta(I) = I$ makes $\phi(x,y) = \phi(x)$ and so using Remark 4.11 gives the two parameters family solution of the ODE (4.56)

$$c_1 \cosh y \sinh x - c_2 \cosh y \cosh\ \ x = \sinh y. \tag{4.60}$$

Example 4.16 We consider the nonlinear second order ODE

$$y'' - 2y'^2 - e^{2y} = 0, \tag{4.61}$$

that describes the geodesics on pseudosphere.

Theorem 4.2 shows that the coefficients of the equation satisfy $S_1 = 0, S_2 = 0$. It follows from Theorem 4.8 that ODE (4.61) is S-linearizable if and only if

$$g'' + g'^2 = 0, \tag{4.62}$$

for some auxiliary function $g(x)$. The solution of ODE (4.62) can be given, by the change of variable $v = g'$ as

$$g(x) = \int \frac{dx}{x + c_1} = \ln|x + c_1| + c_2$$

For simplicity, choose a particular solution of (4.62) to be $g(x) = \ln|x|$, so by (4.29) we have $h = \ln x - 2y$ and hence using Theorem 4.10, we can get the first integral $w(x, y, y') = A(x, y)y' + B(x, y)$ where

$$\begin{aligned} A(x,y) &= x\ e^{-2y}, \\ B(x,y) &= \tfrac{1}{2}(e^{-2y} - x^2). \end{aligned}$$

Finally, the Sundman transformations can be constructed using Theorem 4.1 as follows:

$$\begin{aligned} \psi(x,y) &= \eta(I(x,y)), \\ \phi(x,y) &= -\tfrac{2}{x^2}\ \eta'(I(x,y)), \end{aligned} \tag{4.63}$$

where $I(x,y) = \frac{e^{-2y}}{x} + x$.

Now choosing $\eta(I) = I$ makes $\phi(x,y) = \phi(x)$ and so using Remark 4.11 gives the two parameters family solution of the ODE (4.61)

$$e^{-2y} + x^2 = c_1 x + 2c_2. \tag{4.64}$$

4.3 Conclusion

The recent Muriel-Romero characterization, Theorem 4.1, of the class of S-linearizable equations identifies these as the class of equations that admit first integrals of the form $A(x,y)y' + B(x,y)$. In this chapter, a new characterization of S-linearizable equations in terms of the coefficients and one auxiliary function is given in Theorem 4.8. This criterion is used to directly provide explicit general solutions for the auxiliary functions

A and B Theorem 4.10. So, using Theorem 4.1, the linearizing generalized Sundman transformations can be constructed by solving the first order ODE (4.11). Finally, it is shown in [37] that an equation of the form (4.3) is S-linearizable and linearizable via a point transformation if and only if $S_1 = S_2 = 0$. It is also known that the generalized Sundman transformation is a point transformation if and only if $\phi = \phi(x)$. So, by Remark 4.11, the generalized Sundman transformation leads to the general solution $\psi(x, y) = c_1 + c_2 \int \phi(x)dx$ if and only if $S_1 = S_2 = 0$.

Our method is illustrated in examples where we recover the Sundman transformations of Muriel and Romero in [37]. Furthermore, the system of geodesic equations for surfaces of revolution is expressed as a single second order ODE. It is noticed that this ODE is S-linearizable for surfaces of revolution with constant curvature. The method is applied to find the general solution of these geodesics in a unified manner.

CHAPTER 5

RELATIONSHIPS BETWEEN FIRST INTEGRAL AND λ-SYMMETRIES

5.1 First Integral of the Form $A(x,y)y'+B(x,y)$ and λ-Symmetries

In this section several relationships between first integral and λ-Symmetries will be derived. Let us review the main theorem

Theorem 5.1 *[39] Consider an equation of the form*

$$y'' + F_2(x,y){y'}^2 + F_1(x,y)y' + F(x,y) = 0. \tag{5.1}$$

and the coefficients $F(x,y), F_1(x,y),$ and $F_2(x,y)$ *satisfy the system*

$$AF_2 = A_y, \quad AF_1 = B_y + A_x, \quad AF = B_x. \tag{5.2}$$

and let S_1, S_2, S_3 *and* S_4 *be the functions defined by (4.13) and (4.14). The equation is such that either* $S_1 = S_2 = 0$ *or* $S_3 = S_4 = 0$ *if and only if* $\frac{\partial}{\partial y}$ *is a* λ*-Symmetry of (5.1) for some* $\lambda = -F_2(x,y)y' + \beta(x,y)$.

Proof. The vector field $\frac{\partial}{\partial y}$ is a λ-symmetry of (5.1) if and only if λ is a solution of the equation

$$M_y + \lambda M_{y'} = \lambda_x + y'\lambda_y + M\lambda_{y'} + \lambda^2 \tag{5.3}$$

Assume that the coefficients $F(x,y), F_1(x,y)$ and $F_2(x,y)$ of (5.1) are such that either

- $S_1 = S_2 = 0$, or
- $S_1 \neq 0$ and $S_3 = S_4 = 0$

equation (5.3) has solution of the form $\lambda = \alpha(x,y)y' + \beta(x,y)$. In this case the following system must be compatible

$$\alpha_y + \alpha^2 + F_2\alpha + F_{2y} = 0, \tag{5.4}$$

$$\beta_y + 2\left(F_2 + \alpha\right)\beta + F_{1y} + \alpha_x = 0, \tag{5.5}$$

$$\beta_x + \beta^2 + F_1\beta + F_y - F\alpha = 0. \tag{5.6}$$

It is clear that $\alpha(x,y) = -F_2(x,y)$ is a particular solution of equation(5.4). For this α , equations (5.5)-(5.6) are

$$\beta_y + F_{1y} - F_{2x} = 0, \tag{5.7}$$

$$\beta_x + \beta^2 + F_1\beta + F_y + FF_2 = 0. \tag{5.8}$$

- Case I. If $S_1 = S_2 = 0$ then from (4.13) we have

  $$F_{2x} = \frac{1}{2}F_{1y}$$

  Hence rewriting (5.7) gives

  $$\beta_y + \frac{1}{2}F_{1y} = 0$$

  and the solution of this equation is given by

  $$\beta = -\frac{1}{2}F_1 + h(x) \tag{5.9}$$

  substituting formula (5.9) in equation (5.8) gives

  $$h'(x) + h^2(x) + FF_2 + F_y - \frac{1}{2}F_{1x} - \frac{1}{4}{F_1}^2 = 0$$

  we conclude that $\beta = h(x) - \frac{1}{2}F_1$ satisfies (5.7) where $h(x)$ is any solution of the Riccati equation

  $$h'(x) + h^2(x) + f(x) = 0$$

  where f defined by (4.17) is such that $f = f(x)$, since $S_1 = S_2 = 0$. Therefore

$\lambda = -F_2(x, y)y' + \beta(x, y)$, in this case, such that $\frac{\partial}{\partial y}$ is a λ-Symmetry of (5.1).

- Case II. If $S_1 \neq 0$ and $S_3 = S_4 = 0$, from (4.14) we get

$$F_{2x} = F_{1y} + \left(\frac{S_2}{S_1}\right)_y$$

Hence, equation (5.7) becomes

$$\beta_y - \left(\frac{S_2}{S_1}\right)_y = 0 \tag{5.10}$$

integrating equation (5.10) leads to

$$\beta = \left(\frac{S_2}{S_1}\right) + g(x)$$

substituting this formula in equation (5.8) gives

$$g'(x) + 2\left(\frac{S_2}{S_1}\right) g(x) + g^2(x) + g(x)F_1(x) = 0 \tag{5.11}$$

note that $g(x) = 0$ is a particular solution of equation (5.11). So

$$\beta = \left(\frac{S_2}{S_1}\right)$$

is a solution of (5.7)-(5.8). Therefore $\frac{\partial}{\partial y}$ is a λ-Symmetry of (5.1) for

$$\lambda = -F_2 y' + \frac{S_2}{S_1}$$

Conversely, Suppose that $\frac{\partial}{\partial y}$ is a λ-Symmetry of (5.1) for some functions λ of the form $\lambda = -F_2(x,y)y' + \beta(x,y)$. Then β satisfies equations (5.7)-(5.8). If we define

$$\gamma(x,y) = \beta(x,y) + \frac{1}{2}F_1(x,y) \tag{5.12}$$

then γ satisfies the following system

$$\gamma_y + \frac{1}{2}S_1 = 0, \tag{5.13}$$

$$\gamma_x + \gamma^2 + f = 0 \tag{5.14}$$

where f defined by (4.17) and S_1 is defined by (4.13). Now since

$$\gamma_{xy} = \gamma_{yx}$$

Then

$$-2\,\gamma\,\gamma_y = -\frac{1}{2}S_{1x} + f_y \tag{5.15}$$

Starting from formula (5.12) the left hand side of equation (5.15) can be written as

$$\begin{aligned}
-2\,\gamma\,\gamma_y &= -2\left(\beta+\frac{1}{2}F_1\right)\left(\beta_y+\frac{1}{2}F_y\right)\\
&= (-F_{1y}-2\beta_y)\left(\beta+\frac{1}{2}F_1\right)\\
&= (F_{1y}-2\beta_y-2F_{1y})\left(\beta+\frac{1}{2}F_1\right)\\
&= (F_{1y}-2F_{2x})\left(\beta+\frac{1}{2}F_1\right)\\
&= S_1\gamma
\end{aligned}$$

So we have the

$$S_1\gamma = -\frac{1}{2}S_{1x}+f_y. \tag{5.16}$$

As before this equation leads to consider two cases,

- Case I. If $S_1 = 0$ Then (5.13) implies

  $$\gamma_y = 0$$

  differentiating (5.14) with respect to y implies that $f_y = 0$. Therefore $f = f(x)$ and $S_2 = f_y = 0$.

- Case II. If $S_1 \neq 0$ then by (5.16)

  $$\gamma = \frac{1}{S_1}\left(-\frac{1}{2}S_{1x}+f_y\right)$$

.Therefore

$$\begin{aligned}\beta(x,y) &= \gamma(x,y) - \tfrac{1}{2}F_1(x,y) \\ &= \tfrac{1}{S_1}\left[-\tfrac{1}{2}S_{1x} + f_y - \tfrac{1}{2}F_1(x,y)S_1\right] \\ &= \tfrac{1}{S_1}\left[\left(F_0F_2 + F_{0y}\right)_y + \left(F_{2x} - F_{1y}\right)_x + \left(F_{2x} - F_{1y}\right)F_1\right] \\ &= \tfrac{S_2}{S_1}\end{aligned}$$

and since $\beta(x,y) = \frac{S_2}{S_1}$ satisfies (5.7) and (5.8), it is necessary that $\beta(x,y)$ satisfies (4.14); i.e.

$$S_3 = S_4 = 0$$

$$\blacksquare$$

We can Sum up our former results in the following theorem

Theorem 5.2 *[39] The following conditions on an ODE of the form (5.1) are equivalent:*

1. *Equation $y'' = f(x,y,y')$ admit a first integral of the form $A(x,y)y' + B(x,y)$.*
2. *Equation $y'' = f(x,y,y')$ is of the form (5.1) and there exist two functions $A(x,y)$ and $B(x,y)$ that satisfy the system (5.2).*
3. *Equation $y'' = f(x,y,y')$ is of the form (5.1) and its coefficients are such that either $S_1 = S_2 = 0$ or $S_3 = S_4 = 0$*
4. *Equation $y'' = f(x,y,y')$ is of the form (5.1) and $\frac{\partial}{\partial y}$ is a λ-Symmetry for some function λ of the form $\lambda = -F_2(x,y)y' + \beta(x,y)$.*

5.2 Complete integrability in Case I ($S_1 = S_2 = 0$)

We note that ODEs in case I admit two independent first integrals of the form $A(x,y)y' + B(x,y)$. The following theorem provide the relation between ODEs in this class and the complete integrability.

Theorem 5.3 *[39] The following conditions on an equation of the form (5.1) are equivalent:*

1. *The equation admits two functionally independent first integrals of the form* $A(x,y)y' + B(x,y)$.
2. *The vector field* $\frac{\partial}{\partial y}$ *is a* λ_1*-symmetry and a* λ_2*-symmetry for some functions* $\lambda_1 = -F_2 y' + \beta_1$ *and* $\lambda_2 = -F_2 y' + \beta_2$ *with* $\beta_1 \neq \beta_2$.
3. $S_1 = S_2 = 0$.

Proof.

- (1 → 2) Assume that the equation (5.1) has two functionally independent first integrals of the form $A(x,y)y' + B(x,y)$, namely

  $$w_1(x,y,y') = A_1(x,y)y' + B_1(x,y) \text{ and } w_2(x,y,y') = A_2(x,y)y' + B_2(x,y)$$

  Then, the functions $A_i, B_i, i = 1,2$, must satisfy system (5.2)

  $$\begin{aligned} F_2 &= \tfrac{A_{1y}}{A_1} = \tfrac{A_{2y}}{A_2} \\ F_1 &= \tfrac{B_{1y}+A_{1x}}{A_1} = \tfrac{B_{2y}+A_{2x}}{A_2} \\ F &= \tfrac{B_{1x}}{A_1} = \tfrac{B_{2x}}{A_2} \end{aligned}$$

The vector field $\frac{\partial}{\partial y}$ is λ_i-symmetry for

$$\lambda_i = -\frac{w_{iy}}{w_{iy'}} = -F_2 y' - \frac{B_{iy}}{A_i} \qquad (i = 1, 2).$$

Since w_1 and w_2 are functionally independent, then

$$\begin{vmatrix} w_{1y} & w_{1y'} \\ w_{2y} & w_{2y'} \end{vmatrix} = \begin{vmatrix} A_{1y}y' + B_{1y} & A_1 \\ A_{2y}y' + B_{2y} & A_2 \end{vmatrix} = A_1 A_2 \begin{vmatrix} F_2 y' + \frac{B_{1y}}{A_1} & 1 \\ F_2 y' + \frac{B_{2y}}{A_2} & 1 \end{vmatrix} \neq 0 \qquad (5.17)$$

which definitely implies $\beta_1 = -\frac{B_{1y}}{A_1} \neq -\frac{B_{2y}}{A^2} = \beta^2$.

- $(2 \to 3)$ If $\beta_1 \neq \beta_2$ then the system (5.7)-(5.8) has two different solutions. This cannot happen in case $S_3 = S_4 = 0$ since in this case the function β such that $\frac{\partial}{\partial y}$ is a λ-symmetry for $\lambda = -F_2(x, y)y' + \beta(x, y)$ is uniquely determined by $\beta(x, y) = \frac{S_2}{S_1}$. Therefore $S_1 = S_2 = 0$.

- $(3 \to 1)$ In case $S_1 = S_2 = 0$, equation (4.16) has two linearly independent solutions q_1 and q_2 which provides two functions Q_1 and Q_2 satisfy the system (4.18)

$$Q_{ix} = F\, q_i\, e^p, \quad Q_{iy} = \left(\frac{1}{2}F_1 - \frac{q_i{}'}{q_i}\right) q_i\, e^p, \qquad (i = 1, 2).$$

Therefore the two first integrals of equation (5.1) are given by

$$w_1 = q_1\, e^p\, y' + Q_1, \quad w_2 = q_2\, e^p\, y' + Q_2$$

and since

$$\begin{vmatrix} w_{1y} & w_{1y'} \\ w_{2y} & w_{2y'} \end{vmatrix} = e^{2p}. W(q_1, q_2) \neq 0$$

where $W(q_1, q_2)$ denotes to the Wronskian of the functions q_1 and q_2. This proves that w_1 and w_2 are two functionally independent first integrals of equation $y'' = f(x, y, y')$.

∎

The relation between local and nonlocal linearizable was shown as follow.

Remark 5.4 *[38] An equation of the form*

$$y'' + F_2(x, y){y'}^2 + F_1(x, y)y' + F(x, y) = 0.$$

can be linearized by means of local and sundman transformations if and only if $S_1 = S_2 = 0$.

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