POLYNOMIAL SOLUTIONS OF DIFFERENTIAL EQUATIONS

BY

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To my dear family

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THESIS ABSTRACT

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TITLE OF STUDY: Polynomial Solutions of Differential Equations

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Polynomial solutions of linear differential equations have been investigated by many authors. In this thesis, we study a new approach based on elementary linear algebra for investigating polynomial solutions of differential equations $L(y) = \sum_{k=0}^{N} a_k(x)y^{(k)} = 0$ *with polynomial coefficients. Any differential operator of the form* $L(y) = \sum_{k=0}^{N} a_k(x)y^{(k)}$*, where* a_k *is a polynomial of degree* $\leq k$*, on the space of polynomials over an infinite field* F*, has all eigenvalues in* F*. If these eigenvalues are distinct, then there is a unique monic polynomial of degree* n *which is an eigenfunction of the operator* L*. We also carry out a study of orthogonality of such eigenfunctions. Further for the general case of operators* $L(y) = \sum_{k=0}^{N} a_k(x)y^{(k)}$ *where degree of the polynomials* a_k *is arbitrary, an algorithmic procedure is presented for determining the existence of polynomial solutions*

as well as for constructing these solutions. An implementation of the algorithmic procedure is carried out through Maple codes which are applied to obtain polynomial solutions of different types of differential equations of current interest in Physics.

ملخص الرسالة

الاسم الكامل: وليد أحمد سويدان الخليفي

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في هذه الرسالة نعرض طريقة جديدة معتمدة على مبادئ الجبر الخطي لدراسة كثيرات الحدود كحلول لمعادلات تفاضلية من الشكل $L(y) = \sum_{k=0}^{N} a_k(x) y^{(k)}$ حيث $a_k(x)$هي كثيرة حدود. وسنثبت عندما تكون درجة $a_k(x) \geq k$ فإن أي مؤثر خطي على شكل $L(y)$ ويؤثر على فضاء متجه من كثيرات الحدود على حقل غير منته F لها قيم ذاتية كلها تنتمي إلى الحقلF. إذا كانت هذه القيم الذاتية مختلفة فحينئذ توجد كثيرة حدود واحدية ووحيدة من الدرجة n كدالة ذاتية للمؤثر الخطي $L(y)$.
وكذلك سنناقش تعامد هذه الحلول. ونتناول المؤثرات الخطية التي تكون درجة كثيرة الحدود $a_k(x)$ عشوائية. وسنعرض اللوغاريتمية باستخدام Maple لإيجاد الشروط الضرورية لوجود كثيرات الحدود مع حسابها. وسنطبق هذه اللوغاريتمية على عدة معادلات تفاضلية المتعلقة بتطبيقات فيزيائية حديثة.

CHAPTER 1

INTRODUCTION

Polynomial solutions of linear differential equations is a classical topic. Many authors have extensively studied it, especially in relation to the orthogonal properties of these solutions; see for example [10, 13, 21] and the references therein. In 1884, Routh [20] gave a classification of second order differential equations with polynomial coefficients and discussed the orthogonality of these solutions. Bochner (1929) [4] also gave a classification of these linear differential equations but he missed some cases namely those yielding Romanovski polynomials. In 1930, Brenke [6] investigated polynomial solutions of linear differential equations using an approach similar to the one presented here. In his classification of second-order ODEs that have polynomial solutions, he discussed mainly the self-adjoint case. Polynomial solutions are also of current interest in Physics. Indeed, many authors, e.g. [11, 23] have studied their existence in relation to some specific problems, for example Schrödinger's equation with Coulomb potential, which is useful as a model potential in atomic and molecular physics [12]. Azad et al. [1, 2] approached the subject using elementary linear algebra, and recovered all the classical orthogonal polynomials as well as some non-classical ones. In this thesis we will study in detail this approach and apply it to several ODEs that are of current interest, and provide an algorithmic procedure which can be used to compute polynomial solutions of arbitrary degree, when they exist, depending of course on the available computational power.

We now give a summary of the contents of the thesis. In Chapter 2 we give the details of the theoretical results of this approach. Chapter 3 presents the algorithms using Maple. In Chapter 4 we use the Maple codes on examples that arise in Physics.

CHAPTER 2

MAIN RESULTS

This is the fundamental chapter of this thesis, we classify all linear differential operators of N-th order, which have polynomial coefficients and operate on the space of polynomials over an infinite field, into two cases which are presented in two sections. In each section we deal with the general form of the operators, then we focus our attention to second-order operators because of their importance in applications.

Throughout, $\mathbb{P}$ is the space of all polynomials over an infinite field F and $\mathbb{P}_n$ is the subspace of polynomials of degree at most n. Let $L : \mathbb{P} \to \mathbb{P}$ be the Nth-order operator defined by $L(y) = \sum_{k=0}^{N} a_k(x) D^k y$ where D is the usual differential operator and $a_k(x)$ is a polynomial of degree d_k $(0 \le k \le N)$. We will distinguish and study two cases for the operator L depending on whether $d_j \le j$ for all $0 \le j \le N$, and which we call **non-defective** operator, or $d_j > j$ for some j $(0 \le j \le N)$ which we call **defective** operator.

2.1 Non-defective operators

Following [1], we will show that non-defective operators have all their eigenvalues in the field F and in case the eigenvalues are distinct, there is exactly one monic polynomial in every degree which is an eigenfunction of L. In the first part, we consider general operators of order N, in particular $N = 2$, while, in the second part, we restrict our attention to second-order self-adjoint operators.

2.1.1 General case

In this case we can assume that $a_k(x) = \sum_{h\geq 0} a_{kh}x^h$ where $a_{kh} = 0$ if $k < h$. Let

$$L(y) = \sum_{k=1}^{N} a_k(x)D^k y. \tag{2.1}$$

It is obvious that $\mathbb{P}_n$ is L-invariant for each non-negative integer n. For $1 \leq j \leq n$, $L(x^j)$ is a scalar multiple of x^j plus lower order terms, from this we see that the matrix representation A_n of L, with respect to the standard basis $B_n = \{1, x, \ldots, x^n\}$, is upper triangular and the eigenvalues are the coefficients of x^j in $L(x^j)$. The following example will illustrate the pattern of entries of the matrix A_n

Example 2.1 *Let P_4 be the space of all polynomials of degree at most 4 and consider the 3^{rd} order operator L such that $Ly = \sum_{k=1}^{3} a_k(x)D^k y$. The matrix* *representation of L relative to the standard basis $B_4 = \{1, x, x^2, x^3, x^4\}$ is*

$$A_4 = \begin{pmatrix} 0 & a_{10} & 2a_{20} & 6a_{30} & 0 \\ 0 & a_{11} & 2a_{10} + 2a_{21} & 6a_{20} + 6a_{31} & 24a_{30} \\ 0 & 0 & 2a_{11} + 2a_{22} & 3a_{10} + 6a_{21} + 6a_{32} & 12a_{20} + 24a_{31} \\ 0 & 0 & 0 & 3a_{11} + 6a_{22} + 6a_{33} & 4a_{10} + 12a_{21} + 24a_{32} \\ 0 & 0 & 0 & 0 & 4a_{11} + 12a_{22} + 24a_{33} \end{pmatrix} \blacksquare$$

Now, when L is operating on $\mathbb{P}_n$, the $(n+1) \times (n+1)$ matrix representation in the standard basis of $\mathbb{P}_n$ will be of the form:

$$A_n = \left[\sum_{k \geq 1} (j-k)_k a_{k,k+i-j} \right]_{1 \leq i,j \leq n+1}$$

where $(j-k)_k = (j-1)(j-2)\ldots(j-k)$ and $a_{kh} = 0$ when $k < h$. Clearly A_{n+1} is obtained from A_n by adding one row and one column at the end, and so all the eigenvalues of the operator L are in F and are given by

$$\lambda_0 = 0, \lambda_n = na_{11} + n(n-1)a_{22} + \cdots + n!a_{nn} \text{ for } n \geq 1$$

where $a_{nn} = 0$ if $n > N$. From the invariance of $\mathbb{P}_n$ under L, each λ_n has an eigenfunction of the form $y_n = y_{n0} + y_{n1}x + \cdots + y_{nn}x^n$ whose vector representation is $(y_{n0}, y_{n1}, \ldots, y_{nn})^T$ in the standard basis B_n and can be computed from the homogeneous triangular system

$$(A_n - \lambda_n I)(y_{n0}, y_{n1}, \ldots, y_{nn})^T = 0 \tag{2.2}$$

We shall use the following definitions and lemma.

Definition 2.2 *Let λ be an eigenvalue of an operator L on a finite dimensional vector space V. The geometric multiplicity of λ is the dimension of its eigenspace while its algebraic multiplicity (or multiplicity for short) is its multiplicity as a zero of the characteristic polynomial of L.*

Definition 2.3 *The eigenvalue λ is said to be*

(1) simple if its algebraic multiplicity is equal to 1.

(2) semisimple if its algebraic and geometric multiplicities are equal.

(3) defective if its algebraic multiplicity is greater than its geometric multiplicity.

Lemma 2.4 *Geometric multiplicity $\leq$ Algebraic multiplicity*

Proof. Suppose that $dim(V) = n$ and let E_λ be the eigenspace corresponding to λ where $dim(E_\lambda) = m$. Let $\{v_1, v_2, \ldots, v_m\}$ be a basis for E_λ and extend this basis to a basis $\mathbb{B}$ for V by adding vectors $v_{m+1}, v_{m+2}, \ldots, v_n$. We see that the matrix representation of L, with respect to the basis $\mathbb{B}$, is

$$A = [L]_{\mathbb{B}} = \begin{bmatrix} \lambda I_m & C \\ 0 & D \end{bmatrix}$$

where $L(v_i) = \lambda v_i$ for $i = 1, \ldots, m$ while C and D are block matrices. Now the characteristic polynomial $p_A(x)$ of A is

$$p_A(x) = (\lambda - x)^m p_D(x)$$

where $p_D(x)$ is the characteristic polynomial of D. Hence, the geometric multiplicity is less than or equal to the algebraic multiplicity. ∎

Now back to the study of the operator $L(y)$ on the space of polynomials. We can prove the following result:

Corollary 2.5 *Let $L : \mathbb{P} \to \mathbb{P}$ be an operator given by $L(y) = \sum_{k=1}^{N} a_k(x) D^k y$, where $a_k(x)$ is a polynomial of degree at most k. If all the eigenvalues $\lambda_0, \lambda_1, \ldots, \lambda_n$ (for some n) are distinct, then L has (up to a constant) a unique polynomial for each degree r $(0 \leq r \leq n)$ corresponding to λ_r as an eigenfunction.*

Proof. Without loss of generality, suppose there are at least two different monic polynomials p ad q of different degrees as eigenfunctions of λ_r. Since p and q are linearly independent, the dimension of the eigenspace that contains p and q should be at least 2, but λ_r has multiplicity 1 which contradicts Lemma 2.4 (when applied to $L|_{\mathbb{P}_n}$). ∎

Now, we summarize the above in the following proposition

Proposition 2.6 *[1, Proposition 2.1] Let $L : \mathbb{P} \to \mathbb{P}$ be an operator given by $L(y) = \sum_{k=1}^{N} a_k(x) D^k y$, where $a_k(x)$ is a polynomial of degree at most k. For each k $(1 \leq k \leq N)$, let c_k be the coefficient of x^k in $a_k(x)$. Then all the eigenvalues of L are in the field F and are $\mathbb{Z}$-linear combinations of the c_k. If all the eigenvalues are distinct, then L has, up to a constant, a unique polynomial for each degree as an eigenfunction.*

Some observations concerning the eigenvalues and their multiplicity can be pointed out. First, let

$$f(x) = c_1 x + c_2 x(x-1) + \cdots + c_N x(x-1)\ldots(x-N+1) \tag{2.3}$$

where as in Proposition 2.6, $c_k = a_{kk}$ is the coefficient of x^k in a_k. Then each eigenvalue λ_n of L is just $f(n)$ $(n \geq 0)$. This immediately gives an $(N+1)$-term recurrence relation between the eigenvalues.

Lemma 2.7 *Let* $g(x) = \sum_{i=0}^{N} a_i x^i$ *and let* E *be the shift operator given by* $Eg(x) = g(x+1)$*, then* $(E-1)^{N+1} g(x) = 0$

Proof. By induction, when $N = 1$ suppose that $g(x) = a_0 + a_1 x$ then

$$\begin{aligned}
(E-1)^2 g(x) &= (E-1)(E-1)g(x) \\
&= (E-1)(Eg(x) - g(x)) \\
&= (E-1)(g(x+1) - g(x)) \\
&= (E-1)(a_0 + a_1 x + a_1 - a_0 - a_1 x) \\
&= (E-1)a_1 \\
&= 0
\end{aligned}$$

Assume that $(E-1)^N g(x) = 0$ if $deg(g) < N$. If $deg(g) = N$ then

$$(E-1)^{N+1} g(x) = (E-1)^N (E-1) g(x)$$

but $deg((E-1)g(x)) < N$, so from induction hypothesis we conclude that

$$(E-1)^{N+1}g(x) = 0$$

∎

From Lemma 2.7 we derive $(E-1)^{N+1}f(n) = 0$. When all the c_k are zero (i.e. all eigenvalues are equal to zero), then f is identically zero and one can get eigenfunctions of L by considering the $(N-1)$-st order operator obtained from L by replacing Dy by y. We therefore assume that f is not the zero polynomial. Suppose that an eigenvalue is repeated r times, say

$\lambda_{n_1} = \lambda_{n_2} = \cdots = \lambda_{n_r}$, where $0 \leq n_1 < n_2 < \cdots < n_r$. Here, f takes on the same value at r different non-negative integers, and so $r \leq deg(f) \leq N$. If the field is $\mathbb{R}$ then we have the following lemma

Lemma 2.8 *Let $f(x)$ be a non-constant polynomial over $\mathbb{R}$. Then for every $a \in \mathbb{R}$, the set $\{x : f(x) = a\}$ is finite.*

Proof. For fixed real number $a \in \mathbb{R}$, assume the set $\{x : f(x) = a\}$ is infinite. Consider $g(x) = f(x) - a$. This would have more than N distinct zeros, where N is the degree of f, which is impossible. ∎

Since f is not constant, there exists an interval $[k, \infty)$, where k is a positive integer, over which f is monotonic. This means that only finitely many eigenvalues λ_n of L have multiplicity greater than 1, and, if any exist, they must all lie between the largest local maximum and the smallest local minimum of f for $x \geq 0$. Another fact worth noticing occurs when $N = 2$. Suppose that not both coefficients c_1 and c_2 are zero and that an eigenvalue has algebraic multiplicity 2, say $\lambda_n = \lambda_{n'}$ for some non-negative integers $n < n'$. Then, from Equation (2.3), $c_1 n + c_2 n(n-1) = c_1 n' + c_2 n'(n'-1)$ implies $c_1(n-n') + c_2(n-n')(n+n'-1) = 0$. Since $n - n' \neq 0$, this leads to $c_1 + c_2(n + n' - 1) = 0$ and so

$$c_1(n + n') + c_2(n + n')(n + n' - 1) = 0$$

which means $\lambda_{n+n'} = 0$. Since the multiplicity of the eigenvalue zero cannot exceed 2, we obtain that for each integer $k > n + n'$, the eigenvalue λ_k has multiplicity 1. We also see that if $n_1 + n_2 = n + n'$, where $n_1 < n_2$, by reversing the steps made previously we get $\lambda_{n_1} = \lambda_{n_2}$ meaning that the number of eigenvalues that have multiplicity 2 is $\lceil \frac{n+n'}{2} \rceil$ where $\lceil x \rceil = min\{n \in \mathbb{Z} | n \geq x\}$.

Proposition 2.9 *[1, Proposition 2.2] Let the field be $\mathbb{R}$. Then, with the above notation, either all eigenvalues of the N-th order operator L are equal to 0 or all have multiplicity 1 except finitely many of them which will then have multiplicity at most N. In case $N = 2$, there will be eigenvalues with multiplicity 2 precisely when a non-negative integer k exists for which $c_1 + kc_2 = 0$, and then the number of such eigenvalues is $\lceil \frac{k+1}{2} \rceil$.*

Now, we focus on second-order operators. Let $L(y) = a(x)y'' + b(x)y'$, where $deg(a) = 2$, $deg(b) \leq 1$. By using scaling and translation on $a(x)$ we can assume that $a(x) = x^2 - 1$, $x^2 + 1$ or x^2. To see this let $p(x) = x^2 + bx + c$. Then $p(x)$ can be classified into one of the three possible cases:

Case 1: $p(x)$ has a repeated zero m, so one can write $p(x) = (x - m)^2$. If we use the translation $\mu = x - m$ then $\bar{p}(\mu) = \bar{p}(x - m) = p(x)$ where $\bar{p}(\mu) = \mu^2$.

Case 2: $p(x)$ has two distinct real zeros say m_1 and m_2, then

$p(x) = (x - m_1)(x - m_2) = (x + (\frac{-m_1-m_2}{2}))^2 - (\frac{m_1-m_2}{2})^2$

Let $\alpha = \frac{-m_1-m_2}{2}$ and $\beta = \frac{m_1-m_2}{2}$, then

$p(x) = (x + \alpha)^2 - \beta^2$, the translation $\beta\mu = x + \alpha$ gives $\bar{p}(\mu) = \beta^2(\mu^2 - 1)$ where $\beta \neq 0$.

The last case where $p(x)$ has two distinct complex zeros can be treated in the same way as Case 2 to obtain $\bar{p}(\mu) = \beta^2(\mu^2 + 1)$.

Proposition 2.10 *[1, Proposition 2.3] (i) The equation $(x^2+\epsilon)y'' + (\alpha x+\beta)y' + \lambda y = 0$, $\epsilon = 0, 1, -1$ has unique monic polynomial solutions in every degree if $\alpha > 0$ or if $\alpha < 0$ and it is not an integer. If $\alpha = -(n + m - 1)$ for $0 \leq m \leq (n - 1)$, then the eigenvalue $\lambda = n(n - 1) + \alpha n = -nm$ has algebraic multiplicity 2 and eigenpolynomials can only be of degree n or m. An eigenpolynomial $y = \sum_{k=0}^{n} a_k x^k$ is of degree n if and only if $\epsilon a_{m+2}(m + 2)(m + 1) + \beta a_{m+1}(m + 1) = 0$ in which case the λ eigenspace in $\mathbb{P}_n$ is two-dimensional; otherwise the λ eigenspace is one-dimensional.*

(ii) The equation $xy'' + (\alpha x+\beta)y' + \lambda y = 0$ has unique monic polynomial solutions in every degree if $\alpha \neq 0$.

(iii) The equation $y'' + (\alpha x + \beta)y' + \lambda y = 0$ *has unique monic polynomial solutions in every degree if* $\alpha \neq 0$.

Proof. (i) Let $L(y) = (x^2 + \epsilon)y'' + (\alpha x + \beta)y'$, where $\epsilon = 0, -1, 1$. Using Equation (2.3) with $c_1 = \alpha$ and $c_2 = 1$ or noticing that the eigenvalues are given by the coefficients of x^n in $L(x^n)$ we obtain the eigenvalues $\lambda_n = n(n-1) + \alpha n$. Suppose such an eigenvalue is a repeated eigenvalue (i.e. there is $m \neq n$ such that $\lambda_m = \lambda_n$). Then, $n(n-1) + \alpha n = m(m-1) + \alpha m$ which gives $\alpha = -(n+m-1)$. This means that if α is not an integer or if α is a positive integer, then the operator L has distinct eigenvalues. From Proposition 2.6, there is, up to a scalar, only one polynomial in every degree which is an eigenfunction of L.

Now let $\alpha = -(n + m - 1)$ where n and m are non-negative integers and $\lambda = n(n-1) + \alpha n = -nm$. We may assume that $n > m$. Suppose there is another integer k such that $\lambda_k = \lambda_n$ with $k \neq n$. Then $\alpha = -(n+m-1) = -(n+k-1)$ gives $k = m$. Therefore, if there is a repeated eigenvalue, it is of multiplicity 2 and eigenpolynomials can only be of degree m and n. Moreover, if $\alpha = -(n+m-1) = -(i+j-1)$ then the eigenvalue $-ij$ is also repeated.

Always assuming that $\alpha = -(n+m-1)$ where $n > m \geq 0$, let $y = \sum_{k=0}^{n} a_k x^k$ then

$$\begin{aligned} L(y) &= \sum_{k=0}^{n-2} [a_k k(k-1) + \epsilon a_{k+2}(k+2)(k+1) + \alpha a_k k + \beta a_{k+1}(k+1)]x^k \\ &\quad + [a_{n-1}(n-1)(n-2) + \alpha a_{n-1}(n-1) + \beta a_n n]x^{n-1} + [n(n-1) + \alpha n]a_n x^n \end{aligned}$$

The solutions of $L(y) = \lambda y = (n(n-1) + \alpha n)y = -(nm)y$ satisfy therefore

$$a_k k(k-1) + \epsilon a_{k+2}(k+2)(k+1) + \alpha a_k k + \beta a_{k+1}(k+1) = \lambda a_k, \quad (k = 0, \ldots, n) \tag{2.4}$$

Where $a_{n+1} = 0 = a_{n+2}$ and $a_n \neq 0$. The following is the system of equations we obtain from (2.4).

$$\begin{bmatrix} 2\epsilon a_2 + \beta a_1 \\ \vdots \\ \epsilon a_{m+2}(m+2)(m+1) + \beta a_{m+1}(m+1) + a_m(m(m-1) + \alpha m) \\ \vdots \\ a_{n-1}(n-1)(n-2+\alpha) + \beta a_n n \\ \lambda a_n \end{bmatrix} = \begin{bmatrix} \lambda a_0 \\ \vdots \\ \lambda a_m \\ \vdots \\ \lambda a_{n-1} \\ \lambda a_n \end{bmatrix} \tag{2.5}$$

From Equation (2.4) we can solve for a_k in terms of a_{k+1}, a_{k+2} provided $k(k-1) + \alpha k - \lambda \neq 0$. Therefore, we can solve for all a_k with $k > m$ in terms of a_n. For $k = m$, Equation (2.4) becomes

$$\epsilon a_{m+2}(m+2)(m+1) + \beta a_{m+1}(m+1) = [\lambda - m(m-1) - \alpha m]a_m = 0 \tag{2.6}$$

Since a_{m+2} and a_{m+1} can be written in terms of a_n, Equation (2.6) has the form $g(\alpha, \beta, \epsilon)a_n = 0$ where $g(\alpha, \beta, \epsilon)$ is a linear combination of the parameters α, β and ϵ.

If $g(\alpha, \beta, \epsilon) = 0$ then a_m can be arbitrary and every a_k for $k < m$ is determined in terms of a_m and a_n. In this case, the λ eigenspace is two-dimensional. If $a_n = 0$ then $a_k = 0$ for $n \geq k > m$ which means that there is no eigenpolynomial of degree $> m$. In this case, there will be a unique monic eigenpolynomial of degree m.

(ii) and (iii) Here the proof follows from the fact that the eigenvalues of the operator L in both cases are αn so, for $\alpha \neq 0$, all the eigenvalues are distinct. ▌

Proposition 2.10 shows that there are cases where the algebraic and geometric multiplicities are equal to 2 and cases where the algebraic multiplicity is 2 and the geometric multiplicity is 1 (cf. [4]).

Corollary 2.11 *[1, Corollary 2.4] Let* $L(y) = x^2y'' + (\alpha x + \beta)y'$*.*

(i) If α is not a non-positive integer then all the eigenvalues of L are distinct.

(ii) If $\alpha = -(n+m-1)$ where $n > m \geq 0$ then all eigenvalues λ except $\lambda = -nm$, are distinct and the eigenvalue $-nm$ has multiplicity 2.

In this case if $\beta = 0$ then all the eigenvalues are semisimple with eigenpolynomials x^k $(k = 0, 1, \ldots)$.

If $\beta \neq 0$ then the repeated eigenvalue $-nm$ is defective with eigenpolynomial

$$\sum_{l=0}^{m}(-\beta)^m \binom{m}{l} \frac{(n-m)!}{(n-l)!} \left(-\frac{x}{\beta}\right)^l$$

Proof. (i) and the first statement of (ii) follow directly from Proposition 2.10. Putting $\alpha = -(n+m-1)$ where $0 \leq m \leq n-1$, $\lambda = n(n-1) + \alpha n = -nm$, $\epsilon = 0$ in Equation (2.4) we get

$$
\begin{aligned}
& a_k k(k-1) - (n+m-1)a_k k + \beta a_{k+1}(k+1) = (-nm)a_k \\
i.e \quad & \beta a_{k+1}(k+1) = a_k(k - k^2 + kn + km - k - nm)
\end{aligned}
$$

Which leads to

$$
\beta a_{k+1}(k+1) = a_k(n-k)(k-m) \tag{2.7}
$$

We now have a system of equations

$$
\begin{bmatrix} \beta a_1 \\ \vdots \\ \beta a_m m \\ \beta a_{m+1}(m+1) \\ \beta a_{m+2}(m+2) \\ \vdots \\ \beta a_n n \\ \beta a_{n+1}(n+1) \end{bmatrix} = \begin{bmatrix} a_0(-nm) \\ \vdots \\ a_{m-1}(m-n-1) \\ 0 \\ a_{m+1}(n-m-1) \\ \vdots \\ a_{n-1}(n-1-m) \\ 0 \end{bmatrix} \tag{2.8}
$$

If $\beta = 0$, then $a_k(n-k)(k-m) = 0$. Therefore, all a_k are zero except for $k = n$ and $k = m$. The general solution is therefore a linear combination of x^n and x^m. The eigenvalue $\lambda = -nm$ in this case is of geometric multiplicity 2.

Now, let $\beta \neq 0$. For $k = n$ and $k = m$ Equation (2.7) gives $\beta a_{n+1}(n+1) = 0$ and $\beta a_{m+1}(m+1) = 0$. Since a_{n+1} is zero, if there is an eigenpolynomial of degree n then we must have $a_{m+1} = 0$ and therefore all $a_k = 0$ for $(k = m+1, \ldots, n)$. This means that there cannot be an eigenpolynomial of degree n.

Now from Equation (2.7) we get

$$a_k = \frac{\beta a_{k+1}(k+1)}{(n-k)(k-m)} \quad 0 \leq k \leq m-1$$

Taking $a_m = 1$ one can evaluate the coefficients

$$\begin{aligned}
a_{m-1} &= \frac{-m\beta}{(n-(m-1))} \\
a_{m-2} &= \frac{(-1)^2 m(m-1)\beta^2}{2(n-(m-1))(n-(m-2))} \\
a_{m-3} &= \frac{(-1)^3 m!\beta^3 (n-m)!}{3!(m-3)!(n-(m-3))!} = (-1)^3\beta^3\binom{m}{3}\frac{(n-m)!}{(n-(m-3))!} \\
&\vdots \\
a_{m-k} &= (-1)^k\beta^k\binom{m}{k}\frac{(n-m)!}{(n-(m-k))!}
\end{aligned}$$

Substituting $l = m - k$ we get

$$a_l = (-1)^{m-l}\beta^{m-l}\binom{m}{l}\frac{(n-m)!}{(n-l)!}$$

Which gives the monic eigenpolynomial of degree l

$$\sum_{l=0}^{m}(-\beta)^m\binom{m}{l}\frac{(n-m)!}{(n-l)!}(-\frac{x}{\beta})^l$$

∎

2.1.2 Self-adjoint case

After we established the basic results of this approach we shall investigate necessary and sufficient conditions for a second-order operator to be self-adjoint. We give a complete classification of second-order operators which are self-adjoint with respect to some weight function, using an inner product on $\mathbb{P}$ of the form

$$(u,v)=\int_I \rho(x)u(x)v(x)dx$$

where $u,v\in\mathbb{P}$, $\rho>0$ is the weight function and I is a finite or infinite interval. This includes all the classical orthogonal polynomials and some non-classical ones. In the last part of this subsection, we briefly discuss recurrence relations for orthogonal polynomials. We have the following general definitions

Definition 2.12 *Two integrable real-valued functions u and v are orthogonal with weight function $\rho>0$ on an interval I if and only if*

$$(u,v)=\int_I \rho(x)u(x)v(x)dx=0$$

Definition 2.13 *Let T be an operator on an inner product space V. Then we say that T is self-adjoint if*

$$(T(y), u) = (y, T(u))$$

for all y and u in V.

Now, the following proposition shows that both the weight and general boundary conditions (for the interval I) are forced upon us as soon as we demand that the operator $L(y) = a(x)y'' + b(x)y' + c(x)y$ should be self-adjoint for some weight function ρ. Throughout, we shall denote by $E(x)|_I$, where $E(x)$ is a function of x and $I = (a, b)$ (finite or infinite) interval, the expression $\lim_{x\to b^-} E(x) - \lim_{x\to a^+} E(x)$.

Proposition 2.14 *[1, Proposition 2.5] Let L be the operator defined by $Ly = a(x)y'' + b(x)y' + c(x)y$ on a linear space C of functions which are at least twice differentiable on a finite interval I. Define a bilinear function on C by $(y, u) = \int_I \rho yu dx$, where $\rho \in C$ is nonnegative and does not vanish identically in any open subinterval of I. Then*

$$(Ly, u) - (y, Lu) = \rho a(uy' - u'y)|_I \text{ for all } y, u \in C \text{ if and only if } (\rho a)' = \rho b$$

Proof.

$$\begin{aligned}
(Ly,u)-(y,Lu) &= \int\limits_I \rho u(ay''+by'+cy)dx - \int\limits_I \rho y(au''+bu'+cu)dx \\
&= \int\limits_I \rho a(uy''-u''y)dx + \int\limits_I \rho b(uy'-yu')dx \\
&= \int\limits_I \rho a(uy''+u'y'-u'y'-u''y)dx + \int\limits_I \rho b(uy'-u'y)dx \\
&= \int\limits_I \rho a(uy'-u'y)'dx + \int\limits_I \rho b(uy'-u'y)dx \qquad (2.9)
\end{aligned}$$

If $(\rho a)' = \rho b$, then

$$\begin{aligned}
(Ly,u)-(y,Lu) &= \int\limits_I \rho a(uy'-u'y)'dx + \int\limits_I (\rho a)'(uy'-u'y)dx \\
&= \int\limits_I [\rho a(uy'-u'y)]'dx \\
&= (\rho a)(uy'-u'y)|_I
\end{aligned}$$

For the converse, assume that $(Ly,u)-(y,Lu) = (\rho a)(uy'-u'y)|_I$, from integration by parts we have $\int\limits_I (\rho a)(uy'-u'y)'dx = (\rho a)(uy'-u'y)|_I - \int\limits_I (\rho a)'(uy'-u'y)dx$ i.e.

$$(\rho a)(uy'-u'y)|_I = \int\limits_I (\rho a)(uy'-u'y)'dx + \int\limits_I (\rho a)'(uy'-u'y)dx \qquad (2.10)$$

since the left-hand sides of Equations (2.10), (2.9) are equal we get

$$\int_I (\rho a)'(uy' - u'y)dx = \int_I \rho b(uy' - u'y)dx$$
$$i.e. \quad \int_I ((\rho a)' - \rho b)(uy' - u'y)dx = 0$$

Putting $w = (\rho a)' - \rho b$, $u = 1$, and choosing y so that $y' = (\rho a)' - \rho b$, we get

$$\int_I w^2 dx = 0$$

Suppose first that I is an open interval (α, β). Then, from $\lim\limits_{\substack{s\to\alpha^+ \\ t\to\beta^-}} \int_s^t w^2 dx = 0$, we get for each subinterval $[\sigma, \tau]$ of I, $\int_\sigma^\tau w^2 dx = 0$. From Mean Value Theorem for integrals there is $d \in [\sigma, \tau]$ such that $w^2(d) = \frac{1}{\tau - \sigma} \int_\sigma^\tau w^2 dx = 0$ for each $\sigma, \tau \in I$ where $\sigma < \tau$, which implies $w^2 = 0$ on I and so w is identically zero on I. The case when one or both endpoints of I are in I is similarly dealt with. ▌

From Proposition 2.14, the operator L would be self-adjoint if there is no contribution from the boundary terms: this is ensured if the product $a(x)p(x)$ vanishes at the endpoints of the interval (finite or infinite) on which the natural weight function $\rho(x)$ associated to L is integrable on the entire interval and it is determined from Proposition 2.14:

Let $z = \rho a$ then

$$
\begin{aligned}
(\rho a)' = \rho b \quad &\Rightarrow \quad z' = \rho b \\
&\Rightarrow \quad a z' = (a\rho) b = zb \\
&\Rightarrow \quad \int \frac{dz}{z} = \int \frac{b}{a} dx \\
&\Rightarrow \quad \ln z = \int \frac{b}{a} dx \\
&\Rightarrow \quad \ln \rho = \int \frac{b}{a} dx - \ln a = \int \frac{b - a'}{a} dx
\end{aligned}
$$

So the weight function is of the form

$$
\rho = e^{\int \frac{b-a'}{a} dx} \qquad or \qquad \rho = \frac{e^{\int \frac{b}{a} dx}}{|a|}, \qquad a \neq 0 \tag{2.11}
$$

Now the following lemma yields immediately orthogonal polynomials

Lemma 2.15 *Let L be the operator defined in* (2.1) *on the inner product space $\mathbb{P}_n$. If L is self-adjoint then eigenpolynomials corresponding to distinct eigenvalues are orthogonal.*

Proof. Let y_m and y_n be the eigenpolynomials corresponding to the eigenvalues λ_m and λ_n ($\lambda_m \neq \lambda_n$). From the self-adjointness of L we see that

$$
0 = (L(y_m), y_n) - (y_m, L(y_n)) = (\lambda_m y_m, y_n) - (y_m, \lambda_n y_n) = (\lambda_m - \lambda_n)(y_m, y_n)
$$

Since the eigenvalues are distinct, we obtain

$$(y_m, y_n) = 0$$

$\blacksquare$

From Lemma 2.15 we see that when L is self-adjoint (with distinct eigenvalues), we obtain orthogonal eigenpolynomials. Let us summarize the requirements for the operator L to be self-adjoint

1. the leading term $a(x)$ is non-zero of degree at most 2, the degree of $b(x)$ is at most 1 and $c(x)$ is a constant.

2. the natural weight function associated to L is integrable on the interval I determined by roots of $a(x)$.

3. $\rho(x)a(x)$ vanishes at the endpoints of I and, in case there is an end point at infinity, the product $\rho(x)a(x)P(x)$ should vanish at infinity for all polynomials $P(x)$.

4. all polynomials should have finite norm on the interval I with the weight $\rho(x)$.

Now, we study the possible cases for the operator L with respect to the leading term $a(x)$

Case 1: The polynomial $a(x)$ has two distinct real roots.

By scaling and translation we may assume that $a(x) = 1 - x^2$ and so the roots are 1 and -1 and it is non-negative on the interval $[-1, 1]$. Let $b(x) = \alpha x + \beta$, so

$$\frac{b(x)}{a(x)} = \frac{\alpha x + \beta}{(1-x)(1+x)} = \frac{\frac{\beta+\alpha}{2}}{1-x} + \frac{\frac{\beta-\alpha}{2}}{1+x}$$

From Equation (2.11), the weight function $\rho(x)$ is

$$\begin{aligned} \rho(x) = \frac{1}{1-x^2} e^{\int (\frac{\frac{\beta+\alpha}{2}}{1-x} + \frac{\frac{\beta-\alpha}{2}}{1+x}) dx} &= (1-x)^{-1}(1+x)^{-1}(1-x)^{-\frac{\beta+\alpha}{2}}(1+x)^{\frac{\beta-\alpha}{2}} \\ &= \frac{(1+x)^{\frac{\beta-\alpha-2}{2}}}{(1-x)^{\frac{\beta+\alpha+2}{2}}} \end{aligned}$$

It is clear that the weight function $\rho(x)$ is finite in the interval $(-1, 1)$ and is integrable if $\frac{\beta-\alpha-2}{2} + 1 > 0$ and $\frac{-\beta-\alpha-2}{2} + 1 > 0$ i.e. $\beta - \alpha > 0$ and $\beta + \alpha < 0$. Thus $\alpha < \beta < -\alpha$, so $\alpha < 0$.

Case 2: The polynomial $a(x)$ has repeated roots.

By scaling and translation we assume that $a(x) = x^2$. Let $b(x) = \alpha x + \beta$, so the weight function is

$$\rho(x) = \frac{1}{x^2} e^{\int \frac{\alpha x + \beta}{x^2} dx} = \frac{1}{x^2} e^{\alpha \ln |x|} \; e^{-\frac{\beta}{x}} = \frac{|x|^\alpha}{x^2} e^{-\frac{\beta}{x}}$$

We can take the interval $I = (0, \infty)$, since $\int_0^\infty x^n \rho dx$ must be finite for every non-negative integer n, the integral $\int_M^\infty x^{n+\alpha-2} e^{-\frac{\beta}{x}} dx$ must also be finite for arbitrary positive M. On any subinterval $[M, M']$ of $[M, \infty)$, $e^{-\frac{\beta}{x}}$ has a positive minimum,

k say. This implies

$$\int_{M}^{\infty} x^{n+\alpha-2} e^{-\frac{\beta}{x}} \geq k \int_{M}^{M'} x^{n+\alpha-2} dx = k \frac{M'^{n+\alpha-1} - M^{n+\alpha-1}}{n+\alpha-1}$$

which goes to ∞ for sufficiently large n and M', so this case does not arise.

Case 3: The polynomial $a(x)$ is linear.

By scaling and translation we can assume that $a(x) = x$. Let $b(x) = \alpha x + \beta$, so the weight function is

$$\rho(x) = \frac{1}{|x|} e^{\int \frac{\alpha x+\beta}{x} dx} = |x|^{\beta-1} e^{\alpha x}.$$

This is integrable near zero if and only if $\beta > 0$. From the fact that

$$\int_{0}^{\infty} e^{\alpha x} x^{\epsilon} dx \tag{2.12}$$

where $\epsilon > 0$, is finite only if $\alpha < 0$ we only choose one the intervals $[0, \infty)$ or $(-\infty, 0]$. Without loss of generality let the interval $I = [0, \infty)$, so the weight function is $\rho = x^{\beta-1} e^{\alpha x}$ with $\alpha < 0$ and $\beta > 0$. From Equation (2.12) all the eigenpolynomials $P(x)$ have finite norm and the product $xP(x)\rho(x)$ vanishes at 0 and ∞. Hence the equation $xy'' + (\alpha x + \beta)y' + \lambda y = 0$, where the eigenvalue is $\lambda = -\alpha n$, has polynomial solutions in every degree and all are orthogonal (in different degrees).

Case 4: When $a(x) = 1$.

Here, the weight function is

$$\rho(x) = e^{\int \frac{\alpha x+\beta}{1} dx} = e^{\frac{\alpha x^2}{2}} e^{\beta x}.$$

For all polynomials $P(x)$, the product $\rho(x)P(x)$ will vanish at the endpoints of the interval I if $\alpha < 0$, and therefore I must be $(-\infty, \infty)$.

Case 5: When $a(x)$ has no real roots.

We can assume that $a(x) = x^2 + 1$ and $b(x) = \alpha x + \beta$. Then the weight function is

$$\rho(x) = \frac{1}{x^2+1} e^{\int \frac{\alpha x+\beta}{x^2+1} dx} = (x^2+1)^{\frac{\alpha}{2}-1} e^{\beta \tan^{-1}(x)} \tag{2.13}$$

When $\alpha < 0$ the product $a(x)\rho(x) = (x^2+1)^{\frac{\alpha}{2}} e^{\beta \tan^{-1}(x)}$ vanishes at the endpoints of $I = (-\infty, \infty)$. Since $P(x)\rho(x)$ must be integrable on I for all polynomials $P(x)$, we must have $(x^2+1)^{n-1+\frac{\alpha}{2}} e^{\beta \tan^{-1}(x)}$ integrable where $n = deg(P(x))$. For all x and β, $\beta \tan^{-1}(x) \geq -|\beta|\frac{\pi}{2}$, so that $e^{\beta \tan^{-1}(x)} \geq e^{-|\beta|\frac{\pi}{2}}$, hence $(x^2+1)^{n-1+\frac{\alpha}{2}}$ must be integrable on I, which is impossible for sufficiently large n.

Now we provide examples of classical orthogonal polynomials and non-classical polynomials and discuss them using the ideas and the results obtained so far.

I) **Classical orthogonal polynomials**

1) **Laguerre polynomials**

Consider the following differential equation

$$xy'' + (1-x)y' = \lambda y.$$

The operator L is given by $L(y) = xy'' + (1-x)y'$. The weight function is $\rho(x) = (\frac{1}{|x|})e^{\int \frac{1-x}{x}dx} = e^{-x}$. Hence the weight is integrable over the interval $I = [0, \infty)$. As $\int\limits_0^\infty x^n e^{-x} dx = n!$, all polynomials have finite norm. As $x\rho(x)$ vanishes at $x = 0$ and at infinity, the operator L is self-adjoint on the space of all polynomials. For every non-negative integer n, there must be a polynomial of degree n which is an eigenfunction of L. The corresponding eigenvalue is given by the coefficient of x^n in $L(x^n)$ which is $\lambda = -n$. Therefore, the ODE for Laguerre polynomials is $xy'' + (1-x)y' + ny = 0$.

2) **Hermite polynomials**

Here the differential equation is

$$y'' - 2xy' + \lambda y = 0.$$

The weight function is $\rho(x) = e^{\int -2x dx} = e^{-x^2}$ and the interval is $I = (-\infty, \infty)$. Thus all polynomials have finite norm relative to this weight. As $x^n\rho(x)$ vanishes at ∞ and $-\infty$, the operator L is self-adjoint with the eigenvalues $\lambda = -2n$. Hence there is a unique monic polynomial of every degree n which is an eigenfunction of L.

3) **Jacobi polynomials**

Consider the following differential equation

$$(1-x^2)y'' + (\alpha x + \beta)y' + \lambda y = 0.$$

Here the operator is $L(y) = (1-x^2)y'' + (\alpha x + \beta)y'$ and the weight function is

$$\rho(x) = \frac{1}{1-x^2}e^{\int \frac{\frac{\beta+\alpha}{2}}{1-x} + \frac{\frac{\beta-\alpha}{2}}{1+x}dx} = \frac{(1+x)^{\frac{\beta-\alpha-2}{2}}}{(1-x)^{\frac{\beta+\alpha+2}{2}}}$$

So $\int_{-1}^{1} \rho(x)dx$ would be finite if $\beta + \alpha < 0$ and $\beta - \alpha > 0$, that is, if $\alpha < \beta < -\alpha$, so $\alpha < 0$.

Moreover,

$$\rho(x)a(x) = (1-x^2)\frac{(1+x)^{\frac{\beta-\alpha-2}{2}}}{(1-x)^{\frac{\beta+\alpha+2}{2}}} = (1-x)^{\frac{-(\beta+\alpha)}{2}}(1+x)^{\frac{\beta-\alpha}{2}} \qquad (2.14)$$

From (2.14) we see that $\rho(x)a(x)$ vanishes at the endpoints -1 and 1. It is easy to verify that $(L(x^m), x^n) = (x^m, L(x^n))$ for all non-negative integers m, n. Hence, L is a self-adjoint operator on the vector space of all polynomials of degree at most n and so, there must be, up to scalar, a unique polynomial which is an eigenfunction of L for eigenvalue

$\lambda = -n(n-1) + \alpha n$. So the polynomials of different degrees that satisfy the equation

$$(1-x^2)y'' + (\alpha x + \beta)y' + (n(n-1) - n\alpha)y = 0 \qquad (2.15)$$

are all pairwise orthogonal with respect to the given inner product.

Remark If we substitute specific values for α and β in (2.15) then the polynomial solutions will be special cases of Jacobi polynomials, Table 2.1 shows some standard examples because of their importance in applications.

α	β	Polynomials
-2	0	Legendre
-1	0	Chebyshev (the first kind)
-3	0	Chebyshev (the second kind)
$-(2\nu+1)$	0	Gegenbauer

Table 2.1

II) **Some non-classical polynomials**

1) **The equation** $t(1-t)y'' + (1-t)y' + \lambda y = 0$

This equation was obtained in [7] where the polynomial solutions were determined by machine computations. Here, we apply the results we had before to find the eigenvalues and to study the orthogonality of the polynomial solutions of this equation.

Let $L(y) = t(1-t)y'' + (1-t)y'$ operate on $\mathbb{P}_n$ where $\mathbb{P}_n$ is the space of all polynomials of degree at most n. The eigenvalues λ_n of L are the coefficients of t^n in the expression

$$t(1-t)(n(n-1)t^{n-2}) + (1-t)(nt^{n-1})$$

so that $\lambda_n = -n^2$. As we can see, the eigenvalues are distinct, so from Proposition 2.6 there is, up to a constant, a unique polynomial of degree n as an eigenfunction of L.

The weight function is $\rho(t) = \frac{1}{|t(1-t)|}e^{\int \frac{1-t}{t(1-t)}dx} = \frac{1}{1-t}$ on the interval $(0,1)$, and it is not integrable over this interval. For every polynomial that is a multiple of $(1-t)$, say $y = (1-t)\psi(t)$ we have

$$L(y) = t(1-t)\left[((1-t)y'' - 2y']\right) + (1-t)\left((1-t)y' - y)\right)$$

which is also a polynomial multiple of $(1-t)$. So L maps the space V of all polynomials that are multiples of $(1-t)$ into itself. Moreover, the norm of each polynomial in V is finite:

$$\int_0^1 \rho(t)((1-t)\psi(t))^2 dt = \int_0^1 (1-t)\psi^2(t)dt$$

From Proposition 2.14, for every ξ and η in V we have

$$(L(\xi),\eta) - (\xi, L(\eta)) = \rho(t)t(1-t)(\xi\eta' - \xi'\eta)\big|_0^1 = t(\xi\eta' - \xi'\eta)\Big|_0^1 = 0$$

since ξ and η vanish at 1, the operator L is self-adjoint on V.

Now let $V_n = (1-t)\mathbb{P}_n$, where $\mathbb{P}_n$ is the space of all polynomials of degree at most n. As the codimension of V_n in V_{n+1} is 1, the operator L must have an eigenvector in V_n for all the degrees from 1 to $n+1$.

Therefore, there is, up to a scalar, a unique eigenfunction of degree $n+1$ which is a multiple of $1-t$ for each n and all these functions are orthogonal with respect to the weight $\rho(t) = \frac{1}{1-t}$.

2) **Romanovski polynomials**

These polynomials are investigated in [19, 22] and their finite orthogonality is also proven there. We are going to apply the results of the approach used here on these polynomials. Consider the following operator

$$L(y) = (1+x^2)y'' + (\alpha x + \beta)y'.$$

From Proposition 2.10 Part (i), when $\alpha > 0$ or when $\alpha < 0$ and it is not an integer there is a unique monic polynomial in every degree which is an eigenfunction of L. If α is a non-positive integer then the eigenspaces corresponding to the eigenvalues of L can be two-dimensional for certain values of β.

The weight function will be as in (2.13):

$$\rho(x) = (x^2+1)^{\frac{\alpha-2}{2}} e^{\beta \tan^{-1}(x)}.$$

Therefore, a polynomial of degree N is integrable over $I = (-\infty, \infty)$ with weight ρ if and only if $N < 1 - \alpha$. If the product of two nonconstant polynomials P, Q is integrable, then the polynomials are themselves integrable for the weight ρ since $deg(P) + deg(Q) = deg(PQ)$.

For polynomials P and Q and for sufficiently large $|x|$

$$(x^2+1)^{\frac{\alpha}{2}} e^{\beta \tan^{-1}(x)} \left(PQ' - P'Q\right) = |x|^{\alpha}(1 + \frac{1}{x^2})^{\frac{\alpha}{2}} e^{\beta \tan^{-1}(x)} H(x)$$

where $H(x)$ is a polynomial of degree at most $deg(P) + deg(Q) - 1$. Since $\lim\limits_{x \to \pm\infty} |x|^{\alpha} H(x) = 0$ when $\alpha + deg(P) + deg(Q) - 1 < 0$ we get

$$(L(P), Q) - (P, L(Q)) = \left[|x|^{\alpha}(1 + \frac{1}{x^2})^{\frac{\alpha}{2}} e^{\beta \tan^{-1}(x)} H(x) \right]_{-\infty}^{\infty} = 0$$

Therefore, if P, Q are integrable eigenfunctions of L with different eigenvalues and $\alpha + deg(P) + deg(Q) - 1 < 0$, then P, Q are orthogonal.

For further reading concerning applications for Romanovski polynomials in Physics, the reader is referred to [19].

As we shall see, the coefficients of the three-term recurrence relation (2.16) for orthogonal polynomials can be obtained from their norms.[1] To do that, we prove the following well-known proposition.

Proposition 2.16 *[3, p.356] [1, Proposition 2.6] If P_n is a sequence of orthogonal polynomials, then in the expression*

$$xP_n = \sum_{j=0}^{n+1} k_j P_j,$$

all the coefficients are 0 except for $j = n-1, n, n+1$.

Proof.

$$\begin{aligned}
k_j(P_j, P_j) &= (k_j P_j, P_j) \\
&= ((xP_n - \sum_{\substack{i=0 \\ i\neq j}}^{n+1} k_i P_i), P_j) \\
&= (xP_n, P_j) \quad \textit{since} \quad (P_i, P_j) = 0 \quad for \quad i \neq j \\
&= (P_n, xP_j) \\
&= (P_n, (\sum_{i=0}^{j+1} k_i P_i))
\end{aligned}$$

If $j+1 < n$, then $k_j(P_j, P_j) = 0$ which means $k_j = 0$ for $j = 0, 1, \ldots, n-2$. ∎

[1] *We point out that there is a purely linear algebraic approach to classical orthogonal polynomials (given an inner product) using Gram-Schmidt orthogonalization procedure; we refer the reader to [15].*

Now the expression in Proposition 2.16 can be written as

$$xP_n = a_nP_{n+1} + b_nP_n + c_nP_{n-1} \tag{2.16}$$

for some constants a_n, b_n, c_n. Let us assume that all eigenfunctions (or eigenpolynomials) are normalized to be monic. So that $a_n = 1$ and $P_n = xP_{n-1} + q(x)$, for some polynomial $q(x)$ of degree less than n. Then

$$\begin{aligned}(P_n, P_n) = (xP_{n-1} + q(x), P_n) &= (xP_{n-1}, P_n) \\ &= (P_{n-1}, xP_n) \\ &= (P_{n-1}, a_nP_{n+1} + b_nP_n + c_nP_{n-1}) \\ &= (P_{n-1}, c_nP_{n-1}) = c_n(P_{n-1}, P_{n-1})\end{aligned}$$

since $(P_j, P_i) = 0$ for all j and i such that $j \neq i$. We can evaluate the coefficients c_n as follows

$$c_n = \frac{(P_n, P_n)}{(P_{n-1}, P_{n-1})} \tag{2.17}$$

Also it is clear that

$$(P_n, P_n) = c_nc_{n-1}\ldots c_1(P_0, P_0)$$

where (P_0, P_0) is the integral of the weight function ρ over an appropriate interval. The values of b_n, c_n for classical monic polynomials (i.e. $a_n = 1$) are given in Table 2.2 below.

Polynomial	b_n	c_n
Legendre	0	$\frac{n^2}{(2n+1)(2n-1)}$
Hermite	0	$\frac{n}{2}$
Laguerre	$2n+1$	n^2
Chebyshev	0	$\frac{1}{4}$ (for $n \geq 2$)
Jacobi	$\frac{-\beta(2+\alpha)}{(2n-2-\alpha)(2n-\alpha)}$	$\frac{n(n-\alpha-2)(2n-(\beta+\alpha+2))(2n+(\beta-\alpha-2))}{(2n-\alpha-3)(2n-2-\alpha)^2(2n-\alpha-1)}$ (for $n \geq 2$)
	$b_1 = \frac{\beta(2+\alpha)}{\alpha(2-\alpha)}$	$c_1 = \frac{(\alpha-\beta)(\alpha+\beta)}{(1-\alpha)\alpha^2}$: here $\alpha < \beta < -\alpha$

Table 2.2

As an example we will evaluate b_n and c_n for Laguerre polynomials. These polynomials are orthogonal on $(0, \infty)$ with respect to the weight e^{-x} and they forms an orthonormal basis for $\mathbb{P}$. These polynomials are of the form

$$L_n(x) = \sum_{i=0}^{n} (-1)^i \binom{n}{n-i} \frac{x^i}{i!},$$

when we normalize them to be monic we get

$$L_n^*(x) = (-1)^n n! L_n(x).$$

Now

$$(L_n^*(x), L_n^*(x)) = \frac{(n!)^2}{(-1)^{2n}}(L_n(x), L_n(x)) = (n!)^2$$

Therefore, using Equation (2.17) we get

$$c_n = \frac{(L_n^*(x), L_n^*(x))}{(L_{n-1}^*(x), L_{n-1}^*(x))} = \frac{(n!)^2}{((n-1)!)^2} = n^2 \tag{2.18}$$

The first two leading terms of $L_n^*(x)$, $L_{n-1}^*(x)$, $L_{n+1}^*(x)$ and $xL_n^*(x)$ are obtained as follows

$$\begin{aligned} L_n^*(x) &= (-1)^n n! L_n(x) = \frac{n!}{(-1)^n}\sum_{i=0}^{n}(-1)^i \binom{n}{n-i}\frac{x^i}{i!} = x^n - n^2 x^{n-1} + \ldots \\ L_{n-1}^*(x) &= x^{n-1} - (n-1)^2 x^{n-2} + \ldots \\ xL_n^*(x) &= x^{n+1} - n^2 x^n + \ldots \\ L_{n+1}^*(x) &= x^{n+1} - (n+1)^2 x^n + \ldots \end{aligned} \tag{2.19}$$

Now substituting (2.18) and (2.19) in Equation (2.16) (with $a_n = 1$) and equating the coefficients of x^n gives

$$-n^2 = -(n+1)^2 + b_n + 0$$

which leads to

$$b_n = 2n + 1$$

2.2 Defective operators

In [2], linear differential equations of arbitrary order with polynomial coefficients that are of arbitrary degrees are discussed and necessary and sufficient conditions for the existence of polynomial solutions of these differential equations are given. In this section we present details of these results. Let $L(y) = \sum_{k=0}^{N} p_k(x) D^k y$ where $p_k(x) = \sum_{h \geq 0} p_{kh} x^h$ is a polynomial of degree d_k (with the convention that the zero polynomial has degree $-\infty$ and that $D^0 y = y$). We no longer assume, as we did in the previous section, that $d_k \leq k$ for all k. in this case, it is clear that $\mathbb{P}_n$ is not necessarily invariant under L, and so we cannot apply directly the eigenvalue approach used in Section 2.1. To address this issue we will do the following.

Let $m = \max_{0 \leq i \leq N} (d_i - i)$ and put $y = D^m z$ to get

$$H(z) = \sum_{k=1}^{N+m} a_k(x) D^k z, \tag{2.20}$$

where $a_k(x) = 0$ if $k < m$ and $a_k(x) = p_{k-m}(x)$ if $k \geq m$. Put $a_k(x) = \sum_{h \geq 0} a_{kh} x^h$. The equation $L(y) = 0$ is equivalent to $H(z) = 0$, and $L(y) = 0$ has a polynomial solution of degree $n \geq 0$ if and only if $H(z) = 0$ has a polynomial solution of degree

$n+m$. Clearly, for each non-negative integer n, $\mathbb{P}_n$ is H-invariant, and $H(x^n)$ is a scalar multiple of x^n plus lower order terms. So the matrix representation of H, with respect to the standard basis $B_n = \{1, x, \ldots, x^n\}$ of $\mathbb{P}_n$ is upper triangular and its eigenvalues are the coefficients of x^n in $H(x^n)$. Moreover, the $(n+1)\times(n+1)$ matrix A_n of H operating on $\mathbb{P}_n$ is of the form

$$A_n = \left[\sum_{k\geq 1}(j-k)_k a_{k,k+i-j}\right]_{1\leq i,j\leq n+1}$$

where $(j-k)_k = (j-1)(j-2)\ldots(j-k)$. We note that the first m columns of A_n are zero and A_{n+1} is obtained from A_n by adding one row and one column at the end. Each eigenvalue λ_n has an eigenpolynomial $y_n(x)$ of degree at most n and whose vector representation $(y_{n0}, y_{n1}, \ldots, y_{nn})^T$ in the standard basis B_n can be directly computed from (2.2). Now our task is reduced to finding necessary and sufficient conditions for which the operator H has an eigenpolynomial of degree $n+m$ corresponding to $\lambda_{n+m} = 0$, that is, necessary and sufficient conditions for the homogeneous system

$$A_{n+m}(y_{n+m,0}, \ldots, y_{n+m,n+m})^T = 0 \tag{2.21}$$

to have a solution $(y_{n+m,0}, \ldots, y_{n+m,n+m})^T$ with $y_{n+m,n+m} = 1$. This can be done from the following lemmas.

Lemma 2.17 *[2, Lemma 1] Let A be an $m \times n$ matrix. Then the homogeneous system $AX = 0$ has a solution $X = (x_1, x_2, ..., x_n)^T$ with $x_k \neq 0$ for some k if and* *only if* $rank(A) = rank(A_k)$ *where* (A_k) *is the matrix obtained from* A *by deleting the* k^{th} *column.*

Proof. Let $c_1, c_2, \ldots, c_k, \ldots, c_n$ be the columns of A. It is clear that $rank([A_k : c_k]) = rank(A)$. Hence,

$rank(A) = rank(A_k) \Leftrightarrow rank([A_k : c_k]) = rank(A_k) \Leftrightarrow$ the system $A_k X = -c_k$ is consistent $\Leftrightarrow$ there exists a solution $X = (x_1, x_2, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n)^T$ such that $\sum_{\substack{i=1 \\ i \neq k}}^{n} c_i x_i = -c_k \Leftrightarrow \sum_{i=1}^{n} c_i x_i = 0$ where $(x_k = 1) \Leftrightarrow X = (x_1, x_2, \ldots, x_{k-1}, 1, x_{k+1}, \ldots, x_n)^T$ is a solution for $AX = 0$. ▌

Lemma 2.18 *Let* A *be an* $n \times n$ *real upper triangular matrix and let* A' *be the matrix obtained from* A *by deleting the last column. If* $rank(A) = rank(A')$ *then the last entry* $a_{n,n}$ *of* A *is zero.*

Proof. Assume that $rank(A) = rank(A') = k$ $(0 \leq k \leq n-1)$ then there are k linearly independent columns of A', say $c_1, c_2, \ldots, c_k$. Now, in upper triangular matrices, the last row has all entries 0 except possibly the last entry. If c_n is the last column of A then there exist $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{R}$, such that

$$c_n = \alpha_1 c_1 + \cdots + \alpha_k c_k$$

Since the last entries of the columns $c_1, c_2, \ldots, c_k$ are all 0, we get $a_{n,n} = 0$ ▌

Now we state the main result of this section whose proof is straightforward from the above lemmas.

Proposition 2.19 *With the above notations, let A_{n+m} be the $(n+m+1)\times(n+m+1)$ matrix with $(i,j)^{th}$ entry $\sum_{i=0}^{j-1} a_{t+m,t+m+i-j}(j-t)_{t+m}$ and let A_{n+m-1} be the matrix obtained from A_{n+m} by deleting the last column. Then the differential equation $L(y)=0$ has a polynomial solution of degree $n\geq 0$ if and only if $rank(A_{n+m}) = rank(A_{n+m-1})$.* ■

Let M_n and $M_n^{'}$ be, respectively, the matrices obtained from A_{n+m} and A_{n+m-1} by deleting the first m zero columns. It is obvious that $rank(A_{n+m}) = rank(A_{n+m-1})$ if and only if $rank(M_n) = rank(M_n^{'})$. Moreover, the $(i,j)^{th}$ entry of the $(n+m-1)\times(n+1)$ matrix M_n is $\sum_{t=0}^{j-1} a_{t+m,t+i-j}(j-t)_{t+m} = \sum_{t=0}^{j-1} p_{t,t+i-j}(j-t)_{t+m}$. Now Proposition 2.19 is reduced to the following

Proposition 2.20 *[2, Proposition 2] Let M_n be the $(n+m)\times(n+1)$ matrix with $(i,j)^{th}$ entry $\displaystyle\sum_{t=0}^{j-1} p_{t,\,t+i-j}(j-t)_{t+m}$, and let $M_n^{'}$ be the matrix obtained from M_n by deleting the last column. Then the differential equation $Ly=0$ has a polynomial solution of degree $n\geq 0$ if and only if*

$$rank(M_n) = rank(M_n^{'}). \tag{2.22}$$

If the equation $L(y)=0$ has a polynomial solution of degree $n\geq 0$, then $\lambda_{n+m} = \sum_{t\geq 1} p_{t,t+m}(n-t)_{t+m} = 0$, and since $M_n^{'}$ has n columns, $rank(M_n) = rank(M_n')$ implies that $rank(M_n)\leq n$. The following lemma is useful for determining necessary conditions to have (2.22).

Lemma 2.21 *Let M be a $m \times n$ matrix such that*

$$\textit{rank}(M) \leq k, \text{ where } k \leq \min\{m, n\} - 1$$

then every $(k+1) \times (k+1)$ submatrix of M has zero determinant.

Proof. Suppose $rank(M) = k$ and choose $k+1$ columns $c_1, c_2, \ldots, c_{k+1}$ of M, they will be linearly dependent, so without loss of generality assume

$$c_{k+1} = \alpha_1 c_1 + \alpha_2 c_2 + \cdots + \alpha_k c_k \text{ for some scalars } \alpha_1, \ldots, \alpha_k$$

Let $\bar{M} = [c_1|c_2|\ldots|c_k|c_{k+1}]$. Then by elementary column operations

$$\begin{aligned} \bar{M} \longrightarrow [c_1|c_2|\ldots|c_k|c_{k+1} - \alpha_1 c_1] &\longrightarrow [c_1|c_2|\ldots|c_k|c_{k+1} - \alpha_1 c_1 - \alpha_2 c_2] \\ &\vdots \\ &\longrightarrow [c_1|c_2|\ldots|c_k|0] = \bar{W} \end{aligned}$$

where "$\longrightarrow$" stands for equivalence of matrices. Now, let $\bar{\bar{M}}$ be any $(k+1)\times(k+1)$ submatrix of $\bar{M}$ then clearly $det(\bar{\bar{M}}) = 0$ since the last column of $\bar{W}$ is zero. ∎

Now we summarize the results obtained so far in the following proposition which determines the conditions for existence of polynomial solutions of linear differential equations.

Proposition 2.22 *[2, proposition 3] Let L be the operator defined by $L(y) = \sum_{k=0}^{N} a_k(x) D^k(y)$. Let d_i be the degree of a_i and let*

$$m = \max\{0, d_i : 0 \leq i \leq N\}.$$

Let H be the operator defined by $H(y) = \sum_{k=0}^{N} a_k(x) D^{k+m}(y)$. A necessary condition for the equation $L(y) = 0$ to have a non-zero polynomial solution of degree at most n is that 0 must occur as an eigenvalue of the operator H with multiplicity at least $(m+1)$. Moreover, the eigenvalues of H are the coefficients of x^n in $H(x^n)$ for $n = 0, 1, 2, \dots$.

We next present another way for determining necessary and sufficient conditions to have polynomial solutions, which can be extended to high-order linear differential equations through appropriate modifications.

First, every cubic polynomial must have a real root, so by scaling and translation we can assume that polynomial to be of the form $f(x) = rx^3 + sx^2 + tx$.

Let $y = \sum_{k \geq 0} c_k x^k$ be a solution of the differential equation

$$L(y) = (rx^3 + sx^2 + tx)y'' + (bx^2 + cx + \delta)y' + (\epsilon x + f)y = 0 \tag{2.23}$$

Then

$$\sum_{k \geq 0} (rk(k-1) + bk + \epsilon)c_k x^{k+1} + (sk(k-1) + ck + f)c_k x^k + (tk(k-1) + \delta k)c_k x^{k-1} = 0.$$

For $k \geq 0$, the coefficient of x^k is zero which gives the following relation $\alpha(k)c_{k-1} = \beta(k)c_k + \gamma(k)c_{k+1}$ (and $c_{-1} = 0$) where

$$\begin{aligned} \alpha(k) &= r(k-1)(k-2) + b(k-1) + \epsilon \\ \beta(k) &= -sk(k-1) - ck - f \\ \gamma(k) &= -tk(k+1) - \delta(k+1) \end{aligned}$$

Suppose that y is monic of degree n (i.e. $c_n = 1$), then $\alpha(n+1) = \beta(n+1)c_{n+1} + \gamma(n+1)c_{n+2} = 0$ (since $c_{n+1} = c_{n+2} = 0$). If n is the smallest positive integer for which there is a monic polynomial solution (2.23) with degree n, then $\alpha(n+1) = 0$ and $\alpha(k) \neq 0$ for $1 \leq k \leq n$, so that $c_{k-1} = \frac{\beta(k)}{\alpha(k)}c_k + \frac{\gamma(k)}{\alpha(k)}c_{k+1}$.

For each k $(1 \leq k \leq n)$, let $A_k = \begin{bmatrix} \frac{\beta(k)}{\alpha(k)} & \frac{\gamma(k)}{\alpha(k)} \\ 1 & 0 \end{bmatrix}$. Then

$$A_k \begin{bmatrix} c_k \\ c_{k+1} \end{bmatrix} = \begin{bmatrix} \frac{\beta(k)}{\alpha(k)}c_k + \frac{\gamma(k)}{\alpha(k)}c_{k+1} \\ c_k \end{bmatrix} = \begin{bmatrix} c_{k-1} \\ c_k \end{bmatrix}$$

so that

$$\begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = A_1 \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = A_1 A_2 \begin{bmatrix} c_2 \\ c_3 \end{bmatrix} = A_1 A_2 \ldots A_n \begin{bmatrix} c_n \\ c_{n+1} \end{bmatrix} = \left(\prod_{k=1}^{n} A_k \right) \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

If we substitute $y = \sum_{i=0}^{n} c_i x^i$ in (2.23) and then we put $x = 0$ we get

$$\delta c_1 + f c_0 = 0 \Rightarrow \begin{bmatrix} f & \delta \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} f & \delta \end{bmatrix} \left(\prod_{k=1}^{n} A_k \right) \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0.$$

Up to now we proved that if the ODE (2.23) has a monic polynomial solution of smallest degree n then $\alpha(n+1) = 0$ and $\begin{bmatrix} f & \delta \end{bmatrix} \left(\prod_{k=1}^{n} A_k \right) \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0.$

For the converse, assume there exists a positive integer n such that $\alpha(n+1) = 0$ and $\begin{bmatrix} f & \delta \end{bmatrix} \left(\prod_{k=1}^{n} A_k \right) \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0$. Define a sequence $(c_j)_{j \geq 0}$ by

$$\begin{aligned} c_j &= 0 \quad if \quad j > n \\ c_n &= 1 \\ c_{j-1} &= \frac{\beta(j)}{\alpha(j)} c_j + \frac{\gamma(j)}{\alpha(j)} c_{j+1} \quad if \quad 0 \leq j \leq n-1 \end{aligned}$$

It is easy to verify, by reversing the argument above, that the monic polynomial $\sum_{j \geq 0} c_j x^j$ of degree n is a solution of the ODE (2.23). We summarize this in the following result.

Proposition 2.23 *[2, Proposition 4] With above notation, the ODE* (2.23) *has a monic polynomial solution of degree n and so no other monic polynomial solution of smaller degree if and only if* $\alpha(n+1) = 0$ *and* $\begin{bmatrix} f & \delta \end{bmatrix} \left(\prod_{k=1}^{n} A_k \right) \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0.$

This monic solution is given by $y = x^n + \sum_{k=0}^{n-1} \begin{bmatrix} 1 & 0 \end{bmatrix} \left(\prod_{k=1}^{n} A_k \right) \begin{bmatrix} 1 \\ 0 \end{bmatrix} x^k.$

CHAPTER 3

THE ALGORITHMIC PROCEDURE

In this chapter we provide Maple codes for two algorithms that will be used to compute polynomial solutions of given degree n of any ODE of the type $L(y) = \sum_{k=0}^{N} a_k(x)D^k y$. The main difference between these two algorithms is in solving the rank equation

$$rank(A_{n+m}) = rank(A_{n+m-1})$$

where A_{n+m} is the matrix representation of $H(z) = L(\frac{d^m z}{dz^m})$ and (A_{n+m-1}) is the matrix obtained from A_{n+m} by deleting the last column. We illustrate each algorithm with an example using Maple.

3.1 1^{st} Approach

Consider the linear second order ODE arising in the study of one dimensional Schrödinger problems [14]. The investigation of Krylov and Robnik [14] about polynomial solutions of one dimensional Schrödinger problems leads to finding polynomial solutions of the following differential equation

$$x^3 \frac{d^2y}{dx^2} + a(x^2 - 1)\frac{dy}{dx} + (\epsilon x + f)y = 0 \tag{3.1}$$

Here we apply the results we obtained so far algorithmically. The set of Maple commands given below can be used for any given value of a. For illustration we take $a = \frac{-15}{2}$ and look for solution of degree $n = 6$. The Maple code below determines $\epsilon = 15$, $f = 3(750)^{\frac{1}{4}}$ and computes the corresponding polynomial solution of degree 6 of ODE (3.1) as

$$\begin{aligned} y(x) &= 7\,x^6 + \frac{7}{30}\,1080^{3/4}x^5 + 21\,\frac{\sqrt{1080}\left(-65 + 2\sqrt{1080}\right)x^4}{11\sqrt{1080} - 360} + 420\,\frac{\sqrt[4]{1080}\left(\sqrt{1080} - 30\right)x^3}{11\sqrt{1080} - 360} \\ &+ 1575\,\frac{\left(-72 + \sqrt{1080}\right)x^2}{11\sqrt{1080} - 360} + 630\,\frac{\left(-72 + \sqrt{1080}\right)750^{3/4}x}{\left(11\sqrt{1080} - 360\right)\left(-60 + \sqrt{750}\right)} \\ &- 1575\,\frac{\left(-72 + \sqrt{1080}\right)\sqrt{750}}{\left(11\sqrt{1080} - 360\right)\left(-60 + \sqrt{750}\right)} \end{aligned} \tag{3.2}$$

The Maple code with explanations is provided below:

restart:

```
with(LinearAlgebra):
ode := x^3(diff(y(x), x, x)) + a(x^2 - 1)(diff(y(x), x)) + (εx + f)y(x)
N := 2
a := -15/2
pcoeff := Array(0 .. N)
pcoeff[0] := εx + f
pcoeff[1] := a(x^2 - 1)
pcoeff[2] := x^3
```

The commands above are defining the ODE, its order N, the value of a and the polynomials $p_k(x)$ from the operator $\sum_{k=0}^{N} p_k(x)D^k y$. The commands below determine the value m (which is the maximum difference between the indices and the orders of each term) and then we define the polynomials $a_k(x)$ of the operator $\sum_{k=0}^{N+m} a_k(x)D^k y$

```
Vm := Array(0..N) :
dk := Array(0..N) :
for i from 0 to N do
   dk[i] := degree(pcoeff[i], x)
   Vm[i]:= degree(pcoeff[i],x)-i end do:
m := max(Vm)
dkmax := max(dk)
acoeff := Array(1 .. m+N):
```

for i from m to (m+N) do

acoeff[i]:= pcoeff[i-m] end do:

Next we set the degree of the polynomial solution as we desired and let us call it "soldegree".

soldegree := 6:

n := soldegree+m:

Now we compute the matrices A_{n+m}, A_{n+m-1} by executing the following commands where we denote them by *An* and *Anprime* respectively.

```
cAM := max(dkmax, n+m+N):
rAM := m+N:
AM := Array(1 .. rAM, 0 .. cAM):
for i from m to rAM do
   AM[i,0]:= coeff(x acoeff[i],x^1) end do:
for i from m to rAM do
   for j from 1 to cAM do
      AM[i, j] := coeff(acoeff[i], x^j)
   end do
end do
An := Matrix(n+1,n+1):
for i from 1 to n+1 do
   for j from 1 to n+1 do
```

```
      for k from 1 to rAM do
        if (k+i-j) ≥ 0 and (k-j+1) ≤ 0
          then An[i,j]:= An[i,j]+AM[k,k+i-j] (j-1)!/(j-k-1)!
        end if
      end do
    end do
end do
Anprime := Matrix(n+1,n)
for i from 1 to n+1 do
    for j from 1 to n do
      Anprime[i, j] := An[i, j]
    end do
end do
```

At this stage the matrix An (or A_{n+m}) should be

$$\begin{bmatrix} 0 & f & 15 & 0 & 0 & 0 & 0 & 0 \\ 0 & \epsilon & 2f & 45 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2\epsilon - 15 & 3f & 90 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3\epsilon - 39 & 4f & 150 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4\epsilon - 66 & 5f & 225 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5\epsilon - 90 & 6f & 315 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6\epsilon - 105 & 7f \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7\epsilon - 105 \end{bmatrix}$$

and the matrix $Anprime$ (or A_{n+m-1}) is the same as the matrix An after deleting the last column. If we check the ranks of the matrices at this stage we see that $Rank(An) \neq Rank(Anprime)$. Next we determine the values of parameters so that $Rank(An)$ equals $Rank(Anprime)$. The first condition employed is vanishing of the last diagonal entry of the upper triangular matrix An which determines ϵ by the following commands.

ansbeta := solve(An[n+1, n+1] = 0, ϵ*):*

ϵ *:= ansbeta*

Here the ranks still not equal which we have to put another condition. Implementing Lemma 2.21 by taking the 7×7 submatrix, obtained by deleting the last zero row and the first zero column of An, and have its determinant equal to zero, provides the value of f which can be computed via the commands below.

with(linalg) :

Andet := Matrix(submatrix(An, [1,2,3,4,5,6,7],[2,3,4,5,6,7,8])):

Determinant(Andet):

ansdet := solve(Determinant(Andet) = 0, f)

This leads to seven roots. At this stage a root needs to be chosen before checking the rank condition. For illustration we choose $f = 3(750)^{1/4}$.

f:= $3(750)^{1/4}$:

Rank(An)

Rank(Anprime)

As the rank condition is satisfied (i.e. $Rank(An) = Rank(Anprime)$) so the desired polynomial solution can be obtained by the following set of commands.

kern := NullSpace(An)

The output of the above command contains two vectors $kern_1$ and $kern_2$ with $kern_2 = (1, 0, 0, 0, 0, 0, 0)^T$ so the vector $kern_1$ is used as below to find the solution.

Vkern := $kern_1$:

solm := 0:

for i from 1 to (n+1) do

solm:= solm + Vkern[i] x^{i-1} *end do:*

soln := diff(solm, x)

The output *soln* provides the solution given in (3.2).

3.2 2^{nd} Approach

This approach differs with the first one on reducing the rank of An. To demonstrate this approach we consider the following differential equation related to the investigation of the radial Schrödinger equation with shifted Coulomb potential which has been discussed in [11, 12].

$$\begin{aligned} L(y) &= x(x+\beta)\frac{d^2}{dx^2}y(x) + \left(-2\alpha x^2 + 2(K+1-\alpha\beta)x + 2\beta(K+1)\right)\frac{d}{dx}y(x) \\ &+ \left((-2\alpha(K+1)+2Z)x - 2\alpha\beta(K+1)\right)y(x) = 0 \end{aligned} \tag{3.3}$$

For illustration, we fix $K = -3$ and compute polynomial solution of ODE (3.3) of degree 7 (so we set "soldegree" equal to 7). Here we apply the same commands as in the 1^{st} approach until we reach the following

ansbeta := solve(An[n+1, n+1] = 0, α):

α *:= ansbeta*

The above commands provide the first condition for existence of the desired polynomial solution which is $\alpha = \frac{1}{5}Z$. Now the matrix An reads

$$\begin{bmatrix} 0 & \frac{4}{5}\beta Z & -8\beta & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{14}{5}Z & \frac{4}{5}\beta Z - 8 & -18\beta & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{24}{5}Z & -18 & -24\beta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6\,Z & -\frac{8}{5}\beta Z - 24 & -20\beta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{32}{5}Z & -4\beta\,Z - 20 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6\,Z & -\frac{36}{5}\beta Z & 42\beta & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{24}{5}Z & -\frac{56}{5}\beta\,Z + 42 & 112\beta \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{14}{5}Z & -16\beta Z + 112 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The idea of this approach is to make the entries $An(1, j)$ zero using $An(j, j)$ for $j = 2, 3, \ldots, 8$ which can be done by the following commands

for k from m+1 to n do

for i from 1 to m do

mult $:= \frac{An[i,k]}{An[k,k]}$

for j from m+1 to n+1 do

*An[i,j]:= An[i,j]- mult*An[k,j]:*

end do

end do

end do

which gives

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A(1,9) \\ 0 & \frac{14}{5}Z & \frac{4}{5}\beta Z-8 & -18\beta & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{24}{5}Z & -18 & -24\beta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6\,Z & -\frac{8}{5}\beta Z-24 & -20\beta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{32}{5}Z & -4\beta\,Z-20 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6\,Z & -\frac{36}{5}\beta Z & 42\beta & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{24}{5}Z & -\frac{56}{5}\beta\,Z+42 & 112\beta \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{14}{5}Z & -16\beta Z+112 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where $An(1,9) = \frac{100}{7}\frac{\beta^2\left(\beta^2Z^2-50\,\beta\,Z-75\right)\left(-60\,\beta\,Z+175+4\,\beta^2Z^2\right)}{Z^5}$. To reduce the rank of An we need $An(1,9)=0$, and this leads to the following condition

$$\beta = 0, 0, \frac{5(5\pm 2\sqrt{7})}{Z}, \frac{5(3+\sqrt{2})}{2Z}.$$

As an example we take $\beta = \frac{5(5+2\sqrt{7})}{Z}$, then we continue the commands as in 1^{st} approach where we evaluate the null space of An which gives the polynomial solution as

$$\begin{aligned} y(x) \;&=\; 8\,x^7+80\,\frac{\left(9+5\,\sqrt{7}\right)x^6}{Z}+300\,\frac{\left(277+100\,\sqrt{7}\right)x^5}{Z^2}+2500\,\frac{\left(5+2\,\sqrt{7}\right)\left(159+56\,\sqrt{7}\right)x^4}{Z^3} \\ &+\; 12500\,\frac{\left(3+\sqrt{7}\right)\left(5+2\,\sqrt{7}\right)\left(159+56\,\sqrt{7}\right)x^3}{Z^4} \\ &+\; 75000\,\frac{\left(5+2\,\sqrt{7}\right)\left(159+56\,\sqrt{7}\right)\left(8+3\,\sqrt{7}\right)x^2}{Z^5} \\ &+\; 31250\,\frac{\left(5+2\,\sqrt{7}\right)\left(159+56\,\sqrt{7}\right)\left(193+73\,\sqrt{7}\right)x}{Z^6} \\ &+\; 156250\,\frac{\left(5+2\,\sqrt{7}\right)\left(159+56\,\sqrt{7}\right)\left(193+73\,\sqrt{7}\right)}{Z^7} \end{aligned}$$

CHAPTER 4

APPLICATIONS

In this chapter we provide several examples as applications for the results of the algorithmic approach presented in Chapter 2 & 3. Two of the applications here, have been dealt with in [2] and the others arise in the study of solutions of linear second-order differential equations [16, 23]. Here we show the effectiveness of the algorithmic procedure provided in Chapter 3 by determining the conditions for the existence of polynomial solutions and calculating the corresponding polynomial solutions.

4.1 A Bohr Hamiltonian

Boztosun et al [5] studied the analytical solutions of the Bohr Hamiltonian for the Davidson potential which can be rewritten as [11, Eq.22]

$$L(y) = x\frac{d^2}{dx^2}y(x) - (2x^2 - 2\mu - 2)\frac{d}{dx}y(x) - (2\mu + 3 - \epsilon)xy(x) = 0 \tag{4.1}$$

The operator $L(y)$ is of defect 1 ($m = 1$), so the substitution $y = \frac{dz}{dx}$ in ODE (4.1) gives

$$H(z) = x\frac{d^3z}{dx^3} - (2x^2 - 2\mu - 2)\frac{d^2z}{dx^2} - (2\mu + 3 - \epsilon)x\frac{dz}{dx} = 0 \tag{4.2}$$

The necessary condition to have polynomial solution of degree $n+1$ for ODE (4.2), i.e. of degree n for ODE (4.1) is when $\lambda_{n+1} = 0$ or when the coefficient of x^{n+1} in $H(x^{n+1})$ is equal to zero, which is

$$\epsilon = 2\mu + 3 + 2n$$

Using the Maple code provided in Chapter 3 for ODE 4.1 readily generates polynomial solutions of a given degree n. The Tables 4.1 and 4.2 show some examples for solutions of even as well as odd degrees respectively.

n	ϵ	Polynomial solutions of ODE (4.1) of degree $n = 2m$
0	$2\mu+3$	1
2	$2\mu+7$	$2x^2-(2\mu+3)$
4	$2\mu+11$	$4x^4-4(2\mu+5)x^2+(2\mu+3)(2\mu+5)$
6	$2\mu+15$	$8x^6-12(2\mu+7)x^4+6(2\mu+5)(2\mu+7)x^2$ $-(2\mu+3)(2\mu+5)(2\mu+7)$
8	$2\mu+19$	$16x^8-32(2\mu+9)x^6+24(2\mu+7)(2\mu+9)x^4$ $-8(2\mu+5)(2\mu+7)(2\mu+9)x^2$ $+(2\mu+3)(2\mu+5)(2\mu+7)(2\mu+9)$
10	$2\mu+23$	$32x^{10}-80(2\mu+11)x^8+80(2\mu+9)(2\mu+11)x^6$ $-40(2\mu+7)(2\mu+9)(2\mu+11)x^4$ $+10(2\mu+5)(2\mu+7)(2\mu+9)(2\mu+11)x^2$ $-(2\mu+3)(2\mu+5)(2\mu+7)(2\mu+9)(2\mu+11)$
12	$2\mu+27$	$64x^{12}-192(2\mu+13)x^{10}+240(2\mu+11)(2\mu+13)x^8$ $-160(2\mu+9)(2\mu+11)(2\mu+13)x^6$ $+60(2\mu+7)(2\mu+9)(2\mu+11)(2\mu+13)x^4$ $-12(2\mu+5)(2\mu+7)(2\mu+9)(2\mu+11)(2\mu+13)$ $(2\mu+3)(2\mu+5)(2\mu+7)(2\mu+9)(2\mu+11)(2\mu+13)$

Table 4.1

From the Table 4.1 one can find the general form for polynomial solutions of degree $n = 2m$ which is

$$y(x) = 2^m x^{2m} + \sum_{i=1}^{m} [(-1)^i 2^{m-i} \binom{m}{i} x^{2(m-i)} \prod_{j=m-i+1}^{m} (2\mu + 2j + 1)]$$

n	ϵ	μ	Polynomial solutions of ODE (4.1) of degree $n = 2m + 1$
1	$2\mu + 5$	-1	x
3	$2\mu + 9$	-1	$x^3 - \frac{3}{2}x$
		-2	x^3
5	$2\mu + 13$	-1	$x^5 - 5x^3 + \frac{15}{4}x$
		-2	$x^5 - \frac{5}{2}x^3$
		-3	x^5
7	$2\mu + 17$	-1	$x^7 - \frac{21}{2}x^5 + \frac{105}{4}x^3 - \frac{105}{8}x$
		-2	$x^7 - 7x^5 + \frac{35}{4}x^3$
		-3	$x^7 - \frac{7}{2}x^5$
		-4	x^7
9	$2\mu + 21$	-1	$x^9 - 18x^7 + \frac{189}{2}x^5 - \frac{315}{2}x^3 + \frac{945}{16}x$
		-2	$x^9 - \frac{27}{2}x^7 + \frac{189}{4}x^5 - \frac{315}{8}x^3$
		-3	$x^9 - 9x^7 + \frac{63}{4}x^5$
		-4	$x^9 - \frac{9}{2}x^7$
		-5	x^9

Table 4.2

An observation can made from Table 4.2 for finding family of polynomial solutions of degree $n = 2m + 1$. For $k = 0, 1, 2, 3$ ODE (4.1) admits the following class of polynomial solutions of degree $n = 2m+1$ $(m \geq k)$ with $\mu = -(m+1-k)$ and $\epsilon = 2m + 2k + 3$.

- If $k = 0$ $(m \geq 0)$

$$y = x^{2m+1}$$

- If $k = 1$ $(m \geq 1)$

$$y = x^{2m+1} - \frac{2m+1}{2} x^{2m-1}$$

- If $k = 2, 3$ $(m \geq k)$

$$\begin{aligned} y &= x^{2m+1} + k \sum_{i=1}^{k-1} \frac{(-1)^i (2m+1)(2m-1)\ldots(2m+1-2(i-1))}{2^i} x^{2m+1-2i} \\ &+ \frac{(-1)^k (2m+1)(2m-1)\ldots(2m+1-2(k-1))}{2^k} x^{2m+1-2k} \end{aligned}$$

4.2 Coulomb diamagnetic problem

In their study of the polynomial solutions of the planar Coulomb diamagnetic problem, Chhajlany and Malnev [8] studied the following differential equation:

$$L(y) = \frac{d^2}{dx^2}y(x) + (p - 2x^2)\frac{d}{dx}y(x) + (\delta x + \alpha)y(x) = 0 \tag{4.3}$$

Ciftci et al. [11, Eq.18,19] provided conditions for the existence of polynomial solutions of ODE (4.3). If we apply Proposition 2.22 then the necessary condition to have polynomial solution of degree n is when $A_{n+m}(n+1, n+1) = 0$ where A_{n+m} is the matrix representation of $H(z) = L(\frac{dz}{dx})$ which gives $\delta = 2n$.

Now, using the Maple commands provided in Chapter 3 we can obtain polynomial solutions for ODE (4.3). The Table 4.3 shows some examples of these polynomials for $\alpha \neq 0$. The conditions on the parameters δ and p, for having these solutions, are also determined.

n	δ	p	Polynomial solutions of ODE (4.3) of degree n
1	2	$\frac{\alpha^2}{2}$	$2x-\alpha$
2	4	$\frac{\alpha^3+16}{8\alpha}$	$x^2-\frac{\alpha}{2}x+\frac{\alpha^3-16}{16\alpha}$
3	6	$\frac{5}{18}\alpha^2 \pm \frac{2}{9}\sqrt{\alpha^4-54\alpha}$	$4x^3-2\alpha x^2-\frac{1}{3}\frac{\left(\pm 5\alpha^3+4\sqrt{\alpha(\alpha^3-54)}\alpha\mp 216\right)\alpha x}{\pm\alpha^2+2\sqrt{\alpha(\alpha^3-54)}}$ $+\frac{1}{54}\frac{\pm 41\alpha^5+40\alpha^3\sqrt{\alpha(\alpha^3-54)}\mp 1728\alpha^2-432\sqrt{\alpha(\alpha^3-54)}}{\pm\alpha^2+2\sqrt{\alpha(\alpha^3-54)}}$
4	8	$\frac{1}{64}\frac{5\alpha^3+192\pm 3A}{\alpha}$	$5x^4-\frac{5\alpha}{2}x^3+\frac{15}{32}\frac{(\mp\alpha^6\pm 768\alpha^3+\alpha^3A\mp 4096-64A)x^2}{\alpha(\pm 3\alpha^3\mp 192+5A)}$ $\pm\frac{5}{64}\frac{(\mp\alpha^9\mp 128\alpha^6+\alpha^6A\pm 8192\alpha^3-1728\alpha^3A\pm 262144+4092A)x}{(\pm 3\alpha^3\mp 192+5A)(\mp\alpha^3\pm 64+A)}$ $\mp\frac{5}{2048}\frac{B}{\alpha^2(\pm 3\alpha^3\mp 192+5A)(\mp\alpha^3\pm 64+A)}$ where $A=\sqrt{\alpha^6-384\alpha^3+4096}$ and $B=\mp\alpha^{12}\mp 2816\alpha^9+\alpha^9A\pm 524288\alpha^6-5184\alpha^6A$ $\pm 19922944\alpha^3+200704\alpha^3A\mp 150994944-2359296A.$

Table 4.3

Using the algorithms provided in Chapter 3 one can compute polynomial solution of ODE (4.3) of any given degree n, of course depending on the computational power. The following are examples for polynomial solutions of degrees 12 and 30.

- For $\alpha=0$ and $n=12$ implies $\delta=24$, $p=\frac{1}{20}(741810)^{\frac{1}{3}}$ and the polynomial solution of degree 12 of ODE (4.3) is

$$
\begin{aligned}
y(x) &= 13x^{12}-\frac{39}{20}\sqrt[3]{741810}x^{10}-286x^9+\frac{39}{320}741810^{2/3}x^8+\frac{1521}{50}\sqrt[3]{741810}x^7\\
&- \frac{415233}{320}x^6-\frac{156}{125}741810^{2/3}x^5-\frac{2842749}{51200}\sqrt[3]{741810}x^4+\frac{24033139}{1600}x^3\\
&+ \frac{9232509}{5120000}741810^{2/3}x^2-\frac{9232509}{128000}\sqrt[3]{741810}x-\frac{382104339643}{40960000}
\end{aligned}
$$

- For $\alpha = 0$, $n = 33$ with $\delta = 66$ and $p = 0$ the polynomial solution of ODE (4.3) of degree 33 is

$$\begin{aligned} y(x) &= 34x^{33} - 5984x^{30} + 433840x^{27} - 16919760x^{24} + 389154480x^{21} \\ &- 5448162720x^{18} + 46309383120x^{15} - 231546915600x^{12} + 636754017900x^{9} \\ &- 849005357200x^{6} + 424502678600x^{3} - 38591152600 \end{aligned}$$

4.3 A Schrödinger wave function with Coulomb potential

In this example we consider the following ODE that appeared in the study of system of two electrons, interacting via Coulomb potential, but constrained on the surface of $\mathcal{D}$-sphere ($\mathcal{D} = 1, 2, \dots$) of radius R [16, 23].

$$\left[\frac{u^2}{4R^2} - 1\right]\frac{d^2y}{du^2} + \left[\frac{(2\mathcal{D}-1)u}{4R^2} - \frac{\mathcal{D}-1}{u}\right]\frac{dy}{du} + \frac{y}{u} = Ey,$$

where E is the energy. If we made change of variable $u = 2Rx$ then the above ODE becomes a Heun's equation

$$L(y) = x(x^2-1)\frac{d^2y}{dx^2} + ((2\mathcal{D}-1)x^2 - \mathcal{D} + 1)\frac{dy}{dx} + (2R - 4R^2Ex)y = 0. \quad (4.4)$$

At this stage, Loos [16] and Zhang [23] adopted different methods in finding the polynomial solutions of (4.4). Here we implement the results of this approach algorithmically and we determine the necessary conditions for existence of polynomial solutions as well as we find these solutions.

The operator L is of defect 1 (i.e $m = 1$), so the necessary condition for existence of polynomial solutions of degree n is when the coefficient of x^{n+1} in $L(\frac{d}{dx}(x^{n+1}))$ is zero, that is

$$E = \frac{n}{4R^2}(n + 2\mathcal{D} - 2)$$

Now adopting the algorithms presented in Chapter 3 can generate polynomial solutions of ODE (4.4) along with determining the sufficient condition on R and $\mathcal{D}$ for the existence of these solutions. Some examples of solutions up to degree 4 generated by this approach are presented in Table 4.4. It is worth mentioning that Loos [16] and Zhang [23] produced only solutions up to degree 2.

n	E	R	Polynomial solutions of ODE (4.4) of degree n
1	$\frac{2\mathcal{D}-1}{2\mathcal{D}^2-3\mathcal{D}+1}$	$\frac{1}{2}\sqrt{2\mathcal{D}^2-3\mathcal{D}+1}$	$\frac{2\mathcal{D}-2}{\sqrt{2\mathcal{D}^2-3\mathcal{D}+1}}+2x$
2	$\frac{2\mathcal{D}}{4\mathcal{D}^2-\mathcal{D}}$	$\frac{1}{2}\sqrt{8\mathcal{D}^2-2\mathcal{D}}$	$\frac{3\mathcal{D}-3}{2\mathcal{D}+1}+3\frac{\sqrt{2}\sqrt{\mathcal{D}(4\mathcal{D}-1)}x}{2\mathcal{D}+1}+3x^2$
3	$\frac{3+6\mathcal{D}}{3+10\mathcal{D}^2+10\mathcal{D}\pm\sqrt{A}}$	$\frac{1}{2}\sqrt{3+10\mathcal{D}^2+10\mathcal{D}\pm\sqrt{A}}$	$\pm\frac{(24\mathcal{D}-24)\mathcal{D}(1+\mathcal{D})}{\sqrt{3+10\mathcal{D}^2+10\mathcal{D}\pm\sqrt{A}}\left(\pm6\mp4\mathcal{D}^2\pm5\mathcal{D}+\sqrt{A}\right)}\pm\frac{24\mathcal{D}(1+\mathcal{D})x}{\pm6\mp4\mathcal{D}^2\pm5\mathcal{D}+\sqrt{A}}$ $+\frac{(12+12\mathcal{D})\left(\pm6\pm4\mathcal{D}^2\pm13\mathcal{D}+\sqrt{A}\right)x^2}{\sqrt{3+10\mathcal{D}^2+10\mathcal{D}\pm\sqrt{A}}\left(\pm6\mp4\mathcal{D}^2\pm5\mathcal{D}+\sqrt{A}\right)}+4x^3$ where $A=64\mathcal{D}^4+128\mathcal{D}^3+169\mathcal{D}^2+132\mathcal{D}+36$
4	$\frac{8+8\mathcal{D}}{20\mathcal{D}^2+45\mathcal{D}+28\pm3\sqrt{B}}$	$\frac{1}{2}\sqrt{20\mathcal{D}^2+45\mathcal{D}+28\pm3\sqrt{B}}$	$\pm\frac{(10\mathcal{D}-10)\mathcal{D}(1+\mathcal{D})}{(5+2\mathcal{D})\left(\pm9\mathcal{D}\pm12+\sqrt{B}\right)}\pm\frac{10\mathcal{D}(1+\mathcal{D})\sqrt{20\mathcal{D}^2+45\mathcal{D}+28\pm3\sqrt{B}}x}{(5+2\mathcal{D})\left(\pm9\mathcal{D}\pm12+\sqrt{B}\right)}$ $+\frac{(15+15\mathcal{D})\left(\pm4\mathcal{D}^2\pm15\mathcal{D}\pm12+\sqrt{B}\right)x^2}{(5+2\mathcal{D})\left(\pm9\mathcal{D}\pm12+\sqrt{B}\right)}+\frac{5\sqrt{20\mathcal{D}^2+45\mathcal{D}+28\pm3\sqrt{B}}x^3}{5+2\mathcal{D}}+5x^4$ where $B=16\mathcal{D}^4+72\mathcal{D}^3+185\mathcal{D}^2+264\mathcal{D}+144$

Table 4.4

4.4 Schrödinger equation from the kink stability analysis of ϕ^6-type field Theory

Consider the linear second order ODE arising in the study of the Schrödinger equation that comes from the stability analysis of ϕ^6-type field theory [23]. Zhang transformed the problem into the following ODE

$$\left(x^4+\left(\frac{1}{\epsilon^2}-1\right)x^2-\frac{1}{\epsilon^2}\right)y''-\left(5x^3-\left(\frac{1}{\epsilon^2}+6\right)x\right)y'$$
$$+\left(\left(\frac{4E}{\mu^2}+5\right)x^2+\frac{4E}{\epsilon^2\mu^2}-\frac{1}{\epsilon^2}-6\right)y=0$$

where ϵ is a real constant, $E \geq 0$ and $\mu \neq 0$. We rewrite the ODE into the form,

$$L(y)=\left(\mu^2\epsilon^2x^4+\left(\mu^2-\mu^2\epsilon^2\right)x^2-\mu^2\right)y''+\left(-5\,\mu^2\epsilon^2x^3+\left(\mu^2+6\mu^2\epsilon^2\right)x\right)y'$$
$$+\left(\left(4E\epsilon^2+5\mu^2\epsilon^2\right)x^2+4E-\mu^2-6\mu^2\epsilon^2\right)y=0 \tag{4.5}$$

The operator L is of defect 2 (i.e. $m=2$). The necessary condition for (4.5) to have a non-zero polynomial solution of degree n is when the coefficient of x^{n+2} in $H(x^{n+2})$ is zero where $H(z)=L(\frac{d^2y}{dz^2})$, that is

$$\epsilon^2\mu^2(n+2)(n+1)n(n-1)-5\epsilon^2\mu^2(n+2)(n+1)n+(4E\epsilon^2+5\epsilon^2\mu^2)(n+2)(n+1)=0$$

which implies

$$\mu^2(n^2 - n) - 5\mu^2 n + 4E + 5\mu^2 = 0$$

so the first condition we have is

$$E = \frac{\mu^2}{4}(n-1)(5-n) \tag{4.6}$$

since $E \geq 0$ we have $n = 1, 2, 3, 4, 5$. Although it appears that one gets 5 different polynomial solutions for (4.5), corresponding to $n = 1, \ldots, 5$, this in fact is not the case (Cf. [23] where the author states that there are five non-negative solutions to the equation). First, the following table shows the first and second polynomial solutions with the given values E and ϵ

n	E	ϵ	Polynomial solutions of ODE (4.5) of degree n
1	0	no constraint	$6x$
2	$\frac{3}{4}\mu^2$	$\pm\frac{1}{2}\sqrt{2}$	$12x^2 - 24$

Table 4.5

When $n = 3$ then $E = \mu^2$, let $A_{n+2} = A_5$ be the matrix representation of H

then

$$A_5 = \begin{bmatrix} 0 & 0 & 6\,\mu^2 - 12\,\mu^2\epsilon^2 & 0 & -24\,\mu^2 & 0 \\ 0 & 0 & 0 & 24\,\mu^2 & 0 & -120\,\mu^2 \\ 0 & 0 & 18\,\mu^2\epsilon^2 & 0 & 84\,\mu^2 + 48\,\mu^2\epsilon^2 & 0 \\ 0 & 0 & 0 & 24\,\mu^2\epsilon^2 & 0 & 240\,\mu^2 + 120\,\mu^2\epsilon^2 \\ 0 & 0 & 0 & 0 & 12\,\mu^2\epsilon^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

According to Proposition 2.20, Equation (4.5) has a polynomial solution of degree 3 if and only if $rank(A_5) = rank(A_5')$ if and only if $\mu = 0$ (and therefore $E = 0$) where A_5' is the matrix obtained from A_5 after deleting the last column, so this solution does not arise.

Similarly, for $n = 4$ we have $E = \frac{3}{4}\mu^2$ and

$$A_6 = \begin{bmatrix} 0 & 0 & 4\,\mu^2 - 12\,\mu^2\epsilon^2 & 0 & -24\,\mu^2 & 0 & 0 \\ 0 & 0 & 0 & 18\,\mu^2 & 0 & -120\,\mu^2 & 0 \\ 0 & 0 & 16\,\mu^2\epsilon^2 & 0 & 72\,\mu^2 + 48\,\mu^2\epsilon^2 & 0 & -360\,\mu^2 \\ 0 & 0 & 0 & 18\,\mu^2\epsilon^2 & 0 & 220\,\mu^2 + 120\,\mu^2\epsilon^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 540\,\mu^2 + 180\,\mu^2\epsilon^2 \\ 0 & 0 & 0 & 0 & 0 & -20\,\mu^2\epsilon^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Notice that the eigenvalue zero is repeated four times (as $\lambda_0 = \lambda_1 = \lambda_4 = \lambda_6 = 0$). Now $rank(A_6) = rank(A_6')$ if $\epsilon = \pm\frac{1}{2}\sqrt{2}$. and the polynomial solution is $y(x) = 12x^2 - 24$ with no constraint on μ. With the same way when $n = 5$ we get $E = 0$

and the polynomial solution is $y(x) = 6x$.

4.5 Planar Dirac electron in Coulomb and magnetic fields

As a last example, we consider the second-order differential equation arising in the study of Dirac equation for an electron in two spatial dimensions in the Coulomb and homogeneous magnetic fields [9, 23]. After certain transformations made in [23], the problem is reduced to the question of finding solutions of

$$\frac{d^2 f}{dx^2} + \left(\frac{2\xi + 1}{x} - \frac{1}{x + r_0} - eBx\right)\frac{df}{dx} + \frac{c_2 x^2 + c_1 x + c_0}{x(x + r_0)} f = 0$$

where

$$\begin{aligned} c_2 &= E^2 - m_e^2 - eB(\xi + L + \frac{3}{2}), \\ c_1 &= 2EZ\alpha + \left[E^2 - m_e^2 - eB(\xi + L + \frac{5}{2})\right] r_0, \\ c_0 &= 2EZ\alpha r_0 + L + \frac{1}{2} - \xi. \end{aligned}$$

where α and e are constants, ξ is related to the parameter Z. We put the above ODE into the following form

$$L(y) = x(x+r_0)\frac{d^2 f}{dx^2} + \big((2\xi+1)(x+r_0) - x - eBx^2(x+r_0)\big)\frac{df}{dx} + (c_2x^2 + c_1x + c_0)f = 0. \tag{4.7}$$

In [23], Zhang provided the conditions for existence of solution $f(x) = 1$ (i.e. $n = 0$). Here, we apply the algorithms of this approach to determine the conditions for existence of polynomial solutions as well as evaluate them.

Since $m = 2$, the substitution $f = \frac{d^2 z}{dx^2}$ in ODE (4.7) yields

$$H(z) = x(x+r_0)\frac{d^4 z}{dx^4} + \big((2\xi+1)(x+r_0) - x - eBx^2(x+r_0)\big)\frac{d^3 z}{dx^3} + (c_2x^2 + c_1x + c_0)\frac{d^2 z}{dx^2} = 0. \tag{4.8}$$

The necessary condition for the existence of polynomial solution of degree n of ODE (4.7) is obtained when the coefficient of x^{n+2} in $H(x^{n+2})$ is zero, that is

$$E^2 = m_e^2 + eB(\xi + L + \frac{3}{2} + n). \tag{4.9}$$

Now, when $n = 0$ then the matrix representation A_{0+2} of H relative to the basis $\{1, x, x^2, \dots, x^n\}$ is

$$\begin{bmatrix} 0 & 0 & 4\,EZ\alpha\,r_0 + 2\,L + 1 - 2\,\xi \\ 0 & 0 & 4\,EZ\alpha + 2\,r_0\left(E^2 - {m_e}^2 - eB\left(\xi + L + \frac{5}{2}\right)\right) \\ 0 & 0 & 0 \end{bmatrix} \tag{4.10}$$

and according to Proposition 2.19 the ODE (4.7) has a polynomial solution of degree $n = 0$ (i.e. the constant solution) if and only if $rank(A_2) = 0$ if and only if the parameters E, Z, B satisfy the following relations

$$\begin{gathered} 4\,EZ\alpha\,r_0 + 2\,L + 1 - 2\,\xi = 0 \\ 4\,EZ\alpha + 2\,r_0\left(E^2 - {m_e}^2 - eB\left(\xi + L + \frac{5}{2}\right)\right) = 0 \end{gathered} \tag{4.11}$$

and Equation (4.9).

Following the 1^{st} approach of algorithms presented in Chapter 3, the existence of polynomial solutions of degree $n = 1$ of ODE (4.7) is ensured if the parameters E, m_e, B obey the following relations

$$E^2 = m_e^2 + eB(\xi + L + \frac{5}{2})$$
$$4\,E^2Z^2\alpha^2{r_0}^2 + 4\,EZ\alpha\,r_0(L + \xi) + (L + \frac{1}{2})^2 - \xi^2 = 0$$
$$-4\,E^2Z^2\alpha^2(L + \xi + \frac{5}{2}) + 4\,EZ\alpha\,r_0(E^2 - m_e^2) + (E^2 - m_e^2)(L + \xi + \frac{1}{2}) = 0$$

and is given by

$$f(x) = -12\,\frac{(2\,\xi+1)\,r_0}{\sqrt{-8\,L\xi+8\,\xi^2-1-4\,L}+1}+6\,x.$$

The question of finding polynomial solutions of degree ≥ 2 can be treated similarly, however the computational complexity may increase for finding polynomial solutions of higher degree.

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