BOUNDARY CONTROL OF NONLINEAR PARTIAL DIFFERENTIAL EQUATION

BY

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DEDICATION

To my parents, who instilled in me the values of life and saw me grow to this stage and made me what I am.

To my wife, my lovely daughter and my wonderful son, whose presence beside me is a great strength, and in whose well-being lies my happiness.

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THESIS ABSTRACT

Name: Mohammed Jamal Maqatif

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The object in this work is to find a stabilizing boundary control for a class of nonlinear parabolic PDE system by using a transformation (transmutation operator). We transform the original system into a new one whose stability has been proved. We shall work out a few examples to illustrate the boundary control of system exhibiting blow up in finite time, thus showing the effectiveness of the method.

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الموضوع في هذا العمل هو إيجاد السيطرة المستقرة لمعادلة تفاضلية جزئية مكافئة غير خطية باستخدام تحويل تفاضلي خطي قابل للانعكاس ثم تحويل النظام الأصلي عبر نظام مستقر مثبت مسبقا.

في الأخير تم عرض أمثلة للاستفادة من فعالية هذه الطريقة.

ماجستير رياضيات

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الظهران – المملكة العربية السعودية

Chapter 1

INTRODUCTION

1.1 Introduction

Partial differential equations (PDEs) appear frequently in all areas of physics and engineering. Moreover, in recent years we have seen a dramatic increase in the use of PDEs in areas such as biology, chemistry, computer sciences and in economics (finance), beside the classical engineering areas (electrical, mechanical, aerospace engineering). The control of such infinite dimensional systems has been an active area of research since at least 1960s. Control of PDEs comes about from two types of settings depending on where the sensors and actuators are located. In domain control, where the actuation penetrates inside the domain of the PDE system or is distributed everywhere in the domain and control, where the actuation and sensing are applied only at the boundary points. Over the years, a few methods have been developed for boundary control problems of PDEs, and most of researchers either do not cover the boundary control or dedicate only a small fractions of their coverage to boundary control. There have been many successes of the theory of adaptive control for over three decades now, including the development of stability and an understanding of the dynamical properties of adaptive schemes. The area of adaptive control has grown to be one of the richest in terms of algorithms, design techniques and analytical tool. Several books and studies already exist on the topics of parameter estimation and adaptive control. The advances in stability theory and the progress in control theory in the 1960s improved the understanding of adaptive control and contributed to a strong renewed interest in the field in the 1970s. Around 2000, Krstic and Smyshlyaev began to extend backstepping to partial differential equations in the context of boundary

control. State space techniques and stability theory on Lyapunov theory were introduced. Backstepping method is different from any other methods previously developed for control of PDEs. Appropriate references in this area are the books by J.L Lions [34], Curtain and Zwart [17], Lasiecka and Triggiani [33], Bensoussan et al. [8], and Christofides [16]. Many of application books on control of PDEs have been dedicated to problems that arise from flexible structures [39], [31], [32], [7], [20] and from flow control [1], [23].

Due to the curse of the dimensionality which arise when one tries numerically to solve such problems, the backstepping method has been introduced to tackle such problems. It is an adaptation to the PDE case of a method initially used for ODE systems.

Some of the advantages of the method are

- its simplicity
- ease of implementation, one has to find a continuous inevertible transformation mapping the original PDE system to a new one which is known to be stable.

As for the disadvantages we can list,

- it is not necessarily optimal in any sense.
- one has to find an appropriate transformation for each given system.

1.2 Literature overview

Past efforts include the book [16], which solves problems of nonlinear parabolic PDE control but for inside the domain actuation, rather than with boundary control, and developments to solve the problem of motion planning for boundary controlled nonlinear parabolic PDEs [40] (using flatness and formal power series) and structural systems [30] (with a flatness/passivity approach). For example, [29] for the viscous Burgers equation; the Korteweg-de Vries equation [35] and the Kuramoto-Sivashinsky equation [37]. The result by Banaszuk, Hauksson, and Mesic [6] has involved one step of backstepping on a system consisting of a nonlinear PDE and two ODEs.

Adaptive control of nonlinear PDEs has received some attention. Liu and Krstic [36] and Kobayashi [26] considered a Burgers equation with various parametric uncertainties. Javanovic and Bamieh [25] designed adaptive controllers for nonlinear systems on lattices.

Research on control of linear/quasi-linear parabolic PDEs has been extensive in the past and has mainly focused on systems with fixed spatial domains, for example [4] and the book [16]. The main feature of parabolic PDE systems is that the eigenspectrum of the spatial differential operator can be partitioned into a finite-dimensional slow one and an infinite-dimensional stable fast complement [22]. This implies that the dominant dynamic behavior of such systems can be approximately described by finite-dimensional systems. Therefore, the standard approach to the control of linear/quasi-linear parabolic PDE systems (e.g., [5] and [15]) involves the application of standard Galerkin's method to the parabolic PDE system to derive

ordinary differential equation (ODE) systems that accurately describe the dominant dynamics of the PDE system, which are subsequently used as the basis for controller synthesis. Unfortunately, the developed control methods for quasi-linear parabolic PDE systems cannot be directly employed for the design of low-dimensional controllers for systems that include nonlinear spatial differential operators. The reason is that, in general, the eigenvalue problem of nonlinear spatial differential operators can not be solved analytically, and thus, it is difficult to choose a priori (without having any information about the solution of the system) an optimal (in the sense that it will lead to a low-dimensional ODE system) basis to expand the solution of the PDE system. An approximate way to address this problem [43] is to linearize the nonlinear spatial differential operator around a steady state and address the controller design problem on the basis of the resulting quasi-linear system. However, this approach is only valid in a small neighborhood of the steady state where the linearization takes place. Despite the recent progress on nonlinear control of parabolic PDE systems with fixed spatial domains, few results are available on control and estimation of parabolic PDE systems with time-dependent spatial domains. Important contributions include work on the synthesis of linear optimal controllers (e.g., [52, 53]), as well as the synthesis of nonlinear distributed state estimators using stochastic methods in [44], and the design of nonlinear and robust controllers on the basis of finite-dimensional models obtained using a combination of Galerkin's method with approximate inertial manifolds [2, 3]. Recently, Chanane and AL-Qarni used the transmutation approach (backstepping method) to control linear parabolic PDEs with coefficient depending

on time [12] and space and time [13]. This work is an extension to the nonlinear case of the approach presented in [12, 13].

1.3 Objectives

To tackle the problem of boundary control of nonlinear PDE

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) + \lambda u(x,t) + f(u) \quad , \quad 0 < x < 1 \ , \ t > 0 \\ u(0,t) = 0 \quad , \quad t > 0 \\ u(1,t) = U(t) \quad , \quad t > 0 \\ u(x,0) = u_0(t) \quad , \quad t > 0 \end{cases} \tag{1.1}$$

where U is the boundary control and u_0 is the initial condition, Krstic and Coworker [28] had to embed the problem into a problem with Volterra series nonlinearity with its sequence of higher dimensional kernels. Although, they succeeded in their approach, the question which arose was whether we can tackle this problem without resorting to Volterra series operator but using the original Volterra operator (transmutation). Hence the objective, in this work, is to use the transmutation approach to stabilize the system (1.1), thus avoiding the increased complexity introduced by the case of the Volterra series operator.

1.4 Methodology

Using the transmutation operator, we transform (1.1) into the target system

$$\begin{cases} v_t(x,t) = v_{xx}(x,t) + F(v) \ , \quad 0 < x < 1 \ , \ t > 0 \\ v(0,t) = 0 \quad , \quad t > 0 \\ v(1,t) = 0 \quad , \quad t > 0 \\ v(x,0) = v_0(x) \quad , \quad 0 < x < 1 \end{cases} \tag{1.2}$$

whose stability will be assessed. Note at this point that we have now Dirichlet boundary condition. Next, we consider the associated linear PDE system

$$\begin{cases} w_t(x,t) = w_{xx}(x,t) + g(x,t) \quad , \ 0 < x < 1 \quad , \ t > 0 \\ w(0,t) = 0 \quad , \ t > 0 \\ w(1,t) = 0 \quad , \ t > 0 \\ u(x,0) = w_0(x) \quad , \ 0 < x < 1 \end{cases} \tag{1.3}$$

By replacing $g(x,t)$ by $F(v)$ we get an integral equation satisfied by the solution v of (1.2). Solving this nonlinear integral equation allows us to find v, then using the inverse transformation we obtain u solution to (1.1) and the associated boundary control $U(t)$.

We shall introduce a few examples to illustrate the usefulness of the approach.

The thesis is organized as follows: In chapter two, we present the basic definitions, lemmata, properties and notation needed later in this work. In chapter three, we present the backstepping (transmutation) method to design boundary controllers sta-

bilizing the PDE system. In chapter four, we present our contribution specifically we revisit the backstepping method and consider boundary control of nonlinear parabolic systems.

Chapter 2

PRELIMINARIES

2.1 Terminology

In this work we shall be using the following function spaces

Name	Description	Norm
$C^{(n)}[a,b]$	$f, f', ..., f^{(n)}$ continuous functions on (a,b)	$\|f\|_\infty = \max_x \|f(x)\|$
$L^1(a,b)$	Integrable function: $\int \|f(x)\|\, dx < +\infty$	$\|f\|_{L^1} = \int \|f(x)\|\, dx$
$L^2(a,b)$	Square integrable function: $\int \|f(x)\|^2\, dx < +\infty$	$\|f\|_{L^2} = \left[\int \|f(x)\|^2\, dx\right]^{\frac{1}{2}}$
$H^1(a,b)$	Sobolev space: $f \in L^2$, $f' \in L^2$	$\|f\|^2_{H^1} = \|f\|^2_{L^2} + \|f'\|^2_{L^2}$

and in general

$$H^m(a,b) = \left\{f : f, f', ..., f^{(m)} \in L^2(a,b)\right\}$$

with the norm

$$\|f\|^2_{H^m} = \sum_{j=0}^{m} \left\|f^{(j)}(x)\right\|^2_{L^2} = \|f\|^2_{H^{m-1}} + \left\|f^{(m)}\right\|^2_{L^2} = \|f'\|^2_{H^{m-1}} + \|f\|^2_{L^2}\,.$$

The inner product in $L^2(a,b)$ and $H^1(a,b)$ are defined as

$$\langle f, g\rangle = \int_a^b f(x)\bar{g}(x)dx$$

and

$$\langle f, g\rangle = \int_a^b \left[f(x)\bar{g}(x) + f'(x)\bar{g'}(x) \right] dx$$

respectively.

2.2 Lyapunov Stability

2.2.1 Definition of stability

Before we study the stability for PDEs under consideration, we mention some of the basics of stability analysis for linear ODEs. We consider system described by ordinary differential equations of the form

$$\dot{z} = Az \tag{2.1}$$

with $z \in R^n$ and A is an $n \times n$ real matrix. (2.1) is said to be exponentially stable at $z = 0$ if there exist positive constants M , α and T such that

$$\|z(t)\| \leq Me^{-\alpha t} \|z(t)\| \quad \text{for all} \quad t \geq T \tag{2.2}$$

where $\|.\|$ denotes a vector norm. We can test the exponential stability by verifying that all the eigenvalues of the matrix A have negative real parts. But this test is not always practical, so we can use the Lyapunov second method, which is presented next.

The system (2.1) is exponentially stable in the sense of definition (2.2) if and only if for a positive definite $n \times n$ matrix Q there exists a positive definite and symmetric

matrix P such that

$$PA + A^T P = -Q \tag{2.3}$$

Along with this test comes the concept of a Lyapunov Function

$$v(z) = z^T P z \ .$$

2.2.2 Normalization of the Basic Parabolic PDE

The aim here is to develop a basic "non-dimensionalized" PDE model, which will be the starting point for many of the analysis and control design considerations in this study.

Consider a thermally conducting rod of length L whose temperature $T(\xi, \tau)$ is a function of the spatial variable ξ and time τ. The initial temperature distribution is $T(\xi)$ and the ends of the rod are kept at constant temperatures T_1 and T_2 . The evolution of the temperature profile is described by the heat equation

$$T_\tau(\xi, \tau) = \varepsilon T_{\xi\xi}(\xi, \tau), \tag{2.4}$$

$$T(0, \tau) = T_1, \tag{2.5}$$

$$T(1, \tau) = T_2, \tag{2.6}$$

$$T(\xi, 0) = T_0(\xi) \tag{2.7}$$

where ε denotes the thermal diffusivity.

To normalize the above system we have to make a change of variables such that

$$T(\xi, \tau) = T(x, t),$$

then we scale ξ to normalize the length

$$x = \frac{\xi}{L}, \tag{2.8}$$

and scale time to normalize the thermal diffusivity:

$$t = \frac{\varepsilon}{L^2}\tau, \tag{2.9}$$

which gives

$$T_{\xi\xi}(\xi, \tau) = \frac{1}{L^2} T_{xx}(x, t)$$

And also

$$T_\tau(\xi, \tau) = \frac{\varepsilon}{L^2} T_t(x, t)$$

By using (2.4) ,(2.5) and (2.6) we get

$$T_t(x,t) = T_{xx}(x,t) \quad , \quad 0 < x < 1 \quad , \quad t > 0 \tag{2.10}$$

$$T(0,t) = T_1 \quad , \quad t > 0 \tag{2.11}$$

$$T(1,t) = T_2 \quad , \quad t > 0 \tag{2.12}$$

Now we introduce the new variable

$$w = T - \overline{T} \tag{2.13}$$

where

$$\overline{T}(x) = T_1 + x(T_2 - T_1)$$

is the steady-state profile and is a solution to the two-point boundary -value ODE

$$\begin{cases} \overline{T}''(x) = 0 \\ \overline{T}(0) = T_1 \\ \overline{T}(1) = T_2 \end{cases}$$

Finally, by using (2.10) and (2.13) we obtain

$$\begin{cases} w_t(x,t) = w_{xx}(x,t) \quad , \quad 0 < x < 1, t > 0 \\ w(0,t) = 0 \quad , \quad t > 0 \\ w(1,t) = 0 \quad , \quad t > 0 \end{cases} \tag{2.14}$$

where the initial distribution of the temperature fluctuation is

$$w_0(x) = w(x,0)$$

The following are the basic types of boundary conditions for PDE in one dimension:

- Dirichlet: $w(0,t) = 0$ (fixed temperature at x=0)

- Neumann :$w_x(0,t) = 0$ (fixed heat flux at x=0)

- Robin :$w_x(0,t) + qw(0,t) = 0$ (mixed)

2.3 Stability of the PDE system

Consider the initial boundary value problem

$$\begin{cases} w_t(x,t) = w_{xx}(x,t) \quad , \quad 0 < x < 1, t > 0 \\ w(0,t) = 0 \quad , \quad t > 0 \\ w(1,t) = 0 \quad , \quad t > 0 \end{cases} \tag{2.15}$$

and initial condition

$$w(x,0) = w_0(x)$$

We want to show that this system is exponentially stable in the sense of an $L2\_norm$ of the state $w(x,t)$ with respect to x. To analyse the stability of the system, we have two approaches, finding the exact solution (easy in this case but more difficult in general problems) or use Lyapanov theory , hence avoiding solving the PDE.

Consider the Lyapunov function candidate

$$V(t) = \frac{1}{2}\int_0^1 w^2(x,t)dx \ .$$

Taking the time derivative of V we obtain,

$$\frac{dV(t)}{dt} = \int_0^1 w(x,t)w_t(x,t)dx$$

By using (2.15), we get

$$\frac{dV(t)}{dt} = \int_0^1 w(x,t)w_{xx}(x,t)dx$$

integrating by parts we get

$$\begin{aligned} \frac{dV(t)}{dt} &= w(x,t)w_x(x,t)\Big|_0^1 - \int_0^1 w_x^2(x,t)dx \\ &= -\int_0^1 w_x^2(x,t)dx < 0 \end{aligned} \tag{2.16}$$

This mean only that V is decreasing (not necessary to zero as $t \to \infty$)

We shall find an upper bound to the right-hand side of (2.16) in terms of V .

For this, we shall recall the following results,

1- Young's Inequality

$$ab \leq \frac{\alpha}{2}a^2 + \frac{1}{2\alpha}b^2 \tag{2.17}$$

2- Cauchy-Schwarz Inequality

$$\int_0^1 uwdx \leq \|u\| \|w\| \leq \frac{\alpha}{2}\|u\|^2 + \frac{1}{2\alpha}\|w\|^2 \tag{2.18}$$

The following lemma establishes the relationship between the L_2-norms of w and

w_x

Lemma 1 *(Poincare Inequality)*

For any w , continuously differentiable on $[0,1]$

$$\begin{aligned} \textstyle\int_0^1 w^2 dx &\leq 2w^2(1,t) + 4\textstyle\int_0^1 w_x^2 dx \\ \textstyle\int_0^1 w^2 dx &\leq 2w^2(0,t) + 4\textstyle\int_0^1 w_x^2 dx \end{aligned} \tag{2.19}$$

Now, return to equation (2.16), using the Poincare inequality along with boundary conditions, we get

$$\begin{aligned} \frac{dV}{dt} &= -\int_0^1 w_x^2(x,t)dx \leq -\frac{1}{4}\int_0^1 w^2(x,t) \\ &= -\frac{1}{2}V \end{aligned}$$

then

$$\frac{dV(t)}{dt} \leq -\frac{1}{2}V(t) \tag{2.20}$$

and that gives

$$V(t) \leq V(0)e^{-\frac{t}{2}} \tag{2.21}$$

But

$$V(t) = \frac{1}{2}\left\|w(x,t)\right\|^2 \tag{2.22}$$

Thus

$$\left\|w(t)\right\| \leq e^{-\frac{t}{4}}\left\|w_0\right\| \tag{2.23}$$

where $w_0 = w(x,0)$, leading to $\|w(x,t)\| \to 0$ as $t \to \infty$. That is, the system (2.15) is exponentially stable in L_2 sense.

2.4 Pointwise Stability

We established that

$$\|w(.,t)\| \to 0 \text{ as } t \to \infty$$

in $L_2 - norm$ but not necessarily in the $\sup - norm$, therefore we may have $w(x,t) \to \infty$ as $t \to \infty$ for some x. Thus we consider the $\sup - norm$

$$\|w(.,t)\|_\infty = \max_{x\in(0,1)} |w(x,t)|$$

It would be desirable to prove that

$$\|w(.,t)\|_\infty \leq e^{-\frac{t}{4}} \|w(.,0)\|_\infty \tag{2.24}$$

so that

$$\lim_{t\to\infty} \|w(.,t)\|_\infty = 0$$

However, we shall prove a more restrictive result , given by

$$\|w(.,t)\|_\infty \leq Ke^{-\frac{t}{2}} \|w(.,0)\|_{H_1} \tag{2.25}$$

for some $K > 0$, where the H_1 is a space of a square integrable functions whose derivative are square integrable, with $H_1 - norm$ defined by

$$\|w\|_{H_1}^2 := \int_0^1 w^2 dx + \int_0^1 w_x^2 dx \tag{2.26}$$

To prove (2.25), we need the following result [28]

Lemma 2 *(Agmon's Inequality)*

For a function $w \in H_1$, the following inequalities hold:

$$\max_{x\in[0,1]} |w(x,t)|^2 \leq w(0,t)^2 + 2\|w(t)\|\|w_x(t)\| \tag{2.27}$$

$$\max_{x\in[0,1]} |w(x,t)|^2 \leq w(1,t)^2 + 2\|w(t)\|\|w_x(t)\| \tag{2.28}$$

Now, to prove (2.25), we use the following Lyapunov function candidate :

$$\begin{aligned} V &= \frac{1}{2}\int_0^1 w^2 dx + \frac{1}{2}\int_0^1 w_x^2 dx \\ &= \frac{1}{2}\left(\|w(.,t)\|^2 + \|w_x(.,t)\|^2\right) \end{aligned} \tag{2.29}$$

The time derivative of (2.29) is given by

$$\begin{aligned} \frac{dV(t)}{dt} &= \int_0^1 w(x,t)w_t(x,t)dx + \int_0^1 w_x(x,t)w_{xt}(x,t)dx \\ &= \int_0^1 w(x,t)w_{xx}(x,t)dx + \int_0^1 w_x(x,t)w_{xt}(x,t)dx \end{aligned}$$

$$
\begin{aligned}
\frac{dV(t)}{dt} &= w(x,t)w_x(x,t)\Big|_{x=0}^{1} - \int_0^1 w_x^2(x,t)dx + w_x(x,t)w_t(x,t)\Big|_{x=0}^{1} \\
&\quad - \int_0^1 w_{xx}(x,t)w_t(x,t)dx \\
&= -\int_0^1 w_x^2(x,t)dx - \int_0^1 w_{xx}^2(x,t)dx
\end{aligned}
$$

Using Poincare Inequality we obtain,

$$
\begin{aligned}
\frac{dV(t)}{dt} &\leq -\frac{1}{4}\int_0^1 w^2(x,t)dx - \frac{1}{4}\int_0^1 w_x^2(x,t)dx \\
&= -\frac{1}{4}\left(\|w\|^2 + \|w_x\|^2\right) \\
&= -\frac{1}{2}V
\end{aligned}
$$

Thus,

$$
\frac{dV(t)}{dt} \leq -\frac{1}{2}V(t)
$$

which implies

$$
V(t) \leq V(0)e^{\frac{-t}{2}}
$$

use of (2.29) gives

$$
\|w\|^2 + \|w_x\|^2 \leq e^{\frac{-t}{2}}\left(\|w_0\|^2 + \|w_{0_x}\|^2\right)
$$

That is,

$$
\|w\|_{H_1} \leq e^{\frac{-t}{2}}\|w_0\|_{H_1}
$$

where $w_0 = w(x,0)$ is the initial condition. Finally, using Young's inequality, we get

$$\begin{aligned}
\max_{x\in[0,1]} |w(x,t)| &\leq 2\,\|w\|\,\|w_x\| \\
&\leq \|w\|^2 + \|w_x\|^2 \\
&\leq e^{\frac{-t}{2}} \left(\|w_0\|^2 + \|w_{0_x}\|^2\right)
\end{aligned}$$

Hence

$$\|w(x,t)\|_\infty \leq e^{\frac{-t}{2}} \|w(x,0)\|_{H_1}$$

from which we get $w(x,t) \to 0$ as $t \to \infty$ for all $x \in [0,1]$.

2.5 Exact solution to PDEs

We shall use the separation of variables to find the exact solution of a PDE system.

Consider the diffusion equation which includes a reaction term with boundary and initial conditions

$$\begin{cases}
u_t(x,t) = u_{xx}(x,t) + \lambda u(x,t) \quad , \quad 0 < x < 1 \\
u(0,t) = 0 \quad , \quad t > 0 \\
u(1,t) = 0 \quad , \quad t > 0 \\
u(x,0) = u_0(x) \quad , \quad t > 0
\end{cases} \tag{2.30}$$

where u_0 is a continuous function over $(0,1)$. Let us find the solution to this initial

boundary value problem and determine for which values of parameter λ this system is unstable .

Let us assume that the solution $u(x,t)$ can be written as a product of function of x and a function of t,

$$u(x,t) = X(x)T(t) \tag{2.31}$$

substitution (2.31) in the partial differential equation in (2.30) gives

$$X(x)T^{'}(t) = X^{''}(x)T(t) + \lambda X(x)T(t)$$

Assuming no x_0 , t_0 exist for which $X(x_0) = 0$, $T(t_0) = 0$, we obtain

$$\frac{T^{'}(t)}{T(t)} = \frac{X^{''}(x) + \lambda X(x)}{X(x)}.$$

Since the function on the left depends only on time, and function on the right depends only on the spatial variable, the equality can hold only if both functions are equal to the same constant. Let us denote this constant by σ . We then get,

$$T^{'}(t) = \sigma T(t) \ , \ t > 0 \tag{2.32}$$

and

$$X^{''}(x) + (\lambda - \sigma)X(x) = 0 \tag{2.33}$$

together with the boundary conditions

$$X(0) = X(1) = 0 \tag{2.34}$$

So, the solution to (2.33) has the form

$$X(x) = A\sin\left(\sqrt{\lambda - \sigma}x\right) + B\cos(\sqrt{\lambda - \sigma}x) \tag{2.35}$$

where A and B are constant that can be determined by using (2.34) which gives:

$$\begin{cases} B = 0 \\ A\sin(\sqrt{\lambda - \sigma}) = 0 \end{cases}$$

Since $A \neq 0$ because we get a trivial solution then the equality can hold only if $\sqrt{\lambda - \sigma} = \pi n$ for $n = 1, 2, 3, ...$, so that

$$\sigma_n = \lambda - \pi^2 n^2 \quad , \quad n = 1, 2, 3, ... \tag{2.36}$$

are the eigenvalues with corresponding the eigenfunctions

$$X_n(x) = A_n \sin(\pi n x) \tag{2.37}$$

As for T we have

$$T(t) = T_{0n} e^{\sigma_n t} \qquad , \qquad n \geq 1 \tag{2.38}$$

substituting (2.38) and (2.37) into (2.31) yields

$$u_n(x,t) = T_{0n} A_n e^{(\lambda - \pi^2 n^2)t} \sin(\pi n x) \ , \ n = 1, 2, 3, ... \tag{2.39}$$

where A_n is a constant . Superposition principle gives

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{(\lambda - \pi^2 n^2)t} \sin(\pi n x), \tag{2.40}$$

where $C_n = A_n T_{0n}$ are constants. Using the initial condition

$$u(x,0) = u_0(x),$$

we get

$$u_0(x) = \sum_{n=1}^{\infty} C_n \sin(\pi n x)$$

which is the Fourier sine series whose coefficients are given by

$$C_n = 2 \int_0^1 u_0(x) \sin(\pi n x) dx \tag{2.41}$$

Substituting C_n in (2.40), we get

$$u(x,t) = 2 \sum_{n=1}^{\infty} e^{(\lambda - \pi^2 n^2)t} \sin(\pi n x) \int_0^1 \sin(\pi n \overline{x}) u_0(\overline{x}) d\overline{x} \tag{2.42}$$

That is

$$u(x,t) = \int_0^1 g(x,\overline{x},t)u_0(\overline{x})d\overline{x}$$

where

$$g(x,y) = 2\sum_{n=1}^{\infty} e^{(\lambda-\pi^2n^2)t}\sin(\pi nx)\sin(\pi ny),$$

is the Green's function for the problem.

This solution consists of the following element:

- eigenvalues: $\lambda - \pi^2 n^2$.

- eigenfunctions: $\sin(\pi nx)$.

-effect of initial conditions: $\int_0^1 \sin(\pi n\overline{x})u_0(\overline{x})d\overline{x}$

The eigenvalue whose index is the index of the first nonzero Fourier sine series coefficient of the initial condition u_0 , dictates the rate of decay of the solution. Hence, if $C_n = 0$ for $n = 1, 2, ..., n_0 - 1$ and $c_{n_0} \neq 0$ then the system is stable for

$$\lambda - (n_0\pi)^2 < 0$$

which mean the system (2.30) is stable if

$$\lambda < (n_0\pi)^2$$

Chapter 3

BOUNDARY CONTROL Of PARABOLIC PDEs

3.1 Backstepping: The main idea

In this chapter we present the first designs of feedback laws for stabilization of parabolic PDEs using boundary control and introduce the method of backstepping.

The main feature of backstepping is that it is capable of eliminating destabilizing effect terms that appear throughout the domain while the control is acting only from the boundary.

Backstepping has been proved to be a remarkable method for designing controllers for PDE systems. In addition, it achieves stabilization of unstable PDEs in a physically appealing way, that is , the destabilizing terms are eliminated through a change of the PDE and boundary feedback.

Let us start with the simplest unstable PDE, the reaction-diffusion equation

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) + \lambda u(x,t) \quad , \quad 0 < x < 1 \quad , \quad t > 0 \\ u(0,t) = 0 \quad , \quad t > 0 \\ u(1,t) = U(t) \quad , \quad t > 0 \\ u(x,0) = u_0(x) \quad , \quad 0 < x < 1 \end{cases} \tag{3.1}$$

where λ is an arbitrary constant , $U(t)$ is the control input and u_0 is a continuous function. The open loop system $(u(1,0) = 0)$ is unstable with arbitrary many unstable eigenvalues for sufficiently large λ .

The term λu is the source of instability, so we need to eliminate this term using the backstepping method. The main idea of the backstepping method is to use the

coordinate transformation :

$$w(x,t) = u(x,t) - \int_0^x k(x,y)u(y,t)dy \tag{3.2}$$

along with feedback control

$$u(1,t) = \int_0^1 k(1,y)u(y,t)dy \ . \tag{3.3}$$

We map the system (3.1) into the exponentially stable target system

$$\begin{cases} w_t(x,t) = w_{xx}(x,t) \ \ , \ \ 0 < x < 1 \ \ , \ \ t > 0 \\ w(0,t) = 0 \ \ , \ \ t > 0 \\ w(1,t) = 0 \ \ , \ \ t > 0 \\ w(x,0) = w_0(x) \ \ , \ \ 0 < x < 1 \end{cases} \tag{3.4}$$

The transformation (3.2) is called a Volterra integral transformation of the second kind.

The important property of the Volterra transformation is that it is invertible, so that stability of the target system translates into stability of the closed-loop system consisting of the plant plus boundary feedback.

The function k , (which is call the "gain kernel"), makes the system (3.1) with the controller behave as the target system (3.4)

3.2 Gain Kernel PDE

To find out what conditions k has to satisfy, we simply substitute the transformation (3.2) into the target system (3.4) and use the system equation (3.1). To do that , we need to differentiate both sides of (3.2) with respect to x and t . We recall the Leibnitz differentiation rule

$$\frac{d}{dx}\int_0^x f(x,y)dy = f(x,x) + \int_0^x f_x(x,y)dy.$$

Let kernel k be twice continuous differentiable in $0 \leq y \leq x \leq 1$.

We have

$$k_x(x,x) = \tfrac{\partial}{\partial x}k(x,y)\,|_{y=x}$$

$$k_y(x,x) = \tfrac{\partial}{\partial y}k(x,y)\,|_{y=x}$$

$$\tfrac{d}{dx}k(x,x) = k_x(x,x) + k_y(x,x)$$

Differentiating (3.2) with respect to x gives:

$$\begin{aligned} w_x(x,t) &= u_x(x,t) - k(x,x)u(x,t) - \int_0^x k_x(x,y)u(y,t)dy \\ w_{xx}(x,t) &= u_{xx}(x,t) - u(x,t)\frac{d}{dx}k(x,x) - k(x,x)u_x(x,t) - \\ &\quad k_x(x,x)u(x,t) - \int_0^x k_{xx}(x,y)u(y,t)dy \end{aligned} \tag{3.5}$$

Next, we differentiate (3.2) with respect to time:

$$\begin{aligned} w_t(x,t) &= u_t(x,t) - \int_0^x k(x,y)u_t(x,t)dy \\ &= u_{xx}(x,t) + \lambda u(x,t) - \int_0^x k(x,y)\left[u_{yy}(y,t) + \lambda u(y,t)\right]dy \\ &= u_{xx}(x,t) + \lambda u(x,t) - \int_0^x k(x,y)u_{yy}(y,t)dy - \\ &\quad \int_0^x \lambda k(x,y)u(y,t)dy \end{aligned}$$

Integrating by parts gives:

$$\begin{aligned} w_t(x,t) &= u_{xx}(x,t) + \lambda u(x,t) - k(x,x)u_x(x,t) + k(x,0)u_x(0,t) + \\ &\quad \int_0^x k_y(x,y)u_y(y,t)dy - \int_0^x \lambda k(x,y)u(y,t)dy \\ &= u_{xx}(x,t) + \lambda u(x,t) - k(x,x)u_x(x,t) + k(x,0)u_x(0,t) + \\ &\quad k_y(x,x)u(x,t) - k_y(x,0)u(0,t) - \int_0^x k_{yy}(x,y)u(y,t)dy - \\ &\quad \int_0^x \lambda k(x,y)u(y,t)dy \end{aligned} \tag{3.6}$$

Subtracting (3.5) from (3.6), we get

$$\begin{aligned} w_t(x,t) - w_{xx}(x,t) &= \left[\lambda + k_y(x,x) + \frac{d}{dx}k(x,x) + k_x(x,x)\right]u(x,t) + \\ &\quad k(x,0)u_x(x,t) + \int_0^x [k_{xx}(x,y) - k_{yy}(x,y) - \\ &\quad \lambda k(x,y)u(y,t)dy] \end{aligned}$$

$$
\begin{aligned}
w_t(x,t) - w_{xx}(x,t) &= \left[\lambda + 2\frac{d}{dx}k(x,x)\right] u(x,t) + k(x,0)u_x(0,t) + \\
&\quad \int_0^x \left[k_{xx}(x,y) - k_{yy}(x,y) - \lambda k(x,y)\right] u(y,t)dy
\end{aligned}
$$

We shall take k satisfying :

$$
\begin{cases}
k_{xx}(x,y) - k_{yy}(x,y) = \lambda k(x,y) \quad , \quad 0 < y < 1 \\
k(x,0) = 0 \quad , \quad 0 < x < 1 \\
k(x,x) = -\frac{\lambda}{2}x \quad , \quad 0 < x < 1
\end{cases}
\tag{3.7}
$$

so that we get

$$
w_t(x,t) - w_{xx}(x,t) = 0
$$

3.3 Converting the Gain Kernel PDE into an Integral Equation

To find the solution of the PDE (3.7), we first convert into an integral equation. Introducing the change of variables

$$
\xi = x + y \qquad , \qquad \eta = x - y
$$

we have,

$$
\begin{aligned}
k(x,y) &= G(\xi,\eta) \\
k_x &= G_\xi + G_\eta \\
k_y &= G_\xi - G_\eta \\
k_{xx} &= G_{\xi\xi} + 2G_{\xi\eta} + G_{\eta\eta} \\
k_{yy} &= G_{\xi\xi} - 2G_{\xi\eta} + G_{\eta\eta}
\end{aligned}
$$

The Gain Kernel PDE becomes:

$$
\begin{cases}
G_{\xi\eta}(\xi,\eta) = \frac{\lambda}{4}G(\xi,\eta), \\
G(\xi,\xi) = 0, \\
G(\xi,0) = -\frac{\lambda}{4}\xi \quad .
\end{cases}
\tag{3.8}
$$

Now, integrating (2.8) with respect to η gives:

$$
\begin{aligned}
G_\xi(\xi,\eta) &= G_\xi(\xi,0) + \int_0^\eta \frac{\lambda}{4}G(\xi,s)ds \\
&= -\frac{\lambda}{4} + \int_0^\eta \frac{\lambda}{4}G(\xi,s)ds \quad .
\end{aligned}
\tag{3.9}
$$

Next, we integrate (3.9) with respect to ξ from η to ξ to get

$$
G(\xi,\eta) = G(\eta,\eta) - \frac{\lambda}{4}(\xi-\eta) + \frac{\lambda}{4}\int_\eta^\xi \int_0^\eta G(\tau,s)dsd\tau \quad ,
$$

which gives

$$G(\xi,\eta) = -\frac{\lambda}{4}(\xi-\eta) + \frac{\lambda}{4}\int_{\eta}^{\xi}\int_{0}^{\eta} G(\tau,s)dsd\tau \tag{3.10}$$

the integral equation, which is equivalent to PDE (3.7) in the sense that every solution of (3.7) is a solution of (3.10) and vice-versa.

3.4 Method of Successive Approximations

Let us start with an initial guess for a solution of the integral equation, substitute it into the right-hand side of the equation, then use the obtained expression as the next guess in the integral equation and repeat the process. Eventually this process results in a solution of the integral equation. More precisely, let us start with an initial guess

$$G_0(\xi,\eta) = -\frac{\lambda}{4}(\xi-\eta)$$

and define,

$$G_n(\xi,\eta) = \frac{\lambda}{4}\int_{\eta}^{\xi}\int_{0}^{\eta} G_{n-1}(\tau,s)dsd\tau \quad , \quad n \geq 1$$

If this converges, we can write the solution $G(\xi,\eta)$ as

$$G(\xi,\eta) = \lim_{n\to\infty} G_n(\xi,\eta).$$

We want to prove by induction that

$$G_n(\xi,\eta) = -\left(\frac{\lambda}{4}\right)^{n+1} \frac{\xi^n\eta^n}{n!(n+1)!}(\xi-\eta)\ ,\ n \geq 0$$

indeed, the relation is true for $n = 0$

$$G_0(\xi,\eta) = -\frac{\lambda}{4}(\xi-\eta).$$

Assume the relation is true for n .We want to prove the relation is true for $n+1$

$$\begin{aligned}
G_{n+1}(\xi,\eta) &= \frac{\lambda}{4}\int_\eta^\xi \int_0^\eta G_n(\tau,s)dsd\tau \\
&= \frac{\lambda}{4}\int_\eta^\xi \int_0^\eta -\left(\frac{\lambda}{4}\right)^{n+1} \frac{\tau^n s^n}{n!(n+1)!}(\tau-s)dsd\tau \\
&= \frac{\lambda}{4}\int_\eta^\xi -\left(\frac{\lambda}{4}\right)^{n+1} \frac{1}{n!(n+1)!}\int_0^\eta (\tau^{n+1}s^n - \tau^n s^{n+1})dsd\tau \\
&= \frac{\lambda}{4}\int_\eta^\xi -\left(\frac{\lambda}{4}\right)^{n+1} \frac{1}{n!(n+1)!}\left[\frac{\tau^{n+1}s^{n+1}}{n+1} - \frac{\tau^n s^{n+2}}{n+2}\right]_{s=0}^{\eta} d\tau \\
&= \frac{\lambda}{4}\int_\eta^\xi -\left(\frac{\lambda}{4}\right)^{n+1} \frac{1}{n!(n+1)!}\left[\frac{\tau^{n+1}\eta^{n+1}}{n+1} - \frac{\tau^n \eta^{n+2}}{n+2}\right] d\tau \\
&= -\left(\frac{\lambda}{4}\right)^{n+2} \frac{1}{n!(n+1)!}\left[\frac{\tau^{n+2}\eta^{n+1}}{(n+1)\,(n+2)} - \frac{\tau^{n+1}\eta^{n+2}}{(n+1)\,(n+2)}\right]_{\tau=\eta}^{\xi} \\
&= -\left(\frac{\lambda}{4}\right)^{n+2} \frac{1}{(n+1)!(n+2)!}\left(\xi^{n+2}\eta^{n+1} - \xi^{n+1}\eta^{n+2}\right) \\
&= -\left(\frac{\lambda}{4}\right)^{n+2} \frac{\xi^{n+1}\eta^{n+1}}{(n+1)!(n+2)!}\,(\xi-\eta)
\end{aligned}$$

Thus the relation is true for $n+1$. Hence, the relation is true for all $n \geq 0$ .We can

write the solution as,

$$G(\xi,\eta) = -\sum_{n=0}^{\infty} \frac{\xi^n \eta^n (\xi-\eta)}{(n+1)!(n+2)!} \left(\frac{\lambda}{4}\right)^{n+1}. \tag{3.11}$$

By using the ratio test we can prove that this series is convergent absolutely. We can write (3.11) by using modified Bessel Function

$$I_1(x) = \sum_{n=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2n+1}}{n!(n+1)!} \ ,$$

Now,

$$\begin{aligned}
G(\xi,\eta) &= -\sum_{n=0}^{\infty} \frac{\xi^n \eta^n (\xi-\eta)}{n!(n+1)!} \left(\frac{\lambda}{4}\right)^{n+1} \\
&= -\frac{\lambda}{4}(\xi-\eta) \sum_{n=0}^{\infty} \frac{\xi^n \eta^n \left(\frac{\lambda}{4}\right)^n}{n!(n+1)!} \\
&= -\frac{\lambda}{4}(\xi-\eta) \sum_{n=0}^{\infty} \frac{\left(\frac{\lambda}{4}\xi\eta\right)^n}{n!(n+1)!} \\
&= -\frac{\lambda}{4}(\xi-\eta) \sum_{n=0}^{\infty} \frac{\left(\sqrt{\frac{\lambda}{4}\xi\eta}\right)^{2n+1}}{n!(n+1)!\sqrt{\frac{\lambda}{4}\xi\eta}} \\
&= -\frac{\lambda}{4}(\xi-\eta) \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\sqrt{\lambda\xi\eta}\right)^{2n+1}}{n!(n+1)!\sqrt{\frac{\lambda}{4}\xi\eta}} \\
&= -\frac{\lambda}{4}(\xi-\eta) \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\sqrt{\lambda\xi\eta}\right)^{2n+1}}{n!(n+1)!\frac{\sqrt{\lambda\xi\eta}}{2}} \\
&= -\frac{\lambda}{2}(\xi-\eta) \frac{I_1\left(\sqrt{\lambda\xi\eta}\right)}{\sqrt{\lambda\xi\eta}}
\end{aligned}$$

Returning to the original x, y variables,

$$k(x,y) = -\lambda y \frac{I_1\left(\sqrt{\lambda\left(x^2-y^2\right)}\right)}{\sqrt{\lambda\left(x^2-y^2\right)}} \tag{3.12}$$

for $x > y$ and $k(x,x) = -\frac{\lambda}{2}x$.

3.5 Inverse Transformation

Since the Volterra integral operator of the second kind is invertible, we shall find the inverse transformation in the form

$$u(x,t) = w(x,t) + \int_0^x l(x,y)w(y,t)dy \tag{3.13}$$

where $l(x,y)$ is the transformation kernel.

We follow the same approach that led us to the kernel PDE for $k(x,y)$; We differentiate (3.13) with respect to x and t and use the plant and the target system to obtain the PDE for $l(x,y)$.

$$\begin{aligned}
u_t(x,t) &= w_t(x,t) + \int_0^x l(x,y)w_t(y,t)dy \\
&= w_{xx}(x,t) + \int_0^x l(x,y)w_{yy}(y,t)dy \\
&= w_{xx}(x,t) + l(x,x)w_x(x,t) - l(x,0)w_x(0,t) - \\
&\quad \int_0^x l_y(x,y)w_y(y,t)dy
\end{aligned}$$

$$
\begin{aligned}
u_t(x,t) &= w_{xx}(x,t) + l(x,x)w_x(x,t) - l(x,0)w_x(0,t) - \\
&\quad l_y(x,x)w(x,t) + l_y(x,0)w(0,t) + \int_0^x l_{yy}(x,y)w(y,t)dy
\end{aligned}
$$

which gives

$$
\begin{aligned}
u_t(x,t) &= w_{xx}(x,t) + l(x,x)w_x(x,t) - l(x,0)w_x(0,t) - \\
&\quad l_y(x,x)w(x,t) + \int_0^x l_{yy}(x,y)w(y,t)dy \qquad (3.14)
\end{aligned}
$$

Differentiating twice (3.13) with respect to x gives

$$
\begin{aligned}
u_x(x,t) &= w_x(x,t) + l(x,x)w(x,t) + \int_0^x l_x(x,y)w(y,t)dy \\
u_{xx}(x,t) &= w_{xx}(x,t) + w(x,t)\frac{d}{dx}l(x,x) + l(x,x)w_x(x,t) + \\
&\quad l_x(x,x)w(x,t) + \int_0^x l_{xx}(x,y)w(y,t)dy \qquad (3.15)
\end{aligned}
$$

Subtracting (3.15) from (3.14) gives

$$
\begin{aligned}
u_t(x,t) - u_{xx}(x,t) &= \left[-l_y(x,x) - \tfrac{d}{dx}l(x,x) - l_x(x,x)\right] w(x,t) - \\
&\quad l(x,0)w_x(0,t) + \textstyle\int_0^x \left[l_{yy}(x,y) - l_{xx}(x,y)\right] w(y,t)dy \\
&= -2w(x,t)\tfrac{d}{dx}l(x,x) - l(x,0)w_x(0,t) + \\
&\quad \textstyle\int_0^x \left[l_{yy}(x,y) - l_{xx}(x,y)\right] w(y,t)dy
\end{aligned}
$$

But

$$u_t(x,t) - u_{xx}(x,t) = \lambda u(x,t)$$

Then,

$$\lambda w(x,t) + \lambda \int_0^x l(x,y)w(y,t)dy = -2w(x,t)\frac{d}{dx}l(x,x) -$$

$$l(x,0)w_x(0,t) + \int_0^x \left[l_{yy}(x,y) - l_{xx}(x,y)\right] w(y,t)dy$$

Taking,

$$\begin{aligned} l_{xx}(x,y) - l_{yy}(x,y) &= -\lambda l(x,y) \\ l(x,0) &= 0 \\ \lambda + 2\frac{d}{dx}l(x,x) &= 0 \end{aligned}$$

yields

$$\begin{cases} l_{xx}(x,y) - l_{yy}(x,y) = -\lambda l(x,y) \quad , \quad 0 < y < x < 1 \\ l(x,0) = 0 \quad , \quad 0 < x < 1 \\ l(x,x) = -\frac{\lambda}{2}x \quad , \quad 0 < x < 1 \end{cases} \tag{3.16}$$

Comparing this PDE with the PDE (3.7) for $k(x,y)$ we see that

$$l(x,y;\lambda) = -k(x,y,-\lambda) \tag{3.17}$$

From (3.12), we have

$$\begin{aligned} l(x,y) &= -\lambda y \frac{I_1\left(\sqrt{-\lambda\left(x^2-y^2\right)}\right)}{\sqrt{-\lambda\left(x^2-y^2\right)}} \\ &= -\lambda y \frac{I_1\left(j\sqrt{\lambda\left(x^2-y^2\right)}\right)}{j\sqrt{\lambda\left(x^2-y^2\right)}} \end{aligned}$$

where $j=\sqrt{-1}$ .By using the properties of the Bessel functions

$$I_n(x) = i^{-n} J_n(ix) \quad , \quad I_n(ix) = i^n J_n(x)$$

we obtain,

$$l(x,y) = -\lambda y \frac{J_1\left(\sqrt{\lambda\left(x^2-y^2\right)}\right)}{\sqrt{\lambda\left(x^2-y^2\right)}} \tag{3.18}$$

Hence the inverse transformation is

$$u(x,t) = w(x,t) - \int_0^x \lambda y \frac{J_1\left(\sqrt{\lambda\left(x^2-y^2\right)}\right)}{\sqrt{\lambda\left(x^2-y^2\right)}} w(y,t)dy \tag{3.19}$$

while the direct transformation is,

$$w(x,t) = u(x,t) + \int_0^x \lambda y \frac{I_1\left(\sqrt{\lambda\left(x^2-y^2\right)}\right)}{\sqrt{\lambda\left(x^2-y^2\right)}} u(y,t)dy \tag{3.20}$$

and the boundary controller is therefore,

$$U(t) = u(1,t) = -\int_0^1 \lambda y \frac{I_1\left(\sqrt{\lambda\left(1-y^2\right)}\right)}{\sqrt{\lambda\left(1-y^2\right)}} u(y,t)dy$$

To summarize, we have the following results [13], for the PDE

$$u_t(x,t) = u_{xx}(x,t) + \lambda u \ , \quad 0 < x < 1 \ , \ t > 0$$

with one of boundary conditions below,

- Dirichlet: $u(0,t) = 0$, $u(1,t) = U(t)$
- Neumann: $u(0,t) = 0$, $u_x(1,t) = U(t)$
- Robin: $u(0,t) + u_x(0,t) = 0$, $u(1,t) + u_x(1,t) = U(t)$.

we have the following stabilizing boundary controllers,

- Dirichlet:

$$U(t) = \int_0^1 k_D(1,y)u(y,t)dy$$

where the kernel $k_D(1,y)$ is defined by

$$k_D(1,y) = -\lambda y \frac{I_1\left(\sqrt{\lambda\,(1-y^2)}\right)}{\sqrt{\lambda\,(1-y^2)}}$$

- Neumann:

$$U(t) = -\frac{\lambda}{2}u(1,t) + \int_0^1 k_N(1,y)u(y,t)dy$$

where the kernel $k_N(1,y)$ is defined by

$$k_N(1,y) = -\lambda y \frac{I_2\left(\sqrt{\lambda\,(1-y^2)}\right)}{1-y^2}$$

- Robin:

$$U(t) = -\frac{\lambda}{2}u(1,t) + \int_0^1 k_R(1,y)u(y,t)dy$$

where the kernel $k_R(1,y)$ is defined by

$$k_R(1,y) = -\lambda y \left[\frac{I_1\left(\sqrt{\lambda\left(1-y^2\right)}\right)}{\sqrt{\lambda\left(1-y^2\right)}} + \frac{I_2\left(\sqrt{\lambda\left(1-y^2\right)}\right)}{\lambda\left(1-y^2\right)} \right] .$$

where I_1 and I_2 are the modified Bessel function.

Chapter 4

PROBLEM STATEMENT

4.1 Boundary Control of Nonlinear Parabolic PDE

4.1.1 Introduction

It is well known that control of nonlinear parabolic equation is difficult due to the possibility of blow up of the solution when it exists. Kristic [28] considered the boundary control of such a problem and his first task was to turn the PDE with some analytic nonlinearity into a PDE with Volterra type series nonlinearity. He looked for a boundary control using the backstepping method given by a Volterra series, whose kernels had to be identified. Specifically, he considered

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) + f(u) \quad , \quad 0 < x < 1 \ , \ t > 0 \\ u(0,t) = 0 \quad , \quad t > 0 \\ u(1,t) = U(t) \quad , \quad t > 0 \end{cases} \tag{4.1}$$

He transformed the system into

$$\begin{cases} v_t(x,t) = v_{xx}(x,t) + (Tv)(x,t) \ , \quad 0 < x < 1 \ , \ t > 0 \\ v(0,t) = 0 \quad , \quad t > 0 \\ v(1,t) = 0 \quad , \quad t > 0 \end{cases} \tag{4.2}$$

where, Tv is a Volterra series given by

$$\begin{aligned}(Tv)(x) &= v(x,t) + \int_0^x T_1(x,x_1)v(x_1,t)dx_1 \\ &\quad + \sum_{k\geq 2} \int_0^x \int_0^{x_k} ... \int_0^{x_2} T_k(x,x_1,...,x_k)v(x_1,t)...v(x_k,t)dx_1...dx_k\end{aligned}$$

and exhibited a backstepping transformation in the form of another Volterra series

$$\begin{aligned}w(x,t) &= v(x,t) + \int_0^x k_1(x,x_1)v(x_1,t)dx_1 \\ &\quad + \sum_{j\geq 2} \int_0^x \int_0^{x_k} ... \int_0^{x_2} k_j(x,x_1,...,x_j)v(x_1,t)...v(x_j,t)dx_1...dx_j\end{aligned}$$

mapping (4.2) into

$$\begin{cases} w_t(x,t) = v_{xx}(x,t) \ , \quad 0 < x < 1 \ , \ t > 0 \\ w(0,t) = 0 \ , \quad t > 0 \\ w(1,t) = 0 \ , \quad t > 0 \end{cases} \tag{4.3}$$

so that the boundary control is given by

$$\begin{aligned}V(t) &= v(1,t) = -\int_0^1 k_1(1,x_1)v(x_1,t)dx_1 \\ &\quad - \sum_{j\geq 2} \int_0^1 \int_0^{x_k} ... \int_0^{x_2} k_j(1,x_1,...,x_j)v(x_1,t)...v(x_j,t)dx_1...dx_j. \end{aligned} \tag{4.4}$$

Note that the kernels have to satisfy PDEs of increasing complexity, even in a

"simple" structure, where $f(u) = u^2$.

4.2 Main result

The backstepping transformation

In this work we shall consider the nonlinear problem below,

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) + \lambda u(x,t) + f(u) \quad , \quad 0 < x < 1 \ , \ t > 0 \\ u(0,t) = 0 \quad , \quad t > 0 \\ u(1,t) = U(t) \quad , \quad t > 0 \\ u(x,0) = u_0(x) \quad , \quad 0 < x < 1 \end{cases} \tag{4.5}$$

Let K and L be the Volterra operators defined by

$$(\mathbf{K}u)(z,t) = \int_0^z k(z,y)u(y,t)dy, \tag{4.6}$$

and

$$(\mathbf{L}v)(z,t) = \int_0^z l(z,y)v(y,t)dy, \tag{4.7}$$

where the twice continuously differentiable kernels k and l satisfy the Goursat problems,

$$\begin{cases} k_{xx}(x,y) - k_{yy}(x,y) = \lambda k(x,y) \quad , \quad 0 < x < 1 \quad , \quad 0 < y < 1 \\ k(x,0) = 0, \quad , \quad 0 < x < 1 \\ k(x,x) = -\frac{\lambda}{2}x \quad , \quad 0 < x < 1 \end{cases} \tag{4.8}$$

and

$$\begin{cases} l_{xx}(x,y) - l_{yy}(x,y) = -\lambda l(x,y) \quad , \quad 0 < x < 1 \quad , \quad 0 < y < 1 \\ l(x,0) = 0, \quad , \quad 0 < x < 1 \\ l(x,x) = -\frac{\lambda}{2}x \quad , \quad 0 < x < 1 \end{cases} \tag{4.9}$$

respectively.

Theorem 3 *The transformation* $(1-\mathbf{K})$ *defined by* $v = (1-\mathbf{K})u$*, from* $H^i \to H^i$ *,* $i = 1, 2$ *with*

$$v(x,t) = u(x,t) - \int_0^x k(x,y)u(y,t)dy \tag{4.10}$$

maps (4.5) into the target system

$$\begin{cases} v_t(x,t) = v_{xx}(x,t) + F(v) \ , \quad 0 < x < 1 \ , \ t > 0 \\ v(0,t) = 0 \quad , \quad t > 0 \\ v(1,t) = 0 \quad , \quad t > 0 \\ v(x,0) = v_0(x) \quad , \quad 0 < x < 1 \end{cases} \tag{4.11}$$

when

$$U(t) \triangleq u(1,t) = \int_0^1 k(1,y)u(y,t)dy. \tag{4.12}$$

Here F *and* v_0 *are defined by*

$$F(v) = (1-\mathbf{K})f((1-\mathbf{K})^{-1}v), \tag{4.13}$$

$$v_0 = (1-\mathbf{K})u_0 \tag{4.14}$$

Furthermore

$$(1-\mathbf{K})^{-1} = 1+\mathbf{L} \tag{4.15}$$

and

$$u = (1+\mathbf{L})v$$

i.e. ,

$$u(x,t) = v(x,t) + \int_0^x l(x,y)v(y,t)dy \tag{4.16}$$

Proof. That the transformation $1-\mathbf{K} : H^i \to H^i$, $(i = 1, 2)$ has been shown in [45]. Applying $(1-\mathbf{K})$ to both sides of PDE (4.5) we get

$$(1-\mathbf{K})u_t(x,t) = (1-\mathbf{K})\left(u_{xx}(x,t) + \lambda u(x,t)\right) + (1-\mathbf{K})f(u)$$

$$\frac{\partial}{\partial t}(1-\mathbf{K})u(x,t) = (1-\mathbf{K})\left(u_{xx}(x,t) + \lambda u(x,t)\right) + (1-\mathbf{K})f((1-\mathbf{K})^{-1}v)$$

then,

$$v_t(x,t) = (1-\mathbf{K})\left(u_{xx}(x,t) + \lambda u(x,t)\right) + F(v). \tag{4.17}$$

Now, we want to prove that

$$(1-\mathbf{K})\left(u_{xx}(x,t) + \lambda u(x,t)\right) = v_{xx}(x,t).$$

So,

$$
\begin{aligned}
(1-\mathbf{K})\left(u_{xx}(x,t)+\lambda u(x,t)\right) &= u_{xx}(x,t)+\lambda u(x,t)-\int_0^x k(x,y)\left[u_{yy}(y,t)+\lambda u(y,t)\right]dy \\
&= u_{xx}(x,t)+\lambda u(x,t)-\int_0^x k(x,y)u_{yy}(y,t)dy \\
&\quad -\int_0^x \lambda k(x,y)u(y,t)dy.
\end{aligned}
$$

Integrating by parts gives:

$$
\begin{aligned}
(1-\mathbf{K})\left(u_{xx}(x,t)+\lambda u(x,t)\right) &= u_{xx}(x,t)+\lambda u(x,t)-k(x,x)u_x(x,t)+k(x,0)u_x(0,t)+ \\
&\quad \int_0^x k_y(x,y)u_y(y,t)dy-\int_0^x \lambda k(x,y)u(y,t)dy \\
&= u_{xx}(x,t)+\lambda u(x,t)-k(x,x)u_x(x,t)+k(x,0)u_x(0,t)+ \\
&\quad k_y(x,x)u(x,t)-k_y(x,0)u(0,t)-\int_0^x k_{yy}(x,y)u(y,t)dy- \\
&\quad \int_0^x \lambda k(x,y)u(y,t)dy. \qquad (4.18)
\end{aligned}
$$

Next, differentiating both sides of (4.10) with respect to x, gives

$$
v_x(x,t)=u_x(x,t)-k(x,x)u(x,t)-\int_0^x k_x(x,y)u(y,t)dy
$$

and

$$
\begin{aligned}
v_{xx}(x,t) &= u_{xx}(x,t)-u(x,t)\frac{d}{dx}k(x,x)-k(x,x)u_x(x,t) \\
&\quad -k_x(x,x)u(x,t)-\int_0^x k_{xx}(x,y)u(y,t)dy \qquad (4.19)
\end{aligned}
$$

Subtracting (4.19) from (4.18), we get

$$\begin{aligned}(1-\mathbf{K})\left(u_{xx}(x,t)+\lambda u(x,t)\right)-v_{xx}(x,t) &= \left[\lambda+2\frac{d}{dx}k(x,x)\right]u(x,t)+k(x,0)u_x(0,t)\\ &\quad+\int_0^x\left[k_{xx}(x,y)-k_{yy}(x,y)-\lambda k(x,y)\right]u(y,t)dy,\end{aligned}$$

using (4.8), we obtain

$$(1-\mathbf{K})\left(u_{xx}(x,t)+\lambda u(x,t)\right)=v_{xx}(x,t).$$

Therefore,

$$v_t(x,t)=v_{xx}(x,t)+F(v).$$

From the boundary conditions in (4.5), we get

$$v(0,t)=0.$$

And from (4.12), we obtain,

$$v(1,t)=0$$

For $t=0$,we get

$$v_0{=}(1{-}\mathbf{K})u_0.$$

Therefore, the solution of the system (4.5) is

$$\begin{aligned} u &= (1+\mathbf{L})v, \\ &= v + \mathbf{L}v. \end{aligned}$$

That is

$$u(x,t) = v(x,t) + \int_0^x l(x,y)v(y,t)dy,$$

hence the theorem. ■

4.3 The linear system

We shall associate to (4.11) the linear system,

$$\begin{cases} w_t(x,t) = w_{xx}(x,t) + g(x,t) \ , \quad 0 < x < 1 \ , \ t > 0 \\ w(0,t) = 0 \quad , \quad t > 0 \\ w(1,t) = 0 \quad , \quad t > 0 \\ w(x,0) = \overline{w}(x) = v_0(x) \quad , \quad 0 < x < 1 \end{cases} \tag{4.20}$$

Theorem 4 *The solution of the linear problem (4.20) is given by*

$$w(x,t) = A(x,t) + \int_0^t \int_0^1 G(x,y,t-\tau)g(y,\tau)dyd\tau \tag{4.21}$$

where, the function A is given by

$$A(x,t)=\sum_{k\geq 1} 2\int_0^1 \overline{w}(\xi)\sin(k\pi\xi)d\xi e^{-(k\pi)^2 t}\sin(k\pi x), \tag{4.22}$$

and G is the Green's function of (4.20), given by

$$G(x,y,t-\tau)= 2\sum_{k\geq 1} e^{-(k\pi)^2(t-\tau)}\sin(k\pi x)\sin(k\pi y) \tag{4.23}$$

Proof. Let $\left\{-(k\pi)^2, \sin(k\pi x)\right\}$, $k\geq 1$ be the pair of eigenvalues and eigenfunctions of the linear operator $\frac{\partial^2}{\partial x^2}$. Let $g(\cdot,t)$ be in $L^2[0,1]$ for each $t\geq 0$ then,

$$g(x,t)=\sum_{k\geq 1} c_k(t)\sin(k\pi x), \tag{4.24}$$

where

$$c_k(t)=2\int_0^1 g(x,t)\sin(k\pi x)dx. \tag{4.25}$$

We shall look for solution of the system (4.20) as

$$w(x,t)=\sum_{k\geq 1} a_k(t)\sin(k\pi x) \tag{4.26}$$

Substituting the initial condition in (4.26) we get

$$w(x,0)=\overline{w}(x)=\sum_{k\geq 1} d_k\sin(k\pi x) \tag{4.27}$$

where,

$$d_k = a_k(0) = 2\int_0^1 \overline{w}(\xi)\sin(k\pi\xi)d\xi \tag{4.28}$$

Substituting (4.24) and (4.26) in the PDE (4.20) we get

$$\sum_{k\geq 1} a_k^{\prime}(t)\sin(k\pi x) = -\sum_{k\geq 1} a_k(t)(k\pi)^2\sin(k\pi x) + \sum_{k\geq 1} c_k(t)\sin(k\pi x)$$

Thus we obtain

$$\begin{cases} a_k^{\prime}(t) = -(k\pi)^2 a_k(t) + c_k(t) \quad , \quad k \geq 1 \\ a_k(0) = d_k \quad , \quad k \geq 1 \end{cases} \tag{4.29}$$

which is an initial value problem , whose solution is,

$$a_k(t) = e^{-(k\pi)^2 t} d_k + \int_0^t e^{-(k\pi)^2(t-\tau)} c_k(\tau)d\tau \tag{4.30}$$

Substituting (4.30) into (4.26) we get

$$\begin{aligned} w(x,t) &= \sum_{k\geq 1}\left[e^{-(k\pi)^2 t} d_k + \int_0^t e^{-(k\pi)^2(t-\tau)} c_k(\tau)d\tau\right].\sin(k\pi x) \\ &= \sum_{k\geq 1} d_k e^{-(k\pi)^2 t}\sin(k\pi x) \\ &\quad + \int_0^t \sum_{k\geq 1} e^{-(k\pi)^2(t-\tau)} c_k(\tau)\sin(k\pi x)d\tau \end{aligned} \tag{4.31}$$

Replacing (4.25) into (4.31) we obtain,

$$\begin{aligned} w(x,t) &= \sum_{k\geq 1} d_k e^{-(k\pi)^2 t}\sin(k\pi x) + \int_0^t \sum_{k\geq 1} e^{-(k\pi)^2(t-\tau)}\sin(k\pi x)\left[2\int_0^1 \sin(k\pi\overline{x})g(\overline{x},t)d\overline{x}\right]d\tau \\ &= \sum_{k\geq 1} d_k e^{-(k\pi)^2 t}\sin(k\pi x) + \int_0^t\int_0^1 \left[2\sum_{k\geq 1} e^{-(k\pi)^2(t-\tau)}\sin(k\pi x)\sin(k\pi\overline{x})\right] g(\overline{x},t)d\overline{x}d\tau \end{aligned}$$

That is,

$$w(x,t) = A(x,t) + \int_0^t\int_0^1 G(x,\overline{x},t-\tau)g(\overline{x},\tau)d\overline{x}d\tau \tag{4.32}$$

■

Remark 1 *The Green's function G satisfies*

$$\begin{cases} G_t - G_{xx} = \delta(x-y)\delta(t) \ \ , \ \ 0 < x < 1 \ \ , \ \ -\infty < t < \infty \\ G(x,y,t) = 0 \ \ , \ t < 0 \\ G(0,y,t) = G(1,y,t) = 0 \ \ , \ \ t > 0 \end{cases} \tag{4.33}$$

4.4 The Nonlinear system

Returning to (4.11)

$$\begin{cases} v_t(x,t) = v_{xx}(x,t) + F(v) \ , \ \ 0 < x < 1 \ , \ t > 0 \\ v(0,t) = 0 \ \ , \ \ t > 0 \\ v(1,t) = 0 \ \ , \ \ t > 0 \\ v(x,0) = v_0(x) \ \ , \ \ 0 < x < 1 \end{cases}$$

we have the following theorem which results in replacing $g(x,t)$ in (4.32) by $F(v)$

Theorem 5 *The solution of the nonlinear problem (4.11) satisfies the integral equation,*

$$\begin{aligned} v(x,t) &= A(x,t) + \int_0^t \int_0^1 G(x,\overline{x},t-\tau)F(v)(\overline{x},\tau)d\overline{x}d\tau \\ &\triangleq Tv\,(x,t) \end{aligned} \tag{4.34}$$

where the functions A, G and F are defined in (4.22), (4.23) and (4.13) respectively.

Theorem 6 *Let $x \in (0,1)$ and $\lambda \in (0,16)$ then the condition number of $1-\mathbf{K}$ be greater than* 1

i.e. ,

$$\rho = \|1-\mathbf{K}\|_\infty \|1+\mathbf{L}\|_\infty > 1$$

Proof.

$$\|1-\mathbf{K}\| = \sup_{\|u\|=1} \|(1-\mathbf{K})\,u\|\,,$$

then

$$\begin{aligned} \rho &= \|1-\mathbf{K}\|_\infty \|1+\mathbf{L}\|_\infty = \sup_{\|u\|=1} \|(1-\mathbf{K})\,u\| \sup_{\|v\|=1} \|(1+\mathbf{L})\,v\| \\ &= \sup_{\|u\|=1} \left\| u - \int_0^x k(x,y)u(y,t)dy \right\| \sup_{\|v\|=1} \left\| v + \int_0^x l(x,y)v(y,t)dy \right\| \end{aligned}$$

But, in our case the kernel $k(x,y)$ has the form

$$k(x,y) = -\lambda y \frac{I_1\left(\sqrt{\lambda\,(x^2-y^2)}\right)}{\sqrt{\lambda\,(x^2-y^2)}}, \tag{4.35}$$

where I_1 is a first-order modified Bessel Function given by

$$I_1(x) = \sum_{n=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2n+1}}{n!(n+1)!}. \tag{4.36}$$

Comparing the PDE (4.9) for $l(x, y)$ with the PDE (4.8) for $k(x, y)$, we see that

$$l(x, y; \lambda) = -k(x, y; -\lambda) \tag{4.37}$$

Thus,

$$\begin{aligned}
k(x,y) &= \frac{-\lambda y}{\sqrt{\lambda\left(x^2-y^2\right)}} \sum_{n=0}^{\infty} \frac{\left(\frac{\sqrt{\lambda(x^2-y^2)}}{2}\right)^{2n+1}}{n!(n+1)!} \\
&= \frac{-\lambda y}{2\sqrt{\lambda\left(x^2-y^2\right)}} \sum_{n=0}^{\infty} \frac{\left(\lambda\left(x^2-y^2\right)\right)^n \sqrt{\lambda\left(x^2-y^2\right)}}{2^{2n} n!(n+1)!} \\
&= \frac{-\lambda y}{2} \sum_{n=0}^{\infty} \frac{\lambda^n \left(x^2-y^2\right)^n}{4^n\ n!(n+1)!} \\
&= \frac{-\lambda y}{2} \sum_{n=0}^{\infty} \left(\frac{\lambda}{4}\right)^n \frac{\left(x^2-y^2\right)^n}{n!(n+1)!}
\end{aligned} \tag{4.38}$$

Let $u_0 = 1$, then

$$\begin{aligned}
(1-\mathbf{K})\, u_0 &= 1 - \int_0^x k(x,y)dy \\
&= 1 + \int_0^x \frac{\lambda y}{2} \sum_{n=0}^{\infty} \left(\frac{\lambda}{4}\right)^n \frac{\left(x^2-y^2\right)^n}{n!(n+1)!} dy \\
&= 1 + \frac{\lambda}{2} \sum_{n=0}^{\infty} \left(\frac{\lambda}{4}\right)^n \int_0^x \frac{y\left(x^2-y^2\right)^n}{n!(n+1)!} dy
\end{aligned}$$

$$
\begin{aligned}
&= 1-\frac{\lambda}{4}\sum_{n=0}^{\infty}\left(\frac{\lambda}{4}\right)^{n}\frac{\left(x^{2}-y^{2}\right)^{n+1}}{(n+1)!(n+1)!}\Big|_{y=0}^{x} \\
&= 1+\frac{\lambda}{4}\sum_{n=0}^{\infty}\left(\frac{\lambda}{4}\right)^{n}\frac{\left(x^{2}\right)^{n+1}}{\left[\ (n+1)!\right]^{2}} \\
&= 1+\sum_{n=0}^{\infty}\frac{\left(\frac{\lambda}{4}x^{2}\right)^{n+1}}{\left[\ (n+1)!\right]^{2}}>1
\end{aligned}
$$

Hence, $\|(1-\mathbf{K})\,u_0\|_{\infty}>1$, for $x\in(0,1)$ and $\lambda>0$.

Now, from (4.37) and (4.38) we get

$$
l(x,y)=\frac{-\lambda y}{2}\sum_{n=0}^{\infty}\left(\frac{-\lambda}{4}\right)^{n}\frac{\left(x^{2}-y^{2}\right)^{n}}{n!(n+1)!}
$$

Let $v_0=1$,then

$$
\begin{aligned}
(1+\mathbf{L})v_0 &= 1+\int_{0}^{x}l(x,y)dy \\
&= 1-\int_{0}^{x}\frac{\lambda y}{2}\sum_{n=0}^{\infty}\left(\frac{-\lambda}{4}\right)^{n}\frac{\left(x^{2}-y^{2}\right)^{n}}{n!(n+1)!}dy \\
&= 1+\frac{\lambda}{4}\sum_{n=0}^{\infty}\left(\frac{-\lambda}{4}\right)^{n}\frac{\left(x^{2}-y^{2}\right)^{n+1}}{\left[(n+1)!\right]^{2}}\Big|_{y=0}^{x} \\
&= 1-\frac{\lambda}{4}\sum_{n=0}^{\infty}\left(\frac{-\lambda}{4}\right)^{n}\frac{\left(x^{2}\right)^{n+1}}{\left[(n+1)!\right]^{2}} \\
&= 1-\sum_{n=0}^{\infty}\left(\frac{-\lambda}{4}\right)^{n+1}\frac{\left(x^{2}\right)^{n+1}}{\left[(n+1)!\right]^{2}} \\
&= 1-\sum_{n=0}^{\infty}\frac{\left(\frac{-\lambda}{4}x^{2}\right)^{n+1}}{\left[(n+1)!\right]^{2}} \\
&= 1+\sum_{n=0}^{\infty}\frac{(-1)^{n}\left(\frac{\lambda}{4}x^{2}\right)^{n+1}}{\left[(n+1)!\right]^{2}}
\end{aligned}
$$

Let

$$a_n = \frac{\left(\frac{\lambda}{4}x^2\right)^{n+1}}{\left[(n+1)!\right]^2} \ , \ n \geq 0$$

The sequence $a_n > 0$ is monotonically decreasing for $x \in (0,1)$, then

$$(1+\mathbf{L})v_0 = 1 + S \ ,$$

where

$$S = \sum_{n=0}^{\infty} (-1)^n \, a_n$$

which is an alternating series. Let

$$\begin{aligned} S_m &= \sum_{n=0}^{m} (-1)^n \, a_n \\ S - S_m &= \sum_{n=m+1}^{\infty} (-1)^n \, a_n \end{aligned}$$

Since the sequence a_n is monotonically decreasing then,

$$|S - S_m| \leq a_{m+1}$$

which gives

$$1 + S_m - a_{m+1} < 1 + S < 1 + S_m + a_{m+1}$$

and that is true for every $m \leq n$, so for $m = 0$ we obtain

$$1 + S_0 - a_1 < 1 + S < 1 + S_0 + a_1$$

Thus,

$$1 + S > 1 + \frac{\lambda x^2}{4} - \frac{\lambda^2 x^4}{64}$$

Let

$$\begin{aligned} f_0(x, \lambda) &= 1 + \frac{\lambda x^2}{4} - \frac{\lambda^2 x^4}{64} \\ &= 1 + \frac{\lambda x^2}{4}\left(1 - \frac{1}{4}\frac{\lambda x^2}{4}\right) \end{aligned}$$

For f_0 to be greater than one,

$$0 < \frac{\lambda x^2}{4} < 4$$

for all $x \in (0, 1)$, i.e,

$$0 < \lambda < \frac{\lambda}{x^2}$$

for all $x \in (0, 1)$, leading to

$$0 < \lambda < 16$$

Hence, $(1 + L)v_0 > 1$, for $x \in (0, 1)$ and $\lambda \in (0, 16)$.

Therefore $\|(1 + \mathbf{L})\, v_0\|_\infty > 1$. That is $\rho > 1$ for $x \in (0, 1)$ and $\lambda \in (0, 16)$. ■

Theorem 7 *Let the operator $T \in C^{2,1}\left[(0, 1) \times (0, \infty)\,; \mathbb{R}\right]$ be defined in (4.34) and let K and the function F be defined as in (4.6), (4.13) respectively , and let f be a Lipschitz function with a constant α such that*

$$0 < \alpha < \frac{1}{\rho}$$

where

$$\rho = \|1-K\|_\infty \left\|(1-K)^{-1}\right\|_\infty$$

and

$$\beta = \alpha\rho < \frac{1}{\sup_{\substack{x\in(0,1)\\ t\in(0,\infty)}} \int_0^t \int_0^1 |G(x,\overline{x},t)|\, d\overline{x}d\tau}$$

then:

(i) F is a contraction with Lipschitz constant β.

(ii) The operator T is a contraction.

(iii) The fixed point of T is the unique solution of (4.34).

Proof. (i) From (4.13)

$$F(v) = (1-\mathbf{K})f((1-\mathbf{K})^{-1}v),$$

then,

$$\begin{aligned}\|F(v_1)-F(v_2)\|_\infty &= \left\|(1-\mathbf{K})f((1-K)^{-1}v_1)-(1-\mathbf{K})f((1-\mathbf{K})^{-1}v_2)\right\|_\infty \\ &\leq \|1-\mathbf{K}\|_\infty \left\|f((1-\mathbf{K})^{-1}v_1)-f((1-\mathbf{K})^{-1}v_2)\right\|_\infty\end{aligned}$$

But since f is Lipschitz , then

$$\|F(v_1)-F(v_2)\|_\infty \leq \|1-\mathbf{K}\|_\infty \alpha \left\|(1-\mathbf{K})^{-1}\right\|_\infty \|v_1-v_2\|_\infty$$

That is

$$\|F(v_1) - F(v_2)\|_\infty \leq \beta \|v_1 - v_2\|_\infty ,$$

but $\alpha < \frac{1}{\rho}$, giving $\beta < 1$. Hence F is a contraction.

(ii) From (4.34) we get

$$\begin{aligned}
Tv &= A(x,t) + \int_0^t \int_0^1 G(x,\overline{x},t-\tau) F(v)(\overline{x},\tau) d\overline{x} d\tau \\
Tv_1 - Tv_2 &= \int_0^t \int_0^1 G(x,\overline{x},t-\tau) \left[F(v_1) - F(v_2)\right] d\overline{x} d\tau \\
|Tv_1 - Tv_2| &\leq \int_0^t \int_0^1 |G(x,\overline{x},t-\tau)| \, |F(v_1) - F(v_2)| \, d\overline{x} d\tau
\end{aligned}$$

But from (i) F is a Lipschitz Function with constant β , then

$$\begin{aligned}
|Tv_1 - Tv_2| &\leq \int_0^t \int_0^1 \beta \, |v_1(\overline{x},\tau) - v_2(\overline{x},\tau)| \, |G(x,\overline{x},t-\tau)| \, d\overline{x} d\tau \\
&\leq \beta \sup_{\substack{x\in(0,1)\\ t\in(0,\infty)}} |v_1(\overline{x},\tau) - v_2(\overline{x},\tau)| \int_0^t \int_0^1 |G(x,\overline{x},t)| \, d\overline{x} d\tau
\end{aligned}$$

That is

$$\|Tv_1 - Tv_2\|_\infty \leq \mu \|v_1 - v_2\|_\infty ,$$

where μ

$$\mu = \beta \sup_{\substack{x\in(0,1)\\ t\in(0,\infty)}} \int_0^t \int_0^1 |G(x,\overline{x},t)| \, d\overline{x} d\tau < 1$$

Hence, T is contraction.

(iii) Since T is a contraction, it has a unique fixed point v^*, such that

$$Tv^* = v^*,$$

which is the unique solution of the problem.

Let

$$v_0 = v(x, 0),$$

and define v_n by

$$v_{n+1} = Tv_n$$

for $n \geq 1$, then

$$\lim_{n\to\infty} v_n = v^*$$

uniformly. Explicitly,

$$v_{n+1}(x,t) = A(x,t) + \int_0^t \int_0^1 G(x, \overline{x}, t-\tau) F(v_n)(\overline{x}, \tau) d\overline{x} d\tau, \ n \geq 0 \tag{4.39}$$

■

Theorem 8 *The system (4.11)*

$$\begin{cases} v_t(x,t) = v_{xx}(x,t) + F(v) \ , \quad 0 < x < 1 \ , \ t > 0 \\ v(0,t) = 0 \quad , \quad t > 0 \\ v(1,t) = 0 \quad , \quad t > 0 \\ v(x,0) = v_0(x) \quad , \quad 0 < x < 1 \end{cases}$$

is exponential stable

Proof. From Theorem (4) the solution of the system (4.11) satisfies the integral equation

$$v(x,t) = A(x,t) + \int_0^t \int_0^1 G(x,\overline{x},t-\tau)F(v)(\overline{x},\tau)d\overline{x}d\tau,$$

then

$$\begin{aligned}
|v(x,t)| &\leq |A(x,t)| + \int_0^1 \left| \int_0^t G(x,\overline{x},t-\tau)F(v)(\overline{x},\tau)d\tau \right| d\overline{x}, \\
&\leq |A(x,t)| + \int_0^1 \left| \int_0^{t-\varepsilon} G(x,\overline{x},t-\tau)F(v)(\overline{x},\tau)d\tau + \int_{t-\varepsilon}^t G(x,\overline{x},t-\tau)F(v)(\overline{x},\tau)d\tau \right| d\overline{x}
\end{aligned}$$

so,

$$\begin{aligned}
|v(x,t)| &\leq |A(x,t)| + \int_0^1 \int_0^{t-\varepsilon} |G(x,\overline{x},t-\tau)| \, |F(v)(\overline{x},\tau)| \, d\tau d\overline{x} \\
&\quad + \int_0^1 \left| \int_{t-\varepsilon}^t G(x,\overline{x},t-\tau)F(v)(\overline{x},\tau)d\tau \right| d\overline{x} \\
&\leq |A(x,t)| + \beta \int_0^1 \int_0^{t-\varepsilon} |G(x,\overline{x},t-\tau)| \, |v(\overline{x},\tau)| \, d\tau d\overline{x} + \int_0^1 |F(v)(\overline{x},t)| \, d\overline{x} \\
&\leq |A(x,t)| + \beta \int_0^1 \int_0^{t-\varepsilon} |G(x,\overline{x},t-\tau)| \sup_{x\in(0,1)} |v(\overline{x},\tau)| \, d\tau d\overline{x} + \beta \sup_{x\in(0,1)} |v(x,t)|
\end{aligned}$$

from (4.22) and (4.23) we obtain

$$
\begin{aligned}
|v(x,t)| &\leq 2 \sup_{x\in(0,1)} |v_0(x)| \sum_{k\geq 1} e^{-(k\pi)^2 t} + 2\beta \int_0^1 \int_0^{t-\varepsilon} \sum_{k\geq 1} e^{-(k\pi)^2 (t-\tau)} \sup_{\overline{x}\in(0,1)} |v(\overline{x},\tau)|\, d\tau d\overline{x} \\
&\quad + \beta \sup_{x\in(0,1)} |v(x,t)| \\
\sup_{x\in(0,1)} |v(x,t)| &\leq \gamma_1 e^{-\pi^2 t} + 2\beta\gamma_2 \int_0^{t-\varepsilon} e^{-\pi^2(t-\tau)} \sup_{x\in(0,1)} |v(\overline{x},\tau)|\, d\tau + \beta \sup_{x\in(0,1)} |v(x,t)| \qquad (4.40)
\end{aligned}
$$

where γ_1 and γ_2 are defined as

$$
\begin{aligned}
\gamma_1 &= 2 \sup_{x\in(0,1)} |v_0(x)| \sum_{k\geq 1} e^{-\pi^2(k^2-1)t} \\
\gamma_2 &= \sum_{k\geq 1} e^{-\pi^2(k^2-1)(t-\tau)} \quad , \quad \tau < t
\end{aligned}
$$

Let

$$
w(t) = \sup_{x\in(0,1)} |v(x,t)| \ , \qquad (4.41)
$$

then as $\varepsilon \to 0$ we get

$$
\begin{aligned}
w(t) &\leq \gamma_1 e^{-\pi^2 t} + \int_0^t 2\beta\gamma_2 e^{-\pi^2(t-\tau)} w(\tau) d\tau + \beta w(t) \\
(1-\beta) w(t) &\leq \gamma_1 e^{-\pi^2 t} + \int_0^t 2\beta\gamma_2 e^{-\pi^2(t-\tau)} w(\tau) d\tau \\
w(t) &\leq \frac{\gamma_1}{1-\beta} e^{-\pi^2 t} + \frac{2\beta\gamma_2}{1-\beta} \int_0^t e^{-\pi^2(t-\tau)} w(\tau) d\tau
\end{aligned}
$$

Multiply the inequality by $e^{\pi^2 t}$ we obtain

$$e^{\pi^2 t} w(t) \leq \alpha_1 + \alpha_2 \int_0^t e^{\pi^2 \tau} w(\tau) d\tau,$$

where α_1 and α_2 are

$$\begin{aligned} \alpha_1 &= \frac{\gamma_1}{1-\beta}, \\ \alpha_2 &= \frac{2\beta\gamma_2}{1-\beta} \end{aligned}$$

The use of Gronwall's inequality leads to

$$e^{\pi^2 t} w(t) \leq \alpha_1 e^{\int_0^t \alpha_2 dr} = \alpha_1 e^{\alpha_2 t} \ ,$$

that is,

$$w(t) \leq \alpha_1 e^{-(\pi^2 - \alpha_2)t}$$

as,

$$w(t) = \sup_{x \in (0,1)} |v(x,t)| \leq \alpha_1 e^{-(\pi^2 - \alpha_2)t}$$

Hence, the system (4.11) is exponential stable whenever $\alpha_2 < \pi^2$. ■

Chapter 5

Numerical Examples

5.1 Examples

In this section, we shall work out few examples of systems which exhibit blow up in finite time when there are not controlled. When we apply our boundary controllers the systems are stabilized, thus illustrating the effectiveness of our method.

We shall consider the following system

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) + \lambda u(x,t) + f(u) \quad , \quad 0 < x < 1 \ , \ t > 0 \\ u(0,t) = 0 \quad , \quad t > 0 \\ u(1,t) = \int_0^1 k(1,y)u(y,t)dy \quad , \quad t > 0 \\ u(x,0) = u_0(x) \quad , \quad 0 < x < 1 \end{cases}$$

The kernel k and l are shown in figure (5-1) and (5-2) respectively.

Example 1 $\quad f(u) = u^2 \quad , \quad u_0(x) = \sin(\pi x)$

Example 2 $\quad f(u) = u(1-u) \quad , \quad u_0(x) = \sin(\pi x)$

Example 3 $\quad f(u) = u^2 \quad , \quad u_0(x) = 1000\sin(\pi x)$

Example 4 $\quad f(u) = u^2 \quad , \quad u_0(x) = 10000\sin(\pi x)$

Note that in Example 3 and 4, the initial condition is not small

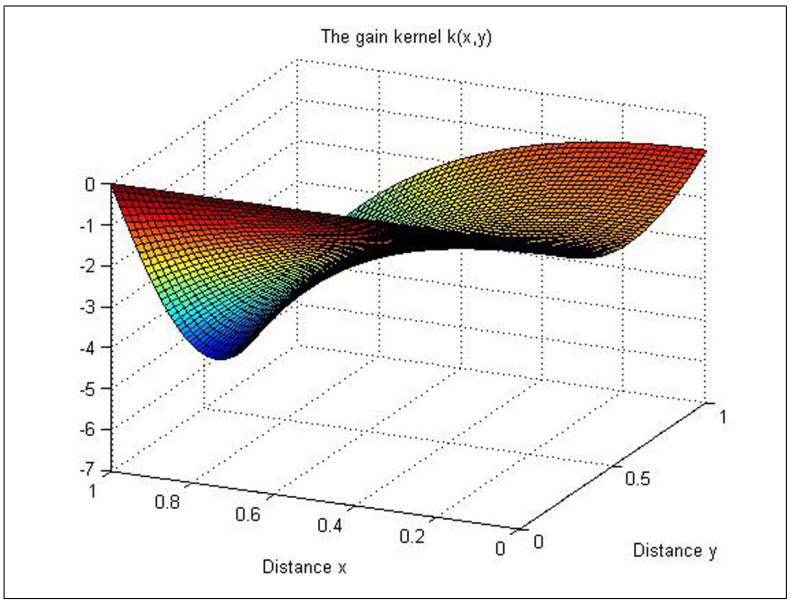


Figure 5-1: Simulation of the gain kernel $k(x, y)$

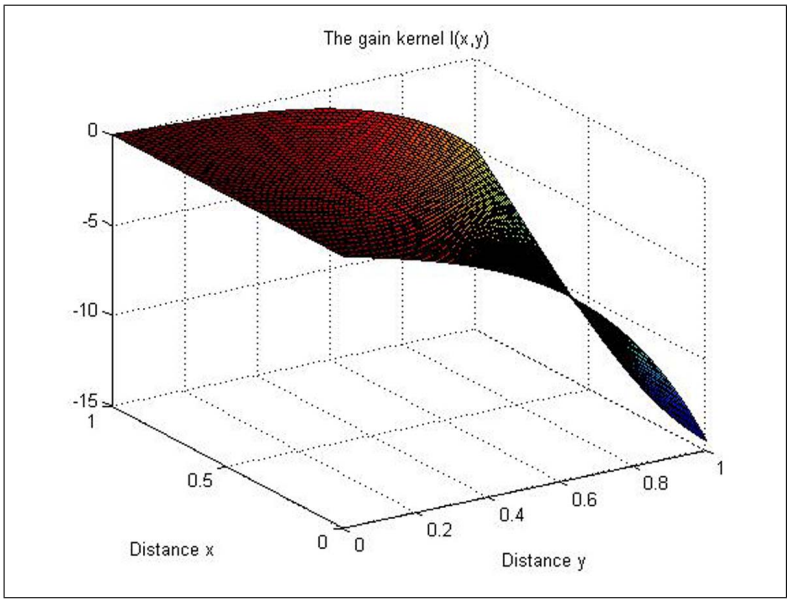


Figure 5-2: Simulation of the gain kernel $l(x, y)$

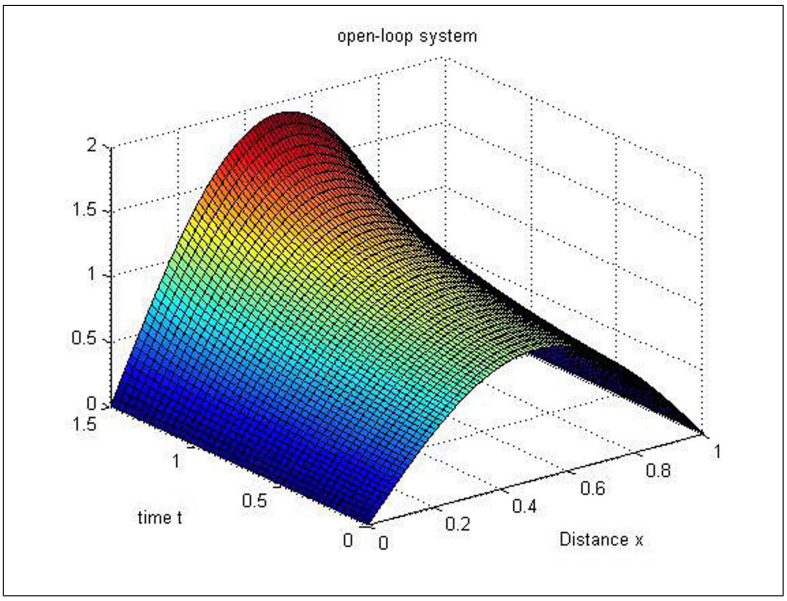


Figure 5-3: Simulation results for the open-loop response for Example 1

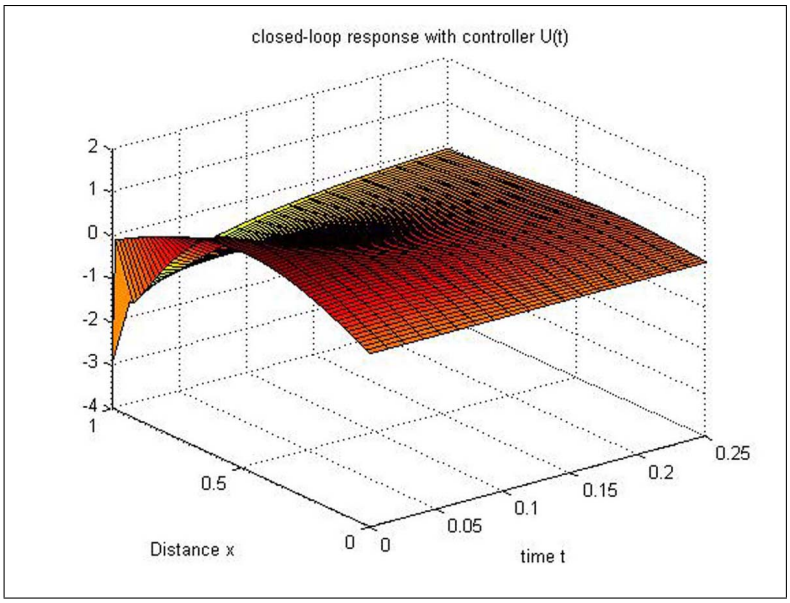


Figure 5-4: Simulation results for the closed-loop response with controller for Example 1

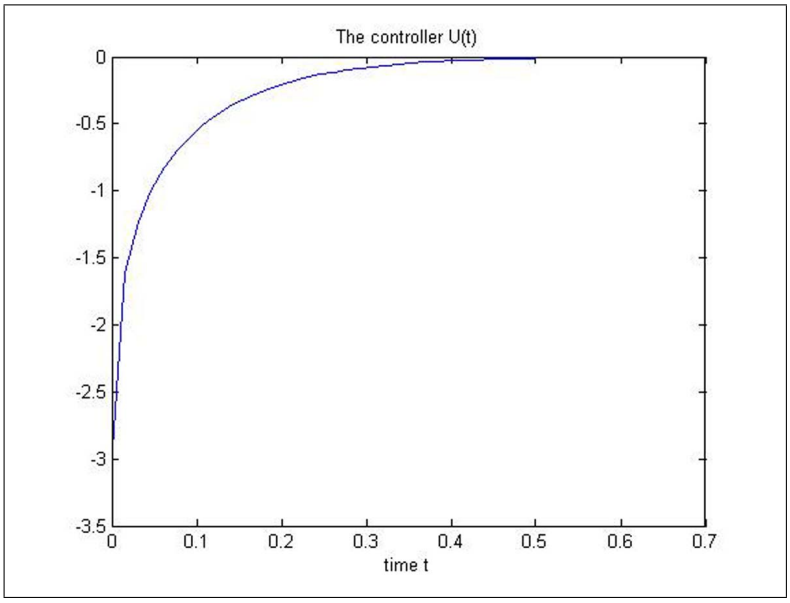


Figure 5-5: The controller of the plant for Example 1

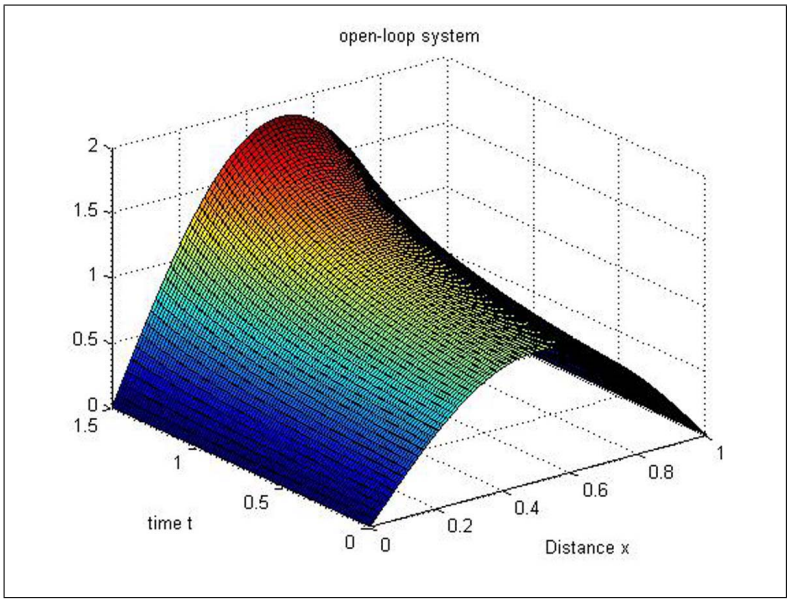


Figure 5-6: Simulation results for the open-loop response for Example 2

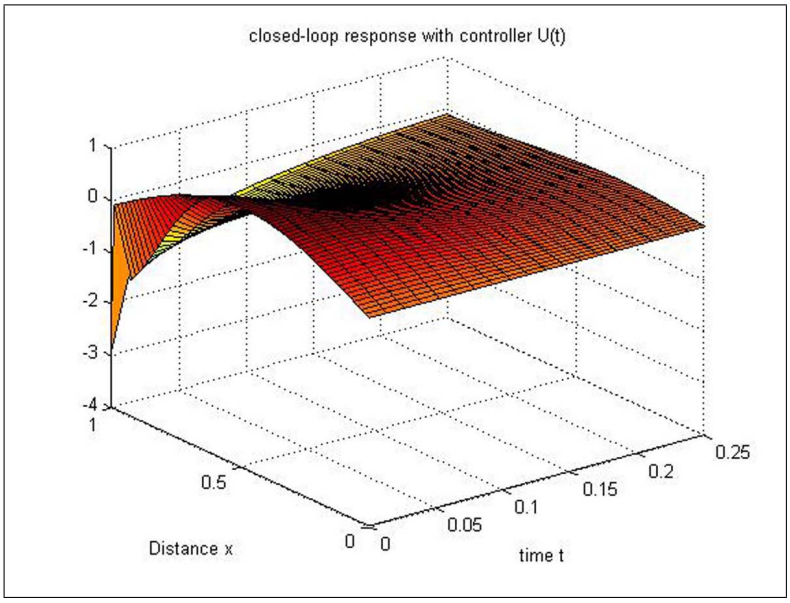


Figure 5-7: Simulation results for the closed-loop response with controller for Example 2

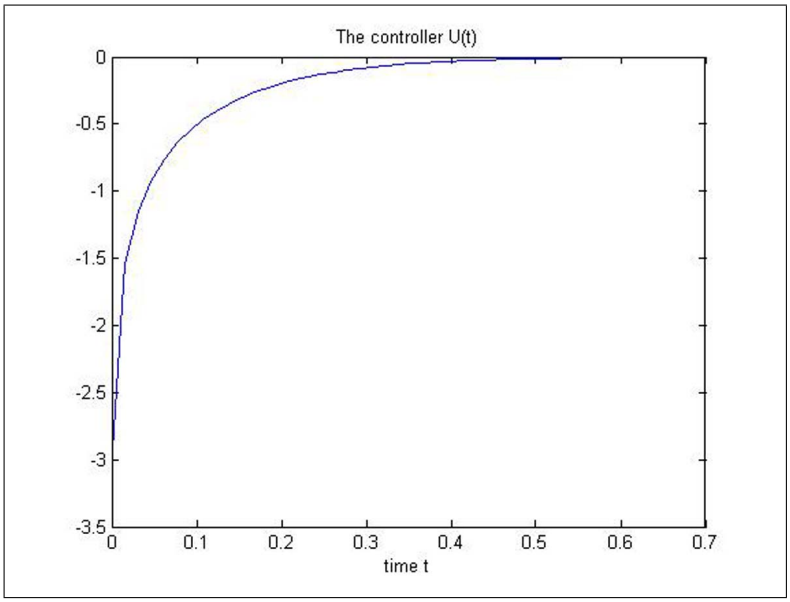


Figure 5-8: The controller of the plant for Example 2

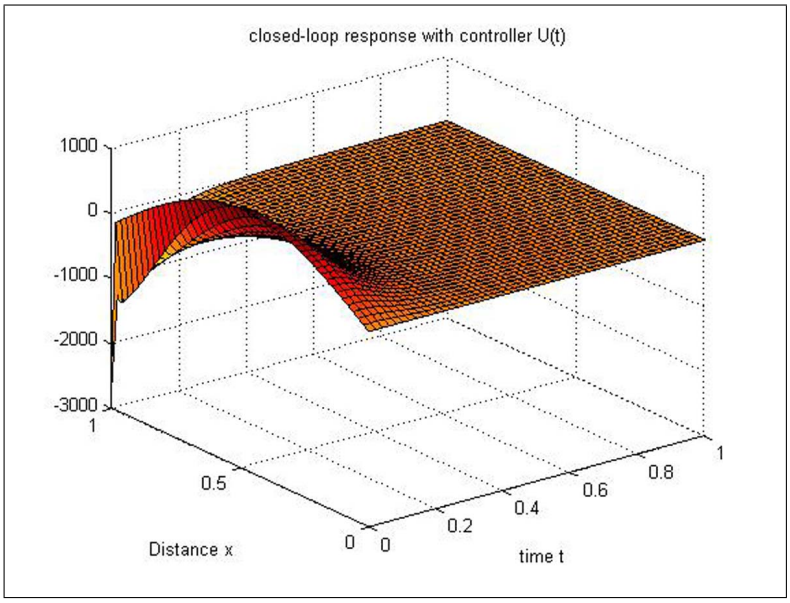
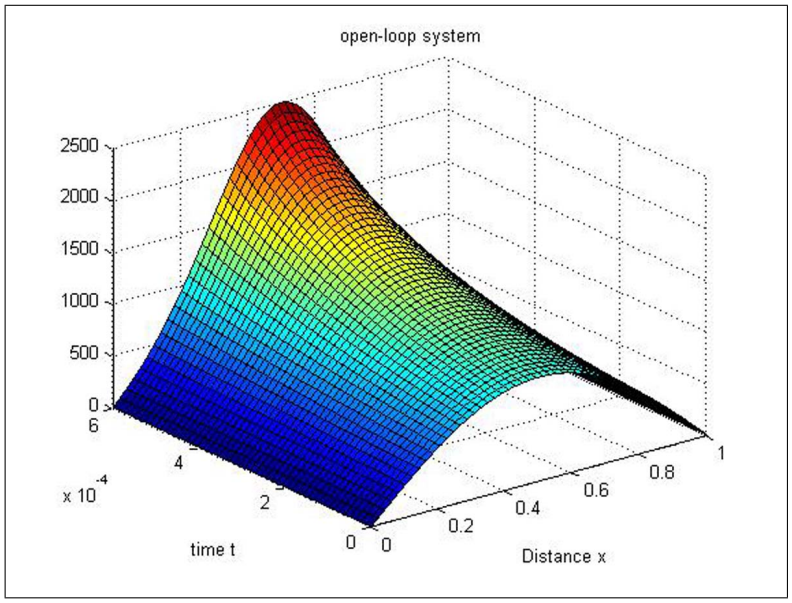


Figure 5-9: Simulation results for the closed-loop response with controller for Example 3



Simulation results for the open-loop response for

Example 3

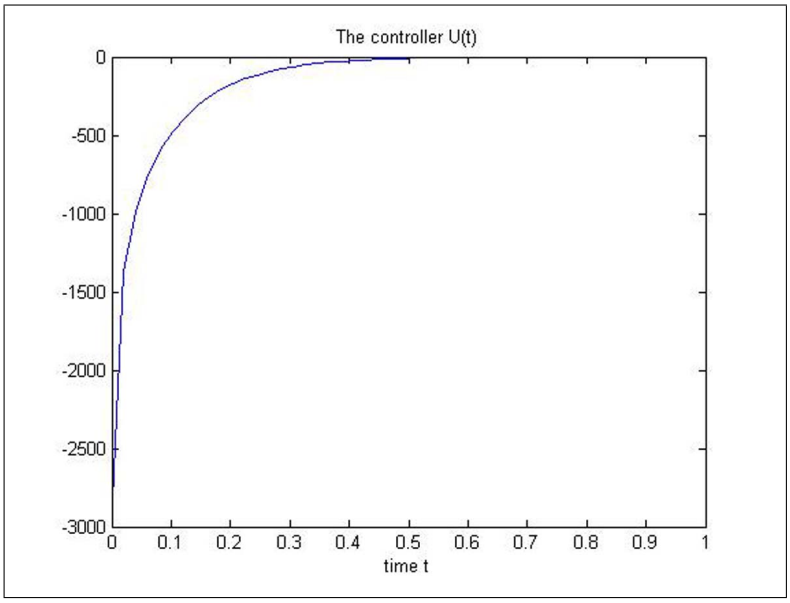
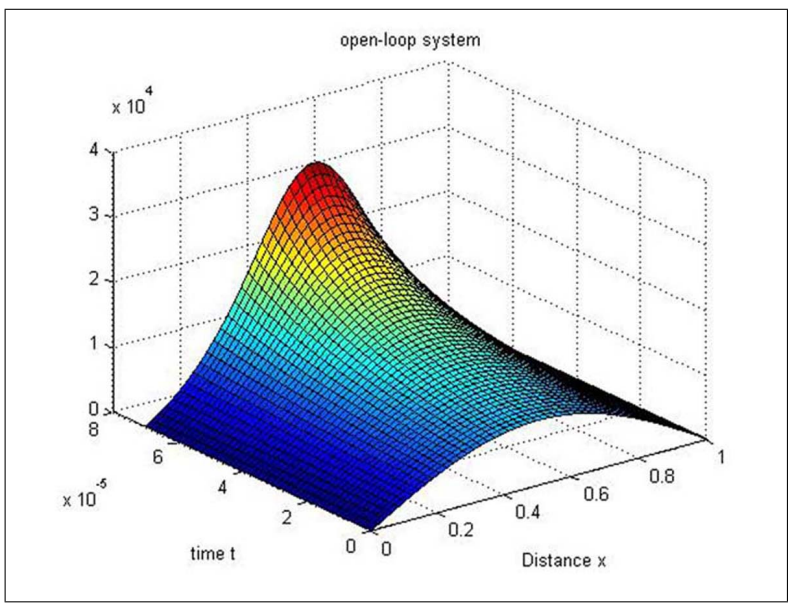
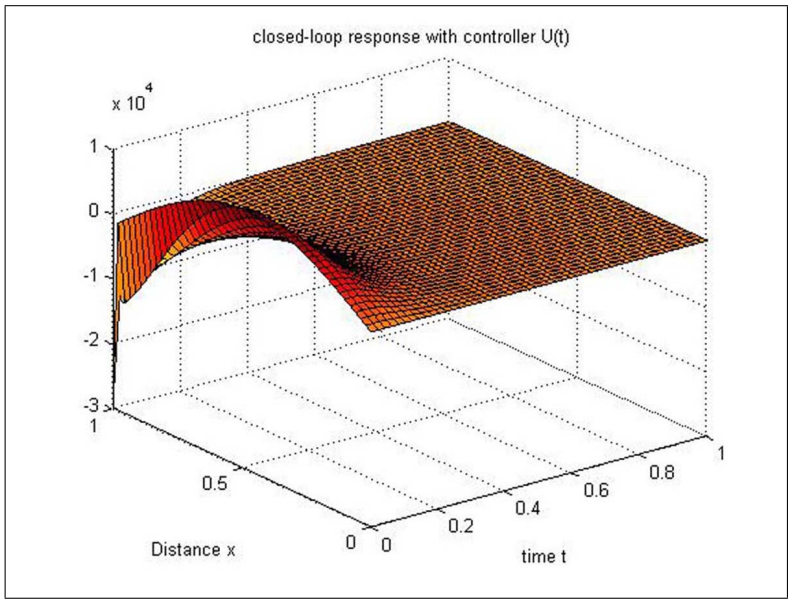


Figure 5-10: The controller of the plant for Example 3



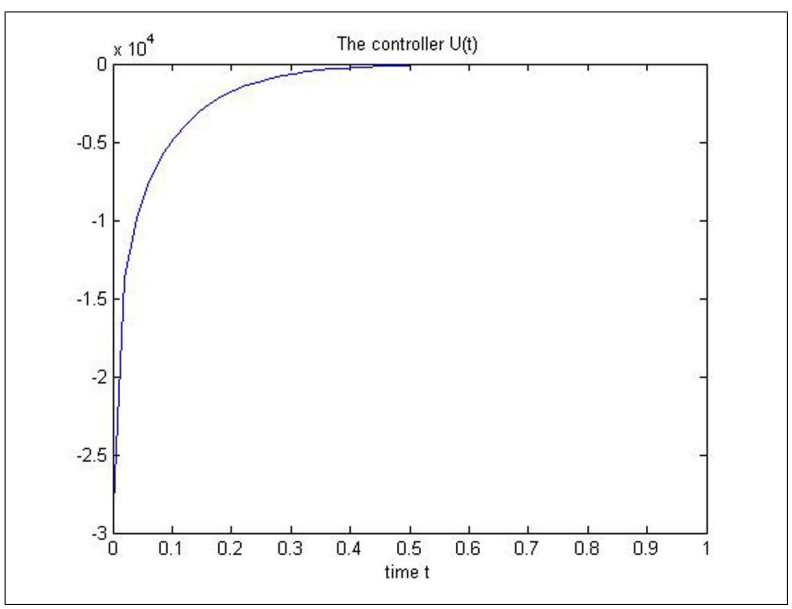
Simulation results for the open-loop response for

Example 4



Simulation results for the closed-loop response with

controller for Example 4



The controller of the plant for Example 4

Chapter 6

Conclusion and Recent and Future Work

6.1 Conclusion

The backstepping method is well known. It has been introduced by Krstic and Coworker to define a stabilizing boundary controller for 1D linear parabolic PDE. When it comes to nonlinear 1D parabolic, Krstic and Coworker had to introduce a backstepping transformation based on infinite Volterra series operator instead of the usual Volterra operator of the second kind. Although they succeeded in doing so, they had to solve a sequence of PDEs with increasing dimensions in order to obtain the multi-dimensional kernels of the Volterra series operator describing the boundary controller.

In this work we have succeeded in stabilizing such nonlinear parabolic PDE system

from the boundary using the original backstepping method. System which, if left uncontrolled, would have exhibited blow up in finite time.

6.2 Recent and Future Work

This work has been submitted for publication [14] as well as the work done in the context of boundary control of impulsive parabolic PDEs [11]. Future problem could be boundary control of impulsive nonlinear parabolic PDEs, boundary control of system of parabolic PDEs, adaptive boundary control of nonlinear parabolic PDEs. All these problems are challenging and very important.

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