Various Mathematical Properties of the Generalized Incomplete Gamma Functions with Applications

by

Bader Ahmed Al-Humaidi

A Dissertation Presented to the

DEANSHIP OF GRADUATE STUDIES

In Partial Fulfillment of the Requirements

for the Degree

DOCTOR OF PHILOSOPHY

IN

MATHEMATICS

KING FAHD UNIVERSITY
OF PETROLEUM & MINERALS
DHAHRAN, SAUDI ARABIA

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This dissertation, written by BADER AHMED AL HUMAIDI under the direction of his thesis advisors and approved by his thesis committee, has been presented to and accepted by the Dean of Graduate Studies, in partial fulfillment of the requirements for the degree of DOCTOR OF PHILOSOPHY IN MATHEMATICS.

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To my parents, wife, kids, brothers and sisters

Acknowledgements

First, praise be to ALLAH who helped and guided me to accomplish this work. I ask him to inspire me and grant me a successful future.

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DISSERTATION ABSTRACT

Name: Bader Ahmed Al-Humaidi

Title of study: Various Mathematical Properties of the Generalized Incomplete Gamma Functions with Applications

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In this dissertation, we study the generalized incomplete gamma function and investigate its properties and connections with the other special functions. We provide a generalization of the well known Euler's reflection formula in terms of the generalized incomplete gamma function. As a result of this generalization, various relations between the generalized incomplete gamma function and other special functions (including the complementary error, the integral exponential, and the Macdonald's functions) has been obtained.

We introduced the generalized error functions and studied their properties using the relationship between the generalized incomplete gamma functions and the complementary error functions. Furthermore, the iterated integrals of the generalized complementary error function has been studied. We also established their integral and series representations, recurrence relation and partial differential equation and deduce the classical result of the iterated integrals of the complementary error function as special cases of our study.

Finally, we solve analytically a class of heat conduction problems via Laplace transform where the special functions play an important role in finding the closed form solutions. Some numerical and graphical representations of the constructed solutions are given.

ملخص بحث

درجة الدكتوراة في الفلسفة

الاســــــــــــم: بدر احمد محمد الحميدي

عنوان الرسالة: الخواص الرياضية المتنوعة لدوال قاما المعممة الناقصة و تطبيقاتها .

التخصــــــص: الريــــاضيـــــات

تاريخ التخرج: مايو 2011

لقد قمنا في هذه الرسالة بدراسة دالة قاما المعممة الناقصة و بحث خواصها المختلفة بالإضافة إلى علاقتها مع بعض الدوال الخاصة الأخرى . تمكنا من إيجاد تعميم لنظرية اويلر الإنعكاسية بدلالة دالة قاما المعممة الناقصة و كنتيجة من هذا التعميم ، استخرجنا علاقات متنوعة تربط دالة قاما المعممة الناقصة مع بعض الدوال الخاصة الأخرى (كدالة الخطأ ، الدالة التكاملية الأسية، ودوال ماكدونالد).

و تمكنا ايضاً من تعميم دوال الخطأ ودراسة خواصها الرياضية وذلك من خلال استخدام العلاقة بين دوال قاما الناقصة و دوال الخطأ. بالإضافة إلى ذلك قمنا بدراسة دالة التكامل المتكرر لدالة الخطأ المعممة و تضمنت هذه الدراسة استخراج خواص هذة الدالة كالتمثيل التكاملي و التسلسلي، علاقات التفاضل المتتابع، والمعادلة التفاضلية الجزئية، ثم قمنا باستخراج الخواص التقليدية لدالة التكامل المتكرر لدالة الخطأ كحالات خاصة من هذة الدراسة.

وأخيرا قمنا بدراسة معادلات حرارية مختلفة بطريقة لابلاس و اوضحنا أهمية الدوال الخاصة في إيجاد حلول هذه المسائل و دعّمنا الدراسة بتمثيل هذه الدوال والحلول هندسيا .

Introduction

Modern engineering and physics applications demand a more thorough knowledge of applied mathematics than ever before. In particular, it is important to have a good understanding of the basic properties of *special functions*. These functions commonly arise in such areas of application as heat conduction, communication system, nonlinear wave propagation, electromagnetic theory, quantum mechanics, approximation theory, probability theory, and electric circuit theory, among others. The subject of special functions is quite rich and expanding continuously with the emergence of new problems in areas of applications in engineering and applied mathematics. The development of computational techniques and the advent of computers have increased the importance of the special functions and their formulas. However, it has been noticed that there are several problems in heat conduction and astrophysics where even these most general classes of special functions are not sufficient to accommodate their solutions. This leads to development of the well known special functions. Chaudhry and Zubair [10] introduce a class of special functions found useful in the analytic study of several heat conduction problems. The purpose of this theses is to give a closed study of the generalized incomplete gamma function and investigate new properties and relations of this function and study some heat conduction problems.

Chapter 1 deals with the generalized gamma function.In the first section, we give some historical background of the gamma function. For completeness, we present the basic definitions and properties of the Euler gamma function in Sections 2 and 3. In Section 4 we introduce the digamma function. The definition and properties of the incomplete gamma functions are given in Section 5.

Chapter 2 presents the incomplete generalized gamma functions introduced by [10]. In Section 1 we introduce the definitions of the generalized gamma functions. Then we study the properties of the generalized gamma functions in Section 2. In the Section 3 we presented the generalized digamma function with its properties. The definition of the generalized incomplete gamma function and their properties are presented in Section 4. The connection between the generalized incomplete gamma functions with other special functions are presented in Section 5.

Chapter 3 presents a generalization of the well known Euler'r reflection identity in terms of the generalized incomplete gamma function. Some useful identities and relations that connect the generalized incomplete gamma function and other special functions are also given.

Chapter 4 discusses the iterated integrals of the generalized complementary error function. In Section 1 we introduce the definition of the generalized error functions in terms of the generalized incomplete gamma function. The properties of the generalized error functions are presented in Section 2. We define the iterated integrals of the generalized complementary error function and study its properties in Section 3. Numerical computations and graphical representations are provided in Section 4.

Chapter 5 present the problem of heating two layer systems using Laplace integral transform method. This system is composed of a thin film of thickness; for example, d deposited on a thick substrate. In Section 1 we give some literature review for the use of laser in heat conduction problems. The mathematical formulation of the problem will be presented in Section 2. In Section 3 the closed form solution for the temperature profile in the thin film and the substrate region is presented. In Section 4 we present the surface temperature as well as the temperature profiles for two different materials with different laser flux densities in graphical form. In Section 5 some remarks about this new formulation are given.

In Chapter 6 we discuss the solution of a heat conduction in a semi-infinite solid when subjected to an instantaneous laser source. In this chapter we give a brief introduction of the functions $E(x,t)$ and $F(x,t)$. These two functions were introduce by Chaudhry and Zubair [10]. The solution of the heat conduction problem will be given in terms of these special functions. Finally, we discuss some limiting cases of our solution and give some graphical representations of the temperature profile and heat flux for different time levels.

Chapter 1

The Generalized Gamma Function

In this chapter we introduce the gamma function and its basic properties. In the first section we give some historical background about the gamma function. The definitions of the gamma and beta functions are introduced in the second section. In the third section, we present some basic properties of the gamma function. For the seek of completeness, we give the complete proof of these properties. In the fourth section, we introduce the diagamma function and some of its properties. In the last section we give a study of the incomplete gamma function.

1.1 The Historical Background of the Gamma Function

The problem of extending the definition of $x!$ was the starting point that lead to the gamma function. Wallis worked on the development of the gamma function. Euler is given credit for the creation of the gamma function. He mention it in a letter to Goldback in 1729 and then in 1730 in one of his papers. Legendre, in the 19th Century, called the gamma function Eulerian integral and gave it the symbol of Greek letter gamma. Weierstrass expressed the gamma function as an infinite product and Gauss further established the role of the gamma function in complex analysis, starting from an infinite product representation. Gauss also proved the multiplication theorem of the gamma function and investigated the connection between the gamma function and elliptic integrals. This function was given a mathematical definition by Goldbach (1690-1764) for more details we refer to [10] and [13].

1.2 The Gamma and Beta Function

There are different representations for the gamma function. We present the gamma function by its integral representation in the following definition.

Definition 1.2.1. *The gamma function is defined by*

$$\Gamma(\alpha) := \int_0^\infty t^{\alpha-1} e^{-t}\, dt \qquad (Re(\alpha) > 0). \tag{1.1}$$

The notation $\Gamma(\alpha)$ and the name gamma function were introduced by Legendre (1752-1833). One important functional relation for the gamma function is given in the following theorem.

Theorem 1.2.2.

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha) \qquad (Re(\alpha) > 0). \tag{1.2}$$

Proof. By using integration by parts we find that

$$\begin{aligned}
\Gamma(\alpha+1) &= \int_0^\infty t^\alpha e^{-t} dt = -e^{-t} t^\alpha \Big|_0^\infty + \int_0^\infty \alpha t^{\alpha-1} e^{-t} dt \\
&= \alpha \int_0^\infty t^{\alpha-1} e^{-t}\, dt = \alpha\Gamma(\alpha) \qquad (Re(\alpha) > 0).
\end{aligned}$$

□

From the definition of the gamma function we have

$$\Gamma(1) = \int_0^{\infty} e^{-t}\, dt = -e^{-t}\Big|_0^{\infty} = 1. \tag{1.3}$$

The functional relation (1.2) and equation (1.3) give the following relation between gamma function and factorial that is;

$$\Gamma(n+1) = n!, \qquad (n = 0, 1, 2, \ldots),$$

and this shows why the gamma function can be seen as an extension of the factorial function.

The functional relation given by Theorem (1.2.2) can be used to find an analytic continuation of the gamma function for $Re(\alpha) \leq 0$. Let $-1 < Re(\alpha) \leq 0$, then we have $Re(\alpha+1) > 0$. Hence, $\Gamma(\alpha+1)$ is defined by the integral representation (1.1). Now we define

$$\Gamma(\alpha) = \frac{\Gamma(\alpha+1)}{\alpha}, \quad (-1 < Re(\alpha) \leq 0,\ \alpha \neq 0),$$

Then the gamma function $\Gamma(\alpha)$ is analytic for $Re(\alpha) > -1$ except $\alpha = 0$. For $\alpha = 0$ we have

$$\lim_{\alpha \to 0} \alpha \Gamma(\alpha) = \lim_{\alpha \to 0} \Gamma(\alpha+1) = \Gamma(1) = 1.$$

This implies that $\Gamma(\alpha)$ has a simple pole at $\alpha = 0$ with residue 1. This process can be repeated for $-2 < Re(\alpha) \leq -1$, $-3 < Re(\alpha) \leq -2$, etcetera. Then the gamma function turns out to be an analytic function on the whole complex plane $\mathbb{C}$ except for single poles at $\alpha = 0, -1, -2, \ldots$. The residue at $\alpha = -n$ equals

$$\begin{aligned}
\lim_{\alpha\to -n}(\alpha+n)\Gamma(\alpha) &= \lim_{\alpha\to -n}(\alpha+n)\frac{\Gamma(\alpha+1)}{\alpha} \\
&= \lim_{\alpha\to -n}(\alpha+n)\frac{1}{\alpha}\ \frac{1}{\alpha+1}\cdots\frac{1}{\alpha+n-1}\ \frac{\Gamma(\alpha+n+1)}{\alpha+n} \\
&= \frac{\Gamma(1)}{(-n)(-n+1)\dots(-1)} = \frac{(-1)^n}{n!} \qquad (n=0,1,2,\dots).
\end{aligned}$$

Definition 1.2.3. *The beta function is defined by*

$$B(\alpha,\beta) := \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt \qquad (Re(\alpha)>0\ Re(\beta)>0). \tag{1.4}$$

From the definition we easily obtain the symmetry

$$B(\alpha,\beta) = B(\beta,\alpha)$$

Since we have by using the substitution $t = 1-s$

$$\begin{aligned}
B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt &= -\int_1^0 (1-s)^{\alpha-1}s^{\beta-1}\,ds \\
&= \int_0^1 s^{\beta-1}(1-s)^{\alpha-1}\,ds \\
&= B(\beta,\alpha).
\end{aligned}$$

The connection between the beta function and the gamma function is given by the following theorem [10]:

Theorem 1.2.4.

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \quad (Re(\alpha)>0,\ Re(\beta)>0). \tag{1.5}$$

Proof. First, we make the substitution $t = x^2$ in (1.1), we find

$$\Gamma(\alpha) = 2\int_0^{\infty} e^{-x^2} x^{2\alpha-1}\, dx \qquad (Re(\alpha) > 0). \tag{1.6}$$

Multiplying two such integrals together, we find

$$\Gamma(\alpha)\Gamma(\beta) = 4\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)} x^{2\alpha-1} y^{2\beta-1}\, dx\; dy, \quad (Re(\alpha) > 0,\ Re(\beta) > 0). \tag{1.7}$$

If we transfer to polar coordinates in the double integral, we then find

$$\begin{aligned}\Gamma(\alpha)\Gamma(\beta) = &\\ & 4\int_0^{\infty}\int_0^{\frac{\pi}{2}} e^{-r^2} r^{2(\alpha+\beta)-1} (\cos\theta)^{2\alpha-1} (\sin\theta)^{2\beta-1}\, d\theta\; dr \\ & = 4\left(\int_0^{\infty} e^{-r^2} r^{2(\alpha+\beta)-1}\, dr\right)\left(\int_0^{\frac{\pi}{2}} (\cos\theta)^{2\alpha-1} (\sin\theta)^{2\beta-1}\, d\theta\right).\end{aligned} \tag{1.8}$$

From (1.6) and (1.8), we find

$$\int_0^{\frac{\pi}{2}} (\cos\theta)^{2\alpha-1} (\sin\theta)^{2\beta-1}\, d\theta = \frac{\Gamma(\alpha)\Gamma(\beta)}{2\Gamma(\alpha+\beta)}, \quad (Re(\alpha) > 0,\ Re(\beta) > 0). \tag{1.9}$$

Substituting $x = \cos^2\theta$ in (1.9), we obtain

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\, dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \quad (Re(\alpha) > 0,\ Re(\beta) > 0). \tag{1.10}$$

Note that the left hand side is $\beta(\alpha,\beta)$ as desired □

We note that the substitution $t = \frac{x}{1-x}$ in (1.10) give another representation of the beta function that is;

$$B(\alpha,\beta) := \int_0^\infty \frac{t^{\alpha-1}}{(1+t)^{\alpha+\beta}}\,dt, \qquad (Re(\alpha) > 0, Re(\beta) > 0). \tag{1.11}$$

1.3 Properties of the Gamma Function

In this section we present some properties of the gamma function that are related to our work. For other properties we refer to [10], [26] and [37].

One important property of the gamma function is called Legendre's duplication formula. This formula is presented in the following theorem:

Theorem 1.3.1. *(Duplication Formula)*

$$\Gamma(\alpha)\Gamma\left(\alpha+\frac{1}{2}\right) = 2^{1-2\alpha}\sqrt{\pi}\,\Gamma(2\alpha) \qquad (Re(\alpha) > 0). \tag{1.12}$$

Proof. First, we note that from equation (1.9) we have

$$B(\alpha,\beta) = 2\int_0^{\frac{\pi}{2}} (\cos\theta)^{2\alpha-1}(\sin\theta)^{2\beta-1}\,d\theta \tag{1.13}$$

Now, if we set $\alpha = \beta = \frac{1}{2}$ and use Theorem (1.2.2) we get $\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = \pi$. Hence $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. Also, if we set $\alpha = \beta$ in equation(1.13) and make the substitution $t = 2\theta$, we get

$$\begin{aligned} B(\alpha,\alpha) &= 2\int_0^{\frac{\pi}{2}} (\cos\theta)^{2\alpha-1}(\sin\theta)^{2\alpha-1}\,d\theta \\ &= 2.2^{1-2\alpha}\int_0^{\frac{\pi}{2}} (\sin 2\theta)^{2\alpha-1}\,d\theta \\ &= 2^{1-2\alpha}\int_0^{\pi} (\sin t)^{2\alpha-1}\,dt \\ &= 2^{1-2\alpha}\cdot 2\int_0^{\frac{\pi}{2}} (\sin t)^{2\alpha-1}\,dt \\ &= 2^{1-2\alpha}B\left(\alpha,\frac{1}{2}\right). \end{aligned}$$

Now we apply Theorem (1.2.2) to obtain

$$\frac{\Gamma(\alpha)\Gamma(\alpha)}{\Gamma(2\alpha)} = B(\alpha,\alpha) = 2^{1-2\alpha}B\left(\alpha,\frac{1}{2}\right) = 2^{1-2\alpha}\frac{\Gamma(\alpha)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\alpha+\frac{1}{2}\right)} \qquad (Re(\alpha>0)$$

Finally, by substituting $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, we find

$$\Gamma(\alpha)\Gamma\left(\alpha+\frac{1}{2}\right) = 2^{1-2\alpha}\sqrt{\pi}\,\Gamma(2\alpha) \qquad (Re(\alpha)>0).$$

□

Note that if α is an integer n, then the duplication formula is written as:

$$\Gamma\left(n+\frac{1}{2}\right) = \frac{\sqrt{\pi}(2n)!}{2^{2n}n!} \quad (n=0,1,2,3,\ldots). \tag{1.14}$$

Legendre's duplication formula can be generalized to Gauss's multiplication formula:

$$\Gamma(\alpha)\prod_{k=1}^{n-1}\Gamma\left(\alpha+\frac{k}{n}\right) = n^{\frac{1}{2}-n\alpha}(2\pi)^{\frac{(n-1)}{2}}\,\Gamma(n\alpha) \quad (n=1,2,3,\ldots). \tag{1.15}$$

The case $n = 1$ is trivial and the case $n = 2$ is Legendre's duplication formula.

Another important property of the gamma function is the Euler's reflection formula. This property is presented in the following theorem:

Theorem 1.3.2. *(Reflection Formula)*

$$\Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin(\pi\alpha)} \qquad (\alpha \neq 0, \pm 1, \pm 2, \ldots). \tag{1.16}$$

Proof. This can be shown by using contour integration. First we restrict to real values of α, say $\alpha = x$ with $0 < x < 1$. By using (1.9) and (1.10) we

$$\Gamma(x)\Gamma(1-x) = B(x, 1-x) = \int_0^\infty \frac{t^{x-1}}{1+t}\, dt \tag{1.17}$$

In order to compute this integral we consider the Contour integral

$$\int_C \frac{z^{x-1}}{1-z} dz$$

where the contour C consists of two circles about the origin of radii R and ε respectively, which are joined along the negative real axis from $-R$ to $-\varepsilon$. Move a long the outer circle with radius R in positive (counter clockwise) direction and along the inner circle with radius ε in the negative (clockwise) direction. By the residue theorem we have

$$\int_C \frac{z^{x-1}}{1-z} dz = -2\pi i$$

where z^{x-1} has its principal value. This implies that

$$-2\pi i = \int_{C_1} \frac{z^{x-1}}{1-z} dz + \int_{C_2} \frac{z^{x-1}}{1-z} dz + \int_{C_3} \frac{z^{x-1}}{1-z} dz + \int_{C_4} \frac{z^{x-1}}{1-z} dz,$$

where C_1 denotes the outer circle with radius R, C_2 denote the line segment from $-R$ to

$-\varepsilon, C_3$ denotes the inner circle with radius ε and C_4 denotes the line segment from $-\varepsilon$ to $-R$.

Then we have by writing $z = Re^{i\theta}$ for the outer circle

$$\int_{C_1} \frac{z^{x-1}}{1-z} = \int_{-\pi}^{\pi} \frac{R^{x-1}\, e^{i(x-1)\theta}}{1-R\,e^{i\theta}}\, iR\,e^{i\theta}\ d\theta = \int_{-\pi}^{\pi} \frac{i\,R^x\, e^{ix\theta}}{1-Re^{i\theta}}\, d\theta$$

For the line segment from $-R$ to $-\varepsilon$ we have by writing $z = -t = te^{i\pi}$

$$\int_{C_2} \frac{z^{x-1}}{1-z}\, dz = \int_{R}^{\varepsilon} \frac{t^{x-1} e^{i(x-1)\pi}}{1+t} \cdot e^{i\pi}\, dt = \int_{R}^{\varepsilon} \frac{t^{x-1}\, e^{i\pi x}}{1+t}\, dt$$

In the same way we have by writing $z = -t = te^{\pi i}$

$$\int_{C_4} \frac{z^{x-1}}{1-z}\, dz = \int_{\varepsilon}^{R} \frac{t^{z-1} e^{-i\pi x}}{1+t}\, dt$$

Since $0 < x < 1$ we have

$$\lim_{R\to\infty} \int_{-\pi}^{\pi} \frac{iR^x\, e^{ix\theta}}{1-R\,e^{i\theta}}\ d\theta = 0 \text{ and } \lim_{\varepsilon\to 0} \int_{\pi}^{-\pi} \frac{i\varepsilon^x\, e^{ix\theta}}{1+t}\ d\theta = 0$$

Hence we have

$$-2\pi i = \int_{\infty}^{0} \frac{t^{x-1}\, e^{ix\pi}}{1+t}\, dt + \int_{0}^{\infty} \frac{t^{x-1}\, e^{-ix\pi}}{1+t}\, dt,$$

or

$$-2\pi i = \left(e^{-ix\pi} - e^{ix\pi}\right) \int_{0}^{\infty} \frac{t^{x-1}}{1+t}\, dt,$$

which gives

$$\int_{0}^{\infty} \frac{t^{x-1}}{1+t}\, dt = \frac{2\pi i}{e^{ix\pi} - e^{-ix\pi}} = \frac{\pi}{\sin \pi x}.$$

□

This proves the theorem for real values of α, say $\alpha = x$ with $0 < x < 1$. The full result follows by analytic continuation. That is, if the result holds for each values of α with $0 < \alpha < 1$, then it holds for all complex α with $0 < Re(\alpha) < 1$ by analyticity. Then it also holds for $Re(\alpha) = 0$ with $\alpha \neq 0$ by continuity. Finally, the full result follows for α shifted by integers using (1.2) and $\sin(\alpha + \pi) = -\sin\alpha$.

Note that (1.14) holds for all complex values of α with $(\alpha \neq 0, -1, -2, \ldots)$. Instead of (1.14) we may write

$$\frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} = \frac{\sin\pi\alpha}{\pi}, \tag{1.18}$$

which holds for all $\mathbb{C}$.

Another important property of the gamma function is the asymptotic formula which is due to Stirling.

Theorem 1.3.3. *(Asymptotic Behavior for Large x)*

$$\Gamma(x+1) \sim x^{x+\frac{1}{2}} e^{-x} \sqrt{2\pi}, \qquad (x \to \infty). \tag{1.19}$$

Here x denotes a real variable.

Proof. This can be proved as follows. Consider

$$\Gamma(x+1) = \int_0^\infty t^x e^{-t}\, dt,$$

where $x \in \mathbb{R}$. Then we obtain by using the transformation $t = x(1+u)$

$$\begin{aligned} \Gamma(x+1) &= \int_{-1}^\infty e^{-x(1+u)} x^x (1+u)^x x\, du \\ &= x^{x+1} e^{-x} \int_{-1}^\infty e^{-xu} (1+u)^x\, du \\ &= x^{x+1} e^{-x} \int_{-1}^\infty e^{x(-u+\ln(1+u))}\, du. \end{aligned}$$

The function $f(u) = -u + \ln(1+u)$ equals to zero for $u = 0$. For other values of u we have

$f(u) < 0$. This implies that the integrand of the last integral equals 1 at $u = 0$ and that this integrand becomes very small for large values of x at other values of u. So for large values of x at other values of u. We only have to deal with the integrand near $u = 0$. Note that we have

$$f(u) = -u + \ln(1+u) = -\frac{1}{2}u^2 + \bigcirc(u^3) \text{ for } \quad (u \to 0).$$

This implies

$$\int_{-1}^{\infty} e^{x(-u+\ln(1+u))} du \sim \int_{-\infty}^{\infty} e^{-\frac{xu^2}{2}} du \text{ for } \quad (x \to \infty).$$

If we set $u = t\sqrt{\frac{2}{x}}$ we have

$$\int_{-\infty}^{\infty} e^{-\frac{xu^2}{2}} du = x^{-\frac{1}{2}} \sqrt{2} \int_{-\infty}^{\infty} e^{-t^2} dt = x^{-\frac{1}{2}} \sqrt{2\pi}.$$

Hence we have

$$\Gamma(x+1) \sim x^{x+\frac{1}{2}} e^{-x} \sqrt{2\pi} \qquad (x \to \infty).$$

□

One can find a generalization of this formula. For complex variable [10] and [31]. We conclude this section by the following theorem that gives the Log-convexity of the gamma function.

Theorem 1.3.4. *(Log-Convex Property)*

For $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$ we have

$$\Gamma\left(\frac{x}{p} + \frac{y}{q}\right) \leq (\Gamma(x))^{\frac{1}{p}} \, (\Gamma(y))^{\frac{1}{q}}, \qquad (x > 0, y > 0). \tag{1.20}$$

Proof. by using the definition (1.1) of the gamma function we have

$$\Gamma\left(\frac{x}{p}+\frac{y}{q}\right)=\int_0^\infty \left(e^{-t}t^{x-1}\right)^{\frac{1}{p}}\left(e^{-t}t^{y-1}\right)^{\frac{1}{q}}dt \qquad (x<0,y>0)$$

Now by using Hölder inequality, we obtain

$$\Gamma\left(\frac{x}{p}+\frac{y}{q}\right)\leq\left(\int_0^\infty e^{-t}t^{x-1}\,dt\right)^{\frac{1}{p}}\left(\int_0^\infty e^{-t}t^{y-1}\,dt\right)^{\frac{1}{q}}, \tag{1.21}$$

or

$$\Gamma\left(\frac{x}{p}+\frac{y}{q}\right)\leq(\Gamma(x))^{\frac{1}{p}}(\Gamma(y))^{\frac{1}{q}}, \tag{1.22}$$

which shows that the gamma function is Log-convex. □

1.4 The Digamma Function $\psi(z)$.

In this section we introduce another function that is related to the gamma function called the digamma function or Psi function. This function is denoted by $\psi(z)$.

Definition 1.4.1. *The digamma function $\psi(z)$ is defined by*

$$\psi(z):=\frac{d}{dz}\{\ln\Gamma(z)\}=\frac{\Gamma'(z)}{\Gamma(z)}. \tag{1.23}$$

Other definitions of this function can be found in literature in terms of limits and series representations [10, p. 20].

Now we present two properties of the digamma function and List other properties with some references where someone can find the proof of these properties.

Theorem 1.4.2. *(Functional Equation)*

$$\psi(z+1)=\psi(z)+\frac{1}{z} \tag{1.24}$$

Proof. by using (1.2) we have

$$\psi(z+1) = \frac{d}{dz}\ln\Gamma(z+1) = \frac{d}{dz}\ln(z\Gamma(z)) = \frac{d}{dz}\ln z + \frac{d}{dz}\ln\Gamma(z) = \frac{1}{z} + \psi(z).$$

□

Note that iteration of (1.22) leads to

$$\psi(z+n) = \psi(z) + \sum_{j=0}^{n-1} \frac{1}{j+z}, \ (n = 1,2,3,\ldots) \tag{1.25}$$

Another property of the digamma function is given by

Theorem 1.4.3. *(Reflection Formula)*

$$\psi(z) - \psi(1-z) = -\frac{\pi}{\tan \pi z}, \quad (z \neq 0, \pm 1, \pm 2, \ldots). \tag{1.26}$$

Proof. The proof of this theorem is based on the reflection formula of the gamma function (1.16). We have

$$\begin{aligned}
\psi(z) - \psi(1-z) &= \frac{d}{dz}\ln\Gamma(z) + \frac{d}{dz}\ln\Gamma(1-z) \\
&= \frac{d}{dz}\ln(\Gamma(z)\Gamma(1-z)) \\
&= \frac{d}{dz}\ln\frac{\pi}{\sin \pi z} = \frac{\sin \pi z}{\pi} \cdot \frac{-\pi^2 \cos \pi z}{(\sin \pi z)^2} \\
&= -\frac{\pi}{\tan \pi z}.
\end{aligned}$$

□

We now list some properties of the digamma function for sake of completeness. The digamma function has several integral representations [10] and [26]. These are a few:

$$\psi(z+1) := -\gamma + \int_0^1 \frac{1-t^z}{1-t}\,dt \quad (Re\,z > -1) \tag{1.27}$$

$$\psi(z) := \int_0^\infty \left(e^{-t} - \frac{1}{(1+t)^z}\right)\frac{dt}{t} \qquad (Re\,z > 0) \tag{1.28}$$

$$\psi(z) := \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-tz}}{1-e^{-t}}\right) dt \qquad (Re\,z > 0). \tag{1.29}$$

where

$$\gamma := \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{1}{k} - \ln(n)\right) = 0.57721566..., \tag{1.30}$$

is known as Euler's constant.

The digamma function satisfies several functional relations, including the following [26] and [37] .

$$\psi\left(\frac{1}{2} - z\right) = \psi\left(\frac{1}{2} + z\right) - \pi \tan \pi z, \tag{1.31}$$

$$\psi\left(\frac{1}{2} + n\right) = -\gamma - \ln 4 + \sum_{j=1}^{n} \frac{2}{2j-1} \qquad (n = 0, 1, 2, \ldots) \tag{1.32}$$

$$\psi(nz) = \ln(n) + \frac{1}{n}\sum_{j=0}^{n-1} \psi\left(z + \frac{j}{n}\right) \qquad (n = 2, 3, 4, \ldots) \tag{1.33}$$

The asymptotic expansion of the digamma function, valid for large values of x, is given by

$$\psi(x) \sim \ln x - \frac{1}{2x} - \sum_{n=1}^{\infty} \frac{B_{2n}}{2n x^{2n}} = \ln x - \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4} - \frac{1}{252x^6} + \ldots \qquad (x \to \infty),$$

where B_n is a Bernoulli number.

1.5 The Incomplete Gamma Function

The closed-form solution to a considerable number of problems in applied mathematics, astrophysics, nuclear and molecular Physics, Statistics and Engineering problems can be expressed in terms of incomplete gamma function:

$$\gamma(\alpha,x) := \int_0^x t^{\alpha-1} e^{-t}\, dt \qquad (Re(\alpha) > 0, \;\; |arg(\alpha)| < \pi), \tag{1.34}$$

$$\Gamma(\alpha,x) := \int_x^\infty t^{\alpha-1} e^{-t}\, dt \qquad (|arg(\alpha)| < \pi). \tag{1.35}$$

These functions were first investigated for real x by Legendre. The functional behavior of these functions and the decomposition formula

$$\gamma(\alpha,x) + \Gamma(\alpha,x) = \Gamma(\alpha),$$

was studied by Prym [39], [47] in 1977. The older theory of incomplete gamma functions and reference to literature are given by Nielsen and Böhmer [3], [7], [13], [22], [23] and [47]. Gautsehi [23] has summarized some of the recent developments and gave an extensive list of references to the current literature on the incomplete gamma functions.

The function $\gamma(\alpha,x)$ has the inconvenience of not only having poles at the nonpositive integers $\alpha = 0,-1,-2,...$, but also it is a multivalued function of the complex parameter x, owing to the fractional power in the integrand. These inconveniences can be avoided by introducing, as Tricomi does in [48] and Böhmer before him in [7], the function

$$\gamma^*(\alpha,x) = \frac{x^{-\alpha}}{\Gamma(\alpha)} \gamma(\alpha,x), \tag{1.36}$$

which is an entire function in α as well as in x and real valued for real α and real x. The following theorem gives the recurrence formula for $\gamma^*(\alpha,x)$.

Theorem 1.5.1.

$$\Gamma^*(\alpha+1,x) = \frac{\gamma^*(\alpha,x)}{x} - \frac{e^{-x}}{x\,\Gamma(\alpha+1)} \quad (x \neq 0). \tag{1.37}$$

Proof. The result follows from the definition of $\gamma^*(\alpha,x)$ and $\gamma(\alpha,x)$ and by using integration by parts. Indeed,

$$\begin{aligned}
\gamma^*(\alpha+1,x) &= \frac{x^{-\alpha-1}}{\Gamma(\alpha+1)}\,\gamma(\alpha+1,x) = \frac{x^{-\alpha-1}}{\Gamma(\alpha+1)}\int_0^x t^{\alpha}\,e^{-t}\,dt \\
&= \frac{x^{-\alpha-1}}{\Gamma(\alpha+1)}\left[-t^{\alpha}\,e^{-t}\Big|_0^x + \alpha\int_0^x t^{\alpha-1}\,e^{-t}\,dt\right] \\
&= \frac{x^{-\alpha-1}}{\Gamma(\alpha+1)}\left[-x^{\alpha}\,e^{-x} + \alpha\,\gamma(\alpha)\right] \\
&= \frac{\alpha\,x^{-\alpha-1}}{\Gamma(\alpha+1)}\,\gamma(\alpha) - \frac{e^{-x}}{x\Gamma(\alpha+1)} \\
&= \frac{x^{-\alpha}}{x\Gamma(\alpha)}\,\gamma(\alpha) - \frac{e^{-x}}{x\Gamma(\alpha+1)} \\
&= \frac{\gamma^*(\alpha,x)}{x} - \frac{e^{-x}}{x\Gamma(\alpha+1)}.
\end{aligned}$$

□

The connection between $\gamma^*(\alpha,x)$ and the complementary error function is given in the following theorem.

Theorem 1.5.2.

$$\gamma^*\left(\frac{1}{2},x\right) = \frac{1}{\sqrt{x}}\ \mathrm{erf}(\sqrt{x}). \tag{1.38}$$

Proof.

$$\begin{aligned}
\gamma^*\left(\frac{1}{2},x\right) &= \frac{x^{-\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)}\gamma\left(\frac{1}{2},x\right) = \frac{x^{-\frac{1}{2}}}{\sqrt{\pi}}\int_0^x t^{-\frac{1}{2}}\,e^{-t}\,dt \\
&= \frac{1}{\sqrt{x}}\cdot\frac{1}{\sqrt{\pi}}\int_0^x t^{-\frac{1}{2}}\,e^{-t}\,dt.
\end{aligned}$$

The substitution, $t = u^2 \Rightarrow dt = 2u\,du$, in the integral gives

$$\begin{aligned}\gamma^*\left(\frac{1}{2},x\right) &= \frac{1}{\sqrt{x}}\cdot\frac{1}{\sqrt{\pi}}\int_0^{\sqrt{x}}\frac{e^{-u^2}}{u}(2u)\,du \\ &= \frac{1}{\sqrt{x}}\cdot\frac{2}{\sqrt{\pi}}\int_0^{\sqrt{x}} e^{-u^2}\,du = \frac{1}{\sqrt{x}}\operatorname{erf}(\sqrt{x}).\end{aligned}$$

□

The following theorem gives the connection between the incomplete gamma functions and the error functions.

Theorem 1.5.3.

$$i)\ \gamma\left(\frac{1}{2},x\right) = \sqrt{\pi}\operatorname{erf}(\sqrt{x}) \tag{1.39}$$

$$ii)\ \Gamma\left(\frac{1}{2},x\right) = \sqrt{\pi}\operatorname{erfc}(\sqrt{x}) \tag{1.40}$$

Proof. We prove only (1.40) since the proof of (1.39) follow similarly.

$$\Gamma\left(\frac{1}{2},x\right) = \int_x^{\infty} t^{-\frac{1}{2}}\,e^{-t}\,dt. \tag{1.41}$$

The substitution $t = u^2$ in (1.41) yields

$$\Gamma\left(\frac{1}{2},x\right) = \int_{\sqrt{x}}^{\infty}\frac{1}{u}e^{-u^2}(2u)\,du = 2\int_{\sqrt{x}}^{\infty} e^{-u^2}\,du = \sqrt{\pi}\ \operatorname{erfc}\ (\sqrt{x}).$$

□

It is noted that some important special cases of (1.34) and (1.35) are obtained when $\alpha = 1 \pm n$ is an integer. These cases are summarized in the following theorem.

Theorem 1.5.4. *For* $n \geq 0$,

$$
\begin{aligned}
i)\ \gamma(1+n,x) &= n!\left[1-e^{-x}e_n(x)\right], && (1.42)\\
ii)\Gamma(1+n,x) &= n!\left[e^{-x}e_n(x)\right], && (1.43)\\
iii)\Gamma(1-n,x) &= x^{1-n}E_n(x), && (1.44)
\end{aligned}
$$

where the function

$$e_n(x) := \int_1^\infty e^{-xt}\,t^{-n}\,dt, \tag{1.45}$$

and

$$e_n(x) := 1+x+\frac{x^2}{2!}+\ldots+\frac{x^n}{n!} \quad (n=0,1,2,\ldots). \tag{1.46}$$

Proof.

i) by using the definition of $\gamma(1+n,x)$ and repeated integration by parts we get,

$$
\begin{aligned}
\gamma(1+n,x) &= \int_0^x t^n\,e^{-t}\,dt\\
&= -t^n\,e^{-t}|_0^x + \int_0^x n\,t^{n-1}\,e^{-t}\,dt\\
&= -x^n e^{-x} + n\int_0^x t^{n-1}\,e^{-t}\,dt\\
&= -x^n e^{-x} + n\left[-t^{n-1}e^{-t}|_0^x + \int_0^x (n-1)t^{n-2}e^{-t}\,dt\right]\\
&= -x^n e^{-x} - n\,x^{n-1}\,e^{-x} - \ldots - n!\,e^{-x} + n!\\
&= n!\left[1-e^{-x}\,e_n(x)\right].
\end{aligned}
$$

ii) by substituting $\alpha = 1+n$ in (1.35) and use integration by parts we have

$$\begin{aligned}
\Gamma(1+n,x) &= \int_x^\infty t^n e^{-t}\,dt \\
&= -t^n e^{-t}\Big|_x^\infty + \int_x^\infty n t^{n-1} e^{-t}\,dt \\
&= x^n e^{-x} + n x^{n-1} e^{-x} + n(n-1)e^{-x} x^{n-2} + \ldots + n!\, e^{-x} \\
&= n! e^{-x}\left[\frac{x^n}{n!} + \frac{x^{n-1}}{(n-1)!} + \ldots + x + 1\right] \\
&= n!\, e^{-x} e_n(x).
\end{aligned}$$

iii) By substituting $\alpha = 1-n$ in (1.35) we get

$$\Gamma(1-n,x) = \int_x^\infty t^{-n} e^{-t}\,dt. \tag{1.47}$$

Making the substitution $t = ux$ in (1.47) yields

$$\begin{aligned}
\Gamma(1-n,x) &= \int_1^\infty (ux)^{-n} e^{-ux} x\,du \\
&= x^{1-n} \int_1^\infty e^{-ux} u^{-n}\,du = x^{1-n} E_n(x).
\end{aligned}$$

□

These functions have several identities that are useful in operational calculus [10, p. 41]. One of these identities is given in the following theorem

Theorem 1.5.5.

$$\frac{d^n}{dx^n}\left(x^{-\alpha}\Gamma(\alpha,x)\right) = (-1)^n x^{-\alpha-n}\,\Gamma(\alpha+n,x). \tag{1.48}$$

Proof. The result is shown by using mathematical induction. Indeed, for $n=1$ we have

$$\begin{aligned}
\frac{d}{dx}\left(x^{-\alpha}\Gamma(\alpha,x)\right) &= -\alpha x^{-\alpha-1}\Gamma(\alpha,x)+x^{-\alpha}\frac{d}{dx}(\Gamma(\alpha,x)) \\
&= -\alpha x^{-\alpha-1}\Gamma(\alpha,x)+x^{-\alpha}\left(-x^{\alpha-1}e^{-x}\right) \\
&= -x^{-\alpha-1}\left[\alpha\Gamma(\alpha,x)+x^{\alpha}e^{-x}\right] \\
&= (-1)x^{-\alpha-1}\Gamma(\alpha+1,x).
\end{aligned}$$

So, (1.48) is true for $n=1$. Assume (1.48) is true for $n=k$. We need to show that (1.48) is true for $n=k+1$. Indeed, by using induction hypothesis we have

$$\frac{d^k}{dx^k}\left(x^{-\alpha}\Gamma(\alpha,x)\right)=(-1)^k x^{-\alpha-k}\Gamma(\alpha+k,x).$$

This implies

$$\begin{aligned}
\frac{d^{k+1}}{dx^{k+1}}\left(x^{-\alpha}\Gamma(\alpha,x)\right) &= (-1)^k(-\alpha-k)x^{-\alpha-k-1}\Gamma(\alpha+k,x) \\
&\qquad +(-1)^k x^{-\alpha-k}\left(-x^{\alpha+k-1}e^{-x}\right) \\
&= (-1)^{k+1}x^{-\alpha-(k+1)}\left[(\alpha+k)\Gamma(\alpha+k,x)+x^{\alpha+k}e^{-x}\right] \\
&= (-1)^{k+1}x^{-\alpha-(k+1)}\Gamma(\alpha+k+1,x).
\end{aligned}$$

As desired □

We conclude this section by the following series expansion of $\gamma(\alpha,x)$. Series expansions for $\Gamma(\alpha,x)$ follows from the decomposition formula for the incomplete gamma functions.

Theorem 1.5.6.

$$\gamma(\alpha,x)=e^{-x}\sum_{n=0}^{\infty}\frac{x^{\alpha+n}}{(\alpha)_{n+1}}=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{x^{\alpha+n}}{\alpha+n} \tag{1.49}$$

Proof. The first series arises after we make the transformation $t=x(1-u)$ in (1.34). This gives

$$\gamma(\alpha,x) = x^{\alpha} e^{-x} \int_0^1 (1-u)^{\alpha-1} e^{ux} \, du. \tag{1.50}$$

Expanding the exponential function, and using the definition of the beta function (1.4), and using the relation between the gamma function and the beta function (1.5) we get

$$\begin{aligned}
\gamma(\alpha,x) &= x^{\alpha} e^{-x} \sum_{n=0}^{\infty} \int_0^1 (1-u)^{\alpha-1} \frac{(ux)^n}{n!} \, du \\
&= x^{\alpha} e^{-x} \sum_{n=0}^{\infty} \frac{x^n}{n!} \int_0^1 (1-u)^{\alpha-1} u^n \, du \\
&= x^{\alpha} e^{-x} \sum_{n=0}^{\infty} \frac{x^n}{n!} B(\alpha, n+1) \\
&= x^{\alpha} e^{-x} \sum_{n=0}^{\infty} \frac{x^n}{n!} \frac{\Gamma(\alpha)\Gamma(n+1)}{\Gamma(\alpha+n+1)} \\
&= e^{-x} \sum_{n=0}^{\infty} x^{n+\alpha} \frac{\Gamma(\alpha)}{\Gamma(\alpha+n+1)} \\
&= e^{-x} \sum_{n=0}^{\infty} \frac{x^{n+\alpha}}{(\alpha)_{n+1}}. \qquad (1.51)
\end{aligned}$$

Where the last series in 1.51 is obtained from the relation between the Pochhammer symbol and the gamma function that is given by

$$(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}. \tag{1.52}$$

The second series representation is obtained by expanding the function e^{-t} in the integral of (1.34). Indeed,

$$
\begin{aligned}
\gamma(\alpha,x) &= \int_0^x t^{\alpha-1} e^{-t}\, dt \\
&= \int_0^x \left(t^{\alpha-1} \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{n!} \right) dt \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^x t^{\alpha+n-1}\, dt \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left. \frac{t^{\alpha+n}}{\alpha+n} \right|_0^x \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{x^{\alpha+n}}{\alpha+n}. \qquad (1.53)
\end{aligned}
$$

□

Chapter 2

The Generalized Incomplete Gamma Functions

In this chapter we summarize the properties of the generalize gamma function. The definition of the generalized gamma function will be given in the first section. In the second section we study the properties of the generalized gamma function and deduce the properties of the Euler gamma function as special cases. The generalized Psi function with its properties will be presented in the third section. The fourth section will introduce the definition of the generalized incomplete gamma functions and study their basic properties. In the last section, we study the connection between the incomplete gamma functions and other special functions.

2.1 Definition of the Generalized Gamma Function

It is possible to extend the classical gamma function in infinitely many ways. Some of these extensions could be useful in certain types of problems. However, it is desirable to find an extension of the gamma function that meets the requirement that the previous results for the function are naturally and simply extended.

It is also required that the results for the extension should be no less elegant than those for the original function. Recently, Chaudhry and Zubair gave a generalization of the Euler gamma function. This generalization is given by:

Definition 2.1.1. *The generalized gamma function is defined by*

$$\Gamma_b(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t-b/t}\, dt \qquad (Re(b) > 0; b = 0,\ Re(\alpha) > 0). \tag{2.1}$$

The factor $e^{-\frac{b}{t}}$ in the integral (2.1) plays the role of a regularizer. For $Re(b) > 0$, $\Gamma_b(\alpha)$ is defined in the complex plane and for $b = 0$, the function $\Gamma_b(\alpha)$ coincides with the classical gamma function. The integral in (2.1) can be simplified in terms of the Macdonald function to give

$$\Gamma_b(\alpha) = 2b^{\frac{\alpha}{2}} K_\alpha(2\sqrt{b}) \quad (Re(b) > 0, |arg(\sqrt{b})| < \pi). \tag{2.2}$$

2.2 Properties of the Generalized Gamma Function

Several properties of the generalized gamma function can be proved by using the representation (2.2) together with the properties of the Macdonald function. However, these proofs are cumbersome and hence do not reflect the spirit of the generalization.

Theorem 2.2.1. *(The Difference Formula)*

$$\Gamma_b(\alpha + 1) = \alpha\,\Gamma_b(\alpha) + b\Gamma_b(\alpha - 1). \tag{2.3}$$

Proof. Let M be the Mellin transform operator as defined by

$$M\{f(t);\alpha\} := \langle t_+^{\alpha-1}, f(t)\rangle := \int_0^\infty t^{\alpha-1} f(t)\, dt. \tag{2.4}$$

Then, $\Gamma_b(\alpha)$ is simply the Mellin transform of $f(t) = e^{-t-bt^{-1}}$ in α. That is,

$$\Gamma_b(\alpha) := M\left\{e^{-t-bt^{-1}};\alpha\right\}. \tag{2.5}$$

Recalling the relationship,

$$M\{f'(t);\alpha\} = -(\alpha-1)\ M\{f(t);\alpha-1\} \tag{2.6}$$

between the Mellin transform of a function and its derivative, we find

$$-(\alpha-1)\ \Gamma_b(\alpha-1) = M\left\{\left(-1+bt^{-2}\right)e^{-t-bt^{-1}};\alpha\right\}, \tag{2.7}$$

which simplifies to give

$$-(\alpha-1)\Gamma_b(\alpha-1) = -\Gamma_b(\alpha) + b\Gamma_b(\alpha-2). \tag{2.8}$$

Replacing α by $\alpha+1$ in (2.8) we get the proof of (2.3). □

We note that if we put $b = 0$ in (2.3) we recover the functional relation for the classical gamma function given by (1.2).

Theorem 1.3.4 gives the Log-convex property of the classical gamma function. The following theorem shows that the generalized gamma function is also Log-convex.

Theorem 2.2.2. *(Log-convex property). Let* $1 < p < \infty$ *and* $\left(\frac{1}{p}\right) + \left(\frac{1}{q}\right) = 1$, *then*

$$\Gamma_b\left(\frac{x}{p}+\frac{y}{q}\right) \le (\Gamma_b(x))^{\frac{1}{p}}\,(\Gamma_b(y))^{\frac{1}{q}} \qquad (b \ge 0, x > 0, y > 0). \tag{2.9}$$

Proof. Taking $\alpha = \frac{x}{p} + \frac{y}{q}$ in (2.1) and noting that $\frac{1}{p} + \frac{1}{q} = 1$ we find

$$\begin{aligned} \Gamma_b\left(\frac{x}{p}+\frac{y}{q}\right) &= \int_0^\infty t^{\frac{x}{p}+\frac{y}{q}-1}\, e^{-t-\frac{b}{t}}\, dt \\ &= \int_0^\infty \left(t^{x-1}\, e^{-t-\frac{b}{t}}\right)^{\frac{1}{p}} \left(t^{y-1}\, e^{t-\frac{b}{t}}\right)^{\frac{1}{q}} dt. \end{aligned}$$

Using the Hölder inequality, we find

$$\Gamma_b\left(\frac{x}{p}+\frac{y}{q}\right) \le \left(\int_0^\infty t^{x-1}\, e^{-t-\frac{b}{t}}\, dt\right)^{\frac{1}{p}} \left(\int_0^\infty t^{y-t}\, e^{-t-\frac{b}{t}}\, dt\right)^{\frac{1}{q}},$$

as desired. □

We note that Theorem 1.3.4 is recovered from Theorem 2.2.2 by setting $b = 0$ in (2.9). Moreover, some interesting special cases can be gained from (2.9). For example, setting $p = q = 2$ in (2.9) and using the fact that the arithmetic mean of two positive numbers is greater than or equal to their geometric mean, we find

$$\Gamma_b\left(\frac{x+y}{2}\right) \le \sqrt{\Gamma_b(x)\,\Gamma_b(y)} \le \frac{1}{2}\left(\Gamma_b(x)+\Gamma_b(y)\right) \qquad (x>0,\, y>0,\, b\ge 0). \tag{2.10}$$

The following result follows from Theorem 2.2.2 and the definition of the generalized gamma function given by (2.2).

Corollary 2.2.3. *For* $1 < p < \infty$ *and* $\left(\frac{1}{p}\right) + \left(\frac{1}{q}\right) = 1$, *then*

$$K_{\frac{\alpha}{p}+\frac{\beta}{q}}(t) \le \left(K_\alpha(t)\right)^{\frac{1}{p}} \left(K_\beta(t)\right)^{\frac{1}{q}} \qquad (\alpha>0, \beta>0, t>0). \tag{2.11}$$

Proof. Replacing the generalized gamma functions in (2.9) by their representations in (2.2) we find

$$K_{\frac{\alpha}{p}+\frac{p}{q}}(2\sqrt{b}) \leq \left(K_\alpha(2\sqrt{b})\right)^{\frac{1}{p}} \left(K_\beta(2\sqrt{b})\right)^{\frac{1}{q}} \quad (\alpha > 0, \beta > 0, b > 0). \tag{2.12}$$

The substitution $t = 2\sqrt{b}$ in (2.12) yield the proof of (2.11). □

Another important property of the generalized gamma function is the following reflection formula:

Theorem 2.2.4. *(The Reflection Formula)*

$$b^\alpha \Gamma_b(-\alpha) = \Gamma_b(\alpha) \qquad (Re(b) > 0) \tag{2.13}$$

Proof. The substitutions $t = bu^{-1}$ and $dt = -bu^{-2}\,du$, $(Re(b) > 0)$, in (2.1) yield

$$\Gamma_b(\alpha) = b^\alpha \int_0^\infty u^{-\alpha-1}\, e^{-u-bu^{-1}}\, du,$$

which is exactly (2.13). □

In Theorem 1.2.4 it has been shown that the product of two Euler gamma functions leads to the relation between the gamma function and the beta function. It would be interesting to investigate whether this is the case when two generalized gamma functions are multiplied by each other. The next theorem discuss this multiplication from which Theorem 1.2.4 will be deduced as a special case.

Theorem 2.2.5. *(Product Formula)*

$$\Gamma_b(\alpha)\Gamma_b(\beta) = 2\int_0^\infty r^{2(\alpha+\beta)-1}\, exp(-r^2) B\left(\alpha, \beta; \frac{b}{r^2}\right) dr, \tag{2.14}$$

where

$$B(\alpha,\beta,b) := \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\, exp\left(\frac{-b}{t(1-t)}\right) dt, \tag{2.15}$$

is the extended beta function.

Proof. The transformation $t = x^2$ in (2.1) yields

$$\Gamma_b(\alpha) = 2\int_0^\infty x^{2\alpha-1}\, e^{-x^2-bx^{-2}}\, dx \qquad (Re(b) > 0; b = 0, Re(\alpha) > 0). \tag{2.16}$$

Multiplying $\Gamma_b(\alpha)\Gamma_b(\beta)$ by using (2.16), we find

$$\Gamma_b(\alpha)\Gamma_b(\beta) = 4\int_0^\infty\int_0^\infty x^{2\alpha-1}y^{2\beta-1}\, exp\left\{-\left(x^2+y^2\right) - b\left(\frac{x^2+y^2}{x^2y^2}\right)\right\} dx\ dy. \tag{2.17}$$

If we transfer (2.17) to polar coordinates, we find

$\Gamma_b(\alpha)\Gamma_b(\beta)$

$$= 2\int_0^\infty r^{2(\alpha+\beta)-1}\, e^{-r^2}\left\{2\int_0^{\frac{\pi}{2}}(\cos\theta)^{2\alpha-1}(\sin\theta)^{2\beta-1}\, exp\left(\frac{-4b}{r^2}\csc 2\theta\right) d\theta\right\} dr. \tag{2.18}$$

However, the inner integral in (2.17) is expressible in terms of the extended beta function to give

$$B\left(\alpha,\beta;\frac{b}{r^2}\right) = 2\int_0^{\frac{\pi}{2}}(\cos\theta)^{2\alpha-1}\, exp\left(-\frac{4b}{r^2}\csc 2\theta\right) d\theta. \tag{2.19}$$

From (2.19) and (2.18), we get the proof of (2.14). □

As a result of Theorem 2.2.5, we have the following corollary.

Corollary 2.2.6.

$$\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} = B(\alpha,\beta) \qquad (Re(\alpha) > 0,\ Re(\beta) > 0). \tag{2.20}$$

Proof. This result follows from Theorem 2.2.5 by setting $b = 0$ in (2.14). □

The generalized gamma function satisfies a second-order partial differential equation. This result is given in the following theorem.

Theorem 2.2.7. *(Differential Equation) The generalized gamma function* $\Gamma_b(\alpha)$ *satisfies the second-order partial differential equation*

$$b\frac{\partial^2\Gamma_b(\alpha)}{\partial b^2}+(1-\alpha)\frac{\partial\Gamma_b(\alpha)}{\partial b}-\Gamma_b(\alpha)=0 \qquad (Re(b)>0). \tag{2.21}$$

Proof. First, we not that for $n=0,1,2,3,\ldots$

$$\frac{\partial^n}{\partial b^n}\{\Gamma_b(\alpha)\}=(-1)^n\Gamma_b(\alpha-n) \qquad (Re(b)>0). \tag{2.22}$$

Replacing α by $\alpha-1$ in the difference equation (2.3) of the generalized gamma function, we get

$$b\Gamma_b(\alpha-2)-(1-\alpha)\Gamma_b(\alpha-1)-\Gamma_b(\alpha)=0.$$

This implies by using (2.22)

$$b\frac{\partial^2\Gamma_b(\alpha)}{\partial b^2}+(1-\alpha)\frac{\partial\,\Gamma_b(\alpha)}{2b}-\Gamma_b(\alpha)=0,$$

as desired □

The Mellin transform of the generalized gamma function is studied in the following Theorem.

Theorem 2.2.8. *(Melllin transform representation)*

$$M\{\Gamma_b(\alpha);s\}=\Gamma(s)\Gamma(\alpha+s) \qquad (Re(s)>0, Re(\alpha+s)>0). \tag{2.23}$$

Proof. According to the definition of the Mellin transform of $\Gamma_b(\alpha)$ in S, we find

$$M\{\Gamma_b(\alpha);s\} := \langle b_+^{s-1}, \Gamma_b(\alpha)\rangle \tag{2.24}$$

Replacing the generalized gamma function by its representation (2.5) we find

$$M\{\Gamma_b(\alpha);s\} = \langle b_+^{s-1}, \langle x_+^{\alpha-1}, e^{-x-bx^{-1}}\rangle\rangle. \tag{2.25}$$

An application of Fubini theorem [10, p. 458] yields

$$M\{\Gamma_b(\alpha);s\} = \langle x_+^{\alpha-1}, e^{-x}\langle b_+^{s-1}, e^{-bx^{-1}}\rangle\rangle. \tag{2.26}$$

However, according to (1.1) it can be easily shown that

$$\langle b_+^{s-1}, e^{-bx^{-1}}\rangle = x^s\,\Gamma(s) \qquad (Res > 0). \tag{2.27}$$

Therefore,

$$M\{\Gamma_b(\alpha);s\} = \Gamma(s)\langle x_+^{\alpha+s-1}, e^{-x}\rangle = \Gamma(s)\Gamma(\alpha+s)$$

$$(Res(s) > 0, Re(\alpha+s) > 0),$$

as desired. □

We conclude this section by the following theorem that gives the asymptotic representation of the generalized gamma function.

Theorem 2.2.9. *(Asymptotic Behavior for small b)*

$$\Gamma_b(\alpha) \sim \sum_{n=0}^{\infty} \Gamma(\alpha-n)\frac{(-1)^n}{n!}b^n + b^{\alpha}\sum_{n=0}^{\infty}\Gamma(-\alpha-n)\frac{(-1)^n}{n!}b^n$$

$$(b \to 0^+, \quad 0 < Re(\alpha) < 1). \tag{2.28}$$

Proof. Taking the inverse Mellin transform of both sides in (2.23) we find

$$\Gamma_b(\alpha) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(s)\Gamma(\alpha+s)\, b^{-s}\, ds \qquad (0 < c < 1). \tag{2.29}$$

The integrand in (2.29) has simple poles at $-n$ and $-n-\alpha$ $(n = 0, -1, -2, \ldots)$. The residues of these poles are, respectively,

$$\begin{aligned} Res\{f, -n\} &= \frac{(-1)^n}{n!}\Gamma(\alpha-n)\, b^n, \\ Res\{f; -n-\alpha\} &= \frac{(-1)^n}{n!}\Gamma(-n-\alpha)\, b^{n+\alpha}. \end{aligned}$$

Summing over all these residues yields the asymptotic representation (2.28). □

Another asymptotic representation can be obtained from (2.2) that come directly from the asymptotic representations of $K_\alpha(z)$ that are well studied in the literature [34], [35], [36] and [47]. This representation is given by

$$\Gamma_b(\alpha) \sim \frac{1}{2}\sqrt{\pi}\, b^{-\frac{1}{4}}\, e^{-2\sqrt{b}} \sum_{m=0}^{\infty} \frac{(\alpha, m)}{(4\sqrt{b})^m} \left(b \to \infty, |arg(\sqrt{b})| < \frac{3\pi}{2}\right), \tag{2.30}$$

where

$$(\alpha, m) := \frac{\Gamma(\alpha+m+\frac{1}{2})}{m!\,\Gamma\left(\alpha-m+\frac{1}{2}\right)}, \tag{2.31}$$

is the Hankel symbole [51].

2.3 Generalization of the Psi (Digamma) Function

In the first section of this chapter, we introduced the generalized gamma function. The generalized gamma function has been found to be a simple and natural generalization of the Euler gamma function. This generalization leads to a generalization of the Psi (digamma) function. Analogous to the definition of the diagamma function, the generalized diagamma function is defined as the logarithmic derivative of the generalized gamma function. In this section we shall prove some of the properties of the generalized diagamma function and establish different integral representations of this function. The classical representations of the diagamma function will be deduced as special cases.

Definition 2.3.1. *The generalized digamma function is defined by*

$$\psi_b(\alpha) := \frac{d}{d\alpha}\left\{\ln\left(\Gamma_b(\alpha)\right)\right\} = \frac{1}{\Gamma_b(\alpha)}\frac{d}{d\alpha}\left\{\Gamma_b(\alpha)\right\}. \tag{2.32}$$

From the integral representation (2.1) of the generalized gamma function we have

$$\psi_b(\alpha) := \frac{1}{\Gamma_b(\alpha)}\int_0^\infty t^{\alpha-1}(\ln t)e^{-t-bt^{-1}}\,dt \qquad (Re(b)>0; b=0, Re(\alpha)>0). \tag{2.33}$$

The reflection formula and recurrence relation of the generalized digamma function are presented in following theorems.

Theorem 2.3.2. *(Reflection Formula)*

$$\psi_b(-\alpha) = \ln b - \psi_b(\alpha) \qquad (Re(b)>0). \tag{2.34}$$

Proof. Replacing α by $-\alpha$ in (2.33), we find

$$\psi_b(-\alpha) = \frac{1}{\Gamma_b(-\alpha)} \int_0^\infty t^{-\alpha-1} (\ln t)\, e^{-t-bt^{-1}}\, dt. \tag{2.35}$$

Making the substitutions $t = bx^{-1}$, $dt = -bx^{-2}\, dx$ in (2.35) yield

$$\psi_b(-\alpha) = \frac{b^{-\alpha}}{\Gamma_b(-\alpha)} \int_0^\infty (\ln b - \ln x)\, x^{\alpha-1}\, e^{-x-bx^{-1}}\, dx. \tag{2.36}$$

Using the reflection formula (2.13) for the generalized gamma function, we get

$$\begin{aligned} \psi_b(-\alpha) &= \frac{1}{\Gamma_b(\alpha)} \int_0^\infty (\ln b - \ln x)\, x^{\alpha-1}\, e^{-x-bx^{-1}}\, dx, \\ &= \ln b - \psi_b(\alpha). \end{aligned}$$

□

Theorem 2.3.3. *(Recurrence Relation)*

$$\left(\frac{\Gamma_b(\alpha+1)}{\alpha\Gamma_b(\alpha)}\right)\psi_b(\alpha+1) - b\left(\frac{\Gamma_b(\alpha-1)}{\alpha\,\Gamma_b(\alpha)}\right)\psi_b(\alpha-1) = \frac{1}{\alpha} + \psi_b(\alpha)$$

$$(Re(b) \geq 0, Re(\alpha) > 0). \tag{2.37}$$

Proof. According to ([9], [26]) we have

$$\int_0^\infty \ln t\left(t^\alpha - \alpha t^{\alpha-1} - bt^{\alpha-2}\right) e^{-t-bt^{-1}}\, dt = 2b^{\frac{\alpha}{2}}\, k_\alpha(2\sqrt{b}) \quad (Re(b) > 0), \tag{2.38}$$

which is simplified in terms of the generalized gamma and digamma functions by using (2.2) and (2.33) to give

$$\Gamma_b(\alpha+1)\psi_b(\alpha+1) - \alpha\,\Gamma_b(\alpha)\psi_b(\alpha) - b\,\Gamma_b(\alpha-1)\psi_b(\alpha-1) = \Gamma_b(\alpha). \tag{2.39}$$

Dividing both sides of (2.39) by $\alpha\Gamma_b(\alpha)$ and rearranging the terms we get (2.37). □

The recurrence relation (1.24) is a special case of (2.37) when we set $b=0$. Now we give integral representations of the generalized digamma function and deduce the representations (1.28) and (1.29) of the digamma function as special cases

Theorem 2.3.4.

$$\psi_b(\alpha)=\int_0^\infty \left\{e^{-x}-(1+x)^{-\alpha}\frac{\Gamma_{b(1+x)}(\alpha)}{\Gamma_b(\alpha)}\right\}\frac{dx}{x} \quad (Re(b)\geq 0; b=0, Re(\alpha)>0). \tag{2.40}$$

Proof. Consider the double integral

$$I=\int_0^\infty\int_0^\infty t^{\alpha-1}e^{-\frac{b}{t}}\left\{\frac{e^{-t-x}-e^{-t(1+x)}}{x}\right\}dt\,dx. \tag{2.41}$$

If we integrate the double integral with respect to t we have

$$I=\int_0^\infty\left\{e^{-x}\int_0^\infty t^{\alpha-1}e^{-t-bt^{-1}}dt-\int_0^\infty t^{\alpha-1}e^{-t(1+x)-bt^{-1}}dt\right\}\frac{dx}{x}. \tag{2.42}$$

The inner integrals are the standard form of the generalized gamma function (2.1). Thus, we find

$$I=\int_0^\infty \left\{e^{-x}\Gamma_b(x)-(1-x)^{-\alpha}\Gamma_{b(1+x)}(\alpha)\right\}\frac{dx}{x}. \tag{2.43}$$

However, if we integrate the double integral (2.41) with respect to x, we get

$$I=\int_0^\infty t^{\alpha-1}e^{-t-bt^{-1}}\left\{\int_0^\infty\frac{e^{-x}-e^{-tx}}{x}dx\right\}dt. \tag{2.44}$$

The inner integral in (2.44) is the integral representation of lnt [3]. Thus we have

$$I=\int_0^\infty t^{\alpha-1}(\ln t)e^{-t-bt^{-1}}dt=\frac{d}{d\alpha}\left(\int_0^\infty t^{\alpha-1}e^{-t-bt^{-1}}dt\right)=\frac{d}{d\alpha}\left(\Gamma_b(\alpha)\right). \tag{2.45}$$

From (2.43) and (2.45), we find

$$\frac{d}{d\alpha}\left(\Gamma_b(\alpha)\right) = \int_0^\infty \left\{e^{-x}\Gamma_b(\alpha) - (1+x)^{-\alpha}\Gamma_{b(1+x)}(\alpha)\right\}\frac{dx}{x}. \tag{2.46}$$

Dividing both sides of (2.46) by $\Gamma_b(\alpha)$ we get the proof of (2.43). □

Note that integral representation (1.28) of the classical digamma function can be recovered by setting $b = 0$ in (2.40).

Theorem 2.3.5.

$$\psi_b(\alpha) = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{\Gamma_{be^t}(\alpha)}{\Gamma_b(\alpha)}\frac{e^{-\alpha t}}{1-e^{-t}}\right) dt$$

$$(Re(b) > 0; b = 0, Re(\alpha) > 0). \tag{2.47}$$

Proof. From (2.40), we get

$$\psi_b(\alpha) = \lim_{\delta\to 0}\left[\int_\delta^\infty \frac{e^{-x}}{x}dx - \int_\delta^\infty \frac{(1+x)^{-\alpha}}{x}\cdot\frac{\Gamma_{b(1+x)}(\alpha)}{\Gamma_b(\alpha)}dx\right]. \tag{2.48}$$

The transformation $x = e^t - 1$ in the second integral of the right-hand side of (2.48) yields

$$-\int_0^\infty \frac{(1+x)^\alpha}{x}\frac{\Gamma_{b(1+x)}(\alpha)}{\Gamma_b(\alpha)}dx = \int_{\ln(1+\delta)}^\infty \frac{\Gamma_{be^t}(\alpha)}{\Gamma_b(\alpha)}\frac{e^{-t\alpha}}{1-e^{-t}}dt. \tag{2.49}$$

From (2.48) and (2.49), we get

$$\psi_b(\alpha) = \lim_{\delta\to 0^+}\left[\int_\delta^{\ln(1+\delta)}\frac{e^{-t}}{t}dt + \int_{\ln(1+\delta)}^\infty\left\{\frac{e^{-t}}{t} - \frac{\Gamma_{be^t}(\alpha)}{\Gamma_b(\alpha)}\frac{e^{-t\alpha}}{1-e^{-t}}\right\}dt\right.$$

$$= \int_0^\infty\left(\frac{e^{-t}}{t} - \frac{\Gamma_{be^t}(\alpha)}{\Gamma_b(\alpha)}\frac{e^{-t\alpha}}{1-e^{-t}}\right)dt. \tag{2.50}$$

Since

$$\left|\int_\delta^{\ln(1+\delta)}\frac{e^{-t}}{t}dt\right| \leq \int_{\ln(1+\delta)}^\delta \frac{1}{t}dt = \ln\left(\frac{\delta}{\ln(1+\delta)}\right) \to 0\,\text{as}\,\delta\to 0^+.$$

This completes the proof of (2.47). □

Also we note that the integral representation (1.29) of the digamma function is gained when we put $b = 0$ in (2.47).

2.4 Generalized Incomplete Gamma Functions

The generalization of the gamma functions leads to a generalization of the incomplete gamma functions. Recently, Chaudhry and Zubair have shown that the closed-form solutions to several problems in heat conduction can be expressed in terms of the generalized incomplete gamma functions [10, pp. 357–440].

In this section we give the definition and state some relations of these functions.

Definition 2.4.1. *The generalized Incomplete gamma functions are defined by*

$$\gamma(\alpha,x;b) := \int_0^x t^{\alpha-1}\, exp\,(-t-bt^{-1})\,dt, \tag{2.51}$$

$$\Gamma(\alpha,x;b) := \int_x^\infty t^{\alpha-1}\, exp\,(-t-bt^{-1})\,dt, \tag{2.52}$$

where α, x are complex parameters and b is a complex variable. When the argument b vanishes, the functions (2.51) and (2.52) reduces the ordinary incomplete gamma functions (1.34) and (1.35).

Like incomplete gamma functions, the generalized incomplete gamma functions satisfy several relations found useful in applications. We state these relations in the following theorems.

Theorem 2.4.2. *(Decomposition Theorem)*

$$\gamma(\alpha,x;b) + \Gamma(\alpha,x;b) = \Gamma_b(\alpha) \qquad (Re(b) \geq 0). \tag{2.53}$$

Proof. This follows when we add the incomplete integrals (2.51) and (2.52) to get the complete integral (2.1) which is the right-hand side of (2.53). □

Theorem 2.4.3. *(Recurrence relation)*

$$\Gamma(\alpha+1,x;b) = \alpha\Gamma(\alpha,x;b) + b\Gamma(\alpha-1,x;b) + x^{\alpha} e^{-x-bx^{-1}}. \tag{2.54}$$

Proof. Put

$$f(t) := H(t-x)\, exp\,(-t-bt^{-1}). \tag{2.55}$$

Where $H(t)$ is the Heaviside function defined by

$$H(t) = \begin{cases} 1 & \text{if} \quad t > 0 \\ 0 & \text{if} \quad t < 0 \end{cases}. \tag{2.56}$$

A "formal" differentiation of $f(t)$ with respect to t yields

$$f'(t) = \delta(t-x)\exp(-t-bt^{-1}) + H(t-x)\left\{-1+bt^{-2}\right\} exp\left(-t-bt^{-1}\right). \tag{2.57}$$

However, the generalized incomplete gamma function has the Mellin transform representation

$$\Gamma(\alpha,x;b) := M\{f(t);\alpha\}, \tag{2.58}$$

that yields

$$-(\alpha-1)\Gamma(\alpha-1,x;b) = M\{f'(t);\alpha\}. \tag{2.59}$$

From (2.57) and (2.59), we get

$$-(\alpha-1)\Gamma(\alpha-1,x;b) = x^{\alpha-1}\, exp\,(-x-bx^{-1}) - \Gamma(\alpha,x;b) + b\Gamma(\alpha-2,x;b). \tag{2.60}$$

Rearranging the terms of (2.60) gives

$$\Gamma(\alpha,x;b) = (\alpha-1)\Gamma(\alpha-1,x;b) + b\,\Gamma(\alpha-2,x;b) + x^{\alpha-1}\,exp\,(-x-bx^{-1}). \qquad (2.61)$$

Finally replacing α by $\alpha+1$ in (2.61) we get (2.54) □

Corollary 2.4.4.

$$\Gamma(\alpha+1,x) = \alpha\,\Gamma(\alpha,x) + x^{\alpha}\,e^{-x}. \qquad (2.62)$$

Proof. This follows directly from (2.54) when we put $b=0$. □

One important representation of the generalized incomplete gamma function is its Mellin transform representation. This representation is given in the following theorem.

Theorem 2.4.5.

$$\Gamma(\alpha,x;b) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(s)\Gamma(\alpha+s,x)\,b^{-s}\,ds \quad (Re(b)>0, 0<c<1). \qquad (2.63)$$

Proof. Multiplying both side in (2.52) by b^{s-1} and then integrating it with respect to b from $b=0$ to $b=\infty$, we get

$$\int_0^\infty b^{s-1}\,\Gamma(\alpha,x;b)\,db = \int_x^\infty t^{\alpha-1}\,e^{-t}\left(\int_0^\infty b^{s-1}\,e^{-\frac{b}{t}}\,db\right)dt. \qquad (2.64)$$

The inner integral in (2.64) is $\Gamma(s)t^s$ for $Re(s)>0$.

Hence,

$$\int_0^\infty b^{s-1}\Gamma(\alpha,x;b)\,db = \Gamma(s)\int_x^\infty t^{\alpha+s-1}\,e^{-t}\,dt, \qquad (2.65)$$

which gives

$$\int_0^\infty b^{s-1}\Gamma(\alpha,x;b)\,db = \Gamma(s)\,\Gamma(\alpha+s,x). \qquad (2.66)$$

Taking the inverse Mellin transform of both sides in (2.66). We get (2.63). □

The next theorem gives the series representation of the generalized incomplete gamma function in terms of the incomplete gamma function.

Theorem 2.4.6.

$$\Gamma(\alpha,x;b) = \sum_{n=0}^{\infty} (-1)^n \Gamma(\alpha-n,x) \frac{b^n}{n!}. \tag{2.67}$$

Proof. First, we note that a formal differentiation of (2.52) with respect to the parameter b yields

$$\frac{\partial^n}{\partial b^n}\{\Gamma(\alpha,x;b)\} = (-1)^n \Gamma(\alpha-n,x;b) \qquad (n=0,1,2,\ldots) \tag{2.68}$$

Now, if we let

$$\Gamma(\alpha,x;b) = \sum_{n=0}^{\infty} c_n b^n, \tag{2.69}$$

by the Maclurin series of $\Gamma(\alpha,x;b)$. Then

$$c_n = \frac{\partial^n}{\partial b^n}\{\Gamma(\alpha,x;0)\}/n!. \tag{2.70}$$

From (2.68) and (2.70), we get

$$c_n = (-1)^n \Gamma(\alpha-n,x)/n!. \tag{2.71}$$

Finally, substituting (2.71) in (2.69) we get (2.67). □

2.5 Connection with other special Functions

It is important to find out possible relations of a given special function with other special functions. This is helpful to classify the function and to find the solution of the related applied problems under different conditions in terms of the tabulated special functions. In this section we give the relations of the generalized incomplete gamma function with other functions.

Theorem 2.5.1.

$$\Gamma\left(\frac{1}{2},x;b\right)=\frac{\sqrt{\pi}}{2}\left[e^{-2\sqrt{b}}\operatorname{erfc}\left(\sqrt{x}-\frac{\sqrt{b}}{\sqrt{x}}\right)+e^{2\sqrt{b}}\operatorname{erfc}\left(\sqrt{x}+\frac{\sqrt{b}}{\sqrt{x}}\right)\right]. \tag{2.72}$$

Proof. The generalized incomplete gamma function $\Gamma(\alpha,x;b)$ can be represented as

$$\Gamma(\alpha,x;b)=e^{-2\sqrt{b}}\int_x^{\infty}t^{\alpha-1}\,e^{-\left(\sqrt{t}-\frac{\sqrt{b}}{\sqrt{t}}\right)^2}\,dt, \tag{2.73}$$

and

$$\Gamma(\alpha,x;b)=e^{2\sqrt{b}}\int_x^{\infty}t^{\alpha-1}e^{-\left(\sqrt{t}+\frac{\sqrt{b}}{\sqrt{t}}\right)^2}\,dt. \tag{2.74}$$

Replacing α by $\alpha+\frac{1}{2}$ in (2.73) and (2.74) and rearranging the terms, we get

$$\Gamma(\alpha+\frac{1}{2},x;b)=e^{-2\sqrt{b}}\int_x^{\infty}t^{\alpha}\left(\frac{1}{\sqrt{t}}+\frac{\sqrt{b}}{t\sqrt{t}}\right)e^{-\left(\sqrt{t}-\frac{\sqrt{b}}{\sqrt{t}}\right)^2}\,dt$$

$$-\sqrt{b}\int_x^{\infty}t^{\alpha-\frac{3}{2}}\,e^{-t-bt^{-1}}\,dt, \tag{2.75}$$

and

$$\Gamma(\alpha+\frac{1}{2},x;b)=e^{2\sqrt{b}}\int_x^{\infty}t^{\alpha}\left(\frac{1}{\sqrt{t}}-\frac{\sqrt{b}}{t\sqrt{t}}\right)e^{-\left(\sqrt{t}+\frac{\sqrt{b}}{\sqrt{t}}\right)^2}\,dt$$

$$+\sqrt{b}\int_x^{\infty}t^{\alpha-\frac{3}{2}}\,e^{-t-bt^{-1}}\,dt. \tag{2.76}$$

Adding (2.75) and (2.76), we get

$$2\Gamma(\alpha+\frac{1}{2},x;b) = e^{-2\sqrt{b}}\int_x^\infty t^\alpha\left(\frac{1}{\sqrt{t}}+\frac{\sqrt{b}}{t\sqrt{t}}\right)e^{-\left(\sqrt{t}-\frac{\sqrt{b}}{\sqrt{t}}\right)^2}dt$$

$$+e^{2\sqrt{b}}\int_x^\infty t^\alpha\left(\frac{1}{\sqrt{t}}-\frac{\sqrt{b}}{t\sqrt{t}}\right)e^{-\left(\sqrt{t}+\frac{\sqrt{b}}{\sqrt{t}}\right)^2}dt. \tag{2.77}$$

Making the substitutions

$$u = \sqrt{t}-\frac{\sqrt{b}}{\sqrt{t}} \text{ and } v = \sqrt{t}+\frac{\sqrt{b}}{\sqrt{t}} \tag{2.78}$$

in (2.77) and taking $\alpha = 0$, we get

$$\Gamma\left(\frac{1}{2},x;b\right) = e^{-2\sqrt{b}}\int_{\sqrt{x}-\frac{\sqrt{b}}{\sqrt{x}}}^\infty e^{-u^2}du + e^{2\sqrt{b}}\int_{\sqrt{x}+\frac{\sqrt{b}}{\sqrt{x}}}^\infty e^{-v^2}dv, \tag{2.79}$$

which directly gives (2.72). □

In Theorem 1.5.3, we gave the relation between the incomplete gamma function with the complementary error function. This result can be recovered from Theorem 2.5.1 by putting $b = 0$ in (2.72).

Corollary 2.5.2.

$$\Gamma\left(\frac{1}{2},x\right) = \sqrt{\pi}\ \text{erfc}\ (\sqrt{x}). \tag{2.80}$$

Corollary 2.5.3.

$$\Gamma\left(-\frac{1}{2};x;b\right) = \frac{\sqrt{\pi}}{2\sqrt{b}}\left[e^{-2\sqrt{b}}\ \text{erfc}\ \left(\sqrt{x}-\frac{\sqrt{b}}{\sqrt{x}}\right) - e^{2\sqrt{b}}\ \text{erfc}\ \left(\sqrt{x}+\frac{\sqrt{b}}{\sqrt{x}}\right)\right] \tag{2.81}$$

Proof. (2.81) can be obtained directly from (2.72) by differentiating both sides of (2.72) with respect to the parameter b. □

We note that the function $\Gamma\left(\frac{1}{2},x;b\right)$ and $\Gamma\left(-\frac{1}{2},x,b\right)$ can further be expressed in terms of the incomplete gamma and confluent hypergeometric functions. For more details one can refer to [10].

Chapter 3

A Generalization of the Euler's Reflection Formula

In this chapter we prove a generalization of the well known and important Euler's reflection identity (1.16) in terms of the generalized incomplete gamma function (2.52). From this generalization we will deduce some other generalizations that are known in the literature as special cases. Also, we discuss a class of integral representations involving incomplete gamma function, generalized incomplete gamma function, error functions, exponential integral function and Macdonald's function.

3.1 Introduction

The classical gamma function defined by

$$\Gamma(s) := \int_0^\infty t^{s-1} e^{-t} dt \qquad (s := \sigma + i\tau, 0 < \sigma < \infty), \tag{3.1}$$

plays an important basic role in the subject of Special Functions. The properties and historical background of the function is given in the first Chapter. One of the classical identities useful in several applications of "Operational Calculus", and Engineering applications is the closed form representation,

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s} = \int_0^\infty \frac{t^{s-1}}{1+t} dt \qquad (0 < \sigma < 1). \tag{3.2}$$

The representation

$$\beta(s, a-s) = \Gamma(s)\Gamma(a-s)/\Gamma(a) = \int_0^\infty \frac{t^{s-1}}{(1+t)^a} dt \qquad (0 < \sigma < a), \tag{3.3}$$

of the beta function extend the above identity (3.2) in a natural way as one can recover it by taking $a = 1$. It may be noted that there are several generalizations of the classical identity (3.2) available in literature ([4], [29] and [47]). One of these generalizations of (3.2) is the representation [22, p. 355]

$$\Gamma(s)\Gamma(1-s, x) = \int_0^\infty \frac{t^{s-1} e^{-(1+t)x}}{1+t} dt \qquad (0 < \sigma < 1, 0 \leq x < \infty), \tag{3.4}$$

that involves the incomplete gamma function. The above identity can be further generalized to

$$\Gamma(s)\Gamma(a-s, x) = \int_0^\infty \frac{t^{s-1}\Gamma(a, x(1+t))}{(1+t)^a} dt \qquad (0 < \sigma < a, 0 \leq x < \infty), \tag{3.5}$$

which is useful in finding the asymptotic representations of the incomplete gamma function [47]. Putting $a = 1+n$ in (3.5) and using $\Gamma(1+n, z) = n! e^{-z} e_n(z)$ we find that

$$\Gamma(s)\Gamma(1+n-s, x)/n! = \int_0^\infty \frac{t^{s-1} e^{-x(1+t)} e_n(x(1+t))}{(1+t)^{n+1}} dt$$

$$(0 < \sigma < 1, x \geq 0, n = 0, 1, 2, 3, \ldots). \tag{3.6}$$

The identity (3.4) is recovered from (3.6) when $n = 0$. Chaudhry and Zubair[10] introduced the generalized incomplete gamma functions defined by

$$\Gamma(s,x;b) := \int_x^\infty t^{s-1} e^{-t-bt^{-1}}\, dt \qquad (b \geq 0, 0 \leq x < \infty), \tag{3.7}$$

$$\gamma(s,x;b) := \int_0^x t^{s-1} e^{-t-bt^{-1}}\, dt \qquad (b \geq 0, 0 \leq x < \infty), \tag{3.8}$$

where they defined the generalized gamma function as

$$\Gamma_b(s) := \Gamma(s,0;b) = \int_0^\infty t^{s-1} e^{-t-bt^{-1}} dt = 2b^{s/2} K_s\left(2\sqrt{b}\right)$$

$$(0 < b < \infty,\ -\infty < \sigma < \infty;\ b = 0,\ 0 < \sigma < \infty). \tag{3.9}$$

These functions are found useful in several engineering applications ([10], [54], [55]) [56]). The asymptotic expansion of the function $K_s(b)$ is well known [4, p. 223]

$$K_s(b) \sim \sqrt{\frac{\pi}{2b}} e^{-b} \left[1 + \sum_{n=1}^{\infty} \frac{(s,n)}{(2b)^n}\right] \quad (b \to \infty), \tag{3.10}$$

where (s,n) is defined by

$$(s,n) := \frac{(-1)^n (1/2+s)_n}{(1/2-s)_n} = (-1)^n \frac{\Gamma(1/2-s)}{\Gamma(1/2+s)} \left[\frac{\Gamma(1/2+s+n)}{\Gamma(1/2-s+n)}\right]. \tag{3.11}$$

In particular, we have

$$K_s(b) \sim \sqrt{\frac{\pi}{2b}} e^{-b} \qquad (b \to \infty). \tag{3.12}$$

Similarly [47, p. 234]

$$K_s(2b) \sim \frac{1}{2} \Gamma(s) (b)^{-s} \qquad (\sigma \to \infty, b > 0). \tag{3.13}$$

We prove a generalization of the classical identity (3.2) in Section 3.2 that involves the generalized incomplete gamma function. Our result extends naturally the known identity (3.5) as it can be recovered from the result by substituting $b = 0$.

Some special cases of the main result that involves complementary error functions are discussed in Section 3.3. The results may be found useful in the analytic study of temperature distribution in a variety of heat conduction problem [10, pp. 385 – 413]. One of the important features of the main result is that, as the generalized incomplete gamma function reduces to Macdonald's function when $x = 0$, we find a useful identities involving the Macdonald's function.

3.2 The Generalized Identity

In this section we present the main theorem that generalize the classical reflection identity in terms of the generalized incomplete gamma function defined by (2.52).

Theorem 3.2.1. *Let $\Gamma(s,x;b)$ be the generalized incomplete gamma function. Then*

$$\Gamma(s)\Gamma(a-s,x;b) = \int_0^\infty \frac{t^{s-1}\Gamma(a,x(1+t);(1+t)b)}{(1+t)^a}\,dt$$

$$(s = \sigma + i\tau, 0 < \sigma < a, x \geq 0, b \geq 0). \tag{3.14}$$

Proof. First we note that for $b = 0$ the Mellin transform integral (3.14) reduces to (3.5). Hence, we assume that $0 < b < \infty$. Using the asymptotic representation [10, p. 6]

$$|\Gamma(s)| = \sqrt{2\pi}|\tau|^{\sigma-1/2}\exp\left(-\frac{\pi}{2}|\tau|\right)(1+O(1/|\tau|)$$

$$(|\tau| \to \infty, -\infty < A \leq \sigma < B < \infty), \tag{3.15}$$

of the gamma function , we find that for all $b > 0$,

$$|\Gamma(s)\Gamma(a-s,x,b)| \leq C|\tau|^{a} e^{(-\pi|\tau|)}$$

$$(-\infty < A < \sigma < B < \infty, 0 \leq x < \infty, |\tau| \to \infty). \tag{3.16}$$

Hence, the inverse Mellin transform (IMT)integral

$$I_a(t,x;b) := \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(s)\Gamma(a-s,x;b)t^{-s}\, ds$$

$$(0 < c < a, 0 \leq x < \infty, 0 < t < \infty), \tag{3.17}$$

is uniformly convergent for all $t\,(0 < t_0 \leq t \leq t_1 < \infty)$. Moreover, for $0 < t < 1$ the asymptotic relation (3.16) shows that the integral is absolutely convergent and, it can be evaluated by Cauchy's residue theorem. The integrand has simple poles at $s = -n\,(n = 0,1,2,3,\ldots)$ leading to the series representation

$$I_a(t,x;b) = \sum_{n=0}^{\infty} \frac{\Gamma(a+n,x;b)(-t)^n}{n!} \qquad (0 < t < 1, 0 \leq x < \infty, 0 \leq b < \infty). \tag{3.18}$$

Replacing the generalized incomplete gamma function in (3.18) by its integral representation, we find

$$\begin{aligned} I_a(t,x;b) &= \sum_{n=0}^{\infty} \frac{(-t)^n}{n!}\left(\int_x^{\infty} y^{a+n-1}\, e^{-y-\frac{b}{y}}\, dy\right) \\ &= \int_x^{\infty} y^{a-1}\, e^{-y-\frac{b}{y}} \left(\sum_{n=0}^{\infty} \frac{(-ty)^n}{n!}\right) dy \\ &= \int_x^{\infty} y^{a-1}\, e^{-(1+t)y-\frac{b}{y}}\, dy \\ &= \frac{1}{(1+t)^a}\Gamma(a,(1+t)x;(1+t)b) \\ &\quad (0 \leq x < \infty, 0 \leq b < \infty), \end{aligned} \tag{3.19}$$

It is to be remarked that despite the fact that the requirement $0 < t < 1$ is necessary for the convergence of the series (3.18), the representation (3.19) remains well defined for all $t > 0$. Hence (3.19) is an analytic continuation of (3.18). From (3.17) and (3.19), we find

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(s)\Gamma(a-s,x;b)t^{-s}ds = \frac{1}{(1+t)^a}\Gamma(a,(1+t)x;(1+t)b)$$

$$(0 < c < a, 0 \leq x < \infty, 0 < t < \infty). \tag{3.20}$$

Inverting the relation (3.20), we arrive at (3.14). □

3.3 Special Cases of the Generalized Identity

We discuss special cases of the general result in this section. First, putting $b = 0$ in (3.14), we arrive at (3.5). However, if we set $x = 0$, in (3.14), we find

$$\Gamma(s)\Gamma_b(a-s) = \int_0^\infty \frac{\Gamma_{b(1+t)}(a)}{(1+t)^a}t^{s-1}\,dt \quad (0 < \sigma < a,\, b \geq 0). \tag{3.21}$$

Replacing the generalized gamma function in (3.21) by its representation (3.9) in terms of Macdonald's function, we get

$$2\Gamma(s)\,b^{\frac{(a-s)}{2}}K_{a-s}(2\sqrt{b}) = 2\int_0^\infty \frac{((1+t)b)^{\frac{a}{2}}\,K_a\left(2\sqrt{(1+t)\,b}\right)}{(1+t)^a}\,t^{s-1}\,dt$$

$$(0 < \sigma < a, b \geq 0). \tag{3.22}$$

Letting $x = 2\sqrt{b}$ in (3.22) and simplifying we get the following useful identity

$$K_{a-s}(x) = \left(\frac{x}{2}\right)^s \frac{1}{\Gamma(s)} \int_0^\infty \frac{K_a\left(x\sqrt{1+t}\right)}{(1+t)^{\frac{a}{2}}} t^{s-1}\, dt,$$

$$(0 < \sigma < a, 0 \le x < \infty). \tag{3.23}$$

Now if we put $b = 0$ in (3.14), we find

$$\Gamma(s)\Gamma(a-s) = \Gamma(a) \int_0^\infty \frac{t^{s-1}}{(1+t)^a}\, dt \quad (0 < \sigma < a), \tag{3.24}$$

that can be rewritten in terms of the beta function as

$$\beta(s, a-s) = \Gamma(s)\Gamma(a-s)/\Gamma(a) = \int_0^\infty \frac{t^{s-1}}{(1+t)^a}\, dt \qquad (0 < \sigma < a), \tag{3.25}$$

which is a generalization of the well known identity

$$\beta(s, 1-s) = \Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s} = \int_0^\infty \frac{t^{s-1}}{(1+t)}\, dt \qquad (0 < \sigma < 1). \tag{3.26}$$

Now, if we let $b = 0, a = 1+n$ in (3.14) we get

$$\frac{\Gamma(s)\Gamma(1-n-s,x)}{n!} = \int_0^\infty \frac{t^{s-1} e^{-x(1+t)} e_n(x(1+t))}{(1+t)^{n+1}}\, dt$$

$$(0 < \sigma < 1, 0 \le x < \infty, n = 0, 1, 2, 3 \ldots), \tag{3.27}$$

which is (3.6). Also, if we set $b = 0$, $a = 1-n$ in (3.14) we get

$$\frac{\Gamma(s)\Gamma(1-n-s,x)}{x^{n-1}} = \int_0^\infty t^{s-1} E_n(x(1+t))dt$$

$$(0 < \sigma < 1, x \ge 0, n = 0, 1, 2, 3 \ldots), \tag{3.28}$$

where $E_n(x) := \int_1^\infty e^{-xt} t^{-n}\, dt$, [10, p. 40] is the exponential integral function. Another special case of (3.14) that involves the complementary error function can be obtained by letting $s = a - \frac{1}{2}, a > \frac{1}{2}$ and using relation [10, p. 51(2.120)]. This gives

$$\begin{aligned} & \int_0^\infty \frac{\Gamma(a, x(1+t); (1+t)b)}{(1+t)^a} t^{a-3/2}\, dt \\ = \;& \frac{1}{2}\sqrt{\pi}\,\Gamma(a - \frac{1}{2}) \left[e^{-2\sqrt{b}} \mathrm{erfc}\left(\sqrt{x} - \frac{\sqrt{b}}{\sqrt{x}}\right) + e^{2\sqrt{b}} \mathrm{erfc}\left(\sqrt{x} + \frac{\sqrt{b}}{\sqrt{x}}\right)\right], \end{aligned}$$

$$(0 \le b < \infty, 0 < x < \infty, \frac{1}{2} < a < \infty). \qquad (3.29)$$

For $b = 0$ in (3.29) we find a useful identity

$$\int_0^\infty \frac{\Gamma(a, x(1+t))}{(1+t)^a} t^{a-3/2}\, dt = \sqrt{\pi}\,\Gamma\left(a - \frac{1}{2}\right) \mathrm{erfc}\left(\sqrt{x}\right),$$

$$(0 < x < \infty, \frac{1}{2} < a < \infty). \quad (3.30)$$

that does not seems to have been realized in the literature. Putting $a = \frac{3}{2}$ in (3.30), we find

$$\int_0^\infty \frac{\Gamma\left(\frac{3}{2}, x(1+t)\right)}{(1+t)^{\frac{3}{2}}}\, dt = \sqrt{\pi}\,\mathrm{erfc}\left(\sqrt{x}\right) \qquad (0 < x < \infty). \qquad (3.31)$$

The substitution $u = x(1+t)$ in (3.31) leads to

$$\int_x^\infty \frac{\Gamma\left(\frac{3}{2}, u\right)}{u^{\frac{3}{2}}}\, du = \sqrt{\pi/x}\,\mathrm{erfc}\left(\sqrt{x}\right) \qquad (0 < x < \infty). \qquad (3.32)$$

The integrand in (3.32) can be simplified in terms of the complementary error function [10, p. 45] to get

$$\int_x^\infty \frac{\frac{\sqrt{\pi}}{2}\mathrm{erfc}\left(\sqrt{u}\right) + \sqrt{u}\, e^{-u}}{u^{\frac{3}{2}}}\, du = \sqrt{\pi/x}\,\mathrm{erfc}\left(\sqrt{x}\right) \qquad (0 < x < \infty). \qquad (3.33)$$

The formula (3.33) is important in the sense as the integral in the left hand side can be rewritten as the difference of the integral of the complementary error function and the exponential integral function [10, p. 43]. This leads to the formula

$$\int_x^\infty \frac{\operatorname{erfc}\left(\sqrt{u}\right)}{u^{3/2}}\,du = \frac{2}{\pi}E_i(-x) + \frac{2}{x}\operatorname{erfc}\left(\sqrt{x}\right) \qquad (0 < x < \infty), \tag{3.34}$$

which is useful in finding the solution of problems of heat conduction in materials with cylindrical symmetry [8, p. 261].

Remark: We note that our technique in proving the main result leads to another useful relation for the Macdonal's function. Indeed, if we set $x = 0$ in (4.24) and (4.25) we have

$$\frac{1}{(1+t)^a}\Gamma(a,0;(1+t)b) = \sum_{n=0}^{\infty} \frac{\Gamma(a+n,0;b)(-t)^n}{n!}$$

$$(0 < t < 1, 0 \le b < \infty). \tag{3.35}$$

Replacing the generalized gamma function in (3.35) by its representation (3.9) in terms of Macdonald's function, we get

$$\frac{2}{(1+t)^a}\left((1+t)\,b\right)^{a/2} K_a\left(2\sqrt{(1+t)b}\right) = \sum_{n=0}^{\infty} \frac{2\,b^{(a+n)/2} K_{a+n}\left(2\sqrt{b}\right)(-t)^n}{n!}$$

$$(0 < t < 1, 0 \le b < \infty). \tag{3.36}$$

Letting $x = 2\sqrt{b}$ in (3.36) and simplifying we get the following useful identity

$$K_a\left(x\sqrt{(1+t)}\right) = (1+t)^{a/2} \sum_{n=0}^{\infty} \frac{\left(\frac{x}{2}\right)^n K_{a+n}(x)\,(-t)^n}{n!}$$

$$(0 < t < 1, 0 \le x < \infty). \tag{3.37}$$

3.4 Concluding Remarks

Operational calculus has found applications in several areas of mathematics and engineering sciences. The most notable applications are found in electrical engineering problems for the calculation of transients in linear circuits. From the development of the subject by Gottfried Leibnitz and Euler to Olver Heaviside and Jan Mikusinski the gamma function $\Gamma(s)$ and the basic identity $\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}$ have played an important role in the development of the subject. A generalization of the basic identity is expected to play as well an important role. We have noticed that the generalization proved in this note leads to a class of integral representations involving incomplete gamma, generalized incomplete gamma, error functions and Macdonald's function.

Chapter 4

Iterated Integrals of the Generalized Complementary Error Function

In Section 4.1 of this chapter we give a generalization of the classical error functions based on the generalized incomplete gamma functions defined by (2.51) and (2.52). The properties of the generalized error functions will be studied in Section 4.2. In Section 4.3, we study the iterated integrals of the generalized error functions and deduce the classical results as special cases. In Section 4.4 we present some numerical computations and graphical representations of these new functions that may help in some application problems.

4.1 Introduction

The error functions defined by

$$\operatorname{erf}(x) := \frac{2}{\sqrt{\pi}} \int_0^x exp\,(-t^2)\,dt, \tag{4.1}$$

$$\operatorname{erfc}(x) := \frac{2}{\sqrt{\pi}} \int_x^\infty exp\,(-t^2)\,dt, \tag{4.2}$$

satisfy the decomposition formula

$$\mathrm{erf}(x) + \mathrm{erfc}(x) = 1. \tag{4.3}$$

These functions were originally introduced by Kramp [30]. However, they also called Gauss error functions or probability integrals. These functions occur in probability, statistics and partial differential equations. For example, the solution to the heat conduction problem

$$\begin{aligned} \frac{\partial T}{\partial t} &= \alpha \frac{\partial^2 T}{\partial x^2} \qquad (x \geq 0, t \geq 0), \\ T(x,0) &= 0 \\ T(0,t) &= T_0, \end{aligned} \tag{4.4}$$

is given by

$$T(x,t) = T_0 \,\mathrm{erfc}\left(\frac{x}{\sqrt{4\alpha t}}\right). \tag{4.5}$$

The error functions are expressible in terms of the incomplete gamma functions as

$$\mathrm{erf}(x) = \frac{1}{\sqrt{\pi}}\, \gamma\left(\frac{1}{2}, x^2\right), \tag{4.6}$$

$$\mathrm{erfc}(x) = \frac{1}{\sqrt{\pi}}\, \Gamma\left(\frac{1}{2}, x^2\right). \tag{4.7}$$

Moreover, the inverse error function has the closed form representation

$$\mathrm{erf}^{-1}(x) = \frac{\sqrt{\pi}}{2}\left(x + \frac{\pi}{12}x^3 + \frac{7\pi^2}{480}x^5 + \frac{127\pi^3}{40320}x^7 + \dots\right), \tag{4.8}$$

which is useful in solving inverse problems in applied sciences and engineering.

Chaudhry and Zubair [10] introduced the generalized incomplete gamma functions defined by

$$\Gamma(\alpha,x;b)=\int_x^\infty t^{\alpha-1}e^{-t-\frac{b}{t}}\,dt \quad (b\geq 0, x\geq 0), \tag{4.9}$$

$$\gamma(\alpha,x;b)=\int_0^x t^{\alpha-1}e^{-t-\frac{b}{t}}\,dt \quad (b\geq 0, x\geq 0). \tag{4.10}$$

It is to be noted that these function can naturally extend the classical error functions as follows:

$$\operatorname{erf}(x;b):=\frac{1}{\sqrt{\pi}}\gamma\left(\frac{1}{2},x^2;b\right), \tag{4.11}$$

$$\operatorname{erfc}(x;b):=\frac{1}{\sqrt{\pi}}\Gamma\left(\frac{1}{2},x^2;b\right). \tag{4.12}$$

For $b=0$, we find

$$\operatorname{erf}(x;0)=\operatorname{erf}(x), \tag{4.13}$$

$$\operatorname{erfc}(x;0)=\operatorname{erfc}(x). \tag{4.14}$$

Replacing the GIGF in (4.11) and (4.12) by their integral representation, we find

$$\operatorname{erf}(x;b):=\frac{1}{\sqrt{\pi}}\int_0^{x^2} exp\left(-t-\frac{b}{t}\right)\frac{dt}{\sqrt{t}}, \tag{4.15}$$

$$\operatorname{erfc}(x;b):=\frac{1}{\sqrt{\pi}}\int_{x^2}^{\infty} exp\left(-t-\frac{b}{t}\right)\frac{dt}{\sqrt{t}}. \tag{4.16}$$

The transformation $t=\tau^2$ in (4.15) and (4.16) yields

$$\operatorname{erf}(x;b)=\frac{2}{\sqrt{\pi}}\int_0^{x} exp\left(-\tau^2-b\,\tau^{-2}\right)d\tau, \tag{4.17}$$

$$\operatorname{erfc}(x;b)=\frac{2}{\sqrt{\pi}}\int_x^{\infty} exp\left(-\tau^2-b\,\tau^{-2}\right)d\tau. \tag{4.18}$$

From the decomposition formula of the generalized incomplete gamma functions (2.53), we can see that the extended error functions satisfy the decomposition formula

$$\mathrm{erf}(x;b)+\mathrm{erfc}(x;b)=e^{-2\sqrt{b}}. \tag{4.19}$$

Other properties of these functions will be investigated in following section.

4.2 Properties of The Generalized Error Functions

In this section, we investigate some properties of the generalized error functions and present a heat conduction problem whose solution is written in terms of these functions.

Theorem 4.2.1. *(Connection with error functions I)*

$$\mathrm{erfc}\,(x;b)=\frac{1}{2}\left[e^{-2\sqrt{b}}\mathrm{erfc}\left(x-\frac{\sqrt{b}}{x}\right)+e^{2\sqrt{b}}\mathrm{erfc}\left(x+\frac{\sqrt{b}}{x}\right)\right]. \tag{4.20}$$

Proof. Using the identity between the GIGF and the complementary error functions given by [10], we have

$$\Gamma\left(\frac{1}{2},x;b\right)=\frac{\sqrt{\pi}}{2}\left[e^{-2\sqrt{b}}\mathrm{erfc}\left(\sqrt{x}-\frac{\sqrt{b}}{\sqrt{x}}\right)+e^{2\sqrt{b}}\mathrm{erfc}\left(\sqrt{x}+\frac{\sqrt{b}}{\sqrt{x}}\right)\right].$$

This implies

$$\frac{1}{\sqrt{\pi}}\Gamma\left(\frac{1}{2},x^2;b\right)=\frac{1}{2}\left[e^{-2\sqrt{b}}\mathrm{erfc}\left(x-\frac{\sqrt{b}}{x}\right)+e^{2\sqrt{b}}\mathrm{erfc}\left(x+\frac{\sqrt{b}}{x}\right)\right]$$

and this directly gives

$$\mathrm{erfc}\,(x;b)=\frac{1}{2}\left[e^{-2\sqrt{b}}\mathrm{erfc}\left(x-\frac{\sqrt{b}}{x}\right)+e^{2\sqrt{b}}\mathrm{erfc}\left(x+\frac{\sqrt{b}}{x}\right)\right].$$

□

Theorem 4.2.2. *(Connection with error function II)*

$$\sqrt{b}\frac{\partial}{\partial b}\{\mathrm{erfc}(x;b)\} = \frac{1}{2}\left[e^{2\sqrt{b}}\mathrm{erfc}\left(x+\frac{\sqrt{b}}{x}\right) - e^{-2\sqrt{b}}\mathrm{erfc}\left(x-\frac{\sqrt{b}}{x}\right)\right]. \tag{4.21}$$

Proof. Differentiating (4.18) with respect to b, we find

$$\frac{\partial}{\partial b}\{\mathrm{erfc}\,(x;b)\} = \frac{-1}{\sqrt{\pi}}\Gamma\left(-\frac{1}{2},x^2;b\right). \tag{4.22}$$

Since

$$\Gamma\left(-\frac{1}{2},x;b\right) = \frac{\sqrt{\pi}}{2\sqrt{b}}\left[e^{-2\sqrt{b}}\mathrm{erfc}\left(\sqrt{x}-\frac{\sqrt{b}}{\sqrt{x}}\right) - e^{2\sqrt{b}}\mathrm{erfc}\left(\sqrt{x}+\frac{\sqrt{b}}{\sqrt{x}}\right)\right], \tag{4.23}$$

we replace the GIGF in (4.22) by its representation in (4.23), arrive at (4.21). □

Corollary 4.2.3.

$$\left(\sqrt{b}\frac{\partial}{\partial b}+1\right)\{\mathrm{erfc}(x;b)\} = \mathrm{erfc}\left(x+\frac{\sqrt{b}}{x}\right). \tag{4.24}$$

Proof. (4.24) is obtained by adding (4.20) and (4.21). □

Corollary 4.2.4.

$$\left(\sqrt{b}\frac{\partial}{\partial b}-1\right)\{\mathrm{erfc}(x;b)\} = -\mathrm{erfc}\left(x-\frac{\sqrt{b}}{x}\right). \tag{4.25}$$

Proof. (4.25) is obtained by subtracting (4.20) from (4.21). □

Applications to heat conduction problem.

The heat conduction equation

$$\frac{\partial T}{\partial t} = \alpha\frac{\partial^2 T}{\partial x^2} \quad (x \geq 0, t \geq 0) \tag{4.26}$$

subject to conditions

$$T(x,0) = 0 \quad \text{and} \quad T(0,t) = F(t)$$

governs the variation of the temperature of a semi-infinite solid, initially at temperature zero, with the plane face at temperature $F(t)$. The solution to the problem is given by [27]

$$T = \frac{2}{\sqrt{\pi}} \int_{\frac{x}{\sqrt{4\alpha t}}}^{\infty} F\left(t - \frac{x^2}{4\alpha\beta^2}\right) \exp\left(-\beta^2\right) d\beta. \tag{4.27}$$

In particular, when $F(t) = T_0 \exp(\lambda t)$, we find

$$T = T_0 e^{\lambda t} \operatorname{erfc}\left(\frac{x}{\sqrt{4\alpha t}}; \frac{\lambda x^2}{4\alpha}\right). \tag{4.28}$$

4.3 Iterated Integral of the Generalized Complementary Error Function

Hartree [26] investigated the iterated integrals

$$i^{-1} \operatorname{erfc}(x) := \frac{-2}{\sqrt{\pi}} \exp\left(-x^2\right)$$

$$i^{0} \operatorname{erfc}(x) := \operatorname{erfc}(x),$$

$$i^{n} \operatorname{erfc}(x) := \int_x^{\infty} i^{n-1} \operatorname{erfc}(t)\, dt.$$

Some properties of the iterated integrals of the complementary error function are presented in ([25], [31],[37], [44]). One important property is the integral representation

$$i^{n} \operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} \frac{(t-x)^n}{n!} \exp(-t^2)\, dt. \tag{4.29}$$

The general recurrence relation is

$$i^n \operatorname{erfc}(x) = \frac{-x}{n}\; i^{n-1} \operatorname{erfc}(x) + \frac{1}{2n}\; i^{n-2} \operatorname{erfc}(x). \tag{4.30}$$

These functions are infinitely differentiable and their derivative is given by

$$\frac{d}{dx}\left(i^n \operatorname{erfc}(x)\right) = -i^{n-1} \operatorname{erfc}(x) \qquad (n = 0, 1, 2, \ldots). \tag{4.31}$$

The series representation is found to be

$$i^n \operatorname{erfc}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{2^{n-k} k!\, \Gamma\left(1 + \frac{n-k}{2}\right)} \qquad (n = 0, 1, 2, \ldots). \tag{4.32}$$

This can directly give for $x = 0$

$$i^n \operatorname{erfc}(0) = \frac{1}{2^n \Gamma\left(1 + \frac{n}{2}\right)} \qquad (n = 0, 1, 2, \ldots). \tag{4.33}$$

It is to be noted that these functions satisfy the following differential equation:

$$\left(D^2 + 2xD - 2n\right)\{i^n \operatorname{erfc}(x)\} = 0 \qquad (n = 0, 1, 2, \ldots). \tag{4.34}$$

The asymptotic expansions are

$$i^n \operatorname{erfc}(x) = \frac{e^{-\frac{1}{2}x^2} e^{-x(2n)^{\frac{1}{2}}}}{2^n \Gamma\left(1 + \frac{n}{2}\right)} \left[1 + 0\left(n^{-\frac{1}{2}}\right)\right] \tag{4.35}$$

$$(x \quad \text{bounded}, \quad n \to \infty),$$

$$i^n \operatorname{erfc}(x) \sim \frac{2}{\sqrt{\pi}} \frac{e^{-x^2}}{(2x)^{n+1}} \sum_{m=0}^{\infty} \frac{(-1)^m (2m+n)!}{n!\, m!\, (2x)^{2m}} \tag{4.36}$$

$$\left(x \to \infty, |\arg x| < \frac{3\pi}{4}\right).$$

The generalized error function provides a closed form solution to some heat conduction problem as shown in (4.28). The iterated integrals of the CEF are also useful in finding the closed form solutions of a variety of partial differential equations. For example, the solution to the heat conduction boundary value problem

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{\alpha} \frac{\partial u}{\partial t} + \frac{A_0}{k} = 0 \qquad (x > 0, t > 0), \tag{4.37}$$

with zero initial and surface temperature such that the heat is produced at a constant rate A_0 per unit time per unit volume for $t > 0$ in the region $a < x < b$, is discussed in [8, p. 79]. The temperature gradient at the surface is given by

$$u = \frac{A_0}{k} \sqrt{4\alpha t} \left[i\operatorname{erfc}\left(\frac{a}{\sqrt{4\alpha t}}\right) - i\operatorname{erfc}\left(\frac{b}{\sqrt{4\alpha t}}\right) \right]. \tag{4.38}$$

We refer to [8, pp. 79 – 80] for several other cases of the above problem. One does not find the iterated integrals of the GCEF in the literature. It is expected that the repeated integrals of the GCEF will also be found useful in scientific and engineering applications. In this part, we define and explore the properties of the iterated integrals of the GCEF. In analogy to the iterated integrals of the complementary error functions, one can define the iterated integrals of the GCEF as follows:

$$i^{-1} \operatorname{erfc}(x;b) := \frac{-2}{\sqrt{\pi}} \exp\left(-x^2 - \frac{b}{x^2}\right), \tag{4.39}$$

$$i^0 \operatorname{erfc}(x;b) := \operatorname{erfc}(x;b), \tag{4.40}$$

$$i^n \operatorname{erfc}(x;b) := \int_x^\infty i^{n-1} \operatorname{erfc}(t;b)\, dt. \tag{4.41}$$

The classical iterated integrals of the complementary error function $i^n\operatorname{erfc}(x)$ satisfy the important integral representation given by (4.29). One would like to know if the iterated integrals of the GCEF have an integral representation which reduces to the classical one as a special case.

Theorem 4.3.1. *(Integral Representation)*

$$i^n \operatorname{erfc}(x;b) = \frac{2}{\sqrt{\pi}} \int_x^\infty \frac{(t-x)^n}{n!} \exp\left(-t^2 - \frac{b}{t^2}\right) dt. \tag{4.42}$$

Proof. We show the result using mathematical induction. For $n = 1$,

$$\begin{aligned}
i^1 \operatorname{erfc}(x;b) &= \int_x^\infty \operatorname{erfc}(s;b)\, ds \\
&= \frac{2}{\sqrt{\pi}} \int_x^\infty \int_s^\infty \exp\left(-t^2 - \frac{b}{t^2}\right) dt\ ds \\
&= \frac{2}{\sqrt{\pi}} \int_x^\infty \int_x^t \exp\left(-t^2 - \frac{b}{t^2}\right) ds\ dt \\
&= \frac{2}{\sqrt{\pi}} \int_x^\infty \frac{(t-x)^1}{1!} \exp\left(-t^2 - \frac{b}{t^2}\right) dt.
\end{aligned}$$

Now, we assume (4.42) is true for $n = k$. This gives

$$i^k \operatorname{erfc}(x;b) = \frac{2}{\sqrt{\pi}} \int_x^\infty \frac{(t-x)^k}{k!} e^{-t^2 - \frac{b}{t^2}}\, dt$$

and so

$$
\begin{aligned}
i^{k+1}\operatorname{erfc}(x,b) &= \frac{2}{\sqrt{\pi}}\int_x^\infty\int_s^\infty \frac{(t-s)^k}{k!}\,e^{-t^2-\frac{b}{t^2}}\,dt\,ds \\
&= \frac{2}{\sqrt{\pi}}\int_x^\infty\int_x^t \frac{(t-s)^k}{k!}\,e^{-t^2-\frac{b}{t^2}}\,ds\,dt \\
&= \frac{2}{\sqrt{\pi}}\int_x^\infty \frac{(t-x)^{k+1}}{(k+1)!}\,e^{-t^2-\frac{b}{t^2}}\,dt.
\end{aligned}
$$

□

Theorem 4.3.1 can be used to get the following differentiation formula which reduces to (4.31) if we put $b = 0$.

Theorem 4.3.2. *(Differentiation Formula)*

$$
\frac{\partial}{\partial x}\left(i^n\operatorname{erfc}(x;b)\right) = -i^{n-1}\operatorname{erfc}(x;b) \qquad (n = 0,1,2,\ldots). \tag{4.43}
$$

Proof. Using Theorem 4.3.1, we have

$$
\begin{aligned}
\frac{\partial}{\partial x}\left(i^n\operatorname{erfc}(x;b)\right) &= \frac{\partial}{\partial x}\left(\frac{2}{\sqrt{\pi}}\int_x^\infty \frac{(t-x)^n}{n!}\exp\left(-t^2-\frac{b}{t^2}\right)dt\right) \\
&= \frac{2}{\sqrt{\pi}}\int_x^\infty \frac{\partial}{\partial x}\left(\frac{(t-x)^n}{n!}\exp\left(-t^2-\frac{b}{t^2}\right)\right)dt \\
&= -\frac{2}{\sqrt{\pi}}\int_x^\infty \frac{(t-x)^{n-1}}{(n-1)!}\exp\left(-t^2-\frac{b}{t^2}\right)dt \\
&= -i^{n-1}\operatorname{erfc}(x;b),
\end{aligned}
$$

where the interchange between the derivative and integral is allowed since the integral is absolutely convergent and the integrand vanishes at $t = x$. □

The following result gives a general series representation for the iterated integrals of GCEF which gives (4.32) and (4.33) as special cases.

Theorem 4.3.3. *(Series Representation)*

$$i^n \operatorname{erfc}(x,b) = \sum_{k=0}^{\infty} \frac{(-x)^k}{k!} \; \frac{{}_0F_1\left(-, \frac{1}{2} - \frac{1}{2}(n-k), b\right)}{2^{n-k}\Gamma\left(\frac{1}{2}(n-k)+1\right)}, \tag{4.44}$$

where ${}_0F_1(-,m,b)$ is the hypergeometric series defined by

$${}_0F_1(-,m,b) := \sum_{k=0}^{\infty} \frac{b^k}{k!\,(m)_k}.$$

Proof. Let $i^n \operatorname{erfc}(x,b) = \sum_{k=0}^{\infty} c_k x^k$, and the task is to determine c_k. Now the coefficients of the Maclaurin series are given by

$$c_k = \frac{1}{k!}\frac{\partial^k}{\partial x^k}\left\{i^n \operatorname{erfc}\right\}(0,b),$$

which implies

$$c_0 = i^n \operatorname{erfc}(0,b) = \frac{2}{\sqrt{\pi}n!}\int_0^{\infty} t^n\, e^{-t^2-\frac{b}{t^2}}\,dt.$$

Let $t^2 = y$, then $t = \sqrt{y}$ and $dt = \frac{1}{2\sqrt{y}}\,dy$. This gives

$$c_0 = \frac{1}{\sqrt{\pi}\,n!}\int_0^{\infty} y^{\frac{1}{2}n+\frac{1}{2}-1} e^{-y-\frac{b}{y}}\,dy = \frac{1}{\sqrt{\pi}n!}\Gamma_b\left(\frac{1}{2}n+\frac{1}{2}\right), \tag{4.45}$$

where, $\Gamma_b(\alpha)$ is the generalized gamma function. We claim:

$$\frac{1}{\sqrt{\pi}n!}\Gamma_b\left(\frac{1}{2}n+\frac{1}{2}\right) = \frac{1}{2^n\Gamma\left(\frac{1}{2}n+1\right)}\,{}_0F_1\left(-,\frac{1}{2}-\frac{n}{2},b\right).$$

To establish the claim, we use the identity

$$\Gamma_b(\alpha) = \sum_{k=0}^{\infty} \frac{(-b)^k}{k!} \Gamma(\alpha - k),$$

[10, page 47 (2.91)] with $\alpha = \frac{1}{2}n + \frac{1}{2}$. This gives

$$\frac{\Gamma_b\left(\frac{1}{2}n + \frac{1}{2}\right)}{\sqrt{\pi}\ n!} = \sum_{k=0}^{\infty} \frac{(-b)^k}{k!} \cdot \frac{\Gamma\left(\frac{1}{2}n + \frac{1}{2} - k\right)}{\sqrt{\pi}\, n!}.$$

Now, using the formula see [10, page 6 (1.51)]

$$\Gamma(\alpha - k) = \frac{\Gamma(\alpha)(-1)^k}{(1-\alpha)_k}$$

with $\alpha = \frac{1}{2}n + \frac{1}{2}$, we have

$$\Gamma\left(\frac{1}{2}n + \frac{1}{2} - k\right) = \frac{\Gamma\left(\frac{1}{2}n + \frac{1}{2}\right)(-1)^k}{\left(\frac{1}{2} - \frac{1}{2}n\right)_k}.$$

This implies

$$\frac{\Gamma\left(\frac{1}{2}n + \frac{1}{2} - k\right)}{\sqrt{\pi}\, n!} = \frac{(-1)^k}{\left(\frac{1}{2} - \frac{1}{2}n\right)_k} \cdot \frac{\Gamma\left(\frac{1}{2}n + \frac{1}{2}\right)}{\sqrt{\pi}\, n!}.$$

Now, by the duplication formula,

$$\frac{\Gamma\left(\frac{1}{2}n + \frac{1}{2}\right)}{\sqrt{\pi} \quad n!} = \frac{1}{2^n \Gamma\left(\frac{1}{2}n + 1\right)},$$

we find

$$\frac{\Gamma\left(\frac{1}{2}n + \frac{1}{2} - k\right)}{\sqrt{\pi}\, n!} = \frac{(-1)^k}{\left(\frac{1}{2} - \frac{1}{2}n\right)_k} \cdot \frac{1}{2^n \Gamma\left(\frac{1}{2}n + 1\right)}.$$

This implies

$$\begin{aligned}
\frac{\Gamma_b\left(\frac{1}{2}n+\frac{1}{2}\right)}{\sqrt{\pi}\,n!} &= \sum_{k=0}^{\infty}\frac{(-b)^k}{k!}\frac{(-1)^k}{\left(\frac{1}{2}-\frac{1}{2}n\right)_k}\frac{1}{2^n\Gamma\left(\frac{1}{2}n+1\right)} \\
&= \frac{1}{2^n\Gamma\left(\frac{1}{2}n+1\right)}\sum_{k=0}^{\infty}\frac{b^k}{k!\left(\frac{1}{2}-\frac{1}{2}n\right)_k} \\
&= \frac{1}{2^n\Gamma\left(\frac{1}{2}n+1\right)} \quad {}_0F_1\left(-,\frac{1}{2}-\frac{1}{2}n,b\right).
\end{aligned}$$

This proves the claim. Hence from (4.45), we have

$$c_0 = \frac{{}_0F_1\left(-,\frac{1}{2}-\frac{1}{2}n,b\right)}{2^n\Gamma\left(\frac{1}{2}n+1\right)}.$$

Now, using (3.37)

$$\frac{\partial}{\partial x}\left(i^n\operatorname{erfc}(x,b)\right) = -i^{n-1}\operatorname{erfc}(x,b) \quad (n=0,1,2,\ldots),$$

we get,

$$\frac{\partial^k}{\partial x^k}\left(i^n\operatorname{erfc}(x,b)\right) = (-1)^k i^{n-k}\operatorname{erfc}(x,b).$$

This gives

$$c_k = \frac{(-1)^k\, i^{n-k}\operatorname{erfc}(0,b)}{k!}.$$

Hence,

$$c_k = \frac{(-1)^k}{k!}\frac{{}_0F_1\left(-,\frac{1}{2}-\frac{1}{2}(n-k),b\right)}{2^{n-k}\Gamma\left(\frac{1}{2}(n-k)+1\right)},$$

□

Corollary 4.3.4.

$$i^n \operatorname{erfc}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{2^{n-k} k! \Gamma\left(1 + \frac{n-k}{2}\right)} \qquad (n = 0, 1, 2, \ldots).$$

Proof. Take $b = 0$ in (4.44) and note that ${}_0F_1\left(-, \frac{1}{2} - \frac{1}{2}(n-k), 0\right) = 1$. □

Corollary 4.3.5.

$$i^n \operatorname{erfc}(0; b) = \frac{{}_0F_1\left(-, \frac{1}{2} - \frac{1}{2}n; b\right)}{2^n \Gamma\left(1 + \frac{n}{2}\right)} \qquad (n = 0, 1, 2, \ldots).$$

Proof. Take $x = 0$ in (4.44). □

Corollary 4.3.6.

$$i^n \operatorname{erfc}(0, 0) = \frac{1}{2^n \Gamma\left(1 + \frac{n}{2}\right)} \qquad (n = 0, 1, 2, \ldots).$$

Proof. Take $x = 0$ and $b = 0$ in (4.44). □

The recurrence relation given in (4.30) for the iterated integrals of the complementary error function is extended for the GCEF in the following result.

Theorem 4.3.7. *(Recurrence Relation)*

Let

$$A = \left(1 - b\frac{\partial^2}{\partial b^2}\right)$$

be the differential operator. Then

$$2nAi^n \operatorname{erfc}(x; b) = -2Axi^{n-1} \operatorname{erfc}(x; b) + i^{n-2} \operatorname{erfc}(x; b). \tag{4.46}$$

Proof. By replacing $i^n \operatorname{erfc}(x;b)$ with its integral representation given by Theorem 4.29, we have

$$\begin{aligned}
\text{L.H.S.} &= 2nAi^n \operatorname{erfc}(x;b) \\
&= \frac{2}{\sqrt{\pi}}\,\frac{2n}{n!}\left(1-b\frac{\partial^2}{\partial b^2}\right)\int_x^\infty (t-x)^n\, e^{-t^2-\frac{b}{t^2}}\,dt \\
&= \frac{2}{\sqrt{\pi}}\frac{2}{(n-1)!}\left(1-b\frac{\partial^2}{\partial b^2}\right)\int_x^\infty (t-x)(t-x)^{n-1}\, e^{-t^2-\frac{b}{t^2}}\,dt \\
&= \frac{2}{\sqrt{\pi}}\frac{2}{(n-1)!}\left(1-b\frac{\partial^2}{\partial b^2}\right)\int_x^\infty t(t-x)^{n-1}\, e^{-t^2-\frac{b}{t^2}}\,dt \\
&\quad -\frac{2}{\sqrt{\pi}}\frac{2}{(n-1)!}\left(1-b\frac{\partial^2}{\partial b^2}\right)\int_x^\infty x(t-x)^{n-1}e^{-t^2-\frac{b}{t^2}}\,dt \\
&= I_n - 2Ax\,i^{n-1}\operatorname{erfc}(x;b),
\end{aligned}$$

where

$$I_n = \frac{2}{\sqrt{\pi}}\frac{2}{(n-1)!}\left(1-b\frac{\partial^2}{\partial b^2}\right)\int_x^\infty t(t-x)^{n-1}\, e^{-t^2-\frac{b}{t^2}}\,dt.$$

To complete the proof, we need to show that $I_n = i^{n-2}\operatorname{erfc}(x;b)$. Indeed, applying the differential operator to the integral and then using integration by parts, we have

$$\begin{aligned}
I_n &= \frac{2}{\sqrt{\pi}}\frac{1}{(n-1)!}\left[\int_x^\infty 2t(t-x)^{n-1}\, e^{-t^2-\frac{b}{t^2}}\,dt - \int_x^\infty \frac{2b}{t^3}(t-x)^{n-1}\, e^{-t^2-\frac{b}{t^2}}\,dt\right] \\
&= \frac{2}{\sqrt{\pi}}\cdot\frac{1}{(n-1)!}\left[-\int_x^\infty\left(-2t+\frac{2b}{t^3}\right)e^{-t^2-\frac{b}{t^2}}(t-x)^{n-1}\,dt\right] \\
&= \frac{-2}{\sqrt{\pi}}\cdot\frac{1}{(n-1)!}\left[e^{-t^2-\frac{b}{t^2}}(t-x)^{n-1}\Big|_{t=x}^{\infty} - \int_x^\infty (n-1)(t-x)^{n-2}e^{-t^2-\frac{b}{t^2}}\,dt\right] \\
&= \frac{2}{\sqrt{\pi}}\int_x^\infty \frac{(t-x)^{n-2}}{(n-2)!}\,e^{-t^2-\frac{b}{t^2}}\,dt \\
&= i^{n-2}\operatorname{erfc}(x;b).
\end{aligned}$$

□

One would like to find a differential equation whose solution can be expressed in terms of the iterated integrals of the GCEF. This will enhance the applications of these functions in the areas where such type of differential equations may arise. We prove in the following theorem that these functions satisfy a partial differential equation. Moreover, from the new partial differential equation, we prove that for $b = 0$, the classical differential equation is recovered.

Theorem 4.3.8. *(Differential Equation)*

$$\left[D^2 + 2xD\left(1 - b\frac{\partial^2}{\partial b^2}\right) - 2n\left(1 - b\frac{\partial^2}{\partial b^2}\right)\right] i^n \operatorname{erfc}(x;b) = 0. \tag{4.47}$$

Proof. In order to show (3.16), it suffices to show that

$$\left[D^2 + 2xD\left(1 - b\frac{\partial^2}{\partial b^2}\right)\right] i^n \operatorname{erfc}(x;b) = 2n\left(1 - b\frac{\partial^2}{\partial b^2}\right) i^n \operatorname{erfc}(x;b). \tag{4.48}$$

Note that by using Theorem 4.3.2, we have

$$D^2(i^n \operatorname{erfc}(x;b)) = i^{n-2} \operatorname{erfc}(x;b), \tag{4.49}$$

and

$$D(i^n \operatorname{erfc}(x;b)) = -i^{n-1} \operatorname{erfc}(x;b). \tag{4.50}$$

Now by substituting (4.49) and (4.50) in the left hand side of (4.48), the result follows directly from Theorem 4.35. □

Remark 4.3.9. *One of the important features of the above theorem is that, as expected, we recover the classical differential equation satisfied by the iterated integrals of the complementary error function. This is achieved from Theorem 4.3.8 when we set $b = 0$.*

4.4 Numerical Computations and Graphical Representations

In this section, we present some graphical and tabular representations of the generalized error function, GCEF, and the iterated integrals of GCEF for scientific and engineering applications.

Numerical Computation

In order to compute the GCEF and its iterated integrals, we use their integral representations given by (4.29) and (3.36), respectively. To compute the generalized error function we use (4.19). In order to evaluate the infinite integral,

$$\operatorname{erfc}(x;b) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} exp\left(-t^2 - bt^{-2}\right) dt, \tag{4.51}$$

we decompose the integral

$$\int_x^{\infty} exp\left(-t^2 - bt^{-2}\right) dt, \tag{4.52}$$

into

$$\int_x^{20} exp\left(-t^2 - bt^{-2}\right) dt + \int_{20}^{\infty} exp\left(-t^2 - bt^{-2}\right) dt. \tag{4.53}$$

To compute the integral $\int_x^{20} exp\left(-t^2 - bt^{-2}\right) dt$, we use 100-points Gauss Quadrature rule with 100 nodes [14, 365]. That is, we approximate

$$\int_x^{20} exp\left(-t^2 - bt^{-2}\right) dt \tag{4.54}$$

by

$$\sum_{i=1}^{100} w_i \, exp\left(-t^2 - bt^{-2}\right) \tag{4.55}$$

where w_i and t_i are the weights and the nodes, respectively. It is to be noted that w_i and t_i can be determined by solving the following system of equations:

$$\int_x^{20} t^m\,dt = \sum_{i=1}^{100} w_i t_i^m \tag{4.56}$$

for m = 0, 1,..., 199. However, a simple manipulation shows that

$$\int_{20}^{\infty} exp\left(-t^2 - bt^{-2}\right) dt \le 10^{-8} \tag{4.57}$$

and

$$\int_{20}^{\infty} \frac{(t-x)^n}{n!} \exp\left(-t^2 - \frac{b}{t^2}\right) dt \le 10^{-8}. \tag{4.58}$$

Remark: The r-points Gauss Quadrature is exact for all polynomials of degree $\le 2r-1$. We use the same procedure to evaluate the integral:

$$i^n \operatorname{erfc}(x;b) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} \frac{(t-x)^n}{n!} \exp\left(-t^2 - \frac{b}{t^2}\right) dt. \tag{4.59}$$

From Figure 4.3– Figure 4.7 of the iterated integrals of the GCEF we note that, for each fixed x, the function is decreasing as b increases. Moreover, all graphs of the iterated integrals of the GCEF approaching zero for sufficiently large x . It is also observed that the rate of decaying increases as b decreases and n increases. For example, for $b = 0$ and $x = 0.5$, if x increases by 10 % the decay rate of $i\operatorname{erfc}(x;0)$ is 11.5%. However, for the same value of $x = 0.5$ the decay rate of $i\operatorname{erfc}(x;0.25)$ reduces to 10%. It is important to note that for $n > 1$ the decay rate becomes faster. For instance, when $n = 2$ at $x = 0.5$, an increase of 10% in the value of x leads to 13.4% decay in the value of $i^2\operatorname{erfc}(x;0)$ and 12.7% for the function $i^2\operatorname{erfc}(x;0.25)$.

Table 4.1: Some representative values of erfc (x, b)

$x\backslash b$	0.00	0.25	0.5	0.75	1.00	1.25	1.50
0.00	1.0000	0.3679	0.2431	0.1769	0.1353	0.1069	0.0863
0.01	0.9887	0.3679	0.2431	0.1769	0.1353	0.1069	0.0863
0.02	0.9774	0.3679	0.2431	0.1769	0.1353	0.1069	0.0863
0.03	0.9662	0.3679	0.2431	0.1769	0.1353	0.1069	0.0863
0.04	0.9549	0.3679	0.2431	0.1769	0.1353	0.1069	0.0863
0.05	0.9436	0.3679	0.2431	0.1769	0.1353	0.1069	0.0863
0.06	0.9324	0.3679	0.2431	0.1769	0.1353	0.1069	0.0863
0.07	0.9211	0.3679	0.2431	0.1769	0.1353	0.1069	0.0863
0.08	0.9099	0.3679	0.2431	0.1769	0.1353	0.1069	0.0863
0.09	0.8987	0.3679	0.2431	0.1769	0.1353	0.1069	0.0863
0.10	0.8875	0.3679	0.2431	0.1769	0.1353	0.1069	0.0863
0.15	0.8320	0.3679	0.2431	0.1769	0.1353	0.1069	0.0863
0.20	0.7773	0.3679	0.2431	0.1769	0.1353	0.1069	0.0863
0.25	0.7237	0.3674	0.2431	0.1769	0.1353	0.1069	0.0863
0.30	0.6714	0.3654	0.2430	0.1769	0.1353	0.1069	0.0863
0.35	0.6206	0.3607	0.2426	0.1769	0.1353	0.1069	0.0863
0.40	0.5716	0.3524	0.2411	0.1766	0.1353	0.1069	0.0863
0.45	0.5245	0.3406	0.2382	0.1759	0.1351	0.1068	0.0863
0.50	0.4795	0.3257	0.2332	0.1742	0.1345	0.1066	0.0863
0.55	0.4367	0.3085	0.2263	0.1714	0.1334	0.1062	0.0861
0.60	0.3961	0.2895	0.2173	0.1672	0.1314	0.1052	0.0856
0.65	0.3580	0.2694	0.2067	0.1616	0.1285	0.1037	0.0848
0.70	0.3222	0.2487	0.1948	0.1547	0.1245	0.1014	0.0835
0.75	0.2888	0.2280	0.1820	0.1467	0.1195	0.0983	0.0815
0.80	0.2579	0.2076	0.1685	0.1379	0.1137	0.0944	0.0790
0.85	0.2293	0.1878	0.1548	0.1284	0.1071	0.0899	0.0759
0.90	0.2031	0.1689	0.1412	0.1185	0.1000	0.0848	0.0722
0.95	0.1791	0.1510	0.1278	0.1086	0.0926	0.0792	0.0680
1.00	0.1573	0.1343	0.1149	0.0987	0.0849	0.0733	0.0635
1.50	0.0339	0.0311	0.0286	0.0263	0.0242	0.0223	0.0205
2.00	0.0047	0.0044	0.0042	0.0040	0.0038	0.0036	0.0034
2.50	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003	0.0003
3.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
3.50	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
4.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
4.50	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
5.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 4.2: Some representative values of erf (x, b)

$x\backslash b$	0.00	0.25	0.5	0.75	1.00	1.25	1.50
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.01	0.0113	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.02	0.0226	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.03	0.0338	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.04	0.0451	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.05	0.0564	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.06	0.0676	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.07	0.0789	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.08	0.0901	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.09	0.1013	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.10	0.1125	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.15	0.1680	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.20	0.2227	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.25	0.2763	0.0005	0.0000	0.0000	0.0000	0.0000	0.0000
0.30	0.3286	0.0024	0.0001	0.0000	0.0000	0.0000	0.0000
0.35	0.3794	0.0072	0.0006	0.0001	0.0000	0.0000	0.0000
0.40	0.4284	0.0155	0.0020	0.0003	0.0000	0.0000	0.0000
0.45	0.4755	0.0273	0.0050	0.0011	0.0002	0.0001	0.0000
0.50	0.5205	0.0421	0.0099	0.0027	0.0008	0.0002	0.0001
0.55	0.5633	0.0594	0.0168	0.0055	0.0019	0.0007	0.0003
0.60	0.6039	0.0784	0.0258	0.0097	0.0039	0.0016	0.0007
0.65	0.6420	0.0985	0.0364	0.0153	0.0069	0.0032	0.0015
0.70	0.6778	0.1191	0.0483	0.0222	0.0108	0.0055	0.0029
0.75	0.7112	0.1399	0.0612	0.0302	0.0158	0.0086	0.0048
0.80	0.7421	0.1603	0.0746	0.0391	0.0217	0.0125	0.0073
0.85	0.7707	0.1800	0.0883	0.0485	0.0282	0.0170	0.0105
0.90	0.7969	0.1990	0.1019	0.0584	0.0353	0.0221	0.0142
0.95	0.8209	0.2169	0.1153	0.0684	0.0428	0.0277	0.0183
1.00	0.8427	0.2336	0.1282	0.0783	0.0504	0.0335	0.0228
1.50	0.9661	0.3367	0.2145	0.1506	0.1111	0.0846	0.0658
2.00	0.9953	0.3634	0.2389	0.1729	0.1315	0.1033	0.0829
2.50	0.9996	0.3675	0.2427	0.1766	0.135	0.1065	0.0860
3.00	1.0000	0.3679	0.2431	0.1769	0.1353	0.1069	0.0863
3.50	1.0000	0.3679	0.2431	0.1769	0.1353	0.1069	0.0863
4.00	1.0000	0.3679	0.2431	0.1769	0.1353	0.1069	0.0863
4.50	1.0000	0.3679	0.2431	0.1769	0.1353	0.1069	0.0863
5.00	1.0000	0.3679	0.2431	0.1769	0.1353	0.1069	0.0863

Table 4.3: Some representative values of i^1erfc (x, b)

$x\backslash b$	0.00	0.25	0.5	0.75	1.00	1.25	1.50
0.00	0.5642	0.3396	0.2507	0.1958	0.1578	0.1300	0.1088
0.01	0.5542	0.3359	0.2483	0.1940	0.1565	0.1289	0.1079
0.02	0.5444	0.3322	0.2458	0.1922	0.1551	0.1278	0.1070
0.03	0.5347	0.3286	0.2434	0.1905	0.1538	0.1268	0.1062
0.04	0.5251	0.3249	0.2410	0.1887	0.1524	0.1257	0.1053
0.05	0.5156	0.3212	0.2385	0.1869	0.1511	0.1246	0.1044
0.06	0.5062	0.3175	0.2361	0.1852	0.1497	0.1236	0.1036
0.07	0.4970	0.3138	0.2337	0.1834	0.1483	0.1225	0.1027
0.08	0.4878	0.3102	0.2312	0.1816	0.1470	0.1214	0.1018
0.09	0.4788	0.3065	0.2288	0.1798	0.1456	0.1204	0.1010
0.10	0.4698	0.3028	0.2264	0.1781	0.1443	0.1193	0.1001
0.15	0.4268	0.2844	0.2142	0.1692	0.1375	0.1139	0.0958
0.20	0.3866	0.2660	0.2021	0.1604	0.1308	0.1086	0.0915
0.25	0.3491	0.2476	0.1899	0.1515	0.1240	0.1033	0.0872
0.30	0.3142	0.2293	0.1778	0.1427	0.1172	0.0979	0.0829
0.35	0.2819	0.2111	0.1656	0.1339	0.1105	0.0926	0.0785
0.40	0.2521	0.1933	0.1535	0.1250	0.1037	0.0872	0.0742
0.45	0.2247	0.1760	0.1415	0.1162	0.0969	0.0819	0.0699
0.50	0.1996	0.1593	0.1297	0.1074	0.0902	0.0766	0.0656
0.55	0.1767	0.1434	0.1182	0.0988	0.0835	0.0712	0.0613
0.60	0.1559	0.1285	0.1071	0.0903	0.0769	0.0659	0.0570
0.65	0.1371	0.1145	0.0965	0.0821	0.0704	0.0607	0.0527
0.70	0.1201	0.1015	0.0865	0.0742	0.0640	0.0556	0.0485
0.75	0.1048	0.0896	0.0771	0.0666	0.0579	0.0506	0.0444
0.80	0.0912	0.0787	0.0683	0.0595	0.0521	0.0458	0.0404
0.85	0.0790	0.0688	0.0602	0.0529	0.0466	0.0412	0.0365
0.90	0.0682	0.0599	0.0528	0.0467	0.0414	0.0368	0.0328
0.95	0.0587	0.0519	0.0461	0.0410	0.0366	0.0327	0.0293
1.00	0.0503	0.0448	0.0400	0.0358	0.0321	0.0289	0.0260
1.50	0.0086	0.0081	0.0075	0.0070	0.0066	0.0061	0.0057
2.00	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0007
2.50	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001
3.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
3.50	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
4.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
4.50	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
5.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 4.4: Some representative values of i^2erfc (x, b)

$x\backslash b$	0.00	0.25	0.5	0.75	1.00	1.25	1.50
0.00	0.2500	0.1839	0.1467	0.1208	0.1015	0.0865	0.0745
0.01	0.2444	0.1806	0.1442	0.1189	0.0999	0.0852	0.0734
0.02	0.2389	0.1772	0.1418	0.1170	0.0984	0.0839	0.0723
0.03	0.2335	0.1739	0.1393	0.1150	0.0968	0.0826	0.0712
0.04	0.2282	0.1707	0.1369	0.1132	0.0953	0.0814	0.0702
0.05	0.2230	0.1674	0.1345	0.1113	0.0938	0.0801	0.0691
0.06	0.2179	0.1642	0.1321	0.1094	0.0923	0.0789	0.0681
0.07	0.2129	0.1611	0.1298	0.1076	0.0908	0.0776	0.0671
0.08	0.2080	0.1579	0.1275	0.1057	0.0893	0.0764	0.0660
0.09	0.2031	0.1549	0.1252	0.1039	0.0878	0.0752	0.0650
0.10	0.1984	0.1518	0.1229	0.1021	0.0864	0.0740	0.0640
0.15	0.1760	0.1371	0.1119	0.0935	0.0794	0.0682	0.0591
0.20	0.1557	0.1234	0.1015	0.0852	0.0726	0.0626	0.0544
0.25	0.1373	0.1105	0.0917	0.0774	0.0663	0.0573	0.0500
0.30	0.1207	0.0986	0.0825	0.0701	0.0602	0.0523	0.0457
0.35	0.1058	0.0876	0.0739	0.0632	0.0546	0.0475	0.0417
0.40	0.0925	0.0775	0.0659	0.0567	0.0492	0.0430	0.0379
0.45	0.0806	0.0683	0.0585	0.0507	0.0442	0.0388	0.0343
0.50	0.0700	0.0599	0.0517	0.0451	0.0395	0.0348	0.0309
0.55	0.0606	0.0523	0.0455	0.0399	0.0352	0.0311	0.0277
0.60	0.0523	0.0455	0.0399	0.0352	0.0312	0.0277	0.0247
0.65	0.0449	0.0395	0.0348	0.0309	0.0275	0.0245	0.0220
0.70	0.0385	0.0341	0.0303	0.0270	0.0241	0.0216	0.0195
0.75	0.0329	0.0293	0.0262	0.0234	0.0211	0.0190	0.0171
0.80	0.0280	0.0251	0.0225	0.0203	0.0183	0.0166	0.0150
0.88	0.0238	0.0214	0.0193	0.0175	0.0159	0.0144	0.0131
0.90	0.0201	0.0182	0.0165	0.0150	0.0137	0.0125	0.0114
0.95	0.0169	0.0154	0.0140	0.0128	0.0117	0.0107	0.0098
1.00	0.0142	0.0130	0.0119	0.0109	0.0100	0.0092	0.0084
1.50	0.0020	0.0019	0.0018	0.0017	0.0016	0.0015	0.0014
2.00	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002
2.50	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
3.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
3.50	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
4.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
4.50	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
5.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 4.5: Some representative values of i^3erfc (x, b)

$x\backslash b$	0.00	0.25	0.5	0.75	1.00	1.25	1.50
0.00	0.0940	0.0764	0.0643	0.0550	0.0477	0.0418	0.0368
0.01	0.0916	0.0746	0.0628	0.0538	0.0467	0.0409	0.0361
0.02	0.0891	0.0728	0.0614	0.0527	0.0457	0.0401	0.0353
0.03	0.0868	0.0710	0.0600	0.0515	0.0447	0.0392	0.0346
0.04	0.0845	0.0693	0.0586	0.0504	0.0438	0.0384	0.0339
0.05	0.0822	0.0676	0.0572	0.0492	0.0428	0.0376	0.0332
0.06	0.0800	0.0660	0.0559	0.0481	0.0419	0.0368	0.0325
0.07	0.0779	0.0643	0.0546	0.0471	0.0410	0.0360	0.0318
0.08	0.0758	0.0627	0.0533	0.0460	0.0401	0.0352	0.0312
0.09	0.0737	0.0612	0.0520	0.0449	0.0392	0.0345	0.0305
0.10	0.0717	0.0596	0.0508	0.0439	0.0383	0.0337	0.0299
0.15	0.0623	0.0524	0.0449	0.0390	0.0342	0.0302	0.0268
0.20	0.0541	0.0459	0.0396	0.0346	0.0304	0.0269	0.0240
0.25	0.0467	0.0401	0.0348	0.0305	0.0269	0.0239	0.0214
0.30	0.0403	0.0348	0.0304	0.0268	0.0238	0.0212	0.0190
0.35	0.0346	0.0302	0.0265	0.0235	0.0209	0.0187	0.0168
0.40	0.0297	0.0261	0.0230	0.0205	0.0183	0.0164	0.0148
0.45	0.0254	0.0224	0.0199	0.0178	0.0160	0.0144	0.0130
0.50	0.0216	0.0192	0.0172	0.0154	0.0139	0.0125	0.0114
0.55	0.0184	0.0164	0.0147	0.0133	0.0120	0.0109	0.0099
0.60	0.0155	0.0140	0.0126	0.0114	0.0104	0.0094	0.0086
0.65	0.0131	0.0119	0.0107	0.0098	0.0089	0.0081	0.0074
0.70	0.0110	0.0100	0.0091	0.0083	0.0076	0.0070	0.0064
0.75	0.0092	0.0084	0.0077	0.0071	0.0065	0.0060	0.0055
0.80	0.0077	0.0071	0.0065	0.0060	0.0055	0.0051	0.0047
0.85	0.0064	0.0059	0.0055	0.0050	0.0046	0.0043	0.0040
0.90	0.0053	0.0049	0.0046	0.0042	0.0039	0.0036	0.0034
0.95	0.0044	0.0041	0.0038	0.0035	0.0033	0.0030	0.0028
1.00	0.0036	0.0034	0.0031	0.0029	0.0027	0.0025	0.0024
1.50	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003	0.0003
2.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
2.50	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
3.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
3.50	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
4.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
4.50	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
5.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 4.6: Some representative values of i^4erfc (x, b)

$x\backslash b$	0.00	0.25	0.5	0.75	1.00	1.25	1.50
0.00	0.0312	0.0268	0.0234	0.0206	0.0183	0.0164	0.0147
0.01	0.0303	0.0261	0.0228	0.0201	0.0179	0.0160	0.0143
0.02	0.0294	0.0253	0.0222	0.0196	0.0174	0.0156	0.0140
0.03	0.0285	0.0246	0.0215	0.0190	0.0169	0.0152	0.0136
0.04	0.0277	0.0239	0.0210	0.0185	0.0165	0.0148	0.0133
0.05	0.0268	0.0232	0.0204	0.0180	0.0161	0.0144	0.0130
0.06	0.0260	0.0226	0.0198	0.0175	0.0156	0.0140	0.0126
0.07	0.0252	0.0219	0.0193	0.0171	0.0152	0.0137	0.0123
0.08	0.0245	0.0213	0.0187	0.0166	0.0148	0.0133	0.0120
0.09	0.0237	0.0207	0.0182	0.0161	0.0144	0.0130	0.0117
0.10	0.0230	0.0200	0.0177	0.0157	0.0140	0.0126	0.0114
0.15	0.0197	0.0173	0.0153	0.0136	0.0122	0.0110	0.0100
0.20	0.0168	0.0148	0.0132	0.0118	0.0106	0.0096	0.0087
0.25	0.0142	0.0126	0.0113	0.0102	0.0092	0.0083	0.0076
0.30	0.0121	0.0108	0.0097	0.0087	0.0079	0.0072	0.0066
0.35	0.0102	0.0092	0.0083	0.0075	0.0068	0.0062	0.0057
0.40	0.0086	0.0078	0.0070	0.0064	0.0058	0.0053	0.0049
0.45	0.0072	0.0065	0.0060	0.0054	0.0050	0.0045	0.0042
0.50	0.0060	0.0055	0.0050	0.0046	0.0042	0.0039	0.0036
0.55	0.0050	0.0046	0.0042	0.0039	0.0036	0.0033	0.0030
0.60	0.0042	0.0039	0.0035	0.0033	0.0030	0.0028	0.0026
0.65	0.0035	0.0032	0.0030	0.0027	0.0025	0.0023	0.0022
0.70	0.0029	0.0027	0.0025	0.0023	0.0021	0.0020	0.0018
0.75	0.0024	0.0022	0.0020	0.0019	0.0018	0.0016	0.0015
0.80	0.0020	0.0018	0.0017	0.0016	0.0015	0.0014	0.0013
0.85	0.0016	0.0015	0.0014	0.0013	0.0012	0.0011	0.0011
0.90	0.0013	0.0012	0.0011	0.0011	0.0010	0.0009	0.0009
0.95	0.0011	0.0010	0.0009	0.0009	0.0008	0.0008	0.0007
1.00	0.0009	0.0008	0.0008	0.0007	0.0007	0.0006	0.0006
1.50	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001
2.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
2.50	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
3.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
3.50	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
4.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
4.50	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
5.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 4.7: Some representative values of i^5erfc (x, b)

$x\backslash b$	0.00	0.25	0.5	0.75	1.00	1.25	1.50
0.00	0.0094	0.0083	0.0075	0.0067	0.0061	0.0055	0.0050
0.01	0.0091	0.0081	0.0072	0.0065	0.0059	0.0054	0.0049
0.02	0.0088	0.0078	0.0070	0.0063	0.0057	0.0052	0.0048
0.03	0.0085	0.0076	0.0068	0.0061	0.0056	0.0051	0.0046
0.04	0.0082	0.0073	0.0066	0.0059	0.0054	0.0049	0.0045
0.05	0.0080	0.0071	0.0064	0.0058	0.0052	0.0048	0.0043
0.06	0.0077	0.0069	0.0062	0.0056	0.0051	0.0046	0.0042
0.07	0.0074	0.0066	0.0060	0.0054	0.0049	0.0045	0.0041
0.08	0.0072	0.0064	0.0058	0.0052	0.0048	0.0043	0.0040
0.09	0.0069	0.0062	0.0056	0.0051	0.0046	0.0042	0.0039
0.10	0.0067	0.0060	0.0054	0.0049	0.0045	0.0041	0.0037
0.15	0.0056	0.0051	0.0046	0.0042	0.0038	0.0035	0.0032
0.20	0.0047	0.0043	0.0039	0.0036	0.0033	0.0030	0.0027
0.25	0.0040	0.0036	0.0033	0.0030	0.0028	0.0025	0.0023
0.30	0.0033	0.0030	0.0028	0.0025	0.0023	0.0021	0.0020
0.35	0.0028	0.0025	0.0023	0.0021	0.0020	0.0018	0.0017
0.40	0.0023	0.0021	0.0019	0.0018	0.0016	0.0015	0.0014
0.45	0.0019	0.0017	0.0016	0.0015	0.0014	0.0013	0.0012
0.50	0.0016	0.0014	0.0013	0.0012	0.0012	0.0011	0.0010
0.55	0.0013	0.0012	0.0011	0.0010	0.0010	0.0009	0.0008
0.60	0.0010	0.0010	0.0009	0.0008	0.0008	0.0007	0.0007
0.65	0.0009	0.0008	0.0007	0.0007	0.0007	0.0006	0.0006
0.70	0.0007	0.0007	0.0006	0.0006	0.0005	0.0005	0.0005
0.75	0.0006	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004
0.80	0.0005	0.0004	0.0004	0.0004	0.0004	0.0003	0.0003
0.85	0.0004	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003
0.90	0.0003	0.0003	0.0003	0.0003	0.0002	0.0002	0.0002
0.95	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002
1.00	0.0002	0.0002	0.0002	0.0002	0.0002	0.0001	0.0001
1.50	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
2.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
2.50	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
3.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
3.50	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
4.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
4.50	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
5.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

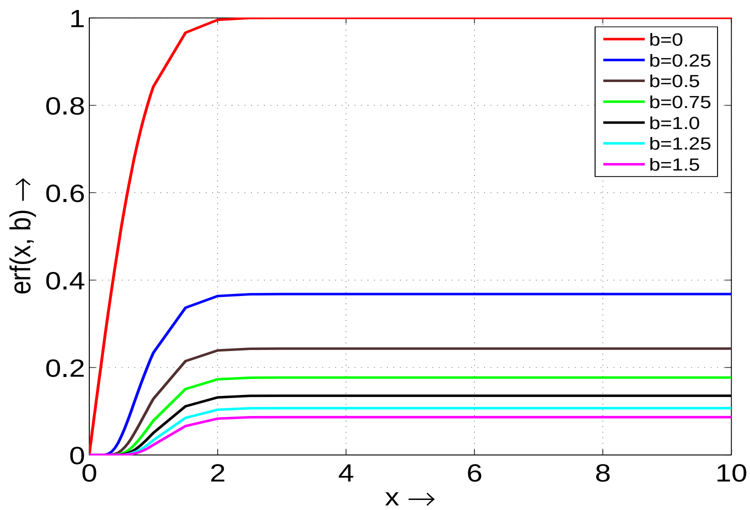

Figure 4.1: The graphical representation of the generalized error function for different values of b

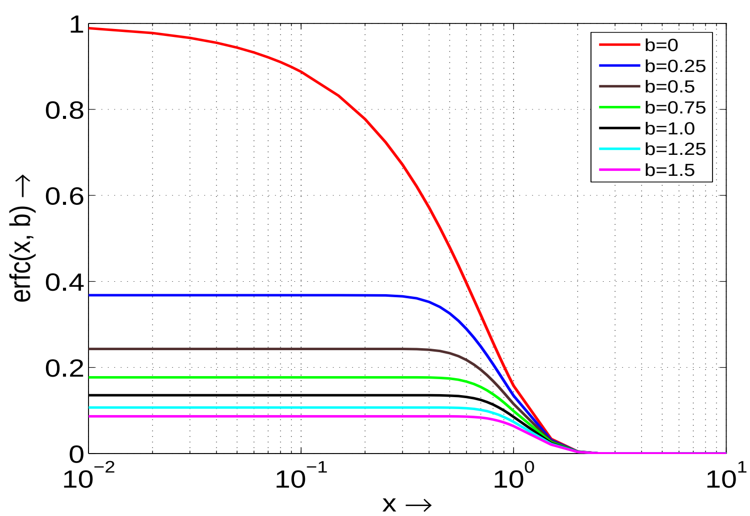

Figure 4.2: The graphical representation of the generalized complementary error function for different values of b

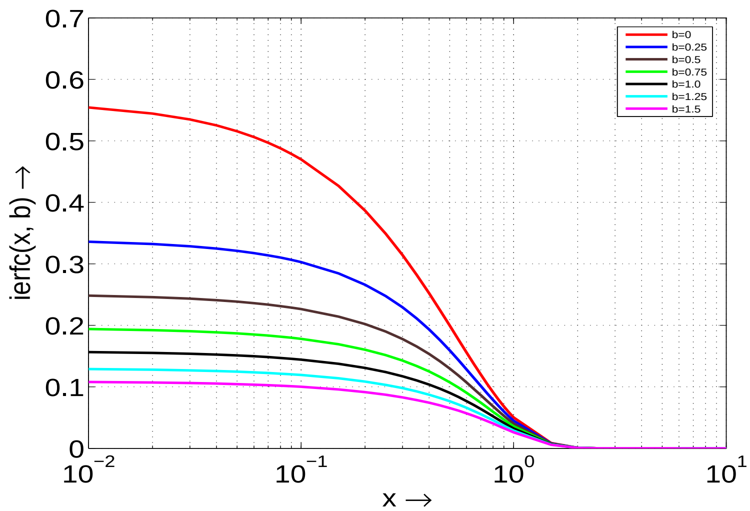

Figure 4.3: The graphical representation of the i^1erfc (x, b)function for different values of b

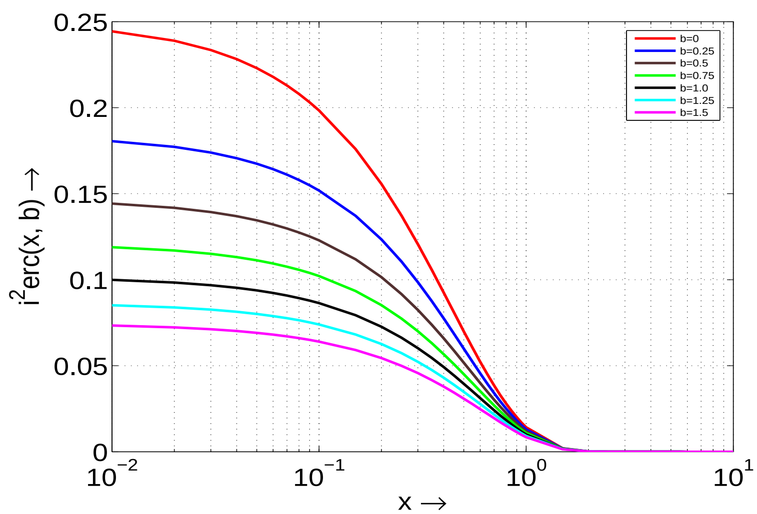

Figure 4.4: The graphical representation of the i^2erfc (x, b) function for different values of b

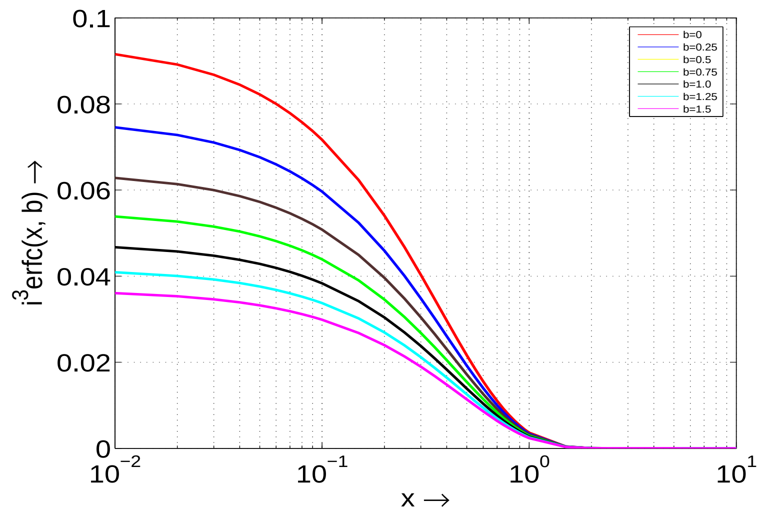

Figure 4.5: The graphical representation of the i^3erfc (x, b) function for different values of b

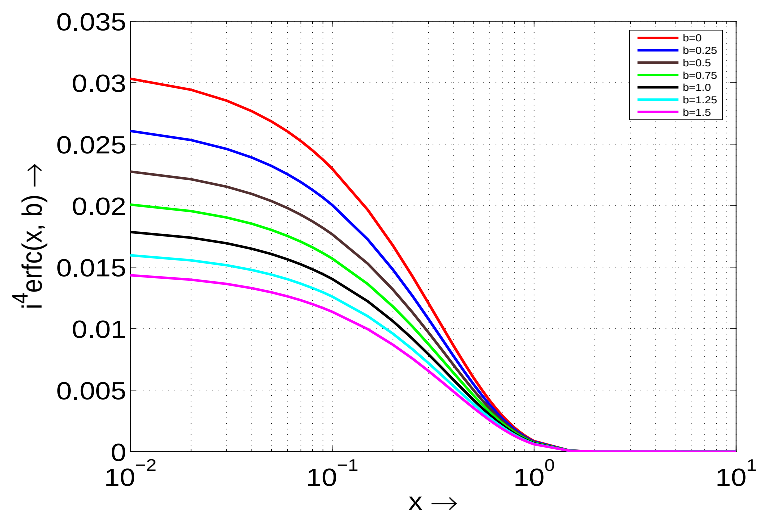

Figure 4.6: The graphical representation of the i^4erfc (x, b) function for different values of b

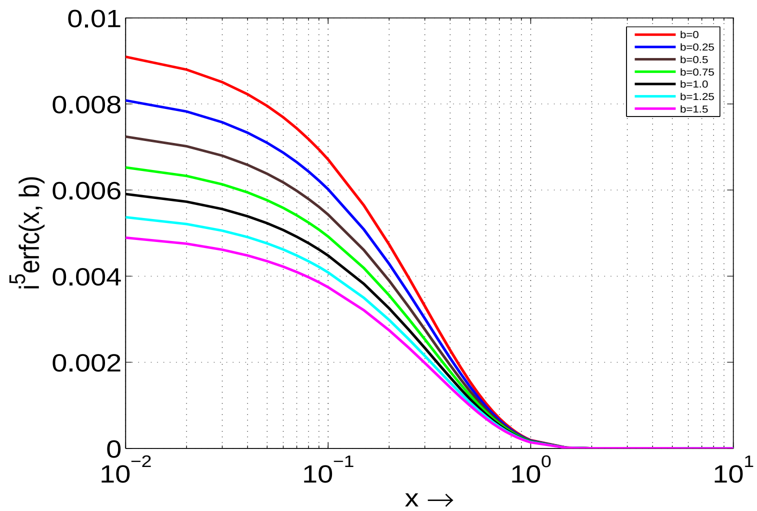

Figure 4.7: The graphical representation of the i^5erfc (x, b) function for different values of b

Chapter 5

Heat Conduction of a two-Layer System Due to Laser Source

In this chapter we present an analytical solution to the problem of heating two layer system using Laplace integral transform method. In Section 5.1, we give some literature review for the use of laser in heat conduction problem. In Section 5.2, we discuss the mathematical formulation of the problem. In Section 5.3, we present the closed form solution for the temperature profile in the thin film and the substrate region. The surface temperature as well as the temperature profiles for two different materials with different pulse shape and different laser flux densities are presented in a graphical form in Section 5.4. In Section 5.5, we give some remarks about this new formulation of the problem.

5.1 Introduction

Lasers have been widely used in the material processing industry for the heat treatment of metals and semiconductors [1] and [41]. An advantage of laser heating is that the incident energy can be carefully controlled to limit the heating to a small, localized area with a temperature increase as much as 100 K/s for a few nanoseconds to hundreds of milliseconds [1]. Laser–material interaction has a considerable interest for many investigator [1], [6], [11], [12], [33], [50], [52], [53], [54] and [55] for a variety of applications. The problem has different industrial applications such as laser drilling, machining, and heat treatment [2], [24], [28], [32], [43] and [49].

Different models and techniques are used to obtain solutions for laser heating of materials under different boundary conditions. In particular, several authors presented the closed-form solutions to one dimensional heat conduction in a semi-infinite solid subjected to laser heating. Each study considered laser source as a definite function of time and position. For instance, Chaudhry and Zubair [11] considered an instantaneous laser source; however, the initial temperature profile is an exponentially decaying. They also considered a time dependent source in [54], [55]. Yilbas [52] considered an exponentially varying time dependent laser source. Laser heating mechanism including evaporation process during laser drilling of metallic substrates was also investigated analytically by Yilbas et al. [53]. They found the closed form solution for the temperature rise due to a step input intensity pulse and determined the drilling efficiency. Also, heat conduction in a moving semi-infinite solid subjected to a pulse laser irradiation was studied analytically by Modest and Abakians [33]. They obtained a closed form solution for the temperature distribution inside the substrate material.

The problem of heating a homogenous slab of material subjected to time-dependent laser irradiance, i.e. $q_0 = q_0(t)$ was also studied be El-Adawi et al. [17]. They obtained an exact solution for the temperature distribution of slab by using Fourier series expansion technique. The rate of melting of a solid slab induced by constant laser irradiance for time interval less than or equal to the transit time was studied by El-Adawi [16] However, El-Adawi et al. [18] considered the problem of melting a solid but for time interval greater than or equal to the transit time. They also investigated the laser heating and melting of a thin film with two different time intervals in [19] and [20].

Laser heating of a two-layer system was formulated by El-Adawi et al. [21] using Laplace transform method where they computed the time required for the melting of the films situated on a glass substance. In their study they assumed that the laser source is continuously operating and has a constant value. However, in certain applications of material processing it is important to expose the materials only for certain time intervals. The objective of our study is to present an analytical solution for a two-layer system when subjected desired to have a laser source applied in a time interval [a, b]. This improved formulation enables us to study the temperature profile of the thin film and substrate under different operating conditions.

5.2 Mathematical Formulation

In setting up the problem, it is assumed that the incident laser irradiance for a finite time interval is applied on the front surface of the two-layer system. This system is composed of a thin film of thickness d on a thick substrate such as glass. The two layers are in perfect thermal contact. The laser source can be modeled in terms of the Heaviside function, $H(t)$ such as,

$$q_0[H(t-a)-H(t-b)]. \tag{5.1}$$

It is important to note that with the introduction of the Heaviside function, one can easily control the incident radiation in a time interval [a, b]. In actual application some part of this radiation is absorbed, while the other part is reflected. The absorbed energy flux at the surface can be simply be written as,

$$q_0A_f[H(t-a)-H(t-b)], \tag{5.2}$$

Where A_f is the absorptance of the thin film, which is normally a temperature independent. One $x-$ axis normal to the free surface of the considered system , along the direction of the free surface of the considered system is used to described the problem. In this regard the boundary $x = 0$ represents the front surface of the thin film, while $x = d$ represents the interface between the two layers. For a one dimensional heat flow in the direction of incident radiation, the heat diffusion equations for both the thin film and substrate can be written, respectively, as

$$\frac{\partial T_f(x,t)}{\partial t} = \alpha_f \frac{\partial^2 T_f(x,t)}{\partial x^2}, \quad t > 0, \quad 0 \leq x \leq d; \tag{5.3}$$

$$\frac{\partial T_p(x,t)}{\partial t} = \alpha_p \frac{\partial^2 T_p(x,t)}{\partial x^2}, \quad d \leq x \leq \infty, \tag{5.4}$$

where T is the excess temperature with respect to the ambient temperature T_0, α is the normal diffusivity in terms of the thermal conductivity λ, and the heat capacity per unit volume (pc_p). In the present formulation, the physical parameters of the thin film and the substrate are assumed to be temperature independent.

For the case of no plasma formation at the front surface for the considered values of the incident laser flux and a negligible energy loss due to radiation and multi-reflections within the considered system, the system of equations (5.3) and (5.4) is subjected to the following initial and boundary conditions:

$$T_f(x,0) = 0 \tag{5.5}$$

$$T_p(x,0) = 0 \tag{5.6}$$

The condition at the free surface $x = 0$ can be written as,

$$-\lambda_f \frac{\partial T_f}{\partial x} = q_0 A_f [H(t-a) - H(t-b)]. \tag{5.7}$$

The condition at the interface between the thin film and substrate $x = d$ is

$$T_f(d,t) = T_p(d,t), \tag{5.8}$$

$$-\lambda_f \frac{\partial T_f(d,t)}{\partial x} = -\lambda_p \frac{\partial T_p(0,t)}{\partial x}, \tag{5.9}$$

For the substrate one more condition is given as follows:

$$T_p(\infty,t) = 0. \tag{5.10}$$

If we take the Laplace transform with respect to the time variable for both equations (5.3) and (5.4) and applying the boundary conditions (5.5) and (5.6), this results in,

$$\frac{\partial^2 \overline{T_f}(x,s)}{\partial x^2} - \frac{s}{\alpha_f}\overline{T_f}(x,s) = 0, \tag{5.11}$$

$$\frac{\partial^2 \overline{T_p}(x,s)}{\partial x^2} - \frac{s}{\alpha_p}\overline{T_p}(x,s) = 0 \tag{5.12}$$

where $\overline{T_f}(x,s)$ and $\overline{T_p}(x,s)$, denote the Laplace transform of T in the film and substrate region, respectively. Taking the Laplace transform of the boundary conditions (5.7), (5.8), (5.9) and (5.10), gives

$$-\lambda_f \frac{\partial \overline{T_f}(0,s)}{\partial x} = \frac{q_0 A_f}{s}[\exp(-as) - \exp(-bs)] \tag{5.13}$$

$$\overline{T_f}(d,s) = \overline{T_p}(0,s) \tag{5.14}$$

$$-\lambda_f \frac{\partial \overline{T_f}(d,s)}{\partial x} = -\lambda_p \frac{\partial \overline{T_p}(0,s)}{\partial x} \tag{5.15}$$

$$\overline{T_p}(\infty,s) = 0. \tag{5.16}$$

The solutions of equations (5.11) and (5.12) can be written in the form

$$\overline{T}_f(x,s) = c_1 \exp\left(\sqrt{s/\alpha_f}\,x\right) + c_2 \exp\left(-\sqrt{s/\alpha_f}\,x\right) \tag{5.17}$$

$$\overline{T}_p(x,s) = c_3 \exp\left(\sqrt{s/\alpha_p}\,x\right) + c_4 \exp\left(-\sqrt{s/\alpha_p}\,x\right). \tag{5.18}$$

Condition (5.16) gives $c_3 = 0$; therefore, the substrate temperature distribution is written as,

$$\overline{T}_p(x,s) = c_4 \exp\left(-\sqrt{s/\alpha_p}\,x\right). \tag{5.19}$$

The boundary conditions (5.13), (5.14), and (5.15), when used in equations (5.17), and (5.19), respectively, result in the following equations:

$$-\lambda_f \sqrt{\frac{s}{\alpha_f}}(c_1 - c_2) = \frac{q_0 A_f}{s}[\exp(-as) - \exp(-bs)], \tag{5.20}$$

$$c_4 = c_2 \exp\left(-\sqrt{s/\alpha_f}\;d\right) + c_1 \exp\left(\sqrt{s/\alpha_f}\;d\right).,\tag{5.21}$$

$$c_4 \frac{\lambda_p}{\lambda_f} \frac{\sqrt{s/\alpha_p}}{\sqrt{s/\alpha_f}} + c_2 \exp\left(-\sqrt{s/\alpha_f}\;d\right) - c_1 \exp\left(\sqrt{s/\alpha_f}\;d\right). \tag{5.22}$$

Solving the above algebraic equations, we get the constants, c_1, c_2 and c_4 in the following form:

$$c_1 = \frac{B\,q_0 A_f \exp\left(-2\sqrt{s/\alpha_f}\;d\right)[\exp(-as) - \exp(-bs)]}{\lambda_f\, s\sqrt{s/\alpha_f}\left[1 - B\exp\left(-2\sqrt{s/\alpha_f}\;d\right)\right]} \tag{5.23}$$

$$c_2 = \frac{q_0\,A_f\,[\exp(-as) - \exp(-bs)]}{\lambda_f\, s\sqrt{s/\alpha_f}\left[1 - B\exp\left(-2\sqrt{s/\alpha_f}\;d\right)\right]} \tag{5.24}$$

$$c_4 = \frac{2q_0\,A_f\,[\exp\left(-\sqrt{s/\alpha_f}\;d\right)[\exp(-as) - \exp(-bs)]}{\lambda_f\, s\sqrt{s/\alpha_f}\,(1+\varepsilon)\left[1 - B\,\exp\left(-2\sqrt{s/\alpha_f}\;d\right)\right]}, \tag{5.25}$$

Where

$$\varepsilon = \frac{\lambda_p}{\lambda_f} \frac{\sqrt{s/\alpha_p}}{\sqrt{s/\alpha_f}} = \frac{\lambda_p}{\lambda_f} \frac{\sqrt{\alpha_f}}{\sqrt{\alpha_p}}, \tag{5.26}$$

and

$$B = \frac{1-\varepsilon}{1+\varepsilon}. \tag{5.27}$$

Thus the temperature distribution can be written in the form:

$$\overline{T}_f(x,s) = \left\{ \frac{q_0\, A_f\, [\exp(-a\,s) - \exp(-b\,s)]}{\lambda_f\; s\; \sqrt{s/\alpha_f}\, \left[1 - B\exp\left(-2\,\sqrt{s/\alpha_f}\; d\right)\right]} \right\} \times$$

$$\left\{ B\exp\left[-\sqrt{s/\alpha_f}(2d-x)\right] + \exp\left(-\sqrt{s/\alpha_f}\; x\right) \right\} \tag{5.28}$$

and

$$\overline{T}_p(x,s) = \frac{2\,q_0\, A_f \exp\left[-\sqrt{s/\alpha_f}\left(x\sqrt{\alpha_f/\alpha_p} + d\right)\right] [\exp(-as) - \exp(-bs)]}{\lambda_f\, s\; \sqrt{s/\alpha_f}(1+\varepsilon)\left[1 - B\exp\left(-2\sqrt{s/\alpha_f}\; d\right)\right]} \tag{5.29}$$

Using the well known geometric series identity [40]

$$\frac{1}{1-a} = \sum_{n=0}^{\infty} a^n, |a| < 1, \tag{5.30}$$

we can write

$$\frac{1}{1 - B\exp\left(-2\sqrt{s/\alpha_f}\; d\right)} = \sum_{n=0}^{\infty} \left[B^n \exp\left(-2n\sqrt{s/\alpha_f}\; d\right)\right]$$

$$\left(\left|B\exp(-2\,\sqrt{s/\alpha_f}\,d)\right| < 1\right) \tag{5.31}$$

Substituting equation (5.31) into equations (5.28) and (5.29), we get

$$\overline{T}_f(x,s) = \sum_{n=0}^{\infty} \frac{q_0 A_f[\exp(-as)-\exp(-bs)]}{\lambda_f s\sqrt{s/\alpha_f}} \times$$

$$\left\{B^{n+1}\exp\left(-\sqrt{s/\alpha_f}[2(n+1)d-x]\right)+B^n\exp\left[-\sqrt{s/\alpha_f}(2nd+x)\right]\right\}, \tag{5.32}$$

and

$$\overline{T}_p(x,s) =$$
$$\frac{2q_0\,A_f\,[\exp(-as)-\exp(-bs)]}{\lambda_f s\,\sqrt{s/\alpha_f}(1+\varepsilon)}\sum_{n=0}^{\infty} B^n\exp\left(-\sqrt{s/\alpha_f}\left[(1+2n)d+x\sqrt{\alpha_f/\alpha_p}\right]\right). \tag{5.33}$$

If we let

$$F(s) = \frac{1}{s\sqrt{s/\alpha_f}}\exp\left(-\sqrt{s/\alpha_f}\;x\right), \tag{5.34}$$

then the inverse Laplace transform of $F(s)$ is given by [12]

$$L^{-1}\{F(s)\} = 2\sqrt{\frac{\alpha_f t}{\pi}}\exp\left(\frac{-x^2}{4\alpha_f t}\right) - x\,\mathrm{erfc}\left(\frac{x}{\sqrt{4\alpha_f t}}\right) \quad x>0,\,\alpha_f>0 \tag{5.35}$$

Substituting equation (5.35) into equations (5.32) and using the well known formula

$$L\{f(t-a)H(t-a)\} = \exp(-as)F(s), \tag{5.36}$$

we get after some manipulation,

$$T_f(x,t) = \sum_{n=0}^{\infty} \frac{q_0 A_f}{\lambda_f} B^{n+1} \left(\sqrt{\frac{4\alpha_f(t-a)}{\pi}} \exp\left\{ -\frac{[2d(1+n)-x]^2}{4\alpha_f(t-a)} \right\} \right.$$

$$\left. -[2d(1+n)-x] \operatorname{erfc}\left(\frac{2d(1+n)-x}{\sqrt{4\alpha_f(t-a)}} \right) \right) H(t-a)$$

$$+\sum_{n=0}^{\infty} \frac{q_0 A_f}{\lambda_f} B^{n} \left(\sqrt{\frac{4\alpha_f(t-a)}{\pi}} \exp\left[-\frac{(2nd+x)^2}{4\alpha_f(t-a)} \right] \right.$$

$$\left. (-2nd+x) \operatorname{erfc}\left(\frac{(2nd+x)}{\sqrt{4\alpha_f(t-a)}} \right) \right) H(t-a)$$

$$-\sum_{n=0}^{\infty} \frac{q_0 A_f}{\lambda_f} B^{n+1} \left(\sqrt{\frac{4\alpha_f(t-b)}{\pi}} \exp\left\{ -\frac{[2d(1+n)-x]^2}{4\alpha_f(t-b)} \right\} \right.$$

$$\left. -[2d(1+n)-x] \operatorname{erfc}\left(\frac{2d(1+n)-x}{\sqrt{4\alpha_f(t-b)}} \right) \right) H(t-b)$$

$$-\sum_{n=0}^{\infty} \frac{q_0 A_f}{\lambda_f} B^{n} \left(\sqrt{\frac{4\alpha_f(t-b)}{\pi}} \exp\left[-\frac{(2nd+x)^2}{4\alpha_f(t-b)} \right] \right.$$

$$\left. -(2nd+x) \operatorname{erfc}\left(\frac{(2nd+x)}{\sqrt{4\alpha_f(t-b)}} \right) \right) H(t-b) \tag{5.37}$$

Similarly, substituting equation (5.35) into equation (5.33), results in

$$T_p(x,t) = \sum_{n=0}^{\infty} \frac{2q_0 A_f}{\lambda_f} \frac{B^n}{1+\varepsilon} \left(\frac{\sqrt{4\alpha_f(t-a)}}{\pi} \exp\left\{ \frac{-\left[x\sqrt{\alpha_f/\alpha_p} + (1+2n)d\right]^2}{4\alpha_f(t-a)} \right\} \right.$$

$$\left. - \left[x\sqrt{\alpha_f/\alpha_p} + (1+2n)d\right] \operatorname{erfc}\left\{ \frac{\left[x\sqrt{\alpha_f/\alpha_p} + (1+2n)d\right]}{\sqrt{4\alpha_f(t-a)}} \right\} \right) H(t-a)$$

$$- \sum_{n=0}^{\infty} \frac{2q_0 A_f}{\lambda_f} \frac{B^n}{1+\varepsilon} \left(\frac{\sqrt{4\alpha_f(t-b)}}{\pi} \exp\left\{ \frac{-\left[x\sqrt{\alpha_f/\alpha_p} + (1+2n)d\right]^2}{4\alpha_f(t-b)} \right\} \right.$$

$$\left. - \left[x\sqrt{\alpha_f/\alpha_p} + (1+2n)d\right] \operatorname{erfc}\left\{ \frac{\left[x\sqrt{\alpha_f/\alpha_p} + (1+2n)d\right]}{\sqrt{4\alpha_f(t-b)}} \right\} \right) H(t-b). \qquad (5.38)$$

By substituting $x = d$ into equation (5.37) and equation (5.38) one can easily confirm that condition (5.8) is satisfied.

It is interesting to examine the time-dependent temperature distribution of the front surface by substituting $x = 0$ in equation (5.37). This result in,

$$T_f(0,t) = \sum_{n=0}^{\infty} \frac{q_0 A_f}{\lambda_f} B^{n+1} \left(\sqrt{\frac{4\alpha_f(t-a)}{\pi}} \exp\left\{ -\frac{[2d(1+n)]^2}{4\alpha_f(t-a)} \right\} \right.$$

$$\left. -[2d(1+n)-x] \operatorname{erfc}\left(\frac{2d(1+n)}{\sqrt{4\alpha_f(t-a)}} \right) \right) H(t-a)$$

$$+ \sum_{n=0}^{\infty} \frac{q_0 A_f}{\lambda_f} B^n \left(\sqrt{\frac{4\alpha_f(t-a)}{\pi}} \exp\left[-\frac{(2nd)^2}{4\alpha_f(t-a)} \right] \right.$$

$$\left. -(2nd) \operatorname{erfc}\left(\frac{2nd}{\sqrt{4\alpha_f(t-a)}} \right) \right) H(t-a)$$

$$- \sum_{n=0}^{\infty} \frac{q_0 A_f}{\lambda_f} B^{n+1} \left(\sqrt{\frac{4\alpha_f(t-b)}{\pi}} \exp\left\{ -\frac{[2d(1+n)]^2}{4\alpha_f(t-b)} \right\} \right.$$

$$\left. -[2d(1+n)] \operatorname{erfc}\left(\frac{2d(1+n)}{\sqrt{4\alpha_f(t-b)}} \right) \right) H(t-b)$$

$$-\sum_{n=0}^{\infty}\frac{q_0A_f}{\lambda_f}B^n\left(\sqrt{\frac{4\alpha_f(t-b)}{\pi}}\exp\left[-\frac{(2nd)^2}{4\alpha_f(t-b)}\right]\right.$$
$$\left.-(2nd)\,\mathrm{erfc}\left(\frac{(2nd)}{\sqrt{4\alpha_f(t-b)}}\right)\right)H(t-b). \tag{5.39}$$

We note that if we take $a=0$ and letting $b\to\infty$, this implies a continuously operating laser source. If we substitute these values in equations (5.37) and (5.38), the present solution exactly reduces to the temperature distribution for the film and substrate given by El-Adawi et al. [21].

5.3 Results and Discussion

In this section, as illustrative examples, we give some computations for the following two-layer systems (that are studied by E-Adawi [21] for a continuously operating constant heat source) by controlling the time interval of the source. In this regard, we have considered the following examples: aluminum–glass, copper–glass for $a=0$ and different values time b. Each system is subjected to a laser flux of densities 10^{13} and $10^{12}\,Wm^{-2}$ and the metal thickness is taken to be 5μ in all the cases. The thermo-physical properties for the chosen material are tabulated in Table 5.1.

Table 5.1: Thermo-physical properties of the substrate and thin film [15] and [45]

Element	ρ (Kgm^{-3})	λ $(Wm^{-1}K^{-1})$	α (m^2s^{-1})	c_p $(JKg^{-1}K^{-1})$	A_f	T_m (K)
Aluminum	2700	238	8.410×10^{-5}	896	0.056	633
Copper	8954	386	11.25×10^{-5}	383	0.001	1056
Glass	2707	0.76	0.035×10^{-5}	800		

The temperature response for the thin film, $T_f(x,t)$ and the substrate for different boundary conditions are presented in the following paragraphs.

Figure 5.2 presents a typical surface temperature response for a thin Aluminum film when subjected to continuously operating constant laser source ($b \rightarrow \infty$). As expected, we see that surface temperature of the thin film increases with time. The temperature reaches the melting temperature of Aluminum in 67.26×10^{-11} seconds for an exposed laser source of $10^{13} W\,m^{-2}$. By controlling the laser exposure up to b = 40×10^{-11} seconds, Figure 5.3 shows that the surface temperature of thin film, never exceeded the melting temperature. It first increases logarithmically up to the cutoff point of the source and then decreases exponentially. Thus, clearly indicating that the desired surface temperature can be controlled by adjusting the cutoff time of the source.

Figure 5.4 shows the temperature distribution of thin film as a function of the distance from the surface for the situation when the time reaches the melting time, $t_m = 67.26 \times 10^{-11}$ seconds for various values of exposure time b. We find that temperature in the film decreases exponentially with distance for almost all exposure times. These plots for surface temperature, temperature distribution in the thin film and substrate are also presented in Figs. 5.5 - 5.8 for the reduced laser source. It is important to note that for large exposure time in the case of reduced laser source of $10^{12} W m^{-2}$, there is a temperature distribution in the substrate; that is for $x > 5 \times 10^{-6}$ m; however for less exposure time (refer to Figure 5.4), there is no heat penetration in the substrate region.

Figures 5.8–5.13 show the temperature distribution in the thin film and substrate in the case of copper-glass combination for laser flux of 10^{13} and $10^{12} W\,m^{-2}$ respectively. These figures (refer to Figs. 5.10 and 5.13) clearly demonstrate that there is a heat penetration in the substrate (copper) because of large exposure time when compared to Aluminum substrate. In addition, energy storage effect is clearly visible in Figure 5.13 because of large value of volumetric heat capacity of copper as well as large exposure time.

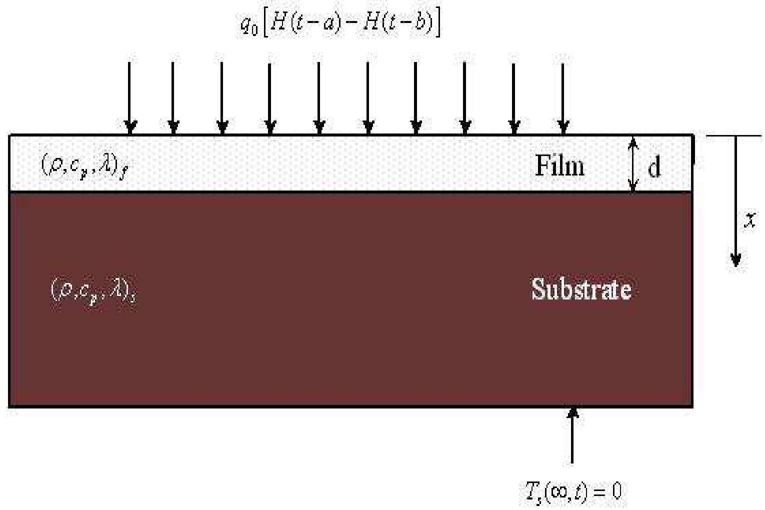

Figure 5.1: The two-layer system consist of a thin film of thickness d and a glass substrate

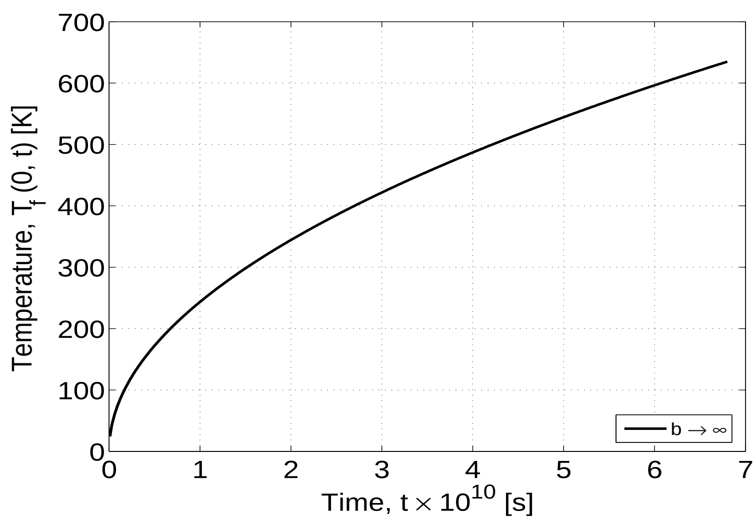

Figure 5.2: The front surface temperature $T_f(0,t)$ against the exposure time t for the two-layer system aluminum on glass where $t_m = 67.26 \times 10^{-11}$ [s] and $q_0 = 10^{13} W m^{-2}$.

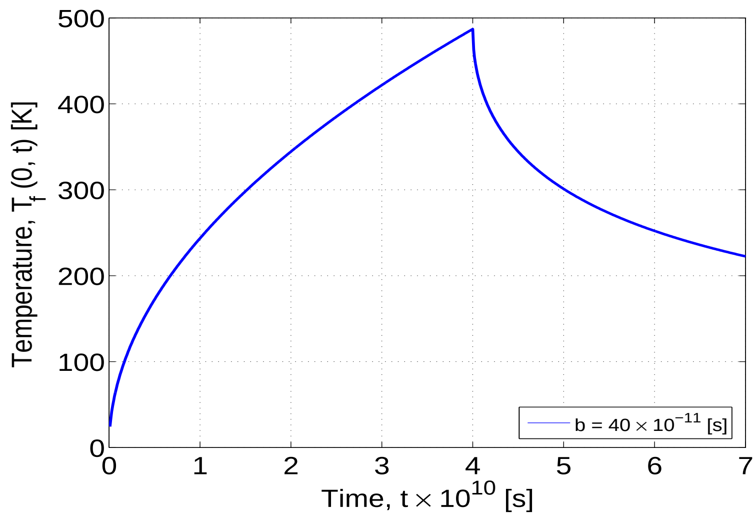

Figure 5.3: The front surface temperature $T_f(0,t)$ against the exposure time t for the two-layer system aluminum on glass where $t_m = 67.26 \times 10^{-11}$ [s] and $q_0 = 10^{13} W m^{-2}$.

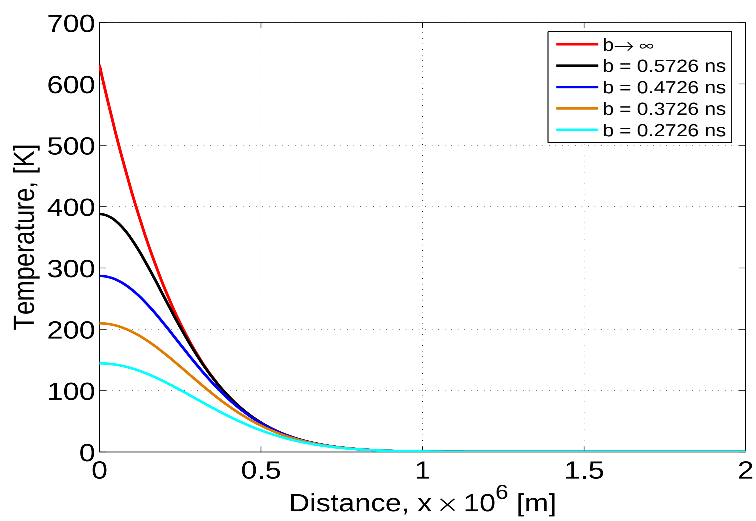

Figure 5.4: Temperature profile within a two-layer system aluminum on glass for different cutoff points b, where $t_m = 67.26 \times 10^{-11}$ [s] and $q_0 = 10^{13} W m^{-2}$.

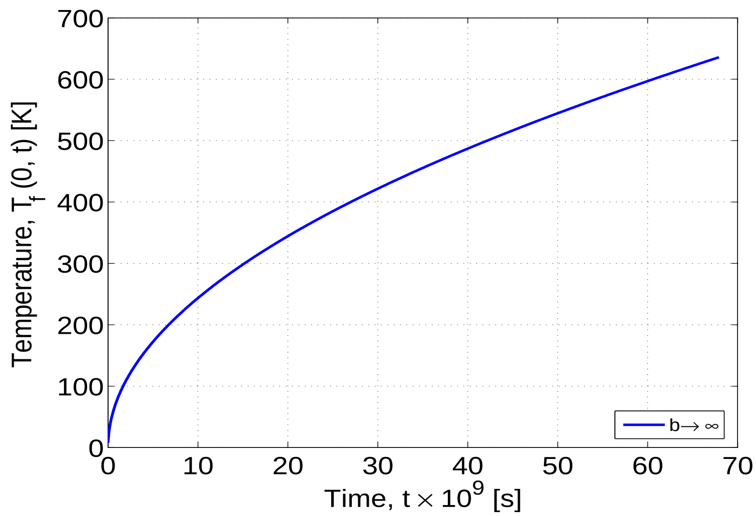

Figure 5.5: The front surface temperature $T_f(0,t)$ against the exposure time t for the two-layer system aluminum on glass where $t_m = 67.26$ ns and $q_0 = 10^{12} W m^{-2}$.

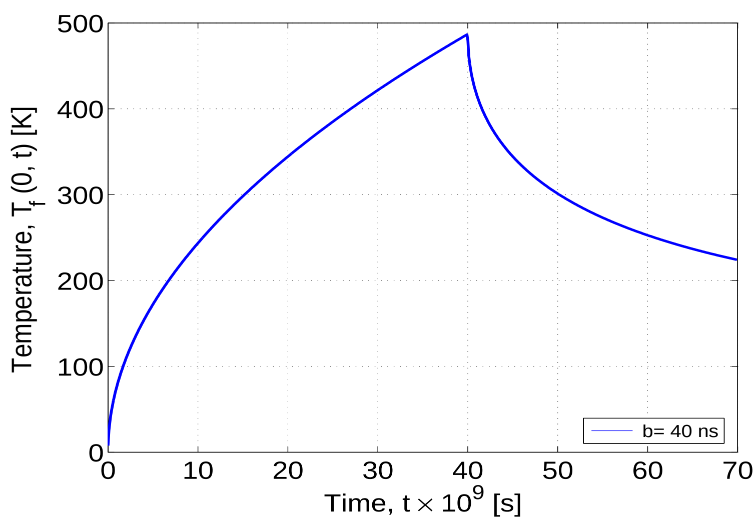

Figure 5.6: The front surface temperature $T_f(0,t)$ against the exposure time t for the two-layer system aluminum on glass where $t_m = 67.26$ ns and $q_0 = 10^{12} W m^{-2}$.

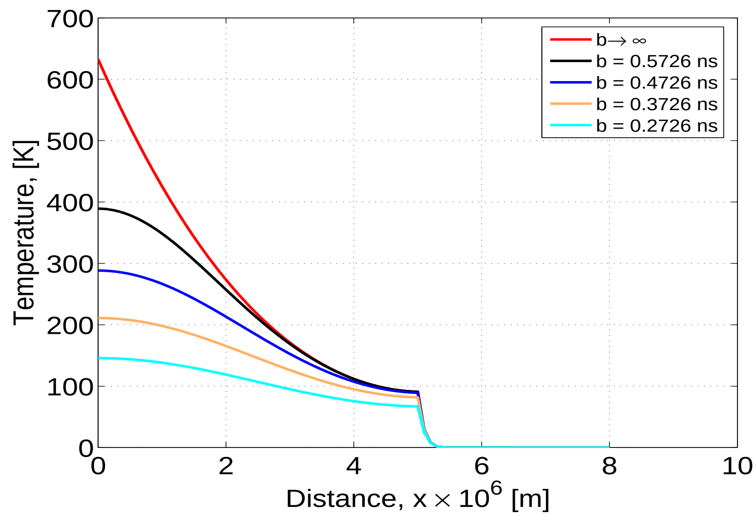

Figure 5.7: Temperature profile within a two-layer system aluminum on glass for different cutoff points b, where $t_m = 67.26$ ns and $q_0 = 10^{12} W m^{-2}$.

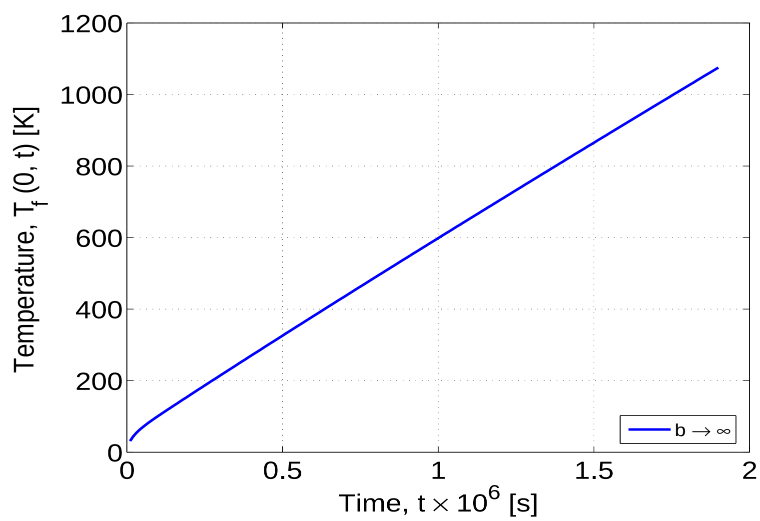

Figure 5.8: The front surface temperature $T_f(0,t)$ against the exposure time t for the two-layer system copper on glass where $t_m = 1.85\,\mu s$ and $q_0 = 10^{13} W m^{-2}$.

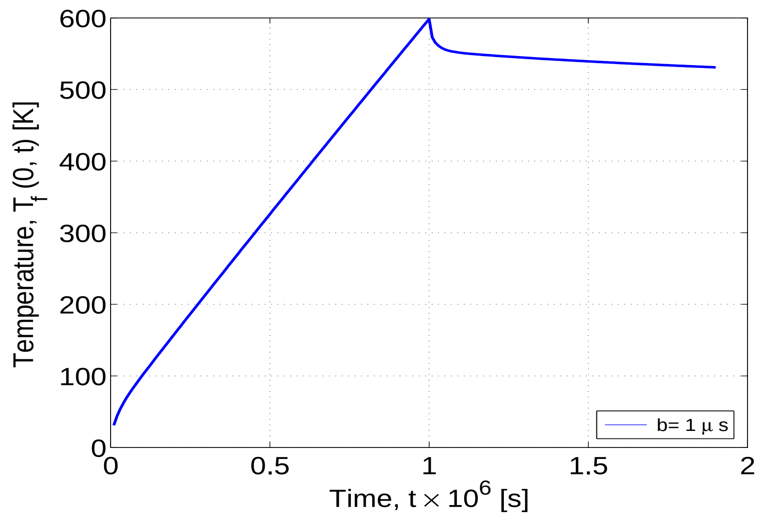

Figure 5.9: The front surface temperature $T_f(0,t)$ against the exposure time t for the two-layer system copper on glass where $t_m = 1.85\,\mu s$ and $q_0 = 10^{13} W m^{-2}$.

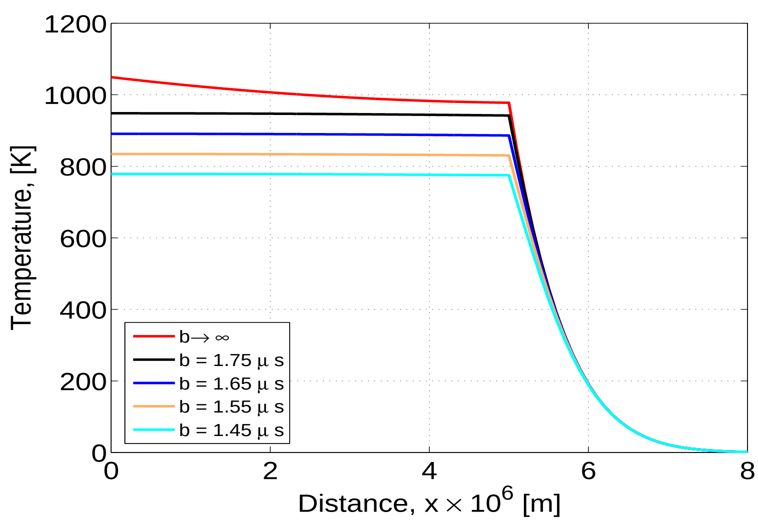

Figure 5.10: Temperature profile within a two-layer system copper on glass for different cutoff points b, where $t_m = 1.85\,\mu s$ and $q_0 = 10^{13} W m^{-2}$.

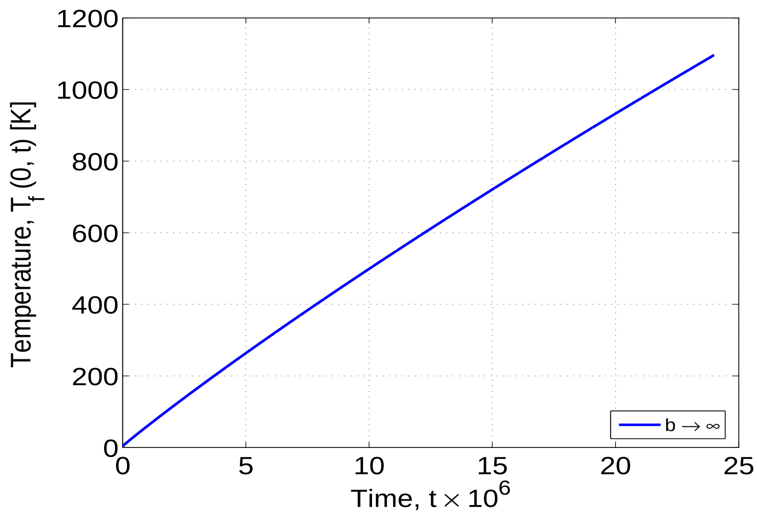

Figure 5.11: The front surface temperature $T_f(0,t)$ against the exposure time t for the two-layer system copper on glass where $t_m = 23.04\,\mu s$ and $q_0 = 10^{12} W m^{-2}$.

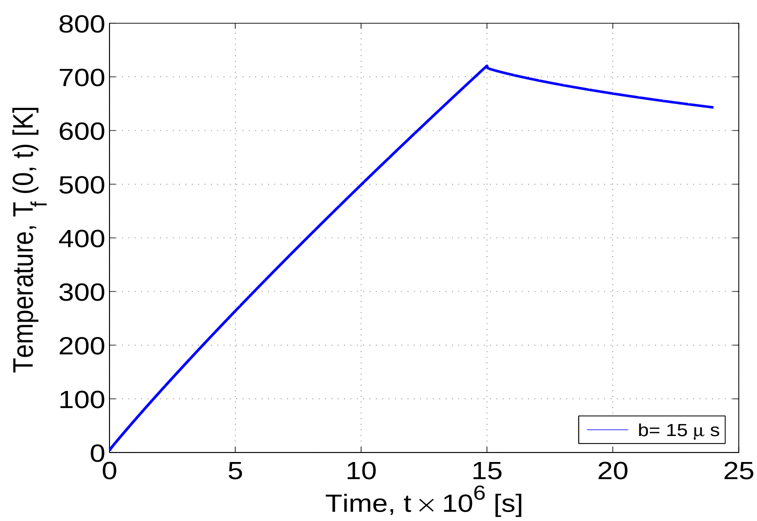

Figure 5.12: The front surface temperature $T_f(0,t)$ against the exposure time t for the two-layer system copper on glass where $t_m = 23.04\,\mu s$ and $q_0 = 10^{12} W m^{-2}$.

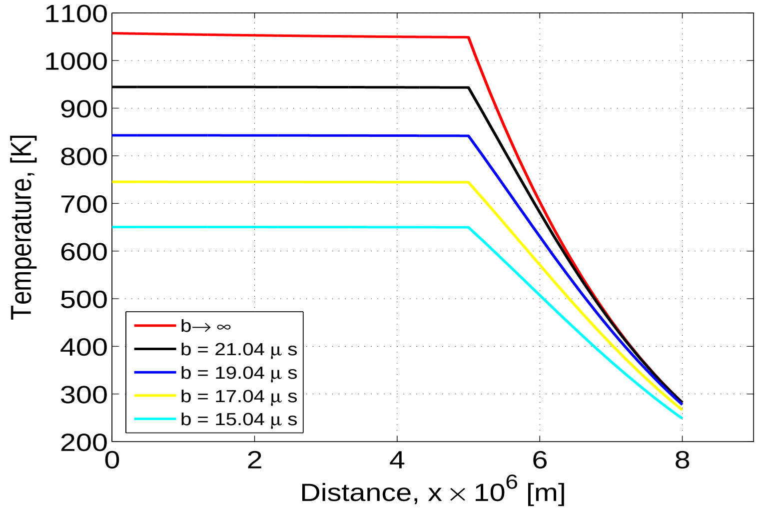

Figure 5.13: Temperature profile within a two-layer system copper on glass for different cutoff points b, where $t_m = 23.04\,\mu s$ and $q_0 = 10^{12} W m^{-2}$.

5.4 Concluding Remarks

The analytical solution of a two-layer material problem subjected to a laser source is further extended by controlling the exposure time of the laser source. It is demonstrated that through mathematical analysis and graphical results that Heaviside function seems to be very useful to control the exposure time. In the graphical results for Aluminum-glass and Copper-glass combination, we find that laser exposure time and strength can help us control the heat penetration in the glass if needed. It is expected that the analytical benchmark solutions presented in this paper will be very useful for further numerical studies of such problems, in which the temperature dependency of the material properties is an important issue.

Chapter 6

Heat Conduction in an infinite Solid when Subjected to an Instantaneous Laser Source

In this chapter we study another heat conduction problem. The problem of heating a semi–infinite solid is introduced in Section 6.1. The closed form solution of the problem is given in terms of two special functions $E(x,t)$ and $F(x,t)$. These functions are introduced in Section 6.2. The mathematical formulation of the problem is given in Section 6.3. In section 6.4 we discussed some special cases of practical interest. The graphical representation of the temperature profile and heat flux distribution are provided in Section 6.5 for different time-levels.

6.1 Introduction

The change of state problems that occurs during the emission or absorption of heat are classified as one of the most important groups of problems. Zubair and Chaudhry [56] study the problem of heating a semi–infinite region $x > 0$ initially at a constant temperature with mixed boundary conditions. They provide an analytical solution to the problem in terms of

$E(x,t)$ and $F(x,t)$ special functions. Moreover, they investigate the temperature and heat flux profiles for a semi–infinite solid when subjected to spatially decaying, instantaneous laser source in [54]. In the present work we study the problem of heat conduction in a semi–infinite solid when subjected to spatially decaying instantaneous laser source applied at different n–time levels. We provide an analytical solution of the temperature and heat flux profiles. The present work clearly extend the solution provided by Zubair and Chaudhry in [54].

6.2 The Functions $E(x,t)$ and $F(x,t)$

A considerable number of change of state problem were solved explicitly in terms of $E(x,t)$ and $F(x,t)$ functions [56], [10]. These functions are defined by:

Definition 6.2.1.

$$E(x,t) := exp\left(x+x^2t\right) \text{ erfc}\left(\frac{1}{2\sqrt{t}}+x\sqrt{t}\right), \tag{6.1}$$

$$F(x,t) := \text{erfc}\left(\frac{1}{2\sqrt{t}}\right) - E(x,t) \tag{6.2}$$

Several properties of these functions follow from that of the complementary error function. Some of these properties [10] are as follows:

$$E(x,t)+F(x,t) = \text{erfc}\left(\frac{1}{2\sqrt{t}}\right), \tag{6.3}$$

$$E(x,\infty)+F(x,\infty) = 1, \tag{6.4}$$

$$E(0,t) = \text{erfc}\left(\frac{1}{2\sqrt{t}}\right), \tag{6.5}$$

$$F(0,t) = 0, \tag{6.6}$$

$$F(x,t) = E(0,t)-F(x,t). \tag{6.7}$$

Some useful differentiation formulas involving these functions include:

$$\frac{\partial}{\partial x}\left\{E\left(bx,\frac{\alpha t}{x^2}\right)\right\}=bE\left(bx,\frac{\alpha t}{x^2}\right)-\frac{1}{\sqrt{\pi\,\alpha\,t}}\,exp\left(-\frac{x^2}{4\alpha t}\right), \tag{6.8}$$

$$\frac{\partial}{\partial x}\left\{F\left(bx,\frac{\alpha t}{x^2}\right)\right\}=-bE\left(bx,\frac{\alpha t}{x^2}\right), \tag{6.9}$$

$$\frac{\partial}{\partial t}\left\{E\left(bx,\frac{\alpha t}{x^2}\right)\right\}=\alpha\left[b^2E\left(bx,\frac{\alpha t}{x^2}\right)+\left(\frac{x}{2\sqrt{\pi}(\alpha t)^{3/2}}-\frac{b}{\sqrt{\pi\,\alpha\,t}}\right)\,exp\left(-\frac{x^2}{4\alpha t}\right)\right], \tag{6.10}$$

$$\frac{\partial}{\partial t}\left\{F\left(bx,\frac{\alpha t}{x^2}\right)\right\}=\alpha b\left[\frac{1}{\sqrt{\pi\alpha t}}\,exp\left(-\frac{x^2}{4\alpha t}\right)-bE\left(bx,\frac{\alpha t}{x^2}\right)\right]. \tag{6.11}$$

These functions have the useful inverse Laplace transform representation [8, p. 495]

$$E\left(bx,\frac{\alpha t}{x^2}\right)=L^{-1}\left\{\frac{exp\left(-x\sqrt{s/\alpha}\right)}{\sqrt{s}\,\left(\sqrt{s}+b\sqrt{\alpha}\right)};t\right\}, \tag{6.12}$$

and

$$F\left(bx,\frac{\alpha t}{x^2}\right)=L^{-1}\left\{\left(\frac{1}{s}-\frac{1}{s+b\sqrt{\alpha\,s}}\right)exp\left(-x\sqrt{s/\alpha}\right);t\right\}\qquad(x\geq 0). \tag{6.13}$$

It is to be noted that $E\left(bx,\frac{\alpha t}{x^2}\right)$ and $F\left(bx,\frac{\alpha t}{x^2}\right)$ are solutions to the heat equation

$$\frac{\partial^2 T}{\partial x^2}=\frac{1}{\alpha}\frac{\partial T}{\partial t}. \tag{6.14}$$

6.3 Mathematical Formulation

The heat conduction equation describing the temperature distribution in a semi-infinite, homogeneous and isotropic body with an energy source term is given by [8], [5]

$$\rho C_p \frac{\partial T}{\partial t} = \lambda \frac{\partial^2 T}{\partial x^2} + q'''. \tag{6.15}$$

We assume that at instants $t = 0, 1, 2 \ldots, n-1$, there is a sudden exposure of laser radiation which is absorbed partially in the surface layers followed by an exponential decay with position in the material itself. This is typically true for organic materials [42], in which the absorption coefficients is considerably smaller and the energy is deposited over a greater thickness. So, the energy source term in Equation 6.15 may be modeled as

$$q'''(x,t) = I_0 \mu (1-R) e^{-\mu x} \left[\sum_{i=0}^{n-1} \delta(t - d_i) \right]. \tag{6.16}$$

Where I_0 is the radiation intensity at the surface, R is the surface reflectance, μ is the material absorption coefficient, and this model assumes no spatial variation of I_0 in the plane normal to the beam and $d_i = i$, $i = 0, 1, ..., n-1$ are the $n-$time levels.

$$T(x,0) = T_s, \tag{6.17}$$

$$-\lambda \frac{\partial T(0,t)}{\partial x} = h \left[T_\infty - T(0,t) \right] \tag{6.18}$$

$$\frac{\partial T(\infty,t)}{\partial x} = 0 \tag{6.19}$$

Defining θ as the temperature rise above the initial temperature $\theta = (T - T_s)$ and substituting in the above equations, result in

$$\frac{\partial \theta}{\partial t} = \alpha \frac{\partial^2 \theta}{\partial x^2} + \frac{(1-R)}{\rho C_p} I_0(t) \mu e^{-\mu x} \left[\sum_{i=0}^{n-1} \delta(t - d_i) \right] \tag{6.20}$$

$$\frac{\partial \theta(0,t)}{\partial x} + \frac{h}{\lambda} \left[(T_\infty - T_s) - \theta(0,t) \right] = 0 \tag{6.21}$$

$$\frac{\partial \theta(\infty,t)}{\partial x} = 0 \tag{6.22}$$

Taking the Laplace transforms of Equations 6.20, 6.21 and 6.22 with respect to t and rearranging the terms we get

$$\frac{\partial^2 \bar{\theta}(x,s)}{dx^2} - \left(\frac{s}{\alpha} \right) \bar{\theta}(x,s) = -\frac{(1-R)}{\alpha \rho C_p} \bar{I}_0(s) \mu e^{-\mu x} \left[\sum_{i=0}^{n-1} e^{-d_i s} \right], \tag{6.23}$$

$$\frac{d\bar{\theta}(0,s)}{dx} + \frac{h}{\lambda} \left[\frac{(T_\infty - T_s)}{s} - \bar{\theta}(0,s) \right] = 0, \tag{6.24}$$

$$\frac{d\bar{\theta}(\infty,s)}{dx} = 0. \tag{6.25}$$

The general solution of Equation 6.23 can be expressed as

$$\bar{\theta}(x,s) = C_1 \exp\left(-x \sqrt{\left(\frac{s}{a} \right)} \right) + C_3 \frac{\exp(-\mu x) \left[\sum_{i=0}^{n-1} e^{-d_i s} \right]}{s - b^2} \tag{6.26}$$

Where

$$C_3 = \frac{(1-R)\mu I_0}{\rho C_p} \text{ and } b^2 = \alpha \mu^2$$

To determine C_1, we use the transformed boundary condition

$$\frac{d\bar{\theta}(0,s)}{dx} + \frac{h}{\lambda} \left[\frac{(T_\infty - T_s)}{s} - \bar{\theta}(0,s) \right] = 0$$

$$\Rightarrow -C_1\sqrt{\frac{s}{\alpha}} - \frac{\mu C_3\left[\sum_{i=0}^{n-1} e^{-d_i s}\right]}{s-b^2} + \frac{h}{\lambda}\left[\frac{(T_\infty - T_s)}{s} - C_1 - \frac{C_3\left[\sum_{i=0}^{n-1} e^{-d_i s}\right]}{s-b^2}\right] = 0$$

$$\Rightarrow C_1\left[-\sqrt{\frac{s}{\alpha}} - \frac{h}{\lambda}\right] + \frac{h}{\lambda}\left[\frac{(T_\infty - T_s)}{s}\right] - C_3\left[\sum_{i=0}^{n-1} e^{-d_i x}\right]\left[\frac{\mu + \frac{h}{\lambda}}{s-b^2}\right] = 0$$

$$\Rightarrow C_1\left[-\sqrt{\frac{s}{\alpha}} - \frac{h}{\lambda}\right] = \frac{h}{\lambda}\left[\frac{(T_\infty - T_s)}{s}\right] - C_3\left[\sum_{i=0}^{n-1} e^{-d_i x}\right]\left[\frac{\mu + \frac{h}{\lambda}}{s-b^2}\right]$$

$$\Rightarrow C_1 = \frac{\frac{h}{\lambda}\left[\frac{T_\infty - T_s)}{s}\right]}{\left[\sqrt{\frac{s}{\alpha}} + \frac{h}{\lambda}\right]} - C_3\left[\sum_{i=0}^{n-1} e^{-d_i s}\right]\left[\frac{\frac{\mu + \frac{h}{\lambda}}{s-b^2}}{\left[\sqrt{\frac{s}{\alpha}} + \frac{h}{\lambda}\right]}\right]$$

$$\Rightarrow C_1 = \left(\frac{T_\infty - T_s}{s}\right)\frac{h\sqrt{\alpha}}{\lambda\sqrt{s} + h\sqrt{\alpha}} - C_3\left[\sum_{i=0}^{n-1} e^{-d_i s}\right]\frac{\mu\lambda + h}{\lambda(s-b^2)}\frac{\lambda\sqrt{\alpha}}{\lambda\sqrt{s} + h\sqrt{\alpha}}$$

If we let

$$a = \frac{h\sqrt{\alpha}}{\lambda}, A_1 = \frac{h\sqrt{\alpha}}{\lambda}(T_\infty - T_s) \text{ and } A_2 = -C_3\sqrt{\alpha}\left(\mu + \frac{h}{\lambda}\right).$$

We get,

$$C_1 = \frac{A_1}{s(\sqrt{s} + a)} + \frac{A_2\left[\sum_{i=0}^{n-1} e^{-d_i s}\right]}{(s-b^2)(\sqrt{s} + a)}.$$

This result in,

$$\overline{\theta}(x,s) = \frac{A_1\exp(-x\sqrt{\frac{s}{\alpha}})}{s(\sqrt{s} + a)} + \frac{A_2\exp(-x\sqrt{\frac{s}{\alpha}})\sum_{i=0}^{n-1} e^{-d_i s}}{(s-b^2)(\sqrt{s} + a)} + \frac{C_3\exp(-\mu x)\sum_{i=0}^{n-1} e^{-d_i s}}{s-b^2} \quad (6.27)$$

Now, using the identity

$$\begin{aligned}
&\frac{1}{(x+a)(x+b)(x+c)} \\
&\qquad = \frac{1}{(b-a)(c-a)}\left[\frac{1}{x+a}-\frac{1}{x+c}\right]+\frac{1}{(b-a)(b-c)}\left[\frac{1}{x+b}-\frac{1}{x+c}\right],
\end{aligned}$$

the second term in (6.27) can be simplified. Indeed,

$$\begin{aligned}
\frac{1}{(s-b^2)(\sqrt{s}+a)} &= \frac{1}{(\sqrt{s}-b)(\sqrt{s}+b)(\sqrt{s}+a)} \\
&= \frac{1}{2b(b+1)}\left[\frac{1}{\sqrt{s}-b}-\frac{1}{\sqrt{s}+a}\right]+\frac{1}{2b(b-a)}\left[\frac{1}{\sqrt{s}+b}-\frac{1}{\sqrt{s}+a}\right] \\
&= \frac{1}{2b(b+1)\sqrt{\alpha}}\frac{1}{\sqrt{\frac{s}{\alpha}}-\frac{b}{\sqrt{\alpha}}}+\frac{1}{2b(b-a)\sqrt{\alpha}}\frac{1}{\sqrt{\frac{s}{\alpha}}+\frac{b}{\sqrt{\alpha}}} \\
&\qquad -\frac{1}{\sqrt{\alpha}}\left[\frac{1}{2b(b+a)}+\frac{1}{2b(b-a)}\right]\frac{1}{\sqrt{\frac{s}{\alpha}}+\frac{b}{\sqrt{\alpha}}}.
\end{aligned}$$

Letting

$$\begin{aligned}
A_{11} &= \frac{A_1}{\sqrt{\alpha}}, \quad A_{21}=\frac{A_2}{2b(b+1)\sqrt{\alpha}}, \quad A_{22}=\frac{A_2}{2b\sqrt{\alpha}(b-a)}, \\
A_{23} &= \frac{-A_2}{\sqrt{\alpha}}\left[\frac{1}{2b(b+1)}+\frac{1}{2b(b-a)}\right], \quad a_1=\frac{a}{\sqrt{\alpha}} \text{ and } \quad b_1=\frac{b}{\sqrt{\alpha}}.
\end{aligned}$$

So $\overline{\theta}(x,s)$ can be written in the following form:

$$
\begin{aligned}
\overline{\theta}(x,s) &= \frac{A_{11}\exp\left(-x\sqrt{\frac{s}{\alpha}}\right)}{s\left(\sqrt{s}+a\right)} + \frac{A_{21}\exp(-x\sqrt{\frac{s}{\alpha}})\sum_{i=0}^{n-1} e^{-d_i s}}{\left(\sqrt{\frac{s}{\alpha}}-b_1\right)} \\
&+ \frac{A_{22}\exp(-x\sqrt{\frac{s}{\alpha}})\sum_{i=0}^{n-1} e^{-d_i s}}{\left(\sqrt{\frac{s}{\alpha}}+b_1\right)} \\
&+ \frac{A_{23}\exp(-x\sqrt{\frac{s}{\alpha}})\sum_{i=0}^{n-1} e^{-d_i s}}{\left(\sqrt{\frac{s}{\alpha}}+a_1\right)} + \frac{C_3\exp(-\mu x)\sum_{i=0}^{n-1} e^{-d_i s}}{s-b^2}.
\end{aligned}
\tag{6.28}
$$

Taking the inverse Laplace Transform of (6.28), we get

$$
\begin{aligned}
\theta(x,t) = \frac{A_{11}}{a_1}&\left[\mathrm{erfc}\left(\frac{\mathrm{x}}{2\sqrt{\alpha\,\mathrm{t}}}\right) - \exp\left(\mathrm{a_1 x}+\alpha \mathrm{a}_1^2\mathrm{t}\right)\mathrm{erfc}\left(\frac{\mathrm{x}}{2\sqrt{\alpha \mathrm{t}}}+\mathrm{a}_1\sqrt{\alpha \mathrm{t}}\right)\right] \\
&+A_{21}\sum_{i=0}^{n-1}\left\{\begin{array}{c}\sqrt{\left(\frac{\alpha}{\pi(t-d_i)}\right)}\exp\left(\frac{-x^2}{4\alpha(t-d_i}\right)+\alpha b_1\exp\left(-b_1x+\alpha b_1^2(t-d_i)\right) \\ \times\mathrm{erfc}\left(\frac{\mathrm{x}}{2\sqrt{\alpha(\mathrm{t}-\mathrm{d_i})}}-\mathrm{b}_1\sqrt{\alpha(\mathrm{t}-\mathrm{d_i})}\right)\end{array}\right\}H(t-d_i) \\
&+A_{22}\sum_{i=0}^{n-1}\left\{\begin{array}{c}\sqrt{\left(\frac{\alpha}{\pi(t-d_i)}\right)}\exp\left(\frac{-x^2}{4\alpha(t-d_i)}\right)-\alpha b_1\exp\left(b_1x+\alpha b_1^2(t-d_i)\right) \\ \times\mathrm{erfc}\left(\frac{\mathrm{x}}{2\sqrt{\alpha(\mathrm{t}-\mathrm{d_i})}}+\mathrm{b}_1\sqrt{\alpha(\mathrm{t}-\mathrm{d_i})}\right)\end{array}\right\}H(t-d_i) \\
&+A_{23}\sum_{i=0}^{n-1}\left\{\begin{array}{c}\sqrt{\left(\frac{\alpha}{\pi(t-d_i)}\right)}\exp\left(\frac{-x^2}{4\alpha(t-d_i)}\right)-\alpha a_1\exp\left(a_1x+\alpha a_1^2(t-d_i)\right) \\ \times\mathrm{erfc}\left(\frac{\mathrm{x}}{2\sqrt{\alpha(\mathrm{t}-\mathrm{d_i})}}-\mathrm{a}_1\sqrt{\alpha(\mathrm{t}-\mathrm{d_i})}\right) \\ +C_3\exp\left(-\mu x+b^2(t-d_i)\right)\end{array}\right\}H(t-d_i).
\end{aligned}
\tag{6.29}
$$

Substituting the constants $A_{11}, A_{21}, A_{22}, A_{23}, a_1, b_1$ and b we get

$$\begin{aligned}
\theta(x,t) = (T_\infty - T_s)&\left[\operatorname{erfc}\left(\frac{\mathrm{x}}{2\sqrt{\alpha \mathrm{t}}}\right) - \exp\left(\frac{\mathrm{h\,x}}{\lambda} + \frac{\alpha \mathrm{h}^2 \mathrm{t}}{\lambda^2}\right)\operatorname{erfc}\left(\frac{\mathrm{x}}{2\sqrt{\alpha \mathrm{t}}} + \frac{\mathrm{h}\sqrt{\alpha \mathrm{t}}}{\lambda}\right)\right] \\
+ C_3 \sum_{i=0}^{n-1} &\left[\left\{ -\frac{1}{2}\exp(-\mu x + \alpha\mu^2(t-d_i))\operatorname{erfc}\left(\frac{\mathrm{x}}{2\sqrt{\alpha(\mathrm{t}-\mathrm{d_i})}} - \mu\sqrt{\alpha(\mathrm{t}-\mathrm{d_i})}\right)\right\}\right. \\
+ &\left\{ \frac{1}{2}\left(\frac{\mu + \frac{h}{\lambda}}{\mu - \frac{h}{\lambda}}\right)\exp\left(\mu x + \alpha\mu^2(t-d_i)\right)\operatorname{erfc}\left(\frac{\mathrm{x}}{2\sqrt{\alpha(\mathrm{t}-\mathrm{d_i})}} + \mu\sqrt{\alpha(\mathrm{t}-\mathrm{d_i})}\right)\right\} \\
- &\left\{ \frac{\frac{h}{\lambda}}{\mu - \frac{h}{\lambda}}\exp\left(\frac{hx}{\lambda} + \frac{\alpha h^2}{\lambda}(t-d_i)\right)\operatorname{erfc}\left(\frac{\mathrm{x}}{2\sqrt{\alpha(\mathrm{t}-\mathrm{d_i})}} + \frac{\mathrm{h}}{\lambda}\sqrt{\alpha(\mathrm{t}-\mathrm{d_i})}\right)\right\} \\
&\left. + \exp\left(-\mu x + \alpha\mu^2(t-d_i)\right)\right] H(t-d_i).
\end{aligned}$$

Substituting for C_3 and rewriting $\theta(x,t)$ in terms of $E(x,t)$ function, we get

$$\begin{aligned}
\theta(x,t) = (T_\infty - T_s)\left[\operatorname{erfc}\left(\frac{\mathrm{x}}{2\sqrt{\alpha \mathrm{t}}} - \mathrm{E}\left(\frac{\mathrm{hx}}{\lambda}, \frac{\alpha \mathrm{t}}{\mathrm{x}^2}\right)\right)\right] - \frac{I_0\mu(1-R)}{QC_p} \times \\
\sum_{i=0}^{n-1}\left\{ \begin{array}{c} E\left(-\mu x, \frac{\alpha(t-d_i)}{x^2}\right) - \frac{(\mu + \frac{h}{\lambda})}{(\mu - \frac{h}{\lambda})}E\left(\mu x, \frac{\alpha(t-d_i)}{x^2}\right) + \\ \left(\frac{\frac{2h}{\lambda}}{\mu - \frac{h}{\lambda}}\right)E\left(hx, \frac{\alpha(t-d_i)}{x^2}\right) - \exp\left(-\mu x + \alpha\mu^2(t-d_i)\right. \end{array}\right\} H(t-d_i) \quad . \qquad (6.30)
\end{aligned}$$

Using the following formulas for differentiation,

$$\frac{d}{dx}[\operatorname{erfc}(\mathrm{x})] = \frac{-2}{\sqrt{\pi}}\exp(-\mathrm{x}^2)$$

and

$$\frac{\partial}{\partial x}\left\{E(bx), \frac{\alpha t}{x^2}\right\} = bE\left(bx, \frac{\alpha t}{x^2}\right) - \frac{1}{\sqrt{\pi\alpha t}}\exp\left(\frac{-x^2}{4\alpha t}\right).$$

We can calculate the Heat flux at any x that is;

$$q''_{x,1} = -\lambda \frac{\partial T}{\partial x} = h(t_\infty - T_s) E\left(\frac{hx}{\lambda}, \frac{\alpha t}{x^2}\right) - \frac{I_0 \mu (1-R)}{2\rho C_p} \times \sum_{i=0}^{n-1} \left\{ \begin{array}{c} E\left(-\mu x, \frac{\alpha(t-d_i)}{x^2}\right) + \frac{\left(\mu + \frac{h}{\lambda}\right)}{\left(\mu - \frac{h}{\lambda}\right)} E\left(\mu x, \frac{\alpha(t-d_i)}{x^2}\right) \\ -\frac{2\left(\frac{h}{\lambda}\right)^2}{\mu\left(\mu - \frac{h}{\lambda}\right)} E\left(\frac{hx}{\lambda}, \frac{\alpha(t-d_i)}{x^2}\right) - 2\exp\left(-\mu x + a\mu^2(t-d_1)\right) \end{array} \right\} H(t-d_i). \quad (6.31)$$

The temperature distribution and the heat flux may be simplified by introducing dimensionless variables $\theta_1, Q_1, F_i, Bi, B_0$ and X. To get

$$\theta_1 = \mathrm{erfc}\left(\frac{1}{2\sqrt{\mathrm{F_0}}}\right) - \mathrm{E}(\mathrm{Bi}, \mathrm{F_0}) - B_0 \sum_{i=0}^{n-1} \left\{ E(-X, F_i) - \frac{(X+Bi)}{X-Bi)} E(X, F_i) + \left(\frac{2Bi}{X-Bi}\right) E(Bi, F_i) - 2\exp\left(-X + X^2 F_i\right) \right\} H(F_i). \quad (6.32)$$

and

$$Q_1 = E(Bi, F_0) - \frac{XB_0}{Bi} \sum_{i=0}^{n-1} \left\{ E(-X, F_i) + \frac{(X+Bi)}{X-Bi)} E(X, F_i) - \frac{2Bi^2}{X(X-Bi)} E(Bi, F_i) - 2\exp\left(-X + F_i X^2\right) \right\} H(F_i), \quad (6.33)$$

where

$$\begin{aligned}
\theta_1 &= \frac{\theta(x,t)}{T_\infty - T_s} = \frac{T(x,t) - T_s}{T_\infty - T_s}, \\
F_i &= \frac{a(t-d_i)}{x^2}, d_i = i, \\
B_i &= \frac{hx}{\lambda}, \\
B_0 &= \frac{I_0 \mu (1-R)}{2\rho C_p (T_\infty - T_s)}, \\
X &= \mu x,
\end{aligned}$$

and

$$Q_1 = \frac{q''_{x,1}}{h(T_\infty - T_s)} = \frac{-\lambda \frac{\partial T}{\partial x}}{h(T_\infty - T_s)}.$$

6.4 Special Cases

The dimensionless wall temperature and heat flux can be determined by evaluating (6.32) and (6.33) at $\frac{1}{F_i} = 0, Bi = 0 \,\text{and}\, X = 0$. However, the products

$$\beta_i = \sqrt{F_i} X = \sqrt{\mu^2 \alpha (t - d_i)},$$

$$\eta_i = \sqrt{F_i} Bi = \frac{h\sqrt{\alpha(t-d_i)}}{\lambda},$$

$$\xi = \frac{X}{Bi} = \frac{\mu \lambda}{h}.$$

remain finite, because the geometric distance has been suppressed. For this reason, the dimensionless temperature and heat flux at the wall are given by the following simplified equations

$$\theta_{11} = 1 - \exp\left(\eta_1^2\right)\mathrm{erfc}(\eta_1) - B_0 \sum_{i=0}^{n-1} \left\{ \begin{array}{c} \exp\left(\beta_i^2\right)\mathrm{erfc}(-\beta_{\mathrm{i}}) - \left(\frac{\xi+1}{\xi-1}\right)\exp\left(\beta_{\mathrm{i}}^2\right)\mathrm{erfc}\left(\beta_{\mathrm{i}}\right) \\ +\left(\frac{2}{\xi-1}\right)\exp\left(\eta_i^2\right)\mathrm{erfc}(\eta_{\mathrm{i}}) - 2\exp\left(\beta_{\mathrm{i}}^2\right) \end{array} \right\} H\left(F_i\right), \quad (6.34)$$

and

$$Q_{11} = 1 - \exp\left(\eta_1^2\right)\mathrm{erfc}(\eta_1) - B_0\xi \sum_{i=0}^{n-1} \left\{ \begin{array}{c} \exp\left(\beta_i^2\right)\mathrm{erfc}(-\beta_{\mathrm{i}}) + \left(\frac{\xi+1}{\xi-1}\right)\exp\left(\beta_{\mathrm{i}}^2\right)\mathrm{erfc}\left(\beta_{\mathrm{i}}\right) \\ -\left(\frac{2}{\xi\,(\xi-1)}\right)\exp\left(\eta_i^2\right)\mathrm{erfc}(\eta_{\mathrm{i}}) - 2\exp\left(\beta_{\mathrm{i}}^2\right) \end{array} \right\} H\left(F_i\right). \quad (6.35)$$

We note that if $\alpha(t-d_i)$ is sufficiently small, the diffusion effects are not important. In particular as $\lambda \to 0$, we have $\eta_i \to \infty$ and $\beta_i \to 0$. This gives

$$\theta_{111} = 1.$$

$$Q_{111} = 0.$$

For sufficiently small time or large value of $''X''$ the dimensionless temperature and heat flux distribution in a semi-infinite solid can be determined by evaluating (6.32) and (6.33) as $F_i \to 0$. This results in

$$\begin{aligned} \theta_{12} &= 2nB_0\, exp(-X), \\ Q_{12} &= \frac{2nXB_0}{Bi}\, exp(-X). \end{aligned}$$

We note that for the case of no heat generation in the solid, the reduced temperature and heat flux can be determined by evaluating (6.32) and (6.33) at $X = 0, B_0 = 0$ this gives

$$\theta_{13} = \mathrm{erfc}\left(\frac{1}{2\sqrt{\mathrm{F_i}}}\right) - \mathrm{E}(\mathrm{Bi}, \mathrm{F_1}),$$

and

$$Q_{13} = E(Bi, F_1).$$

Another important solution can be recovered from the present analysis for the case of constant surface temperature, that is, substituting $\frac{1}{h} = 0$ and $T_\infty = T_s$ in (6.30) and (6.31). This gives

$$\theta_1^* = \sum_{i=0}^{n-1} \left\{-E(-X, F_i - E(X, F_i) + 2\exp\ \left(-X + F_i X^2\right)\right\} H(F_i) \tag{6.36}$$

and

$$Q_1^* = \sum_{i=0}^{n-1} \left\{-E(-X, F_i + E(X, F_i) + 2\exp\ \left(-X + F_i X^2\right)\right\} H(F_i). \tag{6.37}$$

where

$$\theta_1^* = \frac{2\rho\, C_p \theta(x,t)}{I_0\, \mu(1-R)} = \frac{2\rho\, C_p(T(x,t) - T_s)}{I_0\, \mu(1-R)},$$

and

$$Q_1^* = \frac{2\rho C_p\, q''_{x,1}}{I_0\, \mu^2 \lambda(1-R)} = \frac{2\rho C_p \left(-\lambda \frac{\partial T}{\partial x}\right)}{I_0\, \mu^2 \lambda(1-R)}.$$

The graphical representation of equations (6.32), (6.33), (6.34), (6.35), (6.36) and (6.37) is shown in the next section for different time levels.

6.5 Graphical Representations

The graphical representations of equations (6.32) and (6.33) are shown in Figures 6.1–6.2 for different time levels. In these figures reduced temperature and heat flux solutions are presented as a function of dimensionless time parameter F_0, for various values of the dimensionless distance X. All the curves shown in these figures are drawn for dimensionless energy absorption $B_0 = 100.0$, and the Biot number $Bi = 1.0$. We note that the reduced temperature plots (refer to Fig. 6.1) are represented by characteristic Gaussian-type curves. On the other hand, the reduced heat flux plots (refer to Fig. 6.2) illustrate that there is a minimum value of the heat flux at $F_0 \approx 1.5$ for $n = 1$. However, if n is increased we observe that the minimum value occurs at F_0 that approaches 1.

Figures 6.3 and 6.4 represent the dimensionless temperature and heat flux at the wall in terms of dimensionless parameter ξ and η, for $B_0 = 100.0$ and $\beta = 1.00$. We note that for the case of $n = 1$ the reduced temperature and heat flux plots indicate that at $\eta \approx 1$ the effect of dimensionless parameter ξ has been suppressed. However, η is getting smaller than one as the number of pulses is increased. Also it can be seen from these figures that maximum reduced temperatures occur as η approaches zero, but the reduced heat flux increases as η increases.

Figures 6.4 and 6.5 give the graphical representations of equations (6.36) and (6.37). In these figures, the reduced temperature and heat flux are presented in terms of F_0 and X. We note that the temperature plots shown in Fig.6.5 are also represented by the characteristic Gaussian-type curves, and both the temperature and heat flux decreases (in proportional to n) with the increase in the dimensionless time parameter F_0, Moreover, we notice that the initial temperature increases as we increase the number of pulses.

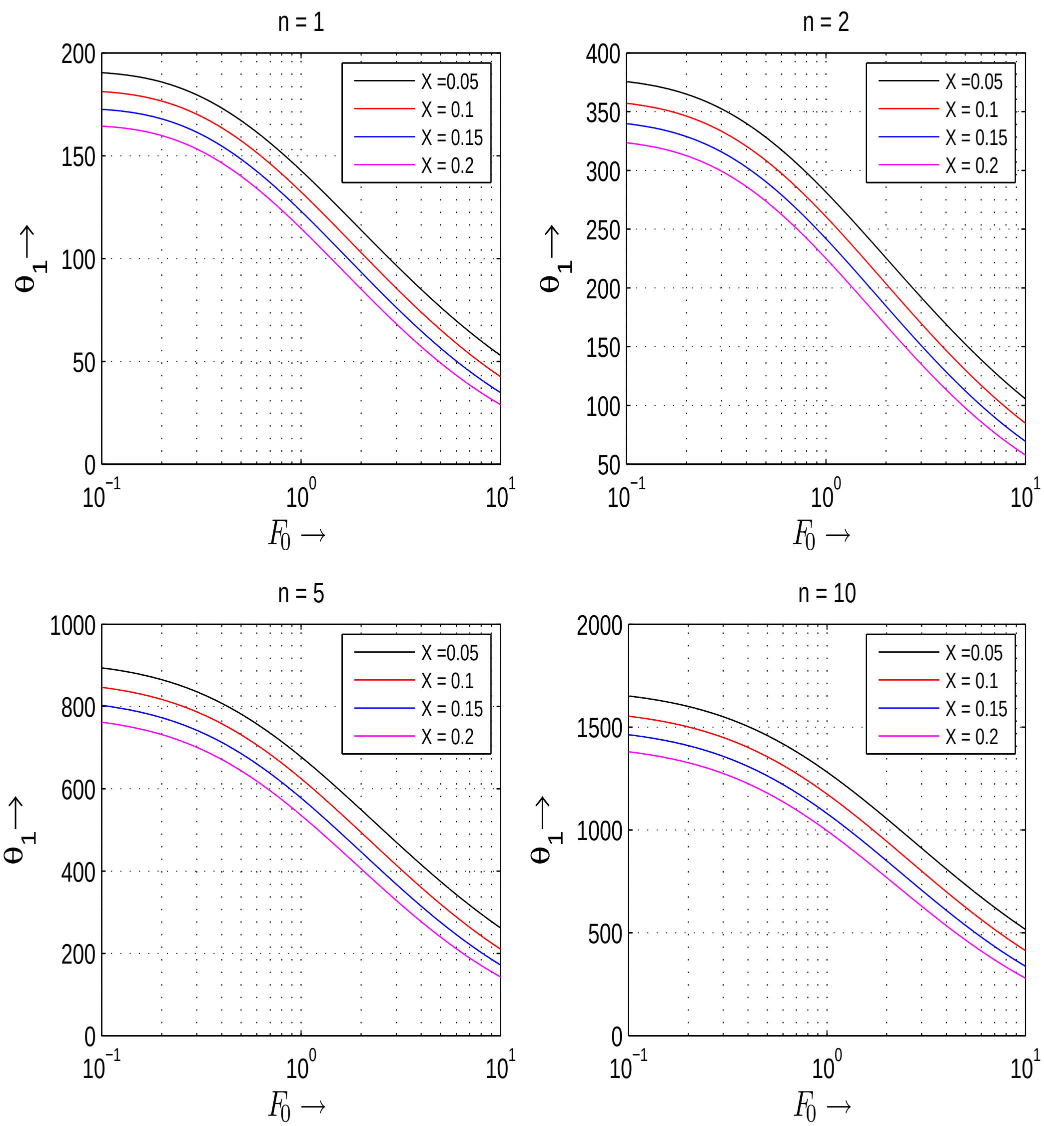

Figure 6.1: Reduced temperature as a function of reduced time and distance for $B_0 = 100$ and $Bi = 1.00$

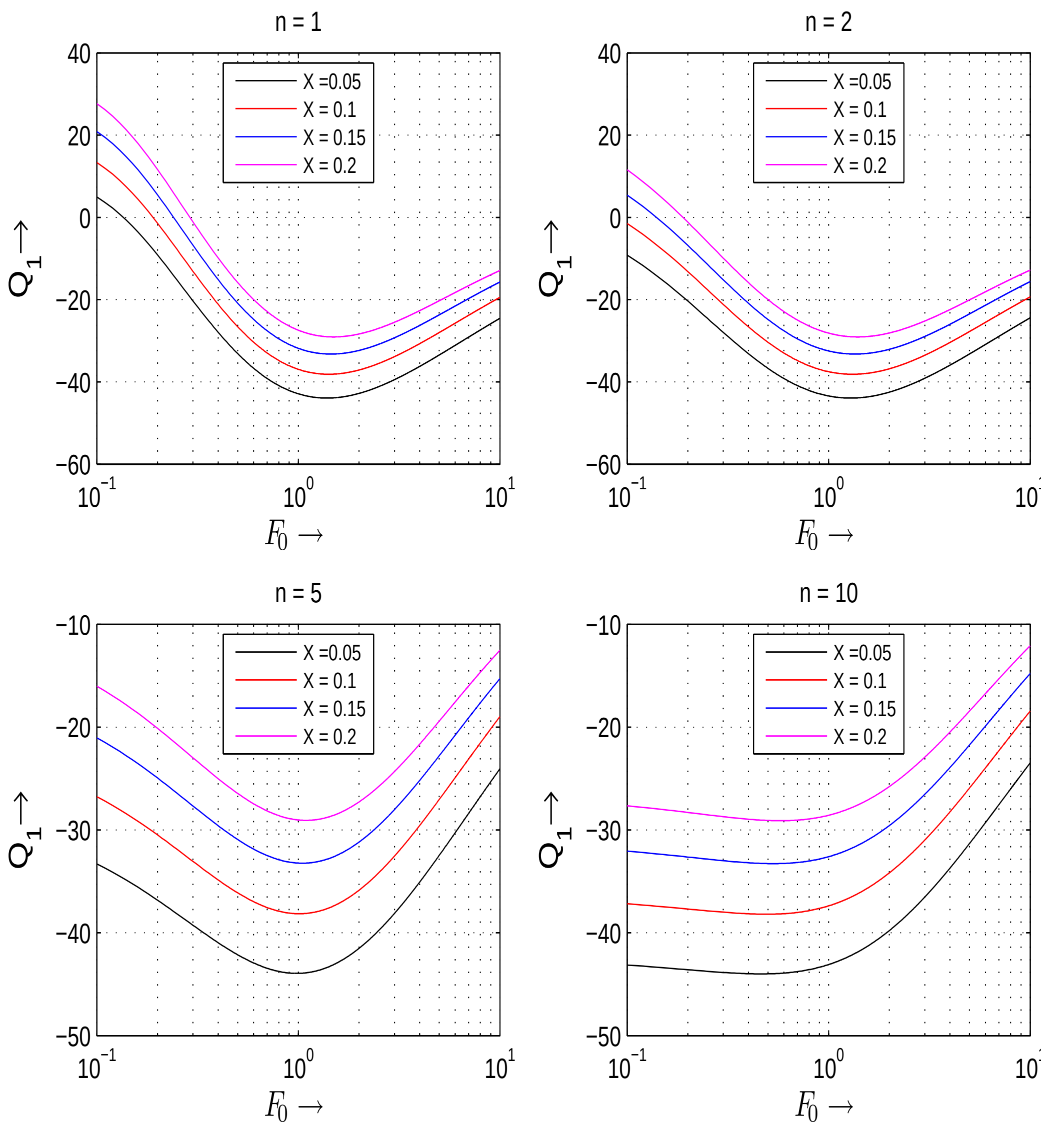

Figure 6.2: Reduced heat flux as a function of reduced time and distance for $B_0 = 100$ and $Bi = 1.00$

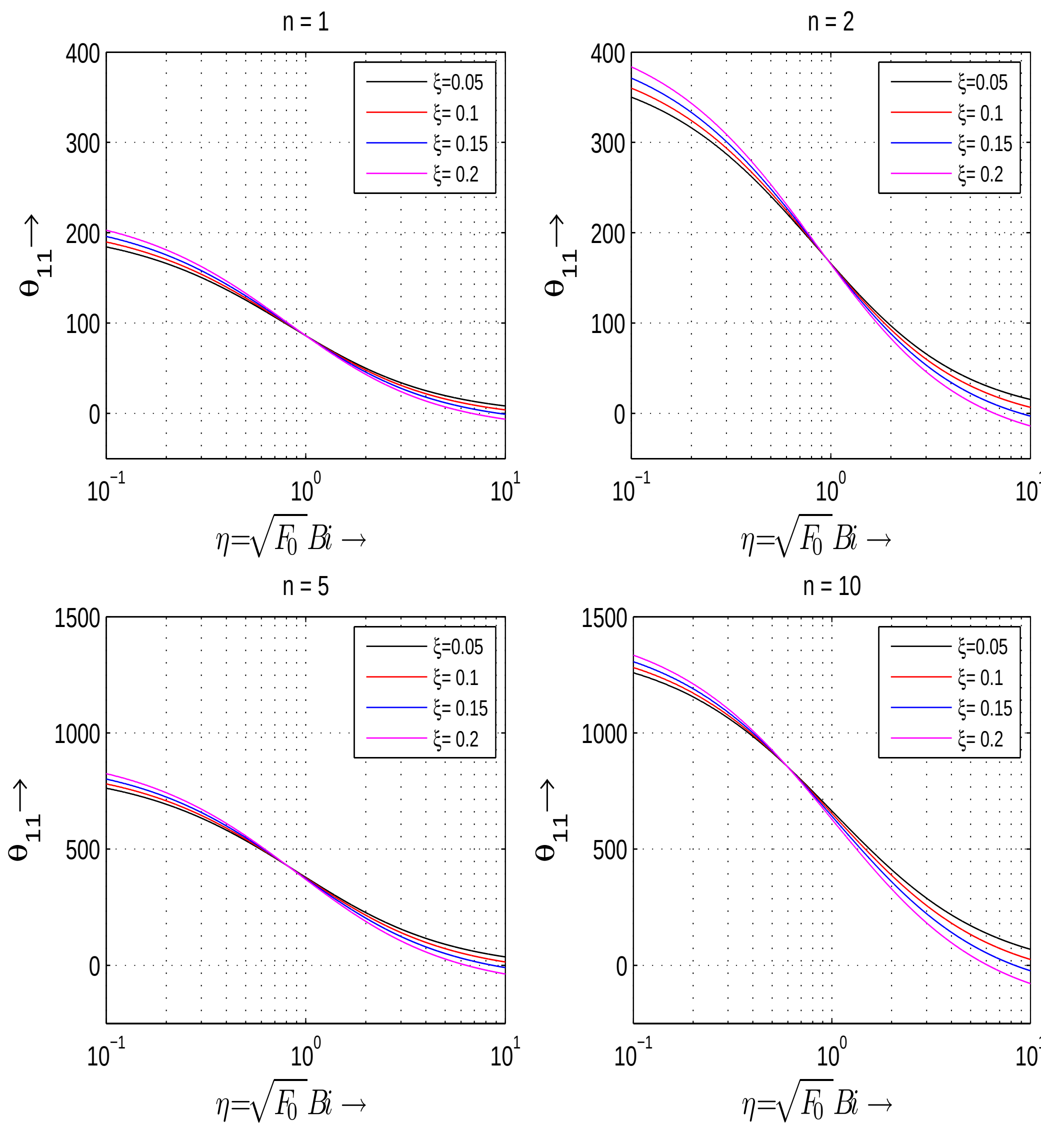

Figure 6.3: Reduced wall temperature as a function of dimensionless parameter η and ζ for $\beta = 1.00$

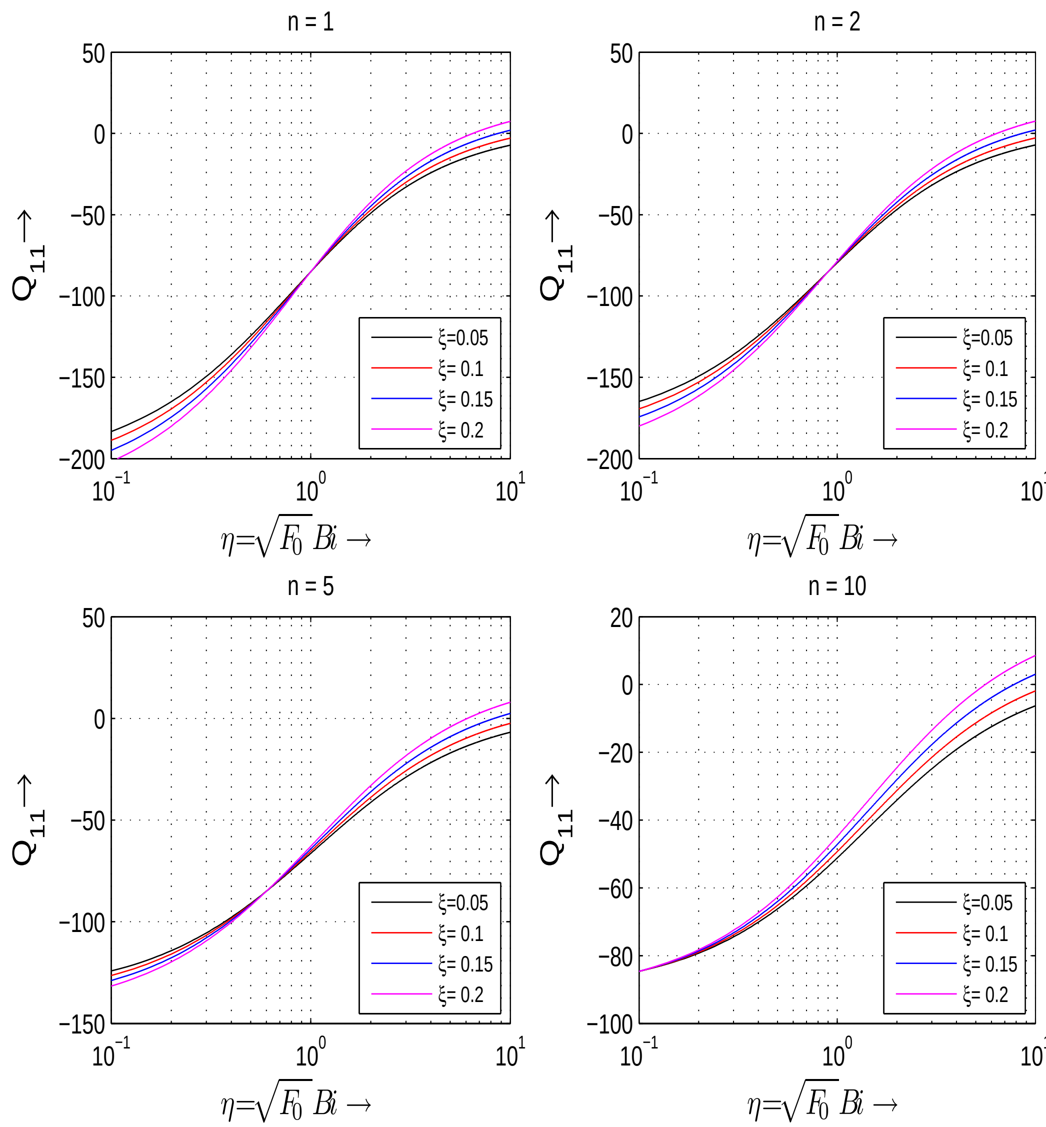

Figure 6.4: Reduced wall flux as a function of dimensionless parameter η and ζ for $\beta = 1.00$

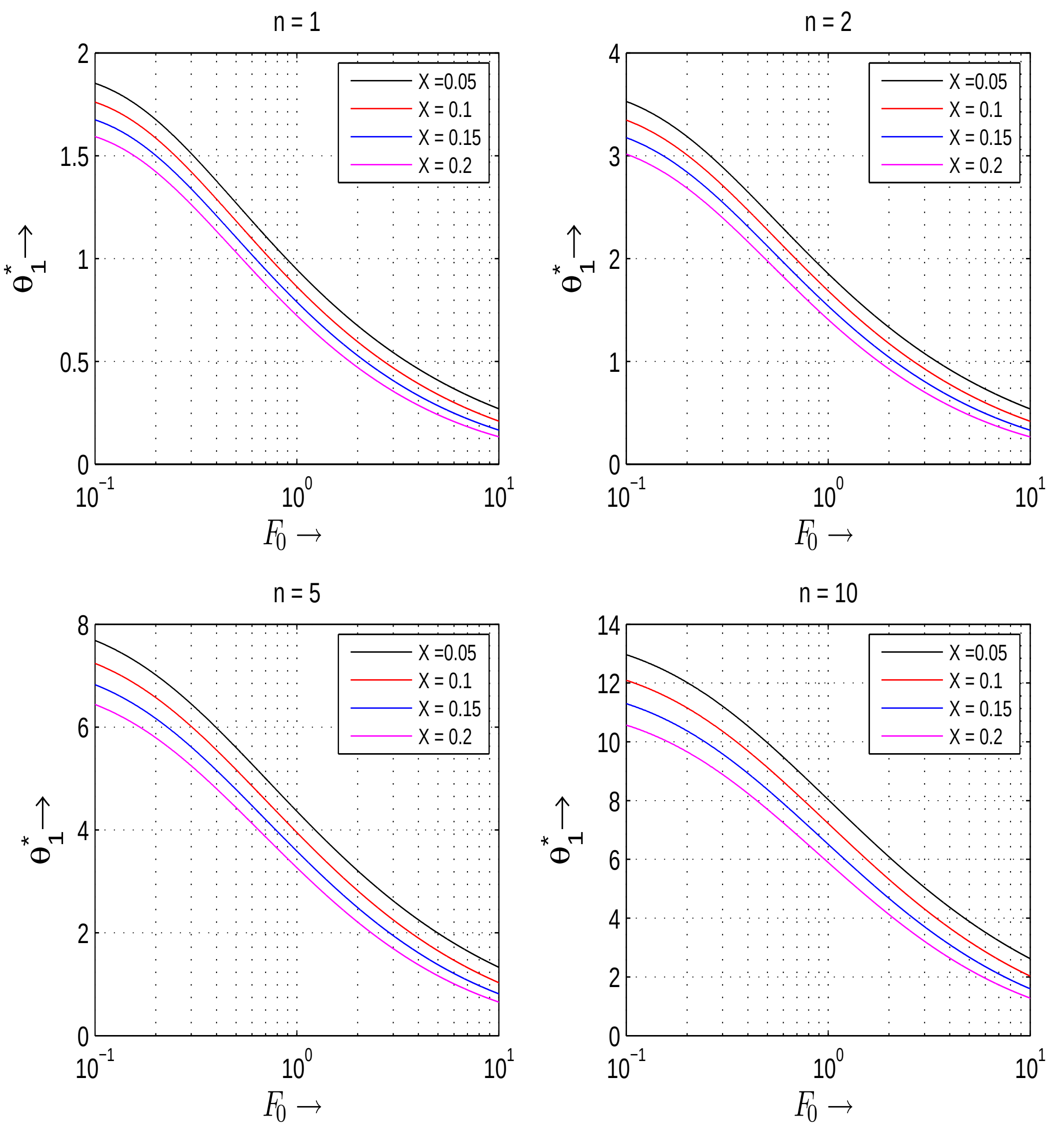

Figure 6.5: Reduced temperature as a function of reduced time and distance for the case of constant surface temperature

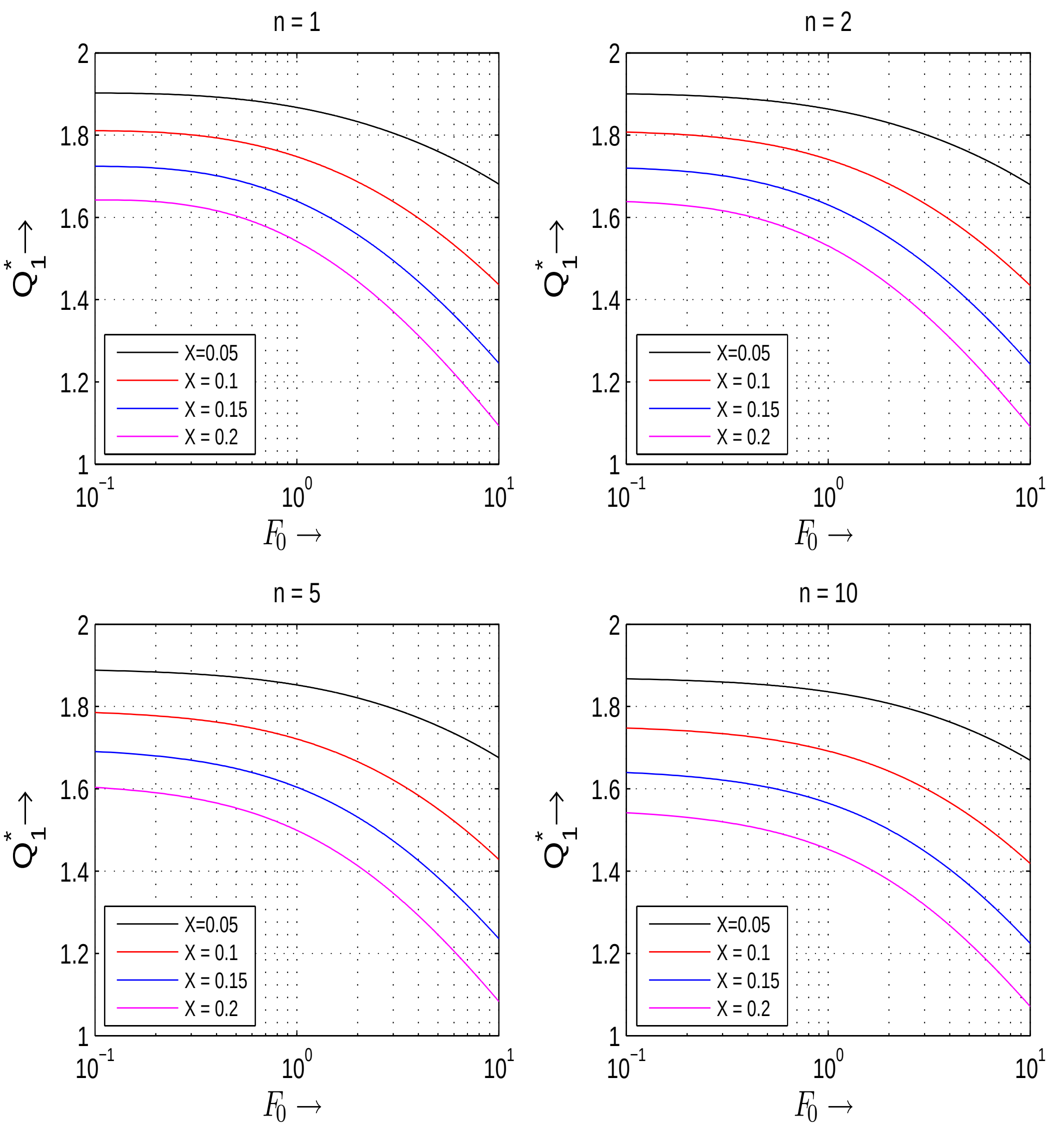

Figure 6.6: Reduced heat flux as a function of reduced time and distance for the case of constant surface temperature

Chapter 7

Table of Indices

Table 7.1: Subscripts for Chapters 5 and 6

f	film layer
p	substrate layer
11	at the wall
111	at the wall when diffusion effects are not important
12	sufficiently small time
13	no heat generation case

Table 7.2: Nomenclature for Chapters 5 and 6

A_f	absorption coefficient
α	thermal diffusivity $[m^2 s^{-1}]$
B	dimensionless parameter
ε	dimensionless parameter
c_p	specific heat $[Jkg^{-1}K^{-1}]$
λ	thermal conductivity $[Wm^{-1}K^{-1}]$
d	thickness of thin film [m]
λ'	wavelength [microns]
q_0	laser flux $[wm^{-2}]$
ρ	density $[kgm^{-3}]$
s	Laplace transform variable
T	excess temperature $[K]$
t	time variable $[s]$
t_m	critical time required to initiate melting $[s]$
x	spatial variable $[m]$
Bi	Biot number
B_0	dimensionless energy absorption
F_0	Fourier number
h	convective heat transfer coefficient $[W/m^2.K]$
I_0	energy released by laser source $[J/m^2]$
Q_1	dimensionless heat flux
q''	heat flux $[W/m^2]$
q'''	rate of energy generation per unit volume $[W/m^3]$
R	surface reflectance
X	dimensionless distance
θ_1	dimensionless temperature

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