GROUP CLASSIFICATION AND SYMMETRY REDUCTIONS OF A CLASS OF NONLINEAR POISSON EQUATIONS ON THE LINE AND CERTAIN SURFACES

BY

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To the memory of Amina, my dear mother

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THESIS ABSTRACT

Name	:	Abdulsattar Ahmed Al-Kubaish
Title	:	Group classification and symmetry reductions for a class of nonlinear Poisson equations on the line and certain surfaces
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Lie symmetry method is a technique to find exact solutions of differential equations. One of the significant applications of Lie symmetry theory is to achieve a complete classification of Lie symmetries and symmetry reductions of differential equations. This project is concerned with carrying out a symmetry analysis of a class of **nonlinear Poisson equations** *of the form*

$$\Delta u = f(u)$$

The Laplacian operator Δ will be considered in four cases: on the line (one dimension), and the two dimensional cases consisting of plane, sphere and helicoid. The function $f(u)$ will be assumed to be nonlinear.

In all four cases, the aim is to

- *find the minimal symmetry algebra*
- *find all forms of $f(u)$ which give larger symmetry algebras and determine these symmetry algebras*
- *find some symmetry reductions and exact solutions for each case of $f(u)$.*

ملخص الرسالة

الاسم : *عبدالستار أحمد آل كبيش*

عنوان الرسالة : *تصنيف الزمر و الاختزالات التناظرية لنوع من معادلات بويسون اللاخطية على الخط المستقيم و بعض السطوح*

مجال التخصص : *رياضيات*

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طريقة تناظر لي هي إحدى الطرق لإيجاد حلول دقيقة للمعادلات التفاضلية ، و أحد أهم التطبيقات لنظرية تناظر لي هو الوصول للتصنيف الكامل لتناظرات لي و الاختزالات التناظرية للمعادلات التفاضلية . هذا البحث معني بالتحليل التماثلي لفئة من معادلات بويسون اللاخطية على الصورة

$$\Delta u = f(u)$$

و سوف ينظر لـ مؤثر لابلاس Δ في أربع حالات : على الخط المستقيم (بعد واحد) و على ثلاثة أسطح ذات بعدين من الخط المستقيم و الكرة و الهيليكويد .
الدالة $f(u)$ يفترض أن تكون لاخطية.

في كل الحالات الأربع الهدف هو

- أن نوجد الجبر التناظري الأصغري
- أن نوجد جميع الصيغ الممكنة لدالة $f(u)$ التي تعطي جبريات تماثلية أكبر مع تحديد هذه الجبريات التماثلية.
- أن نوجد بعض الاختزالات التماثلية و حلول دقيقة لكل حالة من $f(u)$.

CHAPTER 0

INTRODUCTION AND PROBLEM FORMULATION

The mathematical modeling of most of the natural and physical processes leads to such nonlinear differential equations whose analytic solutions are hard to find. Therefore, investigations related to simplifications of nonlinear differential equations and construction of their exact solutions become significant in the analysis of nonlinear differential equations. Lie symmetry method has proven to be a powerful technique for analyzing non-linear ODEs and PDEs. It provides most widely applicable technique to find exact solutions of differential equations and contains, as particular case cf. [33], many efficient methods for solving differential equations like traveling wave solutions, self-similar solutions and exponential self-similar solutions. The classical Lie symmetry theory to study differential equations was developed by Sophus Lie more than a century ago. A modern treatment of the classical Lie symmetry theory was provided by Ovsiannikov [32]. Since the modern treatment by Ovsiannikov, the theory substantially grown and found widespread uses. A large amount of literature about the classical Lie symmetry theory, its applications and its extensions is available, e.g. [2, 6, 7, 8, 11, 15, 17, 18, 19, 20, 21, 30, 31, 32, 37].

The aim of this work is to investigate the group classification problem which, in general, consists of two main steps. The first step is finding the Lie symmetries of a differential equation with arbitrary f. The second step is determining all possible forms of f for which larger symmetry groups exist. The first group classification problem was carried out by Ovsiannikov [32] who classified all forms of the non-linear heat equation $u_t = (f(u)u_x)_x$. Since then a number of articles on symmetry analysis and classification problem for non-linear PDEs have appeared in literature, cf. [3, 4, 5, 9, 10, 12, 14, 16, 22,

23, 24, 25, 26, 27, 28, 34, 35, 38, 39]. In the present work we propose to perform the complete group classification of a class of nonlinear Poisson equations in one space dimension, and in two dimensions on three different surfaces plane, sphere and helicoid. The class of non-linear Poisson equations under our consideration is of the form $\Delta u = f(u)$.

Section 0.1 contains some basic ideas about differential geometry of surfaces, needed for setting up the problems of this research. The problem formulation and a summary of main results of thesis are provided in section 0.2.

0.1. SOME BASIC DEFINITIONS FROM DIFFERENTIAL GEOMETRY

To formulate our problem clearly we need some basic background in differential geometry, cf. [13, 36], specially the definition of Laplacian on surfaces. In this section we define the metric or first fundamental form of surfaces, the Laplacian and Gaussian curvature of surfaces.

Definition:

Let $X(x, y)$ be a coordinate patch or parameterization of a surface M. Then,

$$\mathrm{g} = \mathrm{ds}^2 = E dx^2 + 2F dx dy + G dy^2$$

is called the first fundamental form or Riemannian metric of the surface, where

$$E = X_x \cdot X_x, \quad F = X_x \cdot X_y, \quad G = X_y \cdot X_y.$$

Setting

$$\mathrm{g}_{11} = E = X_x \cdot X_x, \quad \mathrm{g}_{12} = \mathrm{g}_{21} = F = X_x \cdot X_y, \quad \mathrm{g}_{22} = G = X_y \cdot X_y$$

leads to the classical notation of the metric

$$\mathrm{g} = \mathrm{ds}^2 = \mathrm{g}_{11} dx^2 + 2\mathrm{g}_{12} dx dy + \mathrm{g}_{22} dy^2$$

or in the form of a symmetric matrix

$$\mathrm{g} = \begin{bmatrix} \mathrm{g}_{11} & \mathrm{g}_{12} \\ \mathrm{g}_{12} & \mathrm{g}_{22} \end{bmatrix} = (\mathrm{g}_{ij})$$

Note that

$$\det(\mathrm{g}) = \mathrm{g}_{11}\mathrm{g}_{22} - (\mathrm{g}_{12})^2 = \mathrm{EG} - \mathrm{F}^2 = |X_x \times X_y| \neq 0,$$

therefore,

$$\mathrm{g}^{-1} = \begin{bmatrix} \mathrm{g}^{11} & \mathrm{g}^{12} \\ \mathrm{g}^{12} & \mathrm{g}^{22} \end{bmatrix} = \frac{1}{\det(\mathrm{g})} \begin{bmatrix} \mathrm{g}_{22} & -\mathrm{g}_{12} \\ -\mathrm{g}_{12} & \mathrm{g}_{11} \end{bmatrix} = (\mathrm{g}^{ij})$$

The main operator involved in our investigation is Laplacian on surfaces which is defined below.

Definition:

Consider a surface with a metric g. Then the Laplacian on the surface is defined as

$$\boxed{\Delta u = \frac{1}{\sqrt{|\det(\mathrm{g})|}} \frac{\partial}{\partial x^i} \left(\sqrt{|\det(\mathrm{g})|} \mathrm{g}^{ij} \frac{\partial u}{\partial x^j} \right)} \tag{0.1}$$

where the summation is taken over repeated indices.

Since the symmetry analysis of this thesis is carried out on surfaces of different curvatures, it is worth recalling the formulas for calculating curvatures of surfaces [13].

Let $X(x, y)$ be a coordinate patch or parameterization of a surface M.

Setting

$$E = X_x \cdot X_x, \quad F = X_x \cdot X_y, \quad G = X_y \cdot X_y, \quad U = \frac{X_x \times X_y}{|X_x \times X_y|}$$

then the components of the second fundamental form are

$$l = X_{xx} \cdot U, \quad m = X_{xy} \cdot U, \quad n = X_{yy} \cdot U$$

and the Gaussian curvature of the surface is given by

$$\boxed{\kappa = \frac{ln - m^2}{EG - F^2}}$$

0.2. NONLINEAR POISSON EQUATIONS , PROBLEM FORMULATION AND MAIN RESULTS

The objective of the study is to carry out complete symmetry group classification and symmetry analysis of nonlinear Poisson equations of the form

$$\Delta u = f(u) \tag{0.2}$$

where Δ denotes the Laplacian operator defined in the previous section and $f(u)$ is a nonlinear function. Precisely, equation (0.2) will be investigated on

- the line (one dimensional case)

as well as on the following surfaces

- the plane (zero curvature case)
- the sphere (positive curvature case)
- the helicoid (negative curvature case)

0.2.1. Problem for the line (one-dimensional case)

The Laplacian in this case is

$$\Delta u = u_{xx}.$$

Hence the non-linear Poisson equation on the line becomes

$$\boxed{u_{xx} = f(u)} \tag{0.3}$$

The group classification of equation (0.3) is carried out in Chapter 2, where the following result is proved.

Theorem 2.1

The minimal symmetry algebra of nonlinear ODE $y'' = f(y)$ is generated by $X = \frac{\partial}{\partial x}$ and is obtained for all nonlinear arbitrary functions $f(y)$. The larger symmetry algebra exists in the cases given in the table below:

$f(y)$	**Generators of symmetry algebra**
$a(y+b)^c$, $c \neq 0,1,-3$	$X_1 = \frac{\partial}{\partial x}$, $X_2 = x\frac{\partial}{\partial x} - \frac{2}{c-1}(y+b)\frac{\partial}{\partial y}$
ae^{by} , $b \neq 0$	$X_1 = \frac{\partial}{\partial x}$, $X_2 = x\frac{\partial}{\partial x} - \frac{2}{b}\frac{\partial}{\partial y}$
$a(y+b)^{-3}$, $a \neq 0$	$X_1 = \frac{\partial}{\partial x}$, $X_2 = x\frac{\partial}{\partial x} + \frac{1}{2}(y+b)\frac{\partial}{\partial y}$, $X_3 = x^2\frac{\partial}{\partial x} + x(y+b)\frac{\partial}{\partial y}$
$a(y+b)^{-3} + c(y+b)$ $a \neq 0, c > 0$	$X_1 = \frac{\partial}{\partial x}$, $X_2 = e^{2x\sqrt{c}}\frac{\partial}{\partial x} + \sqrt{c}e^{2x\sqrt{c}}(y+b)\frac{\partial}{\partial y}$, $X_3 = e^{-2x}\frac{\partial}{\partial x} - \sqrt{c}\, e^{-2x\sqrt{c}}(y+b)\frac{\partial}{\partial y}$
$a(y+b)^{-3} + c(y+b)$ $a \neq 0, c < 0$	$X_1 = \frac{\partial}{\partial x}$, $X_2 = \sin\left(2\sqrt{-c}\, x\right)\frac{\partial}{\partial x} + \sqrt{-c}(y+b)\cos\left(2\sqrt{-c}\, x\right)\frac{\partial}{\partial y}$, $X_3 = \cos\left(2\sqrt{-c}\, x\right)\frac{\partial}{\partial x} - \sqrt{-c}(y+b)\sin(2\sqrt{-c}\, x)\frac{\partial}{\partial y}$

0.2.2. Problem for plane (zero curvature case)

Consider the xy-plane,

$$z = 0.$$

It can be parameterized as

$$X(x,y) = (x,y,0)$$

To find the first fundamental form, we calculate the following quantities

$$E = X_x \cdot X_x = (1,0,0) \cdot (1,0,0) = 1$$
$$F = X_x \cdot X_y = (1,0,0) \cdot (0,1,0) = 0$$
$$G = X_y \cdot X_y = (0,1,0) \cdot (0,1,0) = 1$$

So, the metric or first fundamental form is obviously

$$\mathrm{g} = dx^2 + dy^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We can check that the **Gaussian curvature** for this metric is 0.

Calculation of the Gaussian curvature:

$$X_x \times X_y = (1,0,0) \times (0,1,0) = (0,0,1)$$
$$U = \frac{X_x \times X_y}{|X_x \times X_y|} = (0,0,1)$$
$$X_{xx} = (0,0,0)$$
$$X_{xy} = (0,0,0)$$
$$X_{yy} = (0,0,0)$$

The components of the second fundamental form are

$$l = X_{xx} \cdot U = 0$$
$$m = X_{xy} \cdot U = 0$$
$$n = X_{yy} \cdot U = 0$$

Gaussian curvature of the plane $= \frac{ln - m^2}{EG - F^2} = \frac{0}{1} = 0$

The Laplacian on the plane is calculated below.

$$\Delta u = \frac{1}{\sqrt{|\det(\mathrm{g})|}} \frac{\partial}{\partial x^i}\left(\sqrt{|\det(\mathrm{g})|}\mathrm{g}^{ij} \frac{\partial u}{\partial x^j}\right) = \frac{\partial}{\partial x}\left(\mathrm{g}^{11} \frac{\partial u}{\partial x} + \mathrm{g}^{12} \frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial y}\left(\mathrm{g}^{21} \frac{\partial u}{\partial x} + \mathrm{g}^{22} \frac{\partial u}{\partial y}\right)$$
$$= \frac{\partial}{\partial x}\left(1 \frac{\partial u}{\partial x} + 0 \frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial y}\left(0 \frac{\partial u}{\partial x} + 1 \frac{\partial u}{\partial y}\right)$$
$$= u_{xx} + u_{yy}$$

Now, equation (0.2) becomes:

$$\boxed{u_{xx} + u_{yy} = f(u)} \tag{0.4}$$

which is nonlinear Poisson equation on plane. Chapter 3 is devoted to group classification and symmetry analysis of PDE (0.4). The main result proved is

Theorem 3.1

The minimal symmetry algebra of nonlinear Poisson equation (0.4) is 3-dimensional generated by

$$P_1 = \frac{\partial}{\partial x}, \qquad P_2 = \frac{\partial}{\partial y}, \qquad P_3 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$

and is obtained for all nonlinear arbitrary functions $f(u)$. The larger symmetry algebra exists in the cases given in the table below:

$f(y)$	Extra Generators of symmetry algebra
$a(u+b)^c$, $c \neq 0,1$	$P_4 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \frac{2}{c-1}(u+b)\frac{\partial}{\partial u}$
ae^{bu} , $a, b \neq 0$	Infinite dimensional algebra generated by $P_\infty = \xi(x,y)\frac{\partial}{\partial x} + \tau(x,y)\frac{\partial}{\partial y} - \frac{2}{b}\tau_y(x,y)\frac{\partial}{\partial u}$ where $\tau_{xx} + \tau_{yy} = 0,$ $\xi_x = \tau_y,$ $\xi_y = -\tau_x$

0.2.3. Problem for sphere (positive curvature case)

Consider the sphere

$$x^2 + y^2 + z^2 = 1$$

with parameterization

$$X(x,y) = (\cos x\,, \sin x \cos y\,, \sin x \sin y)$$

which implies

$$X_x = (-\sin x\,, \cos x \cos y\,, \cos x \sin y)$$

$$X_y = (0, -\sin x \sin y\,, \sin x \cos y)$$

The components of the first fundamental form are given by

$$E = X_x \cdot X_x = (-\sin x\,, \cos x \cos y\,, \cos x \sin y) \cdot (-\sin x\,, \cos x \cos y\,, \cos x \sin y) = 1$$

$$F = X_x \cdot X_y = (-\sin x\,, \cos x \cos y\,, \cos x \sin y) \cdot (0, -\sin x \sin y\,, \sin x \cos y) = 0$$

$$G = X_y \cdot X_y = (0, -\sin x \sin y\,, \sin x \cos y) \cdot (0, -\sin x \sin y\,, \sin x \cos y) = \sin^2 x$$

So, the metric is obviously

$$\mathrm{g} = dx^2 + \sin^2 x\, dy^2 = \begin{bmatrix} 1 & 0 \\ 0 & \sin^2 x \end{bmatrix}$$

and

$$\det(\mathrm{g}) = \sin^2 x$$

which implies

$$\mathrm{g}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \csc^2 x \end{bmatrix}$$

We can check that the Gaussian curvature for this metric is positive.

$$X_x \times X_y = (-\sin x\,, \cos x \cos y\,, \cos x \sin y) \times (0, -\sin x \sin y\,, \sin x \cos y)$$

$$= (\sin x \cos x\,, \sin^2 x \cos y\,, \sin^2 x \sin y)$$

$$U = \frac{X_x \times X_y}{|X_x \times X_y|} = \frac{(\sin x \cos x, \sin^2 x \cos y, \sin^2 x \sin y)}{\sin x} = (\cos x\,, \sin x \cos y\,, \sin x \sin y)$$

$$X_{xx} = (-\cos x\,, -\sin x \cos y\,, -\sin x \sin y)$$

$$X_{xy} = (0, -\cos x \sin y\,, \cos x \cos y)$$

$$X_{yy} = (0, -\sin x \cos y\,, -\sin x \sin y)$$

Then, the components of the second fundamental form are given by

$$l = X_{xx} \cdot U = -\cos^2 x - \sin^2 x \cos^2 y - \sin^2 x \sin^2 y = -1$$

$$m = X_{xy} \cdot U = 0$$

$$n = X_{yy} \cdot U = -\sin^2 x \cos^2 y - \sin^2 x \sin^2 y = -\sin^2 x$$

Gaussian curvature of the sphere

$$= \frac{ln - m^2}{EG - F^2} = \frac{\sin^2 x}{\sin^2 x} = 1$$

which means that the Gaussian curvature of the sphere is positive.

The Laplacian on the sphere

$$\Delta u = \frac{1}{\sqrt{|\det(\mathrm{g})|}} \frac{\partial}{\partial x^i}\left(\sqrt{|\det(\mathrm{g})|}\,\mathrm{g}^{ij} \frac{\partial u}{\partial x^j}\right)$$

$$= \frac{1}{\sin x} \frac{\partial}{\partial x}\left(\sin x\, 1 \frac{\partial u}{\partial x} + \sin x\,(0) \frac{\partial u}{\partial y}\right) + \frac{1}{\sin x} \frac{\partial}{\partial y}\left(\sin x\,(0) \frac{\partial u}{\partial x} + \sin x \csc^2 x \frac{\partial u}{\partial y}\right)$$

$$= \frac{1}{\sin x} \frac{\partial}{\partial x}\left(\sin x \frac{\partial u}{\partial x}\right) + \frac{1}{\sin x} \frac{\partial}{\partial y}\left(\csc x \frac{\partial u}{\partial y}\right)$$

$$= u_{xx} + (\cot x) u_x + (\csc^2 x) u_{yy}$$

Hence, the Poisson equation on sphere becomes:

$$\boxed{u_{xx} + (\cot x)u_x + (\csc^2 x)u_{yy} = f(u)} \tag{0.5}$$

In Chapter 4, we present complete group classification, symmetry reduction and some exact solutions of PDE (0.5). The main result of Chapter 4 is precisely the following classification is

***Theorem* 4.1**

The minimal symmetry algebra of nonlinear PDE (0.5) is three dimensional and is generated by

$$S_1 = \sin y \frac{\partial}{\partial x} + \cot x \cos y \frac{\partial}{\partial y}, \qquad S_2 = \cos y \frac{\partial}{\partial x} - \cot x \sin y \frac{\partial}{\partial y}, \qquad S_3 = \frac{\partial}{\partial y}$$

and is obtained for all nonlinear arbitrary functions $f(u)$. Infinite dimensional symmetry algebra exists in the case

$$f(u) = ae^{bu} + \frac{2}{b}, \ a, b \neq 0$$

which is generated by

$$X = \xi(x,y)\frac{\partial}{\partial x} + \tau(x,y)\frac{\partial}{\partial y} + \phi(x,y)\frac{\partial}{\partial u}$$

where $\tau(x,y)$ is a harmonic function on sphere that satisfies the following pde

$$\tau_{xx} + (\cot x)\tau_x + (\csc^2 x)\tau_{yy} = 0,$$

The function $\xi(x,y)$ is given by the following two relations

$$\xi = -\sin^2 x \int \tau_x(x,y)dy + g(x)$$

$$\xi_x - \cot x\, \xi = \tau_y$$

and the function $\phi(x,y)$ is given by

$$\phi = \frac{-2}{b}\xi_x.$$

0.2.4. Problem for helicoids (negative curvature case)

Consider the helicoid with parameterization

$$X(x,y) = (x\cos y\,, x\sin y\,, by).$$

Then,

$$X_x = (\cos y\,, \sin y\,, 0)$$

$$X_y = (-x\sin y\,, x\cos y\,, b)$$

The first fundamental form has components

$$E = X_x \cdot X_x = (\cos y\,, \sin y\,, 0)\cdot(\cos y\,, \sin y\,, 0) = 1$$

$$F = X_x \cdot X_y = (\cos y\,, \sin y\,, 0)\cdot(-x\sin y\,, x\cos y\,, b) = 0$$

$$G = X_y \cdot X_y = (-x\sin y\,, x\cos y\,, b)\cdot(-x\sin y\,, x\cos y\,, b) = x^2 + b^2$$

So the metric is obviously

$$\mathrm{g} = dx^2 + (x^2+b^2)dy^2 = \begin{bmatrix} 1 & 0 \\ 0 & x^2+b^2 \end{bmatrix}$$

which implies

$$\det(\mathrm{g}) = x^2 + b^2$$

and hence,

$$\mathrm{g}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1/(x^2+b^2) \end{bmatrix}$$

We can check that the Gaussian curvature for this metric is negative.

$$X_x \times X_y = (\cos y\,, \sin y\,, 0)\times(-x\sin y\,, x\cos y\,, b) = (b\sin y\,, -b\cos y\,, x)$$

$$U = \frac{X_x \times X_y}{|X_x \times X_y|} = \frac{(b\sin y, -b\cos y, x)}{\sqrt{x^2+b^2}} = \left(\frac{b\sin y}{\sqrt{x^2+b^2}}, \frac{-b\cos y}{\sqrt{x^2+b^2}}, \frac{x}{\sqrt{x^2+b^2}}\right)$$

$$X_{xx} = (0,0,0)$$

$$X_{xy} = (-\sin y\,, \cos y\,, 0)$$

$$X_{yy} = (-x\cos y\,, -x\sin y\,, 0)$$

which give components of the second fundamental form

$$l = X_{xx}\cdot U = 0$$

$$m = X_{xy}\cdot U = \frac{-b}{\sqrt{x^2+b^2}}$$

$$n = X_{yy}\cdot U = 0$$

Gaussian curvature of the helicoids

$$= \frac{ln-m^2}{EG-F^2} = \frac{0-\frac{b^2}{x^2+b^2}}{x^2+b^2-0} = -\frac{b^2}{(x^2+b^2)^2}$$

This means that the Gaussian curvature of the helicoids is negative.

The Laplacian on the helicoids is given by

$$\Delta u = \frac{1}{\sqrt{|\det(g)|}} \frac{\partial}{\partial x^i}\left(\sqrt{|\det(g)|}\, g^{ij} \frac{\partial u}{\partial x^j}\right)$$

$$= \frac{1}{\sqrt{x^2+b^2}} \frac{\partial}{\partial x}\left(\sqrt{x^2+b^2}(1)\frac{\partial u}{\partial x} + \sqrt{x^2+b^2}(0)\frac{\partial u}{\partial y}\right)$$

$$+ \frac{1}{\sqrt{x^2+b^2}} \frac{\partial}{\partial y}\left(\sqrt{x^2+b^2}(0)\frac{\partial u}{\partial x} + \sqrt{x^2+b^2}\left(\frac{1}{x^2+b^2}\right)\frac{\partial u}{\partial y}\right)$$

$$= \frac{1}{\sqrt{x^2+b^2}} \frac{\partial}{\partial x}\left(\sqrt{x^2+b^2}\frac{\partial u}{\partial x}\right) + \frac{1}{\sqrt{x^2+b^2}} \frac{\partial}{\partial y}\left(\sqrt{x^2+b^2}\left(\frac{1}{x^2+b^2}\right)\frac{\partial u}{\partial y}\right)$$

$$= u_{xx} + \frac{1}{\sqrt{x^2+b^2}}\left(\frac{x}{\sqrt{x^2+b^2}}\right)\frac{\partial u}{\partial x} + \left(\frac{1}{x^2+b^2}\right)u_{yy}$$

$$= u_{xx} + \left(\frac{x}{x^2+b^2}\right)u_x + \left(\frac{1}{x^2+b^2}\right)u_{yy}$$

Hence, the Poisson equation on helicoids is

$$\boxed{u_{xx} + \left(\frac{x}{x^2+b^2}\right)u_x + \left(\frac{1}{x^2+b^2}\right)u_{yy} = f(u)} \qquad (0.6)$$

Chapter 5 contains complete symmetry group classification of PDE (0.6) where we prove

Theorem 5.1

The minimal symmetry algebra of nonlinear Poisson equation (0.6) is one-dimensional generated by

$$H_1 = \frac{\partial}{\partial y},$$

and is obtained for all nonlinear arbitrary functions $f(u)$. Three dimensional symmetry algebra exists in the case

$$f(u) = ae^{Bu}\,,\ a, b \neq 0$$

which is generated by

$$H_1 = \frac{\partial}{\partial y}\ ,$$

$$H_2 = (x^2+b^2)\cos y\frac{\partial}{\partial x} + x\sin y\frac{\partial}{\partial y} - \frac{4x}{B}\cos y\frac{\partial}{\partial u}$$

$$H_3 = (x^2+b^2)\sin y\frac{\partial}{\partial x} - x\cos y\frac{\partial}{\partial y} - \frac{4x}{B}\sin y\frac{\partial}{\partial u}$$

This theorem is summarized in the table below:

$f(u)$	Generators of symmetry algebra
any non-linear function	$H_1 = \frac{\partial}{\partial y}$
ae^{Bu} , $a, B \neq 0$	Three dimensional algebra generated by $H_1 = \frac{\partial}{\partial y}$ $H_2 = (x^2 + b^2)\cos y \frac{\partial}{\partial x} + x \sin y \frac{\partial}{\partial y} - \frac{4x}{B}\cos y \frac{\partial}{\partial u}$ $H_3 = (x^2 + b^2)\sin y \frac{\partial}{\partial x} - x \cos y \frac{\partial}{\partial y} - \frac{4x}{B}\sin y \frac{\partial}{\partial u}$

CHAPTER 1

LIE SYMMETRY METHOD FOR DIFFERENTIAL EQUATIONS

This chapter focuses on basic ideas of Lie symmetry method tht should serve the bases for following the research results presented in chapters 2,3,4 and 5. The main aim is to give a short review of the standard baackground in Lie symmetry method for ODEs and PDEs. Since the fundamental results of Lie symmetry methods are well established and have become a standard now, most of the proofs in this short review are omitted. However, the necessry details are presented where these were thought to be essential. This review follows the books [8, 21]. In general, the reader is referred to the standard books [8, 17, 21, 31, 37] for thorough introduction and complete understanding of the subject of Lie symmetry analysis and its applications.

Section 1.1 and section 1.2 present the concepts and methods of Lie symmetries needed to deal with ODEs and PDEs respectively.

1.1. LIE SYMMETRY METHOD FOR ODEs

Beginning with the ideas of one parameter groups and their infinitesimal generators, this section presents the notions of prolongations, symmetry of ODEs as well as the method of finding symmetries of ODEs. The section ends with illustrative examples of reduction of order of ODEs using symmetries.

1.1.1. One parameter group of transformations and infinitesimal generators

The study of Lie symmetries of ODEs involves one parameter groups of transformations in plane.

Definition: (Point transformation in xy −plane)
A point transformation in xy -plane is a function

$$T: R^2 \longrightarrow R^2$$

given by

$$(x, y) \xrightarrow{T} (\bar{x}, \bar{y})$$

where

$$\bar{x} = f(x, y)$$
$$\bar{y} = g(x, y)$$

A transformation in the xy-plane transforms a point (x, y) to another point $(\bar{x}, \bar{y})$.

Examples of transformations in xy-plane:

Translation:

$$\bar{x} = x + 3$$
$$\bar{y} = y - 2$$

Dilation:

$$\bar{x} = 2x$$
$$\bar{y} = 5y$$

Rotation:

$$\bar{x} = x\cos\theta - y\sin\theta$$
$$\bar{y} = x\sin\theta + y\cos\theta$$

Definition: (one parameter transformation)
It is a transformation that depends on one parameter only.
If the parameter is ϵ, then it is of the form:

$$\bar{x} = f(x, y, \epsilon)$$
$$\bar{y} = g(x, y, \epsilon)$$

Examples:

1) The rotation above depends only on the parameter θ.

2) The translation:

$$\bar{x} = x + \epsilon$$
$$\bar{y} = y + 2\epsilon$$

is a one parameter transformation depending on ϵ only.

3) The translation

$$\bar{x} = x + k_1$$
$$\bar{y} = x + k_2$$

is **not** one parameter transformation since it is associated with **two parameters** k_1 and k_2.

Definition: (**one parameter group of transformations**)

The one parameter transformation

$$\bar{x} = f(x, y, \epsilon)$$
$$\bar{y} = g(x, y, \epsilon)$$

is a one parameter group of transformations if the following properties hold:

(i). Identity:

The transformation with $\epsilon = 0$ is the identity transformation.

$$T_0(x, y) = (x, y)$$

(ii). Inverse:

The transformation with $-\mathcal{E}$ gives the inverse transformation

$$T_\varepsilon T_{-\varepsilon} = T_{-\varepsilon} T_\varepsilon = T_0$$

(iii). Closure:

The composition of two transformations in the group is a member of the set of transformation in the group

$$T_a T_b = T_c$$

Remark: the associativity holds always because the composition operation is always associative.

<u>Example</u>: Show that the following transformation is a one parameter group.

$T_\epsilon : R^2 \longrightarrow R^2$ defined by

$$\bar{x} = x + \epsilon$$
$$\bar{y} = y + 2\epsilon$$

Solution:

(i)The identity is clearly

$$T_0(x,y) = (x,y)$$

(ii)Let $a, b\epsilon\mathbb{R}$

$$T_b T_a(x,y) = T_b(x+a, y+2a) = (x+a+b, y+2a+2b)$$
$$= (x+a+b, y+2(a+b)) = T_{a+b}(x,y)$$

So, the set of these transformations is closed.

(iii)The inverse of T_ϵ is

$$T_\epsilon{}^{-1} = T_{-\epsilon}$$

because

$$T_{-\epsilon}T_\epsilon = T_{-\epsilon+\epsilon} = T_0$$

and

$$T_\epsilon T_{-\epsilon} = T_{\epsilon-\epsilon} = T_0$$

Infinitesimal generators of one parameter group of transformations

Given a one parameter group of transformations

$$\bar{x} = f(x,y,\epsilon) \tag{1.1}$$

$$\bar{y} = g(x,y,\epsilon) \tag{1.2}$$

with

$$f(x,y,0) = x \tag{1.3}$$

$$g(x,y,0) = y \tag{1.4}$$

This obviously defines a curve $\alpha(\epsilon) = (\bar{x},\bar{y}) = (f(x,y,\epsilon), g(x,y,\epsilon))$. If the tangent vector to the curve through the point (x,y) is denoted by (ξ,η), then the operator

$$X = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} \tag{1.5}$$

is called **infinitesimal generator** of one parameter group of transformations . This gives directions in which the point will move along the curve

$$\alpha(\epsilon) = (f(x,y,\epsilon), g(x,y,\epsilon)).$$

For practical purpose, it is more convenient to use the infinitesimal generator (or the operator form) of the group. So it is important to learn how to find the generator from a group, and how to find the group from a generator.

Infinitesimal generator of groups

Given a one parameter group of transformations

$$\bar{x} = f(x, y, \epsilon) \tag{1.6}$$

$$\bar{y} = g(x, y, \epsilon) \tag{1.7}$$

with

$$f(x, y, 0) = x$$

$$g(x, y, 0) = y.$$

The associated infinitesimal generator is given by

$$X = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y} \tag{1.8}$$

where

$$\xi(x, y) = \left.\frac{df(x,y,\epsilon)}{d\epsilon}\right|_{\epsilon=0} \tag{1.9}$$

$$\eta(x, y) = \left.\frac{dg(x,y,\epsilon)}{d\epsilon}\right|_{\epsilon=0} \tag{1.10}$$

<u>Example</u>: Find the infinitesimal generator of the group of rotations

$$\bar{x} = x\cos\theta - y\sin\theta$$

$$\bar{y} = x\sin\theta + y\cos\theta$$

<u>Solution</u>:

$$\xi(x, y) = \left.\frac{d\bar{x}}{d\theta}\right|_{\theta=0} = -x\sin 0 - y\cos 0 = -y$$

$$\eta(x, y) = \left.\frac{d\bar{y}}{d\theta}\right|_{\theta=0} = x\cos 0 - y\sin 0 = x$$

The infinitesimal generator associated with this group then is given by

$$X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$

Some important groups with their associated infinitesimals are given in the table below

Action	Group	Infinitesimal generator
Translation in x −axis	$\bar{x} = x + \epsilon$ $\bar{y} = y$	$\frac{\partial}{\partial x}$
Translation in y −axis	$\bar{x} = x$ $\bar{y} = y + \epsilon$	$\frac{\partial}{\partial y}$
Dilation in x −axis	$\bar{x} = e^{\epsilon} x$ $\bar{y} = y$	$x\frac{\partial}{\partial x}$
Dilation in y −axis	$\bar{x} = x$ $\bar{y} = e^{\epsilon} y$	$y\frac{\partial}{\partial y}$
Irregular dilation	$\bar{x} = e^{a\epsilon} x$ $\bar{y} = e^{b\epsilon} y$	$ax\frac{\partial}{\partial x} + by\frac{\partial}{\partial y}$
Rotation	$\bar{x} = x\cos\epsilon - y\sin\epsilon$ $\bar{y} = x\sin\epsilon + y\cos\epsilon$	$-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$
	$\bar{x} = x/(1-\epsilon y)$ $\bar{y} = y/(1-\epsilon y)$	$xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}$

Group corresponding to generator

Given infinitesimal generator

$$X = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} \tag{1.11}$$

The associated one parameter group of transformations

$$\bar{x} = f(x,y,\epsilon) \tag{1.12}$$

$$\bar{y} = g(x,y,\epsilon) \tag{1.13}$$

with

$$f(x,y,0) = x \tag{1.14}$$

$$g(x,y,0) = y \tag{1.15}$$

can be found by solving the system of ODEs given by

$$\frac{d\bar{x}}{d\epsilon} = \xi(\bar{x},\bar{y}) \tag{1.16}$$

$$\frac{d\bar{y}}{d\epsilon} = \eta(\bar{x},\bar{y}) \tag{1.17}$$

with initial conditions

$$\bar{x}|_{\epsilon=0} = x \tag{1.18}$$

$$\bar{y}|_{\epsilon=0} = y \tag{1.19}$$

<u>Example</u>: Find the one parameter group corresponding to the generator

$$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$$

<u>Solution</u>: Need to solve the system

$$\frac{d\bar{x}}{d\epsilon} = \bar{x} \tag{1.20}$$

$$\frac{d\bar{y}}{d\epsilon} = \bar{y} \tag{1.21}$$

with the initial conditions

$$\bar{x}(0) = x \tag{1.22}$$

$$\bar{y}(0) = y \tag{1.23}$$

Solving Eq. (1.20) gives

$$\bar{x} = Ce^{\epsilon} \tag{1.24}$$

Applying the initial condition (1.22) implies

$$C = x$$

Therefore,

$$\bar{x} = e^{\epsilon}x$$

Similarly, solving the second equation (1.21) with initial condition (1.23) gives

$$\bar{y} = e^{\epsilon}y$$

1.1.2. Definition of symmetries of ODEs

Given an ODE

$$F\left(x, y, y', \dots, y^{(n)}\right) = 0\,. \tag{1.25}$$

A one parameter group of transformations

$$\bar{x} = f(x, y, \epsilon) \tag{1.26}$$

$$\bar{y} = g(x, y, \epsilon) \tag{1.27}$$

is called a symmetry of ODE (1.25) if the form of the ODE (1.25) remains unchanged under the transformation (1.26)-(1.27).

i.e. after change of variables (x, y) to $(\bar{x}, \bar{y})$, we get

$$F\left(\bar{x}, \bar{y}, \overline{y'}, \dots, \overline{y^{(n)}}\right) = 0 \tag{1.28}$$

Example: Show that the transformation

$$\bar{x} = e^{\epsilon} x \tag{1.29}$$

$$\bar{y} = e^{\epsilon} y \tag{1.30}$$

is a symmetry of the ODE

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right) \tag{1.31}$$

Solution:

Eq. (1.29) implies

$$d\bar{x} = e^{\epsilon} dx \tag{1.32}$$

and Eq. (1.30) implies

$$d\bar{y} = e^{\epsilon} dy \tag{1.33}$$

Dividing Eq. (1.33) over Eq. (1.32) gives

$$\frac{d\bar{y}}{d\bar{x}} = \frac{e^{\epsilon} dy}{e^{\epsilon} dx} = \frac{dy}{dx} \tag{1.34}$$

Using ODE (1.31), we get

$$\frac{d\bar{y}}{d\bar{x}} = f\left(\frac{y}{x}\right) = f\left(\frac{e^{-\epsilon} y}{e^{-\epsilon} x}\right) = f\left(\frac{\bar{y}}{\bar{x}}\right) \tag{1.35}$$

which is of the same form as the original ODE.

This implies that the transformation forms a symmetry for the ODE.

1.1.3. Prolongation of infinitesimal generators

Let

$$X = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y}$$

Then, using Taylor's series, the infinitesimal transformation of the corresponding group is given by

$$\bar{x} = x + \epsilon\, \xi(x, y) + O(\epsilon^2)$$

$$\bar{y} = y + \epsilon\, \eta(x, y) + O(\epsilon^2)$$

In order to study symmetries of ODEs, we need to know how the derivatives are transformed via the extension of above transformation.

First prolongation

We look at the transformation of the derivative

$$y' = \frac{dy}{dx} \to \overline{y'} = \frac{d\bar{y}}{d\bar{x}}$$

Let

$$\overline{y'} = y' + \epsilon\, \eta^{[1]}(x, y, y') + O(\epsilon^2)$$

Let us find $\eta^{[1]}(x, y, y')$

$$\overline{y'} = \frac{d\bar{y}}{d\bar{x}} = \frac{dy + \epsilon\, d\eta + O(\epsilon^2)}{dx + \epsilon\, d\xi + O(\epsilon^2)} = \frac{\frac{dy}{dx} + \epsilon \frac{d\eta}{dx}}{1 + \epsilon \frac{d\xi}{dx}}$$

$$= \left(y' + \epsilon \frac{d\eta}{dx}\right)\left(1 + \epsilon \frac{d\xi}{dx}\right)^{-1}$$

$$= \left(y' + \epsilon \frac{d\eta}{dx}\right)\left(1 - \epsilon \frac{d\xi}{dx} + O(\epsilon^2)\right)$$

$$= y' + \epsilon \left(\frac{d\eta}{dx} - y' \frac{d\xi}{dx}\right) + O(\epsilon^2)$$

This implies

$$\boxed{\eta^{[1]} = \frac{d\eta}{dx} - y' \frac{d\xi}{dx}} \tag{1.36}$$

or

$$\eta^{[1]} = \eta_x + y'\eta_y - y'(\xi_x + y'\xi_y)$$

or

$$\boxed{\eta^{[1]} = \eta_x + (\eta_y - \xi_x)y' - \xi_y y'^2} \tag{1.37}$$

Thus, the first prolongation is given by

$$X^{[1]} = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y} + \left[\eta_x + (\eta_y - \xi_x)y' - \xi_y y'^2\right]\frac{\partial}{\partial y'} \tag{1.38}$$

which gives infinitesimal transformation for x, y, y'.

Second prolongation

Let

$$X = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}$$

be infinitesimal transformation and

$$X^{[1]} = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} + \eta^{[1]}(x,y,y')\frac{\partial}{\partial y'}$$

be the its first prolongation. Then, the transformation of x, y and $y'$are given by

$$\bar{x} = x + \epsilon\,\xi(x,y) + O(\epsilon^2)$$

$$\bar{y} = y + \epsilon\,\eta(x,y) + O(\epsilon^2)$$

$$\overline{y'} = y' + \epsilon\,\eta^{[1]}(x,y,y') + O(\epsilon^2)$$

We want to find the transformation of y''

$$y'' = \frac{d^2y}{dx^2} \to \overline{y''} = \frac{d\overline{y'}}{d\bar{x}}$$

Let

$$\overline{y''} = y'' + \epsilon\,\eta^{[2]}(x,y,y',y'') + O(\epsilon^2)$$

Let us find $\eta^{[2]}(x,y,y',y'')$. The transformation of y'' is given by

$$\overline{y''} = \frac{d\overline{y'}}{d\bar{x}} = \frac{dy' + \epsilon\, d\eta^{[1]} + O(\epsilon^2)}{dx + \epsilon\, d\xi + O(\epsilon^2)} = \frac{\frac{dy'}{dx} + \epsilon\frac{d\eta^{[1]}}{dx}}{1 + \epsilon\frac{d\xi}{dx}}$$

$$= \left(y'' + \epsilon\frac{d\eta^{[1]}}{dx}\right)\left(1 + \epsilon\frac{d\xi}{dx}\right)^{-1}$$

$$= \left(y'' + \epsilon\frac{d\eta^{[1]}}{dx}\right)\left(1 - \epsilon\frac{d\xi}{dx} + O(\epsilon^2)\right)$$

$$= y'' + \epsilon\left(\frac{d\eta^{[1]}}{dx} - y''\frac{d\xi}{dx}\right) + O(\epsilon^2)$$

This implies

$$\boxed{\eta^{[2]} = \frac{d\eta^{[1]}}{dx} - y''\frac{d\xi}{dx}} \tag{1.39}$$

or

$$\eta^{[2]} = \frac{d[\eta_x + (\eta_y - \xi_x)y' - \xi_y y'^2]}{dx} - y''\frac{d\xi}{dx}$$

$$= \eta_{xx} + y'\eta_{xy} + (\eta_{yx} + y'\eta_y - \xi_{xx} - y'\xi_{xy})y' + (\eta_y - \xi_x)y''$$

$$-(\xi_{yx} + y'\xi_{yy})y'^2 - 2y'y''\xi_y - y''(\xi_x + y'\xi_y)$$

Simplifying gives

$$\boxed{\eta^{[2]} = \eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})y'^2 - \xi_{yy}y'^3 + (\eta_y - 2\xi_x)y'' - 3\xi_y y'y''} \tag{1.40}$$

Thus, the second prolongation is given by

$$X^{[2]} = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} + \eta^{[1]}\frac{\partial}{\partial y'} + \eta^{[2]}\frac{\partial}{\partial y''}$$

nth prolongation

In general, we can show by induction that

$$\boxed{\eta^{[n]} = \frac{d\eta^{[n-1]}}{dx} - y^{(n)}\frac{d\xi}{dx}} \text{ for } n = 1,2,3,\ldots \tag{1.41}$$

where

$$\eta^{[0]} = \eta$$

and from this,

$$X^{[n]} = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} + \sum_{k=1}^{n}\eta^{[k]}\frac{\partial}{\partial y^{(k)}} \tag{1.42}$$

<u>Example</u>: Find the second prolongation of $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$.

Solution:

$$\xi(x,y) = x$$

$$\eta(x,y) = y$$

Then,

$$\eta^{[1]} = \eta_x + (\eta_y - \xi_x)y' - \xi_y y'^2 = 0$$

$$\eta^{[2]} = \eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})y'^2 - \xi_{yy}y'^3 + (\eta_y - 2\xi_x)y'' - 3\xi_y y'y''$$

$$= (\eta_y - 2\xi_x)y'' = -y''$$

Thus,

$$X^{[2]} = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} + \eta^{[1]}\frac{\partial}{\partial y'} + \eta^{[2]}\frac{\partial}{\partial y''} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 0\frac{\partial}{\partial y'} - y''\frac{\partial}{\partial y''}$$

1.1.4. Invariance criteria for finding symmetries of ODEs

Given an nth order ODE

$$F\left(x, y, y', \dots, y^{(n)}\right) = 0 \tag{1.43}$$

A symmetry of ODE (1.42) is a one parameter group infinitesimal transformations:

$$\bar{x} = x + \epsilon\, \xi(x, y)$$

$$\bar{y} = y + \epsilon\, \eta(x, y)$$

that leaves the ODE (1.43) invariant, i. e.

$$F\left(\bar{x}, \bar{y}, \overline{y'}, \dots, \overline{y^{(n)}}\right) = 0 \tag{1.44}$$

where $\overline{y'}$, … , $\overline{y^{(n)}}$ are transformed through prolongations of the group.

Treating equation (1.43) as an algebraic equation in $x, y, y', \dots, y^{(n)}$, invariance criteria can be generalized to the following definition.

<u>Definition</u>:

The infinitesimal transformation

$$\bar{x} = x + \epsilon\, \xi(x, y)$$

$$\bar{y} = y + \epsilon\, \eta(x, y)$$

or equivalently the generator

$$X = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y}$$

is a symmetry of ODE (1.63) if

$$X^{[n]}F = 0$$

when

$$F = 0$$

where $X^{[n]}$ is the nth prolongation of X.

<u>Example</u>: Show that $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ is a symmetry of the ODE

$$x^2y'' + xy'^2 - yy' = 0$$

Solution:

The function associated with the ODE is

$$F(x, y, y', y'') = x^2 y'' + x y'^2 - y y'$$

The second prolongation of X as found in the previous example is given by

$$X^{[2]} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 0\frac{\partial}{\partial y'} - y''\frac{\partial}{\partial y''}$$

Hence,

$$\begin{aligned} X^{[2]}F &= x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} + 0\frac{\partial F}{\partial y'} - y''\frac{\partial F}{\partial y''} \\ &= x(2xy'' + y'^2) + y(-y') - y''(x^2) \\ &= 2x^2 y'' + x y'^2 - y y' - x^2 y'' \\ &= x^2 y'' + x y'^2 - y y' \\ &= 0 \text{ if } F = 0 \end{aligned}$$

This implies that

$$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$$

is a symmetry of the given ODE.

1.1.5. Method of finding the Symmetries of ODEs

The criteria

$$X^{[n]}F\big|_{F=0} = 0 \tag{1.45}$$

is called the invariance criteria for the symmetries of ODE (1.43) of order n. It is the basis for computing the symmetries of ODE (1.43). This will lead us to an over determined system of linear PDEs in $\xi(x,y)$and $\eta(x,y)$. Solving this system will give us all possible functions $\xi(x,y)$ and $\eta(x,y)$ that satisfy the invariance condition.

Steps for finding symmetries of ODE (1.43)

1) Find prolongation $X^{[n]}$ where n is the order of the ODE.
2) Apply the prolongation to the ODE restricted to the validation of ODE.
3) Obtain determining equations from step 2 by comparing coefficients of powers of derivatives.
4) Simplify and solve the determining equations.

<u>Example</u>: (Simple example to show the procedure, the same procedure works for any other ODEs)

Find all symmetries of

$$y'' = 0 \tag{1.46}$$

Solution: Let

$$F(\,x\,,y\,,y'\,,y'') = y'' \tag{1.47}$$

Let the symmetry be of the form

$$X = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} \tag{1.48}$$

<u>Step 1</u>: write the 2nd prolongation

$$\eta^{[1]} = \eta_x + (\eta_y - \xi_x)y' - \xi_y y'^2$$

$$\eta^{[2]} = \eta_{xx} + (2\eta_{xy} - \xi_{xx})\mathrm{y}' + (\eta_{yy} - 2\xi_{xy})y'^2 - \xi_{yy}y'^3 + (\eta_y - 2\xi_x)y'' - 3\xi_y\mathrm{y}'\mathrm{y}''$$

$$X^{[2]}F = X^{[2]}(y'')$$

$$= \xi(x,y)\,.0 + \eta(x,y).(\,0\,) + \eta^{[1]}(\,0\,) + \eta^{[2]}(\,1\,) = \eta^{[2]} \tag{1.49}$$

<u>Step 2</u> : Apply 2nd prolongation to the differential equation

$$X^{[2]}(y'')\big|_{y''=0} = \eta_{xx} + (2\eta_{xy} - \xi_{xx})\mathrm{y}' + (\eta_{yy} - 2\xi_{xy})y'^2 - \xi_{yy}y'^3 \tag{1.50}$$

<u>Step 3</u> : Finding the determining equations as follows

Coefficient of $(y')^3$: $\xi_{yy} = 0$ (E1)

Coefficient of $(y')^2$: $\eta_{yy} - 2\xi_{xy} = 0$ (E2)

Coefficient of $(y')^1$: $2\eta_{xy} - \xi_{xx} = 0$ (E3)

Coefficient of $(y')^0$: $\eta_{xx} = 0$ (E4)

We get a system of linear PDEs.

<u>Step 4</u>: Solving the over determined system of linear PDEs (E1)-(E4)

From equation (E1), ξ is linear in y. So,

$$\xi = A(x)y + B(x) \tag{1.51}$$

From equation E4, η is linear in y. So,

$$\eta = f(y)x + g(y) \tag{1.52}$$

Differentiating equation E2 with respect to y and use equation E1, we get

$$\eta_{yyy} - 2\xi_{xyy} = 0 \tag{1.53}$$

Using equation E1 in Eq. (1.53) gives

$$\eta_{yyy} = 0 \tag{1.54}$$

which implies

$$f'''(y)x + g'''(y) = 0 \tag{1.55}$$

or

$$f'''(y) = 0 \tag{1.56}$$

and

$$g'''(y) = 0 \tag{1.57}$$

From Eq. (1.56), we get

$$f(y) = c_1 + c_2 y + c_{10} y^2 \tag{1.58}$$

From Eq. (1.57), we get

$$g(y) = c_4 + c_5 y + c_6 y^2$$

Substituting the functions $f(y)$ and $g(y)$ in Eq. (1.52) gives

$$\boxed{\eta = (c_1 + c_2 y + c_{10} y^2)x + c_4 + c_5 y + c_6 y^2} \tag{1.59}$$

Substituting equations (1.59) and (1.51) in E2, we get

$$2(c_2 + 2c_{10} y) - \big(A''(x)y + B''(x)\big) = 0 \tag{1.60}$$

Comparing coefficients of powers of y in the previous equation gives

$$A''(x) = 4c_{10} \tag{1.61}$$

$$B''(x) = 2c_2 \tag{1.62}$$

Integrating Eq. (1.61), we get

$$A(x) = c_7 + c_9 x + 2c_3 x^2 \tag{1.63}$$

And integrating Eq. (1.62), we get

$$B(x) = c_8 + c_3 x + c_2 x^2 \tag{1.64}$$

Substituting the previous two equations (1.63)-(1.64) in Eq. (1.51) gives

$$\xi = (c_7 + c_9 x + 2c_{10} x^2)y + (c_8 + c_3 x + c_2 x^2) \tag{1.65}$$

Substituting equations (1.59) and (1.65) in equation E2, we get

$$2c_{10}x + 2c_6 = 2(c_9 + 4c_{10}x)$$

which implies

$$c_9 = c_6$$

$$c_{10} = 0$$

Remove c_{10} and c_9 from the formulas

$$\xi = (c_7 + c_9\, x + 2c_{10}x^2\,)y + (\,c_8 + c_3\, x + c_2 x^2\,)$$

$$= (c_7 + c_6\, x\,)y + (c_8 + c_3 x + c_2 x^2)$$

$$\eta = (c_1 + c_2 y + c_{10} y^2)x + (c_4 + c_5 y + c_6 y^2)$$

$$= (c_1 + c_2 y)x + (c_4 + c_5 y + c_6 y^2)$$

So, we obtain 8 independent constants (parameters). This provides 8 dimensional Lie algebra generated by the symmetries found below.

Putting $c_1 = 1$ and the remaining constants vanish implies

$$X_1 = x\frac{\partial}{\partial y}$$

Putting $c_2 = 1$ and the remaining constants vanish implies

$$X_2 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}$$

Putting $c_3 = 1$ and the remaining constants vanish implies

$$X_3 = x\frac{\partial}{\partial x}$$

Putting $c_4 = 1$ and the remaining constants vanish implies

$$X_4 = \frac{\partial}{\partial y}$$

Putting $c_5 = 1$ and the remaining constants vanish implies

$$X_5 = y\frac{\partial}{\partial y}$$

Putting $c_6 = 1$ and the remaining constants vanish implies

$$X_6 = xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}$$

Putting $c_7 = 1$ and the remaining constants vanish implies

$$X_7 = y\frac{\partial}{\partial x}$$

and putting $c_8 = 1$ and the remaining constants vanish implies

$$X_8 = \frac{\partial}{\partial x}$$

<u>Note</u> : the set of all symmetries of an ODE forms a Lie Algebra with commutator operation

[X , Y] = XY – YX.

We obtain the basis of all symmetries

$$S = \{ X_1 , X_2 , X_3 , X_4 , X_5 , X_6 , X_7 , X_8 \}$$

with commutation relations

	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8
X_1	0	0	$-X_1$	0	X_1	X_2	$X_3 - X_6$	$-X_4$
X_2	0	0	$-X_2$	$-X_1$	0	0	$-X_6$	$-2X_3 - X_5$
X_3	X_1	X_2	0	0	0	0	$-X_7$	$-X_8$
X_4	0	X_1	0	0	X_4	$X_3 + 2X_5$	X_8	0
X_5	$-X_1$	0	0	$-X_4$	0	X_6	X_7	0
X_6	$-X_2$	0	0	$-X_3 - 2X_5$	$-X_6$	0	0	$-X_7$
X_7	$X_6 - X_3$	X_6	X_7	$-X_8$	$-X_7$	0	0	0
X_8	X_4	$2X_3 + X_5$	X_8	0	0	X_7	0	0

1.1.6. Reduction of order of ODEs using symmetries

Given a symmetry

$$X = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}$$

of an ODE

$$F\left(x, y, y', \dots, y^{(n)}\right) = 0.$$

The invariants obtained through

$$X^{[n]} I = 0$$

can be utilized, cf. [8], to reduce the order of ODE by the standard procedure of change of variables. This method is illustrated in examples below.

Example: Reduce

$$F = x^2 y'' + x\, y'^2 - yy' = 0 \tag{1.66}$$

using the symmetry

$$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$$

Solution :

The 2nd prolongation of X is

$$X^{[2]} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 0\frac{\partial}{\partial y'} - y''\frac{\partial}{\partial y''}$$

We find invariants using $X^{[2]}\, I = 0$, whose characteristic system is

$$\frac{dx}{x} = \frac{dy}{y} = \frac{d\,y'}{0} = \frac{d\,y''}{-y''}$$

Solving $\frac{dx}{x} = \frac{dy}{y}$ implies

$$\frac{y}{x} = constant$$

Solving $\frac{dx}{x} = \frac{d\,y'}{0}$

$$y' = constant$$

Thus, we obtain two invariants of $X^{[2]}$ given by

$$u = \frac{y}{x} \tag{1.67}$$

$$v = y' \tag{1.68}$$

To write the given ODE in terms of u and v

$$x^2y'' = x^2\frac{dy'}{dx} = x^2\frac{dv}{dx} = x^2\frac{dv}{du}\frac{du}{dx} = x^2\frac{dv}{du}\frac{x\frac{dy}{dx} - y}{x^2} = \frac{dv}{du}(xv - y) \tag{1.69}$$

Using equations (1.67)-(1.69) in the ODE (1.66) gives

$$x^2y'' + xy'^2 - yy' = (xv - y)\frac{dv}{du} + xv^2 - yv = 0 \tag{1.70}$$

Dividing Eq. (1.70) by x implies

$$(v - u)\frac{dv}{du} + v^2 - uv = 0 \tag{1.71}$$

(i) If $v \neq u$, we get

$$\frac{dv}{du} = -v$$

which is separable 1st order ODE with solution

$$v = ce^{-u}$$

Going back to the original variables gives

$$\frac{dy}{dx} = ce^{-\frac{y}{x}} \tag{1.72}$$

which is homogeneous first order ODE.

Let $V = \frac{y}{x}$, then

$$\frac{dy}{dx} = \frac{xdV + Vdx}{dx} = x\frac{dV}{dx} + V$$

Substituting this in ODE (1.72) leads to separable 1st order ODE

$$x\frac{dV}{dx} + V = ce^{-V}$$

or

$$\frac{dV}{ce^{-V} - V} = \frac{dx}{x}$$

(ii) If $v = u$, then

$$y' = \frac{y}{x}$$

which implies

$$\frac{dy}{y} = \frac{dx}{x}$$

which gives the solution

$$\boxed{y = ax}$$

Example: Reduce the following ODE using symmetry technique

$$xy\frac{d^2y}{dx^2} + x(\frac{dy}{dx})^2 + y\frac{dy}{dx} - xy^2 = 0 \tag{1.73}$$

Solution:

We can check that

$$X = y\frac{\partial}{\partial y}$$

is a symmetry of ODE (1.73).

Next we use the symmetry to reduce the ODE (1.73). The 2nd prolongation of X is

$$X^{[2]} = 0\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + y'\frac{\partial}{\partial y'} + y''\frac{\partial}{\partial y''}$$

Solving the characteristic system of $X^{[2]}I = 0$ gives differential invariants:

$$u = x \tag{1.74}$$

$$v = \frac{y'}{y} \tag{1.75}$$

These variables give

$$y'' = \frac{dy'}{dx} = \frac{d(yv)}{dx} = y\frac{dv}{du} + v\frac{dy}{dx} = y\frac{dv}{du} + v^2y$$

or

$$\frac{y''}{y} = \frac{dv}{du} + v^2$$

Putting this in Equation (1.73) reduces it to the first order ODE

$$xy^2\left(\frac{dv}{du} + v^2\right) + xy^2v^2 + y^2v - xy^2 = 0 \tag{1.76}$$

Dividing Eq. (1.76) by xy^2 leads to

$$u\left(\frac{dv}{du} + v^2\right) + uv^2 + v - u = 0$$

or

$$\frac{dv}{du} = -2v^2 - \frac{1}{u}v + 1$$

The reduced 1st order ODE is in the form of Riccati equation.

1.2. LIE SYMMETRY METHOD FOR PDEs IN TWO INDEPENDENT VARIABLES

This section is devoted to discussion of symmetries of PDEs, the prolongations of their generators and the method of finding symmetries of PDEs. For the sake of clarity we restrict the main discussions to 2nd order PDEs with one dependent variable and two independent variables. This does not hamper the research work in the thesis since the PDEs involved in the research work in chapters 3, 4 and 5 all belong to this class of PDEs. A detailed account of the subject of Lie symmetry for general PDEs is contained in many standard books on the topic cf. [2, 6, 7, 8, 11, 15, 17, 18, 19, 20, 30, 31, 32, 37].

1.2.1. Symmetries of PDEs

Consider a PDE

$$F\left(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, \dots, \eth^n u\right) = 0 \tag{1.77}$$

where $\eth^n u$ denotes the nth partial derivatives of u with respect to x and y,

$$\eth^1 u = \left(u_x, u_y\right)$$

$$\eth^2 u = \left(u_{xx}, u_{xy}, u_{yy}\right)$$

$$\eth^n u = \left(\frac{\partial^n u}{\partial x^n}, \dots, \frac{\partial^n u}{\partial x^{n-i}\,\partial y^i}, \dots, \frac{\partial^n u}{\partial y^n}\right)$$

A one parameter group of transformations

$$\begin{cases} \bar{x} = f(x, y, u, \epsilon) \\ \bar{y} = g(x, y, u, \epsilon) \\ \bar{u} = h(x, y, u, \epsilon) \end{cases} \tag{1.78}$$

is called a **symmetry** of PDE (1.77) if the form of PDE(1.77) remains unchanged under the transformation (1.78), i.e. after change of variables

$$(x, y, u) \to (\bar{x}, \bar{y}, \bar{u}),$$

we get $F\left(\bar{x}, \bar{y}, \bar{u}, \overline{\eth^1 u}, \dots, \overline{\eth^n u}\right) = 0$.

1.2.2. Prolongations of infinitesimal generators of symmetries of PDEs

In this section we learn how to prolong infinitesimal generators involved in studying symmetries of PDEs. For the sake of simplicity, the derivation of prolongation formulas is restricted to symmetries of 2nd order PDEs with one dependent variable u and two independent variables x, y.

We consider the following PDE:

$$F\left(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}\right) = 0 \tag{1.79}$$

with symmetry operator

$$X = \xi(x, y, u)\frac{\partial}{\partial x} + \tau(x, y, u)\frac{\partial}{\partial y} + \phi(x, y, u)\frac{\partial}{\partial u}$$

Then, the infinitesimal transformation group is given by

$$x^* = x + \epsilon\, \xi(x, y, u) + O(\epsilon^2)$$
$$y^* = y + \epsilon\, \tau(x, y, u) + O(\epsilon^2)$$
$$u^* = u + \epsilon\, \phi(x, y, u) + O(\epsilon^2)$$

The following definition of total differentiation operator will considerably simplify the expressions involved in prolongation formulas for vector field X.

<u>Definition</u>: (total differentiation operator)

Given a function $F(x, y, u(x, y), f^1(x, y), f^2(x, y), \dots, f^n(x, y))$.

The total differentiation operators with respect to x and y are respectively defined as

$$D_x = \frac{\partial}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial f^1}{\partial x}\frac{\partial}{\partial f^1} + \frac{\partial f^2}{\partial x}\frac{\partial}{\partial f^2} + \dots + \frac{\partial f^n}{\partial x}\frac{\partial}{\partial f^n}$$
$$D_y = \frac{\partial}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial}{\partial u} + \frac{\partial f^1}{\partial y}\frac{\partial}{\partial f^1} + \frac{\partial f^2}{\partial y}\frac{\partial}{\partial f^2} + \dots + \frac{\partial f^n}{\partial y}\frac{\partial}{\partial f^n}$$

Example: Let $F = F\left(x, y, u, u_x, u_y\right)$. Then,

$$D_x = \frac{\partial}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial u_x}{\partial x}\frac{\partial}{\partial u_x} + \frac{\partial u_y}{\partial x}\frac{\partial}{\partial u_y} = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + u_{yx}\frac{\partial}{\partial u_y}$$
$$D_y = \frac{\partial}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial}{\partial u} + \frac{\partial u_x}{\partial y}\frac{\partial}{\partial u_x} + \frac{\partial u_y}{\partial y}\frac{\partial}{\partial u_y} = \frac{\partial}{\partial y} + u_y\frac{\partial}{\partial u} + u_{xy}\frac{\partial}{\partial u_x} + u_{yy}\frac{\partial}{\partial u_y}$$

Example:

$$D_x(x) = 1$$

$$D_x(y) = 0$$

$$D_x(u_x) = \frac{\partial u_x}{\partial x}\frac{\partial u_x}{\partial u_x} = u_{xx}$$

$$D_x(u_y) = \frac{\partial u_y}{\partial x}\frac{\partial u_y}{\partial u_y} = u_{yx}$$

First Prolongation of X

Given $X = \xi(x,y,u)\frac{\partial}{\partial x} + \tau(x,y,u)\frac{\partial}{\partial y} + \phi(x,y,u)\frac{\partial}{\partial u}$

or equivalently the infinitesimal transformation

$$x^* = x + \epsilon\,\xi(x,y,u) + O(\epsilon^2) \tag{1.80}$$

$$y^* = y + \epsilon\,\tau(x,y,u) + O(\epsilon^2) \tag{1.81}$$

$$u^* = u + \epsilon\,\phi(x,y,u) + O(\epsilon^2) \tag{1.82}$$

We want to find the transformation of the partial derivatives of 1st order, u_x and u_y.

$i.e.$ to find functions $\eta^{[x]}(x,y,u,u_x,u_y)$ and $\eta^{[y]}(x,y,u,u_x,u_y)$ such that

$$u^*_{x^*} = u_x + \epsilon\,\eta^{[x]}(x,y,u,u_x,u_y) + O(\epsilon^2) \tag{1.83}$$

$$u^*_{y^*} = u_y + \epsilon\,\eta^{[y]}(x,y,u,u_x,u_y) + O(\epsilon^2) \tag{1.84}$$

Eq. (1.80) implies

$$\begin{aligned} dx^* &= dx + \epsilon\, d\xi + O(\epsilon^2) \\ &= dx + \epsilon\,[\xi_x dx + \xi_y dy + \xi_u du] + O(\epsilon^2) \\ &= dx + \epsilon\,\left[\xi_x dx + \xi_y dy + \xi_u\left(\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\right)\right] + O(\epsilon^2) \\ &= \left[1 + \epsilon\left(\xi_x + \frac{\partial u}{\partial x}\xi_u\right)\right]dx + \epsilon\,\left[\xi_y + \frac{\partial u}{\partial y}\xi_u\right]dy + O(\epsilon^2) \end{aligned}$$

which implies

$$dx^* = [1 + \epsilon D_x\xi]dx + \epsilon\left[D_y\xi\right]dy + O(\epsilon^2) \tag{1.85}$$

Similarly, equation (1.81) implies

$$dy^* = \epsilon[D_x\tau]dx + \epsilon\left[1 + D_y\tau\right]dy + O(\epsilon^2) \tag{1.86}$$

From equation (1.82), we get

$$du^* = du + \epsilon\, d\phi + O(\epsilon^2)$$

$$= \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \epsilon \left[\phi_x dx + \phi_y dy + \phi_u \left(\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy \right) \right] + O(\epsilon^2)$$

or

$$du^* = \left[\frac{\partial u}{\partial x} + \epsilon\, D_x \phi \right] dx + \left[\frac{\partial u}{\partial y} + \epsilon D_y \phi \right] dy + O(\epsilon^2) \tag{1.87}$$

Also,

$$u^* = u^*(x^*, y^*)$$

implies that

$$du^* = \frac{\partial u^*}{\partial x^*} dx^* + \frac{\partial u^*}{\partial y^*} dy^* \tag{1.88}$$

Using equations (1.85)-(1.87) in Eq. (1.88), we get

$$\left[\frac{\partial u}{\partial x} + \epsilon\, D_x \phi \right] dx + \left[\frac{\partial u}{\partial y} + \epsilon\, D_y \phi \right] dy + O(\epsilon^2)$$

$$= \frac{\partial u^*}{\partial x^*} \left\{ [1 + \epsilon D_x \xi] dx + \epsilon D_y \xi dy + O(\epsilon^2) \right\} + \frac{\partial u^*}{\partial y^*} \left\{ \epsilon D_x \tau dx + \left[1 + \epsilon D_y \tau \right] dy + O(\epsilon^2) \right\}$$

$$= \left\{ \frac{\partial u^*}{\partial x^*} [1 + \epsilon D_x \xi] + \epsilon [D_x \tau] \frac{\partial u^*}{\partial y^*} \right\} dx + \left\{ \frac{\partial u^*}{\partial x^*} \epsilon \left[D_y \xi \right] + \frac{\partial u^*}{\partial y^*} \left[1 + \epsilon D_y \tau \right] \right\} dy + O(\epsilon^2)$$

But since dx and dy are linearly independent, the previous relation implies

$$\frac{\partial u}{\partial x} + \epsilon D_x \phi = \frac{\partial u^*}{\partial x^*} [1 + \epsilon D_x \xi] + \epsilon [D_x \tau] \frac{\partial u^*}{\partial y^*}$$

$$\frac{\partial u}{\partial y} + \epsilon D_y \phi = \frac{\partial u^*}{\partial x^*} \epsilon \left[D_y \xi \right] + \frac{\partial u^*}{\partial y^*} \left[1 + \epsilon D_y \tau \right]$$

We can put the previous two equations in matrix form as follows:

$$\begin{pmatrix} \frac{\partial u}{\partial x} + \epsilon D_x \phi \\ \frac{\partial u}{\partial y} + \epsilon D_y \phi \end{pmatrix} = \begin{pmatrix} 1 + \epsilon D_x \xi & \epsilon D_x \tau \\ \epsilon D_y \xi & 1 + \epsilon D_y \tau \end{pmatrix} \begin{pmatrix} \frac{\partial u^*}{\partial x^*} \\ \frac{\partial u^*}{\partial y^*} \end{pmatrix} \tag{1.89}$$

Set

$$B = \begin{pmatrix} D_x \xi & D_x \tau \\ D_y \xi & D_y \tau \end{pmatrix}$$

and

$$A = \begin{pmatrix} 1 + \epsilon D_x \xi & \epsilon D_x \tau \\ \epsilon D_y \xi & 1 + \epsilon D_y \tau \end{pmatrix}.$$

Then, we have

$$A = I + \epsilon B$$

which implies

$$A^{-1} = (I + \epsilon B)^{-1} = I - \epsilon B + O(\epsilon^2)$$

Then, from equation (1.87), we get

$$\begin{pmatrix} \frac{\partial u^*}{\partial x^*} \\ \frac{\partial u^*}{\partial y^*} \end{pmatrix} = A^{-1} \begin{pmatrix} \frac{\partial u}{\partial x} + \epsilon D_x \phi \\ \frac{\partial u}{\partial y} + \epsilon D_y \phi \end{pmatrix} + O(\epsilon^2)$$

or

$$\begin{pmatrix} \frac{\partial u^*}{\partial x^*} \\ \frac{\partial u^*}{\partial y^*} \end{pmatrix} = (I - \epsilon B) \begin{pmatrix} \frac{\partial u}{\partial x} + \epsilon D_x \phi \\ \frac{\partial u}{\partial y} + \epsilon D_y \phi \end{pmatrix} + O(\epsilon^2)$$

Simplifying, we get

$$\begin{pmatrix} \frac{\partial u}{\partial x} + \epsilon \eta^{[x]}(x, y, u, u_x, u_y) + O(\epsilon^2) \\ \frac{\partial u}{\partial y} + \epsilon \eta^{[y]}(x, y, u, u_x, u_y) + O(\epsilon^2) \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix} + \epsilon \begin{pmatrix} D_x \phi \\ D_y \phi \end{pmatrix} - \epsilon B \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix} + O(\epsilon^2)$$

which implies

$$\begin{pmatrix} \eta^{[x]} \\ \eta^{[y]} \end{pmatrix} = \begin{pmatrix} D_x \phi \\ D_y \phi \end{pmatrix} - \begin{pmatrix} D_x \xi & D_x \tau \\ D_y \xi & D_y \tau \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix} \tag{1.90}$$

Writing Eq. (1.90) in terms of ξ, τ and ϕ gives

$$\begin{aligned} \begin{pmatrix} \eta^{[x]} \\ \eta^{[y]} \end{pmatrix} &= \begin{pmatrix} \phi_x + \phi_u u_x \\ \phi_y + \phi_u u_y \end{pmatrix} - \begin{pmatrix} \xi_x + \xi_u u_x & \tau_x + \tau_u u_x \\ \xi_y + \xi_u u_y & \tau_y + \tau_u u_y \end{pmatrix} \begin{pmatrix} u_x \\ u_y \end{pmatrix} \\ &= \begin{bmatrix} \phi_x + \phi_u u_x - (\xi_x + \xi_u u_x) u_x - (\tau_x + \tau_u u_x) u_y \\ \phi_y + \phi_u u_y - (\xi_y + \xi_u u_y) u_x - (\tau_y + \tau_u u_y) u_y \end{bmatrix} \\ &= \begin{bmatrix} \phi_x + (\phi_u - \xi_x) u_x - \tau_x u_y - \xi_u u_x^2 - \tau_u u_x u_y \\ \phi_y + (\phi_u - \tau_y) u_y - \xi_y u_x - \xi_u u_y u_x - \tau_u u_y^2 \end{bmatrix} \end{aligned}$$

and from this , we conclude that

$$\boxed{\eta^{[x]} = \phi_x + (\phi_u - \xi_x) u_x - \tau_x u_y - \xi_u u_x^2 - \tau_u u_x u_y} \tag{1.91}$$

and

$$\boxed{\eta^{[y]} = \phi_y + (\phi_u - \tau_y) u_y - \xi_y u_x - \xi_u u_y u_x - \tau_u u_y^2} \tag{1.92}$$

Thus, we have got the first prolongation of X which is

$$X^{[1]} = \xi(x, y, u) \frac{\partial}{\partial x} + \tau(x, y, u) \frac{\partial}{\partial y} + \phi(x, y, u) \frac{\partial}{\partial u} + \eta^{[x]} \frac{\partial}{\partial u_x} + \eta^{[y]} \frac{\partial}{\partial u_y}$$

where $\eta^{[x]}$ and $\eta^{[y]}$ are given by equations (1.91) and (1.92) respectively.

Second Prolongation of X:

Let

$$X = \xi(x,y,u)\frac{\partial}{\partial x} + \tau(x,y,u)\frac{\partial}{\partial y} + \phi(x,y,u)\frac{\partial}{\partial u}$$

The first prolongation has been found with

$$X^{[1]} = \xi(x,y,u)\frac{\partial}{\partial x} + \tau(x,y,u)\frac{\partial}{\partial y} + \phi(x,y,u)\frac{\partial}{\partial u} + \eta^{[x]}\frac{\partial}{\partial u_x} + \eta^{[y]}\frac{\partial}{\partial u_y}.$$

We want to prolong X to get

$$X^{[2]} = X^{[1]} + \eta^{[xx]}\frac{\partial}{\partial u_{xx}} + \eta^{[xy]}\frac{\partial}{\partial u_{xy}} + \eta^{[yy]}\frac{\partial}{\partial u_{yy}}$$

So that the functions $\eta^{[xx]}, \eta^{[xy]}$ and $\eta^{[yy]}$ provide transformation for the 2nd derivatives. $i.e.$

$$u^*_{x^*x^*} = u_{xx} + \epsilon\eta^{[xx]}\left(x,y,u,u_x,u_y,u_{xx},u_{xy},u_{yy}\right) + O(\epsilon^2) \tag{1.93}$$

$$u^*_{x^*y^*} = u_{xy} + \epsilon\eta^{[xy]}\left(x,y,u,u_x,u_y,u_{xx},u_{xy},u_{yy}\right) + O(\epsilon^2) \tag{1.94}$$

$$u^*_{y^*y^*} = u_{yy} + \epsilon\eta^{[yy]}\left(x,y,u,u_x,u_y,u_{xx},u_{xy},u_{yy}\right) + O(\epsilon^2) \tag{1.95}$$

Eq. (1.83) implies

$$\begin{aligned} du^*_{x^*} &= du_x + \epsilon d\eta^{[x]} + O(\epsilon^2) \\ &= \frac{\partial u_x}{\partial x}dx + \frac{\partial u_x}{\partial y}dy + \epsilon\left[\frac{\partial\eta^{[x]}}{\partial x}dx + \frac{\partial\eta^{[x]}}{\partial y}dy + \frac{\partial\eta^{[x]}}{\partial u}du + \frac{\partial\eta^{[x]}}{\partial u_x}du_x + \frac{\partial\eta^{[x]}}{\partial u_y}du_y\right] + O(\epsilon^2) \\ &= \frac{\partial u_x}{\partial x}dx + \frac{\partial u_x}{\partial y}dy + \epsilon\left[\frac{\partial\eta^{[x]}}{\partial x}dx + \frac{\partial\eta^{[x]}}{\partial y}dy + \frac{\partial\eta^{[x]}}{\partial u}\left(\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\right)\right. \\ &\qquad \left. + \frac{\partial\eta^{[x]}}{\partial u_x}\left(\frac{\partial u_x}{\partial x}dx + \frac{\partial u_x}{\partial y}dy\right) + \frac{\partial\eta^{[x]}}{\partial u_y}\left(\frac{\partial u_y}{\partial x}dx + \frac{\partial u_y}{\partial y}dy\right)\right] + O(\epsilon^2) \end{aligned}$$

which implies

$$du^*_{x^*} = \left(\frac{\partial u_x}{\partial x} + \epsilon D_x\eta^{[x]}\right)dx + \left(\frac{\partial u_x}{\partial y} + \epsilon D_y\eta^{[x]}\right)dy \ + O(\epsilon^2) \tag{1.96}$$

Using equations (4), (5) and (1.96) in the following formula:

$$du^*_{x^*} = \frac{\partial u^*_{x^*}}{\partial x^*}dx^* + \frac{\partial u^*_{x^*}}{\partial y^*}dy^*$$

gives

$$du^*_{x^*} = u^*_{x^*x^*}\left[(1+\epsilon D_x\xi)dx + \left(\epsilon D_y\xi\right)dy\right] + u^*_{x^*y^*}\left[(\epsilon D_x\tau)dx + \left(1+\epsilon D_y\tau\right)dy\right] + O(\epsilon^2)$$

Using Eq. (1.96) and the independence of dx and dy imply

$$\begin{pmatrix} u_{xx} + \epsilon D_x\eta^{[x]} \\ u_{xy} + \epsilon D_y\eta^{[x]} \end{pmatrix} = \begin{pmatrix} 1+\epsilon D_x\xi & \epsilon D_x\tau \\ \epsilon D_y\xi & 1+\epsilon D_y\tau \end{pmatrix}\begin{pmatrix} u^*_{x^*x^*} \\ u^*_{x^*y^*} \end{pmatrix} + O(\epsilon^2)$$

$$= A\begin{pmatrix} u^*_{x^*x^*} \\ u^*_{x^*y^*} \end{pmatrix} + O(\epsilon^2)$$

or

$$\begin{pmatrix} u_{xx} + \epsilon D_x\eta^{[x]} \\ u_{xy} + \epsilon D_y\eta^{[x]} \end{pmatrix} = (I + \epsilon B)\begin{pmatrix} u^*_{x^*x^*} \\ u^*_{x^*y^*} \end{pmatrix} + O(\epsilon^2) \tag{1.97}$$

which implies

$$\begin{pmatrix} u^*_{x^*x^*} \\ u^*_{x^*y^*} \end{pmatrix} = \ (I + \epsilon B)^{-1}\begin{pmatrix} u_{xx} + \epsilon D_x\eta^{[x]} \\ u_{xy} + \epsilon D_y\eta^{[x]} \end{pmatrix} + O(\epsilon^2)$$

$$= (I - \epsilon B)\begin{pmatrix} u_{xx} + \epsilon D_x\eta^{[x]} \\ u_{xy} + \epsilon D_y\eta^{[x]} \end{pmatrix} + O(\epsilon^2)$$

$$= \begin{pmatrix} u_{xx} \\ u_{xy} \end{pmatrix} + \epsilon\left[\begin{pmatrix} D_x\eta^{[x]} \\ D_y\eta^{[x]} \end{pmatrix} - B\begin{pmatrix} u_{xx} \\ u_{xy} \end{pmatrix}\right] + O(\epsilon^2)$$

Using equations (1.93) and (1.94), we get

$$\begin{pmatrix} \eta^{[xx]} \\ \eta^{[xy]} \end{pmatrix} = \begin{pmatrix} D_x\eta^{[x]} \\ D_y\eta^{[x]} \end{pmatrix} - B\begin{pmatrix} u_{xx} \\ u_{xy} \end{pmatrix} \tag{1.98}$$

where

$$B = \begin{pmatrix} D_x\xi & D_x\tau \\ D_y\xi & D_y\tau \end{pmatrix}$$

Writing $\eta^{[xx]}$ and $\eta^{[xy]}$ in terms of ξ , τ and ϕ

$$\because \ \eta^{[x]} = \eta^{[x]}(x, y, u, u_x, u_y)$$

$$\therefore \ D_x\eta^{[x]} = \left(\frac{\partial}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial u_x}{\partial x}\frac{\partial}{\partial u_x} + \frac{\partial u_y}{\partial x}\frac{\partial}{\partial u_y}\right)\eta^{[x]}$$

$$= \left(\frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + u_{yx}\frac{\partial}{\partial u_y}\right)\eta^{[x]}$$

Using formula (1.91) in the previous equation, we get

$$\begin{aligned} D_x\eta^{[x]} = &\left[\phi_{xx} + (\phi_{ux} - \xi_{xx})u_x - \tau_{xx}u_y - \xi_{ux}u_x^2 - \tau_{ux}u_xu_y\right] \\ &+u_x\left[\phi_{xu} + (\phi_{uu} - \xi_{xu})u_x - \tau_{xu}u_y - \xi_{uu}u_x^2 - \tau_{uu}u_xu_y\right] \\ &+u_{xx}\left[\phi_u - \xi_x - 2\xi_u u_x - \tau_u u_y\right] + u_{yx}[-\tau_x - \tau_u u_x] \end{aligned} \tag{1.99}$$

Also,

$$D_x\xi = \frac{\partial\xi}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial\xi}{\partial u} = \xi_x + u_x\xi_u \tag{1.100}$$

$$D_y\xi = \frac{\partial\xi}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial\xi}{\partial u} = \xi_y + u_y\xi_u \tag{1.101}$$

$$D_x\tau = \frac{\partial\tau}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial\tau}{\partial u} = \tau_x + u_x\tau_u \tag{1.102}$$

$$D_y\tau = \frac{\partial\tau}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial\tau}{\partial u} = \tau_y + u_y\tau_u \tag{1.103}$$

Now, from Eq. (1.98), we have

$$\eta^{[xx]} = D_x\eta^{[x]} - u_{xx}D_x\xi - u_{xy}D_x\tau$$

From (1.99), (1.100) and (1.102), we get

$$\begin{aligned}
\eta^{[xx]} &= \left[\phi_{xx} + (\phi_{ux} - \xi_{xx})u_x - \tau_{xx}u_y - \xi_{ux}u_x^2 - \tau_{ux}u_xu_y\right] \\
&\quad + u_x\left[\phi_{xu} + (\phi_{uu} - \xi_{xu})u_x - \tau_{xu}u_y - \xi_{uu}u_x^2 - \tau_{uu}u_xu_y\right] \\
&\quad + u_{xx}\left[\phi_u - \xi_x - 2\xi_u u_x - \tau_u u_y\right] + u_{yx}[-\tau_x - \tau_u u_x] - u_{xx}[\xi_x + u_x\xi_u] \\
&\quad - u_{xy}[\tau_x + u_x\tau_u] \\
&= \left[\phi_{xx} + (2\phi_{ux} - \xi_{xx})u_x - \tau_{xx}u_y + (\phi_{uu} - 2\xi_{xu})u_x^2 - 2\tau_{ux}u_xu_y\right] - \xi_{uu}u_x^3 \\
&\quad + \left[-\tau_{uu}u_x^2u_y\right] + u_{xx}[\phi_u - 2\xi_x] + u_{xx}u_x[-3\xi_u] + u_{xx}u_y[-\tau_u] + u_{yx}[-2\tau_x] \\
&\quad + u_{xy}u_x[-2\tau_u]
\end{aligned}$$

$$\boxed{\begin{aligned}\eta^{[xx]} &= \phi_{xx} + (2\phi_{ux} - \xi_{xx})u_x - \tau_{xx}u_y + (\phi_{uu} - 2\xi_{xu})u_x^2 - 2\tau_{ux}u_xu_y - \xi_{uu}u_x^3 \\ &\quad - \tau_{uu}u_x^2u_y + (\phi_u - 2\xi_x)u_{xx} - 3\xi_u u_{xx}u_x - \tau_u u_{xx}u_y - 2\tau_x u_{yx} - 2\tau_u u_{xy}u_x\end{aligned}} \tag{1.104}$$

Similarly, we can find $\eta^{[xy]}$ as follows:

$$\begin{aligned}
D_y\eta^{[x]} &= \left(\frac{\partial}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial}{\partial u} + \frac{\partial u_x}{\partial y}\frac{\partial}{\partial u_x} + \frac{\partial u_y}{\partial y}\frac{\partial}{\partial u_y}\right)\eta^{[x]} \\
&= \left(\frac{\partial}{\partial y} + u_y\frac{\partial}{\partial u} + u_{xy}\frac{\partial}{\partial u_x} + u_{yy}\frac{\partial}{\partial u_y}\right)\eta^{[x]}
\end{aligned}$$

Using Eq. (1.91) in the previous relation, we get

$$\begin{aligned}
D_y\eta^{[x]} &= \left[\phi_{xy} + (\phi_{uy} - \xi_{xy})u_x - \tau_{xy}u_y - \xi_{uy}u_x^2 - \tau_{uy}u_xu_y\right] \\
&\quad + u_y\left[\phi_{xu} + (\phi_{uu} - \xi_{xu})u_x - \tau_{xu}u_y - \xi_{uu}u_x^2 - \tau_{uu}u_xu_y\right] \\
&\quad + u_{xy}\left[\phi_u - \xi_x - 2\xi_u u_x - \tau_u u_y\right] + u_{yy}[-\tau_x - \tau_u u_x] \\
&= \left[\phi_{xy} + (\phi_{uy} - \xi_{xy})u_x + (\phi_{xu} - \tau_{xy})u_y - \xi_{uy}u_x^2 + (\phi_{uu} - \xi_{xu} - \tau_{uy})u_xu_y\right] \\
&\quad - \tau_{xu}u_y^2 + \left[-\xi_{uu}u_x^2u_y - \tau_{uu}u_xu_y^2\right] + (\phi_u - \xi_x)u_{xy} + \left[-2\xi_u u_xu_{xy} - \tau_u u_yu_{xy}\right] \\
&\quad + \left[-\tau_x u_{yy} - \tau_u u_xu_{yy}\right]
\end{aligned} \tag{1.105}$$

Using equations (1.105), (1.101) and (1.103) in Eq. (1.98), we get

$$\begin{aligned}\eta^{[xy]} &= D_y\eta^{[x]} - u_{xx}D_y\xi - u_{xy}D_y\tau \\ &= \left[\phi_{xy} + \left(\phi_{uy} - \xi_{xy}\right)u_x + \left(\phi_{xu} - \tau_{xy}\right)u_y - \xi_{uy}u_x^2 + \left(\phi_{uu} - \xi_{xu} - \tau_{uy}\right)u_xu_y\right] \\ &\quad -\tau_{xu}u_y^2 + \left[-\xi_{uu}u_x^2u_y - \tau_{uu}u_xu_y^2\right] + (\phi_u - \xi_x)u_{xy} + \left[-2\xi_u u_x u_{xy} - \tau_u u_y u_{xy}\right] \\ &\quad +\left[-\tau_x u_{yy} - \tau_u u_x u_{yy}\right] - u_{xx}\left(\xi_y + u_y\xi_u\right) - u_{xy}\left(\tau_y + u_y\tau_u\right)\end{aligned}$$

or

$$\boxed{\begin{aligned}\eta^{[xy]} &= \phi_{xy} + \left(\phi_{uy} - \xi_{xy}\right)u_x + \left(\phi_{xu} - \tau_{xy}\right)u_y - \xi_{uy}u_x^2 + \left(\phi_{uu} - \xi_{xu} - \tau_{uy}\right)u_xu_y \\ &\quad -\tau_{xu}u_y^2 - \xi_{uu}u_x^2u_y - \tau_{uu}u_xu_y^2 + \left(\phi_u - \xi_x - \tau_y\right)u_{xy} - 2\xi_u u_x u_{xy} - 2\tau_u u_y u_{xy} \\ &\quad -\tau_x u_{yy} - \tau_u u_x u_{yy} - \xi_y u_{xx} - \xi_u u_y u_{xx}\end{aligned}}$$

The same procedure can be used to find $\eta^{[yy]}$. Eq. (1.84) implies

$$\begin{aligned}du_{y^*}^* &= du_y + \epsilon d\eta^{[y]} + O(\epsilon^2) \\ &= \frac{\partial u_y}{\partial x}dx + \frac{\partial u_y}{\partial y}dy + \epsilon\left[\frac{\partial\eta^{[y]}}{\partial x}dx + \frac{\partial\eta^{[y]}}{\partial y}dy + \frac{\partial\eta^{[y]}}{\partial u}du + \frac{\partial\eta^{[y]}}{\partial u_x}du_x + \frac{\partial\eta^{[y]}}{\partial u_y}du_y\right] + O(\epsilon^2) \\ &= \frac{\partial u_y}{\partial x}dx + \frac{\partial u_y}{\partial y}dy + \epsilon\left[\frac{\partial\eta^{[y]}}{\partial x}dx + \frac{\partial\eta^{[y]}}{\partial y}dy + \frac{\partial\eta^{[y]}}{\partial u}\left(\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\right)\right. \\ &\quad \left. + \frac{\partial\eta^{[y]}}{\partial u_x}\left(\frac{\partial u_x}{\partial x}dx + \frac{\partial u_x}{\partial y}dy\right) + \frac{\partial\eta^{[y]}}{\partial u_y}\left(\frac{\partial u_y}{\partial x}dx + \frac{\partial u_y}{\partial y}dy\right)\right] + O(\epsilon^2)\end{aligned}$$

or

$$du_{y^*}^* = \left(\frac{\partial u_y}{\partial x} + \epsilon D_x\eta^{[y]}\right)dx + \left(\frac{\partial u_y}{\partial y} + \epsilon D_y\eta^{[y]}\right)dy + O(\epsilon^2) \tag{1.106}$$

Using equations (1.86), (1.87) and (1.106) in the following formula:

$$du_{y^*}^* = \frac{\partial u_{y^*}^*}{\partial x^*}dx^* + \frac{\partial u_{y^*}^*}{\partial y^*}dy^*,$$

we get

$$du_{y^*}^* = u_{y^*x^*}^*\left[(1 + \epsilon D_x\xi)dx + \left(\epsilon D_y\xi\right)dy\right] + u_{y^*y^*}^*\left[(\epsilon D_x\tau)dx + \left(1 + \epsilon D_y\tau\right)dy\right] + O(\epsilon^2)$$

Using Eq. (1.106) and the independence of dx and dy imply

$$\begin{pmatrix} u_{yx} + \epsilon D_x\eta^{[y]} \\ u_{yy} + \epsilon D_y\eta^{[y]}\end{pmatrix} = \begin{pmatrix} 1 + \epsilon D_x\xi & \epsilon D_x\tau \\ \epsilon D_y\xi & 1 + \epsilon D_y\tau\end{pmatrix}\begin{pmatrix} u_{y^*x^*}^* \\ u_{y^*y^*}^*\end{pmatrix} + O(\epsilon^2) = A\begin{pmatrix} u_{y^*x^*}^* \\ u_{y^*y^*}^*\end{pmatrix} + O(\epsilon^2)$$

or

$$\begin{pmatrix} u_{yx} + \epsilon D_x\eta^{[y]} \\ u_{yy} + \epsilon D_y\eta^{[y]}\end{pmatrix} = (I + \epsilon B)\begin{pmatrix} u_{y^*x^*}^* \\ u_{y^*y^*}^*\end{pmatrix} + O(\epsilon^2) \tag{1.107}$$

Eq. (1.107) implies

$$\begin{pmatrix} u^*_{y^*x^*} \\ u^*_{y^*y^*} \end{pmatrix} = (I+\epsilon B)^{-1}\begin{pmatrix} u_{yx}+\epsilon D_x\eta^{[y]} \\ u_{yy}+\epsilon D_y\eta^{[y]} \end{pmatrix} + O(\epsilon^2)$$

$$= (I-\epsilon B)\begin{pmatrix} u_{yx}+\epsilon D_x\eta^{[y]} \\ u_{yy}+\epsilon D_y\eta^{[y]} \end{pmatrix} + O(\epsilon^2)$$

$$= \begin{pmatrix} u_{yx} \\ u_{yy} \end{pmatrix} + \epsilon\left[\begin{pmatrix} D_x\eta^{[y]} \\ D_y\eta^{[y]} \end{pmatrix} - B\begin{pmatrix} u_{yx} \\ u_{yy} \end{pmatrix}\right] + O(\epsilon^2)$$

Using equations (1.94) and (1.95), we conclude

$$\begin{pmatrix} \eta^{[yx]} \\ \eta^{[yy]} \end{pmatrix} = \begin{pmatrix} D_x\eta^{[y]} \\ D_y\eta^{[y]} \end{pmatrix} - B\begin{pmatrix} u_{xx} \\ u_{xy} \end{pmatrix} \tag{1.108}$$

But

$$D_y\eta^{[y]} = \left(\frac{\partial}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial}{\partial u} + \frac{\partial u_x}{\partial y}\frac{\partial}{\partial u_x} + \frac{\partial u_y}{\partial y}\frac{\partial}{\partial u_y}\right)\eta^{[y]}$$

$$= \left(\frac{\partial}{\partial y} + u_y\frac{\partial}{\partial u} + u_{xy}\frac{\partial}{\partial u_x} + u_{yy}\frac{\partial}{\partial u_y}\right)\eta^{[y]}$$

Using Eq. (1.92) in the previous relation, we get

$$\begin{aligned} D_y\,\eta^{[y]} = {} & \left[\phi_{yy} + (\phi_{uy}-\tau_{yy})u_y - \xi_{yy}u_x - \xi_{uy}u_yu_x - \tau_{uy}u_y^2\right] \\ & + u_y\left[\phi_{yu} + (\phi_{uu}-\tau_{yu})u_y - \xi_{yu}u_x - \xi_{uu}u_yu_x - \tau_{uu}u_y^2\right] + u_{xy}\left[-\xi_y - \xi_u u_y\right] \\ & + u_{yy}\left[\phi_u - \tau_y - \xi_u u_x - 2\tau_u u_y\right] \end{aligned}$$

or

$$\begin{aligned} D_y\,\eta^{[y]} = {} & \left[\phi_{yy} + (2\phi_{uy}-\tau_{yy})u_y - \xi_{yy}u_x - 2\xi_{uy}u_yu_x + (\phi_{uu}-2\tau_{yu})u_y^2\right] \\ & - \xi_{uu}u_y^2u_x - \tau_{uu}u_y^3 + \left[-\xi_y u_{xy} - \xi_u u_y u_{xy}\right] + (\phi_u - \tau_y)u_{yy} \\ & + \left[-\xi_u u_x u_{yy} - 2\tau_u u_y u_{yy}\right] \end{aligned} \tag{1.109}$$

Using equations (1.109), (1.101) and (1.103) in Eq. (1.108), we get

$$\begin{aligned} \eta^{[yy]} &= D_y\eta^{[y]} - u_{yx}D_y\xi - u_{yy}D_y\tau \\ &= \left[\phi_{yy} + (2\phi_{uy}-\tau_{yy})u_y - \xi_{yy}u_x - 2\xi_{uy}u_yu_x + (\phi_{uu}-2\tau_{yu})u_y^2\right] - \xi_{uu}u_y^2u_x \\ &\quad - \tau_{uu}u_y^3 + \left[-\xi_y u_{xy} - \xi_u u_y u_{xy}\right] + (\phi_u - \tau_y)u_{yy} + \left[-\xi_u u_x u_{yy} - 2\tau_u u_y u_{yy}\right] \\ &\quad - u_{yx}(\xi_y + u_y\xi_u) - u_{yy}(\tau_y + u_y\tau_u) \end{aligned}$$

or

$$\boxed{\begin{aligned} \eta^{[yy]} = {} & \phi_{yy} + (2\phi_{uy}-\tau_{yy})u_y - \xi_{yy}u_x - 2\xi_{uy}u_yu_x + (\phi_{uu}-2\tau_{yu})u_y^2 - \xi_{uu}u_y^2u_x \\ & - \tau_{uu}u_y^3 - 2\xi_y u_{xy} - 2\xi_u u_y u_{xy} + (\phi_u - 2\tau_y)u_{yy} - \xi_u u_x u_{yy} - 3\tau_u u_y u_{yy} \end{aligned}}$$

<u>Example</u>: Find the second prolongation of $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + u\frac{\partial}{\partial y}$.

Solution: Here, $\xi(x,y,u) = x, \qquad \tau(x,y,u) = y, \qquad \phi(x,y,u) = u$. So,

$$\eta^{[x]} = \phi_x + (\phi_u - \xi_x)u_x - \tau_x u_y - \xi_u u_x^2 - \tau_u u_x u_y$$
$$= 0 + (1-1)u_x - 0u_y - 0u_x^2 - 0u_x u_y = 0$$
$$\eta^{[y]} = \phi_y + \left(\phi_u - \tau_y\right)u_y - \xi_y u_x - \xi_u u_y u_x - \tau_u u_y^2$$
$$= 0 + (1-1)u_y - 0u_x - 0u_y u_x - 0u_y^2 = 0$$

$$\eta^{[xx]} = \phi_{xx} + (2\phi_{ux} - \xi_{xx})u_x - \tau_{xx}u_y + (\phi_{uu} - 2\xi_{xu})u_x^2 - 2\tau_{ux}u_x u_y - \xi_{uu}u_x^3$$
$$-\tau_{uu}u_x^2 u_y + (\phi_u - 2\xi_x)u_{xx} - 3\xi_u u_{xx}u_x - \tau_u u_{xx}u_y - 2\tau_x u_{yx} - 2\tau_u u_{xy}u_x$$
$$= 0 + (0-0)u_x - 0u_y + (0-0)u_x^2 - 0u_x u_y - 0u_x^3 - 0u_x^2 u_y$$
$$+(1-2)u_{xx} - 0u_{xx}u_x - 0u_{xx}u_y - 0u_{yx} - 0u_{xy}u_x$$
$$= -u_{xx}$$

$$\eta^{[xy]} = \phi_{xy} + \left(\phi_{uy} - \xi_{xy}\right)u_x + \left(\phi_{xu} - \tau_{xy}\right)u_y - \xi_{uy}u_x^2 + \left(\phi_{uu} - \xi_{xu} - \tau_{uy}\right)u_x u_y$$
$$-\tau_{xu}u_y^2 - \xi_{uu}u_x^2 u_y - \tau_{uu}u_x u_y^2 + \left(\phi_u - \xi_x - \tau_y\right)u_{xy} - 2\xi_u u_x u_{xy} - 2\tau_u u_y u_{xy}$$
$$-\tau_x u_{yy} - \tau_u u_x u_{yy} - \xi_y u_{xx} - \xi_u u_y u_{xx}$$
$$= 0 + (0-0)u_x + (0-0)u_y - 0u_x^2 + (0-0-0)u_x u_y$$
$$-0u_y^2 - 0u_x^2 u_y - 0u_x u_y^2 + (1-1-1)u_{xy} - 0u_x u_{xy} - 0u_y u_{xy}$$
$$-0u_{yy} - 0u_x u_{yy} - 0u_{xx} - 0u_y u_{xx}$$
$$= -u_{xy}$$

$$\eta^{[yy]} = \phi_{yy} + \left(2\phi_{uy} - \tau_{yy}\right)u_y - \xi_{yy}u_x - 2\xi_{uy}u_y u_x + (\phi_{uu} - 2\tau_{yu})u_y^2 - \xi_{uu}u_y^2 u_x$$
$$-\tau_{uu}u_y^3 - 2\xi_y u_{xy} - 2\xi_u u_y u_{xy} + \left(\phi_u - 2\tau_y\right)u_{yy} - \xi_u u_x u_{yy} - 3\tau_u u_y u_{yy}$$
$$= \phi_{yy} + \left(2\phi_{uy} - \tau_{yy}\right)u_y - \xi_{yy}u_x - 2\xi_{uy}u_y u_x + \left(\phi_{uu} - 2\tau_{yu}\right)u_y^2 - \xi_{uu}u_y^2 u_x$$
$$-\tau_{uu}u_y^3 - 2\xi_y u_{xy} - 2\xi_u u_y u_{xy} + \left(\phi_u - 2\tau_y\right)u_{yy} - \xi_u u_x u_{yy} - 3\tau_u u_y u_{yy}$$
$$= 0 + (0-0)u_y - 0u_x - 0u_y u_x + (0-0)u_y^2 - 0u_y^2 u_x$$
$$-0u_y^3 - 0u_{xy} - 0u_y u_{xy} + (1-2)u_{yy} - 0u_x u_{yy} - 0u_y u_{yy}$$
$$= -u_{yy}$$

So, the second prolongation is given by:

$$X^{[2]} = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial y} + \phi\frac{\partial}{\partial u} + \eta^{[x]}\frac{\partial}{\partial u_x} + \eta^{[y]}\frac{\partial}{\partial u_y} + \eta^{[xx]}\frac{\partial}{\partial u_{xx}} + \eta^{[xy]}\frac{\partial}{\partial u_{xy}} + \eta^{[yy]}\frac{\partial}{\partial u_{yy}}$$
$$= x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + u\frac{\partial}{\partial u} + 0\frac{\partial}{\partial u_x} + 0\frac{\partial}{\partial u_y} - u_{xx}\frac{\partial}{\partial u_{xx}} - u_{xy}\frac{\partial}{\partial u_{xy}} - u_{yy}\frac{\partial}{\partial u_{yy}}$$

1.2.3. Definition of symmetries of PDEs using prolongation

Given an nth order ODE

$$F\left(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, \dots, \eth^n u\right) = 0 \tag{1.110}$$

A symmetry of PDE (1.110) is a one parameter group infinitesimal transformations:

$$x^* = x + \epsilon\, \xi(x, y, u) + O(\epsilon^2)$$

$$y^* = y + \epsilon\, \tau(x, y, u) + O(\epsilon^2)$$

$$u^* = u + \epsilon\, \phi(x, y, u) + O(\epsilon^2)$$

or equivalently the infinitesimal generator

$$X = \xi(x, y, u)\frac{\partial}{\partial x} + \tau(x, y, u)\frac{\partial}{\partial y} + \phi(x, y, u)\frac{\partial}{\partial u}$$

that leaves the PDE (1.110) invariant, i e

$$F\left(\bar{x}, \bar{y}, \bar{u}, \overline{\eth^1 u}, \dots, \overline{\eth^n u}\right) = 0\,,$$

where $\overline{\eth^1 u}, \dots, \overline{\eth^n u}$ are transformed through prolongations of the group.

Treating equation (1.110) as an algebraic equation

in x , y , u , u_x , u_y , u_{xx} , u_{xy} , u_{yy} , ... , $\eth^n u$, invariance criteria can be generalized to the following definition

<u>Definition</u>:

The infinitesimal transformation

$$\bar{x} = x + \epsilon\, \xi(x, y, u)$$

$$\bar{y} = y + \epsilon\, \tau(x, y, u)$$

$$\bar{u} = u + \epsilon\, \phi(x, y, u)$$

or equivalently the generator

$$X = \xi(x, y, u)\frac{\partial}{\partial x} + \tau(x, y, u)\frac{\partial}{\partial y} + \phi(x, y, u)\frac{\partial}{\partial u}$$

is a **symmetry** of PDE (1.110) if

$$X^{[n]}F\Big|_{F=0} = 0$$

where $X^{[n]}$ is the nth prolongation of X.

Example: Show that $X = x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y}$ is a symmetry of the PDE

$$u_{xx} = u_y$$

Solution:

Here, from the PDE given,

$$F\left(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}\right) = u_{xx} - u_y$$

and from the generator X, we get

$$\xi(x, y, u) = x,$$

$$\tau(x, y, u) = 2y,$$

$$\phi(x, y, u) = 0$$

which implies

$$\eta^{[x]} = \phi_x + (\phi_u - \xi_x)u_x - \tau_x u_y - \xi_u u_x^2 - \tau_u u_x u_y$$

$$= 0 + (0 - 1)u_x - 0u_y - 0u_x^2 - 0u_x u_y = -u_x$$

$$\eta^{[y]} = \phi_y + \left(\phi_u - \tau_y\right)u_y - \xi_y u_x - \xi_u u_y u_x - \tau_u u_y^2$$

$$= 0 + (0 - 2)u_y - 0u_x - 0u_y u_x - 0u_y^2 = -2u_y$$

$$\eta^{[xx]} = \phi_{xx} + (2\phi_{ux} - \xi_{xx})u_x - \tau_{xx}u_y + (\phi_{uu} - 2\xi_{xu})u_x^2 - 2\tau_{ux}u_x u_y - \xi_{uu}u_x^3$$
$$-\tau_{uu}u_x^2 u_y + (\phi_u - 2\xi_x)u_{xx} - 3\xi_u u_{xx}u_x - \tau_u u_{xx}u_y - 2\tau_x u_{yx} - 2\tau_u u_{xy}u_x$$

$$= 0 + (0 - 0)u_x - 0u_y + (0 - 0)u_x^2 - 0u_x u_y - 0u_x^3 - 0u_x^2 u_y$$
$$+(0 - 2)u_{xx} - 0u_{xx}u_x - 0u_{xx}u_y - 0u_{yx} - 0u_{xy}u_x$$

$$= -2u_{xx}$$

$$\eta^{[xy]} = \phi_{xy} + \left(\phi_{uy} - \xi_{xy}\right)u_x + \left(\phi_{xu} - \tau_{xy}\right)u_y - \xi_{uy}u_x^2 + \left(\phi_{uu} - \xi_{xu} - \tau_{uy}\right)u_x u_y$$
$$-\tau_{xu}u_y^2 - \xi_{uu}u_x^2 u_y - \tau_{uu}u_x u_y^2 + \left(\phi_u - \xi_x - \tau_y\right)u_{xy} - 2\xi_u u_x u_{xy} - 2\tau_u u_y u_{xy}$$
$$-\tau_x u_{yy} - \tau_u u_x u_{yy} - \xi_y u_{xx} - \xi_u u_y u_{xx}$$

$$= 0 + (0 - 0)u_x + (0 - 0)u_y - 0u_x^2 + (0 - 0 - 0)u_x u_y$$
$$-0u_y^2 - 0u_x^2 u_y - 0u_x u_y^2 + (0 - 1 - 2)u_{xy} - 0u_x u_{xy} - 0u_y u_{xy}$$
$$-0u_{yy} - 0u_x u_{yy} - 0u_{xx} - 0u_y u_{xx}$$

$$= -3u_{xy}$$

$$\eta^{[yy]} = \phi_{yy} + \left(2\phi_{uy} - \tau_{yy}\right)u_y - \xi_{yy}u_x - 2\xi_{uy}u_y u_x + (\phi_{uu} - 2\tau_{yu})u_y^2 - \xi_{uu}u_y^2 u_x$$
$$-\tau_{uu}u_y^3 - 2\xi_y u_{xy} - 2\xi_u u_y u_{xy} + \left(\phi_u - 2\tau_y\right)u_{yy} - \xi_u u_x u_{yy} - 3\tau_u u_y u_{yy}$$

$$= 0 + (0 - 0)u_y - 0u_x - 0u_y u_x + (0 - 0)u_y^2 - 0u_y^2 u_x$$
$$-0u_y^3 - 0u_{xy} - 0u_y u_{xy} + (0 - 4)u_{yy} - 0u_x u_{yy} - 0u_y u_{yy}$$
$$= -4u_{yy}$$

Hence, the second prolongation is given by:

$$X^{[2]} = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial y} + \phi\frac{\partial}{\partial u} + \eta^{[x]}\frac{\partial}{\partial u_x} + \eta^{[y]}\frac{\partial}{\partial u_y} + \eta^{[xx]}\frac{\partial}{\partial u_{xx}} + \eta^{[xy]}\frac{\partial}{\partial u_{xy}} + \eta^{[yy]}\frac{\partial}{\partial u_{yy}}$$

$$= x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} + 0\frac{\partial}{\partial u} - u_x\frac{\partial}{\partial u_x} - 2u_y\frac{\partial}{\partial u_y} - 2u_{xx}\frac{\partial}{\partial u_{xx}} - 3u_{xy}\frac{\partial}{\partial u_{xy}} - 4u_{yy}\frac{\partial}{\partial u_{yy}}$$

So,

$$X^{[2]}F = x\frac{\partial F}{\partial x} + 2y\frac{\partial F}{\partial y} + 0\frac{\partial F}{\partial u} - u_x\frac{\partial F}{\partial u_x} - 2u_y\frac{\partial F}{\partial u_y} - 2u_{xx}\frac{\partial F}{\partial u_{xx}} - 3u_{xy}\frac{\partial F}{\partial u_{xy}} - 4u_{yy}\frac{\partial F}{\partial u_{yy}}$$

$$= x(0) + 2y(0) + 0(0) - u_x(0) - 2u_y(-1) - 2u_{xx}(1) - 3u_{xy}(0) - 4u_{yy}(0)$$

$$= 2u_y - 2u_{xx}$$

$$= -2(u_{xx} - u_y)$$

$$= -2F$$

$$= 0 \text{ if } F = 0$$

This implies that X is symmetry of the PDE given.

1.2.4. Method of finding symmetries of PDEs

The equation

$$X^{[n]}F\big|_{F=0} = 0$$

is called the invariance criteria for the symmetries of PDE (1.110) of order n. It is the basis for computing the symmetries of PDE (1.110). This will lead us to an over determined system of linear PDEs in $\xi(x,y,u), \tau(x,y,u)$ and $\phi(x,y,u)$. Solving this system will give us all possible functions $\xi(x,y,u), \tau(x,y,u)$ and $\phi(x,y,u)$ that satisfy the invariants condition.

Steps for finding Symmetries of (1.110)

1) Find prolongation $X^{[n]}F$ where n is the order of the PDE
2) Substitute the constraint (PDE) in the prolongation found in step 1.
3) Obtain determining equations from step 2 by comparing coefficients of derivatives of u.
4) Simplify and solve the determining equations

Example: (Simple example to show the procedure)

Find all symmetries of

$$u_{xx} = u_y$$

Solution: Let

$$F\left(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}\right) = u_{xx} - u_y$$

Let the symmetry be of the form

$$X = \xi(x, y, u)\frac{\partial}{\partial x} + \tau(x, y, u)\frac{\partial}{\partial y} + \phi(x, y, u)$$

Step 1: write the 2nd prolongation

$$\begin{aligned}
\boldsymbol{X}^{[2]}F &= -\eta^{[y]} + \eta^{[xx]} \\
&= -\left[\phi_y + \left(\phi_u - \tau_y\right)u_y - \xi_y u_x - \xi_u u_y u_x - \tau_u u_y^2\right] \\
&\quad + \left[\phi_{xx} + (2\phi_{ux} - \xi_{xx})u_x - \tau_{xx}u_y + (\phi_{uu} - 2\xi_{xu})u_x^2 - 2\tau_{ux}u_x u_y - \xi_{uu}u_x^3\right. \\
&\quad \left. -\tau_{uu}u_x^2 u_y + (\phi_u - 2\xi_x)u_{xx} - 3\xi_u u_{xx}u_x - \tau_u u_{xx}u_y - 2\tau_x u_{yx} - 2\tau_u u_{xy}u_x\right]
\end{aligned}$$

Step 2: Substituting the constraint condition in the prolongation found in step (1) gives

$$\begin{aligned}
X^{[2]}F\big|_{F=0} &= -\left[\phi_y + \left(\phi_u - \tau_y\right)u_y - \xi_u u_y u_x - \tau_u u_y^2\right] \\
&\quad + \left[\phi_{xx} + -\tau_{xx}u_y + (\phi_{uu} - 2\xi_{xu})u_x^2 - 2\tau_{ux}u_x u_y - \xi_{uu}u_x^3\right. \\
&\quad \left. -\tau_{uu}u_x^2 u_y + (\phi_u - 2\xi_x)u_y - 3\xi_u u_y u_x - \tau_u u_y^2 - 2\tau_x u_{yx} - 2\tau_u u_{xy}u_x\right] \\
&= \left(\phi_{xx} - \phi_y\right) + \left(2\phi_{ux} - \xi_{xx} + \xi_y\right)u_x + \left(\tau_y - 2\xi_x - \tau_{xx}\right)u_y \\
&\quad + (\phi_{uu} - 2\xi_{xu})u_x^2 - 2(\xi_u + \tau_{ux})u_y u_x - 2\tau_x u_{yx} - \xi_{uu}u_x^3 \\
&\quad - \tau_{uu}u_x^2 u_y - 2\tau_u u_{xy}u_x
\end{aligned} \tag{1.111}$$

Step 3: Finding the determining equations

Comparing coefficients in Eq. (1.111) of step 2, we get

Constant term	:	$\phi_{xx} - \phi_y = 0$	(1.112)
Coefficient of u_x	:	$2\phi_{ux} - \xi_{xx} + \xi_y = 0$	(1.113)
Coefficient of u_y	:	$\tau_y - 2\xi_x - \tau_{xx} = 0$	(1.114)
Coefficient of u_x^2	:	$\phi_{uu} - 2\xi_{xu} = 0$	(1.115)
Coefficient of $u_y u_x$	:	$\xi_u + \tau_{ux} = 0$	(1.116)
Coefficient of u_{yx}	:	$\tau_x = 0$	(1.117)
Coefficient of u_x^3	:	$\xi_{uu} = 0$	(1.118)

Coefficient of $u_x^2 u_y$: $\tau_{uu} = 0$ (1.119)

Coefficient of $u_{xy} u_x$: $\tau_u = 0$ (1.120)

We get an over determined system of linear PDEs. This system can be simplified to the following determining equations:

Ordering the equations such that $x > y > u$ and $\phi > \tau > \xi$ to get

E_1 : $\xi_u = 0$

E_2 : $\tau_u = 0$

E_3 : $\tau_x = 0$

E_4 : $\tau_y - 2\xi_x = 0$

E_5 : $\phi_{uu} = 0$

E_6 : $2\phi_{ux} - \xi_{xx} + \xi_y = 0$

E_7 : $\phi_{xx} - \phi_y = 0$

<u>Step 4</u>: Solving the over determined system of linear PDEs found in step 3.

The operation $(E_3)_y - (E_4)_x$ leads to

$$\xi_{xx} = 0 \qquad (1.121)\, E_{1-1}$$

Using this equation, equation E_6 will be simplified to the following:

E_6 : $2\phi_{ux} + \xi_y = 0$

The operation $(E_6)_x - 2(E_7)_u$ gives

$$2\phi_{uy} + \xi_{xy} = 0 \qquad (1.122)\, E_{5-1}$$

The updated system is becoming:

E_1 : $\xi_u = 0$

E_{1-1} : $\xi_{xx} = 0$

E_2 : $\tau_u = 0$

E_3 : $\tau_x = 0$

E_4 : $\tau_y - 2\xi_x = 0$

E_5 : $\phi_{uu} = 0$

E_{5-1} : $2\phi_{ux} + \xi_{xy} = 0$

E_6 : $2\phi_{ux} - \xi_{xx} + \xi_y = 0$

E_7 : $\phi_{xx} - \phi_y = 0$

Now, from the first two equations E_1 and E_{1-1} we get

$$\xi = A(y)x + B(y) \tag{1.123}$$

Equations E_2 and E_3 imply

$$\tau = \tau(y) \tag{1.124}$$

Substituting equations (1.123) and (1.124) in equation E_4 gives

$$\tau'(y) = 2A(y) \tag{1.125}$$

From equation E_5, we have

$$\phi = f(x,y)u + g(x,y) \tag{1.126}$$

Using equations (1.123) and (1.126) in E_{5-1} gives

$$2f_y + A'(y) = 0 \tag{1.127}$$

which implies

$$2f(x,y) + A(y) = K(x) \tag{1.128}$$

Using equations (1.123) and (1.126) in equation E_6 , we get

$$2f_x + A'(y)x + B'(y) = 0 \tag{1.129}$$

Differentiating Eq. (1.127) with respect to x and Eq. (1.129) with respect to y give

$$f_{yx} = 0 \tag{1.130}$$

$$2f_{xy} + A''(y)x + B''(y) = 0 \tag{1.131}$$

Substituting Eq. (1.130) in Eq. (1.131) gives

$$A''(y)x + B''(y) = 0 \tag{1.132}$$

which implies that both $A(y)$ and $B(y)$ are linear functions. Let

$$A(y) = k_1 y + k_2 \tag{1.133}$$

$$B(y) = k_3 y + k_4 \tag{1.134}$$

Using Eq. (1.133) in Eq. (1.127) gives

$$2f_y + k_1 = 0$$

or

$$f(x,y) = -\frac{k_1}{2}y + H(x) \tag{1.135}$$

Substituting Eq. (1.135) in Eq. (1.129), we obtain

$$2H'(x) + k_1 x + k_3 = 0$$

which implies

$$H(x) = -\frac{k_1}{4}x^2 - \frac{k_3}{2}x + k_5 \tag{1.136}$$

Using Eq. (1.136) in Eq. (1.135) gives

$$\boxed{f(x,y) = -\frac{k_1}{2}y - \frac{k_1}{4}x^2 - \frac{k_3}{2}x + k_5} \tag{1.137}$$

Using Eq. (1.133) in Eq. (1.125) gives

$$\tau'(y) = 2A(y) = 2a_1 y + 2a_0$$

or

$$\boxed{\tau(y) = k_1 y^2 + 2k_2 y + k_6} \tag{1.138}$$

Using equations (1.126) and (1.137) in equation E_7 , we get

$$g_{xx} - g_y = 0 \tag{1.139}$$

Eq. (1.139) means that the function $g(x,y)$ can be any function that satisfies the heat equation given, which gives an infinite dimensional algebra generated by the solutions of the given heat equation. Substituting equations (1.133) and (1.134) in Eq. (1.123), we get

$$\boxed{\xi = (k_1 y + k_2)x + k_3 y + k_4} \tag{1.140}$$

Ignoring the infinite dimensional term (the term with g), using equations (1.137) - (1.140) and Eq. (1.126), we have 6 independent constants. The Lie symmetries associated with these constants are as follows

$$X_1 = xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y} + \left(-\frac{1}{2}yu - \frac{1}{4}x^2 u\right)\frac{\partial}{\partial u}$$

$$X_2 = x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y}$$

$$X_3 = y\frac{\partial}{\partial x} - \frac{1}{2}xu\frac{\partial}{\partial u}$$

$$X_4 = \frac{\partial}{\partial x}$$

$$X_5 = u\frac{\partial}{\partial u}$$

$$X_6 = \frac{\partial}{\partial y}$$

Putting

$$g(x,y) \neq 0,$$

and the constants

$$k_i = 0, i = 1,2,\dots,6$$

lead to the infinite dimensional Lie algebra generated by

$$\boxed{\boldsymbol{X_g} = \boldsymbol{g(x,y)}\frac{\partial}{\partial u}}$$

Commutation relations for Lie algebra generated by X_1 , X_2 , X_3 , X_4 , X_5 , X_6 are given in the table below

	X_1	X_2	X_3	X_4	X_5	X_6
X_1	0	$-2X_1$	0	$-X_3$	0	$-X_2+\frac{1}{2}X_5$
X_2	$2X_1$	0	X_3	$-X_4$	0	$-2X_6$
X_3	0	$-X_3$	0	$\frac{1}{2}X_5$	0	$-X_4$
X_4	X_3	X_4	$-\frac{1}{2}X_5$	0	0	0
X_5	0	0	0	0	0	0
X_6	$X_2-\frac{1}{2}X_5$	$2X_6$	X_4	0	0	0

1.2.5. Reduction of PDEs using symmetry

Given a PDE

$$F\left(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}\right) = 0$$

with a symmetry

$$X = \xi(x, y, u)\frac{\partial}{\partial x} + \tau(x, y, u)\frac{\partial}{\partial y} + \phi(x, y, u)\frac{\partial}{\partial u}.$$

The standard procedure, cf. [8, 21], is to utilize the similarity variables to reduce the PDE to ODE where the similarity variables are the invariants of X obtained by solving $X(I) = 0$. The steps of this procedure are clarified in examples below.

<u>Example</u> (Simple example to show the procedure)

Reduce the PDE

$$u_{xx} = u_y \tag{1.141}$$

using the symmetry

$$X = X_2 = x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y}$$

Solution:

<u>Step 1</u>: (Invariants)

To find the invariants of $X(I) = 0$, we solve the characteristic system

$$\frac{dx}{x} = \frac{dy}{2y} = \frac{du}{0}$$

The equation

$$\frac{dx}{x} = \frac{dy}{2y}$$

implies

$$\frac{x^2}{y} = c \tag{1.142}$$

and the equation

$$\frac{dx}{x} = \frac{du}{0}$$

implies

$$u = k \tag{1.143}$$

Equations (1.142) and (1.143) provide us with two invariants leading to the similarity variables

$$\xi(x,y) = \frac{x^2}{y} \tag{1.144}$$

$$V(\xi) = u \tag{1.145}$$

<u>Step 2</u>:

To substitute the similarity variables given by equations (1.144) and (1.145) in the original PDE (1.141), we need to find the following using chain rule

$$u_x = V'\xi_x = \frac{2x}{y}V'$$

$$u_{xx} = \frac{2}{y}V' + \frac{2x}{y}V''\left(\frac{2x}{y}\right) = \frac{2}{y}V' + \frac{4x^2}{y^2}V'' \tag{1.146}$$

$$u_y = V'\xi_y = \frac{-x^2}{y^2}V' \tag{1.147}$$

Using equations (1.146) and (1.147) in the original PDE (1.141) gives

$$\frac{2}{y}V' + \frac{4x^2}{y^2}V'' = \frac{-x^2}{y^2}V'$$

Multiplying the last equation by y gives

$$2\,V' + \frac{4x^2}{y}V'' = \frac{-x^2}{y}V'$$

Using Eq. (1.144) in the previous equation yields the second order ODE

$$2V' + 4\xi V'' = -\xi V'$$

or

$$4\xi V'' + (\xi + 2)V' = 0 \tag{1.148}$$

To solve this ODE, we divide it by $\xi V'$ to get

$$\frac{V''}{V'} = -\frac{1}{4} - \frac{1}{2\xi}$$

Integrating gives

$$\ln V' = -\frac{\xi}{4} - \frac{\ln\xi}{2} + c$$

or

$$V' = \frac{c}{\sqrt{\xi}}e^{-\xi/4}$$

Integrating yields

$$V = \int \frac{c}{\sqrt{\xi}}e^{-\xi/4}\,d\xi$$

or

$$V = 2c\sqrt{\pi}\,\mathrm{erf}\left(\frac{\sqrt{\xi}}{2}\right) + k \tag{1.149}$$

Returning to the original variables, we get the exact solution

$$\boxed{u(x,y) = c_1\,\mathrm{erf}\left(\frac{x}{2\sqrt{y}}\right) + c_2}$$

<u>Example</u>:

Reduce the PDE $u_{yy} = u_{xx} + u^n$ using the symmetry $X = y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$.

Solution:

<u>Step 1</u> : (Invariants)

The characteristic system of $X(I) = 0$ is

$$\frac{dx}{y} = \frac{dy}{x} = \frac{du}{0}.$$

Solving the equation

$$\frac{dx}{y} = \frac{dy}{x},$$

gives the constant

$$\boxed{x^2 - y^2 = c} \tag{1.150}$$

and solving the equation

$$\frac{dx}{y} = \frac{du}{0}$$

gives the constnt

$$\boxed{u = k} \tag{1.151}$$

Hence, we have found from equations (1.150) and (1.151) two similarity variables,

$$\xi(x,y) = x^2 - y^2 \tag{1.152}$$

$$V(\xi) = u \tag{1.153}$$

<u>Step 2</u>: Using equations (1.152) and (1.153), we have

$$u_x = V'\xi_x = 2xV'$$

$$u_{xx} = 2V' + 2xV''(2x) = 2V' + 4x^2V''$$

$$u_y = V'\xi_y = -2yV'$$

$$u_{yy} = -2V' - 2yV''(-2y) = -2V' + 4y^2V''$$

Using these in the PDE in our example gives

$$-2V' + 4y^2V'' = 2V' + 4x^2V'' + V^n$$

or

$$0 = 4V' + 4(x^2 - y^2)V'' + V^n$$

Using Eq. (1.152) in the previous equation leads to

$$4\xi V'' + 4V' + V^n = 0 \tag{1.154}$$

which is the reduction of

$$u_{yy} = u_{xx} + u^n$$

via the symmetry

$$X = y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

The reduced ODE (1.154) can be analyzed for different values n.

CHAPTER 2

GROUP CLASSIFICATION AND SYMMETRY REDUCTIONS FOR A CLASS OF NONLINEAR POISSON EQUATIONS ON THE LINE

This chapter is devoted to obtaining a complete classification of symmetries and examples of symmetry reductions for a class of nonlinear Poisson equations in one dimension of the form

$$y'' = f(y) \tag{2.1}$$

where $f(y)$ is an arbitrary nonlinear function.

The group classification of equation (2.1) is carried out in section 2.1. It is based on necessary conditions on $f(y)$ which are obtained through a triangulation of determining equations of Lie symmetries of equation (2.1). An efficient method to obtain such triangulation is the well-known method of Mansfield [29] of generating differential Grobner bases of determining equations. A variant of Mansfield's method will be used to generate cases of $f(y)$ and hence for carrying out the group classification. Precisely, the following result will be proved.

Theorem 2.1

The minimal symmetry algebra of nonlinear ODE $y'' = f(y)$ is generated by $X = \frac{\partial}{\partial x}$ and is obtained for all nonlinear arbitrary functions $f(y)$. The larger symmetry algebra exists in the cases given in the table below:

$f(y)$	**Generators of symmetry algebra**
$a(y+b)^c$, $c \neq 0,1,-3$	$X_1 = \frac{\partial}{\partial x}$, $X_2 = x\frac{\partial}{\partial x} - \frac{2}{c-1}(y+b)\frac{\partial}{\partial y}$
ae^{by} , $b \neq 0$	$X_1 = \frac{\partial}{\partial x}$, $X_2 = x\frac{\partial}{\partial x} - \frac{2}{b}\frac{\partial}{\partial y}$
$a(y+b)^{-3}$, $a \neq 0$	$X_1 = \frac{\partial}{\partial x}$, $X_2 = x\frac{\partial}{\partial x} + \frac{1}{2}(y+b)\frac{\partial}{\partial y}$, $X_3 = x^2\frac{\partial}{\partial x} + x(y+b)\frac{\partial}{\partial y}$
$a(y+b)^{-3} + c(y+b)$ $a \neq 0, c > 0$	$X_1 = \frac{\partial}{\partial x}$, $X_2 = e^{2x\sqrt{c}}\frac{\partial}{\partial x} + \sqrt{c}e^{2x\sqrt{c}}(y+b)\frac{\partial}{\partial y}$, $X_3 = e^{-2x}\frac{\partial}{\partial x} - \sqrt{c}\, e^{-2x\sqrt{c}}(y+b)\frac{\partial}{\partial y}$
$a(y+b)^{-3} + c(y+b)$ $a \neq 0, c < 0$	$X_1 = \frac{\partial}{\partial x}$, $X_2 = \sin\left(2\sqrt{-c}\, x\right)\frac{\partial}{\partial x} + \sqrt{-c}(y+b)\cos\left(2\sqrt{-c}\, x\right)\frac{\partial}{\partial y}$, $X_3 = \cos\left(2\sqrt{-c}\, x\right)\frac{\partial}{\partial x} - \sqrt{-c}(y+b)\sin(2\sqrt{-c}\, x)\frac{\partial}{\partial y}$

For different cases obtained in section 2.1, some examples of reductions and invariant solutions of equation (2.1) are presented in section 2.2. Reduction of the general case of equation (2.1) using symmetry method leads to a solution in integral form. Closed form solutions are found in some cases of functions.

2.1. GROUP CLASSIFICATION OF POISSON EQUATION ON THE LINE

In order to obtain the Lie symmetries of equation (2.1), let us consider the one parameter group of infinitesimal transformations

$$x^* = x + \epsilon\, \xi(x,y) + O(\epsilon^2) \tag{2.2}$$

$$y^* = y + \epsilon\, \phi(x,y) + O(\epsilon^2) \tag{2.3}$$

where ϵ is the group parameter. The corresponding generator of Lie algebra is of the form

$$X = \xi(x,y)\frac{\partial}{\partial x} + \phi(x,y)\frac{\partial}{\partial y} \tag{2.4}$$

If $X^{[2]}$ denotes the second prolongation of X then using the invariance condition

$$X^{[2]}\big(y'' - f(y)\big)\big|_{(2.1)} = 0 \tag{2.5}$$

yields the system of equations to determine $\xi(x,y)$ and $\phi(x,y)$. Here

$$X^{[2]}\big(y'' - f(y)\big) = \eta^{[2]} - \phi\, f_y$$

with

$$\eta^{[2]} = \phi_{xx} + \big(2\,\phi_{xy} - \xi_{xx}\big)y' + \big(\phi_{yy} - 2\,\xi_{xy}\big)y'^2 - \xi_{yy}y'^3 + \big(\phi_y - 2\xi_x\big)y'' - 3\xi_y y''y'$$

This gives

$$X^{[2]}\big(y'' - f(y)\big) = \phi_{xx} + \big(2\phi_{xy} - \xi_{xx}\big)y' + \big(\phi_{yy} - 2\xi_{xy}\big)y'^2 - \xi_{yy}y'^3 + \big(\phi_y - 2\xi_x\big)y'' - 3\xi_y y''y' - \phi f_y$$

Substituting the constraint

$$y'' = f(y),$$

we get

$$\begin{aligned} X^{[2]}\Big(y'' - f(y)\Big)\Big|_{(2.1)} &= \phi_{xx} + \big(2\phi_{xy} - \xi_{xx}\big)y' + \big(\phi_{yy} - 2\xi_{xy}\big)y'^2 - \xi_{yy}y'^3 \\ &\quad + \big(\phi_y - 2\xi_x\big)f - 3\xi_y\, f y' - \phi\, f_y \\ &= \phi_{xx} + \big(\phi_y - 2\xi_x\big)f - \phi f_y + \big(2\phi_{xy} - \xi_{xx} - 3\xi_y f\big)y' \\ &\quad + \big(\phi_{yy} - 2\xi_{xy}\big)y'^2 - \xi_{yy}y'^3 \end{aligned} \tag{2.6}$$

Comparing the coefficients of the power of derivatives of y in Eq. (2.5) yields the following four determining equations:

$$\xi_{yy} = 0 \qquad e_1$$

$$\phi_{yy} - 2\xi_{xy} = 0 \qquad e_2$$

$$2\phi_{xy} - \xi_{xx} - 3\xi_y f = 0 \qquad e_3$$

$$\phi_{xx} + f\phi_y - 2f\xi_x - \phi f_y = 0 \qquad e_4$$

The system of determining equations will be solved utilizing a triangulation procedure based on techniques for obtaining differential Groebner basis developed by Mansfield in [29].

The operation $(e_3)_y - 2(e_2)_x$ implies

$$\xi_{xxy} - \xi_y f_y = 0 \tag{2.7}$$

Differentiating Eq. (2.7) with respect to y, we get

$$\xi_y f_{yy} = 0.$$

Since $f(y)$ is nonlinear, this implies that

$$\xi_y = 0. \tag{2.8}$$

The system of determining equations hence becomes

$$E_1: \quad \xi_y = 0$$

$$E_2: \quad \phi_{yy} = 0$$

$$E_3: \quad 2\phi_{xy} - \xi_{xx} = 0$$

$$E_4: \quad \phi_{xx} + f\phi_y - 2f\xi_x - \phi f_y = 0$$

The two equations E_3 and E_4 can be used via $(E_3)_x - 2(E_4)_y$ to get

$$-\xi_{xxx} + 4f_y\xi_x + 2\phi f_{yy} = 0 \tag{2.9}$$

Differentiating Eq. (2.9) with respect to y yields

$$2f_{yy}\xi_x + \phi_y f_{yy} + \phi f_{yyy} = 0 \tag{2.10}$$

Differentiating Eq. (2.10) with respect to y, we get

$$2f_{yyy}\xi_x + 2\phi_y f_{yyy} + \phi f_{yyyy} = 0 \tag{2.11}$$

Simplifying Eq. (2.11) with respect to Eq. (2.10) via $f_{yyy}(2.11) - f_{yyyy}(2.10) = 0$ implies

$$\phi_y\left(2f_{yyy}^2 - f_{yy}f_{yyyy}\right) + 2\xi_x\left(f_{yyy}^2 - f_{yy}f_{yyyy}\right) = 0 \tag{2.12}$$

Differentiating Eq. (2.12) with respect to y implies

$$\phi_y\left(3f_{yyy}f_{yyyy} - f_{yy}f_{yyyyy}\right) + 2\xi_x\left(f_{yyy}f_{yyyy} - f_{yy}f_{yyyyy}\right) = 0 \tag{2.13}$$

Equations (2.12) and (2.13) can be used to eliminate ϕ_y, giving the equation

$$\boxed{\left[f_{yyy}^3 f_{yyyy} - 2f_{yy}f_{yyy}f_{yyyy}^2 + f_{yy}f_{yyyyy}f_{yyy}^2\right]\xi_x = 0} \tag{2.14}$$

Differentiating Eq. (2.10) with respect to x, gives

$$2f_{yy}\xi_{xx} + \phi_{xy}f_{yy} + \phi_x f_{yyy} = 0 \tag{2.15}$$

But from E_3,

$$\phi_{xy} = \frac{1}{2}\xi_{xx} \tag{2.16}$$

Using Eq. (2.16) in Eq. (2.15) gives the equation

$$5f_{yy}\xi_{xx} + 2\phi_x f_{yyy} = 0 \tag{2.17}$$

Differentiating Eq. (2.17) with respect to y, we get

$$6f_{yyy}\xi_{xx} + 2\phi_x f_{yyyy} = 0 \tag{2.18}$$

Eliminating ϕ_x using equations (2.17) and (2.18), leads to

$$\boxed{\left(5f_{yy}f_{yyyy} - 6f_{yyy}^2\right)\xi_{xx} = 0} \tag{2.19}$$

Next we look for possibilities of the functions $f(y)$. If

$$\xi_x = 0$$

then by E_1 ,

$$\xi = constant$$

From Eq. (2.9) and the nonlinearity of f, we obtain

$$\phi = 0$$

without any restriction on $f(y)$. Hence the minimal symmetry algebra is one dimensional that exists for any choice of $f(y)$ and is generated by $X = \frac{\partial}{\partial x}$.

To search for functions $f(y)$ that may give larger symmetry algebra we assume $\xi_x \neq 0$ and solve the following two equations

$$5f_{yy}f_{yyyy} - 6f_{yyy}^2 = 0 \tag{2.20}$$

$$f_{yyy}^3 f_{yyyy} - 2f_{yy}f_{yyy}f_{yyyy}^2 + f_{yy}f_{yyyyy}f_{yyy}^2 = 0 \tag{2.21}$$

The substitution $H = f_{yy}$ reduces Eq. (2.20) to the following

$$5HH_{yy} - 6H_y^2 = 0$$

Then,

$$H_y = 0$$

implies $f(y)$ is quadratic. Otherwise, dividing by HH_y, we get the following equation

$$5\frac{H_{yy}}{H_y} - 6\frac{H_y}{H} = 0$$

Integrating, we get

$$5\ln H_y - 6\ln H = c$$

This implies

$$(H)^{-6/5}H_y = c$$

Integrating again, we obtain the following

$$(H)^{-1/5} = cy + k$$

Simplifying, we get

$$H = a(y+b)^{-5}$$

Integrating twice with respect to y, we get the following solution

$$f(y) = a(y+b)^{-3} + cy + d$$

Thus, solving Eq. (2.20) for nonlinear functions gives two solutions:

- $f(y) = a(y^2 + by + c\,)\ with\ \ a \neq 0$
- $f(y) = a(y+b)^{-3} + cy + d\ \ with\ \ a \neq 0$

Similarly, the substitution $H = f_{yy}$ reduces equation (2.21) to the following

$$H_y^3 H_{yy} - 2HH_y H_{yy}^2 + HH_{yyy}H_y^2 = 0$$

Taking H_y as a common factor gives

$$H_y\left[H_y^2 H_{yy} - 2HH_{yy}^2 + HH_{yyy}H_y\right] = 0$$

The case

$$H_y = 0$$

implies $f(y)$ is quadratic. Otherwise,

$$H_y^2 H_{yy} - 2HH_{yy}^2 + HH_{yyy}H_y = 0,$$

we conclude that either

$$H_{yy} = 0$$

which implies that $f(y)$ is cubic, or dividing by HH_yH_{yy} gives the following

$$\frac{H_y}{H} - 2\frac{H_{yy}}{H_y} + \frac{H_{yyy}}{H_{yy}} = 0$$

Integrating, we get

$$\ln H - 2\ln H_y + \ln H_{yy} = c$$

This implies

$$\frac{H_{yy}}{H_y} - c\frac{H_y}{H} = 0$$

Integrating, we get

$$\ln H_y - c\ln H = k$$

$$\frac{H_y}{H^c} = k$$

There are two cases:-

If $c = 1$, then

$$\ln H = ky + k1$$

which implies

$$f(y) = c_1 e^{c_2 y} + c_3 y + c_4$$

If $c \neq 1$, then

$$H^{1-c} = ky + k1.$$

This means

$$f_{yy} = a(y + b\ )^n$$

This can be divided into three cases.

If $n \neq -1, -2$ we get

$$f_y = a\frac{(y+b)^{n+1}}{n+1} + d$$

Integrating gives

$$f(y) = a\frac{(y+b)^{n+2}}{(n+1)(n+2)} + dy + e \quad where \quad n \neq -1\ , -2$$

which can be written in the following simpler form

$$f(y) = a(y+b)^m + dy + e \qquad where \quad m \neq 0,1$$

If $n = -2$, we get

$$f_y = a\frac{(y+b)^{-1}}{-1} + d$$

Hence,

$$f(y) = -a\ln(y+b) + dy + e$$

If $n = -1$, we get

$$f_{yy} = a(y+b)^{-1}$$

Integrating, we get

$$f_y = a\ln(y+b) + d$$

Integrating again, the following function is obtained

$$f(y) = a[(y+b)\ln(y+b) - (y+b)] + dy + e$$

Thus, solving Eq. (2.21) for nonlinear functions gives 6 solutions:

- $f(y)\ is\ quadratic$
- $f(y)\ is\ cubic$
- $f(y) = ae^{by} + cy + d \quad where \quad a, b \neq 0$
- $f(y) = a(y+b)^c + dy + e \quad where\ \ a \neq 0,\ \ c\ \neq 0$,1,2,3,
- $f(y) = a\ln(y+b) + dy + e \quad with\ \ a \neq 0$
- $f(y) = a[(y+b)\ln(y+b) - (y+b)] + cy + d \quad with\ \ a \neq 0$

Different possibilities for $f(y)$ are analyzed below to obtain corresponding symmetry algebras.

Case 2.1: $f(y) \neq a(y+b)^{-3} + cy + d$ and f is not quadratic.

There are 5 possibilities for f

$(\ i\ )\ f(y)\ is\ cubic$

$(\ ii\ )\ f(y) = ae^{by} + cy + d \quad where \quad c\ \neq 0$,1,2,3

$(\ iii\ )\ f(y) = a(y+b)^c + dy + e \quad where \quad c\ \neq 0$,1,2,3

$(\ iv\ )\ f(y) = a\ln(y+b) + dy + e$

$(\ v\ )\ f(y) = a[(y+b)\ln(y+b) - (y+b)] + cy + d$

From Eq. (2.19), we have

$$\xi_{xx} = 0.$$

Then, by equation E_1 and the previous relation, we conclude

$$\boxed{\xi = Ax + B} \tag{2.22}$$

Substituting this in Eq. (2.9), we conclude

$$\boxed{\phi = \frac{-2Af_y}{f_{yy}}} \tag{2.23}$$

which is a function of y only. Therefore $\phi_{yy} = 0$ implies that

$$\boxed{\phi = ny + m} \tag{2.24}$$

where n, m are constants.

Case* 2.1(*i*) *f*(*y*) *is cubic

Let

$$f(y) = a(y^3 + by^2 + cy + d)$$

with $a \neq 0$.

Substituting this in Eq. (2.9), we get

$$0 + 4a(3y^2 + 2by + c)\xi_x + 2\phi a(6y + 2b) = 0$$

Using equations (2.22), (2.24) in the previous equation, we get

$$2(3y^2 + 2by + c)A + (ny + m)(6y + 2b) = 0$$

Comparing the coefficients of powers of y, we get

Coefficient of y^2 : $n = -A$

Coefficient of y : $4Ab + 2bn + 6m = 0 \Longrightarrow 2Ab + 6m = 0 \Longrightarrow m = -\frac{bA}{3}$

Coefficient of y^0 : $2cA + 2mb = 0 \Longrightarrow c = -\frac{b}{A}m \Longrightarrow \boxed{c = \frac{b^2}{3}}$

Substituting in Eq. (2.24), we obtain

$$\phi = -Ay - \frac{bA}{3} \tag{2.25}$$

Using equations (2.22) and (2.25) in E_4, we get

$$(y^3 + by^2 + cy + d)(-A - 2A) - \left(-Ay - \frac{bA}{3}\right)(3y^2 + 2by + c) = 0$$

or

$$-9(y^3 + by^2 + cy + d) + (3y + b)(3y^2 + 2by + c) = 0.$$

Comparing coefficients of powers of y, we conclude

$$c = \frac{b^2}{3}$$

and

$$d = \frac{b^3}{27}.$$

Thus, in order to have larger symmetry algebra for cubic f,

$$f(y) = a\left(y^3 + by^2 + \frac{b^2}{3}y + \frac{b^3}{27}\right) = a(y + \frac{b}{3})^3$$

This means that larger symmetry algebra is possible in the case of cubic f only if the function $f(y)$ is a perfect cube i.e.

$$f(y) = a(y + C)^3.$$

The symmetry algebra in this case is two dimensional and is determined by

$$\xi = k_1 x + k_2$$

and

$$\phi = -k_1(y + C)$$

and is generated by

$$X_1 = \frac{\partial}{\partial x} \quad and \quad X_2 = x\frac{\partial}{\partial x} - (y + b)\frac{\partial}{\partial y}.$$

with the following commutation relation

	X_1	X_2
X_1	0	X_1
X_2	$-X_1$	0

***Case* 2.1(*ii*) $\boldsymbol{f(y) = ae^{by} + cy + d}$**

Substituting this function in Eq. (2.9), we get

$$4A[abe^{by} + c] + 2(ny + b)[ab^2 e^{by}] = 0$$

Comparing the coefficients of independent functions, we get the following

Constant term : $4Ac = 0 \Longrightarrow c = 0$ (2.26)

Coefficient of e^{by} : $4Aab + 2ab^2 m = 0 \Longrightarrow m = -\frac{2A}{b}$ (2.27)

Coefficient of ye^{by} : $2ab^2 n = 0 \Longrightarrow n = 0$ (2.28)

Substituting equations (2.27) and (2.28) in Eq. (2.24), we get

$$\phi = -\frac{2A}{b} \tag{2.29}$$

Substituting in Eq. E_4, we get

$$0 + 0 - 2[ae^{by} + d]A + \frac{2A}{b}[abe^{by}] = 0$$

This implies

$$d = 0 \tag{2.30}$$

Hence from equations (2.26) and (2.30), larger symmetry algebra is possible in this case only if f is of the form

$$f(y) = ae^{by}.$$

The symmetry algebra in this case is two dimensional and is given by

$$\xi = k_1 x + k_2$$

and

$\phi = -\frac{2k_1}{b}$.

This Lie symmetry algebra is generated by the following generators

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = x\frac{\partial}{\partial x} - \frac{2}{b}\frac{\partial}{\partial y}$$

and the commutator table is given by

	X_1	X_2
X_1	0	X_1
X_2	$-X_1$	0

***Case* 2.1(*iii*) $f(y) = a(y+b)^c + dy + e$ *where* $c \neq 0, 1, 2, 3, -3$**

Differentiating $f(y)$ with respect to y , we get

$$f_y = ac(y+b)^{c-1} + d$$

$$f_{yy} = ac(c-1)(y+b)^{c-2}$$

$$f_{yyy} = ac(c-1)(c-2)(y+b)^{c-3}$$

Substituting these in Eq. (2.10), we get

$$(2A+n)ac(c-1)(y+b)^{c-2} + (ny+m)ac(c-1)(c-2)(y+b)^{c-3} = 0$$

This implies

$$(2A+n)(y+b) + (ny+m)(c-2) = 0$$

Comparing coefficient of the powers of y, we get

Coefficient of y : $2A + n + n(c-2) = 0 \Longrightarrow \boxed{n = \frac{-2A}{c-1}}$ (2.31)

Coefficient of 1 : $2Ab + nb + m(c-2) = 0$ (2.32)

Substituting the value of n found in relation (2.31) in Eq. (2.32), we get

$$2Ab + \frac{-2A}{c-1}b + m(c-2) = 0$$

This implies

$$\boxed{m = -\frac{2Ab}{c-1}} \tag{2.33}$$

Substituting the values of n and m found in relations (2.31) and (2.33) in Eq. (2.24), we get

$$\boxed{\phi = \frac{-2A}{c-1}(y+b)} \tag{2.34}$$

Substituting Eq. (2.34) in E_4, we get

$$\left(\frac{-2A}{c-1} - 2A\right)[a(y+b)^c + dy + e] + \frac{2A}{c-1}(y+b)[ac(y+b)^{c-1} + d] = 0$$

This implies

$$\left(\frac{-2Ac}{c-1}\right)[a(y+b)^c + dy + e] + \frac{2A}{c-1}(y+b)[ac(y+b)^{c-1} + d] = 0$$

Simplifying, we get

$$\left[\frac{-2Acd}{c-1}y + \frac{-2Ace}{c-1}\right] + \left[\frac{2Ad}{c-1}(y+b)\right] = 0 \tag{2.35}$$

Comparing the coefficients of powers of y in Eq. (2.35), we get

Coefficient of y : $\frac{-2Acd}{c-1} + \frac{2Ad}{c-1} = 0 \Longrightarrow d = 0$ (2.36)

Coefficient of y^0 : $\frac{-2Ace}{c-1} + \frac{2Adb}{c-1} = 0 \Longrightarrow ce = 0 \Longrightarrow e = 0$ (2.37)

Thus, in order to have larger symmetry algebra in this case, f should be of the form

$$f(y) = a(y+b)^c$$

From equations (2.22) and (2.34), the symmetry algebra in this case is two dimensional and is given by

$$\xi = k_1 x + k_2$$

and

$$\phi = \frac{-2k_1}{c-1}(y+b)$$

The Lie symmetry algebra in this case is generated by the following generators

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = x\frac{\partial}{\partial x} - \frac{2}{c-1}(y+b)\frac{\partial}{\partial y}$$

and the commutation relations are given by

	X_1	X_2
X_1	0	X_1
X_2	$-X_1$	0

$\boldsymbol{Case\ 2.1(iv)\ f(y) = a\ln(y+b) + cy + d}$

Differentiating the function $f(y)$ with respect to y, we get the following derivatives

$$f_y = \frac{a}{y+b} + c$$

$$f_{yy} = \frac{-a}{(y+b)^2}$$

$$f_{yyy} = \frac{2a}{(y+b)^3}$$

Substituting these in Eq. (2.10), we get

$$(2A+n)\left[\frac{-a}{(y+b)^2}\right] + (ny+m)\left[\frac{2a}{(y+b)^3}\right] = 0$$

$$-(2A+n)(y+b) + 2(ny+m) = 0$$

Comparing coefficient of powers of y, we get

Coefficient of y : $-2A - n + 2n = 0 \Longrightarrow \boxed{n = 2A}$ (2.38)

Coefficient of y^0 : $-2Ab - nb + 2m = 0 \Longrightarrow -4Ab + 2m = 0 \Longrightarrow \boxed{m = 2Ab}$ (2.39)

Equations (2.38) and (2.39) imply

$$\boxed{\phi = 2A(y+b)}$$

Substituting these in equation E_4, we get

$$0 + (2A - 2A)[a\ln(y+b) + cy + d] - 2A(y+b)\left[\frac{a}{y+b} + c\right] = 0$$

This implies

$$-2A\,[a + c(y+b)] = 0$$

From the previous relation, we conclude

$$A = 0$$

This implies that

$$\xi_x = 0,$$

which leads to minimal symmetry algebra.

$\boldsymbol{Case\ 2.1(v)\ f(y) = a(y+b)\ln(y+b) + cy + d}$

Differentiating the function with respect to y, we get

$$f_y = a\ln(y+b) + a + c$$

$$f_{yy} = \frac{a}{(y+b)}$$

$$f_{yyy} = \frac{-a}{(y+b)^2}$$

Substituting these in Eq. (2.10), we get

$$(2A+n)\left[\frac{a}{(y+b)}\right] + (ny+m)\left[\frac{-a}{(y+b)^2}\right] = 0$$

Simplifying, we get

$$(2A+n)(y+b) - (ny+m) = 0$$

Comparing the coefficient of y, we conclude

$$A = 0$$

This implies that

$$\xi_x = 0,$$

which leads to minimal symmetry algebra.

$\boldsymbol{Case\ 2.2:\ f(y) = a(y+b)^{-3} + cy + d}$

In this case ξ_{xx} is not necessary equals 0. Then,

$$f_y = -3a(y+b)^{-4} + c$$

$$f_{yy} = 12a(y+b)^{-5}$$

$$f_{yyy} = -60a(y+b)^{-6}$$

Substituting these relations in Eq. (2.10), we get

$$\left(2\xi_x + \phi_y\right)[12a(y+b)^{-5}] - 60a(y+b)^{-6}\phi = 0 \tag{2.43}$$

Simplifying Eq. (2.43), we get

$$\left(2\xi_x + \phi_y\right)(y+b) - 5\phi = 0 \tag{2.44}$$

Differentiating Eq. (2.44) with respect to y, we get

$$\left(2\xi_x + \phi_y\right) - 5\phi_y = 0 \tag{2.45}$$

This implies

$$\boxed{\xi_x = 2\phi_y} \tag{2.46}$$

Using Eq. (2.46) in Eq. (2.44), we get

$$\left(2\xi_x + \frac{1}{2}\xi_x\right)(y+b) - 5\phi = 0 \tag{2.47}$$

Equation (2.47) implies

$$\boxed{\phi = \frac{1}{2}\xi_x(y+b)} \tag{2.48}$$

Substituting Eq. (2.48) in E_4, we get

$$\frac{1}{2}\xi_{xxx}(y+b) + \left(\frac{1}{2}\xi_x - 2\xi_x\right)[a(y+b)^{-3} + cy + d]$$

$$-\frac{1}{2}\xi_x(y+b)[-3a(y+b)^{-4} + c] = 0$$

Simplification of the previous equation leads to

$$\xi_{xxx}(y+b) - 3\xi_x[cy+d] - \xi_x(y+b)[c] = 0$$

Collecting similar terms, we get

$$(\xi_{xxx} - 4c\xi_x)y + b\xi_{xxx} - (3d+bc)\xi_x = 0 \tag{2.49}$$

Comparing coefficient of polynomial in y, we get the two equations

$$\xi_{xxx} - 4c\,\xi_x = 0 \tag{2.50}$$

and

$$b\xi_{xxx} - (3d+bc)\xi_x = 0 \tag{2.51}$$

Eliminating ξ_{xxx} from the system (2.50) and (2.51), we come up with the equation

$$(d-bc)\xi_x = 0 \tag{2.52}$$

This means that to get larger symmetry algebra than the minimal, we should have

$$d = bc \tag{2.53}$$

From Eq. (2.53), the function f should be of the form

$$f(y) = a(y+b)^{-3} + c(y+b) \tag{2.54}$$

Then, we have three different cases of the equation (2.50)

$\boldsymbol{Case\ 2.2(i):\ c=0}$

In this case, equation (2.50) becomes

$$\xi_{xxx} = 0$$

This implies

$$\xi = k_3x^2 + k_2x + k_1, \tag{2.55}$$

Then Eq. (2.48) yields

$$\phi = \tfrac{1}{2}\xi_x(y+b) = \tfrac{1}{2}(2k_3x + k_2)(y+b). \tag{2.56}$$

In this case, we get 3-dimensional symmetry algebra generated by

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = x\frac{\partial}{\partial x} + \frac{1}{2}(y+b)\frac{\partial}{\partial y}, \qquad X_3 = x^2\frac{\partial}{\partial x} + x(y+b)\frac{\partial}{\partial y}$$

with the commutator table

	X_1	X_2	X_3
X_1	0	X_1	$2X_2$
X_2	$-X_1$	0	X_3
X_3	$-2X_2$	$-X_3$	0

***Case* 2.2(*ii*) : $c > 0$**

Solving Eq. (2.50) in this case leads to

$$\xi = k_1 + k_2 e^{2x\sqrt{c}} + k_3 e^{-2x\sqrt{c}} \tag{2.57}$$

From Eq. (2.48), this implies

$$\phi = \frac{1}{2}\xi_x(y+b) = \left(k_2\sqrt{c}\, e^{2x\sqrt{c}} - k_3\sqrt{c}\, e^{-2x\sqrt{c}}\right)(y+b) \tag{2.58}$$

From Equations (2.57) and (2.58), we get 3-dimensional symmetry algebra generated by

$$X_1 = \frac{\partial}{\partial x},$$

$$X_2 = e^{2x\sqrt{c}}\frac{\partial}{\partial x} + \sqrt{c}e^{2x\sqrt{c}}(y+b)\frac{\partial}{\partial y},$$

$$X_3 = e^{-2x}\frac{\partial}{\partial x} - \sqrt{c}\, e^{-2x\sqrt{c}}(y+b)\frac{\partial}{\partial y}$$

with the commutation relations

	X_1	X_2	X_3
X_1	0	$2\sqrt{c}X_2$	$-2\sqrt{c}X_3$
X_2	$-2\sqrt{c}X_2$	0	$-4\sqrt{c}X_1$
X_3	$2\sqrt{c}X_3$	$4\sqrt{c}X_1$	0

For example, the symmetry algebra for the equation

$$y'' = y^{-3} + y$$

is generated by

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = e^{2x}\frac{\partial}{\partial x} + ye^{2x}\frac{\partial}{\partial y}, \qquad X_3 = e^{-2x}\frac{\partial}{\partial x} - ye^{-2x}\frac{\partial}{\partial y}$$

***Case* 2.2(*iii*) : $c < 0$**

Solving Eq. (2.50) in this case leads to

$$\xi = k_1 + k_2\sin(2\sqrt{-c}\, x) + k_3\cos(2\sqrt{-c}\, x) \tag{2.59}$$

From Eq. (2.48), this implies

$$\phi = \left[k_2\sqrt{-c}\cos(2\sqrt{-c}\,x) - k_3\sqrt{-c}\sin(2\sqrt{-c}\,x)\right](y+b) \tag{2.60}$$

From equations (2.59) and (2.60), we get 3-dimensional symmetry algebra generated by

$$X_1 = \frac{\partial}{\partial x},$$

$$X_2 = \sin\left(2\sqrt{-c}\,x\right)\frac{\partial}{\partial x} + \sqrt{-c}(y+b)\cos\left(2\sqrt{-c}\,x\right)\frac{\partial}{\partial y},$$

$$X_3 = \cos\left(2\sqrt{-c}\,x\right)\frac{\partial}{\partial x} - \sqrt{-c}(y+b)\sin(2\sqrt{-c}\,x)\frac{\partial}{\partial y}$$

and the commutation relations are

	X_1	X_2	X_3
X_1	0	$2\sqrt{-c}X_3$	$-2\sqrt{-c}X_2$
X_2	$-2\sqrt{-c}X_3$	0	$-2\sqrt{-c}X_1$
X_3	$2\sqrt{-c}X_2$	$2\sqrt{-c}X_1$	0

For example, the symmetry algebra for the equation

$$y'' = y^{-3} - y$$

is generated by

$$X_1 = \frac{\partial}{\partial x}\ ,\ X_2 = \sin(2x)\frac{\partial}{\partial x} + y\cos(2x)\frac{\partial}{\partial y},\ X_3 = \cos(2x)\frac{\partial}{\partial x} - y\sin(2x)\frac{\partial}{\partial y}$$

$\boldsymbol{Case\ 2.3\ \ f(y)\ is\ quadratic}$

Let

$$f(y) = a(y^2 + by + c)\ \ ,$$

with $a \neq 0$.

Substituting this function in Eq. (2.10), we get

$$2(2a)\xi_x + \phi_y(2a) + (0)\phi = 0$$

This implies

$$\phi_y = -2\xi_x \tag{2.61}$$

Substituting this in equation E_3, we get

$$2(-2\xi_{xx}) - \xi_{xx} = 0$$

We conclude

$$\boxed{\xi_{xx} = 0} \tag{2.62}$$

This returns us to the same conditions as in case 2.1 above.

Then, by equation E_1 and the previous relation, we conclude

$$\boxed{\xi = Ax + B} \tag{2.63}$$

Substituting this in Eq. (2.9), we conclude

$$\boxed{\phi = \frac{-2Af_y}{f_{yy}}}$$

Substituting the function in the previous relation, we get

$$\phi = \frac{-2Aa(2y+b)}{2a} = -A(2y+b) \tag{2.64}$$

Substituting equations (2.63) and (2.64) in E_4, we get

$$0 + (-2A)(y^2 + by + c) - 2A(y^2 + by + c) + A(2y+b)(2y+b) = 0$$

This implies

$$-4Ac + Ab^2 = 0$$

or

$$b^2 - 4c = 0 \tag{2.65}$$

This implies that

$$f(y) = a\left(y^2 + by + \frac{b^2}{4}\right) = a\left(y + \frac{b}{2}\right)^2$$

This means that in this case $f(y)$ should be a perfect square to have larger symmetry algebra. Similar to case 2.1(iii), the symmetry algebra is generated by

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = x\frac{\partial}{\partial x} - 2\left(y + \frac{b}{2}\right)\frac{\partial}{\partial y}.$$

with the commutator table

	X_1	X_2
X_1	0	X_1
X_2	$-X_1$	0

2.2. EXAMPLES OF REDUCTIONS AND INVARIANT SOLUTIONS OF $y'' = f(y)$

In this section we give illustrative examples of reducing equation (2.1) using the symmetries found in section 2 and invariant solutions of the equation. The examples are constructed using the standard procedure of finding and utilizing the invariants of the prolongation of the given symmetry X, in order to obtain reduction of order of the ODE (2.1) through the symmetry X. The reader is referred to [11, 13, 18] for the method of reduction of order of ODEs through differential invariants.

2.2.1. The *general case* $y'' = f(y)$.

This equation admits the symmetry $X = \frac{\partial}{\partial x}$.

The first prolongation of X is given by

$$X^{[1]} = \frac{\partial}{\partial x}$$

The characteristic system of

$$X^{[1]} I = 0$$

is given by

$$\frac{dx}{1} = \frac{dy}{0} = \frac{dy'}{0}$$

Solving the characteristic system gives two invariants of $X^{[1]}$ given by

$$u = y \tag{2.66}$$

$$v = y' \tag{2.67}$$

Using these invariant variables of $X^{[1]}$, we have the following

$$y'' = \frac{dy'}{dx} = \frac{dv}{dx} = \frac{du}{dx}\frac{dv}{du} = v\frac{dv}{du} \tag{2.68}$$

Substituting equations (2.66) and (2.68) in the given second order ODE reduces (2.1) to the first order ODE given by

$$v\frac{dv}{du} = f(u) \tag{2.69}$$

The ODE (2.69) is separable and can be solved easily to get

$$v = \pm\sqrt{2\int f(u)du + c}. \tag{2.70}$$

Returning to the original variables gives

$$\frac{dy}{dx} = \pm\sqrt{2\int f(y)dy + c} \tag{2.71}$$

which provides the general solution of Eq. (2.1) given by

$$\boxed{x = \pm \int \frac{dy}{\sqrt{2\int f(y)dy + c}} + k} \tag{2.72}$$

$2.2.2.$ $\boldsymbol{Reduction\ of\ y'' = y^3}$.

This equation

$$y'' = y^3 \tag{2.73}$$

admits Lie symmetry algebra generated by

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$$

$2.2.2.1.$ $\boldsymbol{Using\ X_1 = \frac{\partial}{\partial x}}$.

From Eq. (2.72), we have

$$x = \pm\sqrt{2}\int \frac{dy}{\sqrt{y^4 + c}} + k. \tag{2.74}$$

As an example, for c=0, we get the solution

$$x = \pm\frac{\sqrt{2}}{y} + k$$

This implies

$$y = \pm\frac{\sqrt{2}}{x-k}.$$

$2.2.2.2.$ $\boldsymbol{Using\ X_2 = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}}$.

The first prolongation of $X = X_2$ is given by

$$X_2{}^{[1]} = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} - 2y'\frac{\partial}{\partial y'}$$

The characteristic system of

$$X^{[1]}I = 0$$

is given by

$$\frac{dx}{x} = \frac{dy}{-y} = \frac{dy'}{-2y'}$$

Solving the characteristic system gives two invariants of $X^{[1]}$ given by

$$u = xy \tag{2.75}$$

$$v = x^2y' \tag{2.76}$$

Using these invariant variables of $X^{[1]}$, we have the following

$$\frac{du}{dx} = y + xy' = \frac{u}{x} + \frac{v}{x} \tag{2.77}$$

$$\frac{dv}{dx} = x^2y'' + 2xy' \tag{2.78}$$

Substituting Eq. (2.73) in Eq. (2.77), we get

$$\frac{dv}{dx} = x^2y^3 + 2xy' = \frac{u^3}{x} + 2\frac{v}{x} \tag{2.79}$$

Dividing Eq. (2.79) over Eq. (2.77), we get the reduced first order ODE

$$\frac{dv}{du} = \frac{u^3+2v}{u+v}, \tag{2.80}$$

which is an Abel equation of second type.

$2.2.3.$ $\boldsymbol{Reduction\ of\ y'' = e^{by}}$.

The equation

$$y'' = e^{by} \tag{2.81}$$

admits the symmetries

$$X_1 = \frac{\partial}{\partial x}\ , \qquad X_2 = x\frac{\partial}{\partial x} - \frac{2}{b}\frac{\partial}{\partial y}$$

$2.2.3.1.$ $\boldsymbol{Using\ X_1 = \frac{\partial}{\partial x}}$.

From Eq. (2.72), we have

$$x = \pm\int \frac{dy}{\sqrt{e^{by}+c}} + k. \tag{2.82}$$

Three exact solutions can be found in the following cases for the values of c .

For c=0, we get

$$x = \pm\int \frac{dy}{\sqrt{e^{by}}} + k = \pm\frac{2}{b\sqrt{e^{by}}} + k \tag{2.83}$$

This implies

$$\boxed{y = \frac{-2}{b}\ln\left|\frac{b(x-k)}{2}\right|} \tag{2.84}$$

For positive values of c, we get

$$x = \pm\int \frac{dy}{\sqrt{e^{by}+c}} + k = \pm\frac{2}{b\sqrt{c}}\operatorname{arctanh}\frac{\sqrt{e^{by}+c}}{\sqrt{c}} + k \tag{2.85}$$

This implies

$$\boxed{y = \frac{1}{b}\ln\left[c\,\mathrm{sech}^2\frac{b\sqrt{c}(x-k)}{2}\right]}. \tag{2.86}$$

For negative values of c, we get

$$x = \pm\int\frac{dy}{\sqrt{e^{by}+c}} + k = \pm\frac{2}{b\sqrt{-c}}\arctan\frac{\sqrt{e^{by}+c}}{\sqrt{-c}} + k, \tag{2.87}$$

which implies

$$\boxed{y = \frac{1}{b}\ln\left[-c\sec^2\frac{b\sqrt{-c}(x-k)}{2}\right]} \tag{2.88}$$

$2.2.3.2.$ $Using\ X_2 = x\frac{\partial}{\partial x} - \frac{2}{b}\frac{\partial}{\partial y}$.

The first prolongation of $X = X_2$ is given by

$$X_2{}^{[1]} = x\frac{\partial}{\partial x} - \frac{2}{b}\frac{\partial}{\partial y} - y'\frac{\partial}{\partial y'}$$

The characteristic system of

$$X^{[1]}I = 0$$

is given by

$$\frac{dx}{x} = -b\frac{dy}{2} = \frac{dy'}{-y'}$$

Solving the characteristic system gives two invariants of $X^{[1]}$ given by

$$u = x^2e^{by} \tag{2.89}$$

$$v = xy' \tag{2.90}$$

Using these invariant variables of $X^{[1]}$, we have the following

$$\frac{du}{dx} = 2xe^{by} + bx^2e^{by}y' = \frac{u}{x}(2+bv) \tag{2.91}$$

$$\frac{dv}{dx} = xy'' + y' \tag{2.92}$$

Substituting Eq. (2.81) in Eq. (2.92), we get

$$\frac{dv}{dx} = xe^{by} + y' = \frac{u}{x} + \frac{v}{x} \tag{2.93}$$

Dividing Eq. (2.93) over Eq. (2.91), we get the reduced first order ODE

$$\frac{dv}{du} = \frac{u+v}{u(2+bv)}, \tag{2.94}$$

which is an Abel equation of second type.

$2.2.4$ $\boldsymbol{Reduction\ of\ y'' = y^{-3}}$.

The ODE

$$y'' = y^{-3} \tag{2.95}$$

admits

$$X_1 = \frac{\partial}{\partial x} \ , \qquad X_2 = x\frac{\partial}{\partial x} + \frac{y}{2}\frac{\partial}{\partial y} \ , \qquad X_3 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}$$

$2.2.4.1.$ $\boldsymbol{Using\ X_1 = \frac{\partial}{\partial x}}$.

From Eq. (2.72), we have

$$x = \pm\int \frac{dy}{\sqrt{c-y^{-2}}} + k = \pm\frac{\sqrt{cy^2-1}}{c} + k \tag{2.96}$$

This implies

$$\boxed{y = \pm\sqrt{\frac{1+c^2(x-k)^2}{c}}}. \tag{2.97}$$

$2.2.4.2.$ $\boldsymbol{Using\ X_2 = x\frac{\partial}{\partial x} + \frac{y}{2}\frac{\partial}{\partial y}}$.

The first prolongation of $X = X_2$ is given by

$$X_2{}^{[1]} = x\frac{\partial}{\partial x} + \frac{y}{2}\frac{\partial}{\partial y} - \frac{y'}{2}\frac{\partial}{\partial y'}$$

The characteristic system of

$$X^{[1]}I = 0$$

is given by

$$\frac{dx}{x} = 2\frac{dy}{y} = -2\frac{dy'}{y'}$$

Solving the characteristic system gives two invariants of $X^{[1]}$ given by

$$u = \frac{\sqrt{x}}{y} \tag{2.98}$$

$$v = \sqrt{x}y' \tag{2.99}$$

Using these invariant variables of $X^{[1]}$, we have the following

$$\frac{du}{dx} = \frac{\frac{y}{2\sqrt{x}} - \sqrt{x}y'}{y^2} = \frac{1}{y^2}\left(\frac{1}{2u} - v\right) = \frac{1}{y^2}\left(\frac{1-2uv}{2u}\right) \tag{2.100}$$

$$\frac{dv}{dx} = \sqrt{x}y'' + \frac{y'}{2\sqrt{x}} \tag{2.101}$$

Substituting Eq. (2.95) in Eq. (2.101), we get

$$\frac{dv}{dx} = \sqrt{x}y^{-3} + \frac{y'}{2\sqrt{x}} = \frac{u}{y^2} + \frac{v}{2x} = \frac{u}{y^2} + \frac{v}{2u^2y^2} = \frac{1}{y^2}\left(\frac{2u^3+v}{2u^2}\right) \tag{2.102}$$

Dividing Eq. (2.102) over Eq. (2.100), we get the reduced first order ODE

$$\frac{dv}{du} = \frac{2u^3+v}{u-2vu^2}, \tag{2.103}$$

which is an Abel equation of second type.

$2.2.4.3.$ $Using\ X_3 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}$.

The first prolongation of $X = X_3$ is given by

$$X_3{}^{[1]} = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} + (y - xy')\frac{\partial}{\partial y'}$$

The characteristic system of

$$X^{[1]}I = 0$$

is given by

$$\frac{dx}{x^2} = \frac{dy}{xy} = \frac{dy'}{y-xy'}$$

Solving the equation

$$\frac{dx}{x^2} = \frac{dy}{xy}$$

gives the invariant variable

$$u = \frac{y}{x} \tag{2.104}$$

Solving the equation

$$\frac{dx}{x^2} = \frac{dy'}{y-xy'}$$

implies

$$\frac{dy'}{dx} = \frac{y-xy'}{x^2}$$

Using the first invariant variable =c in the previous equation gives

$$xy'' + y' = \frac{y}{x} = c \tag{2.105}$$

This implies

$$\frac{d(xy')}{dx} = \frac{y}{x} = c \tag{2.106}$$

Integrating, we get

$$xy' = \left(\frac{y}{x}\right)x + k = y + k$$

From the last equation, we get the second invariant variable

$$v = xy' - y \tag{2.107}$$

Using these invariant variables of $X^{[1]}$, we have the following

$$\frac{du}{dx} = \frac{xy'-y}{x^2} = \frac{v}{x^2} \tag{2.108}$$

$$\frac{dv}{dx} = y' + xy'' - y' = xy'' \tag{2.109}$$

Substituting Eq. (2.95) in Eq. (2.109), we get

$$\frac{dv}{dx} = xy^{-3} \tag{2.110}$$

From Eq. (2.104), we have

$$y = xu \tag{2.111}$$

Substituting Eq. (2.111) in Eq. (2.110), we get

$$\frac{dv}{dx} = \frac{1}{x^2u^3} \tag{2.112}$$

Dividing Eq. (2.112) over Eq. (2.108), we get the reduced first order ODE

$$\frac{dv}{du} = \frac{1}{vu^3}, \tag{2.113}$$

Equation (2.113) is separable and can be integrated easily to get the solution

$$v = \pm\sqrt{k - \frac{1}{u^2}} \tag{2.114}$$

Returning to the original variables, we get

$$xy' - y = \pm\sqrt{k - \frac{x^2}{y^2}} \tag{2.115}$$

Dividing by x^2, we get

$$\frac{xy'-y}{x^2} = \pm\frac{1}{x^2}\sqrt{k - \frac{x^2}{y^2}} \tag{2.116}$$

Let

$$q = \frac{y}{x} \tag{2.117}$$

Using Eq. (2.117) in Eq. (2.116), we get

$$\frac{dq}{dx} = \pm\frac{1}{x^2}\sqrt{k - \frac{1}{q^2}}$$

This ODE is separable and the solution is given by

$$q = \pm\sqrt{\frac{1+\left(\frac{k}{x}+c\right)^2}{k}} \tag{2.118}$$

Returning to the original variables using relation (2.117), we get the general solution

$$\boxed{y = \pm\sqrt{\frac{x^2+(k+cx)^2}{k}}} \tag{2.119}$$

which is the same as that found in Eq. (2.97).

2.2.5 *Reduction of* $y'' = y^{-3} + y$.

The equation

$$y'' = y^{-3} + y \tag{2.120}$$

admits three dimensional symmetry algebra generated by

$$X_1 = \frac{\partial}{\partial x}\,, \qquad X_2 = e^{2x}\frac{\partial}{\partial x} + e^{2x}y\frac{\partial}{\partial y}\,, \qquad X_3 = e^{-2x}\frac{\partial}{\partial x} - e^{-2x}y\frac{\partial}{\partial y}$$

2.2.5.1. *Using* $X_1 = \frac{\partial}{\partial x}$.

In front of Eq. (2.72), we get

$$x = \pm\int\frac{dy}{\sqrt{c-y^{-2}+y^2}} + k = \pm\frac{1}{2}\ln\left[\frac{c}{2} + y^2 + \sqrt{y^4 + cy^2 - 1}\right] + k,$$

which gives the solution

$$\boxed{y = \pm\frac{e^x}{2c_1}\sqrt{4 + c_1^4(e^{-2x} + c_2)^2}} = \pm\sqrt{\frac{4e^{2x}+k^2(1+ce^x)^2}{4k}} \tag{2.121}$$

2.2.5.2. *Using* $X_2 = e^{2x}\frac{\partial}{\partial x} + e^{2x}y\frac{\partial}{\partial y}$.

The 1st prolongation of $X = X_2$ can be found as follows

$$\eta^{[1]} = \eta_x + (\eta_y - \xi_x)y' - \xi_y y'^2 = 2e^{2x}y - e^{2x}y'$$

Then,

$$X^{[1]} = e^{2x}\frac{\partial}{\partial x} + e^{2x}y\frac{\partial}{\partial y} + [2e^{2x}y - e^{2x}y']\frac{\partial}{\partial y'}$$

We find invariants using $X^{[1]}F = 0$ through the corresponding characteristic system given by

$$\frac{dx}{e^{2x}} = \frac{dy}{ye^{2x}} = \frac{dy'}{2ye^{2x} - e^{2x}y'} \tag{2.122}$$

Solving

$$\frac{dx}{e^{2x}} = \frac{dy}{ye^{2x}}$$

gives the invariant variable

$$u = x - \ln \mathrm{y} \tag{2.123}$$

Using properties of fractions, the system (2.122) implies

$$\frac{dx}{1} = \frac{dy - d\,y'}{-\,y + y'} \tag{2.124}$$

Solving Eq. (2.124) gives the invariant variable

$$v = x + \ln(y - y') \tag{2.125}$$

From Eq. (2.125), one can write

$$y'' = \frac{dy'}{dx} = \frac{d(y - e^{v-x})}{dx} = y' - e^{v-x}(\frac{dv}{dx} - 1) \tag{2.126}$$

Substituting Eq. (2.126) in Eq. (2.120), we get

$$y' - e^{v-x}\left(\frac{dv}{dx} - 1\right) = y^{-3} + y \tag{2.127}$$

Using Eq. (2.125) in Eq. (2.127), we get

$$-e^{v-x}\left(\frac{dv}{dx} - 1\right) = y^{-3} + e^{v-x}$$

Simplifying the last equation, we get

$$-e^{v-x}\frac{dv}{dx} = y^{-3} \tag{2.128}$$

Using Eq. (1.123) in Eq. (1.128), we get

$$-e^{v-x}\frac{dv}{dx} = (e^{x-u})^{-3} = e^{3u-3x} \tag{2.129}$$

This implies

$$\frac{dv}{dx} = -e^{3u-v-2x} \tag{2.130}$$

Differentiating Eq. (2.123) with respect to x , we get

$$\frac{du}{dx} = 1 - \frac{y'}{y} = \frac{y-y'}{y} \tag{2.131}$$

Using equations (2.123) and (1.125) in Eq. (1.131) gives

$$\frac{du}{dx} = \frac{e^{v-x}}{e^{x-u}} = e^{v+u-2x} \tag{2.132}$$

Dividing Eq. (2.130) over Eq. (2.132), we get the first order separable ODE given by

$$\frac{dv}{du} = -e^{2u-2v} \tag{2.133}$$

Solving equation (2.133) leads to

$$e^{2v} = e^{2u} + k \tag{2.134}$$

Returning to the original variables, we get

$$e^{2x}(y - y')^2 = -\frac{e^{2x}}{y^2} + k$$

This implies

$$(y - y')^2 = \frac{-1}{y^2} + ke^{-2x} \tag{2.135}$$

Using Eq. (2.123), we get

$$y = e^{x-u} \tag{2.136}$$

Differentiating (2.136) with respect to x, we get

$$y' = (1 - u')e^{x-u} \tag{2.137}$$

Subtracting Eq. (2.137) from Eq. (2.136), we get

$$y - y' = u'e^{x-u} \tag{2.138}$$

Using equations (2.136) and (2.138) in equation (2.135), we get

$$u'^2 e^{2x-2u} = -e^{2u-2x} + ke^{-2x} = e^{-2x}(k - e^{2u}) \tag{2.139}$$

Equation (2.139) is separable and can be solved easily as follows

$$\frac{e^{-u}du}{\sqrt{k-e^{2u}}} = e^{-2x}dx$$

Integrating implies

$$-\frac{\sqrt{k-e^{2u}}}{ke^{u}} = -\frac{1}{2}e^{-2x} + c$$

Solving the last algebraic equation for e^{u}, we get

$$e^{u} = \pm\sqrt{\frac{4k}{4+k^2(e^{-2x}+c)^2}}$$

Returning to the variable y using Eq. (2.136), we get

$$\boxed{y = \pm e^{x}\sqrt{\frac{4+k^2(e^{-2x}+c)^2}{4k}}} \tag{2.140}$$

which is the same as the solution found in equation (2.121).

CHAPTER 3

GROUP CLASSIFICATION AND SYMMETRY REDUCTIONS FOR A CLASS OF NONLINEAR POISSON EQUATIONS ON PLANE

The aim of this chapter is to study the complete group classification problem and some symmetry reductions of the nonlinear Poisson equation on the plane which is given by

$$u_{xx} + u_{yy} = f(u) \tag{3.1}$$

where $f(u)$ is an arbitrary nonlinear function.

The group classification of equation (3.1) is carried out in section 3.1. It is based on necessary conditions on $f(u)$ which are obtained through a triangulation of determining equations of Lie symmetries of equation (3.1). An efficient method to obtain such triangulation is the well-known method of Mansfield [29] of generating differential Grobner bases of determining equations. A variant of Mansfield's method will be used to generate cases of $f(u)$ and hence for carrying out the group classification. Precisely, the following result will be proved.

Theorem 3.1

The minimal symmetry algebra of nonlinear Poisson equation (3.1) is 3-dimensional generated by

$$P_1 = \frac{\partial}{\partial x}, \qquad P_2 = \frac{\partial}{\partial y}, \qquad P_3 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$

and is obtained for all nonlinear arbitrary functions $f(u)$. The larger symmetry algebra exists in the cases given in the table below:

$f(y)$	Extra Generators of symmetry algebra
$a(u+b)^c$, $c \neq 0,1$	$P_4 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \frac{2}{c-1}(u+b)\frac{\partial}{\partial u}$
ae^{bu} , $a, b \neq 0$	Infinite dimensional algebra generated by $P_\infty = \xi(x,y)\frac{\partial}{\partial x} + \tau(x,y)\frac{\partial}{\partial y} - \frac{2}{b}\tau_y(x,y)\frac{\partial}{\partial u}$ where $\tau_{xx} + \tau_{yy} = 0,$ $\xi_x = \tau_y,$ $\xi_y = -\tau_x$

Table (3-1)

For different cases obtained in section 3.1, some examples of reductions and invariant solutions of equation (3.1) are presented in section 3.2. Reduction of the general case of equation (3.1) using symmetry method leads to a second order ODE. In some cases of functions, the reduced ODE obtained can be completely solved using classical methods or by symmetry method for ordinary differential equations as described in chapter II. Other cases are reduced to Abel equation of second type.

3.1. GROUP CLASSIFICATION OF NON-LINEAR POISSON EQUATIONS ON PLANE

In order to obtain the Lie symmetries of equation (3.1), let us consider the one parameter group of infinitesimal transformations

$$x^* = x + \epsilon\,\xi(x,y,u) + O(\epsilon^2) \tag{3.2}$$

$$y^* = y + \epsilon\,\tau(x,y,u) + O(\epsilon^2) \tag{3.3}$$

$$u^* = u + \epsilon\,\phi(x,y,u) + O(\epsilon^2) \tag{3.4}$$

where ϵ is the group parameter. The generator corresponding to the given Lie algebra is of the form

$$X = \xi(x,y,u)\frac{\partial}{\partial x} + \tau(x,y,u)\frac{\partial}{\partial y} + \phi(x,y,u)\frac{\partial}{\partial u} \tag{3.5}$$

If $X^{[2]}$ denotes the second prolongation of X then the invariance condition

$$X^{[2]}\left(u_{xx}+u_{yy}-f(u)\right)\Big|_{u_{xx}+u_{yy}=f(u)}=0 \tag{3.6}$$

is used to find the symmetries of Eq. (3.1). Here

$$X^{[2]}\left(u_{xx}+u_{yy}-f(u)\right)=\eta^{[xx]}+\eta^{[yy]}-\phi f_u \tag{3.7}$$

where

$$\begin{aligned}\eta^{[xx]}=&\ \phi_{xx}+(2\phi_{xu}-\xi_{xx})u_x+(\phi_{uu}-2\xi_{xu})u_x^2-\xi_{uu}u_x^3-\tau_{xx}u_y\\&-2\tau_{xu}u_xu_y-\tau_{uu}u_x^2u_y+(\phi_u-2\xi_x)u_{xx}-3\xi_uu_{xx}u_x\\&-\tau_uu_{xx}u_y-2\tau_xu_{xy}-2\tau_uu_xu_{xy}\end{aligned} \tag{3.8}$$

$$\begin{aligned}\eta^{[yy]}=&\ \phi_{yy}+\left(2\phi_{yu}-\tau_{yy}\right)u_y+\left(\phi_{uu}-2\tau_{yu}\right)u_y^2-\tau_{uu}u_y^3-\xi_{yy}u_x\\&-2\xi_{yu}u_xu_y-\xi_{uu}u_y^2u_x+\left(\phi_u-2\tau_y\right)u_{yy}-3\tau_uu_{yy}u_y\\&-\xi_uu_{yy}u_x-2\xi_xu_{xy}-2\xi_uu_yu_{xy}\end{aligned} \tag{3.9}$$

From the conditional equation (3.1), we have

$$u_{yy}=f(u)-u_{xx} \tag{3.10}$$

Substituting Eq. (3.10) in Eq. (3.9), we get

$$\begin{aligned}\eta^{[yy]}=&\ \phi_{yy}+\left(2\phi_{yu}-\tau_{yy}\right)u_y+\left(\phi_{uu}-2\tau_{yu}\right)u_y^2-\tau_{uu}u_y^3-\xi_{yy}u_x\\&-2\xi_{yu}u_xu_y-\xi_{uu}u_y^2u_x+\left(\phi_u-2\tau_y\right)(f-u_{xx})-3\tau_u(f-u_{xx})u_y\\&-\xi_u(f-u_{xx})u_x-2\xi_xu_{xy}-2\xi_uu_yu_{xy}\\=&\ \phi_{yy}+\phi_uf-2\tau_yf-\left(\xi_uf+\xi_{yy}\right)u_x+\left(2\phi_{yu}-\tau_{yy}-3\tau_uf\right)u_y\\&+\left(\phi_{uu}-2\tau_{yu}\right)u_y^2-\tau_{uu}u_y^3-2\xi_{yu}u_xu_y-\xi_{uu}u_y^2u_x+\left(-\phi_u+2\tau_y\right)u_{xx}\\&+3\tau_uu_{xx}u_y+\xi_uu_{xx}u_y-2\xi_xu_{xy}-2\xi_uu_yu_{xy}\end{aligned} \tag{3.11}$$

Using equations (3.11) and (3.8) in Eq. (3.9), we get

$$\begin{aligned}&X^{[2]}\left(u_{xx}+u_{yy}-f(u)\right)\Big|_{(3.1)}\\&=-u_x^3\xi_{uu}-u_x^2u_y\tau_{uu}+u_x^2(\phi_{uu}-2\xi_{xu})+u_x\left(2\phi_{xu}-\xi_{xx}-\xi_{yy}-f\xi_u\right)\\&\quad-2u_xu_y\left(\tau_{xu}+\xi_{yu}\right)-u_y^2u_x\xi_{uu}+u_{xx}u_x(-2\xi_u)+u_{xy}u_x(-2\tau_u)\\&\quad+2u_{xx}u_y(\tau_u)+u_y^2\left(\phi_{uu}-2\,\tau_{yu}\right)-u_y^3\tau_{uu}-2u_{xy}\left(\xi_y+\tau_x\right)\\&\quad+u_{xx}\left(2\tau_y-2\xi_x\right)+u_y\left(2\phi_{yu}-\tau_{xx}-\tau_{yy}-3f\tau_u\right)-2u_{xy}u_y(\xi_u)\\&\quad+\phi_{xx}+\phi_{yy}+f\phi_u-2f\tau_y-\phi f_u=0\end{aligned} \tag{3.12}$$

From Eq. (3.12), we get the following **determining equations**:

$$\boldsymbol{PE_1 \ : \ \xi_u = 0}$$
$$\boldsymbol{PE_2 \ : \ \tau_u = 0}$$
$$\boldsymbol{PE_3 \ : \ \xi_y + \tau_x = 0}$$
$$\boldsymbol{PE_4 \ : \ \tau_y - \xi_x = 0}$$
$$\boldsymbol{PE_5 \ : \ \phi_{uu} = 0}$$
$$\boldsymbol{PE_6 \ : \ 2\phi_{yu} - \tau_{xx} - \tau_{yy} = 0}$$
$$\boldsymbol{PE_7 \ : \ 2\phi_{xu} - \xi_{xx} - \xi_{yy} = 0}$$
$$\boldsymbol{PE_8 \ : \ \phi_{xx} + \phi_{yy} + f\phi_u - 2f\tau_y - \phi f_u = 0}$$

The system of determining equations will be solved utilizing a triangulation procedure based on techniques for obtaining differential Groebner basis developed by Mansfield in [29]. The operation $(PE_3)_y - (PE_4)_x$ gives

$$\xi_{xx} + \xi_{yy} = 0 \tag{3.13}$$

This reduces PE_7 to

$$\phi_{xu} = 0 \tag{3.14}$$

Also, the operation $(PE_3)_x + (PE_4)_y$ implies

$$\tau_{xx} + \tau_{yy} = 0 \tag{3.15}$$

This simplifies Eq. PE_6 to

$$\phi_{yu} = 0 \tag{3.16}$$

Differentiating Eq. PE_8 with respect to u, we get

$$f_u\phi_u - 2f_u\tau_y - \phi_u f_u - \phi f_{uu} = 0$$

which is simplified to

$$f_{uu}\phi + 2f_u\tau_y = 0 \tag{3.17}$$

Differentiating Eq. (3.17) with respect to u, we get

$$f_{uuu}\phi + f_{uu}\phi_u + 2f_{uu}\tau_y = 0 \tag{3.18}$$

Differentiating Eq. (3.18) with respect to u, we get

$$f_{uuuu}\phi + 2f_{uuu}\phi_u + 2f_{uuu}\tau_y = 0 \tag{3.19}$$

Simplifying Eq. (3.19) with respect to Eq. (3.18) via

$$f_{uu}(3.19) - 2f_{uuu}(3.18)$$

implies

$$\phi(f_{uu}f_{uuuu} - 2f_{uuu}^2) - 2f_{uu}f_{uuu}\tau_y = 0 \tag{3.20}$$

Equations (3.19) and (3.20) can be used to eliminate ϕ, giving the equation

$$(f_{uu}f_{uuu} - 2f_{uuu}^2)(3.19) - f_{uu}(3.20)$$

The previous operation implies

$$[f_u(f_{uu}f_{uuu} - 2f_{uuu}^2) + f_{uu}^2 f_{uuu}]\tau_y = 0$$

Simplifying, we get

$$\boxed{\boxed{[\, f_u f_{uu} f_{uuuu} - 2f_u f_{uuu}^2 + f_{uu}^2 f_{uuu}]\tau_y = 0}} \tag{3.21}$$

Differentiating Eq. (3.17) with respect to y, we get

$$f_{uu}\phi_y + 2f_u\tau_{yy} = 0 \tag{3.22}$$

Differentiating Eq. (3.22) with respect to u, we get

$$f_{uuu}\phi_y + 2f_{uu}\tau_{yy} = 0 \tag{3.23}$$

Eliminating ϕ_y using equations (3.22) and (3.23), leads to

$$f_{uu}(3.23) - f_{uuu}(3.22)$$

which implies

$$\boxed{\boxed{(f_{uu}^2 - f_u f_{uuu})\tau_{yy} = 0}} \tag{3.24}$$

Next, we look at possibilities of $f(u)$.

If $\boldsymbol{\tau_y = 0}$, then equations (3.21) and (3.24) are satisfied for any type of functions $f(u)$.

Equation (3.17) and the assumption

$$\tau_y = 0 \tag{3.25}$$

implies.

$$f_{uu}\phi = 0 \tag{3.26}$$

Nonlinearity of f in Eq. (3.26) implies

$$\boxed{\phi = 0}. \tag{3.27}$$

From Eq. (3.15) and the assumption $\tau_y = 0$, one can conclude

$$\tau_{xx} = 0 \tag{3.28}$$

From equations (3.28), PE_2 and the assumption $\tau_y = 0$, we conclude

$$\boxed{\tau = k_1 + k_2 x} \tag{3.29}$$

Also, from Equation PE_4 and the assumption $\tau_y = 0$, we conclude

$$\xi_x = 0 \tag{3.30}$$

From equations PE_1 and (3.30), we have

$$\xi = \xi(y) \tag{3.31}$$

Using Eq. (3.29) in PE_3, we get

$$\xi_y + k_2 = 0 \tag{3.32}$$

which, from Eq. (3.31), implies

$$\boxed{\xi = -k_2 y + k_3} \tag{3.33}$$

All what we deduced are without any restriction on $f(u)$. Hence, the minimal symmetry algebra given by equations (3.33), (3.29) and (3.27) is three dimensional which exists for any choice of $f(u)$ and is generated by

$$X_1 = \frac{\partial}{\partial x}\,, \qquad X_2 = \frac{\partial}{\partial y}\,, \qquad X_3 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$

with commutation relations given in the table below

	X_1	X_2	X_3
X_1	0	0	X_2
X_2	0	0	$-X_1$
X_3	$-X_2$	X_1	0

To look for functions $f(u)$ that may give larger symmetry algebra we assume $\tau_y \neq 0$ and solve the differential equations (3.21) and (3.24). Let us begin by solving the following two equations

$$f_{uu}^2 - f_u f_{uuu} = 0 \tag{3.34}$$

$$f_u f_{uu} f_{uuuu} - 2 f_u f_{uuu}^2 + f_{uu}^2 f_{uuu} = 0 \tag{3.35}$$

Solving Eq. (3.34) for a nonlinear function gives the solution:

- $f(u) = ae^{cu} + b$ (3.36)

Solving Eq. (3.35) for a nonlinear function gives 4 solutions:

- $f(u) = a(u^2 + bu + c) \quad with\ a \neq 0$ (3.37)
- $f(u) = ae^{bu} + c \quad where \quad a, b \neq 0$ (3.38)
- $f(u) = a(u+b)^c + d \quad where\ \ a \neq 0,\ \ c\ \neq 0\ ,1,2$ (3.39)
- $f(u) = a\ln(u+b) + d \quad with\ \ a \neq 0$ (3.40)

Different possibilities for $f(u)$ are analyzed below to obtain corresponding symmetry algebras. Since the only solution of Eq. (3.34) is given by (3.36), we conclude the following implication for nonlinear functions $f(u)$

$$f(u) \neq ae^{bu} + c \Rightarrow f_{uu}^2 - f_u f_{uuu} \neq 0 \tag{3.41}$$

The Symmetry Algebra for different forms of $f(u)$ are summarized in the following cases:

Case 3.1.1. $f(u) \neq ae^{bu} + c$.

From equations (3.24) and (3.41), if $f(u) \neq a\, e^{bu} + c$, then

$$f_{uu}^2 - f_u f_{uuu} \neq 0 \tag{3.42}$$

From equations (3.24) and (3.42), we get

$$\tau_{yy} = 0. \tag{3.43}$$

There are three possibilities of $f(u)$ in this case and are discussed below.

Case 3.1.1(i) $f(u)$ is quadratic

Let

$$f(u) = a(u^2 + bu + c) \tag{3.44}$$

with $a \neq 0$. Substituting the function in Eq. (3.17) gives

$$2a\phi + 2a(2u + b)\tau_y = 0$$

Simplifying the last equation, we get

$$\phi = -(2u + b)\tau_y \tag{3.45}$$

Using equations (3. 45), (3. 43) and (3.15), we get the following results

$$\phi_{xx} = -(2u + b)\tau_{yxx} = (2u + b)\tau_{yyy} = 0 \tag{3.46}$$

$$\phi_{yy} = -(2u + b)\tau_{yyy} = 0 \tag{3.47}$$

$$\phi_u = -\,2\tau_y \tag{3.48}$$

Substituting the equations (3.44)-(3.48) in equation PE_8, we get

$$(b^2 - \ 4\,c)\tau_y = 0. \tag{3.49}$$

Since we assume $\tau_y \neq 0$, we get

$$b^2 - 4c \ = 0 \tag{3.50}$$

which means that $f(u)$ is a perfect square. Now, using Eq. (3.43), we get in this case

$$\tau = k_1 x + k_2 + k_3 y + kxy \tag{3.51}$$

Substituting Eq. (3.51) in (3.48), we get

$$\phi_u = -2\tau_y = -k_3 - kx \tag{3.52}$$

Substituting Eq. (3.52) in Eq. (3.14) implies

$$k = 0 \tag{3.53}$$

From the relations (3.53) and (3.51), we get

$$\boxed{\tau = k_1 x + k_2 + k_3 y} \tag{3.54}$$

Substituting Eq. (3.54) in PE_3, we get

$$\xi_y = -\tau_x = -k_1$$

This implies

$$\xi = -k_1 y + g(x). \tag{3.55}$$

Substituting Eq. (3.54) in PE_4, we get

$$\xi_x = \tau_y = k_3 \tag{3.56}$$

Using Eq. (3.55) in (3.56), we get

$$g'(x) = k_3 \tag{3.57}$$

Integrating Eq. (3.57), we have

$$g(x) = k_3 x + k_4. \tag{3.58}$$

Substituting Eq. (3.58) in Eq. (3.55), we obtain

$$\boxed{\xi = -k_1 y + k_3 x + k_4} \tag{3.59}$$

From equations (3.45) and (3.51), we get

$$\boxed{\phi = -2\left(u + \frac{b}{2}\right) k_3} \tag{3.60}$$

Thus, in the case

$$f(u) = a(u+b)^2 \ , \tag{3.61}$$

we obtain four dimensional Lie algebra generated by the following symmetries

$$X_1 = \frac{\partial}{\partial x},$$

$$X_2 = \frac{\partial}{\partial y},$$

$$X_3 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y},$$

$$X_4 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - 2(u+b)\frac{\partial}{\partial u}$$

with the commutation relations

	X_1	X_2	X_3	X_4
X_1	0	0	X_2	X_1
X_2	0	0	$-X_1$	X_2
X_3	$-X_2$	X_1	0	0
X_4	$-X_1$	$-X_2$	0	0

Case 3.1.1(ii) $\boldsymbol{f(u) = a(u+b)^n + c, \quad with\ \ a \neq 0,\ n\ \neq 0\ ,1\ ,2}$

Substituting the function in Eq. (3.17), we get

$$an(n-1)(u+b)^{n-2}\phi + 2an(u+b)^{n-1}\tau_y = 0$$

This implies

$$\phi = -\frac{2(u+b)}{n-1}\tau_y \tag{3.62}$$

Substituting Eq. (3.62) in equation PE_8, we get

$$(a(u+b)^n + c)\left(\frac{-2}{n-1}\tau_y\right) + \frac{2(u+b)}{n-1}na(u+b)^{n-1}\tau_y - 2(a(u+b)^n + c)\tau_y = 0$$

Taking τ_y as a common factor, we get

$$\tau_y\left[\frac{-2(a(u+b)^n + c)}{n-1} + \frac{2na(u+b)^n}{n-1} - \frac{2(n-1)(a(u+b)^n + c)}{n-1}\right] = 0$$

Simplifying the previous equation, we get

$$\left[\frac{-2nc}{n-1}\right]\tau_y = 0 \tag{3.63}$$

So, if $c \neq 0$, then $\tau_y = 0$ which leads to minimal algebra. Hence, to get larger symmetry algebra, we should have $c = 0$ which means

$$f(u) = a(u+b)^n \tag{3.64}$$

Using equations (3.43), (3.62), (3.15) and (3.14), we conclude

$$\boxed{\tau = k_1 x + k_2 + k_3 y} \tag{3.65}$$

Substituting Eq. (3.65) in Eq. (3.62), we obtain

$$\boxed{\phi = -\frac{2k_3}{n-1}(u+b)} \tag{3.66}$$

Using Eq. (3.65) in PE_3, we have

$$\xi_y = -\tau_x = -k_1 \tag{3.67}$$

Integrating Eq. (3.67) and using equation PE_1, we get

$$\xi = -k_1 y + g(x) \tag{3.68}$$

But from equations (3.68), PE_4 and (3.65), we have

$$g'(x) = \xi_x = \tau_y = k_3 \tag{3.69}$$

Integrating Eq. (3.69), we get

$$g(x) = k_3 x + k_4 \tag{3.70}$$

Substituting Eq. (3.70) in Eq. (3.68), we conclude

$$\boxed{\xi = -k_1 y + k_3 x + k_4} \tag{3.71}$$

Thus, in this case we get four dimensional Lie algebra defined by equations (3.65), (3.66) and (3.71) and is generated by the following symmetries:

$$X_1 = \frac{\partial}{\partial x},$$

$$X_2 = \frac{\partial}{\partial y},$$

$$X_3 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y},$$

$$X_4 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \frac{2}{n-1}(u+b)\frac{\partial}{\partial u}$$

The commutation table of this Lie algebra is as follows

	X_1	X_2	X_3	X_4
X_1	0	0	X_2	X_1
X_2	0	0	$-X_1$	X_2
X_3	$-X_2$	X_1	0	0
X_4	$-X_1$	$-X_2$	0	0

Remark: If we substitute $n = 2$ in this algebra, we get the same generators found in case 3.1.1(i) above. Hence, we can summarize that this algebra is admitted for all nonlinear functions of the form $f(u) = a(u+b)^n$, $n \neq 0,1$.

Case 3.1.1(iii) $f(u) = a\ln(u+b) + c$ with $a \neq 0$

Substituting the function in Eq. (3.17), we get

$$\frac{-a}{(u+b)^2}\phi + 2\left(\frac{a}{u+b}\right)\tau_y = 0 \tag{3.72}$$

Simplifying Eq. (3.72), we get

$$\phi = 2(u+b)\tau_y \tag{3.73}$$

Substituting Eq. (3.73) in PE_8 and using equations (3.43) and (3.15), we have

$$0 + 0 + (a\ln(u+b) + c)(2\tau_y) - 2(u+b)\tau_y\left(\frac{a}{u+b}\right) - 2(a\ln(u+b) + c)\tau_y = 0$$

Simplifying, we get

$$-2a\tau_y = 0$$

which implies

$$\tau_y = 0 \tag{3.74}$$

This leads to minimal algebra.

Case 3.1.2 $f(u) = ae^{bu} + c, \quad a, b \neq 0$

Here τ_{yy} is not necessarily 0.

From Eq. (3.17), we have

$$ab^2e^{bu}\phi + 2abe^{bu}\tau_y = 0$$

Simplifying, we get

$$\phi = \frac{-2}{b}\tau_y \tag{3.75}$$

Substituting Eq. (3.75) in PE_8 and using Eq. (3.15) imply

$$\frac{2}{b}\tau_{yxx} + \frac{-2}{b}\tau_{yyy} + (ae^{bu} + c)(0) + \frac{2}{b}\tau_y(abe^{bu}) - 2(ae^{bu} + c)\tau_y = 0$$

Simplifying the last equation, we get

$$-2c\tau_y = 0. \tag{3.76}$$

So, if $c \neq 0$, then $\tau_y = 0$ which leads to minimal algebra. Thus, to have larger Lie symmetry algebras in this case, f should be of the form

$$f(u) = ae^{bu}. \tag{3.77}$$

Then, equation PE_2 implies

$$\tau = \tau(x, y) \tag{3.78}$$

The function τ should be harmonic according to Eq. (3.15). From Eq. (3.75), we have

$$\phi = \frac{-2}{b}\tau_y \tag{3.79}$$

Given a harmonic function $\tau(x, y)$, we can find ξ by solving equations PE_3 and PE_4 simultaneously and ϕ is found using Eq. (3.79). Thus the symmetry algebra is infinite dimensional, depending upon solutions of Eq. (3.15). Some examples of the symmetries of this infinite dimensional algebra for simpler form of $\tau(x, y)$ are summarized in the following table:

$\boldsymbol{\xi(x, y, u)}$	$\boldsymbol{\tau(x, y, u)}$	$\boldsymbol{\phi(x, y, u)}$
k_2	k_1	0
$-k_1 y + k_3$	$k_1 x + k_2$	0
$k_1 x + k_3$	$k_1 y + k_2$	$\frac{-2}{b} k_1$
$\frac{k_1}{2}(x^2 - y^2) + k_2$	$k_1 xy$	$\frac{-2}{b} k_1 x$
$-2k_1 xy + k_2$	$k_1(x^2 - y^2)$	$\frac{4}{b} k_1 y$
$k_1 e^{kx} \sin ky + k_2$	$k_1 e^{kx} \cos ky$	$\frac{2}{b} k_1 k e^{kx} \sin ky$
$k_1 e^{kx} \cos ky + k_2$	$k_1 e^{kx} \sin ky$	$\frac{-2}{b} k_1 k e^{kx} \cos ky$
$k_1 e^{-kx} \sin ky + k_2$	$k_1 e^{-kx} \cos ky$	$\frac{2}{b} k_1 k e^{-kx} \sin ky$
$-k_1 e^{-kx} \cos ky + k_2$	$k_1 e^{-kx} \sin ky$	$\frac{-2}{b} k_1 k e^{-kx} \cos ky$
$k_1 y^3 + 3k_2 xy^2 - 3k_1 x^2 y - k_2 x^3 + k_3$	$k_1 x^3 - 3k_2 x^2 y - 3k_1 xy^2 + k_2 y^3$	$\frac{6}{b}(k_2 x^2 + 2k_1 xy - k_2 y^2)$
$k_1(-4x^3 y + 4xy^3) + k_2$	$k_1(x^4 - 6x^2 y^2 + y^4)$	$\frac{8}{b} k_1(3x^2 y - y^3)$
$k_1(x^4 - 6x^2 y^2 + y^4) + k_2$	$k_1(4x^3 y - 4xy^3)$	$\frac{-8}{b} k_1(x^3 - 3xy^2)$
$ih(y - ix) - ig(y + ix) + k$	$h(y - ix) + g(y + ix)$	$\frac{-2}{b}[h(y - ix) + g(y + ix)]$

Table (3-2)

Different types of solutions $\tau(x, y)$ which generate the above table are discussed below.

$\boldsymbol{(I)\tau(x, y) = constant = k_1.}$

Let

$$\tau(x, y) = k_1$$

Then, Eq. (3.15)

$$\tau_{xx} + \tau_{yy} = 0$$

is satisfied.

Substituting this value of τ in equation PE_3, we get

$$\xi_y = 0.$$

This implies that

$$\xi = \xi(x).$$

Using this in equation PE_4, we obtain the ode given by

$$\xi_x = 0.$$

This ode implies that

$$\boxed{\xi = k_2}.$$

From equation (3.79), we conclude

$$\boxed{\phi = 0}.$$

$\boldsymbol{(II)\tau(x, y) = \tau(x)}$

Let us solve Eq. (3.15) which can be written as follows

$$\tau_{xx} + \tau_{yy} = 0$$

implies

$$\tau_{xx} = 0$$

Whose solution is

$$\tau(x) = k_1 x + k_2$$

Substituting this value of τ in equation PE_3, we get

$$\xi_y = -k_1.$$

This implies that

$$\xi = -k_1 y + g(x).$$

Using this in equation PE_4, we obtain the ode given by

$$g'(x) = 0.$$

or

$$g(x) = k_3$$

This ode implies that

$$\boxed{\xi = -k_1 y + k_3}.$$

From equation (3.79), we conclude

$$\boxed{\phi = 0}.$$

(III) $\boldsymbol{\tau(x,y) = \tau(y)}$

From Eq. (3.15), we get the following ode

$$\tau_{yy} = 0.$$

This implies that

$$\boxed{\tau(y) = k_1 y + k_2}$$

Substituting this τ in PE_3, we get

$$\xi_y = 0$$

or

$$\xi = \xi(x)$$

Substituting this in PE_4, we get

$$k_1 - \xi_x = 0$$

or

$$\boxed{\xi = k_1 x + k_3}$$

Substituting this in Eq. (3.79), we get

$$\boxed{\phi = \frac{-2}{b} k_1}$$

(IV) $\boldsymbol{\tau(x,y)}$ is separable

Separable solutions of Eq. (3.15): $\boxed{\boldsymbol{\tau_{xx} + \tau_{yy} = 0}}$

Let

$$\tau(x,y) = P(x)Q(y)$$

be a solution of Eq. (3.15).

Then,

$$P''(x)Q(y) + P(x)Q''(y) = 0$$

Dividing by $P(x)Q(y)$, we get

$$\frac{P''(x)}{P(x)} = -\frac{Q''(y)}{Q(y)} = constant$$

There are three cases for solving the previous system

$(i)\ constant = 0$.

In this case we conclude that

$$Q''(y) = 0$$

which implies that

$$\boxed{Q(y) = c_1 y + c_2}.$$

and

$$P''(x) = 0$$

This implies

$$P(x) = c_3 x + c_4$$

Thus,

$$\tau(x, y) = (c_3 x + c_4)(c_1 y + c_2)$$

The new kind is given by

$$\boxed{\tau(x, y) = k_1 xy}$$

Substituting τ in equation PE_3, we get

$$\xi_y + k_1 y = 0$$

This implies

$$\xi_y = -k_1 y$$

Integrating with respect to y, we get

$$\xi = -\frac{1}{2} k_1 y^2 + g(x)$$

Substituting in equation PE_4, we get

$$k_1 x - g'(x) = 0$$

Integrating, we get

$$g(x) = \frac{1}{2}k_1x^2 + k_2$$

This implies

$$\xi = -\frac{1}{2}k_1y^2 + \frac{1}{2}k_1x^2 + k_2$$

or simply,

$$\boxed{\xi = \frac{1}{2}k_1(x^2 - y^2) + k_2}$$

Substituting this in Eq. (3.79), we get

$$\boxed{\phi = \frac{-2}{b}k_1x}$$

(ii) Constant is positive, say constant $= k^2$

We are going to solve the system

$$\frac{P''(x)}{P(x)} = -\frac{Q''(y)}{Q(y)} = k^2$$

Solving

$$-\frac{Q''(y)}{Q(y)} = k^2$$

gives

$$\boxed{Q(y) = k_1 \sin ky + k_2 \cos ky}$$

Solving

$$\frac{P''(x)}{P(x)} = k^2,$$

implies

$$P(x) = k_3e^{kx} + k_4e^{-kx}$$

Thus,

$$\boxed{\tau(x, y) = (k_3e^{kx} + k_4e^{-kx})(k_1 \sin ky + k_2 \cos ky)}$$

Every value of k generates a solution, which leads to infinitely many solutions.

Substituting τ in equation PE_3, we get

$$\xi_y + k(k_3e^{kx} - k_4e^{-kx})(k_1 \sin ky + k_2 \cos ky) = 0$$

Integrating with respect to y, we get

$$\xi = -(k_3e^{kx} - k_4e^{-kx})(-k_1 \cos ky + k_2 \sin ky) + g(x)$$

Substituting in equation PE_4, we get

$$k(k_3e^{kx}+k_4e^{-kx})(k_1\cos ky-k_2\sin ky)$$
$$+k(k_3e^{kx}+k_4e^{-kx})(-k_1\cos ky+k_2\sin ky)-g'(x)=0$$

which implies

$$g'(x)=0$$

or

$$g(x)=k_5$$

Hence,

$$\boxed{\xi=-(k_3e^{kx}-k_4e^{-kx})(-k_1\cos ky+k_2\sin ky)+k_5}$$

Substituting this in Eq. (3.79), we get

$$\boxed{\phi=-\frac{2k}{b}(k_3e^{kx}+k_4e^{-kx})(k_1\cos ky-k_2\sin ky)}$$

(iii) Constant is negative, say constant $=-k^2$

We are going to solve the system

$$\frac{P''(x)}{P(x)}=-\frac{Q''(y)}{Q(y)}=-k^2$$

Solving

$$-\frac{Q''(y)}{Q(y)}=-k^2$$

gives

$$\boxed{Q(y)=k_1e^{ky}+k_2e^{-ky}}$$

And solving

$$\frac{P''(x)}{P(x)}=-k^2,$$

implies

$$\boxed{P(x)=k_3\sin kx+k_4\cos kx}$$

This gives us

$$\boxed{\tau(x,y)=(k_3\sin kx+k_4\cos kx)(k_1e^{ky}+k_2e^{-ky})}$$

Every value of k generates a solution, which leads to infinitely many solutions.

Substituting τ in equation PE_3, we get

$$\xi_y+k(k_3\cos kx-k_4\sin kx)(k_1e^{ky}+k_2e^{-ky})\ =0$$

Integrating with respect to y, we get

$$\xi = -(k_3 \cos kx - k_4 \sin kx)(k_1 e^{ky} - k_2 e^{-ky}) + g(x)$$

Substituting in equation PE_4, we get

$$k(k_3 \sin kx + k_4 \cos kx)(k_1 e^{ky} - k_2 e^{-ky})$$
$$+k(-k_3 \sin kx - k_4 \cos kx)(k_1 e^{ky} - k_2 e^{-ky}) - g'(x) = 0$$

which implies

$$g'(x) = 0$$

or

$$g(x) = k_5$$

Hence,

$$\boxed{\xi = -(k_3 \cos kx - k_4 \sin kx)(k_1 e^{ky} - k_2 e^{-ky}) + k_5}$$

Substituting this in Eq. (3.79), we get

$$\boxed{\phi = -\frac{2k}{b}(k_3 \sin kx + k_4 \cos kx)(k_1 e^{ky} - k_2 e^{-ky})}$$

This implies infinite dimensional symmetry algebra.

(V) $\boldsymbol{\tau(x,y)}$Polynomial solutions of $\boxed{\tau_{xx} + \tau_{yy} = 0}$

For the sake of simplicity, we consider here two types of polynomials.

$(i)\ \tau(x,y) is\ quadratic\ in\ x,y$

Solving Eq. (3.15) leads to

$$\tau(x,y) = c_1 x^2 - c_1 y^2 + c_2 xy + c_3 x + c_4 y + c_5$$

The cases

$$\tau(x,y) = c_2 xy,$$
$$\tau(x,y) = c_3 x,$$
$$\tau(x,y) = c_4 y,$$

and

$$\tau(x,y) = c_5$$

have already been discussed above. Here, we focus on the case

$$\boxed{\tau(x,y) = k_1 x^2 - k_1 y^2}$$

Substituting this value of τ in equation PE_3, we get

$$\xi_y = -2k_1 x.$$

This implies that

$$\xi = -2k_1 xy + g(x).$$

Using this in equation PE_4, we obtain the ode given by

$$g'(x) = 0.$$

or

$$g(x) = k_2$$

This implies that

$$\boxed{\xi = -2k_1 xy + k_2}.$$

From Eq. (3.79), we conclude

$$\boxed{\phi = \frac{4}{b} k_1 y}.$$

(ii) $\tau(x, y)$ is a polynomial of cubic terms in x, y.

Suppose that $\tau(x, y) = k_1 x^3 + k_2 y^3 + c_1 x^2 y + c_2 x y^2$.

Then,

$$\tau_{xx} = 6k_1 x + 2c_1 y$$

$$\tau_{yy} = 6k_2 y + 2c_2 x$$

Substituting in Eq. (3.15) gives

$$\tau_{xx} + \tau_{yy} = (6k_1 + 2c_2)x + (6k_2 + 2c_1)y = 0$$

This implies

$$c_1 = -3k_2$$

and

$$c_2 = -3k_1$$

So,

$$\boxed{\tau(x, y) = k_1 x^3 + k_2 y^3 - 3k_2 x^2 y - 3k_1 x y^2}$$

Substituting this value of τ in equation PE_3, we get

$$\xi_y = -3k_1 x^2 + 6k_2 xy + 3k_1 y^2.$$

This implies that

$$\xi = -3k_1 x^2 y + 3k_2 x y^2 + k_1 y^3 + g(x).$$

Using this in equation PE_4, we obtain the ode given by

$$[3k_2y^2 - 3k_2x^2 - 6k_1xy] - [-6k_1xy + 3k_2y^2 + g'(x)] = 0$$

$$g'(x) = -3k_2x^2.$$

or

$$g(x) = -k_2x^3 + k_3$$

This implies that

$$\boxed{\xi = -3k_1x^2y + 3k_2xy^2 + k_1y^3 - k_2x^3 + k_3}.$$

From Eq. (3.79), we conclude

$$\boxed{\phi = \frac{-2}{b}[3k_2y^2 - 3k_2x^2 - 6k_1xy]}.$$

(iii) $\tau(x, y)$ is a polynomial of quarto terms in x, y.

Suppose that $\tau(x, y) = k_1x^4 + k_2x^3y + c_1x^2y^2 + c_2xy^3 + c_3y^4$.

Then,

$$\tau_{xx} = 12k_1x^2 + 6k_2xy + 2c_1y^2$$

$$\tau_{yy} = 2c_1x^2 + 6c_2xy + 12c_3y^2$$

Substituting in Eq. (3.15) gives

$$\tau_{xx} + \tau_{yy} = (12k_1 + 2c_1)x^2 + 6(k_2 + c_2)xy + (2c_1 + 12c_3)y^2 = 0$$

This implies

$$c_1 = -6k_1$$

$$c_2 = -k_2$$

$$c_3 = k_1$$

So,

$$\boxed{\tau(x, y) = k_1x^4 + k_2x^3y - 6k_1x^2y^2 - k_2xy^3 + k_1y^4}$$

Substituting this value of τ in equation PE_3, we get

$$\xi_y = -4k_1x^3 - 3k_2x^2y + 12k_1xy^2 + k_2y^3.$$

This implies that

$$\xi = -4k_1x^3y - \frac{3}{2}k_2x^2y^2 + 4k_1xy^3 + \frac{1}{4}k_2y^4 + g(x).$$

Using this in equation PE_4, we obtain the ode given by

$$[k_2x^3 - 12k_1x^2y - 3k_2xy^2 + 4k_1y^3] - [-12k_1x^2y - 3k_2xy^2 + 4k_1y^3 + g'(x)] = 0$$

$$g'(x) = k_2x^3.$$

or

$$g(x) = \frac{1}{4}k_2x^4 + k_3$$

This implies that

$$\boxed{\xi = -4k_1x^3y - \frac{3}{2}k_2x^2y^2 + 4k_1xy^3 + \frac{1}{4}k_2y^4 + \frac{1}{4}k_2x^4 + k_3}.$$

From Eq. (3.79), we conclude

$$\boxed{\phi = \frac{-2}{b}[k_2x^3 - 12k_1x^2y - 3k_2xy^2 + 4k_1y^3]}.$$

3.2. EXAMPLES OF REDUCTIONS AND INVARIANT SOLUTIONS OF $\boldsymbol{u_{xx} + u_{yy} = f(u)}$

3.2.1. The general case $\boldsymbol{u_{xx} + u_{yy} = f(u)}$

This equation admits 3-dimesional Lie algebra generated by

$$X_1 = \frac{\partial}{\partial x}\,, \qquad X_2 = \frac{\partial}{\partial y}\,, \qquad X_3 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

	X_1	X_2	X_3
X_1	0	0	X_2
X_2	0	0	$-X_1$
X_3	$-X_2$	X_1	0

3.2.1.1. Using $\boldsymbol{X_1 = \frac{\partial}{\partial x}}$

$$X = \frac{\partial}{\partial x} + 0\frac{\partial}{\partial y} + 0\frac{\partial}{\partial u}$$

The characteristic system is given by

$$\frac{dx}{1} = \frac{dy}{0} = \frac{du}{0}$$

The equation

$$\frac{dx}{1} = \frac{dy}{0}$$

gives

$$y = c \tag{3.80}$$

The equation

$$\frac{dx}{1} = \frac{du}{0}$$

Implies

$$u = k \tag{3.81}$$

Equations (3.80) and (3.81) provide two invariants of X_1 given by

$$\xi(x, y) = y \tag{3.82}$$

$$V(\xi) = u \tag{3.83}$$

From the last two equations, we get the following derivatives

$$u_x = 0$$

$$u_{xx} = 0 \tag{3.84}$$

$$u_y = V'$$

$$u_{yy} = V'' \tag{3.85}$$

Substituting equations (3.84) and (3.85) in Eq. (3.1), we get

$$0 + V'' = f(V)$$

The PDE (3.1) is reduced to the second order ODE:

$$\boxed{V'' = f(V)} \tag{3.86}$$

Using formula (2.72) in solving Eq. (3.86), one writes

$$\xi = \pm \int \frac{dV}{\sqrt{2\int f(V)dV + c}} + k$$

Returning to the original variables, we get the solution

$$\boxed{y = \pm \int \frac{du}{\sqrt{2\int f(u)du + c}} + k} \tag{3.87}$$

3.2.1.2. Using $X_2 = \frac{\partial}{\partial y}$

The symmetry X_2 is given by

$$X = 0\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + 0\frac{\partial}{\partial u}$$

The characteristic system is given by

$$\frac{dx}{0} = \frac{dy}{1} = \frac{du}{0}$$

Solving

$$\frac{dx}{0} = \frac{dy}{1}$$

gives

$$x = c$$

And solving

$$\frac{dy}{1} = \frac{du}{0}$$

gives

$$u = k$$

So, we get two invariants of X_2 given by

$$\xi(x,y) = x \tag{3.88}$$

$$V(\xi) = u \tag{3.89}$$

Using these invariant variables, we get

$$u_x = V'$$

$$u_{xx} = V'' \tag{3.90}$$

$$u_y = 0$$

$$u_{yy} = 0 \tag{3.91}$$

Substituting equations (3.90) and (3.91) in the PDE (3.1) reduces it to the second order ODE given by

$$\boxed{V'' = f(V)} \tag{3.92}$$

Using the formula (2.72) and returning to the original variables, one gets

$$\boxed{x = \pm \int \frac{du}{\sqrt{2\int f(u)du + c}} + k} \tag{3.93}$$

3.2.1.3. Using $X_3 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$

We use the symmetry

$$X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + 0\frac{\partial}{\partial u}$$

to reduce Eq. (3.1).

The characteristic system of $X_3 F = 0$ is given by

$$\frac{dx}{-y} = \frac{dy}{x} = \frac{du}{0}$$

Solving

$$\frac{dx}{-y} = \frac{dy}{x}$$

implies

$$x^2 + y^2 = c \tag{3.94}$$

Solving

$$\frac{dy}{x} = \frac{du}{0}$$

implies

$$u = k \tag{3.95}$$

We have found two invariants of X_3 which are given by

$$\xi(x,y) = x^2 + y^2 \tag{3.96}$$

$$V(\xi) = u \tag{3.97}$$

Using these invariants, we find the following derivatives

$$u_x = 2xV'$$

$$u_{xx} = 4x^2V'' + 2V' \tag{3.98}$$

$$u_y = 2yV'$$

$$u_{yy} = 4y^2V'' + 2V' \tag{3.99}$$

Substituting equations (3.98) and (3.99) in Eq. (3.1) leads to the second order ODE

$$\boxed{4\xi V'' + 4V' = f(V)} \tag{3.100}$$

3.2.1.4. Using $aX_1 + bX_2 = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}$ (linear combination of X_1 and X_2)

$$X = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + 0\frac{\partial}{\partial u}$$

The characteristic system is given by

$$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{0}$$

Solving

$$\frac{dx}{a} = \frac{dy}{b}$$

gives

$$bx - ay = c$$

Solving

$$\frac{dy}{b} = \frac{du}{0}$$

gives

$$u = k$$

We have found two invariants of X given by

$$\xi(x,y) = bx - ay \tag{3.101}$$

and

$$V(\xi) = u \tag{3.102}$$

Using the variables given by equations (3.101) and (3.102), we get

$$u_x = bV'$$

$$u_{xx} = b^2V'' \tag{3.103}$$

$$u_y = -aV'$$

$$u_{yy} = a^2V'' \tag{3.104}$$

Substituting equations (3.103) and (3.104) reduces the PDE (3.1) to the second order ODE given by

$$b^2V'' + a^2V'' = f(V) \tag{3.105}$$

This implies

$$\boxed{V'' = \frac{1}{a^2+b^2} f(V)}$$

Using formula (2.72), one can write

$$\xi = \pm \int \frac{dV}{\sqrt{\frac{2}{a^2+b^2}\int f(V)dV + c}} + k$$

Returning to the original variables, we get

$$\boxed{bx - ay = \pm\sqrt{a^2+b^2} \int \frac{du}{\sqrt{2\int f(u)du + c}} + k} \tag{3.106}$$

This can be analyzed further for different types of functions as discussed in detail in chapter 2.

3.2.2. Reduction of $u_{xx} + u_{yy} = u^n$

This equation admits 4-dimensional symmetry algebra spanned by

$$X_1 = \frac{\partial}{\partial x},\ X_2 = \frac{\partial}{\partial y},\ X_3 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y},\ X_4 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \frac{2u}{n-1}\frac{\partial}{\partial u}$$

Commutator Table

	X_1	X_2	X_3	X_4
X_1	0	0	X_2	X_1
X_2	0	0	$-X_1$	X_2
X_3	$-X_2$	X_1	0	0
X_4	$-X_1$	$-X_2$	0	0

3.2.2.1. Using $X_1 = \frac{\partial}{\partial x}$

From the above Eq. (3.87), we have the following solution

$$\boxed{y = \pm \int \frac{du}{\sqrt{2\int u^n du + c}} + k}$$

This can analyzed for different values of n.

3.2.2.2. Using $X_2 = \frac{\partial}{\partial y}$

From formula (3.93), we get

$$x = \pm \int \frac{du}{\sqrt{2\int u^n du + c}} + k$$

For $n = -3$, we get the following solution as discussed in chapter 2

$$u = \pm\sqrt{\frac{1+c^2(x-k)^2}{c}}$$

For $n = 3$, we get the following solution as discussed in chapter 2

$$x = \pm\sqrt{2}\int \frac{du}{\sqrt{u^4+c}} + k$$

As an example, for c=0, we get the solution

$$u = \pm\frac{\sqrt{2}}{x-k}.$$

For $n = -1$, we get the following solution

$$x = \pm\frac{\sqrt{2}}{2}\int \frac{du}{\sqrt{\ln u + c}} + k.$$

3.2.2.3. Using $X_3 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$

Using Eq. (3.100) in the general case, we reduce the equation to the following ODE

$$\boxed{4\xi V'' + 4V' = V^n} \tag{3.107}$$

where

$$\xi = x^2 + y^2$$

and

$$V = u.$$

The ODE (3.107) admits the symmetry

$$X = -(n-1)\xi\frac{\partial}{\partial \xi} + V\frac{\partial}{\partial V}$$

The first prolongation of X is given by

$$X^{[1]} = -(n-1)\xi\frac{\partial}{\partial \xi} + V\frac{\partial}{\partial V} + nV'\frac{\partial}{\partial V'}$$

The corresponding characteristic system is

$$\frac{d\xi}{-(n-1)\xi} = \frac{dV}{V} = \frac{dV'}{nV'}$$

which gives the two invariants

$$r = \xi V^{n-1} \tag{3.108}$$

$$w = V'\xi^{\frac{n}{n-1}} \tag{3.109}$$

Differentiating Eq. (3.108) with respect to ξ, we get

$$\frac{dr}{d\xi} = V^{n-1} + (n-1)\xi V^{n-2}V' \tag{3.110}$$

Differentiating Eq. (3.109) with respect to ξ, we get

$$\frac{dw}{d\xi} = \frac{n}{n-1}\xi^{\frac{1}{n-1}}V' + \xi^{\frac{n}{n-1}}V'' \tag{3.111}$$

Using Eq. (3.107), we have

$$V'' = \frac{V^n}{4\xi} - \frac{V'}{\xi}$$

or

$$\xi^{\frac{n}{n-1}}V'' = \frac{1}{4}\xi^{\frac{1}{n-1}}V^n - \xi^{\frac{1}{n-1}}V' \tag{3.112}$$

Substituting this in Eq. (3.111), we get

$$\frac{dw}{d\xi} = \frac{n}{n-1}\xi^{\frac{1}{n-1}}V' + \frac{1}{4}\xi^{\frac{1}{n-1}}V^n - \xi^{\frac{1}{n-1}}V' = \frac{1}{n-1}\xi^{\frac{1}{n-1}}V' + \frac{1}{4}\xi^{\frac{1}{n-1}}V^n \tag{3.113}$$

Using equations (3.108) and (3.109) in Eq. (3.113), we get

$$\frac{dw}{d\xi} = \frac{1}{n-1}\xi^{\frac{1}{n-1}}\frac{w}{\xi^{\frac{n}{n-1}}} + \frac{1}{4}\xi^{\frac{1}{n-1}}\left(\frac{r}{\xi}\right)^{\frac{n}{n-1}}$$

Simplifying this gives

$$\frac{dw}{d\xi} = \frac{1}{\xi}\left[\frac{1}{n-1}w + \frac{1}{4}(r)^{\frac{n}{n-1}}\right] \tag{3.114}$$

Using equations (3.108) and (3.109) in Eq. (3.110), we get

$$\frac{dr}{d\xi} = \frac{r}{\xi} + (n-1)\xi\left(\frac{r}{\xi}\right)^{\frac{n-2}{n-1}}\frac{w}{\xi^{\frac{n}{n-1}}}$$

Simplifying this equation, we get

$$\frac{dr}{d\xi} = \frac{1}{\xi}\left[r + (n-1)w(r)^{\frac{n-2}{n-1}}\right] \tag{3.115}$$

Dividing Eq. (3.114) over Eq. (3.115), we get

$$\frac{dw}{dr} = \frac{\frac{1}{n-1}w + \frac{1}{4}(r)^{\frac{n}{n-1}}}{r + (n-1)w(r)^{\frac{n-2}{n-1}}} \tag{3.116}$$

which is an Abel equation of second type.

3.2.2.4. Using $X_4 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \frac{2u}{n-1}\frac{\partial}{\partial u}$

In this section, we are going to reduce the equation

$$u_{xx} + u_{yy} = u^n \tag{3.117}$$

using the symmetry

$$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \frac{2u}{n-1}\frac{\partial}{\partial u}$$

The characteristic system of $X(F) = 0$ is given by

$$\frac{dx}{x} = \frac{dy}{y} = -(n-1)\frac{du}{2u}$$

Solving

$$\frac{dx}{x} = \frac{dy}{y}$$

leads to the invariant variable of X given by

$$\xi(x, y) = \frac{y}{x} \tag{3.118}$$

Solving

$$\frac{dx}{x} = -(n-1)\frac{du}{2u}$$

leads to the invariant variable

$$V(\xi) = x^2 u^{n-1} \tag{3.119}$$

From (3.118), we get the following formulas

$$\xi_y = \frac{1}{x} = \frac{1}{y}\frac{y}{x} = \frac{1}{y}\xi \tag{3.120}$$

$$\xi_x = \frac{-y}{x^2} = \frac{-1}{x}\frac{y}{x} = \frac{-1}{x}\xi \tag{3.121}$$

$$\xi_y^2 + \xi_x^2 = \left(\frac{1}{y^2} + \frac{1}{x^2}\right)\xi^2 = \left(\frac{x^2+y^2}{x^2y^2}\right)\xi^2 = \frac{1+\xi^2}{y^2}\xi^2 = \frac{1+\xi^2}{x^2} \tag{3.122}$$

$$\xi_{yy} = \frac{-1}{y^2}\xi + \frac{1}{y}\xi_y = \frac{-1}{y^2}\xi + \frac{1}{y}\frac{1}{y}\xi = 0 \tag{3.123}$$

$$\xi_{xx} = \frac{1}{x^2}\xi - \frac{1}{x}\xi_x = \frac{1}{x^2}\xi - \frac{1}{x}\frac{-1}{x}\xi = \frac{2}{x^2}\xi \tag{3.124}$$

Differentiating Eq. (3.119): $x^2u^{n-1} = V(\xi)$ with respect to y, we get

$$(n-1)x^2u^{n-2}u_y = V'\xi_y \tag{3.125}$$

Differentiating with respect to y again, we get

$$(n-1)(n-2)x^2u^{n-3}u_y^2 + (n-1)x^2\, u^{n-2}u_{yy} = V''\xi_y^2 + V'\xi_{yy} \tag{3.126}$$

Also, differentiating Eq. (3.119) with respect to x, we get

$$(n-1)x^2u^{n-2}u_x + 2xu^{n-1} = V'\xi_x \tag{3.127}$$

Differentiating again with respect to x again, we get

$$(n-1)(n-2)x^2\, u^{n-3}u_x^2 + 2(n-1)x\, u^{n-2}u_x + (n-1)x^2\, u^{n-2}u_{xx}$$
$$+2u^{n-1} + 2x(n-1)u^{n-2}u_x = V''\xi_x^2 + V'\xi_{xx} \tag{3.128}$$

From Eq. (3.127),

$$(n-1)x^2u^{n-2}u_x = V'\xi_x - 2xu^{n-1} \tag{3.129}$$

This implies

$$x^2u_x = \frac{1}{(n-1)u^{n-2}}(V'\xi_x - 2xu^{n-1}) \tag{3.130}$$

Our equation $u_{xx} + u_{yy} = u^n$

Using equations (3.126) and (3.128) in Eq. (3.117) or

$$(n-1)u^{n-2}x^2u_{xx} + (n-1)u^{n-2}x^2u_{yy} = (n-1)u^{n-2}x^2u^n$$

gives

$$V''\xi_x^2 + V'\frac{2}{x^2}\xi - (n-1)(n-2)x^2\, u^{n-3}u_x^2 - 2(n-1)x\, u^{n-2}u_x$$
$$-2u^{n-1} - 2x(n-1)u^{n-2}u_x + V''\xi_y^2 + 0$$
$$-(n-1)(n-2)x^2u^{n-3}u_y^2 = (n-1)u^{n-2}x^2u^n$$

This implies

$$V''\left[\xi_x^2 + \xi_y^2\right] + \frac{2}{x^2}\xi V' - (n-1)(n-2)x^2u^{n-3}u_x^2 - 2(n-1)xu^{n-2}u_x$$
$$-2u^{n-1} - 2x(n-1)u^{n-2}u_x - (n-1)(n-2)x^2u^{n-3}u_y^2 = (n-1)u^{n-2}x^2u^n$$

From (3.122), this implies

$$V''\left[\frac{1+\xi^2}{x^2}\right] + \frac{2}{x^2}\xi V' - (n-1)(n-2)x^2u^{n-3}u_x^2 - 4(n-1)xu^{n-2}u_x$$
$$-2u^{n-1} - (n-1)(n-2)x^2u^{n-3}u_y^2 = (n-1)u^{n-2}x^2u^n$$

Multiplying by x^2, we get

$$V''[1+\xi^2] + 2\xi V' - (n-1)(n-2)x^2x^2u^{n-3}u_x^2 - 4(n-1)xx^2u^{n-2}u_x$$
$$-2x^2u^{n-1} - (n-1)(n-2)x^2x^2u^{n-3}u_y^2 = (n-1)u^{n-2}x^2x^2u^n$$

Using (3.129), (3.130), (3.125) and (3.127), we get

$$V''[1+\xi^2] + 2\xi V' - (n-1)(n-2)\left[\frac{V'\xi_x - 2xu^{n-1}}{(n-1)u^{n-2}}\right]^2 u^{n-3} - 4x[V'\xi_x - 2xu^{n-1}]$$
$$-2V - (n-1)(n-2)\left[\frac{V'\xi_y}{(n-1)u^{n-2}}\right]^2 u^{n-3} = (n-1)u^{2n-2}x^4$$

This implies

$$V''[1+\xi^2]+2\xi V'-\frac{(n-2)}{(n-1)u^{n-1}}\left[V'^2\xi_x^2-4xV'\xi_x u^{n-1}+4x^2u^{2n-2}\right]$$
$$-4x[V'\xi_x-2xu^{n-1}]-2V-\frac{(n-2)}{(n-1)u^{n-1}}V'^2\xi_y^2=(n-1)V^2$$

Simplifying, we get

$$V''[1+\xi^2]+2\xi V'-\frac{(n-2)V'^2}{(n-1)u^{n-1}}\left[\xi_x^2+\xi_y^2\right]+4x\frac{(n-2)}{(n-1)}V'\xi_x-4\frac{(n-2)}{(n-1)}x^2u^{n-1}$$
$$-4x[V'\xi_x-2xu^{n-1}]-2V=(n-1)V^2$$

This implies

$$V''[1+\xi^2]+2\xi V'-\frac{(n-2)V'^2}{(n-1)u^{n-1}}\left[\frac{1+\xi^2}{x^2}\right]+4x\frac{(n-2)}{(n-1)}V'\xi_x-4\frac{(n-2)}{(n-1)}V$$
$$-4xV'\xi_x+8x^2u^{n-1}-2V=(n-1)V^2$$

Simplifying, we get

$$V''[1+\xi^2]+2\xi V'-\frac{(n-2)V'^2}{(n-1)V}[1+\xi^2]+4x\frac{(n-2)}{(n-1)}V'\xi_x-4\frac{(n-2)}{(n-1)}V$$
$$-4xV'\xi_x+8V-2V=(n-1)V^2$$

Further simplification implies

$$V''[1+\xi^2]-\frac{(n-2)V'^2}{(n-1)V}[1+\xi^2]+2\frac{(n+1)}{(n-1)}V'\xi+2\frac{(n+1)}{(n-1)}V-(n-1)V^2=0 \tag{3.131}$$

This ODE admits the symmetry

$$X=[1+\xi^2]\frac{\partial}{\partial\xi}-2\xi V\frac{\partial}{\partial V}$$

The first prolongation of X is given by

$$X^{[1]}=[1+\xi^2]\frac{\partial}{\partial\xi}-2\xi V\frac{\partial}{\partial V}+(-2V-4\xi V')\frac{\partial}{\partial V'}$$

The characteristic system of the invariant equation

$$X^{[1]}F=0$$

is given by

$$\frac{d\xi}{1+\xi^2}=\frac{dV}{-2\xi V}=\frac{dV'}{-2V-4\xi V'}$$

Solving

$$\frac{d\xi}{1+\xi^2}=\frac{dV}{-2\xi V}$$

leads to

$$V(1+\xi^2)=c$$

which gives the invariant variable of $X^{[1]}$

$$r = V(1+\xi^2) \tag{3.132}$$

Eq. (3.132) implies

$$V = \frac{c}{(1+\xi^2)}$$

Differentiating with respect to ξ gives

$$V' = \frac{-2c\xi}{(1+\xi^2)^2} = \frac{-2V\xi}{(1+\xi^2)}$$

or

$$(1+\xi^2)\frac{V'}{V} + 2\xi = 0$$

which gives an invariant of $X^{[1]}$ given by

$$w = (1+\xi^2)\frac{V'}{V} + 2\xi \tag{3.133}$$

Next to write Eq. (3.131) in terms of r, w.

From Eq. (3.133), we have

$$Vw - 2\xi V = (1+\xi^2)V' \tag{3.134}$$

Differentiating Eq. (3.134) with respect to ξ, we get

$$(1+\xi^2)V' = Vw - 2\xi V$$

$$2\xi V' + (1+\xi^2)V'' = V'w + V\frac{dw}{d\xi} - 2V - 2\xi V' \tag{3.135}$$

Now,

$$\frac{dw}{d\xi} = \frac{dw}{dr}\frac{dr}{d\xi} = \frac{dw}{dr}(V'(1+\xi^2) + 2\xi V) \tag{3.136}$$

Using Eq. (3.134) in Eq. (3.136), we get

$$\frac{dw}{d\xi} = \frac{dw}{dr}\frac{dr}{d\xi} = \frac{dw}{dr}(Vw - 2\xi V + 2\xi V) = Vw\frac{dw}{dr}$$

Substituting this in Eq. (3.135), we get

$$(1+\xi^2)V'' = V^2 w\frac{dw}{dr} + V'w - 2V - 4\xi V' \tag{3.137}$$

Putting equations (3.123), (3.124), (3.125) and (3.128) in Eq. (3.131) implies

$$V^2 w\frac{dw}{dr} + V'w - 2V - 4\xi V' + 6\xi V'$$
$$-\frac{n-2}{n-1}[V'[w-2\xi] + 4\xi V' + 4V] + 6V = (n-1)V^2$$

or

$$V^2 w\frac{dw}{dr} + V'w + 4V + 2\xi V' - \frac{n-2}{n-1}[V'w + 2\xi V' + 4V] = (n-1)V^2$$

Dividing by V^2, we get

$$w\frac{dw}{dr}+\frac{V'}{V^2}w+\frac{4}{V}+2\xi\frac{V'}{V^2}-\frac{n-2}{n-1}\left[\frac{V'}{V^2}w+2\frac{\xi V'}{V^2}+\frac{4}{V}\right]=(n-1) \tag{3.138}$$

Equation (3.132) implies

$$1+\xi^2=\frac{r}{V} \tag{3.139}$$

Equation (3.133) and the previous equation imply

$$w=\frac{r}{V}\frac{V'}{V}+2\xi=r\frac{V'}{V^2}+2\xi$$

or

$$\frac{V'}{V^2}=\frac{w}{r}-2\frac{\xi}{r} \tag{3.140}$$

which implies

$$w\frac{V'}{V^2}+\frac{4}{V}+2\xi\frac{V'}{V^2}=\frac{w^2}{r}-2\frac{\xi w}{r}+\frac{4}{V}+2\xi\frac{w}{r}-4\frac{\xi^2}{r} \tag{3.141}$$

From Eq. (3.139), we have

$$\xi^2=\frac{r}{V}-1$$

So, substituting this in Eq. (3.141)

$$w\frac{V'}{V^2}+\frac{4}{V}+2\xi\frac{V'}{V^2}=\frac{w^2}{r}+\frac{4}{V}-4\frac{\frac{r}{V}-1}{r}$$

or

$$w\frac{V'}{V^2}+\frac{4}{V}+2\xi\frac{V'}{V^2}=\frac{w^2}{r}+\frac{4}{r} \tag{3.142}$$

Using Eq. (3.142) in Eq. (3.138), we get

$$w\frac{dw}{dr}+\frac{w^2}{r}+\frac{4}{r}-\frac{n-2}{n-1}\left[\frac{w^2}{r}+\frac{4}{r}\right]=(n-1)$$

which is a first order ODE. This can be simplified to

$$w\frac{dw}{dr}+\frac{1}{n-1}\left[\frac{w^2}{r}+\frac{4}{r}\right]=(n-1)$$

or

$$\frac{dw}{dr}+\frac{1}{(n-1)r}w=\left[(n-1)-\frac{4}{(n-1)r}\right]w^{-1} \tag{3.143}$$

which is Bernoulli equation, with solution given by the following standard procedures

Let

$$U=w^2 \tag{3.144}$$

Then,

$$\frac{dU}{dr} = 2w\frac{dw}{dr} = 2w\left(\left[(n-1) - \frac{4}{(n-1)r}\right]w^{-1} - \frac{1}{(n-1)r}w\right)$$

$$= \left[2(n-1) - \frac{8}{(n-1)r}\right] - \frac{2}{(n-1)r}w^2$$

Then, using Eq. (3.144), the equation becomes

$$\frac{dU}{dr} + \frac{2}{(n-1)r}U = 2(n-1) - \frac{8}{(n-1)r} \text{ (Linear first order ODE)}$$

Integrating factor is given by

$$e^{\int \frac{2}{(n-1)r}dr} = (r)^{\frac{2}{n-1}}$$

Multiplying the linear ODE by this integrating factor gives

$$(r)^{\frac{2}{n-1}}\frac{dU}{dr} + \frac{2}{(n-1)}(r)^{\frac{2}{n-1}-1}U = 2(n-1)(r)^{\frac{2}{n-1}} - \frac{8}{(n-1)}(r)^{\frac{3-n}{n-1}} \tag{3.145}$$

There are two cases in this equation.

Case 1 ($\boldsymbol{n \neq -1}$)

Integrating Eq. (3.145) gives

$$U(r)^{\frac{2}{n-1}} = \int\left[2(n-1)(r)^{\frac{2}{n-1}} - \frac{8}{(n-1)}(r)^{\frac{3-n}{n-1}}\right]dr = \frac{2(n-1)^2}{n+1}(r)^{\frac{n+1}{n-1}} - 4(r)^{\frac{2}{n-1}} + c$$

This implies

$$U = \frac{2(n-1)^2}{n+1}r - 4 + c(r)^{\frac{-2}{n-1}} \tag{3.146}$$

Returning to w using Eq. (3.144), we get

$$w^2 = \frac{2(n-1)^2}{n+1}r - 4 + c(r)^{\frac{-2}{n-1}} \tag{3.147}$$

Returning to the variables ξ, V using equations (3.132) and (3.133), we get

$$\left[(1+\xi^2)\frac{V'}{V} + 2\xi\right]^2 = \frac{2(n-1)^2}{n+1}V(1+\xi^2) - 4 + c(V(1+\xi^2))^{\frac{-2}{n-1}}$$

This implies

$$\left(\frac{dr}{d\xi}/V\right)^2 = \frac{2(n-1)^2}{n+1}r - 4 + c(r)^{\frac{-2}{n-1}}$$

or

$$\left(\frac{dr}{d\xi}\right)^2 = \left[\frac{2(n-1)^2}{n+1}r - 4 + c(r)^{\frac{-2}{n-1}}\right]\frac{r^2}{(1+\xi^2)^2}$$

or

$$\boxed{\frac{dr}{d\xi} = \pm\frac{r}{(1+\xi^2)}\sqrt{\frac{2(n-1)^2}{n+1}r - 4 + c(r)^{\frac{-2}{n-1}}}} \tag{3.148}$$

Case 2 ($\boldsymbol{n = -1}$)

In this case, Eq. (3.145) becomes

$$(r)^{-1}\frac{dU}{dr} - (r)^{-2}U = -4(r)^{-1} + 4(r)^{-2} \tag{3.149}$$

Integrating gives

$$U(r)^{-1} = \int[-4(r)^{-1} + 4(r)^{-2}]\,dr = -4\ln r - \frac{4}{r} + c$$

Multiplying by r, we get

$$\boxed{U = -4r\ln r - 4 + cr}$$

Returning to w using Eq. (3.144), we get

$$w^2 = -4r\ln r - 4 + cr \tag{3.150}$$

Returning to the variables ξ, V using equations (3.132) and (3.133), we get

$$\left[(1+\xi^2)\frac{V'}{V} + 2\xi\right]^2 = -4V(1+\xi^2)\ln[V(1+\xi^2)] - 4 + cV(1+\xi^2)$$

This implies

$$\left(\frac{dr}{d\xi}/V\right)^2 = -4r\ln r - 4 + cr$$

or

$$\left(\frac{dr}{d\xi}\right)^2 = [-4r\ln r - 4 + cr]\frac{r^2}{(1+\xi^2)^2}$$

or

$$\frac{dr}{d\xi} = \pm\sqrt{-4r\ln r - 4 + cr}\,\frac{r}{1+\xi^2}$$

which is separable and can be written as

$$\frac{dr}{r\sqrt{-4r\ln r - 4 + cr}} = \pm\frac{d\xi}{1+\xi^2}$$

Integrating gives

$$\boxed{\int\frac{dr}{r\sqrt{-4r\ln r - 4 + cr}} = \pm\arctan\xi + k}$$

3.2.3. Reduction of $\boldsymbol{u_{xx} + u_{yy} = e^{2u}}$

This equation admits infinite dimensional Lie algebra as discussed in section 3.1.

We are going to use the following symmetries

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial y}, \qquad X_3 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$

and some symmetries of the form

$$X = \xi(x,y,u)\frac{\partial}{\partial x} + \tau(x,y,u)\frac{\partial}{\partial y} + \phi(x,y,u)\frac{\partial}{\partial u}$$

with selected ξ, τ and ϕ from table (3-2) presented in section 3.1.2.

3.2.3.1. Using $X_1 = \frac{\partial}{\partial x}$

From the general case, we have the following solution

$$y = \pm\int \frac{du}{\sqrt{2\int e^{2u}du + c}} + k = \pm\int \frac{du}{\sqrt{e^{2u}+c}} + k$$

For $c = 0$, we get

$$y = \pm\int e^{-u}\,du + k = \pm e^{-u} + k.$$

This implies that

$$\boxed{u = -\ln|y-k|}.$$

For positive values of c, we get

$$y = \pm\int \frac{du}{\sqrt{e^{2u}+c}} + k = \pm\frac{1}{\sqrt{c}}\operatorname{arctanh}\frac{\sqrt{e^{2u}+c}}{\sqrt{c}} + k$$

or

$$\boxed{u = \frac{1}{2}\ln\left[c\operatorname{sech}^2\sqrt{c}(y-k)\right]}.$$

For negative values of c, we get

$$y = \pm\int \frac{du}{\sqrt{e^{2u}+c}} + k = \pm\frac{1}{\sqrt{-c}}\arctan\frac{\sqrt{e^{2u}+c}}{\sqrt{-c}} + k$$

or

$$\boxed{u = \frac{1}{2}\ln\left[-c\sec^2\sqrt{-c}(y-k)\right]}$$

3.2.3.2. Using $X_2 = \frac{\partial}{\partial y}$

From the general case, we have the following solution

$$x = \pm\int \frac{du}{\sqrt{2\int e^{2u}du + c}} + k = \pm\int \frac{du}{\sqrt{e^{2u}+c}} + k$$

For c=0, we get

$$x = \pm\int e^{-u}\,du + k = \pm e^{-u} + k.$$

This implies that

$$\boxed{u = -\ln|x-k|}.$$

For positive values of c, we get

$$x = \pm \int \frac{du}{\sqrt{e^{2u}+c}} + k = \pm \frac{1}{\sqrt{c}} \operatorname{arctanh} \frac{\sqrt{e^{2u}+c}}{\sqrt{c}} + k$$

or

$$\boxed{u = \frac{1}{2} \ln\left[c \operatorname{sech}^2 \sqrt{c}(x-k)\right]}.$$

For negative values of c, we get

$$x = \pm \int \frac{du}{\sqrt{e^{2u}+c}} + k = \pm \frac{1}{\sqrt{-c}} \arctan \frac{\sqrt{e^{2u}+c}}{\sqrt{-c}} + k$$

or

$$\boxed{u = \frac{1}{2} \ln\left[-c \sec^2 \sqrt{-c}(x-k)\right]}$$

3.2.3.3. Using $X_3 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$

Using the general case, we reduce the equation to the following ODE

$$\boxed{4\xi V'' + 4V' = e^{2V}} \tag{3.151}$$

Where

$$\xi = x^2 + y^2 \tag{3.152}$$

and

$$V = u. \tag{3.153}$$

The ODE (3.151) admits the symmetry

$$X = -2\xi \frac{\partial}{\partial \xi} + \frac{\partial}{\partial V}$$

The first prolongation of X is given by

$$X^{[1]} = -2\xi \frac{\partial}{\partial \xi} + \frac{\partial}{\partial V} + 2V' \frac{\partial}{\partial V'}$$

The characteristic system of the invariant equation

$$X^{[1]}F = 0$$

is given by

$$\frac{d\xi}{-2\xi} = \frac{dV}{1} = \frac{dV'}{2V'}$$

Solving

$$\frac{d\xi}{-2\xi} = \frac{dV}{1}$$

leads to

$$2V + \ln(\xi) = c$$

which gives the invariant variable of $X^{[1]}$

$$r = 2V + \ln(\xi) \tag{3.154}$$

Solving

$$\frac{d\xi}{-2\xi} = \frac{dV'}{2V'}$$

leads to

$$\xi V' = c$$

which gives the invariant variable of $X^{[1]}$

$$w = \xi V' \tag{3.155}$$

Differentiating Eq. (3.154) with respect to ξ, we get

$$\frac{dr}{d\xi} = 2V' + \frac{1}{\xi} = \frac{2w}{\xi} + \frac{1}{\xi} = \frac{2w+1}{\xi} \tag{3.156}$$

Differentiating Eq. (3.155) with respect to ξ, we get

$$\frac{dw}{d\xi} = \xi V'' + V'$$

Using the ODE (3.151) in the previous equation, we get

$$\frac{dw}{d\xi} = \frac{1}{4} e^{2V}$$

Using the ODE (3.154) in the previous equation, we get

$$\frac{dw}{d\xi} = \frac{e^r}{4\xi} \tag{3.157}$$

Dividing Eq. (3.157) over Eq. (3.156) gives

$$\frac{dw}{dr} = \frac{e^r}{4(2w+1)} \tag{3.158}$$

This ODE is separable and its solution is given by

$$4(w^2 + w) = e^r + c$$

or

$$(2w + 1)^2 = e^r + c$$

From Eq. (3.156) and the previous equation, we get

$$\xi^2 \left(\frac{dr}{d\xi}\right)^2 = e^r + c$$

which implies

$$\frac{dr}{\sqrt{e^r + c}} = \pm \frac{d\xi}{\xi} \tag{3.159}$$

There are three different cases for the values of c.

Case 1 $c = 0$

Integrating Eq. (3.159) in this case gives

$$-\frac{2}{\sqrt{e^r}} = \pm \ln \xi + k$$

Returning to the variables ξ, V using Eq. (3.155) gives

$$-\frac{2}{\sqrt{\xi e^{2V}}} = \pm \ln \xi + k$$

Returning to the original variables using equations (3.152) and (3,153) gives

$$-\frac{1}{\sqrt{(x^2+y^2)e^{2u}}} = \pm \ln(x^2 + y^2) + k$$

Solving this equation for u, we get the exact solution

$$\boxed{u = -\frac{1}{2}\ln(x^2 + y^2) - \ln|\ln(x^2 + y^2) + k|} \tag{3.160}$$

Case 2 $c > 0$

Integrating Eq. (3.159) in this case gives

$$-\frac{2}{\sqrt{c}} \operatorname{arctanh}\left(\frac{\sqrt{e^r+c}}{\sqrt{c}}\right) = \pm \ln \xi + k$$

Returning to the variables ξ, V using Eq. (3.184) gives

$$-\frac{2}{\sqrt{c}} \operatorname{arctanh}\left(\frac{\sqrt{\xi e^{2V}+c}}{\sqrt{c}}\right) = \pm \ln \xi + k$$

Returning to the original variables using equations (3.152) and (3,153) gives

$$-\frac{2}{\sqrt{c}} \operatorname{arctanh}\left(\frac{\sqrt{(x^2 + y^2)e^{2u} + c}}{\sqrt{c}}\right) = \pm \ln(x^2 + y^2) + k$$

or

$$(x^2 + y^2)e^{2u} = c\left[\tanh^2\left(\frac{\sqrt{c}}{2}\ln(x^2 + y^2) + k\right) - 1\right]$$

which implies the exact solution

$$\boxed{u = -\frac{1}{2}\ln(x^2 + y^2) + \ln\left|\sqrt{c}\operatorname{sech}\left(\frac{\sqrt{c}}{2}\ln(x^2 + y^2) + k\right)\right|} \tag{3.161}$$

Case 3 $c < 0$

Integrating Eq. (3.159) in this case gives

$$\frac{2}{\sqrt{-c}}\arctan\left(\frac{\sqrt{e^r+c}}{\sqrt{-c}}\right) = \pm\ln\xi + k$$

Returning to the variables ξ, V using Eq. (3.155) gives

$$-\frac{2}{\sqrt{-c}}\arctan\left(\frac{\sqrt{\xi e^{2V}+c}}{\sqrt{-c}}\right) = \pm\ln\xi + k$$

Returning to the original variables using equations (3.152) and (3,153) gives

$$-\frac{2}{\sqrt{-c}}\arctan\left(\frac{\sqrt{(x^2+y^2)e^{2u}+c}}{\sqrt{-c}}\right) = \pm\ln(x^2+y^2) + k$$

or

$$(x^2+y^2)e^{2u} = -c\left[\tan^2\left(\frac{\sqrt{-c}}{2}\ln(x^2+y^2)+k\right)+1\right]$$

which implies the exact solution

$$\boxed{u = -\frac{1}{2}\ln(x^2+y^2) + \ln\left|\sqrt{-c}\sec\left(\frac{\sqrt{-c}}{2}\ln(x^2+y^2)+k\right)\right|} \tag{3.162}$$

3.2.2.4. Using $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \frac{\partial}{\partial u}$

In this section, we are going to reduce the equation

$$u_{xx} + u_{yy} = e^{2u} \tag{3.163}$$

using the symmetry

$$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \frac{\partial}{\partial u}$$

The characteristic system of $X(F) = 0$ is given by

$$\frac{dx}{x} = \frac{dy}{y} = \frac{du}{-1}$$

Solving

$$\frac{dx}{x} = \frac{dy}{y}$$

leads to the invariant variable of X given by

$$\xi(x,y) = \frac{y}{x} \tag{3.164}$$

Solving

$$\frac{dx}{x} = -\frac{du}{1}$$

leads to the invariant variable

$$V(\xi) = u + \ln x \tag{3.165}$$

Differentiating Eq. (3.165) with respect to x gives

$$u_x = V'\left(\frac{-y}{x^2}\right) - \frac{1}{x}$$

$$u_{xx} = V''\left(\frac{y^2}{x^4}\right) + V'\left(\frac{2y}{x^3}\right) + \frac{1}{x^2}$$

$$= \frac{1}{x^2}[\xi^2 V'' + 2\xi V' + 1] \tag{3.166}$$

Differentiating Eq. (3.165) with respect to y gives

$$u_y = V'\left(\frac{1}{x}\right)$$

$$u_{yy} = \frac{1}{x^2}V'' \tag{3.167}$$

Using equations (3.165), (3.166) and (3.167) in the PDE (3.200) leads to

$$\frac{1}{x^2}[\xi^2 V'' + 2\xi V' + 1] + \frac{1}{x^2}V'' = e^{2(V - \ln x)}$$

This can be simplified to the second order ODE

$$(\xi^2 + 1)V'' + 2\xi V' = e^{2V} - 1 \tag{3.168}$$

This ODE admits the symmetry

$$X = [1 + \xi^2]\frac{\partial}{\partial \xi} - \xi\frac{\partial}{\partial V}$$

The first prolongation of X is given by

$$X^{[1]} = [1 + \xi^2]\frac{\partial}{\partial \xi} - \xi\frac{\partial}{\partial V} + (-1 - 2\xi V')\frac{\partial}{\partial V'}$$

The characteristic system of the invariant equation

$$X^{[1]}F = 0$$

is given by

$$\frac{d\xi}{1+\xi^2} = \frac{dV}{-\xi} = \frac{dV'}{-1-2\xi V'}$$

Solving

$$\frac{d\xi}{1+\xi^2} = \frac{dV}{-\xi}$$

leads to

$$2V + \ln(1 + \xi^2) = c$$

which gives the invariant variable of $X^{[1]}$

$$r = 2V + \ln(1 + \xi^2) \tag{3.169}$$

Solving

$$\frac{d\xi}{1+\xi^2} = \frac{dV}{-\xi} = \frac{dV'}{-1-2\xi V'}$$

leads to

$$\xi + (1+\xi^2)V' = 0$$

which gives the invariant

$$w = \xi + (1+\xi^2)V' \tag{3.170}$$

Eq. (3.169) implies

$$\frac{dr}{d\xi} = 2V' + \frac{2\xi}{1+\xi^2} = \frac{2(1+\xi^2)V'+2\xi}{1+\xi^2} = \frac{2w}{1+\xi^2} \tag{3.171}$$

Differentiating Eq. (3.207) with respect to ξ gives

$$\frac{dw}{d\xi} = 1 + 2\xi V' + (1+\xi^2)V'' \tag{3.172}$$

Using Eq. (3.168) in the previous equation, we get

$$\frac{dw}{d\xi} = e^{2V} \tag{3.173}$$

Using Eq. (3.169) in Eq. (3.173), we have

$$\frac{dw}{d\xi} = \frac{e^r}{1+\xi^2} \tag{3.174}$$

Dividing Eq. (3.174) over Eq. (3.171) gives

$$\frac{dw}{dr} = \frac{e^r}{2w}$$

which implies

$$w^2 = e^r + c \tag{3.175}$$

From Eq. (3.171)

$$\frac{(1+\xi^2)^2}{4}\left(\frac{dr}{d\xi}\right)^2 = e^r + c$$

or

$$\frac{dr}{\sqrt{e^r+c}} = \pm\frac{2d\xi}{1+\xi^2} \tag{3.176}$$

There are three cases in the integration of the previous ODE

Case 1 $c = 0$

Integrating Eq. (3.176) in this case gives

$$-\frac{2}{\sqrt{e^r}} = \pm 2\arctan\xi + c$$

Returning to the variables ξ, V using Eq. (3.206) gives

$$r = 2V + \ln(1+\xi^2) \tag{3.169}$$

$$-\frac{2}{\sqrt{(1+\xi^2)e^{2V}}} = \pm 2\arctan\xi + k$$

Returning to the original variables using equations (3.164) and (3,165) gives

$$-\frac{1}{\sqrt{\left(1+\left(\frac{y}{x}\right)^2\right)x^2e^{2u}}} = \pm\arctan\left(\frac{y}{x}\right) + k$$

Solving this equation for u, we get the exact solution

$$\boxed{u = -\frac{1}{2}\ln(x^2+y^2) - \ln\left|\arctan\left(\frac{y}{x}\right) + k\right|} \tag{3.177}$$

Case 2 $c > 0$

Integrating Eq. (3.176) in this case gives

$$-\frac{2}{\sqrt{c}}\operatorname{arctanh}\left(\frac{\sqrt{e^r+c}}{\sqrt{c}}\right) = \pm 2\arctan\xi + c$$

Returning to the variables ξ, V using Eq. (3.169) gives

$$-\frac{2}{\sqrt{c}}\operatorname{arctanh}\left(\frac{\sqrt{(1+\xi^2)e^{2V}+c}}{\sqrt{c}}\right) = \pm 2\arctan\xi + k$$

or

$$(1+\xi^2)e^{2V} = \pm c\left[\tanh^2\left(\sqrt{c}\arctan\xi + k\right) - 1\right]$$

Returning to the original variables

$$\left(1+\left(\frac{y}{x}\right)^2\right)x^2e^{2u} = c\left[\tanh^2\left(\sqrt{c}\arctan\left(\frac{y}{x}\right) + k\right) - 1\right]$$

which implies the exact solution

$$\boxed{u = -\frac{1}{2}\ln(x^2+y^2) + \ln\left|\sqrt{c}\operatorname{sech}\left(\sqrt{c}\arctan\left(\frac{y}{x}\right) + k\right)\right|} \tag{3.178}$$

Case 3 $\boldsymbol{c} < 0$

Integrating Eq. (3.176) in this case gives

$$\frac{2}{\sqrt{-c}}\arctan\left(\frac{\sqrt{e^r+c}}{\sqrt{-c}}\right) = \pm 2\arctan\xi + k$$

Returning to the variables ξ, V using Eq. (3.169) gives

$$\frac{2}{\sqrt{-c}}\arctan\left(\frac{\sqrt{(1+\xi^2)e^{2V}+c}}{\sqrt{-c}}\right) = \pm 2\arctan\xi + k$$

or

$$(1+\xi^2)e^{2V} = -c\left[\tan^2\left(\sqrt{-c}\arctan\xi + k\right) + 1\right]$$

Returning to the original variables

$$\left(1+\left(\frac{y}{x}\right)^2\right)x^2e^{2u} = -c\left[\tan^2\left(\sqrt{-c}\arctan\left(\frac{y}{x}\right) + k\right) + 1\right]$$

which implies the exact solution

$$\boxed{u = -\frac{1}{2}\ln(x^2+y^2) + \ln\left|\sqrt{-c}\sec\left(\sqrt{-c}\arctan\left(\frac{y}{x}\right) + k\right)\right|} \tag{3.179}$$

3.2.2.5. Using $X = (x^2 - y^2)\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y} - 2x\frac{\partial}{\partial u}$

In this section, we are going to reduce the equation

$$u_{xx} + u_{yy} = e^{2u} \tag{3.180}$$

using the symmetry

$$X = (x^2 - y^2)\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y} - 2x\frac{\partial}{\partial u}$$

The characteristic system of $X(F) = 0$ is given by

$$\frac{dx}{x^2-y^2} = \frac{dy}{2xy} = \frac{du}{-2x}$$

The equation

$$\frac{dx}{x^2-y^2} = \frac{dy}{2xy}$$

is homogeneous and can be written in the form

$$\frac{dy}{dx} = \frac{2xy}{x^2-y^2} \tag{3.181}$$

Let

$$H = \frac{y}{x} \tag{3.182}$$

Then,

$$\frac{dy}{dx} = H + x\frac{dH}{dx} \tag{3.183}$$

Substituting equations (3.182) and (3.183) in Eq. (3.181) gives

$$H + x\frac{dH}{dx} = \frac{2H}{1-H^2}$$

Which is separable and is simplified to

$$x\frac{dH}{dx} = \frac{H+H^3}{1-H^2}$$

or

$$\frac{1-H^2}{H(1+H^2)}dH = \frac{dx}{x}$$

Using partial fraction of the left hand side gives

$$\frac{1}{H} - \frac{2H}{1+H^2} = \frac{dx}{x}$$

Integrating gives

$$\ln H - \ln(1+H^2) = \ln x + c$$

Returning to the variables x, y gives

$$\ln\frac{y}{x} - \ln\left(1+\left(\frac{y}{x}\right)^2\right) = \ln x + c$$

which leads to the invariant variable of X given by

$$\xi(x,y) = \frac{y}{x^2+y^2} \tag{3.184}$$

Solving

$$\frac{dy}{2xy} = \frac{du}{-2x}$$

leads to the invariant variable

$$V(\xi) = u + \ln y \tag{3.185}$$

Differentiating Eq. (3.165) with respect to x gives

$$u_x = V'\left(\frac{-2xy}{(x^2+y^2)^2}\right)$$

$$u_{xx} = V''\left(\frac{4x^2y^2}{(x^2+y^2)^4}\right) + V'\left(\frac{6x^2y-2y^3}{(x^2+y^2)^3}\right) \tag{3.186}$$

Differentiating Eq. (3.185) with respect to y gives

$$u_y = V'\left(\frac{x^2-y^2}{(x^2+y^2)^2}\right) - \frac{1}{y}$$

$$u_{yy} = V''\left(\frac{(x^2-y^2)^2}{(x^2+y^2)^4}\right) + V'\left(\frac{-6x^2y+2y^3}{(x^2+y^2)^3}\right) + \frac{1}{y^2} \tag{3.187}$$

Substituting equations (3.186) and (3.187) in the PDE (3.163) gives

$$\frac{1}{(x^2+y^2)^2}V'' + \frac{1}{y^2} = \frac{e^{2V}}{y^2}$$

Multiplying this equation by y^2 gives

$$\xi^2 V'' + 1 = e^{2V} \tag{3.188}$$

The ODE (3.188) admits the symmetries

$$X = \xi\frac{\partial}{\partial\xi}, \qquad Y = \frac{\partial}{\partial\xi} + \frac{1}{\xi}\frac{\partial}{\partial V}$$

Reduction of Eq. (3.188) using $X = \xi \frac{\partial}{\partial \xi}$

The first prolongation of X is given by

$$X^{[1]} = \xi \frac{\partial}{\partial \xi} + 0 \frac{\partial}{\partial V} - V' \frac{\partial}{\partial V'}$$

The characteristic system of the invariant equation

$$X^{[1]} F = 0$$

is given by

$$\frac{d\xi}{\xi} = \frac{dV}{0} = \frac{dV'}{-V'}$$

Solving

$$\frac{d\xi}{\xi} = \frac{dV}{0}$$

leads to

$$V = c$$

which gives the invariant variable of $X^{[1]}$

$$r = V \tag{3.189}$$

Solving

$$\frac{d\xi}{\xi} = \frac{dV'}{-V'}$$

leads to

$$\xi V' = c$$

which gives the invariant

$$w = \xi V' \tag{3.190}$$

Now, differentiating equations (3.189) and (3.190) with respect to ξ , we get

$$\frac{dr}{d\xi} = V' = \frac{w}{\xi} \tag{3.191}$$

and

$$\frac{dw}{d\xi} = V' + \xi V'' = \frac{w}{\xi} + \frac{e^{2V} - 1}{\xi} \tag{3.192}$$

Dividing Eq. (3.192) over Eq. (3.191), we get

$$\frac{dw}{dr} = \frac{w + e^{2r} - 1}{w}$$

which is an Abel equation of second kind.

Next, let us try the second symmetry of Eq. (3.188).

Reduction of Eq. (3.188) using $Y = \frac{\partial}{\partial \xi} + \frac{1}{\xi}\frac{\partial}{\partial V}$

The first prolongation of X is given by

$$X^{[1]} = \frac{\partial}{\partial \xi} + \frac{1}{\xi}\frac{\partial}{\partial V} - \frac{1}{\xi^2}\frac{\partial}{\partial V'}$$

The characteristic system of the invariant equation

$$X^{[1]}F = 0$$

is given by

$$d\xi = \xi dV = -\xi^2 dV'$$

Solving

$$d\xi = \xi dV$$

leads to

$$V - \ln \xi = c$$

which gives the invariant variable of $X^{[1]}$

$$r = V - \ln \xi \tag{3.193}$$

Solving

$$d\xi = \xi dV = -\xi^2 dV'$$

leads to

$$V' - \frac{1}{\xi} = c$$

which gives the invariant

$$w = V' - \frac{1}{\xi} \tag{3.194}$$

Now, differentiating equations (3.193) and (3.194) with respect to ξ , we get

$$\frac{dr}{d\xi} = V' - \frac{1}{\xi} = w \tag{3.195}$$

and

$$\frac{dw}{d\xi} = V'' + \frac{1}{\xi^2} = \frac{e^{2V}-1}{\xi^2} + \frac{1}{\xi^2} = \frac{e^{2(r+\ln \xi)}}{\xi^2} = e^{2r} \tag{3.196}$$

Dividing Eq. (3.196) over Eq. (3.195) gives

$$\frac{dw}{dr} = \frac{e^{2r}}{w}$$

which is separable with solution given by

$$w^2 = e^{2r} + c$$

Returning to the variables ξ, V gives

$$\left(V' - \frac{1}{\xi}\right)^2 = \frac{e^{2V}}{\xi^2} + c$$

Using the variable r in equations (3.193) and (3.195), we get

$$\left(\frac{dr}{d\xi}\right)^2 = e^{2r} + c$$

or

$$\frac{dr}{d\xi} = \pm\sqrt{e^{2r} + c} \tag{3.197}$$

Three cases are to be discussed in solving Eq. (3.197).

Case 1 $c = 0$

Integrating Eq. (3.197) in this case gives

$$-e^{-r} = \pm\xi + k$$

Returning to the variables ξ, V using Eq. (3.193) gives

$$-\xi e^{-V} = \pm\xi + k$$

Returning to the original variables using equations (3.184) and (3,185) gives

$$-\frac{y}{x^2+y^2}\frac{e^{-u}}{y} = \pm\frac{y}{x^2+y^2} + k$$

Solving this equation for u, we get the exact solution

$$\boxed{u = -\ln|y - k(x^2 + y^2)|} \tag{3.198}$$

Case 2 $c > 0$

Integrating Eq. (3.197) in this case gives

$$-\frac{1}{\sqrt{c}}\operatorname{arctanh}\left(\frac{\sqrt{e^{2r}+c}}{\sqrt{c}}\right) = \pm\xi + k$$

Returning to the variables ξ, V using Eq. (3.193) gives

$$-\frac{1}{\sqrt{c}}\operatorname{arctanh}\left(\frac{\sqrt{e^{2V}/\xi^2+c}}{\sqrt{c}}\right) = \pm\xi + k$$

or

$$e^{2V}/\xi^2 = c\left[\tanh^2\left(\sqrt{c}\xi + k\right) - 1\right]$$

Returning to the original variables

$$y^2 e^{2u} = c\left(\frac{y}{x^2+y^2}\right)^2 \operatorname{sech}^2\left(\frac{y\sqrt{c}}{x^2+y^2} + k\right)$$

which implies the exact solution

$$\boxed{u = -\ln(x^2 + y^2) + \ln\left|\sqrt{c}\,\mathrm{sech}\left(\frac{y\sqrt{c}}{x^2+y^2} + k\right)\right|} \tag{3.199}$$

Case 3 $\boldsymbol{c} < 0$

Integrating Eq. (3.234) in this case gives

$$-\frac{1}{\sqrt{-c}}\arctan\left(\frac{\sqrt{e^{2r}+c}}{\sqrt{-c}}\right) = \pm\xi + k$$

Returning to the variables ξ, V using Eq. (3.193) gives

$$-\frac{1}{\sqrt{-c}}\arctan\left(\frac{\sqrt{e^{2V}/\xi^2+c}}{\sqrt{-c}}\right) = \pm\xi + k$$

or

$$e^{2V}/\xi^2 = -c\left[\tan^2\left(\sqrt{-c}\xi + k\right) + 1\right]$$

Returning to the original variables

$$y^2 e^{2u} = -c\left(\frac{y}{x^2+y^2}\right)^2 \sec^2\left(\frac{y\sqrt{-c}}{x^2+y^2} + k\right)$$

which gives the exact solution

$$\boxed{u = -\ln(x^2 + y^2) + \ln\left|\sqrt{-c}\,\sec\left(\frac{y\sqrt{-c}}{x^2+y^2} + k\right)\right|} \tag{3.200}$$

CHAPTER 4

GROUP CLASSIFICATION AND SYMMETRY REDUCTIONS FOR A CLASS OF NONLINEAR POISSON EQUATIONS ON SPHERE

The aim of this chapter is to study the complete group classification problem and some symmetry reductions of the nonlinear Poisson equation on the sphere given by

$$u_{xx} + (\cot x)u_x + (csc^2 x)u_{yy} = f(u), \tag{4.1}$$

with $f(u)$ nonlinear.

Precisely the following classification is obtained

***Theorem* 4.1**

The minimal symmetry algebra of nonlinear PDE (4.1) *is three dimensional and is generated by*

$$S_1 = sin\, y \frac{\partial}{\partial x} + cot\, x\, cos\, y \frac{\partial}{\partial y}\,, \qquad S_2 = cos\, y \frac{\partial}{\partial x} - cot\, x\, sin\, y \frac{\partial}{\partial y}\,, \qquad S_3 = \frac{\partial}{\partial y}$$

and is obtained for all nonlinear arbitrary functions $f(u)$. *Infinite dimensional symmetry algebra exists in the case*

$$f(u) = ae^{bu} + \frac{2}{b}\,,\ a, b \neq 0$$

which is generated by

$$X = \xi(x,y)\frac{\partial}{\partial x} + \tau(x,y)\frac{\partial}{\partial y} + \phi(x,y)\frac{\partial}{\partial u}$$

where $\tau(x,y)$ is a harmonic function on sphere that satisfies the following pde

$$\tau_{xx} + (cot\, x)\tau_x + (csc^2\, x)\tau_{yy} = 0\,,$$

The function $\xi(x,y)$ is given by the following two relations

$$\xi = -\, sin^2\, x \int \tau_x(x,y)dy + g(x)$$

$$\xi_x - cot\, x\, \xi = \tau_y$$

and the function $\phi(x,y)$ is given by

$$\phi = \frac{-2}{b}\xi_x.$$

The proof of theorem (4.1) is contained in section 4.1. Examples of reductions and some exact solutions, corresponding to the cases obtained in section 4.1, are provided in section 4.2.

4.1. GROUP CLASSIFICATION OF POISSON EQUATION ON SPHERE

We are going to obtain the Lie symmetries of the equation (4.1). To do so, we begin by operator. Let

$$X = \xi(x,y,u)\frac{\partial}{\partial x} + \tau(x,y,u)\frac{\partial}{\partial y} + \phi(x,y,u)\frac{\partial}{\partial u}$$

be symmetry of the PDE (4.1).

If $X^{[2]}$ denotes the second prolongation of X then the invariance condition for finding ξ, τ and ϕ is given by

$$X^{[2]}\big(u_{xx} + (\cot x)u_x + (csc^2 x)u_{yy} - f(u)\big)\big|_{(4.1)} = 0 \tag{4.2}$$

where

$$X^{[2]} = X + \eta^{[\,x\,]}\frac{\partial}{\partial u_x} + \eta^{[\,xx\,]}\frac{\partial}{\partial u_{xx}} + \eta^{[\,yy\,]}\frac{\partial}{\partial u_{yy}}$$

whose coefficients in terms of ξ, τ and ϕ are given by

$$\eta^{[\,x\,]} = \phi_x + (\phi_u - \xi_x)u_x - \xi_u u_x^2 - \tau_x u_y - \tau_u u_x u_y \tag{4.3}$$

$$\begin{aligned}\eta^{[\,xx\,]} = {} & \phi_{xx} + (2\phi_{xu} - \xi_{xx})u_x + (\phi_{uu} - 2\xi_{xu})u_x^2 - \xi_{uu}u_x^3 - \tau_{xx}\, u_y \\ & -2\tau_{xu}u_x u_y - \tau_{uu}u_x^2 u_y + (\phi_u - 2\xi_x)u_{xx} - 3\xi_u u_{xx}u_x \\ & -2\tau_x u_{xy} - 2\tau_u u_x u_{xy}\end{aligned} \tag{4.4}$$

$$\eta^{[yy]} = \phi_{yy} + \left(2\phi_{yu} - \tau_{yy}\right)u_y + \left(\phi_{uu} - 2\tau_{yu}\right)u_y^2 - \tau_{uu}u_y^3$$
$$-\xi_{yy}u_x - 2\xi_{yu}u_xu_y - \xi_{uu}u_y^2u_x + \left(\phi_u - 2\tau_y\right)u_{yy}$$
$$-3\tau_u u_{yy}u_y - \xi_u u_{yy}u_x - 2\xi_y u_{xy} - 2\xi_u u_y u_{xy} \tag{4.5}$$

Using equations (4.3)-(4.5), we have

$$X^{[2]}\left(u_{xx} + (\cot x)u_x + (csc^2x)u_{yy} - f(u)\right)\Big|_{(4.1)}$$
$$= \xi\frac{\partial}{\partial x}\left(u_{xx} + (\cot x)u_x + (csc^2x)u_{yy} - f(u)\right) - \phi f_u + \eta^{[xx]}$$
$$+(cot\ x)\eta^{[x]} + (csc^2x)\eta^{[yy]}$$
$$= \xi\left(-(\csc^2 x)u_x - 2(csc^2x)\cot x\, u_{yy}\right) - \phi f_u + \eta^{[xx]}$$
$$+(cot\ x)\eta^{[x]} + (csc^2x)\eta^{[yy]}$$
$$= \xi\left(-(\csc^2 x)u_x - 2(csc^2x)\cot x\, u_{yy}\right) - \phi f_u$$
$$+[\phi_{xx} + (2\phi_{xu} - \xi_{xx})u_x + (\phi_{uu} - 2\xi_{xu})u_x^2 - \xi_{uu}u_x^3$$
$$-\tau_{xx}u_y - 2\tau_{xu}u_xu_y - \tau_{uu}u_x^2u_y + (\phi_u - 2\xi_x)u_{xx}$$
$$-3\xi_u u_{xx}u_x - \tau_u u_{xx}u_y - 2\tau_x u_{xy} - 2\tau_u u_x u_{xy}]$$
$$+(cot\ x)\{\phi_x + (\phi_u - \xi_x)u_x - \xi_u u_x^2 - \tau_x u_y - \tau_u u_x u_y\}$$
$$+(csc^2x)[\phi_{yy} + \left(2\phi_{yu} - \tau_{yy}\right)u_y + \left(\phi_{uu} - 2\tau_{yu}\right)u_y^2 - \tau_{uu}u_y^3 - \xi_{yy}u_x$$
$$-2\xi_{yu}u_xu_y - \xi_{uu}u_y^2u_x + \left(\phi_u - 2\tau_y\right)u_{yy} - 3\tau_u u_{yy}u_y$$
$$-\xi_u u_{yy}u_x - 2\xi_y u_{xy} - 2\xi_u u_y u_{xy}] \tag{4.6}$$

Putting the condition

$$u_{yy} = (\sin^2 x\, f - \sin^2 x\, u_{xx} - \sin x \cos x\, u_x)$$

in the previous expression (4.6), we get

$$X^{[2]}\left(u_{xx} + (\cot x)u_x + (csc^2x)u_{yy} - f(u)\right)\Big|_{(4.1)}$$
$$= \xi\left(-\csc^2 x\, u_x - 2\csc^2 x \cot x\,(\sin^2 x\, f - \sin^2 x\, u_{xx} - \sin x \cos x\, u_x)\right)$$
$$-\phi f_u + [\phi_{xx} + (2\phi_{xu} - \xi_{xx})u_x + (\phi_{uu} - 2\xi_{xu})u_x^2 - \xi_{uu}u_x^3 - \tau_{xx}u_y$$
$$-2\tau_{xu}u_xu_y - \tau_{uu}u_x^2u_y + (\phi_u - 2\xi_x)u_{xx} - 3\xi_u u_{xx}u_x - \tau_u u_{xx}u_y$$
$$-2\tau_x u_{xy} - 2\tau_u u_x u_{xy}] + (cot\ x)[\phi_x + (\phi_u - \xi_x)u_x - \xi_u u_x^2 - \tau_x u_y$$
$$-\tau_u u_x u_y] + (csc^2x\ )[\phi_{yy} + \left(2\phi_{yu} - \tau_{yy}\right)u_y + \left(\phi_{uu} - 2\tau_{yu}\right)u_y^2$$
$$-\tau_{uu}u_y^3 - \xi_{yy}u_x - 2\xi_{yu}u_xu_y - \xi_{uu}u_y^2u_x$$

$$+\left(\phi_u - 2\tau_y\right)(\sin^2 x\, f - \sin^2 x\, u_{xx} - \sin x \cos x\, u_x)$$
$$-3\tau_u(\sin^2 x\, f - \sin^2 x\, u_{xx} - (\sin x \cos x) u_x) u_y$$
$$-\xi_u(\sin^2 x\, f - \sin^2 x\, u_{xx} - \sin x \cos x\, u_x) u_x - 2\xi_y u_{xy}$$
$$-2\xi_u u_y u_{xy}]$$
$$= \xi\left(-\csc^2 x\, u_x - 2\csc^2 x \cot x\, (\sin^2 x\, f - \sin^2 x\, u_{xx} - \sin x \cos x\, u_x)\right)$$
$$-\phi f_u + [\phi_{xx} + (2\phi_{xu} - \xi_{xx}) u_x + (\phi_{uu} - 2\xi_{xu}) u_x^2 - \xi_{uu} u_x^3 - \tau_{xx} u_y$$
$$-2\tau_{xu} u_x u_y - \tau_{uu} u_x^2 u_y + (\phi_u - 2\xi_x) u_{xx} - 3\xi_u u_{xx} u_x - \tau_u u_{xx} u_y$$
$$-2\tau_x u_{xy} - 2\tau_u u_x u_{xy}] + [\phi_x(cot\, x) + (cot\, x)(\phi_u - \xi_x) u_x$$
$$-(cot\, x)\xi_u u_x^2 - (cot\, x)\tau_x u_y - (cot\, x)\tau_u u_x u_y]$$
$$+[\csc^2 x\, \phi_{yy} + \csc^2 x\left(2\phi_{yu} - \tau_{yy}\right) u_y + \csc^2 x\left(\phi_{uu} - 2\tau_{yu}\right) u_y^2$$
$$- \csc^2 x\, \tau_{uu} u_y^3 - \csc^2 x\, \xi_{yy} u_x - 2\csc^2 x\, \xi_{yu} u_x u_y - \csc^2 x\, \xi_{uu} u_y^2 u_x$$
$$+ \csc^2 x\left(\phi_u - 2\tau_y\right)(\sin^2 x\, f - \sin^2 x\, u_{xx} - \sin x \cos x\, u_x)$$
$$-3\csc^2 x\, \tau_u(\sin^2 x\, f - \sin^2 x\, u_{xx} - \sin x \cos x\, u_x) u_y$$
$$- \csc^2 x\, \xi_u(\sin^2 x\, f - \sin^2 x\, u_{xx} - \sin x \cos x\, u_x) u_x$$
$$-2\csc^2 x\, \xi_y u_{xy} - 2\csc^2 x\, \xi_u u_y u_{xy}]$$
$$= -\xi(\csc^2 x) u_x - 2\xi(\cot x) f + 2\xi(\cot x) u_{xx} + 2\xi(\cot^2 x) u_x - \phi f_u$$
$$+[\phi_{xx} + (2\phi_{xu} - \xi_{xx}) u_x + (\phi_{uu} - 2\xi_{xu}) u_x^2 - \xi_{uu} u_x^3 - \tau_{xx} u_y$$
$$-2\tau_{xu} u_x u_y - \tau_{uu} u_x^2 u_y + (\phi_u - 2\xi_x) u_{xx} - 3\xi_u u_{xx} u_x - \tau_u u_{xx} u_y$$
$$-2\tau_x u_{xy} - 2\tau_u u_x u_{xy}]$$
$$+[\phi_x(cot\, x) + (cot\, x)(\phi_u - \xi_x) u_x - (cot\, x)\xi_u u_x^2 - (cot\, x)\tau_x u_y$$
$$-(cot\, x)\tau_u u_x u_y] + (\csc^2 x)\phi_{yy} + (\csc^2 x)\left(2\phi_{yu} - \tau_{yy}\right) u_y$$
$$+(\csc^2 x)\left(\phi_{uu} - 2\tau_{yu}\right) u_y^2 - (\csc^2 x)\tau_{uu} u_y^3 - (\csc^2 x)\xi_{yy} u_x$$
$$-2(\csc^2 x)\xi_{yu} u_x u_y - (\csc^2 x)\xi_{uu} u_y^2\, u_x + \left(\phi_u - 2\tau_y\right) f$$
$$-\left(\phi_u - 2\tau_y\right) u_{xx} - (\cot x)\left(\phi_u - 2\tau_y\right) u_x - 3\tau_u f u_y + 3\tau_u u_{xx} u_y$$
$$+3\tau_u\, (\cot x)\, u_x u_y - \xi_u f u_x + \xi_u u_x u_{xx} + (\cot x) u_x^2$$
$$-2(\csc^2 x)\xi_y u_{xy} - 2(\csc^2 x)\xi_u u_y u_{xy} \qquad (4.7)$$

Now using (4.7) in (4.2), we find the determining equations as follows:

Coefficient of u_x^3 : $\xi_{uu} = 0$

Coefficient of $u_x^2 u_y$: $\tau_{uu} = 0$

Coefficient of u_x^2 : $\phi_{uu} - 2\xi_{xu} - cot\ x\ \xi_u + \xi_u cot\ x = 0$

Coefficient of u_x : $-\xi \csc^2 x + 2\xi\ cot^2\ x + 2\phi_{xu} - \xi_{xx} + \cot x(\phi_u - \xi_x)$
$-csc^2 x\ \xi_{yy} - cot\ x(\phi_u - 2\tau_y) - \xi_u f = 0$

Coefficient of $u_x u_y$: $-2\tau_{xu} - cot\ x\ \tau_u - 2csc^2 x\ \xi_{yu} + 3\tau_u cot\ x = 0$

Coefficient of $u_y^2 u_x$: $-csc^2 x\ \xi_{uu} = 0$

Coefficient of $u_{xx} u_x$: $-3\xi_u + \xi_u = 0$

Coefficient of $u_{xy} u_x$: $-2\tau_u = 0$

Coefficient of $u_{xx} u_y$: $-\tau_u + 3\tau_u = 0$

Coefficient of u_y^2 : $csc^2 x(\phi_{uu} - 2\tau_{yu}) = 0$

Coefficient of u_y^3 : $-csc^2 x\ \tau_{uu} = 0$

Coefficient of u_{xy} : $-2\tau_x - 2csc^2 x\ \xi_y = 0$

Coefficient of u_{xx} : $2\xi \cot x + (\phi_u - 2\xi_x) - (\phi_u - 2\tau_y) = 0$

Coefficient of u_y : $-\tau_{xx} - cot\ x\ \tau_x + csc^2 x\ (2\phi_{yu} - \tau_{yy}) - 3\tau_u f = 0$

Coefficient of $u_{xy} u_y$: $-\ 2csc^2 x\ \xi_u u_y u_{xy} = 0$

Constant term : $-2\xi f cot\ x - \phi f_u + \phi_{xx} + \phi_x \cot x + csc^2 x\ \phi_{yy}$
$+(\phi_u - 2\tau_y) f = 0$

The above system can be simplified to get the following system of determining equations:

SE_1 : $\xi_u = 0$

SE_2 : $\tau_u = 0$

SE_3 : $csc^2 x\ \xi_y + \tau_x = 0$

SE_4 : $\tau_y - \xi_x + \cot x\ \xi = 0$

SE_5 : $\phi_{uu} = 0$

SE_6 : $2\phi_{yu} - \tau_{yy} - \tau_{xx} sin^2 x - \tau_x \sin x \cos x = 0$

SE_7 : $\xi(cot^2 x - 1) + 2\phi_{xu} - \xi_{xx} - \cot x\ \xi_x - \xi_{yy} csc^2 x + 2\tau_y \cot x = 0$

SE_8 : $(\phi_u - 2\tau_y - 2\xi cot\ x) f - \phi f_u + \phi_{xx} + \phi_x cot\ x + \phi_{yy} csc^2 x = 0$

Like previous chapters, this system of determining equations will be solved utilizing a triangulation procedure based on techniques for obtaining differential Groebner basis developed by Mansfield in [29].

The operation : $(SE_3)_y - (SE_4)_x$ implies

$$\xi_{xx} + (csc^2 x)\xi_{yy} - \xi_x(\cot x) + (csc^2 x)\xi = 0 \tag{4.8}$$

Using Eq. (4.8), we can simplify SE_7 to be

$$SE_7 : \quad \phi_{xu} = 0$$

The operation : $(SE_3)_x + (SE_4)_y$ gives

$$\tau_{xx} sin^2 x + \tau_{yy} + \tau_x sin\, x\, cos\, x = 0 \tag{4.9}$$

Using Eq. (4.9), we can simplify SE_6 to be

$$SE_6 : \quad \phi_{yu} = 0$$

Differentiating equation SE_8 with respect to u, we get

$$\left(-2\tau_y - 2\xi cot\, x\right) f_u - \phi f_{uu} = 0$$

From the above equation and SE_4, we can write

$$f_{uu}\phi + 2f_u\xi_x = 0 \tag{4.10}$$

Differentiating Eq. (4.10) with respect to u, we get

$$f_{uuu}\phi + f_{uu}\phi_u + 2f_{uu}\xi_x = 0 \tag{4.11}$$

Differentiating again with respect to u, we have

$$f_{uuuu}\phi + 2f_{uuu}\phi_u + 2f_{uuu}\xi_x = 0 \tag{4.12}$$

Equations (4.12) and (4.11) can be used to eliminate ϕ_u, giving the following equation

$$\phi(f_{uu}f_{uuuu} - 2f_{uuu}^2) - 2f_{uu}f_{uuu}\xi_x = 0 \tag{4.13}$$

Equations (4.10) and (4.13) can be used to eliminate ϕ, giving the following equation

$$[f_u(f_{uu}f_{uuuu} - 2f_{uuu}^2) + f_{uu}^2 f_{uuu}]\xi_x = 0$$

or

$$\boxed{\boxed{(f_u f_{uu} f_{uuuu} - 2f_u f_{uuu}^2 + f_{uu}^2 f_{uuu})\xi_x = 0}} \tag{4.14}$$

Differentiating Eq. (4.10) with respect to x, we get

$$f_{uu}\phi_x + 2f_u\xi_{xx} = 0 \tag{4.15}$$

Differentiating Eq. (4.15) with respect to u, we get

$$f_{uuu}\phi_x + 2f_{uu}\xi_{xx} = 0 \tag{4.16}$$

Using equations (4.15) and (4.16) to eliminate ϕ_x, or $f_{uu}(4.16) - f_{uuu}(4.15)$, implies that

$$\boxed{\boxed{(f_{uu}^2 - f_u f_{uuu})\xi_{xx} = 0}} \tag{4.17}$$

Note that the equations (4.14) and (4.17) are exactly the same as equations (3.21) and (3.24) that were found in chapter 3.

Next, we look at possibilities of $f(u)$.

If $\boldsymbol{\xi_x = 0}$, then from equation (4.12) and the nonlinearity of $f(u)$, we deduce

$$\boxed{\phi = 0}. \tag{4.18}$$

Substituting the assumption $\xi_x = 0$ of this case in Eq. (4.8), we have

$$\xi_{yy} + \xi = 0 \tag{4.19}$$

which using

$$\xi_x = \xi_u = 0$$

implies that

$$\boxed{\xi = k_1 \sin y + k_2 \cos y} \tag{4.20}$$

Substituting ξ in equation SE_4, one gets

$$\tau_y = -\cot x\,(k_1 \sin y + k_2 \cos y)$$

giving

$$\tau = -\cot x\,(-k_1 \cos y + k_2 \sin y) + g(x) \tag{4.21}$$

Using the last relation in equation SE_3, we get

$$csc^2 x(k_1 \cos y - k_2 \sin y) + csc^2 x\,(-k_1 \cos y + k_2 \sin y) + g'(x) = 0$$

This implies

$$g'(x) = 0$$

from which we come up with

$$g(x) = k_3.$$

Using the value of $g(x)$ obtained in equation (4.21), we get

$$\boxed{\tau = \cot x\,(k_1 \cos y - k_2 \sin y) + k_3} \tag{4.22}$$

This is without any restriction on $f(u)$.

Hence, equations (4.20),(4.22)and (4.18) defines the minimal symmetry algebra which is three dimensional which exists for any choice of $f(u)$ and is generated by the following vector fields:

$$S_1 = \sin y \frac{\partial}{\partial x} + \cot x \cos y \frac{\partial}{\partial y}\,, \quad S_2 = \cos y \frac{\partial}{\partial x} - \cot x \sin y \frac{\partial}{\partial y}\,,\ S_3 = \frac{\partial}{\partial y}$$

The commutation relations for this Lie algebra are given in the table below

	S_1	S_2	S_3
S_1	0	S_3	$-S_2$
S_2	$-S_3$	0	S_1
S_3	S_2	$-S_1$	0

To look for functions $f(u)$ that may give larger symmetry algebra we assume

$$\xi_x \neq 0$$

and solve the differential equations

$$f_{uu}^2 - f_u f_{uuu} = 0 \tag{4.23}$$

$$f_u f_{uu} f_{uuuu} - 2 f_u f_{uuu}^2 + f_{uu}^2 f_{uuu} = 0 \tag{4.24}$$

These two equations are exactly the as equations (3.34) and (3.35) that were solved and discussed in detail in chapter III.

As solved before in chapter III, only one nonlinear function solves Eq. (4.23) which is

- $f(u) = ae^{bu} + c,$ with $a \neq 0,\ b \neq 0$

Also, there are four types of nonlinear functions that solve Eq. (4.24) as discussed in chapter III. These solutions are the following functions

- $f(u) = au^2 + bu + c,$ with $a \neq 0$
- $f(u) = ae^{bu} + c,$ with $a \neq 0,\ b \neq 0$
- $f(u) = a(u+b)^c + d\,,$ with $a \neq 0,\ c \neq 0\,,1\,,2$
- $f(u) = a\ln(u+b) + c,$ with $a \neq 0$

Hence from the solution of Eq. (4.23), we see that

$$f(u) \neq ae^{bu} + c \tag{4.25}$$

is equivalent to

$$f_{uu}^2 - f_u f_{uuu} \neq 0$$

Using this equivalence in Eq. (4.17), relation (4.25) implies

$$\xi_{xx} = 0. \tag{4.26}$$

Different possibilities for $f(u)$ are analyzed below to obtain corresponding symmetry algebras. The Symmetry Algebra for different forms of $f(u)$ are summarized in the following cases:

Case 1: $\boldsymbol{f(u) \neq ae^{bu} + c}$

In this case we have

$$\xi_{xx} = 0. \tag{4.27}$$

There are three possibilities of $f(u)$ in this case and are discussed in the following

Case 1(i) $\boldsymbol{f(u)}$ is quadratic

Let

$$f(u) = a(u^2 + bu + c)$$

with $a \neq 0$. From Eq. (4.10), we have

$$2a\phi + 2a(2u + b)\xi_x = 0$$

which implies

$$\phi = -(2u + b)\xi_x \tag{4.28}$$

Since $\xi_{xx} = 0$, differentiating Eq. (4.28) with respect to x, we get

$$\phi_x = \phi_{xx} = 0. \tag{4.29}$$

Also, differentiating twice Eq. (4.28) with respect to y, we get

$$\phi_{yy} = -(2u + b)\xi_{xyy}. \tag{4.30}$$

Using equations (4.8) and (4.27), we get

$$(csc^2 x)\xi_{yy} - \xi_x(\cot x) + (csc^2 x)\xi = 0$$

This implies

$$\xi_{yy} = \xi_x(\cos x \sin x) - \xi = 0$$

Differentiating previous equation with respect to x , we get

$$\xi_{yyx} = \xi_x(\cos^2 x - \sin^2 x) - \xi_x = -2\xi_x \sin^2 x \tag{4.31}$$

Substituting Eq. (4.31) in Eq. (4.30) gives

$$\phi_{yy} = 2\sin^2 x\,(2u + b)\xi_x \tag{4.32}$$

Also, differentiating Eq. (4.28) with respect to u, we get

$$\phi_u = -2\xi_x \tag{4.33}$$

Substituting all of these in equation SE_8, we get

$$a(-2\xi_x - 2\xi_x)(u^2 + b\,u + c\,) + a(2u+b)^2\xi_x + 0 + 0 + 2(2u+b)\xi_x = 0$$

Simplifying the previous equation gives

$$[a(b^2 - 4c) + 4u + 2b]\xi_x = 0$$

which implies

$$\xi_x = 0.$$

This leads us to the minimal symmetry algebra.

Case 1(ii) $\boldsymbol{f(u) = a(u+b)^n + c}$, $\boldsymbol{with\ \ a \neq 0,\ n\ \neq 0\,,1\,,2}$

Substituting the function in Eq. (4.10), we get

$$an(n-1)(\boldsymbol{u}+\boldsymbol{b})^{n-2}\phi + 2an(\boldsymbol{u}+\boldsymbol{b})^{n-1}\xi_x = 0$$

which implies

$$\phi = -\frac{2(\boldsymbol{u}+\boldsymbol{b})}{n-1}\xi_x \tag{4.34}$$

Using equations (4.31) and (4.34), we get

$$\phi_{yy} = 4\sin^2 x\,\frac{(\boldsymbol{u}+\boldsymbol{b})}{n-1}\xi_x \tag{4.35}$$

Substituting equations (4.34) and (4.35) in equation SE_8, we get

$$\left(-\frac{2}{n-1}\xi_x - 2\xi_x\right)(a(u+b)^n + c) + \frac{2(u+b)}{n-1}an(u+b)^{n-1}\xi_x + \frac{4(u+b)}{n-1}\xi_x = 0$$

Simplifying, we get

$$\left[\frac{-2nc+4(u+b)}{n-1}\right]\xi_x = 0$$

This implies

$$\xi_x = 0$$

which leads us again to the minimal algebra.

Case 1(iii) $\boldsymbol{f(u) = a\ln(u+b) + c}$ **with** $\boldsymbol{a \neq 0}$

Substituting the function in Eq. (4.10), we get

$$\frac{-1}{(u+b)^2}\phi + 2\left(\frac{1}{u+b}\right)\xi_x = 0$$

or

$$\phi = 2(u+b)\xi_x \tag{4.36}$$

Using equations (4.31) and (4.36), we get

$$\xi_{yyx} = \xi_x(\cos^2 x - \sin^2 x) - \xi_x = -2\xi_x \sin^2 x \tag{4.31}$$

$$\phi_{yy} = -4\sin^2 x\,(u+b)\xi_x \tag{4.37}$$

Now from equation SE_8 , one obtains

$$(2\xi_x - 2\xi_x)f - 2(u+b)\xi_x\left(\frac{1}{u+b}\right) + 0 + 0 - 4(u+b)\xi_x = 0$$

Simplifying, we have

$$[2 + 4(u+b)]\xi_x = 0.$$

This implies that

$$\xi_x = 0$$

which leads us to minimal symmetry algebra.

This means that for all the three types of functions presented in case 1, $f(u) \neq ae^{bu} + c$, we have not found new symmetries rather than the minimal symmetry algebra.

Case 2: $\boldsymbol{f(u) = ae^{bu} + c, \quad a, b \neq 0}$

Here ξ_{xx} is not necessarily 0.

Using $f(u)$ in Eq. (4.10) gives

$$ab^2e^{bu}\phi + 2abe^{bu}\xi_x = 0.$$

Thus from Eq. (4.10) we have

$$\boxed{\phi = \frac{-2}{b}\xi_x}. \tag{4.38}$$

Substituting this in equation SE_8, one gets

$$(0 - 2\xi_x)(ae^{b\,u} + c) + \frac{2}{b}\xi_x(abe^{bu}) - \frac{2}{b}\xi_{xxx} - \frac{2}{b}\xi_{xx}cot\,x - \frac{2}{b}\xi_{xyy}csc^2x = 0\,.$$

Simplifying gives

$$bc\xi_x + \xi_{xxx} + \xi_{xx}cot\,x + \xi_{xyy}csc^2x = 0 \tag{4.39}$$

From equations (4.8) and SE_1, we get

$$\xi_{yy} = -(sin^2x)\xi_{xx} + (\cos x \sin x)\xi_x - \xi$$

Differentiating the previous equation with respect to x, we get

$$\xi_{yyx} = -\sin x \cos x\,\xi_{xx} - \sin^2 x\,\xi_{xxx} - 2\sin^2 x\,\xi_x \tag{4.40}$$

Substituting Eq. (4.40) in Eq. (4.39), we get

$$bc\xi_x + \xi_{xxx} + \xi_{xx} cot\, x - \cot x\, \xi_{xx} - \xi_{xxx} - 2\xi_x = 0$$

or

$$(bc - 2)\xi_x = 0 \tag{4.41}$$

So, if $bc \neq 2$, then $\xi_x = 0$ which leads to minimal algebra.

Thus, to obtain larger symmetry algebra, we should have $bc = 2$.

Next we consider the case $\boldsymbol{bc = 2}$.

This means that f is of the form

$$\boldsymbol{f(u) = ae^{bu} + \frac{2}{b}} \tag{4.42}$$

for the equation to may have more than the minimal symmetry algebra.

Since

$$\xi_x = 0$$

gives minimal algebra, we assume

$$\xi_x \neq 0.$$

Then from equation SE_1,

$$\xi = \xi(x, y)$$

which implies via Eq. (4.38) that

$$\phi = \phi(x, y)$$

and hence equation SE_5 is satisfied and equation SE_6 becomes

$$\tau_{xx} sin^2 x + \tau_{yy} + \tau_x sin\, x\, cos\, x = 0 \tag{4.9}$$

with

$$\tau = \tau(x, y)$$

because of equation SE_2. It follows that for every solution

$$\tau = \tau(x, y)$$

of Eq. (4.9), equations SE_7 and SE_8 are satisfied, hence every solution

$$\tau = \tau(x, y)$$

of Eq. (4.9) generates a symmetry algebra with ξ and ϕ determined from equations SE_3, SE_4 and (4.38).

So in order to determine the algebra for this case, we start by solving the PDE (4.9) to find τ. Solving equations SE_3 and SE_4 using the solution $\tau(x, y)$ of Eq. (4.9), the function

$\xi(x,y)$ can be found. Then, the function $\phi(x,y)$ is easily found using the obtained $\xi(x,y)$ in Eq. (4.38).

The algebra generated will be an infinite dimensional algebra. Some elements

$$X = \xi(x,y,u)\frac{\partial}{\partial x} + \tau(x,y,u)\frac{\partial}{\partial y} + \phi(x,y,u)\frac{\partial}{\partial u}$$

of this algebra are presented below.

$\boldsymbol{(I)\tau(x,y) = constant = k_1.}$

Let

$$\tau(x,y) = k_1 \tag{4.43}$$

Then, Eq. (4.9)

$$\tau_{xx}sin^2x + \tau_{yy} + \tau_x sin\, x\, cos\, x = 0$$

is satisfied.

Substituting this value of τ in equation SE_3, we get

$$\xi_y = 0.$$

This implies that

$$\xi = \xi(x).$$

Using this in equation SE_4, we obtain the ode given by

$$\xi_x - \cot x\, \xi = 0.$$

This ode implies that

$$\boxed{\xi = k_2 \sin x}. \tag{4.44}$$

From equation (4.38), we conclude

$$\boxed{\phi = \frac{-2}{b} k_2 \cos x}. \tag{4.45}$$

$\boldsymbol{(II)\tau(x,y) = \tau(x)}$

Let us solve Eq. (4.9) which can be written as follows

$$\tau_{xx}sin^2x + \tau_x sin\, x\, cos\, x = 0$$

The case

$$\tau_x = 0$$

implies

$$\tau(x) = constant$$

which has been discussed in (i).

The case $\tau_x \neq 0$ implies

$$\frac{\tau_{xx}}{\tau_x} = -\cot x.$$

Then,

$$\tau_x = k_1 \csc x.$$

This implies that

$$\boxed{\tau = -k_1 \ln(\csc x + \cot x) + k_2}. \tag{4.46}$$

Substituting this in SE_3, we get

$$\csc^2 x \ \xi_y + k_1 \csc x = 0$$

This implies that

$$\xi_y = -k_1 \sin x$$

Then,

$$\xi = -k_1 y \sin x + g(x)$$

Substituting in equation SE_4, we get

$$k_1 y \cos x - g'(x) + \cot x \left(-k_1 y \sin x + g(x)\right) = 0$$

This implies

$$g'(x) - \cot x \, g(x) = 0.$$

Solving for g, we get

$$g(x) = k_3 \sin x.$$

Thus,

$$\boxed{\xi = -k_1 y \sin x + k_3 \sin x} \tag{4.47}$$

Substituting in (4.9), we get

$$\boxed{\phi = \frac{-2}{b}(-k_1 y \cos x + k_3 \cos x)} \tag{4.48}$$

$(\boldsymbol{III})$ $\boldsymbol{\tau(x,y) = \tau(y)}$

From Eq. (4.9), we get the following ode

$$\tau_{yy} = 0.$$

This implies that

$$\boxed{\tau(y) = k_1 y + k_2} \tag{4.49}$$

Substituting this τ in SE_3, we get

$$\csc^2 x\, \xi_y + 0 = 0$$

This implies that

$$\xi_y = 0$$

or

$$\xi = \xi(x)$$

Substituting this in SE_4, we get

$$k_1 - \xi_x + \cot x\, \xi = 0$$

or

$$\xi_x - \cot x\, \xi = k_1$$

Integrating factor is $\csc x$

$$\csc x\, \xi_x - \csc x \cot x\, \xi = k_1 \csc x$$

This implies

$$\frac{d(\xi \csc x)}{dx} = k_1 \csc x$$

Integrating, we get

$$\xi \csc x = -k_1 \ln(\csc x + \cot x) + k_3$$

This implies that

$$\boxed{\xi = -k_1 \sin x \ln(\csc x + \cot x) + k_3 \sin x} \qquad (4.50)$$

Substituting this in Eq. (4.9), we get

$$\phi = \frac{-2}{b}(k_1 + \cot x\, \xi) = \frac{-2}{b}(k_1 + \cot x\, [-k_1 \sin x \ln(\csc x + \cot x) + k_3 \sin x])$$

$$\boxed{\phi = \frac{-2}{b}[k_1 - k_1 \cos x \ln(\csc x + \cot x) + k_3 \cos x]} \qquad (4.51)$$

(IV) Separable solutions of (4.9): $\boxed{\boldsymbol{\tau_{xx} \sin^2 x + \tau_{yy} + \tau_x \sin x \cos x = 0}}$

Let

$$\tau(x, y) = P(x)Q(y) \qquad (4.52)$$

be a solution of (4.9).

Then,

$$P''(x)Q(y) \sin^2 x + P(x)Q''(y) + P'(x)Q(y) \sin x \cos x = 0$$

Dividing by $P(x)Q(y)$, we get

$$\frac{P''(x)\sin^2 x+P'(x)\sin x\cos x}{P(x)} = -\frac{Q''(y)}{Q(y)} = constant \tag{4.53}$$

There are three cases for solving the previous system (4.53).

$(\boldsymbol{i})$ $\boldsymbol{constant = 0}$.

In this case we conclude that

$$Q''(y) = 0$$

which implies that

$$\boxed{Q(y) = c_1 y + c_2}. \tag{4.54}$$

and

$$P''(x)\sin x + P'(x)\cos x = 0$$

This implies

$$\frac{d[P'(x)\sin x]}{dx} = 0 \tag{4.55}$$

Integrating Eq. (4.55), we get

$$P'(x)\sin x = c_3$$

or

$$P'(x) = c_3 \csc x \tag{4.56}$$

Integrating Eq. (4.56), we get

$$\boxed{P(x) = -c_3 \ln(\csc x + \cot x) + c_4} \tag{4.57}$$

Thus,

$$\tau(x,y) = (-c_3 \ln(\csc x + \cot x) + c_4)(c_1 y + c_2)$$

The new kind, not discussed earlier, is given by

$$\boxed{\tau(x,y) = y \ln(\csc x + \cot x)} \tag{4.58}$$

Substituting τ given in Eq. (4.58) in equation SE_3, we get

$$csc^2 x\, \xi_y - y \csc x \ = 0$$

which implies

$$\xi_y = y \sin x \tag{4.59}$$

Integrating Eq. (4.59) with respect to y, we get

$$\xi = \frac{1}{2} y^2 \sin x + g(x) \tag{4.60}$$

Substituting in equation SE_4, we get

$$\ln(\csc x + \cot x) - \frac{1}{2}y^2 \cos x - g'(x) + \cot x \left[\frac{1}{2}y^2 \sin x + g(x)\right] = 0$$

which implies

$$g'(x) - \cot x\,[g(x)] = \ln(\csc x + \cot x) \tag{4.61}$$

Multiplying by the integrating factor $\csc x$ gives

$$\csc x\, g'(x) - \csc x \cot x\, g(x) = \csc x \ln(\csc x + \cot x)$$

This implies

$$\frac{d[\csc x g(x)]}{dx} = \csc x \ln(\csc x + \cot x) \tag{4.62}$$

Integrating Eq. (4.62), we get

$$\csc x\, g(x) = -\frac{1}{2}[\ln(\csc x + \cot x)]^2 + c$$

or

$$g(x) = -\frac{1}{2}\sin x\,[\ln(\csc x + \cot x)]^2 + c \sin x \tag{4.63}$$

Substituting Eq. (4.63) in Eq. (4.60) implies

$$\boxed{\xi = \frac{1}{2}y^2 \sin x - \frac{1}{2}\sin x\,[\ln(\csc x + \cot x)]^2 + c \sin x} \tag{4.64}$$

Substituting this in (4.38), we get

$$\phi = \frac{-1}{b}[y^2 \cos x - \cos x\,[\ln(\csc x + \cot x)]^2 + 2[\ln(\csc x + \cot x)] + 2c \cos x]$$

or

$$\boxed{\phi = \frac{-\cos x}{b}(y^2 - [\ln(\csc x + \cot x)]^2 + 2\sec x\,[\ln(\csc x + \cot x)] + 2c)} \tag{4.65}$$

(ii) Constant is positive, say constant $= k^2$

We are going to solve the system

$$\frac{P''(x)\sin^2 x + P'(x)\sin x \cos x}{P(x)} = -\frac{Q''(y)}{Q(y)} = k^2 \tag{4.66}$$

Solving

$$-\frac{Q''(y)}{Q(y)} = k^2$$

gives

$$\boxed{Q(y) = k_1 \sin ky + k_2 \cos ky} \tag{4.67}$$

Solving

$$\frac{P''(x)\sin^2 x + P'(x)\sin x\cos x}{P(x)} = k^2,$$

implies

$$P''(x)\sin x + P'(x)\cos x - k^2 P(x)\csc x = 0$$

or

$$d[P'(x)\sin x] = k^2 P(x)\csc x\, dx$$

or

$$d[P'(x)\sin x] = k^2 P(x)\csc x \frac{dx}{dP} dP$$

which means that

$$[P'(x)\sin x]d[P'(x)\sin x] = k^2 P dP \tag{4.68}$$

Integrating Eq. (4.68), we get

$$[P'(x)\sin x]^2 = k^2 P^2 + c \tag{4.69}$$

There are three cases in solving Eq. (4.69).

For $c = 0$, Eq. (4.69) becomes

$$P'(x)\sin x = \pm k P(x)$$

or

$$\frac{P'(x)}{P(x)} = \pm k\csc x \tag{4.70}$$

Integrating Eq. (4.70), we get

$$\ln P(x) = \mp k\ln(\csc x + \cot x) + c_3$$

or

$$P(x) = c_3(\csc x + \cot x)^{\mp k} \tag{4.71}$$

Substituting equations (4.71) and (4.67) in Eq. (4.52) gives

$$\boxed{\tau(x,y) = c_3(\csc x + \cot x)^{\mp k}(c_1\sin ky + c_2\cos ky)} \tag{4.72}$$

Every value of k generates a solution, which leads to infinitely many solutions.

Substituting τ given by Eq. (4.72) in Eq. SE_3, we get

$$csc^2 x\, \xi_y \mp c_3 k(\csc x + \cot x)^{\mp k-1}(-\csc x\cot x - csc^2 x)(c_1\sin ky + c_2\cos ky) = 0$$

Simplifying, we get

$$\xi_y = \mp c_3 k(\csc x + \cot x)^{\mp k}\sin x\,(c_1\sin ky + c_2\cos ky) \tag{4.73}$$

Integrating Eq. (4.73) with respect to y, we get

$$\xi = \mp c_3(\csc x + \cot x)^{\mp k}\sin x\,(-c_1\cos ky + c_2\sin ky) + g(x) \tag{4.74}$$

Differentiating Eq. (4.74) with respect to x, we get

$$\xi_x = kc_3(\csc x + \cot x)^{\mp k-1}(-\csc x \cot x - csc^2 x)\sin x\,(-c_1 \cos ky + c_2 \sin ky)$$
$$\mp c_3(\csc x + \cot x)^{\mp k}\cos x\,(-c_1 \cos ky + c_2 \sin ky) + g'(x)$$

Simplifying, we have

$$\xi_x = kc_3(\csc x + \cot x)^{\mp k}(c_1 \cos ky - c_2 \sin ky)$$
$$\mp c_3(\csc x + \cot x)^{\mp k}\cos x\,(-c_1 \cos ky + c_2 \sin ky) + g'(x) \quad (4.75)$$

Differentiating τ given by Eq. (4.72) with respect to y , we get

$$\tau_y = kc_3(\csc x + \cot x)^{\mp k}(c_1 \cos ky - c_2 \sin ky) \quad (4.76)$$

Now substituting τ_y and ξ_x given by equations (4.76) and (4.75) in equation SE_4, we get

$$kc_3(\csc x + \cot x)^{\mp k}(c_1 \cos ky - c_2 \sin ky)$$
$$-kc_3(\csc x + \cot x)^{\mp k}(c_1 \cos ky - c_2 \sin ky)$$
$$\pm c_3(\csc x + \cot x)^{\mp k}\cos x\,(-c_1 \cos ky + c_2 \sin ky) - g'(x)$$
$$\mp c_3(\csc x + \cot x)^{\mp k}\cos x\,(-c_1 \cos ky + c_2 \sin ky) + \cot x\, g(x) = 0$$

Simplifying, we get

$$g'(x) - \cot x\, g(x) = 0 \quad (4.77)$$

Solving Eq. (4.77), the previous ode, gives

$$g(x) = c_4 \sin x \quad (4.78)$$

Substituting the function g given by Eq. (4.78) in Eq. (4.74), we get

$$\boxed{\xi = \mp c_3(\csc x + \cot x)^{\mp k}\sin x\,(-c_1 \cos ky + c_2 \sin ky) + c_4 \sin x} \quad (4.79)$$

Substituting τ_y given by Eq. (4.76) in Eq. (4.38), we get

$$\boxed{\begin{gathered}\phi = \frac{-2}{b}[kc_3(\csc x + \cot x)^{\mp k}(c_1 \cos ky - c_2 \sin ky) \\ \mp c_3(\csc x + \cot x)^{\mp k}\cos x\,(-c_1 \cos ky + c_2 \sin ky) + c_4 \cos x]\end{gathered}} \quad (4.80)$$

(iii) Constant is negative, say constant $= -k^2$

In this case, we are going to solve the system

$$\frac{P''(x)\sin^2 x + P'(x)\sin x \cos x}{P(x)} = -\frac{Q''(y)}{Q(y)} = -k^2$$

Solving

$$-\frac{Q''(y)}{Q(y)} = -k^2$$

gives

$$\boxed{Q(y) = k_1 e^{ky} + k_2 e^{-ky}} \tag{4.81}$$

Solving

$$\frac{P''(x)\sin^2 x + P'(x)\sin x\cos x}{P(x)} = -k^2,$$

implies

$$P''(x)\sin x + P'(x)\cos x = -k^2 P(x)\csc x$$

or

$$d[P'(x)\sin x] = -k^2 P(x)\csc x\, dx$$

or

$$d[P'(x)\sin x] = -k^2 P(x)\csc x \frac{dx}{dP} dP$$

which means that

$$[P'(x)\sin x]d[P'(x)\sin x] = -k^2 P dP \tag{4.82}$$

Integrating Eq. (4.82), we get

$$[P'(x)\sin x]^2 = -k^2 P^2 + c \tag{4.83}$$

For $c = 0$, we have no real solution except the trivial solution

$$P(x) = 0 \tag{4.84}$$

Substituting in Eq. (4.52) gives

$$\tau(x, y) = 0 \tag{4.85}$$

which was discussed in case (I) and we get the symmetry

$$X_2 = \sin x \frac{\partial}{\partial x} - \frac{2}{b}\cos x \frac{\partial}{\partial u}$$

For $c < 0$, we have no real solution for Eq. (4.83).

For $c > 0$, integrating Eq. (4.83) gives

$$P'(x)\sin x = \pm\sqrt{c - k^2 P^2}$$

or

$$\frac{P'(x)}{\sqrt{c - k^2 P^2}} = \pm \csc x \tag{4.86}$$

Integrating Eq. (4.86), we get

$$\frac{1}{k}\arctan\frac{kP}{\sqrt{c - k^2 P^2}} = \mp \ln(\csc x + \cot x) + c_3$$

or

$$\frac{kP}{\sqrt{c-k^2P^2}} = \mp \tan[k \ln(\csc x + \cot x) + c_3] \tag{4.87}$$

Squaring gives

$$\frac{k^2P^2}{c-k^2P^2} = \tan^2[k \ln(\csc x + \cot x) + c_3]$$

Solving for P, we get

$$P(x) = \pm\frac{\sqrt{c}}{k}\sin[k \ln(\csc x + \cot x) + c_3] \tag{4.88}$$

Substituting equations (4.88) and (4.81) in Eq. (4.52) gives

$$\boxed{\tau(x,y) = \pm\frac{\sqrt{c}}{k}\sin[k \ln(\csc x + \cot x) + c_3]\,(k_1e^{ky} + k_2e^{-ky})} \tag{4.89}$$

Every value of k generates a solution, which leads to infinitely many solutions.

Substituting τ given by Eq. (4.89) in SE_3, we get

$$csc^2x\,\xi_y + P'(x)\left(k_1e^{ky} + k_2e^{-ky}\right) = 0$$

Simplifying, we get

$$\xi_y = -P'(x)\sin^2 x\,(k_1e^{ky} + k_2e^{-ky}) \tag{4.90}$$

Integrating Eq. (4.90) with respect to y, we get

$$\xi = -\frac{1}{k}P'(x)\sin^2 x\,(k_1e^{ky} - k_2e^{-ky}) + g(x) \tag{4.91}$$

Differentiating Eq. (4.90) with respect to x, we get

$$\xi_x = -\frac{1}{k}P''(x)\sin^2 x\,(k_1e^{ky} - k_2e^{-ky})$$
$$-\frac{2}{k}P'(x)\sin x\cos x\,(k_1e^{ky} - k_2e^{-ky}) + g'(x) \tag{4.92}$$

Differentiating τ, given by Eq. (4.89), with respect to y gives

$$\tau_y = kP(x)(k_1e^{ky} - k_2e^{-ky}) \tag{4.93}$$

Now substituting ξ, τ_y and ξ_x, given by equations (4.91)-(4.93), in equation SE_4, we get

$$kP(x)(k_1e^{ky} - k_2e^{-ky}) + \frac{1}{k}P''(x)\sin^2 x\,(k_1e^{ky} - k_2e^{-ky})$$
$$+\frac{2}{k}P'(x)\sin x\cos x\,(k_1e^{ky} - k_2e^{-ky}) - g'(x)$$
$$-\frac{1}{k}P'(x)\sin x\cos x\,(k_1e^{ky} - k_2e^{-ky}) + \cot x\,g(x) = 0 \tag{4.94}$$

Using the relation

$$P''(x)\sin x + P'(x)\cos x = -k^2P(x)\csc x$$

in Eq. (4.93) , we get

$$g'(x) - \cot x\,g(x) = 0 \tag{4.95}$$

Solving the previous ode (4.95) gives

$$g(x) = c_4 \sin x \tag{4.96}$$

Using the function $g(x)$ given in Eq. (4.96) in Eq. (4.91) gives

$$\boxed{\xi = -\frac{1}{k} P^{'}(x) \sin^2 x \, (k_1 e^{ky} - k_2 e^{-ky}) + g(x) + c_4 \sin x} \tag{4.97}$$

Substituting τ_y given by Eq. (4.93) in Eq. (4.38), we get

$$\boxed{\begin{gathered} \phi = \frac{-2}{b} [-\frac{1}{k} P^{''}(x) \sin^2 x \, (k_1 e^{ky} - k_2 e^{-ky}) \\ -\frac{2}{k} P^{'}(x) \sin x \cos x \, (k_1 e^{ky} - k_2 e^{-ky}) + c_4 \cos x] \end{gathered}} \tag{4.98}$$

This implies infinite dimensional symmetry algebra

4.2. REDUCTIONS AND INVARIANT SOLUTIONS OF THE EQUATION ON SPHERE

4.2.1. Reduction of the general case $u_{xx} + (\cot x)u_x + (csc^2 x)u_{yy} = f(u)$

We are going to reduce the following equation

$$\boxed{u_{xx} + (\cot x)u_x + (csc^2 x)u_{yy} = f(u)} \tag{4.99}$$

This equation admits three dimensional algebra generated by

$$S_1 = \sin y \frac{\partial}{\partial x} + \cot x \cos y \frac{\partial}{\partial y},$$

$$S_2 = \cos y \frac{\partial}{\partial x} - \cot x \sin y \frac{\partial}{\partial y},$$

$$S_3 = \frac{\partial}{\partial y}$$

with commutation relations

	S_1	S_2	S_3
S_1	0	S_3	$-S_2$
S_2	$-S_3$	0	S_1
S_3	S_2	$-S_1$	0

4.2.1.1. Reduction Using $S_3 = \frac{\partial}{\partial y}$:

This symmetry can be written as

$$X = S_3 = 0\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + 0\frac{\partial}{\partial u}$$

From $\frac{dx}{0} = \frac{dy}{1}$, we have

$$x = c \tag{4.100}$$

and from $\frac{dy}{1} = \frac{du}{0}$, we get

$$u = k \tag{4.101}$$

So, we get two invariants of $X(I) = 0$ and are given by

$$\xi(x, y) = x \tag{4.102}$$

$$V(\xi) = u \tag{4.103}$$

Differentiating equations (4.102) and (4.103) gives

$$u_x = V' \tag{4.104}$$

$$u_{xx} = V'' \tag{4.105}$$

$$u_y = 0$$

$$u_{yy} = 0 \tag{4.106}$$

Substituting equations (4.104)-(4.106) in Eq. (4.99), we get the second order ODE

$$\boxed{V'' + (\cot\xi)V' = f(V)} \tag{4.107}$$

4.2.1.2. Reduction using $S_1 = \sin y \frac{\partial}{\partial x} + \cot x \cos y \frac{\partial}{\partial y}$

Let

$$X = S_1 = \sin y \frac{\partial}{\partial x} + \cot x \cos y \frac{\partial}{\partial y} + 0 \frac{\partial}{\partial u}.$$

The characteristic system of $X(I) = 0$ is given by

$$\frac{dx}{\sin y} = \frac{dy}{\cot x \cos y} = \frac{du}{0} \tag{4.108}$$

From system (4.108), we have

$$\frac{dx}{\sin y} = \frac{dy}{\cot x \cos y}$$

which implies

$$\cot x \, dx = \tan y \, dy$$

or

$$\ln \sin x = -\ln \cos y + c$$

or

$$\sin x \cos y = c \tag{4.109}$$

Also, system (4.108) implies

$$\frac{dy}{\cot x \cos y} = \frac{du}{0}$$

whose solution

$$u = k \tag{4.110}$$

From equations (4.109) and (4.110), two invariants of X are given by

$$\xi(x, y) = \sin x \cos y \tag{4.111}$$

$$V(\xi) = u \tag{4.112}$$

Then,

$$u_x = V' \cos x \cos y \tag{4.113}$$

$$u_{xx} = (V'' \cos x \cos y) \cos x \cos y - V' \sin x \cos y$$

$$= V'' \cos^2 x \cos^2 y - V' \sin x \cos y \tag{4.114}$$

$$u_y = -V' \sin x \sin y$$

$$u_{yy} = -(-V'' \sin x \sin y) \sin x \sin y - V' \sin x \cos y$$

$$= V'' \sin^2 x \sin^2 y - V' \sin x \cos y \tag{4.115}$$

Substituting equations (4.113)-(4.115) in Eq. (4.99) gives

$$V'' \cos^2 x \cos^2 y - V' \sin x \cos y + (\cot x) V' cos\, x \ cos\, y$$

$$+(\csc^2 x)\,[V'' \sin^2 x \sin^2 y - V' sin\, x \ cos\, y] = f(V)$$

or

$$V''[\cos^2 x \cos^2 y + \sin^2 y]$$

$$+V'\left[-\sin x \cos y + \frac{\cos^2 x}{\sin x}\cos y - \frac{1}{\sin x}\cos y\right] = f(V) \tag{4.116}$$

Using equations (4.111) and (4.112) in Eq. (4.116) gives

$$V''[1-\xi^2] + V'[-\xi + \xi \cot^2 x - \xi \csc^2 x] = f(V)$$

or simply

$$\boxed{V''\,[1-\xi^2] + V'[-2\xi] = f(V)} \tag{4.117}$$

4.2.1.3. Reduction using $S_2 = \cos y \frac{\partial}{\partial x} - \cot x \sin y \frac{\partial}{\partial y}$

Let

$$X = S_2 = \cos y \frac{\partial}{\partial x} - \cot x \sin y \frac{\partial}{\partial y} + 0 \frac{\partial}{\partial u}$$

The characteristic system of $X(I) = 0$ is given by

$$\frac{dx}{\cos y} = \frac{dy}{-\cot x \sin y} = \frac{du}{0} \tag{4.118}$$

From system (4.118), we have

$$\frac{dx}{\cos y} = \frac{dy}{-\cot x \sin y}$$

which implies

$$\cot x\, dx = -\cot y\, dy$$

or

$$\ln \sin x = -\ln \sin y + c$$

or

$$\sin x \sin y = c \tag{4.119}$$

Also, from the system (4.118), we have

$$\frac{dx}{\cos y} = \frac{du}{0}$$

which gives the solution

$$u = k \tag{4.120}$$

So, equations (4.119) and (4.120) give two invariants of the symmetry X and they are given by

$$\xi(x, y) = \sin x \sin y \tag{4.121}$$

$$V(\xi) = u \tag{4.122}$$

Then, from these invariants, we get

$$u_x = V' \cos x \sin y \tag{4.123}$$

$$\begin{aligned} u_{xx} &= (V'' \cos x \sin y) \cos x \sin y - V' \sin x \sin y \\ &= V'' \cos^2 x \sin^2 y - V' \sin x \sin y \end{aligned} \tag{4.124}$$

$$u_y = V' \sin x \cos y$$

$$\begin{aligned} u_{yy} &= (V'' \sin x \cos y) \sin x \cos y - V' \sin x \sin y \\ &= V'' \sin^2 x \cos^2 y - V' \sin x \sin y \end{aligned} \tag{4.125}$$

Substituting equations (4.123)-(4.125) in Eq. (4.99), we get

$$\begin{aligned} &V'' \cos^2 x \sin^2 y - V' \sin x \sin y + \cot x \, (V' \cos x \sin y) + \\ &\quad csc^2 x [V'' \sin^2 x \cos^2 y - V' \sin x \sin y] = f(V) \end{aligned}$$

or

$$V''[\cos^2 x \sin^2 y + \cos^2 y] + V' \left[-\sin x \sin y + \frac{\cos^2 x}{\sin x} \sin y - \frac{1}{\sin x} \sin y \right] = f(V) \tag{4.126}$$

Using equations (4.121) and (4.122) in Eq. (4.126), we get

$$V''[1 - \xi^2] + V'[-\xi + \xi \cot^2 x - \xi \csc^2 x] = f(V)$$

or simply

$$\boxed{V''[1 - \xi^2] + V'[-2\xi] = f(V)} \tag{4.127}$$

which is the same equation as the one obtained in the previous section.

4.2.2. Reduction of the equation when $f(u) = e^u + 2$

We are going to reduce the following equation

$$\boxed{u_{xx} + (\cot x)u_x + (csc^2 x)u_{yy} = e^u + 2} \tag{4.128}$$

This equation admits infinite dimensional algebra as discussed in section 4.1.

4.2.2.1. Reduction using $S_3 = \frac{\partial}{\partial y}$:

From Eq. (4.107) in the general case, Eq. (4.128) is reduced to the second order ODE

$$V'' + (\cot \xi)V' = e^u + 2 \tag{4.129}$$

where ξ and V are defined by equations (4.102) and (4.103)

This ode admits the symmetry

$$X = \sin \xi \frac{\partial}{\partial \xi} - 2 \cos \xi \frac{\partial}{\partial V}$$

whose first prolongation is given by

$$X^{[1]} = \sin \xi \frac{\partial}{\partial \xi} - 2 \cos \xi \frac{\partial}{\partial V} + (2 \sin \xi - V' \cos \xi) \frac{\partial}{\partial V'}$$

The characteristic system of $X^{[1]}(I) = 0$ is given by

$$\frac{d\xi}{\sin \xi} = \frac{dV}{-2 \cos \xi} = \frac{dV'}{2 \sin \xi - V' \cos \xi} \tag{4.130}$$

From

$$\frac{d\xi}{\sin \xi} = \frac{dV}{-2 \cos \xi},$$

we get

$$e^V \sin^2 \xi = constant \tag{4.131}$$

and from

$$\frac{d\xi}{\sin \xi} = \frac{dV'}{2 \sin \xi - V' \cos \xi},$$

we get

$$\frac{dV'}{d\xi} + V' \cot \xi = 2. \tag{4.132}$$

Integrating factor of ODE (4.132) can be found as follows

$$e^{\int \cot \xi d\xi} = e^{\ln \sin \xi} = \sin \xi$$

Multiplying ODE (4.132) by the integrating factor, we get

$$\sin\xi\frac{dV'}{d\xi}+V'\cos\xi=2\sin\xi \tag{4.133}$$

which implies that

$$\frac{d(V'\sin\xi)}{d\xi}=2\sin\xi \tag{4.134}$$

Integrating Eq. (4.134) gives

$$V'\sin\xi+2\cos\xi=c \tag{4.135}$$

So, from equations (4.131) and (4.135), we have found two invariants of $X^{[1]}$ and are given by

$$r=e^V\sin^2\xi \tag{4.136}$$

$$w=V'\sin\xi+2\cos\xi \tag{4.137}$$

From Eq. (4.137), we have

$$V'=w\csc\xi-2\cot\xi \tag{4.138}$$

Differentiating Eq. (4.138) with respect to ξ, we get

$$V''=\frac{dw}{d\xi}\csc\xi-w\csc\xi\cot\xi+2\csc^2\xi \tag{4.139}$$

Substituting V', V'' defined by equations (4.138) and (4.139) in the ODE (4.128), we get

$$\frac{dw}{d\xi}\csc\xi-w\csc\xi\cot\xi+2\csc^2\xi+\cot\xi\,(w\csc\xi-2\cot\xi)=e^V+2$$

Simplifying gives

$$\frac{dw}{d\xi}\csc\xi+2=e^V+2$$

or

$$\frac{dw}{d\xi}=e^V\sin\xi \tag{4.140}$$

Using Eq. (4.136), the preceding equation can be written in the form

$$\boxed{\frac{dw}{d\xi}=r\csc\xi} \tag{4.141}$$

Also, let us find $\frac{dr}{d\xi}$ from differentiating Eq. (4.136).

$$\frac{dr}{d\xi}=V'e^V\sin^2\xi+2e^V\sin\xi\cos\xi \tag{4.142}$$

Using equations (4.136) and (4.137) in the previous equation, we get

$$\frac{dr}{d\xi}=e^V\sin^2\xi\,(w\csc\xi-2\cot\xi)+2e^V\sin\xi\cos\xi$$

$$=rw\csc\xi-2r\cot\xi+2r\cot\xi$$

or

$$\boxed{\frac{dr}{d\xi} = rw \csc \xi} \tag{4.143}$$

Thus, dividing Eq. (4.141) over Eq. (4.143), we reduce the second order ODE (4.128) to the simple first order ODE

$$\frac{dw}{dr} = \frac{1}{w} \tag{4.144}$$

Integrating this ODE gives

$$w^2 = 2r + k \tag{4.145}$$

Returning to the variables ξ, V using equations (4.136) and (4.137)

$$V' \sin \xi + 2 \cos \xi = \pm\sqrt{2e^V \sin^2 \xi + k}$$

Returning to the original variables using equations (4.102) and (4.103), we get

$$\frac{du}{dx} \sin x + 2 \cos x = \pm\sqrt{2e^u \sin^2 x + k} \tag{4.146}$$

This ode admits the symmetry

$$X = \sin x \frac{\partial}{\partial x} - 2 \cos x \frac{\partial}{\partial u}$$

One invariant of this symmetry is given by

$$r = e^u \sin^2 x \tag{4.147}$$

Differentiating with respect to x gives the following

$$\frac{dr}{dx} = \frac{du}{dx} e^u \sin^2 x + 2e^u \sin x \cos x \tag{4.148}$$

Using equations (4.146) and (4.147) in Eq. (4.148), we get

$$\frac{dr}{dx} = \left(-2 \cot x \pm \csc x \sqrt{2r + k}\right) r + 2r \cot x$$

or

$$\frac{dr}{dx} = \pm r \csc x \sqrt{2r + k} \tag{4.149}$$

Which is first order separable ODE and can be written in the form

$$\frac{dr}{r\sqrt{2r+k}} = \pm \csc x \, dx \tag{4.150}$$

The ODE (4.150) consists of three cases depending on the sign of the constant k.

For $\boldsymbol{k = 0}$, integration of Eq.(4.150) gives

$$-\frac{\sqrt{2}}{\sqrt{r}} = \pm \ln(\csc x + \cot x) + c$$

Solving for r, we get

$$r = \frac{2}{[\ln(\csc x + \cot x) + c]^2} \tag{4.151}$$

Returning to the original variables, we get

$$e^u \sin^2 x = \frac{2}{[\ln(\csc x + \cot x) + c]^2}$$

Solving the previous equation for u give the exact solution of Eq. (4.128)

$$u = \ln\left|\frac{2\csc^2 x}{[\ln(\csc x + \cot x) + c]^2}\right|$$

or

$$\boxed{u = \ln 2 - 2\ln|\sin x| - 2\ln|\ln(\csc x + \cot x) + c|} \tag{4.152}$$

For **positive** value of k, we get

$$-\frac{2}{\sqrt{k}}\operatorname{arctanh}\frac{\sqrt{2r+k}}{\sqrt{k}} = \pm\ln(\csc x + \cot x) + c$$

or

$$\sqrt{2r+k} = \pm\sqrt{k}\tanh\left[\frac{\sqrt{k}}{2}\ln(\csc x + \cot x) + c\right]$$

Squaring, we get

$$r = \frac{k}{2}\left(\tanh^2\left[\frac{\sqrt{k}}{2}\ln(\csc x + \cot x) + c\right] - 1\right)$$

Returning to the original variables and simplifying, we get

$$e^u \sin^2 x = \frac{k}{2}\operatorname{sech}^2\left[\frac{\sqrt{k}}{2}\ln(\csc x + \cot x) + c\right]$$

This implies that

$$\boxed{u = \ln\left|\frac{k\csc^2 x}{2}\operatorname{sech}^2\left[\frac{\sqrt{k}}{2}\ln(\csc x + \cot x) + c\right]\right|} \tag{4.153}$$

is an exact solution of Eq. (4.128).

For **negative** value of k, we get

$$\frac{2}{\sqrt{-k}}\operatorname{arctanh}\frac{\sqrt{2r+k}}{\sqrt{-k}} = \pm\ln(\csc x + \cot x) + c$$

or

$$\sqrt{2r+k} = \pm\sqrt{-k}\tan\left[\frac{\sqrt{-k}}{2}\ln(\csc x + \cot x) + c\right]$$

Squaring, we get

$$r = \frac{-k}{2}\left(\tan^2\left[\frac{\sqrt{-k}}{2}\ln(\csc x + \cot x) + c\right] + 1\right)$$

Returning to the original variables, we get

$$e^u \sin^2 x = \frac{-k}{2}\sec^2\left[\frac{\sqrt{-k}}{2}\ln(\csc x + \cot x) + c\right]$$

which implies that

$$\boxed{u = \ln\left|\frac{-k\csc^2 x}{2}\sec^2\left[\frac{\sqrt{-k}}{2}\ln(\csc x + \cot x) + c\right]\right|} \tag{4.154}$$

is an exact solution of Eq. (4.128).

4.2.2.2. Reduction using $S_1 = \sin y\frac{\partial}{\partial x} + \cot x\cos y\frac{\partial}{\partial y}$

From the general case, section 4.2.1.2, the ODE (4.128) is reduced to

$$V''\,[1 - \xi^2\,] + V'[-2\xi] = e^V + 2 \tag{4.155}$$

Where

$$\xi(x, y) = \sin x\cos y \tag{4.111}$$

$$V(\xi) = u \tag{4.112}$$

The ode (4.155) admits the symmetry

$$X = (\xi^2 - 1)\frac{\partial}{\partial\xi} - 2\xi\frac{\partial}{\partial V}$$

The first prolongation of this symmetry is given by

$$X^{[1]} = (\xi^2 - 1)\frac{\partial}{\partial\xi} - 2\xi\frac{\partial}{\partial V} - 2(1 + \xi V')\frac{\partial}{\partial V'}$$

Characteristic system of $X^{[1]}(I) = 0$ is given by

$$\frac{d\xi}{\xi^2-1} = \frac{dV}{-2\xi} = \frac{dV'}{-2(1+\xi V')} \tag{4.156}$$

From

$$\frac{d\xi}{\xi^2-1} = \frac{dV}{-2\xi},$$

we get

$$\ln(\xi^2 - 1) = -V + c$$

which implies

$$e^V(\xi^2 - 1) = c \tag{4.157}$$

From

$$\frac{d\xi}{\xi^2-1} = \frac{dV'}{-2(1+\xi V')},$$

we get

$$\frac{dV'}{d\xi} + \frac{2\xi}{\xi^2-1}V' = -\frac{2}{\xi^2-1} \tag{4.158}$$

Multiplying ode (4.158) by its integrating factor $\xi^2 - 1$ gives

$$(\xi^2-1)\frac{dV'}{d\xi}+2\xi V'=-2$$

or

$$\frac{d[V'(\xi^2-1)]}{d\xi}=-2 \tag{4.159}$$

Integrating gives

$$V'(\xi^2-1)+2\xi=constant \tag{4.160}$$

So, we have found two invariants and are given by

$$r=e^V(\xi^2-1) \tag{4.161}$$

$$w=V'(\xi^2-1)+2\xi \tag{4.162}$$

From Eq. (4.162), we have

$$V'=\frac{1}{\xi^2-1}w-\frac{2\xi}{\xi^2-1} \tag{4.163}$$

Differentiating Eq. (4.163) with respect to ξ gives

$$V''=\frac{dw}{d\xi}\frac{1}{\xi^2-1}-w\frac{2\xi}{(\xi^2-1)^2}+\frac{2\xi^2+2}{(\xi^2-1)^2} \tag{4.164}$$

Using equations (4.163) and (4.164) in the ode (4.155), we get

$$-\frac{dw}{d\xi}+w\frac{2\xi}{(\xi^2-1)}-\frac{2\xi^2+2}{(\xi^2-1)}-\frac{2\xi}{\xi^2-1}w+\frac{4\xi^2}{\xi^2-1}=\frac{r}{\xi^2-1}+2$$

Simplifying, we get

$$\boxed{\frac{dw}{d\xi}=\frac{-r}{\xi^2-1}} \tag{4.165}$$

Also, differentiating Eq. (4.161) with respect to ξ gives

$$\frac{dr}{d\xi}=V'e^V(\xi^2-1)+2\xi e^V$$

Using equations (4.162) in the previous equation gives

$$\frac{dr}{d\xi}=e^V w$$

or

$$\boxed{\frac{dr}{d\xi}=\frac{rw}{\xi^2-1}} \tag{4.166}$$

Dividing Eq. (4.165) over Eq. (4.166) implies

$$\frac{dw}{dr}=\frac{-1}{w} \tag{4.167}$$

Integrating ode (4.167) gives

$$w^2=-2r+k \tag{4.168}$$

Returning to variables ξ, V using equations (4.161) and (4.162), we get

$$[V'(\xi^2 - 1) + 2\xi]^2 = -2e^V(\xi^2 - 1) + k \tag{4.169}$$

The ODE (4.169) admits the symmetry

$$X = (\xi^2 - 1)\frac{\partial}{\partial \xi} - 2\xi\frac{\partial}{\partial V}$$

One invariant variable of this symmetry is given by

$$r = e^V(\xi^2 - 1) \tag{4.170}$$

Using this variable reduces the ode (4.169) to the following

$$\frac{(\xi^2-1)^2}{r^2}\left(\frac{dr}{d\xi}\right)^2 = -2r + k$$

or

$$\frac{dr}{r\sqrt{-2r+k}} = \pm\frac{d\xi}{\xi^2-1} \tag{4.171}$$

There are three cases for the value of the constant k in the ode (4.171).

If $\boldsymbol{k = 0}$, integrating Eq. (4.171) gives

$$\frac{-2}{\sqrt{-2r}} = \pm \operatorname{arctanh}\xi + c$$

Squaring, we get

$$\frac{-2}{r} = (\operatorname{arctanh}\xi + c)^2$$

Solving for r, we get

$$r = \frac{-2}{(\operatorname{arctanh}\xi + c)^2} \tag{4.172}$$

Returning to the variables ξ, V gives the following

$$e^V(\xi^2 - 1) = \frac{-2}{(\operatorname{arctanh}\xi + c)^2}$$

Returning to u, x, y

$$e^u(\sin^2 x\cos^2 y - 1) = \frac{-2}{[\operatorname{arctanh}(\sin x\cos y) + c]^2}$$

This implies that

$$\boxed{u = \ln 2 - \ln(1 - \sin^2 x\cos^2 y) - 2\ln[\operatorname{arctanh}(\sin x\cos y) + c]} \tag{4.173}$$

which is an exact solution of the PDE (4.128).

For **positive** value of k, integrating Eq. (4.171) gives

$$\frac{-2}{\sqrt{k}}\operatorname{arctanh}\frac{\sqrt{-2r+k}}{\sqrt{k}} = \pm\operatorname{arctanh}\xi + c$$

or

$$\sqrt{-2r+k} = \sqrt{k}\tanh\left[\frac{\sqrt{k}}{2}\operatorname{arctanh}\xi + c\right]$$

Squaring, we get

$$r = -\frac{k}{2}\left(\tanh^2\left[\frac{\sqrt{k}}{2}\operatorname{arctanh}\xi + c\right] - 1\right)$$

Returning to the original variables, we get

$$e^u(\sin^2 x\cos^2 y - 1) = -\frac{k}{2}\left(\tanh^2\left[\frac{\sqrt{k}}{2}\operatorname{arctanh}(\sin x\cos y) + c\right] - 1\right)$$

or

$$\boxed{u = \ln\left|\frac{k}{2(1-\sin^2 x\cos^2 y)}\operatorname{sech}^2\left[\frac{\sqrt{k}}{2}\operatorname{arctanh}(\sin x\cos y) + c\right]\right|} \qquad (4.174)$$

For **negative** value of k, integrating Eq. (4.171) gives

$$\frac{2}{\sqrt{-k}}\arctan\frac{\sqrt{-2r+k}}{\sqrt{-k}} = \pm\operatorname{arctanh}\xi + c$$

or

$$\sqrt{-2r+k} = \sqrt{-k}\tan\left[\frac{\sqrt{-k}}{2}\operatorname{arctanh}\xi + c\right]$$

Squaring, we get

$$r = \frac{k}{2}\left(\tan^2\left[\frac{\sqrt{-k}}{2}\operatorname{arctanh}\xi + c\right] + 1\right)$$

Returning to the original variables, we get

$$e^u(\sin^2 x\cos^2 y - 1) = \frac{k}{2}\left(\tan^2\left[\frac{\sqrt{-k}}{2}\operatorname{arctanh}(\sin x\cos y) + c\right] + 1\right)$$

This implies the exact solution

$$\boxed{u = \ln\left|\frac{k}{2(1-\sin^2 x\cos^2 y)}\sec^2\left[\frac{\sqrt{-k}}{2}\operatorname{arctanh}(\sin x\cos y) + c\right]\right|} \qquad (4.175)$$

4.2.2.3. Reduction using $S_2 = \cos y\frac{\partial}{\partial x} - \cot x\sin y\frac{\partial}{\partial y}$

From the general case, section 4.2.1.3, the ODE (4.128) is reduced to

$$V''[1 - \xi^2] + V'[-2\xi] = e^V + 2 \qquad (4.176)$$

where

$$\xi(x,y) = \sin x\sin y \qquad (4.121)$$

$$V(\xi) = u \qquad (4.122)$$

which is the same equation as the one obtained in the previous section.

This leads to the three exact solutions

$$\boxed{u = \ln 2 - \ln(1 - \sin^2 x \sin^2 y) - 2\ln[\operatorname{arctanh}(\sin x \sin y) + c]}$$

$$\boxed{u = \ln\left|\frac{k}{2(1-\sin^2 x \sin^2 y)}\operatorname{sech}^2\left[\frac{\sqrt{k}}{2}\operatorname{arctanh}(\sin x \sin y) + c\right]\right|}\ ,\ k > 0$$

$$\boxed{u = \ln\left|\frac{k}{2(1-\sin^2 x \sin^2 y)}\sec^2\left[\frac{\sqrt{-k}}{2}\operatorname{arctanh}(\sin x \sin y) + c\right]\right|}\ ,\ k < 0$$

4.2.2.4. Reduction Using $X = \sin x \frac{\partial}{\partial x} - 2\cos x \frac{\partial}{\partial u}$:

Let

$$X = \sin x \frac{\partial}{\partial x} + 0\frac{\partial}{\partial y} - 2\cos x \frac{\partial}{\partial u}$$

The characteristic system of $X(I) = 0$ is given by

$$\frac{dx}{\sin x} = \frac{dy}{0} = \frac{du}{-2\cos x}$$

The equation

$$\frac{dx}{\sin x} = \frac{dy}{0}$$

gives the invariant of the symmetry X

$$\xi(x, y) = y \tag{4.177}$$

And the equation

$$\frac{dx}{\sin x} = \frac{du}{-2\cos x}$$

gives the invariant of X

$$V(\xi) = e^u \sin^2 x \tag{4.178}$$

Differentiating Eq. (4.178) with respect to x, we get

$$V'\xi_x = u_x e^u \sin^2 x + 2e^u \sin x \cos x$$

or

$$0 = u_x e^u \sin^2 x + 2e^u \sin x \cos x$$

which implies

$$u_x = -2\cot x \tag{4.179}$$

Differentiating Eq. (4.179) with respect to x, we get

$$u_{xx} = 2\csc^2 x \tag{4.180}$$

Differentiating Eq. (4.178) with respect to y, we get

$$V'\xi_y = u_y e^u \sin^2 x$$

Using Eq. (4.177), the previous equation implies

$$u_y = \frac{V'}{V}$$

Differentiating with respect to y, we get

$$u_{yy} = \frac{V''V-(V')^2}{V^2} \tag{4.181}$$

Substituting equations (4.179)-(4.181) in Eq. (4.128), we get the second order ODE

$$2\csc^2 x + (\cot x)(-2\cot x) + (\csc^2 x)\frac{V''V-(V')^2}{V^2} = (\csc^2 x)V + 2$$

which implies the following second order ode

$$\frac{V''V-(V')^2}{V^2} = V \tag{4.182}$$

ODE (4.182) implies

$$d\left(\frac{V'}{V}\right) = Vd\xi$$

Dividing by dV, we get

$$d\left(\frac{V'}{V}\right)/dV = V\frac{d\xi}{dV} = \frac{V}{V'}$$

or

$$\frac{V'}{V}d\left(\frac{V'}{V}\right) = dV \tag{4.183}$$

Integrating Eq. (4.183) gives

$$\left(\frac{V'}{V}\right)^2 = 2V + k$$

which implies

$$(V')^2 = 2V^3 + kV^2 \tag{4.184}$$

Taking the square root of Eq. (4.184), we get

$$\frac{dV}{V\sqrt{2V+k}} = \pm d\xi \tag{4.185}$$

The sign of the constant k divides the ODE (4.185) into three cases

For **positive** value of k, by integrating Eq. (4.185), we get

$$-\frac{2}{\sqrt{k}}\operatorname{arctanh}\frac{\sqrt{2V+k}}{\sqrt{k}} = \pm\xi + c \tag{4.186}$$

or

$$\sqrt{2V+k} = \pm\sqrt{k}\tanh\left[\frac{\sqrt{k}}{2}\xi + c\right] \tag{4.187}$$

Squaring, we get the solution of ode (4.182).

$$V = \frac{k}{2}\left(\tanh^2\left[\frac{\sqrt{k}}{2}\xi + c\right] - 1\right) \tag{4.188}$$

Returning to the original variables and simplifying, we get

$$e^u \sin^2 x = \frac{k}{2}\operatorname{sech}^2\left[\frac{\sqrt{k}}{2}y + c\right] \tag{4.189}$$

which implies the exact solution of Eq. (4.128) given by

$$\boxed{u = \ln\left|\frac{k\csc^2 x}{2}\operatorname{sech}^2\left[\frac{\sqrt{k}}{2}y + c\right]\right|} \tag{4.190}$$

For **negative** value of k, integrating Eq. (4.185) gives

$$\frac{2}{\sqrt{-k}}\operatorname{arctanh}\frac{\sqrt{2V+k}}{\sqrt{-k}} = \pm\xi + c \tag{4.191}$$

Simplifying Eq. (4.191), we get

$$\sqrt{2V+k} = \pm\sqrt{-k}\tan\left[\frac{\sqrt{-k}}{2}\xi + c\right]$$

Squaring, we get a solution of the ode (4.182)

$$V = \frac{-k}{2}\left(\tan^2\left[\frac{\sqrt{-k}}{2}\xi + c\right] + 1\right) \tag{4.192}$$

Returning to the original variables, we get

$$e^u \sin^2 x = \frac{-k}{2}\sec^2\left[\frac{\sqrt{-k}}{2}y + c\right]$$

which implies the exact solution of the PDE (4.128) given by

$$\boxed{u = \ln\left|\frac{-k\csc^2 x}{2}\sec^2\left[\frac{\sqrt{-k}}{2}y + c\right]\right|} \tag{4.194}$$

For $\boldsymbol{k = 0}$, the ode (4.185) becomes

$$\frac{dV}{V\sqrt{2V}} = \pm d\xi \tag{4.195}$$

Integrating Eq. (4.195) gives

$$-\frac{\sqrt{2}}{\sqrt{V}} = \pm\xi + c$$

Solving for V gives

$$V = \frac{2}{[\xi + c]^2} \tag{4.196}$$

Returning to the original variables, we get

$$e^u \sin^2 x = \frac{2}{[y+c]^2}$$

which implies the exact solution of the PDE (4.128) and is given by

$$u = \ln\left|\frac{2\csc^2 x}{[y+c]^2}\right|$$

Simplifying, we obtain

$$\boxed{u = \ln 2 - 2\ln|\sin x| - 2\ln|y + c|} \tag{4.197}$$

4.2.2.5. Reduction Using $A_1 = -y\sin x\,\frac{\partial}{\partial x} - \ln(\csc x + \cot x)\frac{\partial}{\partial y} + \frac{2y}{b}\cos x\frac{\partial}{\partial u}$:

Let

$$X = A_1 = -y\sin x\,\frac{\partial}{\partial x} - \ln(\csc x + \cot x)\frac{\partial}{\partial y} + 2y\cos x\frac{\partial}{\partial u}$$

The characteristic system of $X(I) = 0$ is given by

$$\frac{dx}{-y\sin x} = \frac{dy}{-\ln(\csc x + \cot x)} = \frac{du}{2y\cos x}$$

Solving

$$\frac{dx}{-y\sin x} = \frac{dy}{-\ln(\csc x + \cot x)}$$

leads to

$$-\ln(\csc x + \cot x)\csc x\,dx = -ydy$$

Integrating, we get the solution

$$[\ln(\csc x + \cot x)]^2 + y^2 = c \tag{4.198}$$

Solving

$$\frac{dx}{-y\sin x} = \frac{du}{2y\cos x}$$

implies

$$u = -2\ln(\sin x) + k$$

or

$$e^u\sin^2 x = k \tag{4.199}$$

We have found two invariant variables in equations (4.198)-(4.199) given by

$$\xi(x, y) = [\ln(\csc x + \cot x)]^2 + y^2 \tag{4.200}$$

$$V(\xi) = e^u\sin^2 x \tag{4.201}$$

From Eq. (4.200), we have

$$\xi_x = -2\csc x\ln(\csc x + \cot x) \tag{4.202}$$

$$\xi_y = 2y \tag{4.203}$$

Differentiating Eq. (4.201) with respect to x, we get

$$V'\xi_x = u_x e^u\sin^2 x + 2e^u\sin x\cos x$$

or

$$V'[-2\csc x\ln(\csc x + \cot x)] = u_x e^u\sin^2 x + 2e^u\sin x\cos x$$

Using Eq. (4.201) in the previous equation, we get

$$V'[-2\csc x\ln(\csc x + \cot x)] = u_x V + 2V\cot x$$

This implies that

$$u_x = \frac{V'}{V}[-2\csc x \ln(\csc x + \cot x)] - 2\cot x \tag{4.204}$$

Differentiating Eq. (4.204) with respect to x gives

$$u_{xx} = \frac{V''V-(V')^2}{V^2}[-2\csc x \ln(\csc x + \cot x)]^2$$
$$+\frac{V'}{V}[2\csc x \cot x \ln(\csc x + \cot x) + 2\csc^2 x] + 2\csc^2 x \tag{4.205}$$

Differentiating Eq. (4.201) with respect to y, we get

$$V'\xi_y = u_y e^u \sin^2 x$$

Using equations (4.200) and (4.201) in the previous equation, we get

$$V'[2y] = u_y V$$

This implies that

$$u_y = \frac{V'}{V}[2y]$$

Differentiating with respect to y gives

$$u_{yy} = \frac{V''V-(V')^2}{V^2}[2y]^2 + 2\frac{V'}{V} \tag{4.206}$$

Substituting equations (4.204)-(4.206) into the PDE (4.128), we get

$$\frac{V''V-(V')^2}{V^2}[-2\csc x \ln(\csc x + \cot x)]^2$$
$$+\frac{V'}{V}[2\csc x \cot x \ln(\csc x + \cot x) + 2\csc^2 x] + 2\csc^2 x$$
$$+(\cot x)\left(\frac{V'}{V}[-2\csc x \ln(\csc x + \cot x)] - 2\cot x\right)$$
$$+\csc^2 x\left(\frac{V''V-(V')^2}{V^2}[2y]^2 + 2\frac{V'}{V}\right) = V\csc^2 x + 2$$

Simplifying, we get

$$4\csc^2 x\frac{V''V-(V')^2}{V^2}([\ln(\csc x + \cot x)]^2 + [y]^2)$$
$$+\frac{V'}{V}(4\csc^2 x) + 2 = V\csc^2 x + 2$$

which implies

$$\boxed{4\xi\frac{V''V-(V')^2}{V^2} + 4\frac{V'}{V} = V} \tag{4.207}$$

This ODE admits the symmetry

$$X = \xi \frac{\partial}{\partial \xi} - V \frac{\partial}{\partial V}$$

The first prolongation of this symmetry is given by

$$X^{[1]} = \xi \frac{\partial}{\partial \xi} - V \frac{\partial}{\partial V} - 2V' \frac{\partial}{\partial V'}$$

This gives the similarity variables

$$r = \xi V \tag{4.208}$$

$$w = \xi^2 V' \tag{4.209}$$

From Eq. (4.209), we have

$$V' = \frac{w}{\xi^2}$$

Differentiating with respect to ξ, we get

$$V'' = \frac{\xi^2 \frac{dw}{d\xi} - 2\xi w}{\xi^4} = \frac{1}{\xi^3}\left(\xi \frac{dw}{d\xi} - 2w\right)$$

Substituting this in the ODE (4.207), we get

$$4\xi \frac{\frac{1}{\xi^3}\left(\xi \frac{dw}{d\xi} - 2w\right)\frac{r}{\xi} - \frac{w^2}{\xi^4}}{r^2/\xi^2} + 4\frac{w/\xi^2}{r/\xi} = \frac{r}{\xi}$$

or

$$4\frac{\left(\xi \frac{dw}{d\xi} - 2w\right)r - w^2}{\xi r^2} + 4\frac{w}{\xi r} = \frac{r}{\xi} \tag{4.210}$$

Multiplying Eq. (4.210) by ξ, we get

$$4\frac{\left(\xi \frac{dw}{d\xi} - 2w\right)r - w^2}{r^2} + 4\frac{w}{r} = r \tag{4.211}$$

Multiplying Eq. (4.211) by r^2, we get

$$4\left(\xi \frac{dw}{d\xi} - 2w\right)r - w^2 = r^3 - 4rw$$

or

$$\xi \frac{dw}{d\xi} = \frac{w^2 + r^3 - 4rw}{4r} + 2w$$

or

$$\xi \frac{dw}{d\xi} = \frac{w^2 + r^3 + 4rw}{4r} \tag{4.212}$$

Also, differentiating Eq. (4.208) with respect to ξ, we get

$$\frac{dr}{d\xi} = V + \xi V' \tag{4.213}$$

Multiplying Eq. (4.213) by ξ gives

$$\xi \frac{dr}{d\xi} = \xi V + \xi^2 V' = r + w \tag{4.214}$$

Dividing Eq. (4.212) over Eq. (4.214), we get

$$\frac{dw}{dr} = \frac{w^2+r^3+4rw}{4r(r+w)} \tag{4.215}$$

which is an Abel equation of second kind!

This result is deduced using the method of differential invariance. Next, we try the method of canonical coordinates.

Using canonical coordinate method

$$\boxed{4\xi \frac{V''V-(V')^2}{V^2} + 4\frac{V'}{V} = V} \tag{4.207}$$

We are going to solve ODE (4.207) using its symmetry

$$X = \xi \frac{\partial}{\partial \xi} - V \frac{\partial}{\partial V}$$

Let $R(\xi, V)$ and $S(\xi, V)$ be canonical coordinates of X. Then,

$$X(R) = 0, \tag{4.216}$$

$$X(S) = 1 \tag{4.217}$$

Solving Eq. (4.216) implies the characteristic equation

$$\frac{d\xi}{\xi} = -\frac{dV}{V}$$

or

$$\xi V = c$$

Thus,

$$R(\xi, V) = \xi V \tag{4.218}$$

is a solution of Eq. (4.216).

To find a particular solution of Eq. (4.217), let

$$S(\xi, V) = S(\xi)$$

Substituting this in Eq. (4.201) gives

$$\xi \frac{dS}{d\xi} = 1$$

which gives a solution

$$S(\xi, V) = \ln \xi \tag{4.219}$$

Suppose that S is the independent variable and $R(S)$ is the dependent variable.

Differentiating Eq. (4.219) with respect to ξ gives

$$\frac{dS}{d\xi} = \frac{1}{\xi} \tag{4.220}$$

From Eq. (4.219), we have

$$\xi(S) = e^S \tag{4.221}$$

From Eq. (4.218), we have

$$V = \frac{R}{\xi} \tag{4.222}$$

Differentiating Eq. (4.221) with respect to ξ gives

$$\frac{dV}{d\xi} = \frac{\xi\frac{dR}{d\xi} - R}{\xi^2} = \frac{\xi\frac{dR}{dS}\frac{dS}{d\xi} - R}{\xi^2}$$

Using Eq. (4.220) in the previous equation, we get

$$\frac{dV}{d\xi} = \frac{\frac{dR}{dS} - R}{\xi^2} = \frac{R' - R}{\xi^2} \tag{4.223}$$

Differentiating Eq. (4.223) with respect to ξ gives

$$\frac{d^2V}{d\xi^2} = \frac{\xi^2\frac{d}{d\xi}(R' - R) - 2\xi(R' - R)}{\xi^4}$$

$$= \frac{\xi^2\frac{d(R' - R)}{dS}\frac{dS}{d\xi} - 2\xi(R' - R)}{\xi^4}$$

$$= \frac{\xi^2\frac{d(R' - R)}{dS}\frac{1}{\xi} - 2\xi(R' - R)}{\xi^4}$$

Simplifying gives

$$\frac{d^2V}{d\xi^2} = \frac{\frac{d^2R}{dS^2} - \frac{dR}{dS} - 2R' + 2R}{\xi^3}$$

$$= \frac{\frac{d^2R}{dS^2} - 3\frac{dR}{dS} + 2R}{\xi^3}$$

or

$$\frac{d^2V}{d\xi^2} = \frac{R'' - 3R' + 2R}{\xi^3} \tag{4.224}$$

Substituting equations (4.222)-(4.224) in ODE (4.207), we get

$$4\xi\frac{\frac{R'' - 3R' + 2R}{\xi^3}\frac{R}{\xi} - \left(\frac{R' - R}{\xi^2}\right)^2}{\left(\frac{R}{\xi}\right)^2} + 4\frac{\frac{R' - R}{\xi^2}}{\frac{R}{\xi}} = \frac{R}{\xi}$$

Simplifying gives

$$4\frac{RR''-3RR'+2R^2-\left(R'-R\right)^2}{R^2}+4\frac{R'-R}{R}=R$$

which is an autonomous equation. Simplifying gives

$$4\frac{RR''-3RR'+2R^2-\left(R'^2-2RR'+R^2\right)}{R^2}+4\frac{R'-R}{R}=R$$

or

$$4\frac{RR''-R'^2}{R^2}=R$$

or

$$4RR''-4R'^2-R^3=0 \tag{4.225}$$

The ODE (4.225) admits the symmetry

$$X=\frac{\partial}{\partial S}$$

since it is autonomous.

The first prolongation is given by

$$X^{[1]}=\frac{\partial}{\partial S}+0\frac{\partial}{\partial R}+0\frac{\partial}{\partial R'}$$

The characteristic system of $X^{[1]}F=0$ is given by

$$\frac{dS}{1}=\frac{dR}{0}=\frac{dR'}{0}$$

which gives the invariants

$$P=R \tag{4.226}$$

$$Q=R' \tag{4.227}$$

From Eq. (4.227), we have

$$R''=\frac{dQ}{dS}=\frac{dQ}{dP}\frac{dP}{dS}=Q\frac{dQ}{dP} \tag{4.228}$$

Substituting equations (4.226)-(4.228) in Eq. (4.225) gives

$$4PQ\frac{dQ}{dP}-4Q^2-P^3=0$$

Dividing by $4PQ$, we get

$$\frac{dQ}{dP}-\frac{1}{P}Q=\frac{1}{4}P^2Q^{-1} \tag{4.229}$$

which is Bernoulli equation and can be solved easily.

Putting

$$U=Q^2, \tag{4.230}$$

we get

$$\frac{dU}{dP} = 2Q\frac{dQ}{dP} \tag{4.231}$$

Multiplying Eq. (4.229) by $2Q$, we have

$$2Q\frac{dQ}{dP} - \frac{2}{P}Q^2 = \frac{1}{2}P^2$$

Substituting (4.230) and (4.231) in (4.229), we get

$$\frac{dU}{dP} - \frac{2}{P}U = \frac{1}{2}P^2 \tag{4.232}$$

which is linear first order ODE with integrating factor

$$e^{\int -\frac{2}{P}dP} = e^{-2\ln P} = \frac{1}{P^2}$$

Multiplying Eq. (4.216) by $\frac{1}{P^2}$, we get

$$\frac{1}{P^2}\frac{dU}{dP} - \frac{2}{P^3}U = \frac{1}{2}$$

or

$$\frac{d\left(UP^{-2}\right)}{dP} = \frac{1}{2}$$

or

$$UP^{-2} = \frac{1}{2}P + c$$

Returning to variables P, Q gives

$$Q^2P^{-2} = \frac{1}{2}P + c$$

Returning to variables R, S gives

$$\left(\frac{dR}{ds}\right)^2 = R^2\left(\frac{1}{2}R + c\right)$$

or

$$\frac{dR}{dS} = \pm R\sqrt{\frac{1}{2}R + c}$$

or

$$\frac{dR}{R\sqrt{\frac{1}{2}R+c}} = \pm dS \tag{4.233}$$

Integrating gives three cases

Case 1 $(\boldsymbol{c = 0})$

In this case, Eq. (4.233) becomes

$$\int \frac{dR}{R\sqrt{\frac{1}{2}R}} = \pm\int dS$$

or

$$\frac{-2\sqrt{2}}{\sqrt{R}} = \pm S + k$$

or

$$R = \frac{8}{(S+k)^2}$$

Returning to variables ξ, V using equations (4.202) and (4.203) gives

$$\xi V = \frac{8}{(\ln \xi + k)^2}$$

which implies

$$\boxed{V = \frac{8}{\xi(\ln \xi + k)^2}} \tag{4.234}$$

Returning to the original variables using equations (4.200) and (4.201) gives

$$([\ln(\csc x + \cot x)]^2 + y^2)e^u \sin^2 x = \frac{8}{(k + \ln([\ln(\csc x + \cot x)]^2 + y^2))^2}$$

which implies

$$\boxed{u = \ln \frac{8 \csc^2 x}{([\ln(\csc x + \cot x)]^2 + y^2)(k + \ln([\ln(\csc x + \cot x)]^2 + y^2))^2}} \tag{4.235}$$

or

$$\boxed{u = \ln 8 - 2\ln(\sin x) - \ln([\ln(\csc x + \cot x)]^2 + y^2) - 2\ln(k + \ln([\ln(\csc x + \cot x)]^2 + y^2))}$$

Case 2 ($\boldsymbol{c} > 0$)

In this case, integration of Eq. (4.233) gives

$$\frac{-2}{\sqrt{c}} \operatorname{arctanh} \frac{\sqrt{2R + 4c}}{2\sqrt{c}} = \pm S + k$$

or

$$R = 2c\left[\tanh^2 \frac{\sqrt{c}(S+k)}{2} - 1\right] = 2c \operatorname{sech}^2 \frac{\sqrt{c}(S+k)}{2}$$

Returning to variables ξ, V using equations (4.218) and (4.219) gives

$$\xi V = 2c \operatorname{sech}^2 \frac{\sqrt{c}(\ln \xi + k)}{2}$$

which implies

$$\boxed{V = \frac{2c}{\xi} \operatorname{sech}^2 \frac{\sqrt{c}(\ln \xi + k)}{2}} \tag{4.236}$$

Returning to the original variables using equations (4.200) and (4.201) gives

$$e^u \sin^2 x = \frac{2c}{[\ln(\csc x + \cot x)]^2 + y^2} \operatorname{sech}^2 \frac{\sqrt{c}(k + \ln([\ln(\csc x + \cot x)]^2 + y^2))}{2}$$

which implies the exact solution of Eq. (4.128) given by

$$\boxed{u = \ln\left[\frac{2c\csc^2 x}{[\ln(\csc x+\cot x)]^2+y^2}\operatorname{sech}^2\frac{\sqrt{c}(k+\ln([\ln(\csc x+\cot x)]^2+y^2))}{2}\right]} \tag{4.237}$$

Case 3 ($\boldsymbol{c} < 0$)

In this case, integration of Eq. (4.233) gives

$$\frac{-2}{\sqrt{-c}}\arctan\frac{\sqrt{2R+4c}}{2\sqrt{-c}} = \pm S + k$$

or

$$R = -2c\left[\tan^2\frac{\sqrt{-c}(S+k)}{2} + 1\right] = -2c\sec^2\frac{\sqrt{-c}(S+k)}{2}$$

Returning to variables ξ, V using equations (4.218) and (4.219) gives

$$\xi V = -2c\sec^2\frac{\sqrt{-c}(\ln\xi+k)}{2}$$

which implies

$$\boxed{V = \frac{-2c}{\xi}\sec^2\frac{\sqrt{-c}(\ln\xi+k)}{2}} \tag{4.238}$$

Returning to the original variables using equations (4.200) and (4.201) gives

$$e^u\sin^2 x = \frac{-2c}{[\ln(\csc x+\cot x)]^2+y^2}\sec^2\frac{\sqrt{-c}(k+\ln([\ln(\csc x+\cot x)]^2+y^2))}{2}$$

which gives an exact solution of the PDE (4.128) and is given by

$$\boxed{u = \ln\left[\frac{-2c\csc^2 x}{[\ln(\csc x+\cot x)]^2+y^2}\sec^2\frac{\sqrt{-c}(k+\ln([\ln(\csc x+\cot x)]^2+y^2))}{2}\right]} \tag{4.239}$$

CHAPTER 5

GROUP CLASSIFICATION AND SYMMETRY REDUCTIONS FOR A CLASS OF NONLINEAR POISSON EQUATIONS ON HELICOID

The aim of this chapter is to study the complete group classification problem and some symmetry reductions of the nonlinear Poisson equation on the helicoid which is given by

$$u_{xx} + \frac{x}{x^2+b^2} u_x + \frac{1}{x^2+b^2} u_{yy} = f(u) \tag{5.1}$$

where $f(u)$ is an arbitrary nonlinear function.

The group classification of equation (5.1) is carried out in section 5.1. It is based on necessary conditions on $f(u)$ which are obtained through a triangulation of determining equations of Lie symmetries of equation (5.1). An efficient method to obtain such triangulation is the well-known method of Mansfield [29] of generating differential Grobner bases of determining equations. A variant of Mansfield's method will be used to generate cases of $f(u)$ and hence for carrying out the group classification. Precisely, the following result will be proved.

Theorem 5.1

The minimal symmetry algebra of nonlinear Poisson equation (5.1) is one-dimensional generated by

$$H_1 = \frac{\partial}{\partial y},$$

and is obtained for all nonlinear arbitrary functions $f(u)$. Three dimensional symmetry algebra exists in the case

$$f(u) = ae^{Bu}\ ,\ a, B \neq 0$$

which is generated by

$$H_1 = \frac{\partial}{\partial y},$$

$$H_2 = (x^2 + b^2)\cos y \frac{\partial}{\partial x} + x \sin y \frac{\partial}{\partial y} - \frac{4x}{B}\cos y \frac{\partial}{\partial u}$$

$$H_3 = (x^2 + b^2)\sin y \frac{\partial}{\partial x} - x \cos y \frac{\partial}{\partial y} - \frac{4x}{B}\sin y \frac{\partial}{\partial u}$$

This theorem is summarized in the table below:

$f(u)$	Generators of symmetry algebra
any non-linear function	$H_1 = \frac{\partial}{\partial y}$
ae^{Bu} , $a, b \neq 0$	Three dimensional algebra generated by $H_1 = \frac{\partial}{\partial y}$ $H_2 = (x^2 + b^2)\cos y \frac{\partial}{\partial x} + x \sin y \frac{\partial}{\partial y} - \frac{4x}{B}\cos y \frac{\partial}{\partial u}$ $H_3 = (x^2 + b^2)\sin y \frac{\partial}{\partial x} - x \cos y \frac{\partial}{\partial y} - \frac{4x}{B}\sin y \frac{\partial}{\partial u}$

Table (5-1)

For different cases obtained in section 5.1, some examples of reductions of Eq. (5.1) to ordinary differential equations are presented in section 5.2.

5.1. GROUP CLASSIFICATION OF POISSON EQUATION ON HELICOID

In order to obtain the Lie symmetries of equation (5.1), let us consider the one parameter group of infinitesimal transformations

$$x^* = x + \epsilon\, \xi(x, y, u) + O(\epsilon^2) \tag{5.2}$$

$$y^* = y + \epsilon\, \tau(x, y, u) + O(\epsilon^2) \tag{5.3}$$

$$u^* = u + \epsilon\, \phi(x, y, u) + O(\epsilon^2) \tag{5.4}$$

where ϵ is the group parameter. The generator corresponding to the given Lie algebra is of the form

$$X = \xi(x,y,u)\frac{\partial}{\partial x} + \tau(x,y,u)\frac{\partial}{\partial y} + \phi(x,y,u)\frac{\partial}{\partial u} \tag{5.5}$$

If $X^{[2]}$ denotes the second prolongation of X then the invariance condition

$$X^{[2]}\left(u_{xx} + \frac{x}{x^2+b^2}u_x + \frac{1}{x^2+b^2}u_{yy} - f(u)\right)\Big|_{u_{xx}+\frac{x}{x^2+b^2}u_x+\frac{1}{x^2+b^2}u_{yy}=f(u)} = 0 \tag{5.6}$$

is used to find the symmetries of Eq. (5.1). Here

$$X^{[2]} = X + \eta^{[x]}\frac{\partial}{\partial u_x} + \eta^{[xx]}\frac{\partial}{\partial u_{xx}} + \eta^{[yy]}\frac{\partial}{\partial u_{yy}}$$

whose coefficients in terms of ξ, τ and ϕ are given by

$$\eta^{[x]} = \phi_x + (\phi_u - \xi_x)u_x - \xi_u u_x^2 - \tau_x u_y - \tau_u u_x u_y \tag{5.7}$$

$$\begin{aligned}\eta^{[xx]} = {} & \phi_{xx} + (2\phi_{xu} - \xi_{xx})u_x + (\phi_{uu} - 2\xi_{xu})u_x^2 - \xi_{uu}u_x^3 - \tau_{xx}\,u_y \\ & -2\tau_{xu}u_x u_y - \tau_{uu}u_x^2 u_y + (\phi_u - 2\xi_x)u_{xx} - 3\xi_u u_{xx}u_x \\ & -2\tau_x u_{xy} - 2\tau_u u_x u_{xy}\end{aligned} \tag{5.8}$$

$$\begin{aligned}\eta^{[yy]} = {} & \phi_{yy} + \left(2\phi_{yu} - \tau_{yy}\right)u_y + \left(\phi_{uu} - 2\tau_{yu}\right)u_y^2 - \tau_{uu}u_y^3 \\ & -\xi_{yy}u_x - 2\xi_{yu}u_x u_y - \xi_{uu}u_y^2 u_x + \left(\phi_u - 2\tau_y\right)u_{yy} \\ & -3\tau_u u_{yy}u_y - \xi_u u_{yy}u_x - 2\xi_y u_{xy} - 2\xi_u u_y u_{xy}\end{aligned} \tag{5.9}$$

Using equations (5.7)-(5.9), we have

$$\begin{aligned}& X^{[2]}\left(u_{xx} + \frac{x}{x^2+b^2}u_x + \frac{1}{x^2+b^2}u_{yy} - f(u)\right)\Big|_{(5.1)} \\ & = \xi\frac{\partial}{\partial x}\left(\frac{x}{x^2+b^2}u_x + \frac{1}{x^2+b^2}u_{yy}\right) - \phi f_u + \eta^{[xx]} + \frac{x}{x^2+b^2}\eta^{[x]} + \frac{1}{x^2+b^2}\eta^{[yy]} \\ & = \xi\left(\frac{b^2-x^2}{\left(x^2+b^2\right)^2}u_x - \frac{2x}{\left(x^2+b^2\right)^2}u_{yy}\right) - \phi f_u + \eta^{[xx]} + \frac{x}{x^2+b^2}\eta^{[x]} + \frac{1}{x^2+b^2}\eta^{[yy]} \\ & = \xi\left(\frac{b^2-x^2}{\left(x^2+b^2\right)^2}u_x - \frac{2x}{\left(x^2+b^2\right)^2}u_{yy}\right) - \phi f_u \\ & \quad +[\phi_{xx} + (2\phi_{xu} - \xi_{xx})u_x + (\phi_{uu} - 2\xi_{xu})u_x^2 - \xi_{uu}u_x^3 \\ & \quad -\tau_{xx}u_y - 2\tau_{xu}u_x u_y - \tau_{uu}u_x^2 u_y + (\phi_u - 2\xi_x)u_{xx} \\ & \quad -3\xi_u u_{xx}u_x - \tau_u u_{xx}u_y - 2\tau_x u_{xy} - 2\tau_u u_x u_{xy}] \\ & \quad +\frac{x}{x^2+b^2}\{\phi_x + (\phi_u - \xi_x)u_x - \xi_u u_x^2 - \tau_x u_y - \tau_u u_x u_y\} \\ & \quad +\frac{1}{x^2+b^2}[\phi_{yy} + \left(2\phi_{yu} - \tau_{yy}\right)u_y + \left(\phi_{uu} - 2\tau_{yu}\right)u_y^2 - \tau_{uu}u_y^3 - \xi_{yy}u_x \\ & \quad -2\xi_{yu}u_x u_y - \xi_{uu}u_y^2 u_x + \left(\phi_u - 2\tau_y\right)u_{yy} - 3\tau_u u_{yy}u_y \\ & \quad -\xi_u u_{yy}u_x - 2\xi_y u_{xy} - 2\xi_u u_y u_{xy}]\end{aligned} \tag{5.10}$$

Putting the condition

$$u_{yy} = \left(\left(x^2+b^2\right)f(u) - \left(x^2+b^2\right)u_{xx} - xu_x\right)$$

in the previous expression (5.10), we get

$$X^{[2]}\left(u_{xx} + \frac{x}{x^2+b^2}u_x + \frac{1}{x^2+b^2}u_{yy} - f(u)\right)\Big|_{(5.1)}$$

$$= \xi\left(\frac{b^2-x^2}{(x^2+b^2)^2}u_x - \frac{2x}{(x^2+b^2)^2}\left((x^2+b^2)f(u) - (x^2+b^2)u_{xx} - xu_x\right)\right) - \phi f_u$$

$$+[\phi_{xx} + (2\phi_{xu} - \xi_{xx})u_x + (\phi_{uu} - 2\xi_{xu})u_x^2 - \xi_{uu}u_x^3 - \tau_{xx}u_y$$

$$-2\tau_{xu}u_xu_y - \tau_{uu}u_x^2u_y + (\phi_u - 2\xi_x)u_{xx} - 3\xi_u u_{xx}u_x - \tau_u u_{xx}u_y$$

$$-2\tau_x u_{xy} - 2\tau_u u_x u_{xy}] + \frac{x}{x^2+b^2}\{\phi_x + (\phi_u - \xi_x)u_x - \xi_u u_x^2 - \tau_x u_y - \tau_u u_x u_y\}$$

$$+\frac{1}{x^2+b^2}[\phi_{yy} + \left(2\phi_{yu} - \tau_{yy}\right)u_y + \left(\phi_{uu} - 2\tau_{yu}\right)u_y^2 - \tau_{uu}u_y^3 - \xi_{yy}u_x$$

$$-2\xi_{yu}u_xu_y - \xi_{uu}u_y^2u_x + \left(\phi_u - 2\tau_y\right)\left((x^2+b^2)f(u) - (x^2+b^2)u_{xx} - xu_x\right)$$

$$-3\tau_u\left((x^2+b^2)f(u) - (x^2+b^2)u_{xx} - xu_x\right)u_y$$

$$-\xi_u\left((x^2+b^2)f(u) - (x^2+b^2)u_{xx} - xu_x\right)u_x - 2\xi_y u_{xy} - 2\xi_u u_y u_{xy}]$$

$$= \xi\frac{1}{x^2+b^2}u_x - \xi\frac{2x}{x^2+b^2}f + \xi\frac{2x}{x^2+b^2}u_{xx} - \phi f_u + [\phi_{xx} + (2\phi_{xu} - \xi_{xx})u_x$$

$$+(\phi_{uu} - 2\xi_{xu})u_x^2 - \xi_{uu}u_x^3 - \tau_{xx}u_y - 2\tau_{xu}u_xu_y - \tau_{uu}u_x^2u_y$$

$$+(\phi_u - 2\xi_x)u_{xx} - 3\xi_u u_{xx}u_x - \tau_u u_{xx}u_y - 2\tau_x u_{xy} - 2\tau_u u_x u_{xy}]$$

$$+\left\{\frac{x}{x^2+b^2}\phi_x + \frac{x}{x^2+b^2}(\phi_u - \xi_x)u_x - \frac{x}{x^2+b^2}\xi_u u_x^2 - \frac{x}{x^2+b^2}\tau_x u_y - \frac{x}{x^2+b^2}\tau_u u_x u_y\right\}$$

$$+\frac{1}{x^2+b^2}[\phi_{yy} + \left(2\phi_{yu} - \tau_{yy}\right)u_y + \left(\phi_{uu} - 2\tau_{yu}\right)u_y^2 - \tau_{uu}u_y^3 - \xi_{yy}u_x$$

$$-2\xi_{yu}u_xu_y - \xi_{uu}u_y^2u_x + \left(\phi_u - 2\tau_y\right)(x^2+b^2)f(u) - \left(\phi_u - 2\tau_y\right)(x^2+b^2)u_{xx}$$

$$-\left(\phi_u - 2\tau_y\right)xu_x - 3\tau_u\left(x^2+b^2\right)f(u)u_y + 3\tau_u\left(x^2+b^2\right)u_yu_{xx} + 3x\tau_u u_x u_y$$

$$-(x^2+b^2)f(u)\xi_u u_x + (x^2+b^2)\xi_u u_x u_{xx} + x\xi_u u_x^2 - 2\xi_y u_{xy} - 2\xi_u u_y u_{xy}]$$

$$= \xi\frac{1}{x^2+b^2}u_x + (2\phi_{xu} - \xi_{xx})u_x - \frac{1}{x^2+b^2}\xi_{yy}u_x - \xi_u f u_x - \xi\frac{2x}{(x^2+b^2)}f + \xi\frac{2x}{(x^2+b^2)}u_{xx}$$

$$-\phi f_u + [\phi_{xx} + (\phi_{uu} - 2\xi_{xu})u_x^2 - \xi_{uu}u_x^3 - \tau_{xx}u_y - 2\tau_{xu}u_xu_y - \tau_{uu}u_x^2u_y$$

$$+(\phi_u - 2\xi_x)u_{xx} - 3\xi_u u_{xx}u_x - \tau_u u_{xx}u_y - 2\tau_x u_{xy} - 2\tau_u u_x u_{xy}]$$

$$+\{\frac{x}{x^2+b^2}\phi_x - \frac{x}{x^2+b^2}\xi_u u_x^2 - \frac{x}{x^2+b^2}\tau_x u_y - \frac{x}{x^2+b^2}\tau_u u_x u_y\} + [\frac{1}{x^2+b^2}\phi_{yy}$$

$$+\frac{1}{x^2+b^2}\left(2\phi_{yu} - \tau_{yy}\right)u_y + \frac{1}{x^2+b^2}\left(\phi_{uu} - 2\tau_{yu}\right)u_y^2 - \frac{1}{x^2+b^2}\tau_{uu}u_y^3 - \frac{2}{x^2+b^2}\xi_{yu}u_xu_y$$

$$-\frac{1}{x^2+b^2}\xi_{uu}u_y^2u_x + \left(\phi_u - 2\tau_y\right)f - \left(\phi_u - 2\tau_y\right)u_{xx} - 3\tau_u f u_y + 3\tau_u u_y u_{xx}$$

$$+\frac{3}{x^2+b^2}x\tau_u u_x u_y + \xi_u u_x u_{xx} + \frac{1}{x^2+b^2}x\xi_u u_x^2 - \frac{2}{x^2+b^2}\xi_y u_{xy} - \frac{2}{x^2+b^2}\xi_u u_y u_{xy}]$$

or

$$X^{[2]}\left(u_{xx}+\frac{x}{x^2+b^2}u_x+\frac{1}{x^2+b^2}u_{yy}-f(u)\right)\Big|_{(5.1)}$$

$$\begin{aligned}
&= -\xi_{uu}u_x^3-\tau_{uu}u_x^2u_y+\left(\phi_{uu}-2\xi_{xu}\right)u_x^2\\
&\quad +u_x\left(\xi\frac{1}{x^2+b^2}+\left(2\phi_{xu}-\xi_{xx}\right)-\frac{1}{x^2+b^2}\xi_{yy}-f\xi_u\right)\\
&\quad -2\tau_{xu}u_xu_y-\frac{x}{x^2+b^2}\tau_uu_xu_y+\frac{3x}{x^2+b^2}\tau_uu_xu_y-\frac{2}{x^2+b^2}\xi_{yu}u_xu_y\\
&\quad -\frac{1}{x^2+b^2}\xi_{uu}u_y^2u_x-2\xi_uu_{xx}u_x+3\tau_uu_yu_{xx}-\tau_uu_{xx}u_y\\
&\quad -\left(\phi_u-2\tau_y\right)u_{xx}+\left(\phi_u-2\xi_x\right)u_{xx}+\xi\frac{2x}{\left(x^2+b^2\right)}u_{xx}\\
&\quad -2\tau_uu_xu_{xy}-2\frac{1}{x^2+b^2}\xi_yu_{xy}-2\tau_xu_{xy}-2\frac{1}{x^2+b^2}\xi_uu_yu_{xy}\\
&\quad +\frac{1}{x^2+b^2}\left(\phi_{uu}-2\tau_{yu}\right)u_y^2-\tau_{uu}\frac{1}{x^2+b^2}u_y^3\\
&\quad -\tau_{xx}u_y-\frac{x}{x^2+b^2}\tau_xu_y+\frac{1}{x^2+b^2}\left(2\phi_{yu}-\tau_{yy}\right)u_y-3\tau_uf u_y\\
&\quad +\left(\phi_u-2\tau_y-\xi\frac{2x}{x^2+b^2}\right)f-\phi f_u+\phi_{xx}+\frac{1}{x^2+b^2}\phi_{yy}+\frac{x}{x^2+b^2}\phi_x=0 \qquad (5.11)
\end{aligned}$$

From Eq. (5.11), we get the following **determining equations**:

$$\boldsymbol{HE_1}\ :\quad \boldsymbol{\xi_u=0}$$

$$\boldsymbol{HE_2}\ :\quad \boldsymbol{\tau_u=0}$$

$$\boldsymbol{HE_3}\ :\quad \boldsymbol{\frac{1}{x^2+b^2}\xi_y+\tau_x=0}$$

$$\boldsymbol{HE_4}\ :\quad \boldsymbol{\tau_y-\xi_x+\frac{x}{x^2+b^2}\xi=0}$$

$$\boldsymbol{HE_5}\ :\quad \boldsymbol{\phi_{uu}=0}$$

$$\boldsymbol{HE_6}\ :\quad \boldsymbol{2\phi_{yu}-\left(x^2+b^2\right)\tau_{xx}-x\tau_x-\tau_{yy}=0}$$

$$\boldsymbol{HE_7}\ :\quad \boldsymbol{2\phi_{xu}-\xi_{xx}+\frac{x}{x^2+b^2}\xi_x+\frac{b^2-x^2}{\left(x^2+b^2\right)^2}\xi-\frac{1}{x^2+b^2}\xi_{yy}=0}$$

$$\boldsymbol{HE_8}\ :\quad \boldsymbol{\phi_{xx}+\frac{1}{x^2+b^2}\phi_{yy}+\frac{x}{x^2+b^2}\phi_x+f\phi_u-2f\xi_x-\phi f_u=0}$$

The system of determining equations will be solved utilizing a triangulation procedure based on techniques for obtaining differential Groebner basis developed by Mansfield in [29].

The operation $(HE_3)_y-(HE_4)_x$ gives

$$\boldsymbol{\xi_{xx}+\frac{1}{x^2+b^2}\xi_{yy}-\frac{x}{x^2+b^2}\xi_x-\frac{b^2-x^2}{\left(x^2+b^2\right)^2}\xi=0} \qquad (5.12)$$

This reduces HE_7 to

$$\phi_{xu} = 0 \tag{5.13}$$

Also, the operation $(HE_3)_x + (HE_4)_y$ implies

$$(\boldsymbol{x^2} + \boldsymbol{b^2})\tau_{xx} + \tau_{yy} + x\tau_x = 0 \tag{5.14}$$

This simplifies Eq. HE_6 to

$$\phi_{yu} = 0 \tag{5.15}$$

Differentiating Eq. HE_8 with respect to u, we get

$$f_u\phi_u - 2f_u\xi_x - \phi_u f_u - \phi f_{uu} = 0$$

which is simplified to

$$f_{uu}\phi + 2f_u\xi_x = 0 \tag{5.16}$$

Differentiating Eq. (5.16) with respect to u, we get

$$f_{uuu}\phi + f_{uu}\phi_u + 2f_{uu}\xi_x = 0 \tag{5.17}$$

Differentiating Eq. (5.17) with respect to u, we get

$$f_{uuuu}\phi \ + 2f_{uuu}\phi_u + 2f_{uuu}\xi_x = 0 \tag{5.18}$$

Simplifying Eq. (5.18) with respect to Eq. (5.17) via

$$f_{uu}(5.18) - 2f_{uuu}(5.17)$$

implies

$$\phi(f_{uu}f_{uuuu} - 2f_{uuu}^2) - 2f_{uu}f_{uuu}\xi_x = 0 \tag{5.19}$$

Equations (5.18) and (5.19) can be used to eliminate ϕ, giving the equation

$$(f_{uu}f_{uuu} - 2f_{uuu}^2)(5.18) - f_{uu}(5.19)$$

The previous operation implies

$$[f_u(f_{uu}f_{uuu} - 2f_{uuu}^2) + f_{uu}^2 f_{uuu}]\xi_x = 0$$

Simplifying, we get

$$\boxed{\boxed{[\, f_u f_{uu} f_{uuuu} - 2f_u f_{uuu}^2 + f_{uu}^2 f_{uuu}]\xi_x = 0}} \tag{5.20}$$

Differentiating Eq. (5.16) with respect to x, we get

$$f_{uu}\phi_x + 2f_u\xi_{xx} = 0 \tag{5.21}$$

Differentiating Eq. (5.21) with respect to u, we get

$$f_{uuu}\phi_x + 2f_{uu}\xi_{xx} = 0 \tag{5.22}$$

Eliminating ϕ_x using equations (5.21) and (5.22), leads to

$$f_{uu}(5.22) - f_{uuu}(5.21)$$

which implies

$$\boxed{\boxed{(f_{uu}^2 - f_u f_{uuu})\xi_{xx} = 0}} \tag{5.23}$$

Next, we look at possibilities of $f(u)$

If $\boldsymbol{\xi_x = 0}$, then equations (5.20) and (5.23) are satisfied for any type of functions $f(u)$.

Equation (5.16) and the assumption

$$\xi_x = 0 \tag{5.24}$$

implies

$$f_{uu}\phi = 0 \tag{5.25}$$

Nonlinearity of f in Eq. (5.25) implies

$$\boxed{\phi = 0}. \tag{5.26}$$

From Eq. (5.12) and the assumption $\xi_x = 0$, one can conclude

$$(x^2 + b^2)\xi_{yy} + (x^2 - b^2)\xi = 0 \tag{5.27}$$

In this case,

$$\xi_x = \xi_u = 0$$

which implies that ξ is a function of y only. Using this in Eq. (5.27) gives

$$\boxed{\xi = 0} \tag{5.28}$$

Substituting the zero value of ξ in the determining equations HE_3 and HE_4 gives

$$\tau_x = \tau_y = 0 \tag{5.29}$$

From system (5.29) and equation HE_2, we conclude

$$\boxed{\tau = k_1} \tag{5.30}$$

Equations (5.26), (5.28) and (5.30) define a one dimensional algebra generated by

$$H_1 = \frac{\partial}{\partial y}$$

which is the minimal symmetry algebra for the PDE (5.1).

To look for functions $f(u)$ that may give larger symmetry algebra we assume

$$\xi_x \neq 0$$

and solve the differential equations

$$f_{uu}^2 - f_u f_{uuu} = 0 \tag{5.31}$$

$$f_u f_{uu} f_{uuuu} - 2 f_u f_{uuu}^2 + f_{uu}^2 f_{uuu} = 0 \tag{5.32}$$

These two equations are exactly the same as equations (3.34) and (3.35) that were solved and discussed in detail in chapter 3.

As solved before in chapter 3, only one nonlinear function solves Eq. (5.31) which is

- $f(u) = ae^{bu} + c,$ with $a \neq 0,\ b \neq 0$

Also, there are four types of nonlinear functions that solve Eq. (5.32) as discussed in chapter III. These solutions are the following functions

- $f(u) = au^2 + bu + c,$ with $a \neq 0$
- $f(u) = ae^{bu} + c,$ with $a \neq 0,\ b \neq 0$
- $f(u) = a(u+b)^c + d\ ,$ with $a \neq 0,\ c \neq 0\ , 1\ , 2$
- $f(u) = a\ln(u+b) + c,$ with $a \neq 0$

Hence from the solution of Eq. (5.31), we see that

$$f(u) \neq ae^{bu} + c \tag{5.33}$$

is equivalent to

$$f_{uu}^2 - f_u f_{uuu} \neq 0$$

Using this equivalence in Eq. (5.23), the relation (5.33) implies

$$\xi_{xx} = 0. \tag{5.34}$$

Different possibilities for $f(u)$ are analyzed below to obtain corresponding symmetry algebras. The Symmetry algebras for different forms of $f(u)$ are summarized in the following cases:

Case 5.1.1: $f(u) \neq ae^{bu} + c$

In this case we have

$$\xi_{xx} = 0. \tag{5.35}$$

There are three possibilities of $f(u)$ in this case and are discussed in the following

Case 5.1.1(i) $f(u)$ is quadratic

Let

$$f(u) = a(u^2 + Bu + c) \tag{5.36}$$

with $a \neq 0$. From Eq. (5.16), we have

$$2a\phi + 2a(2u + B)\xi_x = 0$$

which implies

$$\boxed{\phi = -(2u + B)\xi_x} \tag{5.37}$$

Since $\xi_{xx} = 0$, differentiating Eq. (5.37) with respect to x gives

$$\phi_x = \phi_{xx} = 0. \tag{5.38}$$

Also, differentiating twice Eq. (5.37) with respect to y, we get

$$\phi_{yy} = -(2u + B)\xi_{xyy}. \tag{5.39}$$

Using equations (5.12) and (5.35), we get

$$\xi_{yy} = x\xi_x + \frac{b^2 - x^2}{x^2 + b^2}\xi \tag{5.40}$$

Differentiating previous equation with respect to x , with Eq. (5.35) in front, we get

$$\xi_{yyx} = \xi_x + \frac{b^2 - x^2}{x^2 + b^2}\xi_x - \frac{4xb^2}{\left(x^2 + b^2\right)^2}\xi = \frac{2b^2}{x^2 + b^2}\xi_x - \frac{4xb^2}{\left(x^2 + b^2\right)^2}\xi \tag{5.41}$$

Substituting Eq. (5.41) in Eq. (5.39) gives

$$\phi_{yy} = -(2u + B)\left[\frac{2b^2}{x^2 + b^2}\xi_x - \frac{4xb^2}{\left(x^2 + b^2\right)^2}\xi\right] \tag{5.42}$$

Also, differentiating Eq. (5.37) with respect to u, we get

$$\phi_u = -2\xi_x \tag{5.43}$$

Substituting all of these in equation HE_8, we get

$$\frac{-1}{x^2 + b^2}(2u + B)\left[\frac{2b^2}{x^2 + b^2}\xi_x - \frac{4xb^2}{(x^2 + b^2)^2}\xi\right] - 4\xi_x a\left(u^2 + Bu + c\right) + a(2u + B)^2\xi_x = 0$$

which implies

$$\frac{-1}{x^2 + b^2}(2u + B)\left[\frac{2b^2}{x^2 + b^2}\xi_x - \frac{4xb^2}{(x^2 + b^2)^2}\xi\right] - 4\xi_x a(c) + a\left(B^2\right)\xi_x = 0$$

Comparing coefficients of powers of $(2u + B)$, and simplifying gives

$$B^2 - 4c = 0 \tag{5.44}$$

$$\frac{2b^2}{x^2 + b^2}\xi_x - \frac{4xb^2}{(x^2 + b^2)^2}\xi = 0 \tag{5.45}$$

Equation (5.45) implies

$$\xi_x - \frac{2x}{x^2 + b^2}\xi = 0 \tag{5.46}$$

Multiplying by the integrating factor of Eq. (5.46), $\frac{1}{x^2 + b^2}$, we get

$$\frac{1}{x^2 + b^2}\xi_x - \frac{2x}{(x^2 + b^2)^2}\xi = 0$$

or

$$\frac{1}{x^2+b^2}\xi = K(y)$$

or

$$\boxed{\xi = \left(x^2+b^2\right)K(y)} \tag{5.47}$$

Differentiating Eq. (5.47) twice with respect to x gives

$$\xi_{xx} = 2K(y) \tag{5.48}$$

Using Eq. (5.35) in Eq. (5.48), we get

$$\boxed{K(y) = 0} \tag{5.49}$$

Substituting Eq. (5.49) in Eq. (5.47), we get

$$\boxed{\xi = 0} \tag{5.50}$$

which leads to minimal symmetry algebra.

Case 5.1.1(ii) $f(u) = a(u+B)^n + c, \quad with\ \ a \neq 0,\ n\ \neq 0\ ,1\ ,2$

Substituting the function in Eq. (5.16), we get

$$an(n-1)(u+B)^{n-2}\phi + 2an(u+B)^{n-1}\xi_x = 0$$

which implies

$$\phi = -\frac{2(u+B)}{n-1}\xi_x \tag{5.51}$$

Using equations (5.41) and (5.51), we get

$$\phi_{yy} = -\frac{2(u+B)}{n-1}\left[\frac{2b^2}{x^2+b^2}\xi_x - \frac{4xb^2}{\left(x^2+b^2\right)^2}\xi\right] \tag{5.52}$$

Differentiating Eq. (5.51) with respect to x gives

$$\phi_x = \phi_{xx} = 0 \tag{5.53}$$

Differentiating Eq. (5.51) with respect to u gives

$$\phi_u = -\frac{2}{n-1}\xi_x \tag{5.54}$$

Substituting equations (5.51) - (5.54) in equation HE_8, we get

$$-\frac{2(u+B)}{n-1}\frac{1}{x^2+b^2}\left[\frac{2b^2}{x^2+b^2}\xi_x - \frac{4xb^2}{\left(x^2+b^2\right)^2}\xi\right] - \frac{2}{n-1}\xi_x f - 2f\xi_x + \frac{2(u+B)}{n-1}\xi_x f_u = 0$$

Substituting the function f and its derivative in the previous relation gives

$$-\frac{2(u+B)}{n-1}\frac{1}{x^2+b^2}\left[\frac{2b^2}{x^2+b^2}\xi_x-\frac{4xb^2}{(x^2+b^2)^2}\xi\right]-\frac{2n}{n-1}\xi_x[a(u+B)^n+c]$$

$$+n(u+B)^n a\frac{2}{n-1}\xi_x=0$$

Simplifying gives

$$-\frac{2(u+B)}{n-1}\frac{\mathbf{1}}{\boldsymbol{x^2+b^2}}\left[\frac{2b^2}{x^2+b^2}\xi_x-\frac{4xb^2}{(x^2+b^2)^2}\xi\right]-\frac{2nc}{n-1}\xi_x=0 \tag{5.55}$$

which implies

$$c=0 \tag{5.56}$$

and

$$\xi_x-\frac{2x}{x^2+b^2}\xi=0 \tag{5.57}$$

Solving Eq. (5.57) gives

$$\xi=K(y)\left(x^2+b^2\right) \tag{5.58}$$

Differentiating twice with respect to x and comparing with Eq. (5.34), we get

$$\xi_{xx}=2K(y)=0$$

which implies

$$K(y)=0$$

and substituting this in Eq. (5.58), we get

$$\boxed{\xi=0} \tag{5.59}$$

which leads to minimal algebra.

Case 5.1.1(iii) $\boldsymbol{f(u)=a\ln(u+B)+c}$ with $\boldsymbol{a\neq 0}$

Substituting the function in Eq. (5.16), we get

$$\frac{-1}{(u+B)^2}\phi+2\left(\frac{1}{u+B}\right)\xi_x=0$$

or

$$\phi=2(u+B)\xi_x \tag{5.60}$$

Using equations (5.41) and (5.60), we get

$$\phi_{yy}=2(u+B)\left[\frac{2b^2}{x^2+b^2}\xi_x-\frac{4xb^2}{\left(x^2+b^2\right)^2}\xi\right] \tag{5.61}$$

Differentiating Eq. (5.60) with respect to u, x gives

$$\phi_u=2\xi_x \tag{5.62}$$

$$\phi_x = \phi_{xx} = 0 \tag{5.63}$$

Now substituting equations (5.60)-(5.63) in equation HE_8 , one obtains

$$\frac{1}{x^2+b^2}2(u+B)\left[\frac{2b^2}{x^2+b^2}\xi_x - \frac{4xb^2}{(x^2+b^2)^2}\xi\right] + 2\xi_x f - 2f\xi_x - 2(u+B)\xi_x f_u = 0$$

Substituting the function f given in this case gives

$$\frac{1}{x^2+b^2}2(u+B)\left[\frac{2b^2}{x^2+b^2}\xi_x - \frac{4xb^2}{(x^2+b^2)^2}\xi\right] - 2a\xi_x = 0$$

This implies that

$$\xi_x = 0$$

which leads us to minimal symmetry algebra.

This means that for all the three types of functions presented in case 5.1.1,

$$f(u) \neq ae^{Bu} + c,$$

we have not found new symmetries rather than the minimal symmetry algebra.

Case 5.1.2 $\boldsymbol{f(u) = ae^{Bu} + c, \quad a, B \neq 0}$

Here ξ_{xx} is not necessarily 0.

Using $f(u)$ in Eq. (5. 41) gives

$$aB^2e^{Bu}\phi + 2aBe^{Bu}\xi_x = 0.$$

Thus from Eq. (5.41) we have

$$\boxed{\phi = \frac{-2}{B}\xi_x}. \tag{5.64}$$

Substituting this in equation HE_8, one gets

$$\frac{-2}{B}\xi_{xxx} + \frac{1}{x^2+b^2}\frac{-2}{B}\xi_{xyy} + \frac{x}{x^2+b^2}\frac{-2}{B}\xi_{xx} - 2(ae^{Bu}+c)\xi_x - \frac{-2}{B}\xi_x aBe^{Bu} = 0$$

Simplifying gives

$$Bc\xi_x + \xi_{xxx} + \frac{x}{x^2+b^2}\xi_{xx} + \frac{1}{x^2+b^2}\xi_{xyy} = 0 \tag{5.65}$$

From equations (5.12) and HE_1, we get

$$\xi_{yy} = -\left(x^2+b^2\right)\xi_{xx} + x\xi_x + \frac{b^2-x^2}{x^2+b^2}\xi = 0$$

Differentiating the previous equation with respect to x, we get

$$\xi_{yyx} = -\left(x^2+b^2\right)\xi_{xxx} - 2x\xi_{xx} + x\xi_{xx} + \xi_x + \frac{b^2-x^2}{x^2+b^2}\xi_x - \frac{4b^2x}{\left(x^2+b^2\right)^2}\xi$$

Simplifying gives

$$\xi_{yyx} = -\left(x^2 + b^2\right)\xi_{xxx} - x\xi_{xx} + \frac{2b^2}{x^2+b^2}\xi_x - \frac{4b^2x}{\left(x^2+b^2\right)^2}\xi \tag{5.66}$$

Substituting Eq. (5.66) in Eq. (5.65), we get

$$Bc\xi_x + \xi_{xxx} + \frac{x}{x^2+b^2}\xi_{xx} + \left[-\xi_{xxx} - \frac{x}{x^2+b^2}\xi_{xx} + \frac{2b^2}{\left(x^2+b^2\right)^2}\xi_x - \frac{4b^2x}{\left(x^2+b^2\right)^3}\xi\right] = 0$$

or

$$Bc\xi_x + \left[\frac{2b^2}{\left(x^2+b^2\right)^2}\xi_x - \frac{4b^2x}{\left(x^2+b^2\right)^3}\xi\right] = 0$$

or

$$Bc\left(x^2 + b^2\right)\xi_x + \left[\frac{2b^2}{\left(x^2+b^2\right)}\xi_x - \frac{4b^2x}{\left(x^2+b^2\right)^2}\xi\right] = 0 \tag{5.67}$$

Consider that

$$\frac{\partial}{\partial x}\left[\frac{2b^2}{(x^2+b^2)}\xi\right] = \frac{2b^2}{(x^2+b^2)}\xi_x - \frac{4b^2x}{(x^2+b^2)^2}\xi \tag{5.68}$$

Using this in Eq. (5.67), we get

$$Bc\left(x^2 + b^2\right)\xi_x + \frac{\partial}{\partial x}\left[\frac{2b^2}{\left(x^2+b^2\right)}\xi\right] = 0 \tag{5.69}$$

Multiplying Eq. (5.69) by $\frac{2b^2}{\left(x^2+b^2\right)}\xi$, we get

$$2b^2Bc\xi\xi_x + \left[\frac{2b^2}{(x^2+b^2)}\xi\right]\frac{\partial}{\partial x}\left[\frac{2b^2}{(x^2+b^2)}\xi\right] = 0 \tag{5.70}$$

Now, integrating Eq. (5.70) with respect to x, we get

$$b^2Bc\xi^2 + \frac{1}{2}\left[\frac{2b^2}{(x^2+b^2)}\xi\right]^2 = K(y) \tag{5.71}$$

which implies

$$Bc\xi^2 + \frac{2b^2}{\left(x^2+b^2\right)^2}\xi^2 = K(y)$$

or

$$\boxed{\xi = \frac{K(y)\left(x^2+b^2\right)}{\sqrt{Bc(x^2+b^2)^2+2b^2}}} \tag{5.72}$$

Differentiating Eq. (5.72) with respect to x gives

$$\xi_x = \frac{4b^2K(y)x}{[Bc(x^2+b^2)^2+2b^2]^{\frac{3}{2}}} \tag{5.73}$$

Differentiating Eq. (5.72) with respect to y gives

$$\xi_y = \frac{K'(y)\left(x^2+b^2\right)}{\sqrt{Bc(x^2+b^2)^2+2b^2}} \tag{5.74}$$

Then, from Eq. (5.64) we have

$$\boxed{\phi = \frac{-8b^2K(y)x}{B[Bc(x^2+b^2)^2+2b^2]^{3/2}}} \tag{5.75}$$

and from this information, equation HE_8 will be satisfied.

Using Eq. (5.74) in equation HE_3 gives

$$\frac{K'(y)}{\sqrt{Bc(x^2+b^2)^2+2b^2}} + \tau_x = 0$$

or

$$\tau_x = -\frac{K'(y)}{\sqrt{Bc(x^2+b^2)^2+2b^2}} \tag{5.76}$$

which implies

$$\tau_{xy} = -\frac{K''(y)}{\sqrt{Bc(x^2+b^2)^2+2b^2}} \tag{5.77}$$

Also, using equation (5.72) and (5.73) in equationHE_4 gives

$$\tau_y - \frac{4b^2K(y)x}{[Bc(x^2+b^2)^2+2b^2]^{\frac{3}{2}}} + \frac{K(y)x}{\sqrt{Bc(x^2+b^2)^2+2b^2}} = 0$$

or

$$\tau_y = \frac{4b^2K(y)x - K(y)x\left[Bc\left(x^2+b^2\right)^2+2b^2\right]}{[Bc(x^2+b^2)^2+2b^2]^{\frac{3}{2}}}$$

or

$$\tau_y = K(y)x\frac{2b^2 - Bc\left(x^2+b^2\right)^2}{[Bc(x^2+b^2)^2+2b^2]^{\frac{3}{2}}}$$

Then,

$$\tau_{yx} =$$

$$K(y)\left\{\frac{2b^2-Bc(x^2+b^2)^2}{[Bc(x^2+b^2)^2+2b^2]^{\frac{3}{2}}} + x\frac{-4xBc(x^2+b^2)[Bc(x^2+b^2)^2+2b^2]^{\frac{3}{2}} - [2b^2-Bc(x^2+b^2)^2]\frac{3}{2}[Bc(x^2+b^2)^2+2b^2]^{\frac{1}{2}}4xBc(x^2+b^2)}{[Bc(x^2+b^2)^2+2b^2]^3}\right\}$$

or

$$\tau_{yx} =$$

$$K(y)\left\{\frac{2b^2-Bc\left(x^2+b^2\right)^2}{[Bc(x^2+b^2)^2+2b^2]^{\frac{3}{2}}} - 4x^2Bc(x^2+b^2)\sqrt{Bc(x^2+b^2)^2+2b^2}\,\frac{\left[\frac{-1}{2}Bc\left(x^2+b^2\right)^2+5b^2\right]}{[Bc(x^2+b^2)^2+2b^2]^3}\right\}$$

$$= \frac{K(y)}{\sqrt{Bc(x^2+b^2)^2+2b^2}}\left\{\frac{2b^2-Bc\left(x^2+b^2\right)^2}{[Bc(x^2+b^2)^2+2b^2]} - 4x^2Bc(x^2+b^2)\frac{\left[\frac{-1}{2}Bc\left(x^2+b^2\right)^2+5b^2\right]}{[Bc(x^2+b^2)^2+2b^2]^2}\right\}$$

or

$$\tau_{yx} = \frac{K(y)}{\sqrt{Bc(x^2+b^2)^2+2b^2}}\left\{\frac{4b^4-B^2c^2\left(x^2+b^2\right)^4+2x^2B^2c^2\left(x^2+b^2\right)^3-20b^2Bc\left(x^2+b^2\right)}{[Bc(x^2+b^2)^2+2b^2]^2}\right\} \quad (5.78)$$

From equations (5.77) and (5.78), we get

$$-\frac{K''(y)}{\sqrt{Bc(x^2+b^2)^2+2b^2}} =$$

$$\frac{K(y)}{\sqrt{Bc(x^2+b^2)^2+2b^2}}\left\{\frac{4b^4-B^2c^2\left(x^2+b^2\right)^4+2x^2B^2c^2\left(x^2+b^2\right)^3-20b^2Bc\left(x^2+b^2\right)}{[Bc(x^2+b^2)^2+2b^2]^2}\right\}$$

or

$$-\frac{K''(y)}{K(y)} = \left\{\frac{4b^4-B^2c^2\left(x^2+b^2\right)^4+2x^2B^2c^2\left(x^2+b^2\right)^3-20b^2Bc\left(x^2+b^2\right)}{[Bc(x^2+b^2)^2+2b^2]^2}\right\} = constant \quad (5.79)$$

From Eq. (5.79), we have

$$\frac{4b^4-B^2c^2\left(x^2+b^2\right)^4+2x^2B^2c^2\left(x^2+b^2\right)^3-20b^2Bc\left(x^2+b^2\right)}{[Bc(x^2+b^2)^2+2b^2]^2} = constant$$

Simplifying gives

$$\frac{8b^4-\left[Bc\left(x^2+b^2\right)^2+2b^2\right]^2+2x^2B^2c^2\left(x^2+b^2\right)^3+4Bcb^2\left(x^2+b^2\right)^2-20b^2Bc\left(x^2+b^2\right)}{[Bc(x^2+b^2)^2+2b^2]^2} = constant$$

This expression cannot be equal to constant unless

$\boxed{c = 0}$.

In this case, this expression =1. Thus, Eq. (5.79) becomes

$$-\frac{K''(y)}{K(y)} = 1$$

or

$$K''(y) + K(y) = 0$$

which implies

$$\boxed{K(y) = k_1 \sin y + k_2 \cos y} = \frac{1}{\sqrt{2b^2}}(c_1 \sin y + c_2 \cos y) \quad (5.80)$$

Substituting Eq. (5.80) in Eq. (5.72) gives

$$\boxed{\xi = \frac{\left(x^2+b^2\right)(k_1 \sin y+k_2 \cos y)}{\sqrt{2b^2}}} = (x^2 + b^2)(c_1 \sin y + c_2 \cos y) \quad (5.81)$$

Substituting in Eq. (5.76) gives

$$\tau_x = -\frac{(k_1 \cos y-k_2 \sin y)}{\sqrt{2b^2}} = -(c_1 \cos y - c_2 \sin y) \quad (5.82)$$

Integrating Eq. (5.82) gives

$$\tau = -\frac{(k_1 \cos y - k_2 \sin y)x}{\sqrt{2b^2}} + g(y) = -x(c_1 \cos y - c_2 \sin y) + g(y) \tag{5.83}$$

Differentiating Eq. (5.83) with respect to y, we get

$$\tau_y = \frac{(k_1 \sin y + k_2 \cos y)x}{\sqrt{2b^2}} + g'(y) = K(y)x\frac{1}{\sqrt{2b^2}} + g'(y) \tag{5.84}$$

But from equation HE_4, we have got

$$\tau_y = K(y)x\frac{2b^2 - Bc\left(x^2+b^2\right)^2}{[Bc(x^2+b^2)^2+2b^2]^{\frac{3}{2}}} = K(y)x\frac{1}{\sqrt{2b^2}} \tag{5.85}$$

Comparing equations (5.84) and (5.85), we get

$$g'(y) = 0$$

or

$$g(y) = k_3 \tag{5.86}$$

Substituting this in Eq. (5.83), we get

$$\boxed{\tau = -\frac{(k_1 \cos y - k_2 \sin y)x}{\sqrt{2b^2}} + k_3} = -x(c_1 \cos y - c_2 \sin y) + c_3 \tag{5.87}$$

From equations (5.80) and (5.75), we get

$$\boxed{\phi = \frac{-8b^2K(y)x}{B[2b^2]^{3/2}}} = \frac{-4x}{B\sqrt{2b^2}}(k_1 \sin y + k_2 \cos y) = \frac{-4x}{B}(c_1 \sin y + c_2 \cos y) \tag{5.88}$$

Thus for the case where

$$f(u) = ae^{Bu}, \quad a, B \neq 0$$

Eq. (5.1) admits three dimensional algebra generated by

$$H_1 = \frac{\partial}{\partial y},$$

$$H_2 = (x^2 + b^2)\sin y\frac{\partial}{\partial x} - x\cos y\frac{\partial}{\partial y} - \frac{4x}{B}\sin y\frac{\partial}{\partial u}$$

$$H_3 = (x^2 + b^2)\cos y\frac{\partial}{\partial x} + x\sin y\frac{\partial}{\partial y} - \frac{4x}{B}\cos y\frac{\partial}{\partial u}$$

with commutation relations

	H_1	H_2	H_3
H_1	0	H_3	$-H_2$
H_2	$-H_3$	0	b^2H_1
H_3	H_2	$-b^2H_1$	0

5.2. REDUCTION OF THE EQUATION ON HELICOID

5.2.1. Reduction of the general case

We are going to reduce the following equation

$$u_{xx} + \frac{x}{x^2+b^2} u_x + \frac{1}{x^2+b^2} u_{yy} = f(u) \tag{5.1}$$

This equation admits one dimensional algebra generated by

$$H_1 = \frac{\partial}{\partial y}$$

This symmetry can be written as

$$X = H_1 = 0\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + 0\frac{\partial}{\partial u}$$

The characteristic system of $X(I) = 0$ is given by

$$\frac{dx}{0} = \frac{dy}{1} = \frac{du}{0}$$

From

$$\frac{dx}{0} = \frac{dy}{1},$$

we have

$$x = c \tag{5.89}$$

and from

$$\frac{dy}{1} = \frac{du}{0},$$

we get

$$u = k \tag{5.90}$$

So, we get two invariants of $X(I) = 0$ and are given by

$$\xi(x, y) = x \tag{5.91}$$

$$V(\xi) = u \tag{5.92}$$

Differentiating equations (5.91) and (5.92) gives

$$u_x = V' \tag{5.93}$$

$$u_{xx} = V'' \tag{5.94}$$

$$u_y = 0$$

$$u_{yy} = 0 \tag{5.95}$$

Substituting equations (5.91)-(5.95) in Eq. (5.1), we get the second order ODE

$$\boxed{V'' + \frac{\xi}{\xi^2+b^2} V' = f(V)} \tag{5.96}$$

5.2.2. Reduction of the equation when $f(u) = e^u$

We are going to reduce the following equation

$$u_{xx} + \frac{x}{x^2+b^2} u_x + \frac{1}{x^2+b^2} u_{yy} = e^u \tag{5.97}$$

This equation admits 3- dimensional algebra as discussed in section 5.1 and is generated by

$$H_1 = \frac{\partial}{\partial y},$$

$$H_2 = (x^2 + b^2) \sin y \frac{\partial}{\partial x} - x \cos y \frac{\partial}{\partial y} - \frac{4x}{B} \sin y \frac{\partial}{\partial u},$$

$$H_3 = (x^2 + b^2) \cos y \frac{\partial}{\partial x} + x \sin y \frac{\partial}{\partial y} - \frac{4x}{B} \cos y \frac{\partial}{\partial u}.$$

with commutator table given by

	H_1	H_2	H_3
H_1	0	H_3	$-H_2$
H_2	$-H_3$	0	$b^2 H_1$
H_3	H_2	$-b^2 H_1$	0

5.2.2.1. Reduction using $H_1 = \frac{\partial}{\partial y}$:

From Eq. (5.96) in the general case, Eq. (5.97) is reduced to the second order ODE

$$V'' + \left(\frac{\xi}{\xi^2+b^2}\right) V' = e^V \tag{5.98}$$

where ξ and V are defined by equations (5.91) and (5.92).

5.2.2.2. Reduction using $H_2 = (x^2 + 1) \sin y \frac{\partial}{\partial x} - x \cos y \frac{\partial}{\partial y} - 4x \sin y \frac{\partial}{\partial u}$

The characteristic system of $X(I) = 0$ is given by

$$\frac{dx}{(x^2+1)\sin y} = \frac{dy}{-x \cos y} = \frac{du}{-4x \sin y}$$

From $\frac{dx}{(x^2+1)\sin y} = \frac{dy}{-x \cos y}$, we have

$$\frac{2xdx}{(\boldsymbol{x^2+1})} = 2\frac{\boldsymbol{-\sin y dy}}{\boldsymbol{\cos y}}$$

or

$$(\boldsymbol{x^2+1})\sec^2 y = c \tag{5.99}$$

and from $\frac{dy}{\boldsymbol{-x\cos y}} = \frac{du}{\boldsymbol{-4x\sin y}}$, we get

$$\frac{\boldsymbol{\sin y dy}}{\boldsymbol{\cos y}} = \frac{du}{\boldsymbol{4}},$$

or

$$u + 4\ln(\cos y) = c \tag{5.100}$$

So, we get two invariants of $X(I) = 0$ and are given by

$$\xi(x, y) = (\boldsymbol{x^2+1})\sec^2 y \tag{5.101}$$

$$V(\xi) = u + 4\ln(\cos y) \tag{5.102}$$

Differentiating equations (5.101) and (5.102) gives

$$u_x = V'(\boldsymbol{2x}\sec^2 y) \tag{5.103}$$

$$u_{xx} = V''(\boldsymbol{2x}\sec^2 y)^{\boldsymbol{2}} + V'(\boldsymbol{2}\sec^2 y) \tag{5.104}$$

$$u_y = V'[2(\boldsymbol{x^2+1})\sec^2 y\tan y] + 4\tan y$$

$$u_{yy} = V''[2(\boldsymbol{x^2+1})\sec^2 y\tan y]^2$$

$$+V'[4(\boldsymbol{x^2+1})\sec^2 y\tan^2 y + 2(\boldsymbol{x^2+1})\sec^4 y] + 4\sec^2 y \tag{5.105}$$

Substituting equations (5.101)-(5.105) in Eq. (5.97), we get the second order ODE

$$V''(\boldsymbol{2x}\sec^2 y)^{\boldsymbol{2}} + V'(\boldsymbol{2}\sec^2 y)$$

$$+\frac{x}{x^2+1}V'(\boldsymbol{2x}\sec^2 y)$$

$$+\frac{1}{x^2+1}[V''[2(\boldsymbol{x^2+1})\sec^2 y\tan y]^2]$$

$$+\frac{1}{x^2+1}V'[4(\boldsymbol{x^2+1})\sec^2 y\tan^2 y + 2(\boldsymbol{x^2+1})\sec^4 y]$$

$$+\frac{4}{x^2+1}\sec^2 y = e^V\sec^4 y$$

Simplifying gives

$$V''[\boldsymbol{4x^2}\sec^4 y + 4(\boldsymbol{x^2+1})\sec^4 y\tan^2 y]$$

$$+V'\left[\boldsymbol{2}\sec^2 y + \frac{2x^2}{x^2+1}\sec^2 y + 4\sec^2 y\tan^2 y + 2\sec^4 y\right]$$

$$+\frac{4}{x^2+1}\sec^2 y = e^V\sec^4 y \tag{5.106}$$

Dividing Eq. (5.106) by $(\sec^4 y)$ gives

$$V''[\boldsymbol{4x^2} + 4(\boldsymbol{x^2}+\mathbf{1})\sec^2 y - 4(\boldsymbol{x^2}+\mathbf{1})]$$

$$+V'\left[\mathbf{2}\cos^2 y + \frac{2x^2}{x^2+1}\cos^2 y + 4\sin^2 y + 2\right] + \frac{4}{\xi} = e^V \tag{5.107}$$

Simplifying using trigonometry identities and Eq. (5.101) gives

$$V''[4\xi - 4] + V'\left[\frac{2x^2}{\xi} + 6 - 2\frac{x^2+1}{\xi}\right] + \frac{4}{\xi} = e^V$$

or

$$V''[4\xi - 4] + V'\left[6 - \frac{2}{\xi}\right] + \frac{4}{\xi} = e^V \tag{5.108}$$

The last equation is nonlinear second order ODE with no symmetry, and thus cannot be reduced using symmetry method.

5.2.2.2. Reduction using $\boldsymbol{H_3 = (x^2+1)\cos y\frac{\partial}{\partial x} + x\sin y\frac{\partial}{\partial y} - 4x\cos y\frac{\partial}{\partial u}}$

The characteristic system of $X(I) = 0$ is given by

$$\frac{dx}{(\boldsymbol{x^2+1})\cos \boldsymbol{y}} = \frac{dy}{\boldsymbol{x}\sin \boldsymbol{y}} = \frac{du}{-\boldsymbol{4x}\cos \boldsymbol{y}}$$

From

$$\frac{dx}{(\boldsymbol{x^2+1})\cos \boldsymbol{y}} = \frac{dy}{\boldsymbol{x}\sin \boldsymbol{y}},$$

we have

$$\frac{2xdx}{(\boldsymbol{x^2+1})} = 2\frac{\cos y dy}{\sin \boldsymbol{y}}$$

or

$$(\boldsymbol{x^2}+\mathbf{1})\csc^2 y = c \tag{5.109}$$

and from

$$\frac{dy}{\boldsymbol{x}\sin \boldsymbol{y}} = \frac{du}{-\boldsymbol{4x}\cos \boldsymbol{y}},$$

we get

$$\frac{\cos \boldsymbol{y} dy}{\sin \boldsymbol{y}} = -\frac{du}{\mathbf{4}},$$

or

$$u + 4\ln(\sin y) = c \tag{5.110}$$

So, we get two invariants of $X(I) = 0$ and are given by

$$\xi(x, y) = (\boldsymbol{x^2}+\mathbf{1})\csc^2 y \tag{5.111}$$

$$V(\xi) = u + 4\ln(\sin y) \tag{5.112}$$

Differentiating equations (5.111) and (5.112) gives

$$u_x = V'(\mathbf{2}\boldsymbol{x}\csc^2 y) \tag{5.113}$$

$$u_{xx} = V''(\mathbf{2}\boldsymbol{x}\csc^2 y)^{\mathbf{2}} + V'(\mathbf{2}\csc^2 y) \tag{5.114}$$

$$u_y = V'[-2(\boldsymbol{x}^{\mathbf{2}} + \mathbf{1})\csc^2 y \cot y] - 4\cot y$$

$$u_{yy} = V''[2(\boldsymbol{x}^{\mathbf{2}} + \mathbf{1})\csc^2 y \cot y]^2$$

$$+V'[4(\boldsymbol{x}^{\mathbf{2}} + \mathbf{1})\csc^2 y \cot^2 y + 2(\boldsymbol{x}^{\mathbf{2}} + \mathbf{1})\csc^4 y] + 4\csc^2 y \tag{5.115}$$

Substituting equations (5.111)-(5.115) in Eq. (5.97), we get

$$V''(\mathbf{2}\boldsymbol{x}\csc^2 y)^{\mathbf{2}} + V'(\mathbf{2}\csc^2 y)$$

$$+\frac{x}{x^2+1}V'(\mathbf{2}\boldsymbol{x}\csc^2 y)$$

$$+\frac{1}{x^2+1}[V''[2(\boldsymbol{x}^{\mathbf{2}} + \mathbf{1})\csc^2 y \cot y]^2]$$

$$+\frac{1}{x^2+1}V'[4(\boldsymbol{x}^{\mathbf{2}} + \mathbf{1})\csc^2 y \cot^2 y + 2(\boldsymbol{x}^{\mathbf{2}} + \mathbf{1})\csc^4 y]$$

$$+\frac{4}{x^2+1}\csc^2 y = e^V \csc^4 y$$

Simplifying gives

$$V''[\mathbf{4}\boldsymbol{x}^{\mathbf{2}}\csc^4 y + 4(\boldsymbol{x}^{\mathbf{2}} + \mathbf{1})\csc^4 y \cot^2 y]$$

$$+V'\left[\mathbf{2}\csc^2 y + \frac{2x^2}{x^2+1}\csc^2 y + 4\,\mathrm{scsc}^2 y \cot^2 y + 2\csc^4 y\right]$$

$$+\frac{4}{x^2+1}\csc^2 y = e^V \csc^4 y \tag{5.116}$$

Dividing Eq. (5.116) by $(\csc^4 y)$ gives

$$V''[\mathbf{4}\boldsymbol{x}^{\mathbf{2}} + 4(\boldsymbol{x}^{\mathbf{2}} + \mathbf{1})\csc^2 y - 4(\boldsymbol{x}^{\mathbf{2}} + \mathbf{1})]$$

$$+V'\left[\mathbf{2}\sin^2 y + \frac{2x^2}{x^2+1}\sin^2 y + 4\cos^2 y + 2\right] + \frac{4}{\xi} = e^V \tag{5.117}$$

Simplifying using trigonometry identities and Eq. (5.111) gives

$$V''[4\xi - 4] + V'\left[\frac{2x^2}{\xi} + 6 - 2\frac{x^2+1}{\xi}\right] + \frac{4}{\xi} = e^V$$

or

$$V''[4\xi - 4] + V'\left[6 - \frac{2}{\xi}\right] + \frac{4}{\xi} = e^V \tag{5.118}$$

which is nonlinear second order ODE with no symmetry, and thus cannot be reduced using symmetry method.

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