Bazzoni-Glaz Conjecture on the Weak Global Dimension of Gaussian Rings

BY

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To my father, mother, brothers, sisters and fiancee

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Contents

THESIS ABSTRACT

Name: Khalid Waleed Khalid Adarbeh

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In 1969, Osofsky proved that a local arithmetical ring (i.e., chained ring) with zero divisors has infinite weak global dimension [28]. In view of [19, Corollary 4.2.6], this result asserts that the weak global dimension of an arithmetical ring is 0, 1, or ∞. In 2007, Bazzoni and Glaz studied the homological aspects of Prüfer-like rings, with a focus on Gaussian rings. They proved that Osofsky's aforementioned result is valid in the context of coherent Gaussian rings (resp., coherent Prüfer rings) [20, Theorem 3.3] (resp., [5, Theorem 6.1]). They closed their paper with a conjecture sustaining that "the weak global dimension of a Gaussian ring is 0, 1, or ∞" [5]. Recall for convenience that the class of Gaussian rings contains strictly the class of arithmetical rings.

Since 2007, the Bazzoni-Glaz Conjecture is still elusively open and several papers have appeared in the literature featuring partial results, of which some are of relevant significance. This MS thesis plans to track and study all these works dealing with this conjecture from the very origin; that is, 1969 Osofsky's proof of the existence of a module with infinite projective dimension on a local arithmetical ring. Precisely, we will examine all main results published in [1, 3, 5, 12, 20, 28] which tested the validity of the conjecture in subclasses within the class of Gaussian rings or provided large families of commutative rings (emanating from special constructions) sustaining the conjecture. We will also examine Couchot's very recent contribution to the problem via the finitistic weak dimension. Our ultimate goal is to identify new methods and techniques to tackle these problems from different angles which might offer a possible "happy end" to this conjecture and related problems in the future.

Keywords: Prüfer domain; arithmetical ring; chained ring; fqp-ring; Gaussian ring; Prüfer ring; semihereditary ring; IF-ring; semicoherent ring; quasi-projective module; FP-injective module; flat dimension; projective dimension; global dimension; weak global dimension; finitistic projective dimension; finitistic weak dimension; trivial ring extension.

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Abstract (Arabic)

Introduction

All rings considered in this thesis are commutative with identity element and all modules are unital. Let R be a ring and M an R-module. Then the weak (or flat) dimension (resp., projective dimension) of M, denoted by $\mathrm{w.dim}_R(M)$ (resp., $\mathrm{p.dim}_R(M)$), measures how far M is from being a flat (resp., projective) module. It is defined as follows: Let n be a non-negative integer. We have $\mathrm{w.dim}_R(M) \leq n$ (resp., $\mathrm{p.dim}_R(M) \leq n$) if there is a flat (resp., projective) resolution

$$0 \to E_n \to E_{n-1} \to \dots \to E_1 \to E_0 \to M \to 0.$$

If no such resolution exists, $\mathrm{w.dim}_R(M) = \infty$ (resp., $\mathrm{p.dim}_R(M) = \infty$); and if n is the least such integer, $\mathrm{w.dim}_R(M) = n$ (resp., $\mathrm{p.dim}_R(M) = n$). The weak global dimension (resp., global dimension) of R, denoted by $\mathrm{w.gl.dim}(R)$ (resp., $\mathrm{gl.dim}(R)$), is the supremum of $\mathrm{w.dim}_R(M)$ (resp., $\mathrm{p.dim}_R(M)$), where M ranges over all R-modules. The finitistic weak (resp., projective) dimension of R, denoted by $\mathrm{f.w.dim}(R)$ (resp., $\mathrm{FP.dim}(R)$), is the supremum of $\mathrm{w.dim}_R(M)$ (resp., $\mathrm{p.dim}_R(M)$), where M ranges over all R-modules of finite weak (resp., projective) dimension. For more details on all these notions, we refer the reader to [6, 19, 29].

from [19], a ring R is called coherent if every finitely generated ideal of R is finitely presented; equivalently, if $(0 : a)$ and $I \cap J$ are finitely generated for every $a \in R$ and any two finitely generated ideals I and J of R. Examples of coherent rings are Noetherian rings, Boolean algebras, von Neumann regular rings, and semihereditary rings.

The Bazzoni-Glaz Conjecture is concerned with the weak global dimension of Gaussian rings. These belong to the class of Prüfer-like rings which has recently received much attention from commutative ring theorists. A ring R is called Gaussian if for every $f,g \in R[X]$, one has the content ideal equation $c(fg) = c(f)c(g)$ where $c(f)$, the content of f, is the ideal of R generated by the coefficients of f [32]. The ring R is said to be a chained ring (or valuation ring) if its lattice of ideals is totally ordered by inclusion; and R is called arithmetical if $R_{\mathfrak{m}}$ is a chained ring for each maximal ideal $\mathfrak{m}$ of R [15, 24]. Also R is called semihereditary if every finitely generated ideal of R is projective [8]; and R is Prüfer if every finitely generated regular ideal of R is projective [7, 22]. In the domain context, all these notions coincide with the concept of Prüfer domain. Glaz, in [21], constructs examples which show that all these notions are distinct in the context of arbitrary rings. More examples, in this regard, are provided via trivial ring extensions [1, 3].

The following diagram of implications puts the notion of Gaussian ring in perspective within the family of Prüfer-like rings [4, 5, 1]:

Semihereditary ring

$\Downarrow$

Ring with weak global dimension ≤ 1

$\Downarrow$

Arithmetical ring

$\Downarrow$

fqp-Ring

$\Downarrow$

Gaussian ring

$\Downarrow$

Prüfer ring

In 1969, Osofsky proved that a local arithmetical ring (i.e., chained ring) with zero divisors has infinite weak global dimension [28]. In view of [19, Corollary 4.2.6], this result asserts that the weak global dimension of an arithmetical ring is 0, 1, or ∞.

In 2007, Bazzoni and Glaz proved that if R is a coherent Prüfer ring (and, a fortiori, a Gaussian ring), then $\mathrm{w.gl.dim}(R) = 0$, 1, or ∞ [5, Proposition 6.1]. And also they proved that if R is a Gaussian ring admitting a maximal ideal $\mathfrak{m}$ such that the nilradical of the localization $R_{\mathfrak{m}}$ is a nonzero nilpotent ideal, then $\mathrm{w.gl.dim}(R) = \infty$ [5, Theorem 6.4]. They closed their paper with a conjecture sustaining that "the weak global dimension of a Gaussian ring is 0, 1, or ∞" [5].

In 2010, Bakkari, Kabbaj, and Mahdou proved that if $(A,\mathfrak{m})$ is a local ring, E is a nonzero $\frac{A}{\mathfrak{m}}$-vector space, and $R := A \ltimes E$ is the trivial extension of A by E, then:

- R is a total ring of quotients and hence a Prüfer ring.
- R is Gaussian if and only if A is Gaussian.
- R is arithmetical if and only if $A := K$ is a field and $\dim_K E = 1$.
- $\mathrm{w.gl.dim}(R) \gneqq 1$. If, in addition, $\mathfrak{m}$ admits a minimal generating set, then $\mathrm{w.gl.dim}(R) = \infty$.

As an application, they provided an example of a Gaussian ring which is neither arithmetical nor coherent and has an infinite weak global dimension [3, Example 2.7]; which widened the scope of validity of Bazzoni-Glaz conjecture beyond the class of coherent Gaussian rings.

In 2011, Abuhlail, Jarrar, and Kabbaj investigated the correlation of fqp-rings with well-known Prüfer conditions; namely, they proved that the class of fqp-rings stands between the two classes of arithmetical rings and Gaussian rings [1, Theorem 3.1]. They also examined the transfer of the fqp-property to trivial ring extensions in order to build original examples of fqp-rings. Also they generalized Osofsky's result (mentioned above) and extended

Bazzoni-Glaz's result on coherent Gaussian rings by proving that the weak global dimension of an fqp-ring is equal to 0, 1, or ∞ [1, Theorem 3.11]; and then they provided an example of an fqp-ring that is neither arithmetical nor coherent [1, Example 3.9].

In parallel to the work on the weak global dimension, there were attempts to examine the finitistic dimensions of some Prüfer-like rings. Precisely, Bazzoni and Glaz proved, in 2007, that the small finitistic projective dimension of a Gaussian ring is less than or equal to 1 [5, Proposition 5.3]. While in 2011 Couchot proved that the finitistic weak dimension of an arithmetical ring is less than or equal to 2 [12, Theorem 1], the case of Gaussian rings is still open.

Since 2007, Bazzoni-Glaz conjecture is still elusively open and several papers have appeared in the literature featuring partial results, of which some are of relevant significance. This MS thesis plans to track and study all these works dealing with this conjecture from the very origin; that is, 1969 Osofsky's proof of the existence of a module with infinite projective dimension on a local arithmetical ring. Precisely, we will examine all main results published in [1, 3, 5, 12, 20, 28] which tested the validity of the conjecture in subclasses within the class of Gaussian rings or provided large families of commutative rings (emanating from special constructions) sustaining the conjecture. We will also examine Bazzoni-Glaz work on the small finitistic projective dimension of Gaussian rings and Couchot's very recent contribution on the finitistic weak dimension of Gaussian rings. Our ultimate goal is to identify new methods and techniques to tackle these problems from different angles which might offer a possible "happy end" to this conjecture and related problems in the future.

Chapter 1

Weak global dimension of arithmetical rings

In this chapter, we provide a detailed proof of Osofsky's Theorem that the weak global dimension of an arithmetical ring with zero divisors is infinite. In fact, this result enables one to state that the weak global dimension of an arithmetical ring is 0, 1, or ∞. We start by recalling some basic definitions.

Definition 1.1. Let R be a ring and M an R-module. Then:

(1) The weak dimension of M, denoted by $\mathrm{w.dim}(M)$, measures how far M is from being flat. It is defined as follows: Let n be a non-negative integer. Then $\mathrm{w.dim}(M) \leq n$ if there is a flat resolution

$$0 \to E_n \to E_{n-1} \to \dots \to E_1 \to E_0 \to M \to 0.$$

If no such resolution exists, $\mathrm{w.dim}(M) = \infty$; and if n is the least such integer, $\mathrm{w.dim}(M)$ $= n$.

(2) The weak global dimension of R, denoted by $\mathrm{w.gl.dim}(R)$, is the supremum of $\mathrm{w.dim}(M)$, where M ranges over all (finitely generated) R-modules.

Definition 1.2. Let R be a ring. Then:

(1) R is said to be a chained ring (or valuation ring) if its lattice of ideals is totally ordered by inclusion.

(2) R is called an arithmetical ring if if $R_{\mathfrak{m}}$ is a chained ring for each maximal ideal $\mathfrak{m}$ of R.

Fields and $\mathbb{Z}_{(p)}$, where $\mathbb{Z}$ is the ring of integers and p is a prime number, are examples of chained rings. Also, $\mathbb{Z}/n^2\mathbb{Z}$ is an arithmetical ring for any positive integer n. For more examples, see [3]. For a ring R, let $\mathrm{Z}(R)$ denote the set of all zerodivisors of R.

Next we give the main theorem of this chapter.

Theorem 1.3. *Let R be an arithmetical ring. Then* $\mathrm{w.gl.dim}(R) = 0, 1,$ *or* ∞.

To prove this theorem we make the following reductions:

(1) We may assume that R is a chained ring since $\mathrm{w.gl.dim}(R)$ is the supremum of $\mathrm{w.gl.dim}(R_{\mathfrak{m}})$ for all maximal ideal $\mathfrak{m}$ of R [19, Theorem 1.3.14 (1)].

(2) We may assume that R is a chained ring with zerodivisors. Then we prove that $\mathrm{w.gl.dim}(R) = \infty$ since, if R is a valuation domain, then $\mathrm{w.gl.dim}(R) \leq 1$ by [19, Corollary 4.2.6].

(3) Finally, we may assume that $(R,\mathfrak{m})$ is a chained ring with zerodivisors such that $\mathrm{Z}(R) = \mathfrak{m}$, since $\mathrm{Z}(R)$ is a prime ideal, $\mathrm{Z}(R_{\mathrm{Z}(R)}) = \mathrm{Z}(R)R_{\mathrm{Z}(R)}$, and
$\mathrm{w.gl.dim}(R_{\mathrm{Z}(R)}) \leq \mathrm{w.gl.dim}(R)$.

So our task is reduced to prove the following theorem.

Theorem 1.4 ([28, Theorem])**.** *Let $(R,\mathfrak{m})$ be a chained ring with zerodivisors such that $\mathrm{Z}(R) = \mathfrak{m}$. Then* $\mathrm{w.gl.dim}(R) = \infty$.

To prove this theorem we first prove the following lemmas. Throughout, let $(R,\mathfrak{m})$ be a chained ring with $\mathrm{Z}(R) = \mathfrak{m}$, M an R-module, $I = \{x \in R \mid x^2 = 0\}$, and for $x \in M$,

$(0:x)=\{y\in R \mid yx=0\}$. One can easily check that I is a nonzero ideal since R is a chained ring with zerodivisors.

Lemma 1.5 ([28, Lemma 1])**.** *$I^2=0$, and for all $x\in R$, $x\notin I \Rightarrow (0:x)\subseteq I$.*

Proof. To prove that $I^2=0$, it suffices to prove that $ab=0$ for all $a,b\in I$. So let $a,b\in I$. Then either $a\in bR$ or $b\in aR$, so that $ab\in a^2R=0$ or $ab\in b^2R=0$.

Now let $x\in R\setminus I$ and $y\in(0:x)$. Then either $x\in yR$ or $y\in xR$. But $x\in yR$ implies that $x^2\in xyR=0$, absurd. Therefore $y\in xR$, so that $y^2\in xyR=0$. Hence $y\in I$. □

Lemma 1.6 ([28, Lemma 2])**.** *Let $0\neq x\in \mathrm{Z}(R)$ such that $(0:x)=yR$. Then* $\mathrm{w.gl.dim}(R)=\infty$.

Proof. We first prove that $(0:y)=xR$. The inclusion $(0:y)\supseteq xR$ is trivial since $xy=0$. Now to prove the other inclusion let $z\in(0:y)$. Then either $z=xr$ for some $r\in R$ and in this case we are done, or $x=zj$ for some $j\in R$. We may assume $j\in\mathfrak{m}$. Otherwise, j is a unit and then we return to the first case. Since $x\neq 0$, $j\notin(0:z)$, so $jR\nsubseteq(0:z)$ which implies $(0:z)\subseteq jR$, and hence $y=jk$ for some $k\in\mathfrak{m}$. But then $0=zy=zjk=xk$, so $k\in(0:x)=yR$, and hence $k=yr$ for some $r\in R$. Hence $y=kj=yrj$, and as $j\in\mathfrak{m}$ we have the equality $y=y(1-rj)(1-rj)^{-1}=0$, which contradicts the fact that x is a zero divisor. Hence $z\in xR$, and therefore $(0:y)=xR$.

Now let m_x (resp., m_y) denote the multiplication by x (resp., y). Since $(0:x)=yR$ and $(0:y)=xR$ we have the following infinite flat resolution of xR with syzygies xR and yR:

$$\ldots \longrightarrow R \xrightarrow{m_y} R \xrightarrow{m_x} R \xrightarrow{m_y} \ldots \xrightarrow{m_y} R \xrightarrow{m_x} xR \longrightarrow 0$$

We claim that xR and yR are not flat. Indeed, recall that a projective module over a local ring is free [29]. So no projective module is annihilated by x or y. Since xR is annihilated by y and yR is annihilated by x, both xR and yR are not projective. Further, xR and yR are finitely presented in view of the exact sequence $0 \to yR \to R \to xR \to 0$. It follows that xR and yR are not flat (since a finitely presented flat module is projective [29, Theorem 3.61]). □

Corollary 1.7 ([28, Corollary])**.** *If $I = \mathfrak{m}$, then I is cyclic and R has infinite weak global dimension.*

Proof. Assume that $I = \mathfrak{m}$. Then $\mathfrak{m}^2 = 0$. Now let $0 \neq a \in \mathfrak{m}$. We claim that $\mathfrak{m} = aR$. Indeed, let $b \in \mathfrak{m}$. Since R is a chained ring, either $b = ra$ for some $r \in R$ and in this case we are done, or $a = rb$ for some $r \in R$. In the later case, either r is a unit and then $b = r^{-1}a \in aR$, or $r \in \mathfrak{m}$ which implies $a = rb = 0$, which contradicts the assumption $a \neq 0$. Thus $\mathfrak{m} = aR$, as claimed. Moreover, we have $(0 : a) = aR$. Indeed, $(0 : a) \supseteq aR$ since $a \in I$; if $x \in (0 : a)$, then $x \in \mathrm{Z}(R) = \mathfrak{m} = aR$. Hence $(0 : a) = aR$. It follows that R satisfies the conditions of Lemma 1.6 and hence the weak global dimension of R is ∞. □

Throughout, an element x of an R- module M is said to be regular if $(0 : x) = 0$.

Lemma 1.8 ([28, Lemma 3])**.** *Let F be a free module and $x \in F$. Then x is contained in zR for some regular element z of F.*

Proof. Let $\{y_\alpha\}$ be a basis for F and let $x := \sum_{i=1}^{n} y_i r_i \in F$, where $r_i \in R$. Since R is a chained ring, there is $j \in \{1, 2, ..., n\}$ such that $\sum_{i=1}^{n} r_i R \subseteq r_j R$. So that for each $i \in \{1, 2, ..., n\}$, $r_i = r_j s_i$ for some $s_i \in R$ with $s_j = 1$. Hence $x = r_j(\sum_{i=1}^{n}(y_i s_i))$. We claim that $z := \sum_{i=1}^{n} y_i s_i$ is regular. Deny and let $t \in R$ such that $t(\sum_{i=1}^{n} y_i s_i) = 0$. Then $ts_i = 0$ for all $i \in \{1, 2, ..., n\}$. In particular $t = ts_j = 0$, absurd. Therefore z is regular and $x = r_j z$, as desired. □

Lemma 1.9 ([28, Lemma 4])**.** *Assume that $(0 : r)$ is infinitely generated for all $0 \neq r \in \mathfrak{m}$. Let M be an R-submodule of a free module N such that:*

(1) $M = M_1 \bigcup M_2 \bigcup M_3$, *where* $M_1 = \bigcup_{\substack{x\in M\\ x\ regular}} xR$, $M_2 = \bigcup_{i=0}^{\infty} yu_iR$, *with* y *regular in* N, $u_iR \subsetneqq u_{i+1}R$, *and* yu_i *is not in* M_1, *and* $M_3 = \sum v_jR$.

(2) $yu_0R \cap xR$ *is infinitely generated for some regular* $x \in M$.

Let F *be a free* R*-module with basis* $\{y_x \mid x \text{ regular} \in M\} \cup \{z_i \mid i \in \omega\} \cup \{w_j\}$, *and let* $v : F \longrightarrow N$ *be the map defined by:* $v(y_x) = x$, $v(z_i) = yu_i$, *and* $v(w_j) = v_j$. *Then* $K = Ker(v)$ *has properties* $(1), (2)$, *and* M *is not flat.*

Proof. First the map v exists by [25, Theorem 4.1]. (1) By (2), there exist $r, s \in R$ such that $yu_0r = xs \neq 0$. Here $r \in \mathfrak{m}$; otherwise, $yu_0 = xsr^{-1} \in M_1$, contradiction. Since $\mathrm{Z}(R) = \mathfrak{m}$, the expression for any regular element in terms of a basis for N has one coefficient a unit. Indeed, let $(n_\alpha)_{\alpha\in\Delta}$ be a basis for N and z a regular element in N with $z = \sum_{i=0}^{i=k} c_in_i$ where $c_i \in R$. As R is a chained ring, there exists $j \in \{0, ..., k\}$ such that for all $i \in \{0, ..., k\}$, there exists $d_i \in R$ with $c_i = c_jd_i$ and $d_j = 1$. We claim that c_j is a unit. Deny. Then $c_j \in Z(R)$. So there is a nonzero $d \in R$ with $dc_j = 0$, and hence $dz = dc_j \sum_{i=0}^{i=k} d_in_i = 0$. Absurd since z is regular.

Now, let $x = \sum_{\substack{i\in I\\ I\ finite}} a_in_i$ and $y = \sum_{\substack{i\in I\\ I\ finite}} b_in_i$. Then $b_iu_0r = a_is$ for all $i \in I$. Let $i_0 \in I$ such that a_{i_0} is a unit. So $s = u_0rt$, where $t = b_{i_0}a_{i_0}^{-1} \in R$. Note that $b_{i_0} \neq 0$ since $xs \neq 0$. Clearly, $z_0 - y_xu_0t$ is regular in F (since z_0, y_x are part of the basis of F), is not in K (otherwise, $v(z_0 - y_xu_0t) = 0$ yields $yu_0 = xu_0t$, which contradicts (1)), and $(z_0 - y_xu_0t)r \in K$. We claim that $(z_0 - y_xu_0t)r$ is not in $K_1 := \bigcup_{\substack{x'\in K\\ x'\ regular}} x'R$. Deny and assume that $r(z_0 - u_0ty_x) = r'x'$ with $r' \in R$ and x' regular in K. Then $r' \neq 0$ since $r \neq 0$ and as $x' \in K \subseteq F$, there are $a, b, a_i \in R$ such that $x' = az_0 - by_x + x''$, where $x'' = \sum_{\substack{y_x\neq f_i\\ z_0\neq f_i}} a_if_i$. Thus $r = r'a$, $ru_0t = r'b$, and $r'x'' = 0$. Since x' is regular in F and $r'x'' = 0$, a or b is unit. We claim that a is always a unit. Indeed, if b is a unit, then $r(1 - ab^{-1}u_0t) = 0$, so if $a \in \mathfrak{m}$, then $(1 - ab^{-1}u_0t)$ is a unit which implies $r = 0$, absurd. So $a^{-1}x' = z_0 - a^{-1}by_x + a^{-1}x''$, $r' = a^{-1}r$, and $ru_0t = ra^{-1}b$ which implies

$z_0 - u_0ty_x + (u_0t - a^{-1}b)y_x + a^{-1}x'' = a^{-1}x' \in K$. By Lemma 1.8 $(u_0t - a^{-1}b)y_x + a^{-1}x'' = pq$, fore some q regular in F and $p \in R$. But clearly since $r = r'a$, $ru_0t = r'b$, and $r'x'' = 0$, then $rpq = 0$. Hence $rp = 0$. It follows that $(z_0 - y_xu_0t + qp) \in K$, where q is regular in F and $p \in (0:r)$. Thus by applying v we obtain $yu_0 - xu_0t + pv(q) = 0$. But R is a chained ring, so p and u_0t are comparable and since $u_0tr \neq 0$, $p = u_0th$ for some $h \in R$. Hence $yu_0 = (x - hv(q))u_0t$, we show that $(x - hv(q))$ is regular in M which contradicts property (1). First clearly $(x - hv(q)) \in M$ since $x, v(q) \in M$. Now suppose that $a(x - hv(q)) = 0$ for some $a \in \mathfrak{m}$. Either $u_0t = a'a$ for some $a' \in R$, this yields $yu_0 = (x - hv(q))aa' = 0$ also impossible, or $a = u_0tm$ for some $m \in R$, and this yields $mu_0y = (x - hv(q))a = 0$, so $mu_0 = 0$ as y is regular, and hence $a = mu_0t = 0$. We conclude that $(x - hv(q))$ is regular in M and hence $yu_0 \in M_1$, the desired contradiction.

Last, let $yu_0R \cap xR = \langle x_0, x_1, ..., x_n, ...\rangle$, where

$$\langle x_0, x_1, ..., x_i\rangle \subsetneqq \langle x_0, x_1, ..., x_i, x_{i+1}\rangle.$$

For any integer $i \geq 0$, let $x_i = yu_0r_i$ for some $r_i \in R$. It is clear that $r_0R \subsetneqq r_1R \subsetneqq ... \subsetneqq r_iR \subsetneqq r_{i+1}R \subsetneqq ...$. Now, let $y' := z_0 - y_xu_0t$, $u_i' := r_i$ for each $i \in \mathbb{N}$. Then $K = K_1 \bigcup K_2 \bigcup K_3$, where $K_1 := \bigcup\limits_{\substack{x' \in K \\ x' \text{ regular}}} x'R$, $K_2 := \bigcup\limits_{i=0}^{\infty} y'u_i'R$ with y' regular in F and $u_i'R \subsetneqq u_{i+1}'R$, and $K_3 := K \setminus (K_1 \bigcup K_2)$. Thus K satisfy Property (1).

(2) Since $u_0R \subsetneqq u_1R$, $u_0 = u_1m'$ for some $m' \in \mathfrak{m}$. Hence $x' := z_0 - z_1m'$ is regular in K since $v(x') = v(z_0 - z_1m') = yu_0 - yu_1m' = 0$ and z_0, z_1 are basis elements. We claim that $(z_0 - z_1m')R \cap (z_0 - y_xu_0t)r_0R = z_0(0:m')$. Indeed, since z_0, z_1, y_x are basis elements, then $(z_0 - z_1m')R \cap (z_0 - y_xu_0t)r_0 \subseteq z_0R$. Also $(z_0 - z_1m')R \cap z_0R = z_0(0:m')$. For, let $l \in (z_0 - z_1m')R \cap z_0R$. Then $l = (z_0 - z_1m')a = z_0a'$ for some $a, a' \in R$. Hence $a = a'$ and $am' = 0$, whence $l = az_0$ with $am' = 0$. So $l \in z_0(0:m')$. The reverse inclusion is straightforward. Consequently, $(z_0 - z_1m')R \cap (z_0 - y_xu_0t)r_0R \subseteq z_0(0:m')$. To prove the

reverse inclusion, let $k \in (0 : m')$. Then either $k = r_0 k'$ or $r_0 = kk'$, for some $k' \in R$. The second case is impossible since $r_0 u_0 \neq 0$. Hence $z_0 k = (z_0 - y_x u_0 t) r_0 k' \in (z_0 - y_x u_0 t) r_0 R$. Further, $z_0 k \in (z_0 - z_1 m')R$. Therefore our claim is true. But z_0 is regular, so $z_0(0 : m') \cong (0 : m')$ which is infinitely generated by hypothesis. Therefore $y'u_0'R \cap x'R$ is infinitely generated, as desired.

Finally, M is not flat. Deny. By [29, Theorem 3.57], there is an R-map $\theta : F \longrightarrow K$ such that $\theta\ ((z_0 - y_x u_0 t) r_0) = (z_0 - y_x u_0 t) r_0$. Assume that $\theta(z_0) = a z_0 + b y_x + Z_1$ for some $a, b \in R$ and $\theta(y_x) = a' z_0 + b' y_x + Z_2$ for some $a', b' \in R$. Then $r_0 a - r_0 u_0 t a' = r_0$, $r_0 b - r_0 u_0 t b' = -r_0 u_0 t$, and $r_0 Z_1 - r_0 u_0 t Z_2 = 0$. Hence $r_0(1 - a + u_0 t a') = 0$ and since $r_0 \neq 0$, a or a' is a unit. Suppose that a is a unit and without loss of generality we can assume that $a = 1$. Thus we have the equation $z_0 - u_0 t y_x - u_0 t a' z_0 + (u_0 t - u_0 t b' + b) y_x + Z_1 - u_0 t Z_2 = \theta(z_0) - u_0 t \theta(Z_2) \in K$. By Lemma 1.8, $-u_0 t a' z_0 + (u_0 t - u_0 t b' + b) y_x + Z_1 - u_0 t Z_2 = pq$, where q is regular in F and, clearly, $r_0 p = 0$ since $r_0 u_0 t a' = 0$. Thus $z_0 - u_0 t y_x + pq \in K$, absurd (as seen before in the second paragraph of the proof of Lemma 1.9). □

Now we are able to prove Theorem 1.4.

Proof of Theorem 1.4. If $(0 : r)$ is cyclic for some $r \in \mathfrak{m}$, then R has infinite weak global dimension by Lemma 1.6. Next suppose that $(0 : r)$ is not cyclic, for all $0 \neq r \in \mathfrak{m}$. Which is equivalent to assume that $(0 : r)$ is infinitely generated for all $0 \neq r \in \mathfrak{m}$, since R is a chained ring.

Let $0 \neq a \in I$ and $b \in \mathfrak{m} \setminus I$. Note that b exists since $I \neq \mathfrak{m}$ by the proof of Corollary 1.7. Let N be a free R-module on two generators y, y' and let $M := (y - y'b)R + y(0 : a)$. Then:

(A) $M_1 := \bigcup\limits_{\substack{x \in M \\ x\ regular}} xR = \{(yt - y'b)r \mid 1 - t \in (0 : a), r \in R)$. To show this equality, let c be a regular element in M. Then $c = (r_1 + r_2)y - r_1 b y'$ for some $r_1 \in R, r_2 \in (0 : a)$. We claim that r_1 is a unit. Deny. So either $r_1 \in (r_2)$ hence $ac = 0$, or $r_2 = n r_1$ for some $n \in R$ and since $r_1 \in \mathfrak{m} = \mathrm{Z}(R)$, there is $r_1' \neq 0$ such that $r_1 r_1' = 0$, so $r_1' c = 0$. In both cases there is a contradiction with the fact that c is regular. Thus, r_1 is a unit. It follows that $c = (1+r_1^{-1}r_2)yr_1 - by'r_1 \in \{(yt-y'b)r | 1-t \in (0:a), r \in R\}$. Now let $c = yt - y'b$, where $(1-t) \in (0:a)$. Then c is regular. Indeed, if $rc = 0$ for some $r \in R$, then $rt = 0$. Moreover, either $r = na$ for some $n \in R$, and in this case $r(1-t) = na(1-t) = 0$, so $r = rt = 0$ as desired, or $a = nr$ for some $n \in R$, so $a = at = nrt = 0$, absurd.

(B) There exists a countable chain of ideals $u_0R \subsetneqq u_1R \subsetneqq ...$ where $u_i \in (0:a) \setminus (0:b)$. Since $0 \neq a \in I$ and $b \in \mathfrak{m} \setminus I$, $(a) \subseteq (b)$. Thus $(0:b) \subseteq (0:a)$. Moreover $(0:b) \subsetneqq (0:a)$; otherwise, $a \in (0:a) = (0:b)$, and hence $ab = 0$. Hence $b \in (0:a) = (0:b) \subseteq I$ by Lemma 1.5, absurd. Now let $u_0 \in (0:a) \setminus (0:b)$. Since $(0:a)$ is infinitely generated, there are $u_1, u_2, ...$ such that $(u_0) \subsetneqq (u_0, u_1) \subsetneqq ... \subseteq (0:a)$. So $u_0R \subsetneqq u_1R \subsetneqq ...$ and necessarily $u_i \notin (0:b)$ for all $i \geq 1$ since $u_0 \notin (0:b)$.

Note that $yu_i \in M$(since $u_i \in (0:a)$). Also $yu_i \notin M_1$; otherwise, if $yu_i = ytr - y'br$ with $1-t \in (0:a)$ and $r \in R$, then $u_i = tr$ and $br = 0$. Hence $bu_i = btr = 0$ and thus $u_i \in (0:b)$, contradiction. Also note that y is regular in N (part of the basis) and $y \notin M$; if $y = (y-y'b)r_1 + r_2y$ with $r_1 \in R$ and $r_2 \in (0:a)$, then $r_1b = 0$ and $r_1 + r_2 = 1$. So $r_1 \in \mathfrak{m}$, $ar_1 = a$, and hence $a = 0$, absurd.

(A) and (B) imply that (1) of Lemma 1.9 holds.

Let us show that $yu_0R \cap (y-y'b)R = y(0:b)$. Indeed, if $c = yu_0r = (y-y'b)r'$ where $r, r' \in R$, then $u_0r = r'$ and $r'b = 0$. Hence $c \in y(0:b)$. If $c = ry$ where $rb = 0$, then $r = u_0t$ for some $t \in R$ as $u_0 \in (0:a) \setminus (0:b)$. Thus $c = r(y-y'b)$. Now $y(0:b) \cong (0:b)$ is infinitely generated. Therefore (2) of Lemma 1.9 holds.

Since K satisfies the properties of M we can consider it as a new module M, and then there is a free module F_1 and a map $v_1 : F_1 \longrightarrow F$ such that $K_1 = Ker(v_1)$ satisfies the same conditions of K and K_1 is not flat. We can repeat this iteration above to get the infinite flat resolution of M:

$$... \to F_n \to F_{n-1} \to ... \to F_1 \to F_0 \to M \to 0.$$

with no one of the syzygies $K, K_1, K_2, \ldots$ is flat. Therefore R has an infinite weak global dimension. $\square$

Chapter 2

Weak global dimension of Gaussian rings

In 2005, Glaz proved that if R is a Gaussian coherent ring, then $\mathrm{w.gl.dim}(R) = 0$, 1, or ∞ [20]. In this chapter, we will see that the same conclusion holds for the larger class of Prüfer coherent rings and for some contexts of Gaussian rings. We start by recalling the definitions of Gaussian, Prüfer, and coherent rings.

Definition 2.1. Let R be a ring. Then:

(1) R is called a Gaussian ring if for every $f, g \in R[X]$, one has the content ideal equation $c(fg) = c(f)c(g)$, where $c(f)$, the content of f, is the ideal of R generated by the coefficients of f.

(2) R is called a Prüfer ring if every nonzero finitely generated regular ideal is invertible (or, equivalently, projective)

(3) R is called a coherent ring if every finitely generated ideal of R is finitely presented; equivalently, if $(0 : a)$ and $I \cap J$ are finitely generated for every $a \in R$ and any two finitely generated ideals I and J of R.

Recall that Arithmetical ring $\Rightarrow$ Gaussian ring $\Rightarrow$ Prüfer ring. To see the proofs of the above implications and that they cannot be reversed, in general, we refer the reader to [5, 20, 21] and chapter 3 of this thesis.

Noetherian rings, valuation domains, and $K[x_1,x_2,...]$ where K is a field are examples of coherent rings. For more examples, see [19].

Let $Q(R)$ denote the total ring of fractions of R and $\mathrm{Nil}(R)$ its nilradical. The following proposition is the first main result of this chapter.

Proposition 2.2 ([5, proposition 6.1])**.** *Let R be a coherent Prüfer ring. Then* $\mathrm{w.gl.dim}(R) =$ *0, 1, or* ∞.

The proof of this proposition relies on the following lemmas. Recall that a ring R is called regular if every finitely generated ideal of R has a finite projective dimension; and von Neumann regular if every R-module is flat.

Lemma 2.3 ([19, Corollary 6.2.4])**.** *Let R be a coherent regular ring. Then $Q(R)$ is a von Neumann regular ring.* □

Lemma 2.4 ([20, Lemma 2.1])**.** *Let R be a local Gaussian ring and $I = (a_1,...,a_n)$ be a finitely generated ideal of R. Then $I^2 = (a_i^2)$, for some $i \in \{1,2,...,n\}$.*

Proof. We first assume that $I = (a,b)$. Let $f(x) := ax+b$, $g(x) := ax-b$, and $h(x) := bx+a$. Since R is Gaussian, $c(fg) = c(f)c(g)$, so that $(a,b)^2 = (a^2,b^2)$, also $c(fh) = c(f)c(h)$ which implies that $(a,b)^2 = (ab,a^2+b^2)$. Hence $(a^2,b^2) = (ab,a^2+b^2)$, whence $a^2 = rab+s(a^2+b^2)$, for some r and s in R. That is, $(1-s)a^2+rab+sb^2 = 0$. Since R is a local ring, either s or $1-s$ is a unit in R. If s is a unit in R, then $b^2+rs^{-1}ab+(s^{-1}-1)a^2 = 0$. Next we show that $ab \in (a^2)$. Let $k(x) := (b+\alpha a)x-a$, where $\alpha := rs^{-1}$. Then $c(hk) = c(h)c(k)$ implies that $(b(b+\alpha a),\alpha a^2,-a^2) = (a,b)((b+\alpha a),a)$. But clearly $(b(b+\alpha a),\alpha a^2,-a^2) = ((s^{-1}-1)a^2,\alpha a^2,-a^2) = (a^2)$. Thus $(a^2) = (a,b)((b+\alpha a),a)$. In particular, $ab \in (a^2)$ and so does b^2. If $1-s$ is unit, similar arguments imply that ab, and hence $a^2 \in (b^2)$. Thus for any two elements a and b, $ab \in (b^2)$ or (a^2). It follows that $I^2 = (a_1, ..., a_n)^2 = (a_1^2, ..., a_n^2)$. An induction on n leads to the conclusion. □

Recall that a ring R is called reduced if it has no nonzero nilpotent elements.

Lemma 2.5 ([20, Theorem 2.2]). *Let R be a ring. Then* $\mathrm{w.gl.dim}(R) \leq 1$ *if and only if R is a Gaussian reduced ring.*

Proof. Assume that $\mathrm{w.gl.dim}(R) \leq 1$. By [19, Corollary 4.2.6], R_p is a valuation domain for every prime ideal p of R. As valuation domains are Gaussian, R is locally Gaussian, and therefore Gaussian. Further, R is reduced. For, let $x \in R$ be nilpotent. We claim that $x = 0$. Deny and let $n \geq 2$ be an integer such that $x^n = 0$. Then there exists a prime ideal q in R such that $x \neq 0$ in R_q [2, Proposition 3.8]. It follows that $x^n = 0$ in R_q, a contradiction since R_q is a domain.

Conversely, since R is Gaussian reduced, R_p is a local, reduced, Gaussian ring for any prime ideal p of R. We claim that R_p is a domain. Indeed, let a and b in R_p such that $ab = 0$. By Lemma 2.4, $(a,b)^2$=$(b)^2$ or (a^2). Say $(a,b)^2 = (b^2)$. Then $a^2 = tb^2$ for some $t \in R_p$. Thus $a^3 = tb(ab) = 0$. Since R_p is reduced, $a = 0$, and R_p is a domain. Therefore R_p is a valuation domain for all prime ideals p of R. So $\mathrm{w.gl.dim}(R) \leq 1$ by [19, Corollary 4.2.6]. □

Lemma 2.6 ([5, Theorem3.3]). *Let R be a Prüfer ring. Then R is Gaussian if and only if $Q(R)$ is Gaussian.* □

Lemma 2.7 ([5, Theorem 3.12(ii)]). *Let R be a ring. Then* $\mathrm{w.gl.dim}(R) \leq 1$ *if and only if R is a Prüfer ring and* $\mathrm{w.gl.dim}(Q(R)) \leq 1$.

Proof. If $\mathrm{w.gl.dim}(R) \leq 1$, R is Prüfer and, by localization, $\mathrm{w.gl.dim}(Q(R)) \leq 1$. Conversely, assume that R is a Prüfer ring such that $\mathrm{w.gl.dim}(Q(R)) \leq 1$. By Lemma 2.5, $Q(R)$

is a Gaussian reduced ring. So R is reduced and, by Lemma 2.6, R is Gaussian. By Lemma 2.5, $\operatorname{w.gl.dim}(R) \leq 1$. □

Proof of Proposition 2.2. Assume that $\operatorname{w.gl.dim}(R) = n < \infty$ and let I be any finitely generated ideal of R. Then I has a finite weak dimension. Since R is a coherent ring, I is finitely presented. Hence the weak dimension of I equals its projective dimension by [19, Corollary 2.5.5]. Whence, as I is an arbitrary finitely generated ideal of R, R is a regular ring. So, by [19, Corollary 6.2.4], $Q(R)$ is von Neumann regular. By Lemma 2.7, $\operatorname{w.gl.dim}(R) \leq 1$. □

The following is an example of a coherent Prüfer ring with infinite weak global dimension.

Example 2.8. Let $R = \mathbb{R} \ltimes \mathbb{C}$. Then R is coherent by [26, Theorem 2.6], Prüfer by Theorem 3.2, and $\operatorname{w.gl.dim}(R) = \infty$ by Lemma 3.1.

In order to study the weak global dimension of an arbitrary Gaussian ring, we make the following reductions:

(1) We may assume that R is a local Gaussian ring since $\operatorname{w.gl.dim}(R)$ is the supremum of $\operatorname{w.gl.dim}(R_m)$ for all maximal ideal m of R [19, Theorem 1.3.14 (1)].

(2) We may assume that R is a non-reduced local Gaussian ring since every reduced Gaussian ring has weak global dimension at most 1 by Lemma 2.5.

(3) Finally, we may assume that $(R, \mathfrak{m})$ is a local Gaussian ring with the maximal ideal $\mathfrak{m}$ such that $\mathfrak{m} = \operatorname{Nil}(R)$. For, the prime ideals of a local Gaussian ring R are linearly ordered, so that $\operatorname{Nil}(R)$ is a prime ideal, and $\operatorname{w.gl.dim}(R) \geq \operatorname{w.gl.dim}(R_{\operatorname{Nil}(R)})$.

Next we announce the second main result of this chapter.

Theorem 2.9 ([5, Theorem 6.4])**.** *Let R be a Gaussian ring with a maximal ideal* $\mathfrak{m}$ *such that* $\operatorname{Nil}(R_{\mathfrak{m}})$ *is a nonzero nilpotent ideal. Then* $\operatorname{w.gl.dim}(R) = \infty$.

The proof of this theorem involves the following results:

Lemma 2.10. *Consider the following exact sequence of R-modules*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

where M is flat. Then either the three modules are flat or $\mathrm{w.dim}(M'') = \mathrm{w.dim}(M') + 1$.

Proof. Suppose that M'' is flat. Then by the long exact sequence theorem [29, Theorem 8.3] we get the exact sequence

$$0 = \mathrm{Tor}_2(M'', N) \longrightarrow \mathrm{Tor}_1(M', N) \longrightarrow \mathrm{Tor}_1(M, N) = 0$$

for any R-module N. Hence $\mathrm{Tor}_1(M', N) = 0$ which implies that M' is flat.

Next, assume that M'' is not flat. In this case, we claim that $\mathrm{w.dim}(M'') = \mathrm{w.dim}(M') + 1$. Indeed, let $\mathrm{w.dim}(M') = n$. Then we have the exact sequence

$$0 = \mathrm{Tor}_{n+2}(M, N) \longrightarrow \mathrm{Tor}_{n+2}(M'', N) \longrightarrow \mathrm{Tor}_{n+1}(M', N) = 0$$

for any R-module N. Hence $\mathrm{Tor}_{n+2}(M'', N) = 0$ for any R-module N which implies

$$\mathrm{w.dim}(M'') \leq n + 1 = \mathrm{w.dim}(M') + 1$$

Now let $\mathrm{w.dim}(M'') = m$. Then we have the exact sequence

$$0 = \mathrm{Tor}_{m+1}(M'', N) \longrightarrow \mathrm{Tor}_m(M', N) \longrightarrow \mathrm{Tor}_m(M, N) = 0$$

for any R-module N. Hence $\mathrm{Tor}_m(M', N) = 0$ for any R-module N which implies that

$$\mathrm{w.dim}(M'') = m \geq \mathrm{w.dim}(M') + 1$$

Consequently, $\mathrm{w.dim}(M'') = \mathrm{w.dim}(M') + 1$. □

Recall that an exact sequence of R-modules

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

is pure if it remains exact when tensoring it with any R-module. In this case, we say that M' is a pure submodule of M [29].

Lemma 2.11 ([5, Lemma 6.2]). *Let $(R,\mathfrak{m})$ be a local ring which is not a field. Then* $\mathrm{w.dim}(R/\mathfrak{m}) = \mathrm{w.dim}(\mathfrak{m}) + 1$.

Proof. Consider the short exact sequence

$$0 \to \mathfrak{m} \to R \to R/\mathfrak{m} \to 0.$$

Assume that $R/\mathfrak{m}$ is flat. By [19, Theorem 1.2.15 (1,2,3)], $\mathfrak{m}$ is pure and $(aR)\mathfrak{m} = aR \cap \mathfrak{m} = aR$ for all $a \in \mathfrak{m}$. Hence $a\mathfrak{m} = aR$, for all $a \in \mathfrak{m}$, and so by Nakayama's Lemma, $a = 0$, absurd. By Lemma 2.10, $\mathrm{w.dim}(R/\mathfrak{m}) = \mathrm{w.dim}_R(\mathfrak{m}) + 1$. □

Proposition 2.12 ([5, Proposition 6.3]). *Let $(R,\mathfrak{m})$ be a local ring with nonzero nilpotent maximal ideal. Then* $\mathrm{w.dim}(\mathfrak{m}) = \infty$.

Proof. Let n be the minimum integer such that $\mathfrak{m}^n = 0$. We claim that for all $1 \leq k < n$, $\mathrm{w.dim}(\mathfrak{m}^{n-k}) = \mathrm{w.dim}(\mathfrak{m}) + 1$. Indeed, let $k = 1$. Then $\mathfrak{m}^{n-1}\mathfrak{m} = 0$, so $\mathfrak{m}^{n-1}$ is an $(R/\mathfrak{m})$-vector space, hence $0 \neq \mathfrak{m}^{n-1} \cong \bigoplus R/\mathfrak{m}$, implies that $\mathrm{w.dim}_R(\mathfrak{m}^{n-1}) = \mathrm{w.dim}(R/\mathfrak{m}) = \mathrm{w.dim}(\mathfrak{m}) + 1$ by Lemma 2.11 . Now let h be the maximum integer in $\{1, ..., n-1\}$ such that $\mathrm{w.dim}(\mathfrak{m}^{n-k}) = \mathrm{w.dim}(\mathfrak{m}) + 1$ for all $k \leq h$. Assume by way of contradiction that $h < n-1$. Then we have the exact sequence:

$$0 \to \mathfrak{m}^{n-h} \to \mathfrak{m}^{n-(h+1)} \to \mathfrak{m}^{n-(h+1)}/\mathfrak{m}^{n-h} \to 0 \qquad (*)$$

where $\mathfrak{m}^{n-(h+1)}/\mathfrak{m}^{n-h}$ is a nonzero $(R/\mathfrak{m})$-vector space. So by Lemma 2.11, $\mathrm{w.dim}(\mathfrak{m}^{n-(h+1)}/\mathfrak{m}^{n-h}) = \mathrm{w.dim}(\mathfrak{m})+1$. By assumption, $\mathrm{w.dim}(\mathfrak{m}^{n-h}) = \mathrm{w.dim}(\mathfrak{m})+1$. Let us show that $\mathrm{w.dim}(\mathfrak{m}^{n-(h+1)}) = \mathrm{w.dim}(\mathfrak{m})+1$. Indeed, if $l := \mathrm{w.dim}(\mathfrak{m})+1$, then by applying the long exact sequence theorem to $(*)$, we get

$$0 = \mathrm{Tor}_{l+1}(\mathfrak{m}^{n-h}, N) \longrightarrow \mathrm{Tor}_{l+1}(\mathfrak{m}^{n-(h+1)}, N) \longrightarrow \mathrm{Tor}_{l+1}(\tfrac{\mathfrak{m}^{n-(h+1)}}{\mathfrak{m}^{n-h}}, N) = 0$$

for any R-module N. Hence $\mathrm{Tor}_{l+1}(\mathfrak{m}^{n-(h+1)}, N) = 0$ for any R-module N which implies

$$\mathrm{w.dim}(\mathfrak{m}^{n-(h+1)}) \leq l = \mathrm{w.dim}(\mathfrak{m})+1$$

Further, if $\mathrm{w.dim}(\mathfrak{m}^{n-(h+1)}) \lneqq l$, then we have

$$0 = \mathrm{Tor}_{l+1}(\tfrac{\mathfrak{m}^{n-(h+1)}}{\mathfrak{m}^{n-h}}, N) \longrightarrow \mathrm{Tor}_{l}(\mathfrak{m}^{n-h}, N) \longrightarrow \mathrm{Tor}_{l}(\mathfrak{m}^{n-(h+1)}, N) = 0$$

for any R-module N. Hence $\mathrm{Tor}_l(\mathfrak{m}^{n-h}, N) = 0$ for any R-module N which implies that $\mathrm{w.dim}(\mathfrak{m}^{n-h})) \leq l-1$, absurd. Hence $\mathrm{w.dim}(\mathfrak{m}^{n-(h+1)}) = \mathrm{w.dim}(\mathfrak{m})+1$, the desired contradiction. Therefore the claim is true and, in particular, for $k = n-1$, we have $\mathrm{w.dim}(\mathfrak{m}) = \mathrm{w.dim}(\mathfrak{m})+1$, which yields $\mathrm{w.dim}(\mathfrak{m}) = \infty$. □

Proof of Theorem 2.9. Suppose that R is Gaussian and $\mathfrak{m}$ is a maximal ideal in R such that $\mathrm{Nil}(R_{\mathfrak{m}})$ is a nonzero nilpotent ideal. Then $R_{\mathfrak{m}}$ is also Gaussian and $\mathrm{Nil}(R_{\mathfrak{m}})$ is a prime ideal in R. Moreover $\mathrm{Nil}(R_{\mathfrak{m}}) = pR_{\mathfrak{m}} \neq 0$ for some prime ideal p in R. Now, the maximal ideal pR_p of R_p is nonzero since $0 \neq pR_{\mathfrak{m}} \subseteq pR_p$. Also by assumption, there is a positive integer n such that $(pR_{\mathfrak{m}})^n = 0$, whence $p^n = 0$. So $(pR_p)^n = 0$ and hence pR_p is nilpotent. Therefore R_p is a local ring with nonzero nilpotent maximal ideal. By Proposition 2.12, $\mathrm{w.gl.dim}(R_p) = \infty$. Since $\mathrm{w.gl.dim}(R) \geq \mathrm{w.gl.dim}(R_S)$ for any localization R_S of R, we get $\mathrm{w.gl.dim}(R) = \infty$. □

In the previous chapter, we saw that the weak global dimension of an arithmetical ring is 0, 1, or ∞. In this chapter, we saw that the same result holds if R is Prüfer coherent or R is a Gaussian ring with a maximal ideal $\mathfrak{m}$ such that $\mathrm{Nil}(R_{\mathfrak{m}})$ is a nonzero nilpotent ideal. The question of whether this result is true for an arbitrary Gaussian ring is still elusively open. So we close this sectionchapter by announcing the Bazzoni-Glaz conjecture.

Conjecture 2.13. The weak global dimension of a Gaussian ring is 0, 1, or ∞.

Chapter 3

Gaussian rings via trivial ring extensions

In this chapter, we will use trivial ring extensions to construct new examples of non- arithmetical Gaussian rings , non-Gaussian Prüfer rings, and illustrative examples for Theorem 1.4 and Theorem 2.9. Let A be a ring and M an R-module. The trivial ring extension of A by M (also called the idealization of M over A) is the ring $R := A \ltimes M$ whose underlying group is $A \times M$ with multiplication given by

$$(a,x)(a',x') = (aa', ax' + a'x).$$

Recall that if I is an ideal of A and M' is a submodule of M such that $IM \subseteq M'$, then $J := I \ltimes M'$ is an ideal of R; ideals of R need not be of this form [26, Example 2.5]. However, the form of the prime (resp., maximal) ideals of R is $p \ltimes M$, where p is a prime (resp., maximal) ideal of A [23, Theorem 25.1(3)]. Suitable background on trivial extensions is [19, 23, 26].

The following lemma is useful for the construction of rings with infinite weak global dimension.

Lemma 3.1 ([3, Lemma 2.3])**.** *Let K be a field, E a nonzero K-vector space, and $R := K \ltimes E$.*

Then $\mathrm{w.gl.dim}(R) = \infty$.

Proof. First note that $R^{(I)} \cong A^{(I)} \ltimes E^{(I)}$. So let us identify $R^{(I)}$ with $A^{(I)} \ltimes E^{(I)}$ as R-modules. Now let $\{f_i\}_{i\in I}$ be a basis of E and $J := 0 \ltimes E$. Consider the R-map $u : R^{(I)} \longrightarrow J$ defined by $u((a_i, e_i)_{i\in I}) = (0, \sum_{i\in I} a_i f_i)$. Then we have the following short exact sequence of R-modules

$$0 \longrightarrow \mathrm{Ker}(u) \longrightarrow R^{(I)} \stackrel{u}{\longrightarrow} J \longrightarrow 0$$

But $\mathrm{Ker}(u) = 0 \ltimes E^{(I)}$. Indeed, clearly $0 \ltimes E^{(I)} \subseteq \mathrm{Ker}(u)$. Now suppose that $u((a_i, e_i)) = (0,0)$. Then $\sum_{i\in I} a_i f_i = 0$, hence $a_i = 0$ for each i as $\{f_i\}_{i\in I}$ is a basis for E and we have the equality. Therefore, the above exact sequence becomes

$$0 \longrightarrow 0 \ltimes E^{(I)} \longrightarrow R^{(I)} \stackrel{u}{\longrightarrow} J \longrightarrow 0 \qquad (*)$$

We claim that J is not flat. Deny. Then by [29, Theorem 3.55], $0 \ltimes E^{(I)} \bigcap JR^{(I)} = (0 \ltimes E^{(I)})J$. But $(0 \ltimes E^{(I)})J = 0$. By using the above identification, we obtain $0 = 0 \ltimes E^{(I)} \bigcap JR^{(I)} = (J)^{(I)} \bigcap J^{(I)} = J^{(I)} = 0 \ltimes E^{(I)}$, absurd (since $E \neq 0$).

Now, by Lemma 2.10, $\mathrm{w.dim}(J) = \mathrm{w.dim}(J^{(I)}) + 1 = \mathrm{w.dim}(J) + 1$. It follows that $\mathrm{w.gl.dim}(R) = \mathrm{w.dim}(J) = \infty$. □

Next, we announce the main result of this chapter.

Theorem 3.2 ([3, Theorem 3.1])**.** *Let $(A, \mathfrak{m})$ be a local ring, E a nonzero $\frac{A}{\mathfrak{m}}$-vector space, and $R := A \ltimes E$ the trivial ring extension of A by E. Then:*

(1) *R is a total ring of quotients and hence a Prüfer ring.*

(2) *R is Gaussian if and only if A is Gaussian.*

(3) *R is arithmetical if and only if $A := K$ is a field and* $\dim_K(E) = 1$.

(4) $\mathrm{w.gl.dim}(R) \gneqq 1$. *If $\mathfrak{m}$ admits a minimal generating set, then* $\mathrm{w.gl.dim}(R)$ *is infinite.*

Proof. (1) Let $(a,e)\in R$. Then either $a\in\mathfrak{m}$ and in this case $(a,e)(0,e)=(0,ae)=(0,0)$, or $a\notin\mathfrak{m}$ which implies a is a unit and hence $(a,e)(a^{-1},-a^{-2}e)=(1,0)$, the unity of R. Therefore R is total ring of quotients and hence a Prüfer ring.

(2) Suppose that R is Gaussian. Then, since $A\cong\frac{R}{0\ltimes E}$ and the Gaussian property is stable under factor rings, A is Gaussian.

Conversely, assume that A is Gaussian and let $F:=\sum(a_i,e_i)X^i$ be a polynomial in $R[X]$. If $a_i\notin\mathfrak{m}$ for some i, then (a_i,e_i) is invertible as $(a_i,e_i)(a_i^{-1},-a^{-2}e_i)=(1,0)$. We claim that F is Gaussian. Indeed, for any $G\in R[X]$, we have $c(F)c(G)=Rc(G)=c(G)\subseteq c(FG)$. The reverse inclusion always holds. If $a_i\in\mathfrak{m}$ for each i, let $G:=\sum(a'_j,e'_j)X^j\in R[X]$. We may assume, without loss of generality, that $a'_j\in\mathfrak{m}$ for each j (otherwise, we return to the first case) and let $f:=\sum a_iX^i$ and $g:=\sum a'_jX^j$ in $A[X]$. Then $c(FG)=c(fg)\ltimes c(fg)E$. But since E is an $\frac{A}{\mathfrak{m}}$-vector space, $\mathfrak{m}E=0$ yields $c(FG)=c(fg)\ltimes 0=c(f)c(g)\ltimes 0=c(F)c(G)$, since A is Gaussian. Therefore R is Gaussian, as desired.

(3) Suppose that R is arithmetical. First we claim that A is a field. On the contrary, assume that A is not a field. Then $\mathfrak{m}\neq 0$, so there is $a\neq 0\in\mathfrak{m}$. Let $e\neq 0\in E$. Since R is a local arithmetical ring (i.e., chained ring), either $(a,0)=(a',e')(0,e)=(0,a'e)$ for some $(a',e')\in R$ which contradicts $a\neq 0$; or $(0,e)=(a'',e'')(a,0)=(a'a,0)$ for some $(a'',e'')\in R$ which contradicts $e\neq 0$. Hence A is a field. Next, we show that $\dim_K(E)=1$. Let e,e' be two nonzero vectors in E. We claim that they are linearly dependent. Indeed, since R is a local arithmetical ring, either $(0,e)=(a,e'')(0,e')=(0,ae')$ for some $(a,e'')\in R$, hence $e=ae'$; or similarly if $(0,e')\in(0,e)R$. Consequently, $\dim_K(E)=1$.

Conversely, let J be a nonzero ideal in $K\ltimes K$ and $(0,0)\neq(a,b)\in J$. Then $(0,a^{-1})(a,b)=(0,1)\in J$. Hence $0\ltimes K\subseteq J$. But $0\ltimes K$ is maximal since 0 is the maximal ideal in K. So the ideals of $K\ltimes K$ are $(0,0)K\ltimes K$, $0\ltimes K=R(0,1)$, and $K\ltimes K$. Therefore $K\ltimes K$ is a principal ring and hence arithmetical.

(4) First $\operatorname{w.gl.dim}(R)\gneqq 1$. Let $J:=0\ltimes E$ and $\{f_i\}_{i\in I}$ be a basis of the $\frac{A}{\mathfrak{m}}$-vector space

E. Consider the map $u : R^{(I)} \longrightarrow J$ defined by $u((a_i, e_i)_{i\in I}) = (0, \sum_{i\in I} a_i f_i)$. Here we are using the same identification that has been used in Lemma 3.1. Then clearly $\mathrm{Ker}(u) = (\mathfrak{m} \ltimes E)^{(I)}$. Hence we have the short exact sequence of R-modules

$$0 \longrightarrow (\mathfrak{m} \ltimes E)^{(I)} \longrightarrow R^{(I)} \xrightarrow{u} J \longrightarrow 0 \qquad (1)$$

We claim that J is not flat. Otherwise, by [29, Theorem3.55], we have

$$J^{(I)} = (\mathfrak{m} \ltimes E)^{(I)} \cap JR^{(I)} = J(\mathfrak{m} \ltimes E^{(I)}) = 0.$$

Hence, by [29, Theorem 2.44], $\mathrm{w.gl.dim}(R) \gneqq 1$.

Next, assume that $\mathfrak{m}$ admits a minimal generating set. Then $\mathfrak{m} \ltimes E$ admits a minimal generating set (since E is a vector space). Now let $(b_i, g_i)_{i\in L}$ be a minimal generating set of $\mathfrak{m} \ltimes E$. Consider the R-map $v : R^{(L)} \longrightarrow \mathfrak{m} \ltimes E$ defined by $v((a_i, e_i)_{i\in L}) = \sum_{i\in L} (a_i, e_i)(b_i, g_i)$. Then we have the exact sequence

$$0 \longrightarrow \mathrm{Ker}(v) \longrightarrow R^{(L)} \xrightarrow{v} \mathfrak{m} \ltimes E \longrightarrow 0 \qquad (2)$$

We claim that $\mathrm{Ker}(v) \subseteq (\mathfrak{m} \ltimes E)^{(L)}$. On the contrary, suppose that there is $x = ((a_i, e_i)_{i\in L}) \in \mathrm{Ker}(v)$ and $x \notin (\mathfrak{m} \ltimes E)^{(L)}$. Then $\sum_{i\in L} (a_i, e_i)(b_i, g_i) = 0$ and as $x \notin (\mathfrak{m} \ltimes E)^{(L)}$, there is (a_j, e_j) with $a_j \notin \mathfrak{m}$. So (a_j, e_j) is a unit, which contradicts the minimality of $(b_i, g_i)_{i\in L}$. It follows that

$$\mathrm{Ker}(v) = V \ltimes E^{(L)} = (V \ltimes 0) \bigoplus (0 \ltimes E^{(L)}) = (V \ltimes 0) \bigoplus J^{(L)}$$

where $V := \{(a_i)_{i\in L} \in \mathfrak{m}^i \mid \sum_{i\in L} a_i b_i = 0\}$. Indeed, if $x \in \mathrm{Ker}(v)$, then $x = (a_i, b_i)_{i\in L}$ where $a_i \in \mathfrak{m}$, $b_i \in E$, with $\sum_{i\in L} a_i b_i = 0$, hence $\mathrm{Ker}(v) \subseteq V \ltimes E^{(L)}$. The other inclusion is trivial. Now, by Lemma 2.10 applied to (1), we get

$$\mathrm{w.dim}(J) = \mathrm{w.dim}((\mathfrak{m} \ltimes E)^I) + 1 = \mathrm{w.dim}(\mathfrak{m} \ltimes E) + 1.$$

On the other hand, from (2) we obtain

$$\mathrm{w.dim}(J) \leq \mathrm{w.dim}(V \ltimes 0 \oplus J^L) = \mathrm{w.dim}(\mathrm{Ker}(v)) \leq \mathrm{w.dim}(\mathfrak{m} \ltimes E).$$

It follows that

$$\mathrm{w.dim}(J) \leq \mathrm{w.dim}(J) - 1.$$

Consequently, $\mathrm{w.gl.dim}(R) = \mathrm{w.dim}(J) = \infty$. □

Next, we give examples of non-arithmetical Gaussian rings.

Example 3.3. (1) Let p be a prime number. Then $\mathbb{Z}_{(p)}$ is a non-trivial valuation domain. Hence $\mathbb{Z}_{(p)} \ltimes \frac{\mathbb{Z}}{p\mathbb{Z}}$ is a non-arithmetical Gaussian total ring of quotients by Theorem 3.2.

(2) Since $\dim_{\mathbb{R}}(\mathbb{C}) = 2 \gneqq 1$, $\mathbb{R} \ltimes \mathbb{C}$ is a non arithmetical Gaussian total ring of quotient. In general, if K is a field and E is a K-vector space with $\dim_K(E) \gneqq 1$, then $R := K \ltimes E$ is a non-arithmetical Gaussian total ring of quotients by Theorem 3.2.

Next, we provide examples of non-Gaussian total rings of quotients and hence non-Gaussian Prüfer rings.

Example 3.4. Let $(A, \mathfrak{m})$ be a non-valuation local domain. By Theorem 3.2, $R := A \ltimes \frac{A}{\mathfrak{m}}$ is a non-Gaussian total ring of quotients, hence a non-Gaussian Prüfer ring.

The following is an illustrative example for Theorem 1.4.

Example 3.5. Let $R := \mathbb{R} \ltimes \mathbb{R}$. Then R is a local ring with maximal ideal $0 \ltimes \mathbb{R}$ and $\mathrm{Z}(R) = 0 \ltimes \mathbb{R}$. Further, R is arithmetical by Theorem 3.2. By Osofsky's Theorem (Theorem 1.4) or by Lemma 3.1, $\mathrm{w.gl.dim}(R) = \infty$.

Now we give an example of a non-coherent local Gaussian ring with nilpotent maximal ideal and infinite weak global dimension (i.e., an illustrative example for Theorem 2.9).

Example 3.6. Let K be a field and X an indeterminate over K and let $R := K \ltimes K[X]$. Then:

(1) R is a non-arithmetical Gaussian ring since K is Gaussian and $\dim_K(K[X]) = \infty$ by Theorem 3.2.

(2) R is not a coherent ring since $\dim_K(K[X]) = \infty$ by [26, Theorem 2.6].

(3) R is local with maximal ideal $\mathfrak{m} = 0 \ltimes K[X]$ by [23, Theorem 25.1(3)]. Also $\mathfrak{m}$ is nilpotent since $\mathfrak{m}^2 = 0$. Therefore, by Theorem 2.9, $\operatorname{w.gl.dim}(R) = \infty$.

Chapter 4

Weak global dimension of fqp-rings

Recently, Abuhlail, Jarrar, and Kabbaj studied commutative rings in which every finitely generated ideal is quasi-projective (fqp-rings). They investigated the correlation of fqp-rings with well-known Prüfer conditions; namely, they proved that the class of fqp-rings stands strictly between the two classes of arithmetical rings and Gaussian rings [1, Theorem 3.2]. Also they generalized Osofsky's Theorem on the weak global dimension of arithmetical rings (and partially resolved Bazzoni-Glaz's related conjecture on Gaussian rings) by proving that the weak global dimension of an fqp-ring is 0, 1, or ∞ [1, Theorem 3.11]. In this chapter, we will give the proofs of the above mentioned results. Here too, the needed examples in this chapter will be constructed by using trivial ring extensions. We start by recalling some definitions.

Definition 4.1. (1) Let M be an R-module. An R-module M' is M-projective if the canonical map $\psi : \mathrm{Hom}_R(M', M) \longrightarrow \mathrm{Hom}_R(M', \frac{M}{N})$ is surjective for every submodule N of M.

(2) M' is quasi-projective if it is M'-projective.

Definition 4.2. A commutative ring R is said to be an fqp-ring if every finitely generated ideal of R is quasi-projective.

The following theorem establishes the relation between the class of fqp-rings and the two classes of arithmetical and Gaussian rings.

Theorem 4.3 ([1, Theorem 3.2])**.** *For a ring R, we have*

$$R\ arithmetical\ \Rightarrow\ R\ fqp-ring\ \Rightarrow\ R\ Gaussian$$

where the implications are irreversible in general.

The proof of this theorem needs the following results.

Lemma 4.4 ([1, Lemma 2.2])**.** *Let M be a finitely generated R-module. Then M is quasi-projective if and only if M is projective over $\frac{R}{\operatorname{Ann}(M)}$.* □

Lemma 4.5 ([17, Corollary 1.2])**.** *Let $M_{i 1\leq i\leq n}$ be a finite family of R-modules. Then $\bigoplus_{i=1}^{n} M_i$ is quasi-projective if and only if M_i is M_j-projective for all $i, j \in \{1,\ 2,\ ...,\ n\}$.*

Lemma 4.6 ([1, Lemma 3.6])**.** *Let R be an fqp-ring. Then $S^{-1}R$ is an fqp-ring, for any multiplicative closed subset of R.*

Proof. Let J be a finitely generated ideal of $S^{-1}R$. Then $J = S^{-1}I$ for some finitely generated ideal I of R. Since R is an fqp-ring, I is quasi-projective and hence, by Lemma 4.4, I is projective over $\frac{R}{\operatorname{Ann}(I)}$. By [29, Theorem 3.76], $J := S^{-1}I$ is projective over $\frac{S^{-1}R}{S^{-1}\operatorname{Ann}(I)}$. But $S^{-1}\operatorname{Ann}(I) = \operatorname{Ann}(S^{-1}I) = \operatorname{Ann}(J)$ by [2, Proposition 3.14]. Therefore $J := S^{-1}I$ is projective over $\frac{S^{-1}R}{\operatorname{Ann}(S^{-1}I)}$. Again by Lemma 4.4, J is quasi-projective. It follows that $S^{-1}R$ is an fqp-ring. □

Lemma 4.7 ([1, Lemma 3.8]). *Let R be a local ring and $a,\ b$ two nonzero elements of R such that (a) and (b) are incomparable. If $(a,\ b)$ is quasi-projective, then $(a)\cap(b)=0$, $a^2=b^2=ab=0$, and $\operatorname{Ann}(a)=\operatorname{Ann}(b)$.*

Proof. Let $I:=(a,\ b)$ be quasi-projective. Then by [33, Lemma 2], there exist $f_1,\ f_2 \in \operatorname{End}_R(I)$ such that $f_1(I)\subseteq(a)$, $f_2(I)\subseteq(b)$, and $f_1+f_2=1_I$. Now let $x\in(a)\cap(b)$. Then $x=r_1a=r_2b$ for some $r_1,\ r_2\ \in R$. But $x=f_1(x)+f_2(x)=f_1(r_1a)+f_2(r_2b)=r_1f_1(a)+r_2f_2(b)=r_1a'a+r_2b'b=a'x+b'x$ for some $a',\ b'\ \in R$. We claim that a' is a unit. Deny. Since R is local, $1-a'$ is a unit. But $a=f_1(a)+f_2(a)=a'a+f_2(a)$. Hence $(1-a')a=f_2(a)\subseteq(b)$ which implies that $a\in(b)$, absurd since (a) and (b) are incomparable. Similarly, b' is a unit. It follows that $(a'-(1-b'))$ is a unit. But $x=a'x+b'x$ yields $(a'-(1-b'))x=0$. Therefore $x=0$ and $(a)\cap(b)=0$.

Next, we prove that $a^2=b^2=ab=0$. Obviously, $(a)\cap(b)=0$ implies that $ab=0$. So it remains to prove that $a^2=b^2=0$. Since $(a)\cap(b)=0$, $I=(a)\oplus(b)$. By Lemma 4.5, (b) is (a)-projective. Let $\varphi:(a)\longrightarrow\frac{(a)}{a\operatorname{Ann}(b)}$ be the canonical map and $g:(b)\longrightarrow\frac{(a)}{a\operatorname{Ann}(b)}$ be defined by $g(rb)=r\bar{a}$. If $r_1b=r_2b$, then $(r_1-r_2)b=0$. Hence $r_1-r_2\in\operatorname{Ann}(b)$ which implies that $(r_1-r_2)\bar{a}=0$. So $g(r_1b)=g(r_2b)$. Consequently, g is well defined. Clearly g is an R-map. Now, since (b) is (a)-projective, there exists an R-map $f:(b)\longrightarrow(a)$ with $\varphi\circ f=g$. For b, we have $f(b)\in(a)$, hence $f(b)=ra$ for some $r\in R$. Also $(\varphi\circ f)(b)=g(b)$. Hence $f(b)-a\in a\operatorname{Ann}(b)$. Whence $ra-a=at$ for some $t\in\operatorname{Ann}(b)$ which implies that $(t+1)a=ra$. By multiplying the last equality by a we obtain, $(t+1)a^2=ra^2$. But $ab=0$ implies $0=f(ab)=af(b)=ra^2$. Hence $(t+1)a^2=0$. Since $t\in\operatorname{Ann}(b)$ and R is local, $(t+1)$ is a unit. It follows that $a^2=0$. Likewise $b^2=0$.

Last, let $x\in\operatorname{Ann}(b)$. Then $f(xb)=xra=0$. The above equality $(t+1)a=ra$ implies $(t+1-r)a=0$. But $t+1$ is a unit and R is local. So that r is a unit ($b\neq0$). Hence $xa=0$. Whence $x\in\operatorname{Ann}(a)$ and $\operatorname{Ann}(b)\ \subseteq\ \operatorname{Ann}(a)$. Similarly we can show that $\operatorname{Ann}(a)\ \subseteq\ Ann(b)$. Therefore $\operatorname{Ann}(a)\ =\ \operatorname{Ann}(b)$. $\square$

Proof of Theorem 4.3. R arithmetical $\Rightarrow$ R fqp-ring.

Let R be an arithmetical ring, I a nonzero finitely generated ideal of R, and p a prime ideal of R. Then $I_p := IR_p$ is finitely generated. But R is arithmetical, hence R_p is a chained ring and I_p is a principal ideal of R_p. By [27], I_p is quasi-projective. By [35, 19.2] and [36], it suffices to prove that $(\mathrm{Hom}_R(I,\ I))_p \cong \mathrm{Hom}_{R_p}(I_p,\ I_p)$. But $\mathrm{Hom}_{R_p}(I_p,\ I_p) \cong \mathrm{Hom}_R(I,\ I_p)$ by the adjoint isomorphisms theorem [29, Theorem 2.11] (since $\mathrm{Hom}_{S^{-1}R}(S^{-1}N, S^{-1}M) \cong \mathrm{Hom}(N, S^{-1}M)$ where $S^{-1}N \cong N \bigotimes_R S^{-1}R$ and $S^{-1}M \cong \mathrm{Hom}_{S^{-1}R}(S^{-1}R, S^{-1}M)$). So let us prove that $(\mathrm{Hom}_R(I,\ I))_p \cong \mathrm{Hom}_R(I,\ I_p)$. Let

$$\phi : (\mathrm{Hom}_R(I,\ I))_p \longrightarrow \mathrm{Hom}_R(I,\ I_p)$$

be the function defined by $\phi(\frac{f}{s})(x) = \frac{f(x)}{s}$, for each $x \in I$, $\frac{f}{s} \in (\mathrm{Hom}_R(I,\ I))_p$. Clearly ϕ is a well-defined R-map. Now suppose that $\phi(\frac{f}{s}) = 0$. I is finitely generated, so let $I = (x_1,\ x_2, \ldots,\ x_n)$, where n is an integer. Then for every $i \in \{1,\ 2,\ \ldots,\ n\}$, $\phi(\frac{f}{s})(x_i) = \frac{f(x_i)}{s} = 0$, whence there exists $t_i \in R \setminus p$ such that $t_i f(x_i) = 0$. Let $t := t_1 t_2 \ldots t_n$. Clearly, $t \in R \setminus p$ and $tf(x) = 0$, for all $x \in I$. Hence $\frac{f}{s} = 0$. Consequently, ϕ is injective. Next, let $g \in \mathrm{Hom}_R(I,\ I_p)$. Since I_p is principal in R_p, $I_p = aR_p$ for some $a \in I$. But $g(a) \in I_p$. Hence $g(a) = \frac{ca}{s}$ for some $c \in R$ and $s \in R \setminus p$. Let $x \in I$. Then $\frac{x}{1} \in I_p = aR_p$. Hence $\frac{x}{1} = \frac{ra}{u}$ for some $r \in R$ and $u \in R \setminus p$. So there exists $t \in R \setminus p$ such that $tux = tra$. Now, let $f : I \longrightarrow I$ be the multiplication by c. (i.e., for $x \in I$, $f(x) = cx$). Then $f \in \mathrm{Hom}_R(I,\ I)$ and we have

$$\phi(\frac{f}{s})(x) = \frac{f(x)}{s} = \frac{cx}{s} = \frac{c}{s}\frac{x}{1} = \frac{cra}{su} = \frac{r}{u}g(a) = \frac{1}{tu}g(tra) = \frac{1}{tu}g(txu) = g(x).$$

Therefore ϕ is surjective and hence an isomorphism, as desired.

R fqp-ring $\Rightarrow$ R Gaussian

Recall that, if $(R, \mathfrak{m})$ is a local ring with maximal ideal $\mathfrak{m}$, then R is a Gaussian ring if and only if for any two elements a, b in R, $(a,b)^2 = (a^2)$ or (b^2) and if $(a,b)^2 = (a^2)$ and

$ab = 0$, then $b^2 = 0$ [5, Theorem 2.2 (d)].

Let R be an fqp-ring and let P be any prime ideal of R. Then by Lemma 4.6 R_p is a local fqp-ring. Let $a,\ b\ \in R_P$. We investigate two cases. The first case is $(a,\ b) = (a)$ or (b), say (b). So $(a,\ b)^2 = (b^2)$. Now assume that $ab = 0$. Since $a \in (b)$, $a = cb$ for some $c \in R$. Therefore $a^2 = cab = 0$. The second case is $I := (a,\ b)$ with $I \neq (a)$ and $I \neq (b)$. Necessarily, $a \neq 0$ and $b \neq 0$. By Lemma 4.7, $a^2 = b^2 = ab = 0$. Both cases satisfy the conditions that were mentioned at the beginning of this proof (The conditions of [5, Theorem 2.2 (d)]). Hence R_p is Gaussian. But p being an arbitrary prime ideal of R and the Gaussian notion being a local property, then R is Gaussian.

To prove that the implications are irreversible in general, we will use the following theorem to build examples for this purpose.

Theorem 4.8 ([1, Theorem 4.4])**.** *Let $(A,\ \mathfrak{m})$ be a local ring and E a nonzero $\frac{A}{\mathfrak{m}}$-vector space. Let $R := A \ltimes E$ be the trivial ring extension of A by E. Then R is an fqp-ring if and only if* $\mathfrak{m}^2 = 0$.

The proof of this theorem depends on the following lemmas. Trivial ring extensions were defined in this thesis in chapter 3.

Lemma 4.9 ([30, Theorem 2])**.** *Let R be a local fqp-ring which is not a chained ring. Then* $(\mathrm{Nil}(R))^2 = 0$.

Lemma 4.10 ([1, Lemma 4.5])**.** *Let R be a local fqp-ring which is not a chained ring. Then* $\mathrm{Z}(R) = \mathrm{Nil}(R)$.

Proof. We always have $\mathrm{Nil}(R) \subseteq \mathrm{Z}(R)$. Now, let $s \in \mathrm{Z}(R)$. Then there exists $t \neq 0 \in R$ such that $st = 0$. Since R is not chained, there exist nonzero elements $x,\ y\ \in R$ such that (x) and (y) are incomparable. By Lemma 4.7, $x^2 = xy = y^2 = 0$. Either (x) and (s) are incomparable and hence, by Lemma 4.7, $s^2 = 0$. Whence $s \in \mathrm{Nil}(R)$. Or (x) and (s) are comparable. In this case, either $s = rx$ for some $r \in R$ which implies that $s^2 = r^2x^2 = 0$ and hence $s \in \mathrm{Nil}(R)$. Or $x = sx'$ for some $x' \in R$. Same arguments applied to (s) and (y) yield either $s \in \mathrm{Nil}(R)$ or $y = sy'$ for some $y' \in R$. Since (x) and (y) are incomparable, (x') and (y') are incomparable. Hence, by Lemma 4.7, $(x') \cap (y') = 0$. If (x') and (t) are incomparable, then by Lemma 4.7, $\mathrm{Ann}(x') = \mathrm{Ann}(t)$. So that $s \in \mathrm{Ann}(x')$ which implies that $x = sx' = 0$, absurd. If $(t) \subseteq (x')$, then $(t) \cap (y') \subseteq (x') \cap (y') = 0$. So (t) and (y') are incomparable, whence similar arguments as above yield $y = 0$, absurd. Last, if $(x') \subseteq (t)$, then $x' = r't$ for some $r' \in R$. Hence $x = sx' = str' = 0$, absurd. Therefore all the possible cases lead to $s \in \mathrm{Nil}(R)$. Consequently, $\mathrm{Z}(R) = \mathrm{Nil}(R)$. □

Lemma 4.11 ([1, Lemma 4.6])**.** *Let* $(R,\ \mathfrak{m})$ *be a local ring such that* $\mathfrak{m}^2 = 0$*. Then* R *is an fqp-ring.*

Proof. Let I be a nonzero proper finitely generated ideal of R. Then $I \subseteq \mathfrak{m}$ and $\mathfrak{m}I = 0$. Hence $\mathfrak{m} \subseteq \mathrm{Ann}(I)$, whence $\mathfrak{m} = \mathrm{Ann}(I)$ $(I \neq 0)$. So that $\frac{R}{\mathrm{Ann}(I)} \cong \frac{A}{\mathfrak{m}}$ which implies that I is a free $\frac{R}{\mathrm{Ann}(I)}$-module, hence projective over $\frac{R}{\mathrm{Ann}(I)}$. By Lemma 4.4, I is quasi-projective. Consequently, R is an fqp-ring. □

Proof of Theorem 4.8. Assume that R is an fqp-ring. We may suppose that A is not a field. Then R is not a chained ring since $((a,\ 0)$ and $((0,\ e))$ are incomparable where $a \neq 0 \in \mathfrak{m}$ and $e = (1,\ 0,\ 0,\ ...) \in E$. Also R is local with maximal $\mathfrak{m} \ltimes E$. By Lemma 4.10, $\mathrm{Z}(R) = \mathrm{Nil}(R)$. But $\mathfrak{m} \ltimes E = \mathrm{Z}(R)$. For, let $(a,e) \in \mathfrak{m} \ltimes E$. Since E is an $\frac{A}{\mathfrak{m}}$-vector space, $(a,e)(0,e) = (0,ae) = (0,0)$. Hence $\mathfrak{m} \ltimes E \subseteq \mathrm{Z}(R)$. The other inclusion holds since $\mathrm{Z}(R)$ is an ideal. Hence $\mathfrak{m} \ltimes E = \mathrm{Nil}(R)$. By Lemma 4.9, $(\mathrm{Nil}(R))^2 = 0 = (\mathfrak{m} \ltimes E)^2$. Consequently, $\mathfrak{m}^2 = 0$.

Conversely, $\mathfrak{m}^2 = 0$ implies $(\mathfrak{m} \ltimes E)^2 = 0$ and hence by Lemma 4.11, R is an fqp-ring.

Now we can use Theorem 4.8 to construct examples which prove that the implications in Theorem 4.3 cannot be reversed in general. The following is an example of an fqp-ring which is not an arithmetical ring

Example 4.12. $R := \frac{\mathbb{R}[X]}{(X^2)} \ltimes \mathbb{R}$ is an fqp-ring by Theorem 4.8, since R is local with a nilpotent maximal ideal $\frac{(X)}{(X^2)} \ltimes \mathbb{R}$. Also, since $\frac{\mathbb{R}[X]}{(X^2)}$ is not a field, R is not arithmetical by Theorem 3.2.

The following is an example of a Gaussian ring which is not an fqp-ring.

Example 4.13. $R := \mathbb{R}[X]_{(X)} \ltimes \mathbb{R}$ is Gaussian by Theorem 3.2. Also, by Theorem 4.8, R is not an fqp-ring.

Now the natural question is what are the values of the weak global dimension of an arbitrary fqp-ring? The answer is given by the following theorem.

Theorem 4.14 ([1, Theorem 3.11])**.** *Let R be an fqp-ring. Then* $\mathrm{w.gl.dim}(R) = 0, 1,$ *or* ∞.

Proof. Since $\mathrm{w.gl.dim}(R) = \sup\{\mathrm{w.gl.dim}(R_p) \mid p \text{ prime ideal of } R\}$, one can assume that R is a local fqp-ring. If R is reduced, then by Lemma 2.5, $\mathrm{w.gl.dim}(R) \leq 1$. If R is not reduced, then $\mathrm{Nil}(R) \neq 0$. By Lemma 4.9, either $(\mathrm{Nil}(R))^2 = 0$, in this case, $\mathrm{w.gl.dim}(R) = \infty$ by Theorem 2.9 (since an fqp-ring is Gaussian); or R is a chained ring with zerodivisors ($\mathrm{Nil}(R) \neq 0$), in this case $\mathrm{w.gl.dim}(R) = \infty$ by Theorem 1.3. Consequently, $\mathrm{w.gl.dim}(R) = 0$, 1, or ∞. □

It is clear that Theorem 4.14 generalizes Osofsky's Theorem on the weak global dimension of arithmetical rings (Theorem 1.3) and partially resolves Bazzoni-Glaz Conjecture on Gaussian rings (Conjecture 2.13).

Chapter 5

Finitistic projective and weak dimensions

This chapter consists of two parts: In the first part, we will study the small finitistic projective dimension of Gaussian rings. It was proven by Bazzoni and Glaz that the small finitistic projective dimension of a Gaussian ring is less than or equal to 1 [5, Proposition 5.3]. In the second part, we will study the finitistic weak dimension of arithmetical rings. It was proven by Couchot that the only possible values of the finitistic weak dimension of an arithmetical ring are 0, 1, or 2 [12, Theorem 1].

Part 1: The small finitistic projective dimension of Gaussian rings.

Let $\mathrm{mod}(R)$ denote the class of R modules with a projective resolution consisting of finitely generated projective modules. Notice that all modules in this class are finitely generated. We start by recalling the definitions of the projective dimension of a module and the small finitistic projective dimension of a ring.

Definition 5.1. Let M be an R-module. Then:

(1) The projective dimension of M, denoted by $\mathrm{p.dim}_R(M)$, measures how far M is from being projective. It is defined as follows: Let n be a non-negative integer. We have

$\mathrm{p.dim}(M) \leq n$ if there is a projective resolution

$$0 \to P_n \to P_{n-1} \to \ldots \to P_1 \to P_0 \to M \to 0.$$

If no such resolution exists, $\mathrm{p.dim}(M) = \infty$; and if n is the least such integer, $\mathrm{p.dim}(M) = n$.

(2) The small finitistic projective dimension of R, denoted by $\mathrm{fp.dim}(R)$, is the supremum of $\mathrm{p.dim}(M)$, where M ranges over all R-modules in mod(R) with finite projective dimension.

Recall that the (big) finitistic projective dimension of R, denoted $\mathrm{FP.dim}(R)$ is defined as the supremum of $\mathrm{p.dim}_R(M)$, where M ranges over all R-modules with finite projective dimension. In particular, if $\mathrm{gl.dim}(R) < \infty$, then $\mathrm{gl.dim}(R) = \mathrm{FP.dim}(R)$. Recall also, from [19], that $\mathrm{fp.dim}(R) \leq \mathrm{w.gl.dim}(R) \leq \mathrm{gl.dim}(R)$ and $\mathrm{fp.dim}(R) \leq \mathrm{FP.dim}(R) \leq \mathrm{gl.dim}(R)$. Also Jensen proved that $\mathrm{gl.dim}(R) = \sup\{\mathrm{w.gl.dim}(R), \mathrm{FP.dim}(R)\}$ [18]. Gruson proved that if R is a Noetherian ring, then $\mathrm{FP.dim}(R) = \dim(R)$, the Krull dimension of R [34].

The following proposition asserts that the small finitistic projective dimension of a Gaussian ring is at most one.

Proposition 5.2 ([5, Proposition 5.3])**.** *Let R be a Gaussian ring. Then* $\mathrm{fp.dim}(R) = 0$ *or* 1.

To prove this proposition we need the following two lemmas.

Lemma 5.3 ([5, Lemma 5.1])**.** *Let R be a ring, I an ideal contained in the Jacobson radical of R, and $M \in$ mod(R). Assume that* $\mathrm{Tor}_1^R(\frac{R}{I}, M) = 0$. *Then*

$$\mathrm{p.dim}_R(M) = \mathrm{p.dim}_{\frac{R}{I}}(\frac{M}{IM}).$$

Proof. Since $M \in \mathrm{mod}(R)$, M has a projective resolution consisting of finitely generated projective modules, say

$$\ldots \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \ldots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

Trivially, all the syzygies in this resolution are in mod(R). In particular, $K_0 := \mathrm{Ker}(P_0 \longrightarrow M)$ is in mod(R). Consider the short exact sequence

$$0 \longrightarrow K_0 \longrightarrow P_0 \longrightarrow M \longrightarrow 0 \ (1)$$

First, we prove that $\mathrm{p.dim}_R(M) \geq \mathrm{p.dim}_{\frac{R}{I}}(\frac{M}{IM})$ by induction on $n := \mathrm{p.dim}_R(M)$. For $n = 0$, it holds since if M is R-projective, then $\frac{M}{IM} \cong \frac{R}{I} \bigotimes M$ is $\frac{R}{I}$-projective [29]. Now, suppose that the above inequality is true for $\mathrm{p.dim}_R(M) = n-1$ and $n \geq 1$. Then by [19, Theroem 1.3.2], $\mathrm{p.dim}_R(K_0) = n-1$. Since $K_0 \in \mathrm{mod}(R)$, by induction hypothesis, we have $\mathrm{p.dim}_{\frac{R}{I}}(\frac{K_0}{IK_0}) \leq n-1$. Since $\mathrm{Tor}_1^R(\frac{R}{I}, M) = 0$, the following sequence

$$0 \longrightarrow \frac{K_0}{IK_0} \longrightarrow \frac{P_0}{IP_0} \longrightarrow \frac{M}{IM} \longrightarrow 0$$

is exact. Again by [19, Theroem 1.3.2], we have

$$\mathrm{p.dim}_{\frac{R}{I}}(\frac{M}{IM}) \leq n := \mathrm{p.dim}_R(M).$$

Next, we wish to prove that $\mathrm{p.dim}_R(M) \leq \mathrm{p.dim}_{\frac{R}{I}}(\frac{M}{IM})$ by induction on $n := \mathrm{p.dim}_{\frac{R}{I}}(\frac{M}{IM})$. To prove that this inequality is true for $n = 0$, we first prove that if $\frac{M}{IM}$ is $\frac{R}{I}$-free, then M is R-free. Assume that $\frac{M}{IM}$ is $\frac{R}{I}$-free and let M' be the submodule of M generated by $\{m_i\}_{i=1}^k$, where $\{m_i \otimes 1\}_{i=1}^k$ forms a basis for $\frac{M}{IM}$ over $\frac{R}{I}$. Then $M = M' + IM$. By Nakayama's Lemma, $M = M'$. Hence M is generated by $\{m_i\}_{i=1}^k$. Let F be a free module generated by

$\{x_i\}_{i=1}^k$. Consider the short exact sequence

$$0 \longrightarrow K \longrightarrow F \xrightarrow{\sigma} M \longrightarrow 0 \qquad (2)$$

where $\sigma : F \longrightarrow M$ is the map defined by $\sigma(x_i) = m_i$. Since $\mathrm{Tor}_1^R(\frac{R}{I}, M) = 0$, we get the following exact sequence

$$0 \longrightarrow \frac{K}{IK} \longrightarrow \frac{F}{IF} \xrightarrow{\sigma\otimes 1_{\frac{R}{I}}} \frac{M}{IM} \longrightarrow 0.$$

Since $\{m_i \otimes 1\}_{i=1}^k$ forms a basis for $\frac{M}{IM}$, $\sigma \otimes 1_{\frac{R}{I}}$ is an isomorphism. Hence $K = IK$. By applying Schanuel's Lemma [29, Theorem 3.62] to the sequences (1) and (2), we obtain

$$K_0 \oplus F \cong P_0 \oplus K.$$

So K is finitely generated. Whence by Nakayama's Lemma, $K = 0$. It follows that M is R-free. Now, assume that $\frac{M}{IM}$ is $\frac{R}{I}$-projective. Then

$$\frac{F}{IF} \cong \frac{K}{IK} \oplus \frac{M}{IM} \cong (K \oplus M) \otimes \frac{R}{I} \cong \frac{K \oplus M}{I(K \oplus M)}.$$

Hence $\frac{K\oplus M}{I(K\oplus M)}$ is $\frac{R}{I}$-free. By the first step, $K \oplus M$ is R-free. Therefore M is R-projective. This proves that the inequality holds when $n = 0$. Now assume that $\mathrm{p.dim}_{\frac{R}{I}}(\frac{M}{IM}) = n > 0$. Then $\mathrm{p.dim}_{\frac{R}{I}}(\frac{K}{IK}) = n-1$. By induction hypothesis, we obtain $\mathrm{p.dim}_R(K) \leq n-1$. Consequently, $\mathrm{p.dim}_R(M) \leq n = \mathrm{p.dim}_{\frac{R}{I}}(\frac{M}{IM})$. □

Lemma 5.4 ([5, Lemma 5.2])**.** *Let R be a ring and I an ideal contained in the Jacobson radical of R. Then*

$$\mathrm{fp.dim}(R) \leq \mathrm{fp.dim}(\frac{R}{I}) + \mathrm{w.dim}_R(\frac{R}{I}).$$

Proof. If $\mathrm{fp.dim}(\frac{R}{I}) = \infty$ or $\mathrm{w.dim}(\frac{R}{I}) = \infty$, then the inequality is clear. Assume that $\mathrm{fp.dim}(\frac{R}{I}) = k < \infty$ and $\mathrm{w.dim}(\frac{R}{I}) = n < \infty$. Let $M \in \mathrm{mod}(R)$ of finite projective dimension and

$$0 \longrightarrow F_p \longrightarrow F_{p-1} \longrightarrow ... \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

be a free resolution of M and let K_{n-1} be the $(n-1)$st syzygy of this resolution. Since $\mathrm{w.dim}_R(\frac{R}{I}) = n$, $\mathrm{Tor}^R_{n+1}(M, \frac{R}{I}) = \mathrm{Tor}^R_1(K_{n-1}, \frac{R}{I}) = 0$. By Lemma 5.3, $\mathrm{p.dim}_R(K_{n-1}) = \mathrm{p.dim}_{\frac{R}{I}}(\frac{K_{n-1}}{IK_{n-1}})$. Since $\mathrm{p.dim}(M) < \infty$, $\mathrm{p.dim}_{\frac{R}{I}}(\frac{K_{n-1}}{IK_{n-1}}) = \mathrm{p.dim}$
$(K_{n-1}) < \infty$. It follows that $\mathrm{p.dim}(K_{n-1}) \leq k$. In view of the following exact sequence

$$0 \longrightarrow K_{n-1} \longrightarrow F_{n-1} \longrightarrow ... \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0,$$

we have

$$\mathrm{p.dim}(M) = \mathrm{p.dim}(K_{n-1}) + n \leq k + n = \mathrm{fp.dim}(\frac{R}{I}) + \mathrm{w.dim}(\frac{R}{I}).$$

Now, M being an arbitrary module in $\mathrm{mod}(R)$ implies that

$$\mathrm{fp.dim}(R) \leq \mathrm{fp.dim}(\frac{R}{I}) + \mathrm{w.dim}(\frac{R}{I}).$$

□

Proof of Proposition 5.2. There are two steps.

Step 1: Assume R is local with maximal ideal $\mathfrak{m}$. We envisage two cases.

Case 1: Assume $\mathfrak{m} = \mathrm{Z}(R)$. Let $I = (a_1, a_2, ..., a_n)$ be a finitely generated proper ideal of R. By Lemma 2.4, $I^2 = (a_i^2)$ for some $i \in \{1, , 2, ..., n\}$. Hence there exists $c \neq 0 \in R$ such that $ca_i^2 = 0$. Whence $cI^2 = 0 = I(cI)$. If $cI = 0$, then $\mathrm{Ann}(I) \neq 0$. If $cI \neq 0$, then there exists $b \in I$ such that $cb \neq 0$. But $cbI \subseteq cI^2 = 0$. Hence also in this case $\mathrm{Ann}(I) \neq 0$. By

[19, Corollary 3.3.17], $\operatorname{fp.dim}(R) = 0$.

Case 2: Assume $\mathfrak{m}$ contains a regular element a. Since aR is R-free, in view of the short exact sequence

$$0 \longrightarrow aR \longrightarrow R \longrightarrow \frac{R}{aR} \longrightarrow 0,$$

$\operatorname{w.dim}_R(\frac{R}{aR}) \leq 1$. By Lemma 5.4,

$$\operatorname{fp.dim}(R) \leq \operatorname{fp.dim}(\frac{R}{aR}) + \operatorname{w.dim}_R(\frac{R}{aR}) \leq \operatorname{fp.dim}(\frac{R}{aR}) + 1.$$

So to prove that $\operatorname{fp.dim}(R) \leq 1$, it suffices to prove that $\operatorname{fp.dim}(\frac{R}{aR}) = 0$. First note that $\bar{R} := \frac{R}{aR}$ is local Gaussian with maximal ideal $\bar{\mathfrak{m}} = \frac{\mathfrak{m}}{aR}$. By Case 1, it suffices to show that $\bar{\mathfrak{m}} = \mathrm{Z}(\bar{R})$. Let $\bar{b} \neq \bar{0} \in \bar{\mathfrak{m}}$. By Lemma 2.4, $(a,\ b)^2 = (a^2)$ or (b^2). If $(a,\ b)^2 = (a^2)$, then $b^2 \in (a^2) \subseteq (a)$, hence $\bar{b} \in \mathrm{Z}(\bar{R})$. If $(a,\ b)^2 = (b^2)$, then $a^2 \in (b^2)$. Let $t \in R$ such that $a^2 = tb^2$. Hence $\bar{t}\bar{b}^2 = \bar{0}$. If $\bar{t} \neq \bar{0}$ clearly, $\bar{b} \in \mathrm{Z}(\bar{R})$. Next, assume $\bar{t} = \bar{0}$, i.e., $t = ra$ for some $r \in R$. Then $a^2 = rab^2$. Since a is a nonzero divisor, $a = rb^2$. If $\bar{r} \neq \bar{0}$, then $\bar{b} \in \mathrm{Z}(\bar{R})$. If $r = as$ for some $s \in R$, $a = rb^2 = asb^2$. So $1 = sb^2$, a contradiction (since $b \in \mathfrak{m}$). Therefore $\bar{b}$ is a zero divisor. The reverse inclusion $\mathrm{Z}(\bar{R}) \subseteq \bar{\mathfrak{m}}$ holds since $\mathrm{Z}(\bar{R})$ is an ideal (in the local Gaussian ring $\bar{R}$). Consequently, in both cases, $\operatorname{fp.dim}(R) \leq 1$.

Step 2: Assume R is not local. Let $M \in \operatorname{mod}(R)$ with $\operatorname{p.dim}(M) < \infty$ and let $\mathfrak{m}$ be an arbitrary maximal ideal of R. Then $M_{\mathfrak{m}} \in \operatorname{mod}(R_{\mathfrak{m}})$ and $\operatorname{p.dim}_{R_{\mathfrak{m}}}(M_m) \leq \operatorname{p.dim}_R(M) \lneqq \infty$ (localization of a projective is projective). Hence $\operatorname{p.dim}_{R_{\mathfrak{m}}}(M_m) \leq \operatorname{fp.dim}(R_m) \leq 1$ by Step 1. By [19, Corollary 2.5.5], $\operatorname{w.dim}(M) = \operatorname{p.dim}(M)$. Hence $\operatorname{w.dim}(M_{\mathfrak{m}}) \leq 1$. Since

$$\operatorname{w.dim}_R(M) = \sup\{\operatorname{w.dim}_{R_{\mathfrak{m}}}(M_{\mathfrak{m}}) \mid \mathfrak{m} \text{ maximal ideal of R}\} \leq 1,$$

we get $\operatorname{p.dim}_R(M) \leq 1$. It follows that $\operatorname{fp.dim}(R) \leq 1$. This finishes the proof of the theorem. $\square$

Part 2: The finitistic weak dimension of arithmetical rings.

Recall the following definitions.

Definition 5.5. Let R be a ring. Then the finitistic weak dimension of R, denoted by $\mathrm{f.w.dim}(R)$, is the supremum of $\mathrm{w.dim}_R(M)$, where M ranges over all R-modules with finite weak dimension.

Notice that the small finitistic projective dimension of R may stand as a "small finitistic weak dimension" of R since in mod(R), the weak and projective dimensions coincide. Accordingly, we have

$$\mathrm{fp.dim}(R) \leq \mathrm{f.w.dim}(R) \leq \mathrm{w.gl.dim}(R).$$

Definition 5.6. Let R be a ring. Then:

(1) R is called semicoherent if $\mathrm{Hom}_R(E,F)$ is a submodule of a flat R-module for every injective R-modules E and F.

(2) R is called an IF-ring if every injective R-module is flat.

Recall that R is coherent if every finitely generated R-module is finitely presented. The following implications establish the relations between IF, coherent, and semicoherent rings ([9, 16]):

$$\text{IF-ring} \Rightarrow \text{Coherent ring} \Rightarrow \text{Semicoherent ring.}$$

The following theorem is the main result in this part. It states that the finitistic weak dimension of an arithmetical ring is at most 2.

Theorem 5.7 ([12, Theorem 1])**.** *Let R be an arithmetical ring. Then:*

(1) $\mathrm{f.w.dim}(R) = 0$ *if R is locally IF.*

(2) $\mathrm{f.w.dim}(R) = 1$ *if R is locally semicoherent but not locally IF.*

(3) $\mathrm{f.w.dim}(R) = 2$ *if R is not locally semicoherent.*

First, we prove the following theorem.

Theorem 5.8 ([12, Theorem 2])**.** *Let R be a chained ring. Then:*

(1) $\mathrm{f.w.dim}(R) = 0$ *if R is IF.*

(2) $\mathrm{f.w.dim}(R) = 1$ *if R is semicoherent but not IF.*

(3) $\mathrm{f.w.dim}(R) = 2$ *if R is not semicoherent.*

Moreover, an R-module M has a finite weak dimension if and only if $\mathrm{Z}(R) \otimes_R M$ *is flat.*

Throughout, if R is a chained ring, we denote by $\mathfrak{m}$ its maximal ideal, $\mathrm{Z}(R)$ its subset of zerodivisors which is a prime ideal, and $Q = R_{\mathrm{Z}(R)}$ its fraction ring. Also if M is an R-module, denote by $\mathrm{E}(M)$ the injective hull of M.

From [12], we recall the following background. A shot exact sequence $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ is pure if it remains exact when tensoring it with any R-module. In this case, we say that M' is a pure submodule of M. If M is flat, then M'' is flat if and only if M' is a pure submodule of M. An R-module M is FP-injective if $\mathrm{Ext}_1^R(F, M) = 0$ for any finitely presented R-module F. A ring R is self FP-injective if it is FP-injective as an R-module. A module M is FP-injective if and only if it is a pure submodule of every overmodule. A ring R is IF if and only if it is coherent and self FP-injective. The following equivalent statements will be used frequently in the proof of all assertions of this theorem.

Proposition 5.9 ([12, Proposition 4])**.** *Let R be a chained ring. The following conditions are equivalent:*

(1) *R is semicoherent;*

(2) *Q is an IF-ring;*

(3) *Q is coherent;*

(4) *Either* $\mathrm{Z}(R) = 0$ *or* $\mathrm{Z}(R)$ *is not flat;*

(5) *For each nonzero element a of* $\mathrm{Z}(R)$, $\mathrm{E}(\frac{Q}{Qa})$ *is flat.*

Proof. $(1) \Leftrightarrow (2) \Leftrightarrow (4)$ by [14, Corollary 3]. By [11, Theorem 2.8], Q is self FP-injective. Hence by applying [13, Theorem 10] to Q, we obtain $(2) \Leftrightarrow (3) \Leftrightarrow (5)$. Therefore all the statements are equivalent. □

The following proposition ensures that the finitistic weak dimension of an arbitrary IF-ring is 0. So that the first assertion of Theorem 5.8 is an immediate consequence of it.

Proposition 5.10 ([12, Proposition 4])**.** *Let R be an IF-ring. Then* $\mathrm{f.w.dim}(R) = 0$.

Proof. Let M be an R-module satisfying $\mathrm{w.dim}(M) = n < \infty$ and let

$$0 \longrightarrow F_n \longrightarrow F_{n-1} \longrightarrow \dots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

be a flat resolution of M with syzygies $K_{n-1}, \dots, K_0$. We claim that M is flat. Indeed, clearly K_{n-1} is flat. Next consider the exact sequence

$$0 \longrightarrow K_{n-1} \longrightarrow F_{n-1} \longrightarrow K_{n-2} \longrightarrow 0.$$

Since R is IF, K_{n-1} is FP-injective by [31, Lemma 4.1]. Hence K_{n-1} is a pure submodule of F_{n-1}. Whence K_{n-2} is flat. By iteration, it follows that all syzygies $K_{n-1}, K_{n-2}, \dots, K_0$ and, hence, M are flat. Consequently, $\mathrm{f.w.dim}(R) = 0$. □

The next two lemmas are required for the proof of Theorem 7.8 (2).

Lemma 5.11 ([12, Lemma 6]). *Let R be a chained ring. Then for each $a \neq 0 \in \mathfrak{m}$.*

(1) $(0:a)$ *is a Q-module.*

(2) $(0:a)$ *is a flat $\frac{R}{(a)}$-module.*

Proof. (1) Let us show that $S^{-1}(0:a) = (0:a)$, where $S := R \setminus \mathrm{Z}(R)$. Indeed, let $x \in (0:a)$ and s regular. Since R is a chained ring, either $x = sy$ for some $y \in R$ or $s = xy'$ for some $y' \in R$. But $s = xy'$ implies $as = axy' = 0$, absurd. Hence $x = sy$. Moreover, $0 = ax = asy$. Since s is regular, $ay = 0$. Therefore $y \in (0:a)$. Consequently, $S^{-1}(0:a) \subseteq (0:a)$. The other inclusion is trivial.

(2) Trivially, $(0:a)$ is an $\frac{R}{(a)}$-module. Since R is a chained ring, any finitely generated ideal of $\frac{R}{(a)}$ is principal and of the form $(\bar{c})$, where $c \in R$ and $\bar{c}$ denote $c + (a)$. Now, let $0 \longrightarrow (\bar{c}) \stackrel{i}{\longrightarrow} \frac{R}{(a)}$ be an exact sequence with $c \notin (a)$. To prove that $(0:a)$ is a flat $\frac{R}{a}$-module, it suffices to prove that the sequence

$$0 \longrightarrow (\bar{c}) \otimes (0:a) \stackrel{i \otimes 1}{\longrightarrow} \frac{R}{(a)} \otimes (0:a)$$

is exact [29, Theorem 3.53]. Let $e := \sum \overline{x_i c} \otimes a_i \in (\bar{c}) \otimes (0:a)$. Hence $e = \sum \bar{c} \otimes x_i a_i = \bar{c} \sum x_i a_i = \bar{c} \otimes x$, where $x := \sum x_i a_i \in (0:a)$. It follows that any element in $(\bar{c}) \otimes (0:a)$ is of the form $\bar{c} \otimes x$, where $x \in (0:a)$. Now, assume that $(i \otimes 1)(\bar{c} \otimes x) = \bar{c} \otimes x = \bar{1} \otimes cx = cx = 0$ in $\frac{R}{(a)} \otimes (0:a) \cong (0:a)$. Since R is chained, $0 \neq a = ct$ for some $t \in R$ (recall that $c \notin (a)$). Now, $ct \neq 0$ and $cx = 0$ yield $x = ty$ for some $y \in R$. Hence $ay = cty = cx = 0$. Whence $y \in (0:a)$. It follows that

$$\bar{c} \otimes x = \overline{ct} \otimes y = \bar{a} \otimes y = \bar{0} \otimes y = 0.$$

Consequently, $i \otimes 1$ is injective. Therefore $(0:a)$ is $\frac{R}{(a)}$-flat. $\square$

Lemma 5.12 ([12, Lemma 7]). *Let p be a positive integer, R a chained ring, and M an R-module. Then* $\mathrm{w.dim}_R(M) \leq p$ *if and only if* $\mathrm{w.dim}_Q(M_{\mathrm{Z}(R)}) \leq p$.

Proof. Assume that $\mathrm{w.dim}(M) \leq p$. Then M has the following flat resolution:

$$0 \longrightarrow F_p \longrightarrow F_{p-1} \longrightarrow \dots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0.$$

Since flatness is a local property, Q is flat and

$$0 \longrightarrow (F_p)_{\mathrm{Z}(R)} \longrightarrow (F_{p-1})_{\mathrm{Z}(R)} \longrightarrow \dots \longrightarrow (F_1)_{\mathrm{Z}(R)} \longrightarrow (F_0)_{\mathrm{Z}(R)} \longrightarrow M \longrightarrow 0.$$

is a flat resolution of $M_{\mathrm{Z}(R)}$. Hence $\mathrm{w.dim}_Q(M_{\mathrm{Z}(R)}) \leq p$.

Conversely, assume that $\mathrm{w.dim}_Q(M_{\mathrm{Z}(R)}) \leq p$ and let $0 \neq a \in \mathfrak{m}$. Since $(0:a)$ is a Q-module, we have the following two exact sequences:

$$0 \longrightarrow (0:a) \longrightarrow R \xrightarrow{a} R \longrightarrow \frac{R}{Ra} \longrightarrow 0$$

$$0 \longrightarrow (0:a) \longrightarrow Q \xrightarrow{a} Q \longrightarrow \frac{Q}{Qa} \longrightarrow 0$$

From these two exact sequences, we deduce that for all integer $q \geq 1$,

$$\mathrm{Tor}^R_{q+2}(\frac{R}{Ra}, M) \cong \mathrm{Tor}^R_q((0:a), M) \cong \mathrm{Tor}^Q_q((0:a), M_{\mathrm{Z}(R)}) \cong \mathrm{Tor}^Q_{q+2}(\frac{Q}{Qa}, M_{\mathrm{Z}(R)}).$$

The first and the third two isomorphisms hold by [29, Corollary 6.19] and the second isomorphism holds because of the following three reasons: First, if we have a flat resolution of M, then we can generate a flat resolution of $M_{\mathrm{Z}(R)}$ by tensoring with Q. Second, Tor does not depend on the flat resolution. Last, as $(0:a)$ is Q-module, $(0:a) \otimes N \cong (0:a) \otimes N_{\mathrm{Z}(R)}$ for any R-module N. If $\mathrm{w.dim}(M_{\mathrm{Z}(R)}) \leq p > 1$, then $p = q+1$ for some integer $q \geq 1$. Hence $\mathrm{w.dim}(M_{\mathrm{Z}(R)}) \leq p$ implies that $\mathrm{Tor}^Q_{p+1}(\frac{Q}{Qa}, M_{\mathrm{Z}(R)}) = 0$. Whence $\mathrm{Tor}^R_{p+1}(\frac{R}{Ra}, M) =$

0, for all $a \in R$. Consequently, $\mathrm{w.dim}(M) \leq p$. Now, if $\mathrm{w.dim}(M_{\mathrm{Z}(R)}) \leq p = 1$, then $\mathrm{Tor}_2^Q(\frac{Q}{Qa}, M_{\mathrm{Z}(R)}) = 0$. By applying the long exact sequence theorem to both of the following two short exact sequences

$$0 \longrightarrow (0:a) \longrightarrow R \longrightarrow Ra \longrightarrow 0$$

$$0 \longrightarrow (0:a) \longrightarrow Q \longrightarrow Qa \longrightarrow 0$$

and noting that $\mathrm{Tor}_1^R(Qa, M_{\mathrm{Z}(R)}) \cong \mathrm{Tor}_2^Q(\frac{Q}{Qa}, M_{\mathrm{Z}(R)}) = 0$, we get the two exact sequences:

$$0 \longrightarrow \mathrm{Tor}_1^R(Ra, M) \longrightarrow (0:a) \otimes_R M \longrightarrow M$$

$$0 \longrightarrow (0:a) \otimes_Q M_{\mathrm{Z}(R)} \longrightarrow M_{\mathrm{Z}(R)}.$$

Since $(0:a)$ is a Q-module, $(0:a) \otimes_R M \cong (0:a) \otimes_Q M_{\mathrm{Z}(R)}$. So $\mathrm{Tor}_1^R(Ra, M) \cong \mathrm{Tor}_2^R(\frac{R}{Ra}, M)$ $= 0$ for all $a \in R$. Consequently, $\mathrm{w.dim}(M) \leq 1$. This finishes the proof of the lemma. □

Now, we are able to prove the second assertion of Theorem 5.8.

Proof of Theorem 5.8(2). Let R be a chained ring which is semicoherent and not an IF-ring and let M be an R-module with $\mathrm{w.dim}(M) < \infty$. By Lemma 5.12, $\mathrm{w.dim}(M_{\mathbb{Z}(R)}) < \infty$. Since R is semicoherent, Q is IF (Proposition 5.9). It follows that $\mathrm{f.w.dim}(Q) = 0$ (Proposition 5.10). Hence $\mathrm{w.dim}(M_{\mathbb{Z}(R)}) = 0 \leq 1$. Whence $\mathrm{w.dim}(M) \leq 1$ (Lemma 5.12). Consequently, $\mathrm{f.w.dim}(R) \leq 1$. To show that the last inequality is equality we have to find an example of R -module with weak dimension equal to one. Since R is not IF, $R \neq Q$ (Proposition 5.9). Hence $\mathfrak{m} \neq \mathrm{Z}(R)$. So there exists $a \neq 0 \in \mathfrak{m} \setminus \mathrm{Z}(R)$. By [29, Theorem 4.33], the R-module $\frac{R}{Ra}$ is not flat since it is not torsion free, as desired. □

The following lemma is needed to prove the last assertion of Theorem 5.8.

Lemma 5.13 ([12, Lemma 8]). *Let R be a chained ring which is not semicoherent. Then for every R-module M satisfying* $\mathrm{w.dim}(M) \leq 2$, $\mathrm{Z}(R) \bigotimes_R M$ *is flat.*

Proof. We prove this lemma in two steps. Let $0 \neq a \in \mathrm{Z}(R) \subseteq \mathfrak{m}$.

Step 1: $(0:a) \otimes_R M$ is $\frac{R}{Ra}$-flat. To prove this claim, consider the following flat resolution of M,

$$0 \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0.$$

For $a \in \mathrm{Z}(R)$, $\mathrm{Tor}_1^R((0:a), M) \cong \mathrm{Tor}_3^R(\frac{R}{Ra}, M) = 0$. Therefore, if $K = \mathrm{Ker}(F_0 \longrightarrow M)$, then by applying the long exact sequence theorem to both of the following short exact sequences:

$$0 \longrightarrow F_2 \longrightarrow F_1 \longrightarrow K \longrightarrow 0$$

$$0 \longrightarrow K \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

we obtain the exact sequence

$$0 \longrightarrow (0:a) \otimes_R F_2 \longrightarrow (0:a) \otimes_R F_1 \longrightarrow (0:a) \otimes_R F_0 \longrightarrow (0:a) \otimes_R M \longrightarrow 0.$$

Since $(0:a)$ is $\frac{R}{Ra}$-flat, for each $i \in \{0,\ 1,\ 2\}$ $(0:a) \otimes_R F_i$ is $\frac{R}{Ra}$-flat. By [13, Theorem 11(1)], $\frac{R}{Ra}$ is an IF-ring. Hence $\mathrm{f.w.dim}(\frac{R}{Ra}) = 0$ (Proposition 5.10). It follows that $(0:a) \otimes_R M$ is $\frac{R}{Ra}$-flat.

Step 2: $\mathrm{Z}(R) \bigotimes_R M$ is flat. To prove this, consider the exact sequence

$$S_a := 0 \longrightarrow (0:a) \otimes_R K \longrightarrow (0:a) \otimes_R F_0 \longrightarrow (0:a) \otimes_R M \longrightarrow 0.$$

Since $(0:a)$ is $\frac{R}{Ra}$-flat (Lemma 5.11), $(0:a) \otimes_R F_0$ is $\frac{R}{Ra}$-flat. Also, by Step 1, $(0:a) \otimes_R M$ is $\frac{R}{Ra}$-flat. It follows that S_a is pure exact over $\frac{R}{Ra}$. Now, $\mathrm{Z}(R) = \bigcup_{a \in \mathrm{Z}(R) \setminus 0} (0:a) = \varinjlim (0:a)$. Further, the tensor product preserves the direct limit [29, Corollary 2.20] and the direct limit preserves the pure exactness [16]. It follows that the sequence

$$S := \varinjlim S_a = 0 \longrightarrow \mathrm{Z}(R)\otimes_R K \longrightarrow \mathrm{Z}(R)\otimes_R F_0 \longrightarrow \mathrm{Z}(R)\otimes_R M \longrightarrow 0.$$

is pure exact. Since R is not semicoherent, $\mathrm{Z}(R)$ is flat (Proposition 5.9) which implies that $\mathrm{Z}(R)\otimes_R F_0$ is flat. Consequently, $\mathrm{Z}(R)\bigotimes_R M$ is flat. □

Proof of Theorem 5.8(3). First we prove that $\mathrm{f.w.dim}(R) \geq 2$. Assume that R is a chained ring which is not semicoherent. By Proposition 5.9, there exists $a \neq 0 \in \mathrm{Z}(R)$ such that $\mathrm{E}(\frac{Q}{Qa})$ is not flat, also Q is not IF. Now, by applying [13, Proposition 14] to the exact sequence

$$0 \longrightarrow \mathrm{Z}(R) \longrightarrow Q \longrightarrow \frac{Q}{\mathrm{Z}(R)} \longrightarrow 0 \qquad (3)$$

we obtain the following exact sequence for each $a \neq 0 \in \mathrm{Z}(R)$

$$0 \longrightarrow \frac{Q}{\mathrm{Z}(R)} \longrightarrow \mathrm{E}(\frac{Q}{\mathrm{Z}(R)}) \longrightarrow \mathrm{E}(\frac{Q}{Qa}) \longrightarrow 0. \qquad (4)$$

Combine (3) and (4) to get the following exact sequence

$$0 \longrightarrow \mathrm{Z}(R) \longrightarrow Q \longrightarrow \mathrm{E}(\frac{Q}{\mathrm{Z}(R)}) \longrightarrow \mathrm{E}(\frac{Q}{Qa}) \longrightarrow 0. \qquad (5)$$

In the exact sequence (5), $\mathrm{Z}(R)$ is flat since Q is not IF (Proposition 5.9), Q is flat since flatness is a local property, by [13, Proposition 8] $\mathrm{E}(\frac{Q}{\mathrm{Z}(R)})$ is flat, and a was chosen from the beginning so that $\mathrm{E}(\frac{Q}{Qa})$ is not flat. Consequently, $\mathrm{w.dim}(\mathrm{E}(\frac{Q}{Qa})) \leq 2$. On the other hand, since $\frac{Q}{\mathrm{Z}(R)}$ is not flat (not torsion free), $0 \neq \mathrm{Tor}_1^Q(\frac{Q}{\mathrm{Z}(R)}, M) \cong \mathrm{Tor}_2^Q(E(\frac{Q}{Qa}), M)$ for any R-module M. Therefore $\mathrm{w.dim}(\mathrm{E}(\frac{Q}{Qa})) = 2$. It follows that $\mathrm{f.w.dim}(Q) \geq 2$. By Lemma 5.12, $\mathrm{f.w.dim}(R) = \mathrm{f.w.dim}(Q)$. So $\mathrm{f.w.dim}(R) \geq 2$. Also we can assume that $R = Q$ and hence $\mathfrak{m} = \mathrm{Z}(R)$.

Next, we prove that $\operatorname{f.w.dim}(R) \leq 2$. Deny and assume that there exists an R module M satisfying $2 < \operatorname{w.dim}(M) < \infty$. By replacing M with an appropriate syzygy of a flat resolution of it, we can assume, without loss of generality, that $\operatorname{w.dim}(M) = 3$. Consider the short exact sequence

$$0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0 \qquad (6)$$

where F is flat. Since $\operatorname{w.dim}(M) = 3$, $\operatorname{w.dim}(K) \leq 2$. Hence by Lemma 5.13, $\mathfrak{m} \otimes K$ is flat. Recall that $\mathfrak{m}$ is flat by Proposition 5.9. Hence by tensoring (6) with $\mathfrak{m}$ we obtain: $\operatorname{w.dim}(\mathfrak{m} \otimes M) \leq 1$. In view of the exact sequence

$$0 \longrightarrow \mathfrak{m} \longrightarrow R \longrightarrow \frac{R}{\mathfrak{m}} \longrightarrow 0. \qquad (7)$$

$\operatorname{w.dim}(\frac{R}{\mathfrak{m}}) \leq 1$. Since $\frac{R}{\mathfrak{m}}$ is not torsion free, it is not flat. Hence $\operatorname{w.dim}(\frac{R}{\mathfrak{m}}) = 1$. It follows that $\operatorname{w.dim}(\frac{M}{\mathfrak{m}M}) \leq 1$ and $\operatorname{w.dim}(\operatorname{Tor}_1^R((\frac{R}{\mathfrak{m}}), M)) \leq 1$. By applying the long exact sequence theorem to (7), we obtain the exact sequence

$$0 \longrightarrow \operatorname{Tor}_1^R(\frac{R}{\mathfrak{m}}, M) \longrightarrow \mathfrak{m} \otimes M \longrightarrow M \longrightarrow \frac{M}{\mathfrak{m}M} \longrightarrow 0.$$

Notice that $\operatorname{Ker}(M \longrightarrow \frac{M}{\mathfrak{m}M}) = \mathfrak{m}M$. So that we have the two exact sequences:

$$0 \longrightarrow \operatorname{Tor}_1^R(\frac{R}{\mathfrak{m}}, M) \longrightarrow \mathfrak{m} \otimes M \longrightarrow \mathfrak{m}M \longrightarrow 0 \qquad (8)$$

$$0 \longrightarrow \mathfrak{m}M \longrightarrow M \longrightarrow \frac{M}{\mathfrak{m}M} \longrightarrow 0. \qquad (9)$$

By applying the long exact sequence theorem to (8), we obtain

$$0 = \operatorname{Tor}_3^R(\mathfrak{m} \otimes M, N) \longrightarrow \operatorname{Tor}_3^R(\mathfrak{m}M, N) \longrightarrow \operatorname{Tor}_2^R(\operatorname{Tor}_1^R(\frac{R}{\mathfrak{m}}, M), N) = 0$$

for any R-module N. Hence $\mathrm{Tor}_3^R(\mathfrak{m}M,N)=0$ for any R-module N. Whence $\mathrm{w.dim}(\mathfrak{m}M)\leq 2$. Similarly by applying the long exact sequence theorem to (9) we obtain

$$0=\mathrm{Tor}_3^R(\mathfrak{m}M,N)\longrightarrow \mathrm{Tor}_3^R(M,N)\longrightarrow \mathrm{Tor}_3^R(\frac{M}{\mathfrak{m}M},N)=0$$

and hence $\mathrm{w.dim}(M)\leq 2$, contradiction. Therefore $\mathrm{f.w.dim}(R)=2$.

It remains to prove that an R-module M has a finite weak dimension if and only if $\mathrm{Z}(R)\otimes_R M$ is flat. First, assume that M has a finite weak dimension. Since $\mathrm{f.w.dim}(R)\leq 2$, $\mathrm{w.dim}(M)\leq 2$. By Lemma 5.13, $\mathrm{Z}(R)\otimes_R M$ is flat. Conversely Assume that $\mathrm{Z}(R)\otimes M$ is flat. Again we can assume that $R=Q$ and then $\mathfrak{m}=\mathrm{Z}(R)$. By applying the long exact theorem to (8), we obtain $\mathrm{w.dim}(\mathfrak{m}M)\leq 2$. Also by applying the long exact theorem to (9) we obtain, $\mathrm{w.dim}(M)\leq 2$, as desired. □

Proof of Theorem 5.7. (1) Assume that R is arithmetical and locally IF. Then $R_{\mathfrak{m}}$ is a chained and an IF ring for every maximal ideal $\mathfrak{m}$ of R. By Theorem 5.8(1), $\mathrm{f.w.dim}(R_{\mathfrak{m}})=0$, $\forall\ \mathfrak{m}\in\mathrm{Max}(R)$. Now let M be an R-module such that $\mathrm{w.dim}(M)<\infty$. By [19, Theorem 1.3.14], We have

$$\mathrm{w.dim}_R(M)=\sup\{\mathrm{w.dim}_{R_{\mathfrak{m}}}(M_{\mathfrak{m}})\mid \mathfrak{m}\text{ maximal ideal of R}\}. \qquad (*)$$

By (*), $\forall\ \mathfrak{m}\in\mathrm{Max}(R)$, $\mathrm{w.dim}(M_{\mathfrak{m}})<\infty$ and, hence, $\mathrm{w.dim}_{R_{\mathfrak{m}}}(M_{\mathfrak{m}})=0$. Consequently, $\mathrm{w.dim}(M)=0=\mathrm{f.w.dim}(R)$.

(2) Assume that R is arithmetical which is locally semicoherent and not locally IF. Then there exists a maximal ideal $\mathfrak{m}_0$ of R such that $R_{\mathfrak{m}_0}$ is a chained ring which is semicoherent and not IF. By Theorem 5.8, $\mathrm{f.w.dim}(R_{\mathfrak{m}_0})=1$ as well as $\mathrm{f.w.dim}(R_{\mathfrak{m}})\leq 1$, $\forall\ \mathfrak{m}\in\mathrm{Max}(R)$. Now let M be an R-module such that $\mathrm{w.dim}_R(M)<\infty$. Then $\mathrm{w.dim}_{R_{\mathfrak{m}}}(M_{\mathfrak{m}})<\infty$, $\forall\ \mathfrak{m}\in\mathrm{Max}(R)$. This yields $\mathrm{w.dim}_{R_{\mathfrak{m}}}(M_{\mathfrak{m}})\leq 1$, $\forall\ \mathfrak{m}\in\mathrm{Max}(R)$. By (*), we obtain $\mathrm{w.dim}_R(M)\leq$

1; which forces $\mathrm{f.w.dim}(R) \leq 1$. Moreover, since $\mathrm{f.w.dim}(R_{\mathfrak{m}_0}) = 1$, there exists a non-flat $R_{\mathfrak{m}_0}$-module M. Necessarily, M is a non-flat R-module (since $M = M_{\mathfrak{m}_0}$). Therefore $\mathrm{f.w.dim}(R) = 1$.

(3) Similar arguments as above yield, via Theorem 5.8, $\mathrm{f.w.dim}(R_{\mathfrak{m}}) \leq 2$, $\forall\ \mathfrak{m} \in \mathrm{Max}(R)$ and $\exists\ \mathfrak{m}_0 \in \mathrm{Max}(R)$ such that $\mathrm{f.w.dim}(R_{\mathfrak{m}_0}) = 2$. Let M be an $R_{\mathfrak{m}_0}$-module with $\mathrm{w.dim}_{R_{\mathfrak{m}_0}}(M) = 2$ and consider the flat resolution of M over $R_{\mathfrak{m}_0}$

$$0 \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0.$$

This is also a flat resolution of R-modules. Further $K := \mathrm{Ker}(F_0 \longrightarrow M)$ is not flat R-flat since $K = K_{\mathfrak{m}_0}$ is not $R_{\mathfrak{m}_0}$-flat. Therefore $\mathrm{f.w.dim}(R) = 2$. $\square$

Bibliography

[1] J. Abuihlail, M. Jarrar, and S. Kabbaj, Commutative rings in which every finitely generated ideal is quasi-projective, J. Pure Appl. Algebra 215 (2011) 2504–2511.

[2] M. F. Atiyah and I. G. Macdonald, Introduction to Commutative Algebra, Westview Press, 1969.

[3] C. Bakkari, S. Kabbaj, and N. Mahdou, Trivial extensions defined by Prüfer conditions, J. Pure Appl. Algebra 214 (2010) 53–60.

[4] S. Bazzoni and S. Glaz, Prüfer rings, Multiplicative Ideal Theory in Commutative Algebra, 263–277, Springer, New York, 2006.

[5] S. Bazzoni and S. Glaz, Gaussian properties of total rings of quotients, J. Algebra 310 (2007) 180–193.

[6] N. Bourbaki, Commutative Algebra. Chapters 1–7, Springer-Verlag, Berlin, 1998.

[7] H. S. Butts and W. Smith, Prüfer rings, Math. Z. 95 (1967) 196–211.

[8] H. Cartan and S. Eilenberg, Homological Algebra, Princeton University Press, 1956.

[9] R. Colby, Flat injective modules, J. Algebra 35 (1975) 239-252.

[10] F. Couchot, Almost clean rings and arithmetical rings, Commutative Algebra and its Applications, 135-154, Walter de Gruyter, Berlin, 2009.

[11] F. Couchot, Exemples d'anneaux auto fp-injectifs, Comm. Algebra 10 (4) (1982) 339–360.

[12] F. Couchot, Finitistic weak dimension of commutative arithmetical rings, AJSE D-Mathematics, to appear.

[13] F. Couchot, Injective modules and fp-injective modules over valuations rings, J. Algebra 267 (2003) 359–376.

[14] F. Couchot, Localization of injective modules over arithmetical rings, Comm. Algebra 37 (10) (2009) 3418–3423.

[15] L. Fuchs, Über die Ideale Arithmetischer Ringe, Comment. Math. Helv. 23 (1949) 334–341.

[16] L. Fuchs and L. Salce, Modules over Non-Noetherian Domains, Mathematical Surveys and Monographs, 84, American Mathematical Society, Providence, 2001.

[17] K. R. Fuller and D. A. Hill, On quasi-projective modules via relative projectivity, Arch. Math. (Basel) 21 (1970) 369–373.

[18] R. Gilmer and T. Parker, Semigroups as Prüfer rings, Duke Math. J. 41 (1974) 219–230.

[19] S. Glaz, Commutative Coherent Rings, Lecture Notes in Mathematics, 1371, Springer-Verlag, Berlin, 1989.

[20] S. Glaz, The weak dimension of Gaussian rings, Proc. Amer. Math. Soc. 133 (9) (2005) 2507–2513.

[21] S. Glaz, Prüfer conditions in rings with zero-divisors, CRC Press Series of Lectures in Pure Appl. Math. 241 (2005) 272–282.

[22] M. Griffin, Prüfer rings with zero-divisors, J. Reine Angew. Math. 239/240 (1969) 55–67.

[23] J.A. Huckaba, Commutative Rings with Zero-Divisors, Marcel Dekker, New York, 1988.

[24] C. U. Jensen, Arithmetical rings, Acta Math. Hungar. 17 (1966) 115–123.

[25] S. Lang, Algebra, Graduate Texts in Mathematics, Springer, New York, 2002.

[26] S. Kabbaj and N. Mahdou, Trivial extensions defined by coherent-like conditions, Comm. Algebra 32 (10) (2004) 3937–3953.

[27] A. Koehler, Rings for which every cyclic module is quasi-projective, Math. Ann. 189 (1970) 311–316.

[28] B. Osofsky, Global dimension of commutative rings with linearly ordered ideals, J. London Math. Soc. 44 (1969) 183–185.

[29] J. J. Rotman, An Introduction to Homological Algebra, Academic Press, New York, 1979.

[30] S. Singh and A. Mohammad, Rings in which every finitely generated left ideal is quasi-projective, J. Indian Math. Soc. 40 (1-4) (1976) 195–205.

[31] B. Stenström, Coherent rings and FP-injective modules, J. London Math. Soc. 2 (2) (1970) 323–329.

[32] H. Tsang, Gauss's Lemma, Ph.D. thesis, University of Chicago, Chicago, 1965.

[33] A. Tuganbaev, Quasi-projective modules with exchange property, Communications of the Moscow Mathematical Society, 1999.

[34] W. V. Vasconcelos, The Rings of Dimension Two, Lecture Notes in Pure and Appl. Math, 22, Marcel Dekker, New York, 1976.

[35] R. Wisbauer, Modules and Algebras: Bimodule Structure and Group Actions on Algebras, Longman, 1996.

[36] R. Wisbauer, Local-global results for modules over algebras and Azumaya rings, J. Algebra 135 (1990) 440–455.

Index

Vitae

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