Invariant Solutions, Double Reductions and Conservation Laws for Certain Evolution Equations

BY

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This dissertation, written by AHMAD YOUSEF AL-DWEIK under the direction of his thesis advisors and approved by his thesis committee, has been presented to and accepted by the Dean of Graduate Studies, in partial fulfillment of the requirements for the degree of DOCTOR OF PHILOSOPHY IN MATHEMATICS.

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To my parents, wife, brothers and sisters

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All praise and glory to ALLAH almightily who taught man through the pen. And peace and blessing of ALLAH be upon his last messenger Mohammad (Sallallah-Alihe-Wasallam), who brought the absolute truth and wisdom to mankind.

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DISSERTATION ABSTRACT

Name: Ahmad Yousef Al-Dweik

Title of study: Invariant Solutions, Double Reductions and Conservation Laws for Certain Evolution Equations

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Symmetry analysis of a class of nonlinear equations of evolution type is performed. A complete symmetry classification of a nonlinear $(2+1)$ diffusion equation is obtained. Reductions via two dimensional Lie subalgebras to ordinary differential equations are performed. In few interesting cases, exact invariant solutions are found. As for symmetries are concerned, they have important relationship with the conservation laws admitted by the partial differential equations (PDEs). Using this relationship of symmetries with conservation laws, we obtain conservation laws of evolution type equations describing waves in n dimensions involving arbitrary velocity functions. We also provide a generalized double reduction theory to obtain invariant solutions for a system of nonlinear PDEs having more than two independent variables, then we apply it to a nonlinear $(2+1)$ wave equation with different arbitrary functions.

Lastly, we address the issue of using Lie symmetry analysis to a $(3+1)$ wave equation coupled with a spherically symmetric metric and analyze some implications.

ملخص بحث

درجة الدكتوراة في الفلسفة

الاســــــــــــم: **أحمد يوسف الشيخ راتب الدويك**

عنوان الرسالة: **حلول التناظر والتخفيض المزدوج وقوانين الحفظ لبعض معادلات التطور التفاضلية الجزئية .**

التخصــــــص: **الريــــاضيـــــات**

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لقد قمنا في هذه الرسالة بدراسة التحليل التماثلي لفئة من معادلات التطور. حيث اننا أجرينا تصنيفا تماثليا كاملا لفئة من معادلات الانتشار غير الخطية ذات (٢ +١) بعد. و باستخدام محاور التماثل تم تخفيض هذه المعادلة الى معادلات تفاضلية عادية باستخدام جبريات جزئية ذات بعدين و من ثم إيجاد الحلول التامة التحليلية لبعض الحالات.

و نظرا لوجود علاقة قوية بين محاور التماثل و قوانين الحفظ للمعادلات التفاضلية الجزئية فاننا قمنا بدراسة قوانين الحفظ لفئة من معادلات الأمواج غير الخطية ذات (ن +١) بعد، حيث أن اللاخطية قد قدمت من خلال دالة ممثلة لسرعة الموجة.
ثم قمنا بتعميم نظرية التخفيض المزدوج للمعادلات التفاضلية الجزئية لاكثر من متغيرين مستقلين لايجاد الحلول التامة التحليلية باستخدام قوانين الحفظ. ثم طبقنا تعميم النظرية لمعادلات الأمواج غير الخطية ذات (٢ +١) بعد و سرعتين اختياريين في اتجاهين مختلفين.

وأخيرا أجرينا التحليل التماثلي لمعادلة الامواج الموجودة على سطح كروي متماثل ذات (٣ +١) بعد، و طبقنا هذه الدراسة على الأمواج الموجودة في عالمنا المعاصر.

Introduction

Lie symmetry analysis of differential equations was initiated by Sophus Lie. Today, this area of research is actively engaged. Invariant solutions for scalar PDEs were discovered by Lie (1881). Such solutions can be determined from a Lie point symmetry by reduction of the number of independent variables through canonical coordinates.

In this dissertation, we study certain evolution equations by using the symmetry analysis approach. The one-dimensional heat equation is extensively studied from the point of view of its Lie point symmetries by Ibragimov [20], Cantwell [10] and Bluman and Kumei [6]. Since thermal diffusivity of some materials may be a function of temperature, it introduces nonlinearities in the heat equation that models such phenomenon. On the one hand nonlinear heat equation models some of the real world problems, it may not be easy to tackle such problems by usual methods. In higher dimensions Serov [33] gave some conditional symmetries for a nonlinear heat equation. Nonlinear heat equations in one and higher dimensions are also studied in literature by using both the symmetry as well as other methods [14, 13] and an account of some cases is given by Polyanin [32].

As mentioned above the thermal diffusivity of materials such as gases is not a constant, but depends upon the temperature of the body. Physically, it is quite an interesting situation and can be modeled by (2+1)-nonlinear heat equation

$$u_t - f(u)(u_{xx} + u_{yy}) = 0, \tag{1}$$

where $f(u)$ is an arbitrary function of the variable u. The equation (1) models situations where variations in temperature and thermal diffusivity is relatively small so that the product terms $f_u {u_x}^2$ and $f_u {u_y}^2$ can be ignored. A symmetry classification of (1) was presented by Aijaz et. al [1] and using a two-dimensional subalgebra of Lie point symmetry generators a complete classification of the equation was given.

In this dissertation, we extend this work to the (2+1)- nonlinear diffusion equation

$$u_t - \mathbf{div}(f(u)\mathbf{grad}\, u) = 0, \tag{2}$$

by incorporating the assumption that was dropped in (1); namely, $f_u {u_x}^2$ and $f_u {u_y}^2$ are not being zero. Using the Lie symmetry method, a complete symmetry classification of equation (2) is presented. Reductions, via two dimensional Lie subalgebras of the extended equation, to ordinary differential equations are obtained and exact solutions in interesting cases are found.

As for symmetries are concerned, they have important relationship with the conservation laws admitted by the PDEs. It is for this reason that finding conservation laws associated with symmetries has been a topic of great interest (see, e.g. [6, 31, 15, 30, 16]). A systematic way for determining conservation laws associated with variational symmetries for systems of Euler-Lagrange equations is indeed the famous Noether theorem [29]. Direct construction methods for multipliers and hence the conservation laws [3], Lagrangian approach for evolution equations [21] and formula for relationship between symmetries and conservation laws, irrespective of the existence of a Lagrangian of the system [25] have been investigated. Also, a basis of conservation laws was further investigated in [26] for DEs with and without Lagrangian formulation. Kara and Mahomed in [27] presented a new method to construct conservation laws of DEs via operators that are not necessarily symmetry generators of the underlying system. These partial Noether operators which are associated with partial Lagrangians help via an explicit Noether-like formula in the construction of conservation laws of the system which need not be derivable from a variational principle. These systems are referred to as partial Euler-Lagrange equations with respect to partial Lagrangians. This approach provides a systematic way of obtaining conservation laws for systems which have partial Lagrangians.

The nonlinear $(1+1)$ wave equation

$$u_{tt} - \frac{\partial}{\partial x}(f(u)u_x) = 0 \tag{3}$$

describing waves in one dimension involving arbitrary velocity function arises when transmitting a signal on a transmission line with material properties that are changing along the line. Ames et. al [2] obtained a complete group classification for its admitted point symmetries with respect to the wave speed function $f(u)$ and consequence constructed explicit invariant solutions for some specific cases, we study in this dissertation conservation laws of the nonlinear $(n+1)$ wave equation

$$u_{tt} - \mathbf{div}(f(u)\mathbf{grad}\ u) = 0 \tag{4}$$

involving an arbitrary function of the dependent variable. This equation is not derivable from a variational principle. By writing the equation in the partial Euler-Lagrange form, partial Noether operators associated with the partial Lagrangian are obtained for all possible cases of the arbitrary function. These operators help, via a formula, to construct conservation laws of the wave equation. We find conservation laws for different forms of $f(u)$ and develop a relationship between the partial Noether symmetry operators and the Lie symmetries admitted by the equation.

A third direction in the area of present study arises as a consequence of applying Lie point generator to a conserved vector. This provides either (1) conservation law associ-

ated with that symmetry or (2) conservation law that may be trivial, known already or new. A pioneering work in this direction was published by Kara et. al [25, 26, 24]. Sjöberg [34, 35] later showed that when the generated conserved vector is null, i.e. the symmetry is associated with the conserved vector (association defined as in [25]), a double reduction is possible for PDEs with two independent variables. In this double reduction the PDE of order q is reduced to an ODE of order $(q-1)$. Thus the use of one symmetry associated with a conservation law leads to two reductions, *the first being a reduction of the number of independent variables* and *the second being a reduction of the order of the DE*. Sjöberg also constructed the reduction formula for PDEs with two independent variables which transforms the conserved form of the PDE to a reduced conserved form via an associated symmetry. Application of this method to the linear heat, the BBM and the sine-Gordon equation and a system of differential equations from one dimensional gas dynamics are given in [34]. According to the double reduction theory, a PDE of order q with two independent and m dependent variables, which admits a nontrivial conserved form that has at least one associated symmetry can be reduced to an ODE of order $(q-1)$.
In his papers [34, 35] Sjöberg opines that generalizing the double reduction theory to PDEs of higher dimensions is still an open problem and it is not clear how to overcome the problem when not all derivatives of non-local variables are known explicitly. Further, calculations for higher dimensions are quite tedious and much work is needed to generalize (if possible) the theory to PDEs with more than two independent variables.

In this dissertation we discuss *a generalization of the double reduction theory* , with n independent variables. We show that a nonlinear system of qth order PDEs with n independent and m dependent variables, which admits a nontrivial conserved form that has at least one associated symmetry in every reduction from the n reductions (the first step of double reduction), can be reduced to a nonlinear system of $(q-1)$th order ODEs.

Finally, we address in this dissertation the issue of using Lie symmetry analysis to the

$(3+1)$ wave equation coupled with a spherically symmetric metric and analyze some of its implications. There exists in literature similar work done on coupling some PDEs with non-flat geometries and seeing how these coupled PDEs inherit nonlinearities of the geometry, and how the symmetry structure changes with change in geometric properties of the space itself etc [8, 9].

This thesis consists of following seven chapters:

Chapter 1 provides the fundamental notions from the theory of continuous groups, symmetry properties of differential equations, conservation laws and double reduction.

Chapter 2 provides an extension of an earlier work by Aijaz et. al [1] to the (2+1)- nonlinear diffusion equation (1) by incorporating the assumption that $f_u {u_x}^2$ and $f_u {u_y}^2$ are not zero. A complete symmetry classification of (1) is presented. Reductions, via two dimensional Lie subalgebras of the extended equation, to ordinary differential equations are obtained and exact solutions in interesting cases are found.

In Chapter 3, we study conservation laws of the nonlinear $(n+1)$ wave equation (4), describing waves in n dimensions involving arbitrary velocity functions. This equation does not have a Lagrangian and therefore we use a new method to construct conservation laws.

In Chapter 4, we extend Sjöberg work by introducing the generalized double reduction theory, with n independent variables. we show that a nonlinear system of qth order PDEs with n independent and m dependent variables, which admits a nontrivial conserved form with at least one associated symmetry in every reduction can be reduced to a nonlinear system of $(q-1)$ th order ODEs.

In Chapter 5, we study conservation laws of the nonlinear $(2+1)$ wave equation (4) with different arbitrary functions and we find invariant solutions that conserve the fluxes by using these conservation laws and the associated symmetries. Since the equation has 3 independent variables, we therefore use our generalized double reduction theory to solve (4).

In Chapter 6, we study Lie symmetry analysis to deal with $(3+1)$ wave equation coupled with a spherically symmetric metric and analyze some implications.

In Chapter 7, some achievements and recommendations for future work are addressed.

Chapter 1

Preliminaries

1.1 Introduction

This chapter gives the fundamental notions from the theory of continuous groups, symmetry properties of differential equations, conservation laws and double reduction.

1.2 Lie groups

1.2.1 Groups

Definition 1.2.1. Consider $(G,*)$ to be a set with an operation $*$ that assigns to every ordered pair of elements of G a unique element with the following properties:

(1) **Closure**

For all x,y in G, $x*y$ is also in G.

(2) **Associativity**

For all x,y,z in G, $(x*y)*z = x*(y*z)$

(3) **Identity**

In G there exists an element 'e' known as the identity such that $x*e = e*x = x$ for all x in G.

(4) **Inverse**

For every x in G there exists an element y in G known as inverse of x such that $x*y = y*x = e$

where e is the identity element of G with respect to the binary operation $*$.

Definition 1.2.2. (Abelian group) A group G is called Abelian if in addition to the above properties it satisfies:

$x*y = y*x$ for all x, y in G.

1.2.2 One-parameter group of transformations

Definition 1.2.3. The set of transformations given by $x^\star = \chi(x\,;\varepsilon)$ where $x = (x_1, x_2, ..., x_n)$ lie in a region D$\subset \mathbb{R}^n$ is defined for each ε in a set $S \subset \mathbb{R}$ with the operation ϕ, forms a one-parameter group of transformations on D if the following hold:

(1) For all $\varepsilon \in S$ the transformations are one-to-one onto D.

(2) S with ϕ forms a group G.

(3) For all $x \in$ D, $x^\star$ =x when $\varepsilon = \varepsilon_0$ corresponding to the identity e, i.e.,$\chi(x\,;\varepsilon_0) = x$.

(4) If $x^\star = \chi(x\,;\varepsilon)$ and $x^{\star\star} = \chi(x^\star\,;\delta)$, then $x^{\star\star} = \chi(x\,;\phi(\varepsilon,\delta))$.

1.2.3 Lie groups of transformations

Definition 1.2.4. A one-parameter group G of transformations $x^\star = \chi(x\,;\varepsilon)$ with the operation ϕ is said to be a one-parameter Lie group of transformations if :

(1) ε is a continuous parameter, without loss of generality, with the identity element $\varepsilon = 0$. i.e, the set S is an interval in $\mathbb{R}$ which contains zero.

(2) χ is an infinitely differentiable function with respect to x in D and an analytic function of ε in S.

(3) $\phi(\varepsilon, \delta)$ is an analytic function of ε and δ in S.

Example 1.2.5. Group of translation in the plane

$x^\star = x + \varepsilon$

$y^\star = y$, where $\varepsilon \in \mathbb{R}$.

Here $\phi(\varepsilon, \delta) = \varepsilon + \delta$ and the identity element corresponds to $\varepsilon = 0$.

Example 1.2.6. Group of scaling in the plane

$x^\star = \alpha x$

$y^\star = \alpha^2 y$ where $0 < \alpha < \infty$.

Here $\phi(\alpha, \beta) = \alpha \;\; \beta$ and the identity element corresponds to $\alpha = 1$.

1.3 Infinitesimal transformations and generators

Definition 1.3.1 (Infinitesimal transformations). Consider a one parameter (ε) Lie group of transformation $x^\star = \chi(x\,; \varepsilon)$ with the identity $\varepsilon = 0$ and law of composition ϕ. Expanding $x^\star$ about $\varepsilon = 0$, one gets,

$$x^\star = x + \varepsilon \frac{\partial x^\star}{\partial \varepsilon}\bigg|_{\varepsilon=0} + O(\varepsilon^2) \tag{1.1}$$

where $\frac{\partial x^\star}{\partial \varepsilon}\Big|_{\varepsilon=0} = \xi(x)$. The transformation $x^\star = x + \varepsilon \; \xi(x)$ is called the infinitesimal transformation of the Lie group of transformations and the component $\xi(x)$ are called the infinitesimals of the transformation.

Theorem 1.3.2 (**First Fundamental Theorem of Lie** [7]). There exists a parametrization $\tau(\varepsilon)$ such that the Lie group of transformations $x^\star = \chi(x\,;\varepsilon)$ is equivalent to the solution of an initial value problem for the system of first order ordinary differential equations given by

$$\frac{d\,x^\star}{d\,\tau} = \xi(x^\star), \text{ with } \quad x^\star = x \quad \text{ when } \quad \tau = 0. \tag{1.2}$$

Example 1.3.3. The group of translation in x direction
$x^\star = x + \varepsilon$ and $y^\star = y$ where $\varepsilon \in \mathbb{R}$
is equivalent to the solution of an initial value problem
$\frac{dx^\star}{d\varepsilon} = 1, \frac{dy^\star}{d\varepsilon} = 0$ with $x^\star = x, y^\star = y$ at $\varepsilon = 0$

Definition 1.3.4 (**Infinitesimal generator**). The infinitesimal generator of the one-parameter Lie group of transformations $x^\star = \chi(x\,;\varepsilon)$ is the operator

$$\mathbf{X} = \sum_{i=1}^{n} \xi_i(x) \frac{\partial}{\partial x_i}, \tag{1.3}$$

where $\xi_i = \left.\frac{\partial x_i{}^\star}{\partial \varepsilon}\right|_{\varepsilon=0}$.

Theorem 1.3.5. [7] The one-parameter Lie group of transformations $x^\star = \chi(x\,;\varepsilon)$ is equivalent to :

$$\begin{aligned} x^\star &= e^{\varepsilon \mathbf{X}} x \\ &= \sum_{k=0}^{\infty} \frac{\varepsilon^k}{k!} \mathbf{X}^k x, \end{aligned} \tag{1.4}$$

where the operator **X** is the infinitesimal generator of the Lie group.

Remark 1.3.6. In summary there are two ways to find explicitly a one-parameter Lie group of transformations from its infinitesimal transformation:

(i) Express the group in terms of a power series, called a Lie series, which is developed from the infinitesimal generator corresponding to the infinitesimal transformation;

(ii) Solve the initial value problem . Here one first finds the explicit general solution of the system of first order differential equations.

Example 1.3.7. Consider the rotation group:

$$x^\star = x\cos\varepsilon + y\sin\varepsilon, \;\; y^\star = -x\sin\varepsilon + y\cos\varepsilon \tag{1.5}$$

The infinitesimals $\xi(x,y) = \frac{\partial x^\star}{\partial \varepsilon}\Big|_{\varepsilon=0} = y$ and $\eta(x,y) = \frac{\partial y^\star}{\partial \varepsilon}\Big|_{\varepsilon=0} = -x$ define the symmetry generator associated with (1.5) as

$$\mathbf{X} = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} \tag{1.6}$$

Alternatively, given the symmetry generator, one can find the transformation associated with that generator.

Consider the Lie series corresponding to the generator given by

$$(x^\star, y^\star) = (e^{\varepsilon \mathbf{X}}x, e^{\varepsilon \mathbf{X}}y), \tag{1.7}$$

where $\mathbf{X}x = y$, $\mathbf{X}^2 x = -x$ and $\mathbf{X}^3 x = -y$ etc. Then

$$\begin{aligned} x^\star &= e^{\varepsilon \mathbf{X}}x \\ &= \sum_{k=0}^{\infty} \frac{\varepsilon^k}{k¡}\mathbf{X}^k x \\ &= (1 - \frac{\varepsilon^2}{2} + \frac{\varepsilon^4}{4} - \cdots)x + (\varepsilon - \frac{\varepsilon^3}{3} + \frac{\varepsilon^5}{5} - \cdots)y \\ &= x\cos\varepsilon + y\sin\varepsilon. \end{aligned} \tag{1.8}$$

Similarly

$$y^{\star} = e^{\varepsilon \mathbf{X}} y = \sum_{k=0}^{\infty} \frac{\varepsilon^k}{k!} \mathbf{X}^k y = -x \sin \varepsilon + y \cos \varepsilon. \tag{1.9}$$

Theorem 1.3.8 (**Change of coordinates** [7]). The infinitesimal symmetry generator

$$\mathbf{X} = \sum_{i=1}^{n} \xi_i(\alpha) \frac{\partial}{\partial \alpha_i} = \sum_{i=1}^{n} X(\alpha_i) \frac{\partial}{\partial \alpha_i} \tag{1.10}$$

in coordinates α_i can be transformed to new coordinates β_i by the application of infinitesimal symmetry generator to coordinates β_i, through the following formula:

$$\mathbf{X} = \sum_{i=1}^{n} X(\beta_i) \frac{\partial}{\partial \beta_i}. \tag{1.11}$$

Definition 1.3.9 (**Canonical Coordinates**). A change of coordinates $y = (y_1, y_2, ..., y_n)$ defines a set of canonical coordinates for the one-parameter Lie group of transformations $x^{\star} = \chi(x\,; \varepsilon)$ if in terms of such coordinates the group becomes

$$\begin{aligned} y_i^{\star} &= y_i, \qquad i = 1, 2, ..., n-1, \\ y_n^{\star} &= y_n + \varepsilon. \end{aligned} \tag{1.12}$$

Theorem 1.3.10. [7] For any Lie group of transformations $x^{\star} = \chi(x\,; \varepsilon)$ there exists a set of canonical coordinates $y = (y_1, y_2, ..., y_n)$ that make the Lie group equivalent to (1.12).

Theorem 1.3.11. [7] In terms of any set of canonical coordinates $y = (y_1, y_2, ..., y_n)$, the infinitesimal generator of the one-parameter Lie group of transformations $x^{\star} = \chi(x\,; \varepsilon)$ is $\mathbf{Y} = \frac{\partial}{\partial y_n}$.

1.4 Invariance

1.4.1 Invariance of a function

Definition 1.4.1 (Invariance of a function). Let $x^\star = \chi(x\,;\varepsilon)$ be the Lie group of transformations of one parameter ε and let $f(x)$ be an infinitely differentiable function. The function $f(x)$ is said to be an invariant function if and only if

$$f(x^\star) \equiv f(x) \tag{1.13}$$

Theorem 1.4.2. [7] A function $f(x)$ is an invariant of the Lie group of transformation $x^\star = \chi(x\,;\varepsilon)$ if and only if

$$\mathbf{X}f(x) \equiv 0, \tag{1.14}$$

where **X** is the infinitesimal generator of the symmetry transformation.

1.4.2 Invariance of a PDE

Consider a system of PDEs of order n with p-independent and q-dependent variables represented as,

$$F_\mu(x,u,...,u^{(n)}) = 0, \qquad \mu = 1,2,\ldots,k, \tag{1.15}$$

where $x = (x_1,x_2,x_3,\ldots,x_p)$ denotes independent variables, $u = (u^1,u^2,\ldots,u^q)$ denotes dependent variables and $u^{(n)}$ represents the set of all derivatives of order less and equal to n. We denote the derivative of order m by,

$$u_J^\alpha = \frac{\partial^m u^\alpha}{\partial x_{j_1}\partial x_{j_2}\ldots\partial x_{j_m}}\ , \tag{1.16}$$

where $1 \leq j_i \leq p$ for all $i = 1, 2, \ldots, m$ and the order of m-tuple of integers $J = (j_1, j_2, \ldots, j_m)$ indicates the order of the derivative to be taken.

Theorem 1.4.3 (Infinitesimal criterion for the invariance of PDE [7]**).** Let the system,

$$F_\mu(x, u, ..., u^{(n)}) = 0, \qquad \mu = 1, 2, \ldots, k$$

of k differential equations be given. If G is a group of transformations and,

$$X^{(n)} \left\{ F_\mu(x, u, u^{(n)}) \right\} = 0, \qquad \mu = 1, 2, \ldots, k \qquad \text{whenever} \qquad F_\mu(x, u, u^{(n)}) = 0\,, \quad (1.17)$$

for every infinitesimal symmetry generator X of the group G, then G is a symmetry group of the system.

1.5 The r-parameter Lie group of transformations

Definition 1.5.1 (The r-parameter group of transformations)**.** The set of transformations given by $x^\star = \chi(x\,; \varepsilon)$ where $x = (x_1, x_2, ..., x_n)$ lie in a region D$\subset \mathbb{R}^n$ is defined for each $\varepsilon = (\varepsilon_1, \varepsilon_2, ..., \varepsilon_r)$ in set $S \subset \mathbb{R}^r$ with the operation $\varphi(\varepsilon, \delta) = (\phi_1(\varepsilon, \delta), \phi_2(\varepsilon, \delta), ..., \phi_r(\varepsilon, \delta))$, forms r-parameter group of transformation on D if the following hold:

(1) For all $\varepsilon \in S$ the transformations are one-to-one onto D.

(2) S with φ forms a group G.

(3) For all $x \in$ D, $x^\star$ =x when $\varepsilon = \varepsilon_0$ corresponding to the identity e, i.e.,$\chi(x\,; \varepsilon_0) = x$.

(4) If $x^\star = \chi(x\,; \varepsilon)$ and $x^{\star\star} = \chi(x^\star\,; \delta)$, then $x^{\star\star} = \chi(x\,; \varphi(\varepsilon, \delta))$.

Definition 1.5.2 (The r-parameter Lie group of transformation)**.** A r-parameter Group G_r of transformations $x^\star = \chi(x\,; \varepsilon)$, with $x = (x_1, x_2, ..., x_3)$ and parameters $\varepsilon = (\varepsilon_1, \varepsilon_2, ..., \varepsilon_r)$

is called r-parameter Lie group of transformation if

(1) ε is continuous parameter, without loss of generality, with the identity element $\varepsilon = 0$.

(2) χ is an infinitely differentiable function with respect to x in D and an analytic function of ε in S.

(3) The composition law for parameters, denoted by, $\varphi(\varepsilon,\delta) = (\phi_1(\varepsilon,\delta), \phi_2(\varepsilon,\delta), ..., \phi_r(\varepsilon,\delta))$ is an analytic function of $\varepsilon = (\varepsilon_1, \varepsilon_2, ..., \varepsilon_r)$ and $\delta = (\delta_1, \delta_2, ..., \delta_r)$ in S.

Definition 1.5.3 (**Infinitesimal generator** X_j **).** The infinitesimal generator X_j, corresponding to the parameter ε_α of the r-parameter Lie group of transformations $x^\star = \chi(x\ ;\varepsilon)$ is given by

$$\mathbf{X_j} = \sum_{i=1}^{n} \xi_j^i(x) \frac{\partial}{\partial x_i}, \tag{1.18}$$

where $\xi_j^i = \frac{\partial x_i{}^\star}{\partial \varepsilon_j}\Big|_{\varepsilon_j=0}, \quad j = 1,..,r, \quad i = 1,..,n.$

1.6 Lie algebras

Definition 1.6.1 (**Abstract Lie algebra**)**.** A Lie algebra is a vector space L with a given bilinear multiplication law (the product of the elements $a, b \in L$ is usually denoted by $[a,b]$ and is termed the commutator of these elements) which satisfies the skew symmetry property

$[a,b] = -[b,a]$

and the Jacobi identity

$[[a,b],c] + [[b,c],a] + [[c,a],b] = 0$

If the vector space L is finite-dimensional and its dimension is dim L = r, then the corresponding Lie algebra is called an r-dimensional Lie algebra and is denoted by L_r. If

$e_1,...,e_r$ is a basis of the vector space L of the Lie algebra L_r then $[e_i,e_j] = c^k{}_{ij}e_k$; where $c^k{}_{ij}(i,j,k = 1,..,r)$ are real constants called structure constants of the Lie algebra L_r. In what follows only finite-dimensional Lie algebras will be considered, unless otherwise stated.

Theorem 1.6.2. [22] The set G_r of transformations is an r-parameter Lie group of transformation if and only if the vector space of the vector fields ξ is a Lie algebra with respect to the product defined by the formula

$$[\xi\eta]_i(x) = \xi^k\eta_k{}^i - \eta^k\xi_k{}^i, \tag{1.19}$$

This theorem simplifies the study of r-parameter Lie group of transformation by reducing the problem to the study of Lie algebras.

Often it is more convenient to consider a Lie algebra of the corresponding linear operators X instead of a Lie algebra of the vectors ξ. In this case, the linear combination $\lambda X + \mu Y$ of the operators

$$X = \xi^i\frac{\partial}{\partial x^i}, \quad Y = \eta^i\frac{\partial}{\partial x^i} \tag{1.20}$$

corresponds to the linear combination $\lambda\xi + \mu\eta$ of the vectors ξ and η with real constant λ and μ. The commutator of the operators,

$$[X,Y] = XY - YX, \tag{1.21}$$

where X Y is the usual composition of linear operators, corresponds to the multiplication (1.19).

Definition 1.6.3 (Subalgebra)**.** A subset H of Lie algebra L is called a subalgebra of L if it is closed under the commutation operation.

Example 1.6.4. The group of rigid motions in $\mathbb{R}^2$ that preserve distances between any two points in $\mathbb{R}^2$ is the three-parameter Lie group of transformations of rotations and translations in $\mathbb{R}^2$ given by

$$x^\star = x\cos\varepsilon_1 - y\sin\varepsilon_1 + \varepsilon_2 \qquad (1.22)$$
$$y^\star = x\sin\varepsilon_1 + y\cos\varepsilon_1 + \varepsilon_3$$

The corresponding infinitesimal generators are given by

$$\mathbf{X}_1 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}, \qquad (1.23)$$
$$\mathbf{X}_2 = \frac{\partial}{\partial x},$$
$$\mathbf{X}_3 = \frac{\partial}{\partial y}.$$

The commutator table of the above Lie point symmetries is as follows:

Table 1.1: Commutator table

$[\mathbf{X}_i, \mathbf{X}_j]$	$\mathbf{X}_1$	$\mathbf{X}_2$	$\mathbf{X}_3$
$\mathbf{X}_1$	0	$-\mathbf{X}_3$	$\mathbf{X}_2$
$\mathbf{X}_2$	$\mathbf{X}_3$	0	0
$\mathbf{X}_3$	$-\mathbf{X}_2$	0	0

1.7 Structure constants

Theorem 1.7.1 (Second fundamental Theorem of Lie [7])**.** The commutator of any two infinitesimal generators of an r- parameter Lie group G_r of transformations is also an infinitesimal generator. In particular,

$$[\mathbf{X}_\alpha, \mathbf{X}_\beta] = \sum_{r=1}^{r} C^r_{\alpha\beta}\mathbf{X}_r \in L_r, \ \ \forall \ \ \alpha, \beta = 1, .., r. \qquad (1.24)$$

where $C^\gamma_{\alpha\beta}$ are called the structure constants of the Lie algebra L_r.

Definition 1.7.2 (Commutation relations)**.** For an r-parameter Lie group G_r of transformations with the corresponding infinitesimal generators $\mathbf{X}_1, \mathbf{X}_2, \cdots, \mathbf{X}_r$ the relations defined by equation (1.21) are called commutation relations.

Theorem 1.7.3 (Third Fundamental theorem of Lie [7])**.** The structure constants satisfy the relations:

$$\begin{aligned} &1.\ C^{\gamma}_{\alpha\beta} = -C^{\gamma}_{\beta\alpha} && (\textbf{skew symmetry}). \\ &2.\ C^{\rho}_{\alpha\beta}C^{\delta}_{\rho\gamma}+C^{\rho}_{\beta\gamma}C^{\delta}_{\rho\alpha}+C^{\rho}_{\gamma\alpha}C^{\delta}_{\rho\beta}=0 && (\textbf{Jacobi identity}). \end{aligned} \tag{1.25}$$

1.8 Prolongation

Consider the kth-order system of partial differential equations (PDEs) of n independent variables $x=(x^1,x^2,...,x^n)$ and m dependent variables $u=(u^1,u^2,...,u^m)$

$$E^{\alpha}(x,u,u_{(1)},...,u_{(k)})=0, \quad \alpha=1,...,m\ , \tag{1.26}$$

where $u_{(1)},u_{(2)},...,u_{(k)}$ denote the collections of all first, second,..., kth-order partial derivatives,

i.e., $u^{\alpha}_i=D_i(u^{\alpha}),u^{\alpha}_{ij}=D_jD_i(u^{\alpha})$,...respectively, with the total differentiation operator with respect to x^i given by

$$D_i=\frac{\partial}{\partial x^i}+u^{\alpha}_i\frac{\partial}{\partial u^{\alpha}}+u^{\alpha}_{ij}\frac{\partial}{\partial u^{\alpha}_j}+..., \quad i=1,...,n\ , \tag{1.27}$$

in which the summation convention is used.

In order to apply the infinitesimal criterion for the invariance of the kth-order system of (PDEs) of n independent variables and m dependent variables, one needs to extend the infinitesimal symmetry generator to include all the dependent variables and the derivatives of the dependent variables. In this section we discuss the prolongation formula for a kth-order system of PDE which consists of n independent and m dependent variables.

The Lie-Bäcklund operator is

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} \quad \xi^i, \eta^\alpha \in A \ , \tag{1.28}$$

where A is the space of *differentiable functions*. The operator (1.28) is an abbreviated form of the infinite formal sum

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} \zeta^\alpha_{i_1 i_2 \ldots i_s} \frac{\partial}{\partial u^\alpha_{i_1 i_2 \ldots i_s}}, \tag{1.29}$$

where the additional coefficients are determined uniquely by the prolongation formulae [7]

$$\begin{aligned} \zeta^\alpha_i &= D_i(W^\alpha) + \xi^j u^\alpha_{ij} \\ \zeta^\alpha_{i_1 \ldots i_s} &= D_{i_1} \ldots D_{i_s}(W^\alpha) + \xi^j u^\alpha_{j i_1 \ldots i_s}, \quad s > 1, \end{aligned} \tag{1.30}$$

in which W^α is the *Lie characteristic function*

$$W^\alpha = \eta^\alpha - \xi^j u^\alpha_j. \tag{1.31}$$

1.9 Conservation laws

Emmy Noether (1918) showed that if a system of PDEs admits a variational principle (variational system), then a local symmetry leaving invariant the action integral for its Lagrangian density (variational symmetry) yields a conservation law. Conversely, for a given variational system, all local conservation laws arise from variational symmetries. Hence there is a direct one-to-one correspondence between conservation laws and admitted variational symmetries of a variational system. Moreover, one can show that a variational symmetry must be a local symmetry admitted by the variational system; the converse does not hold in general, i.e., there do exist local symmetries of variational systems that are not variational symmetries. Noether's theorem gives a procedure to use symmetry generators with Lagrangian to construct the gauge terms and then the conservation laws.

Definition 1.9.1 (The conserved vector)**.** The n-tuple vector $T = (T^1, T^2, ..., T^n)$, $T^j \in A$, $j = 1, ..., n$ is a conserved vector of (1.26) if T^i satisfies

$$D_i T^i \mid_{(1.26)} = 0. \tag{1.32}$$

Definition 1.9.2 (The Euler-Lagrange operator)**.** The Euler-Lagrange operator for each α, is given by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} (-1)^s D_{i_1} ... D_{i_s} \frac{\partial}{\partial u^\alpha_{i_1 i_2 ... i_s}}, \quad \alpha = 1, ..., m. \tag{1.33}$$

Definition 1.9.3 (Lagrangian and Euler-Lagrange equations)**.** If there exists a function $L = L(x, u, u_{(1)}, ..., u_{(l)}) \in A$, $l \leq k$ such that the system (1.26) can be written as $\delta L / \delta u^\alpha = 0$, then L is called a *Lagrangian* of the system (1.26) and the differential equations of the form

$$\frac{\delta L}{\delta u^\alpha} = 0, \quad \alpha = 1, ..., m, \tag{1.34}$$

are called *Euler-Lagrange equations*.

Example 1.9.4. Consider the Lagrangian,

$$L = \frac{1}{2} y'^2 - \frac{1}{2} y^2. \tag{1.35}$$

The differential equation associated with this Lagrangian is:

$$y'' + y = 0. \tag{1.36}$$

Definition 1.9.5 (The Action Integral)**.** The action integral of a Lagrangian L is given by the following functional

$$J[u] = \int_\Omega L(x,u,u_{(1)},...,u_{(k)})dx, \tag{1.37}$$

where L is defined on a domain Ω in the space $x = (x_1, x_2, ..., x_n)$.

Definition 1.9.6. The functional (1.37) is said to be invariant with respect to the group G_r if for all transformations of the group and all functions $u = u(x)$ the following equality is fulfilled irrespective of the choice of the domain of integration

$$\int_\Omega L(x,u,u_{(1)},...,u_{(k)})dx = \int_{\bar{\Omega}} L(\bar{x},\bar{u},\bar{u}_{(1)},...,\bar{u}_{(k)})d\bar{x}, \tag{1.38}$$

where $\bar{u}$ and $\bar{\Omega}$ are the images of u and Ω, respectively, under the group G_r.

Lemma 1.9.7. [22] The functional (1.37) is invariant with respect to the group G_r with the *Lie-Bäcklund operator* X of the form (1.29) if and only if the following equalities hold

$$W^\alpha \delta L/\delta u^\alpha + D_i(N^i L) \equiv XL + L\, D_i \xi^i = D_i B^i, \tag{1.39}$$

where

$$N^i = \xi^i + W^\alpha \frac{\delta}{\delta u_i^\alpha} + \sum_{s \geq 1} D_{i_1}...D_{i_s}(W^\alpha) \frac{\delta}{\delta u^\alpha_{ii_1 i_2 ... i_s}}, \quad i = 1,...,n\ , \tag{1.40}$$

Remark 1.9.8. N^i is called the *Noether operator associated* with the Lie-Bäcklund operator X and the Euler-Lagrange operators with respect to derivatives of u^α are obtained from (1.33) by replacing u^α by the corresponding derivatives. and the vector B is called the gauge term.

Theorem 1.9.9 (Noether's Theorem)**.** [22] Let the functional (1.37) be invariant with respect to the group G_r with the *Lie-Bäcklund operator* X of the form (1.29). Then the Euler-

Lagrange equations (1.34) have r linearly independent conservation laws $D_i T^i = 0$, where

$$T^i = B^i - N^i L, \quad i = 1, ..., n. \tag{1.41}$$

Example 1.9.10. In order to find Noether symmetries, **X**, of the Lagrangian (1.35) we use the formula given by (1.39). Since we are using the Lagrangian in which y=y(x), the Noether formula takes the form:

$$X\,L + L\,D_x \xi = D_x f, \tag{1.42}$$

where ξ is coefficient of $\frac{\partial}{\partial x}$.

For this Lagrangian, the prolonged symmetry generator X takes the form,

$$X = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \eta^{[1]} \frac{\partial}{\partial y'}, \tag{1.43}$$

where $\eta^{[1]}$ is given by the formula,

$$\eta^{[1]} = \eta_x + y' \eta_y - y' \xi_x - (y')^2 \xi_y, \tag{1.44}$$

and

$$D_x = \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y}. \tag{1.45}$$

Simplifying (1.42) results in the equation:

$$-\eta y + \eta_x y' + y'^2 \eta_y - y'^2 \xi_x - (y')^3 \xi_y + (\frac{1}{2} y'^2 - \frac{1}{2} y^2)(\xi_x + y' \xi_y) = f_x + y' f_y \tag{1.46}$$

Now comparing coefficient of derivatives of y and the constant term from (1.46) we obtain a system of determining equations. Solving this system gives rise to five Noether symmetries associated with the above Lagrangian with the corresponding gauge terms:

$$\begin{aligned} \xi &= \alpha_1 + \alpha_2 cos2x + \alpha_3 \sin 2x, \\ \eta &= (-\alpha_2 \sin 2x + \alpha_3 \cos 2x)y + \alpha_4 \cos x + \alpha_5 \sin x, \\ f(x,y) &= -(\alpha_2 \cos 2x + \alpha_3 \sin 2x)y^2 + (-\alpha_4 \sin x + \alpha_5 \cos x)y + \alpha_6. \end{aligned} \tag{1.47}$$

To find the conservation law corresponding to the Noether symmetry $X_1 = \frac{\partial}{\partial x}$, we use the formula given by (1.41).

$$\begin{aligned} T^x &= -N^x L, \\ &= -(\xi + W\frac{\partial}{\partial y})(\tfrac{1}{2}y'^2 - \tfrac{1}{2}y^2) \\ &= -(\xi - \xi y'\frac{\partial}{\partial y'})(\tfrac{1}{2}y'^2 - \tfrac{1}{2}y^2) \\ &= \tfrac{1}{2}y'^2 + \tfrac{1}{2}y^2 \end{aligned} \tag{1.48}$$

Hence the conservation law corresponding to the Noether symmetry $X_1 = \frac{\partial}{\partial x}$ of the equation $y'' + y = 0$ is

$$D_x(\frac{1}{2}y'^2 + \frac{1}{2}y^2) = 0. \tag{1.49}$$

1.10 Double reduction theory

Sjöberg in [34, 35] discovered a new procedure to find invariant solution for a PDE with two independent variables that conserve a given conservation law.

Consider a scaler PDE (1.26) with $m = 2$ and $(x_1, x_2) = (t, x)$ which admits a symmetry X that is associated with a conservation law $D_t T^t + D_x T^x = 0$.

We find similarity variables r, s, w such that $X = \frac{\partial}{\partial s}$. An invariant solution under the symmetry X has the form $w(r)$ that satisfies an ordinary differential equation of order k.

So that the conservation law is rewritten as $D_rT^r + D_sT^s = 0$, where

$$T^r = \frac{T^tD_t(r) + T^xD_x(r)}{D_t(r)D_x(s) - D_x(r)D_t(s)} \tag{1.50}$$

$$T^s = \frac{T^tD_t(s) + T^xD_x(s)}{D_t(r)D_x(s) - D_x(r)D_t(s)}. \tag{1.51}$$

Sjöberg proved that if X is associated with a conserved vector T, we have $XT^r = 0$ and $XT^s = 0$. So the conservation law in canonical coordinates becomes $D_rT^r = 0$.
Hence we get an ordinary differential equation of order $k-1$ that is $T^r = c$, for some constant c.

Theorem 1.10.1. [34, 35] Suppose that X is any Lie-Bäcklund symmetry of the form (1.33) and
$T^i,\ i = 1, \ldots, n$ are the components of a conserved vector of (1.26). Then

$$T^{*i} = [T^i, X] = X(T^i) + T^iD_j\xi^j - T^jD_j\xi^i, \quad i = 1, \ldots, n. \tag{1.52}$$

constitute the components of a conserved vector of (1.26), i.e. $D_iT^{*i}\mid_{(1.26)} = 0$

Definition 1.10.2. A Lie-Bäcklund symmetry generator X of the form (1.33) is called an associated symmetry with a conserved vector T of the system (1.26) if X and T satisfy the relations

$$[T^i, X] = 0, \quad i = 1, \ldots, n. \tag{1.53}$$

Theorem 1.10.3 (The double reduction theorem)**.** [34, 35] A nonlinear system of q^{th} order PDEs with two independent variables and m dependent variables, which admit a nontrivial conserved form that has at least one associated symmetry can be reduced to a nonlinear system of $(q-1)^{th}$ order ODEs .

Remark 1.10.4. The PDE with two independent variables of order q is reduced to an ODE of order $(q-1)$.

Thus the use of one symmetry which is associated with a conservation law leads to **two reductions**, the first being a reduction of the number of independent variables and the second being a reduction of the order of the DE.

Example 1.10.5 (Sine-Gordon equation)**.**

$$u_{xy} = \sin u \tag{1.54}$$

admits the scaling symmetry

$$X = \frac{\partial}{\partial x} + c\frac{\partial}{\partial y} \tag{1.55}$$

associated with the conservation laws

$$D_x(\cos u) + D_y(\frac{{u_x}^2}{2}) = 0. \tag{1.56}$$

The generator X has a canonical form $X = \frac{\partial}{\partial s}$ when

$$\frac{dx}{1} = \frac{dy}{c} = \frac{du}{0} = \frac{dr}{0} = \frac{ds}{1} = \frac{dw}{0}, \tag{1.57}$$

or

$$s = x,\ r = c\,x - y \text{ and } w(r) = u(x,y). \tag{1.58}$$

In order to find the reduced conserved form, we use the formula given by (1.50).

$$T^r = \frac{1}{2}{u_x}^2 - c\cos u = \frac{1}{2}c^2{w_r}^2 - c\cos w = k_1 \tag{1.59}$$

So we get

$$w_r = \frac{\pm\sqrt{2}}{c}\sqrt{k_1 + c\cos w}. \tag{1.60}$$

This equation is a separable first order ordinary differential equation. Thus

$$\pm\int \frac{c\ dw}{\sqrt{k_1 + c\ cos\ w}} = \sqrt{2}\ r + k_2. \tag{1.61}$$

Finally, the back substituting give us the invariant solution for Sine-Gordon equation.

$$\pm\int \frac{c\ du}{\sqrt{k_1 + c\ cos\ u}} = \sqrt{2}\ (cx - y) + k_2. \tag{1.62}$$

Chapter 2

A symmetry classification, reductions and some exact solutions of certain nonlinear (2+1) diffusion equation

This chapter* provides a complete symmetry classification of certain nonlinear (2+1)-diffusion equation $u_t - \mathbf{div}(f(u)\mathbf{grad}u) = 0$ with variable diffusivity is considered. Using the Lie method, a complete symmetry classification of the equation is presented. Reductions, via two dimensional Lie subalgebras of the equation, to first or second order ordinary differential equations are given. In few interesting cases, exact solutions are found.

2.1 Introduction

The symmetry analysis of one-dimensional diffusion equations has been widely considered by several authors, e.g., Cantwell [10], Ibragimov [20] and Bluman and Kumei [6]. Of particular interest was the work done by Clarkson and Mansfied [12] wherein they considered the (1+1)-heat equation and presented interesting results on classical and nonclassical symmetries possessed by it. Similar studies in this area can be found in Polyanin [32], Serov

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[33], Estevez et al [14] and Doyle and Vassiliou [13]. Recently, Ahmad et al [1] made extensions to those earlier studies wherein they analyzed a narrow class of the (2+1) diffusion equation.

The main interest in the nonlinear diffusion equation comes from the practical notion that for many gases, diffusivity is proportional to the temperature so that a general two-dimensional diffusion equation of interest can be cast as $u_t - \mathbf{div}(f(u)\mathbf{grad}u) = 0$. In this work, a symmetry classification of this equation, viz.,

$$u_t - f(u)(u_{xx} + u_{yy}) - f'(u)(u_x^2 + u_y^2) = 0, \tag{2.1}$$

is presented using the Lie group method. We show that the two dimensional subalgebra of Lie point symmetry generators reduces the equation to first or second ordinary differential equations (odes) which in some cases leads to exact solutions.

2.2 Symmetry generators

In order to derive symmetry generators of (2.1) and obtain closed-form solutions for all $f(u)$, we consider one parameter Lie point transformation that leaves (2.1) invariant, viz.,

$$\tilde{x}^i = x^i + \varepsilon\,\xi^i(x,y,t,u) + O(\varepsilon^2), i = 1,...,4, \tag{2.2}$$

where $\xi^i = \frac{\partial \tilde{x}^i}{\partial \varepsilon}|_{\varepsilon=0}$ defines the symmetry generator associated with (2.2) given by

$$V = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \tau\frac{\partial}{\partial t} + \phi\frac{\partial}{\partial u}. \tag{2.3}$$

In order to determine four components, we prolong V to second order. This prolongation is given by the formula

$$\begin{aligned} V^{(2)} &= V + \phi^x \frac{\partial}{\partial u_x} + \phi^y \frac{\partial}{\partial u_y} + \phi^t \frac{\partial}{\partial u_t} \\ &+ \phi^{xx} \frac{\partial}{\partial u_{xx}} + \phi^{xy} \frac{\partial}{\partial u_{xy}} + \phi^{xt} \frac{\partial}{\partial u_{xt}} + \phi^{yy} \frac{\partial}{\partial u_{yy}} + \phi^{yt} \frac{\partial}{\partial u_{yt}} + \phi^{tt} \frac{\partial}{\partial u_{tt}}, \end{aligned} \tag{2.4}$$

In the above expression every coefficient 'of the prolonged generator' is a function of (x, y, t, u) and can be determined by the formulae,

$$\begin{aligned} \phi^i &= D_i(\phi - \xi u_x - \eta u_y - \tau u_t) + \xi u_{x,i} + \eta u_{y,i} + \tau u_{t,i}, \\ \phi^{ij} &= D_i D_j(\phi - \xi u_x - \eta u_y - \tau u_t) + \xi u_{x,ij} + \eta u_{y,ij} + \tau u_{t,ij}, \end{aligned} \tag{2.5}$$

where D_i represents total derivative and subscripts of u derivative with respect to the respective coordinates. On (2.1), this becomes

$$V^{[2]}[u_t - f(u)(u_{xx} + u_{yy}) - f_u(u)(u_x^2 + u_y^2)] = 0$$

whenever $u_t = f(u)(u_{xx} + u_{yy}) + f_u(u)(u_x^2 + u_y^2)$. That is,

$$\phi^t = f_u(u)(u_{xx} + u_{yy})\phi + f(u)(\phi^{xx} + \phi^{yy}) + f_{uu}(u)(u_x^2 + u_y^2)\phi + 2f_u(u)(\phi^x u_x + \phi^y u_y) \tag{2.6}$$

Substituting the expressions for $\phi^t, \phi^x, \phi^y, \phi^{xx}$ and ϕ^{yy} using (2.5) into (2.6) and then compare the coefficients of the various monomials in derivatives of u. This yields the following reduced system of over-determined partial differential equations

$$
\begin{aligned}
&\tau = \tau(t),\\
&\xi = \xi(x,y,t),\\
&\eta = \eta(x,y,t),\\
&\eta_x = -\xi_y,\\
&\xi_x = \eta_y,\\
&\phi = \alpha(x,y,t)u + \beta(x,y,t),\\
&\phi_t = f(u)(\phi_{xx} + \phi_{yy}),\\
&\xi_t + 2f(u)\phi_{xu} + 2f_u(u)\phi_x = 0,\\
&\eta_t + 2f(u)\phi_{yu} + 2f_u(u)\phi_y = 0,\\
&f_u(u)\phi + f(u)(\tau_t - 2\xi_x) = 0.
\end{aligned}
\tag{2.7}
$$

2.3 Solving the overdetermined system

In this section we attempt to solve the above system starting with (2.7)-(10) rewriting it in the form

$$(\frac{f_u}{f})\phi = (2\xi_x - \tau_t) \tag{2.8}$$

and then considering all possible cases in $f_u \neq 0$. Note that the case $f_u =$ constant is of no interest because this choice reduces the heat equation to a linear one. For this we begin by considering (2.8), which yields following two cases:

I. $\phi = 0$

II. $\phi \neq 0$

We consider these possibilities separately.

3.1. Case I

In this case $f(u)$ is an arbitrary function. Equation (2.8) becomes

$$\tau_t = 2\xi_x \tag{2.9}$$

Now we differentiate (2.7)-(6) w.r.t u to get

$$\alpha(x,y,t) = \beta(x,y,t) = 0 \tag{2.10}$$

By substituting above expressions into (2.7)-(8) and (2.7)-(9) we obtain $\xi_t = 0 = \eta_t$. After some more manipulations one finds that ξ , η and τ become

$$\begin{aligned} \xi &= c_0 + c_1 x + c_2 y, \\ \eta &= c_3 - c_2 x + c_1 y, \\ \tau &= c_4 + 2c_1 t. \end{aligned} \tag{2.11}$$

At this stage we construct the symmetry generators corresponding to each of the constant involved. These are total of five generators given by

$$\begin{aligned} &V_0 = \frac{\partial}{\partial x}, \quad V_1 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2t\frac{\partial}{\partial t}, \quad V_2 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} \\ &V_3 = \frac{\partial}{\partial y}, \quad V_4 = \frac{\partial}{\partial t}. \end{aligned} \tag{2.12}$$

3.2. Case II

Now from (2.7)-(6), (2.8) we obtain

$$(\frac{f_u}{f}) = \frac{(2\xi_x - \tau_t)}{\alpha(x,y,t)u + \beta(x,y,t)} \tag{2.13}$$

Also from (2.7)-(1) and (2.7)-(2), we have the ability to replace $(\frac{f_u}{f})$ by $g(u)$ and $(2\xi_x - \tau_t)$ by $\gamma(x,y,t)$ in (2.13) to obtain the following form

$$g(u) = \frac{\gamma(x,y,t)}{\alpha(x,y,t)u + \beta(x,y,t)} \tag{2.14}$$

which yields following two subcases:

II.a. $\alpha(x,y,t) = 0$

II.b. $\alpha(x,y,t) \neq 0$

We consider these possibilities separately.

Note. In the latter case, if we replace $\frac{\alpha(x,y,t)}{\gamma(x,y,t)}$ by $a(x,y,t)$ and $\frac{\beta(x,y,t)}{\gamma(x,y,t)}$ by $b(x,y,t)$ to obtain the form

$$g(u) = \frac{1}{a(x,y,t)u + b(x,y,t)}, \tag{2.15}$$

then a and b must be constant so that $\alpha = a\gamma$ and $\beta = b\gamma$.

3.2.1. Subcase II.a

In this subcase (2.13) is reduced to

$$\phi = \beta(x,y,t) = (\frac{f}{f_u})(2\xi_x - \tau_t) \tag{2.16}$$

and (2.14) is reduced to

$$g(u) = \frac{\gamma(x,y,t)}{\beta(x,y,t)} \tag{2.17}$$

Since the right side is independent of u and $\gamma(x,y,t) \neq 0$, $\beta(x,y,t) \neq 0$ we obtain

$$(\frac{f_u}{f}) = A \neq 0 \tag{2.18}$$

To determine $f(u)$ we integrate (2.18) with respect to u to obtain

$$f(u) = \sigma e^{Au}, \text{ where } \sigma, \text{ A are nonzero constants.} \tag{2.19}$$

From (2.16), (2.18) we obtain

$$\phi = \frac{2\xi_x - \tau_t}{A}. \tag{2.20}$$

By substituting (2.19), (2.20) into (2.7)-(7,8,9) we obtain

$$\begin{aligned} \xi_{xx} &= \xi_t = 0 \\ \xi_{xy} &= \eta_t = 0 \\ \tau_{tt} &= 0 \end{aligned} \tag{2.21}$$

After some more manipulations one finds that ξ, η and τ become

$$\begin{aligned} \xi &= c_0 + c_1 x + c_2 y \\ \eta &= c_3 - c_2 x + c_1 y \\ \tau &= c_4 + c_5 t \end{aligned} \tag{2.22}$$

Then (2.20) is reduced to

$$\phi = \frac{2c_1 - c_5}{A} \tag{2.23}$$

At this stage we construct the symmetry generators corresponding to each of the constant involved. These are total of six generators given by

$$\begin{array}{lll} V_0 = \frac{\partial}{\partial x}, & V_1 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{2}{A}\frac{\partial}{\partial u}, & V_2 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}, \\ V_3 = \frac{\partial}{\partial y}, & V_4 = \frac{\partial}{\partial t}, & V_5 = t\frac{\partial}{\partial t} - \frac{1}{A}\frac{\partial}{\partial u}. \end{array} \tag{2.24}$$

3.2.2. Subcase II.b

Recall that

$$g(u) = \frac{1}{au+b}, \;\; u \neq -\frac{b}{a}. \tag{2.25}$$

To determine $f(u)$, we integrate (2.25) with respect to u to obtain

$$f(u) = \sigma \mid au+b \mid^{(1/a)}, \tag{2.26}$$

where σ and a are nonzero constants. By substituting the above expression in (2.7)-(6) we obtain

$$\phi = (au+b)\gamma = (au+b)(2\xi_x - \tau_t). \tag{2.27}$$

By substituting (2.25) and (2.27) into (2.7)-(7) we obtain

$$\gamma_t = \gamma_{xx} + \gamma_{yy} = 0. \tag{2.28}$$

Also, by substituting (2.25) and (2.27) into (2.7)-(8) we obtain

$$\xi_t = -2f(u)(1+a)\gamma_x. \tag{2.29}$$

Finally, by substituting (2.25) and (2.27) into (2.7)-(9) we obtain

$$\eta_t = -2f(u)(1+a)\gamma_y, \tag{2.30}$$

which yields the following two subcases:

II.b.1 $a \neq -1$

II.b.2 $a = -1$

We consider these possibilities separately.

3.2.2.1 Subcase II.b.1

In this subcase, since the left side of (2.29) and (2.30) are independent of u, we obtain

$$\gamma_x = \gamma_y = 0 \tag{2.31}$$

By using (2.28)-(2.31), we get

$$\xi_{xx} = \xi_{xy} = \xi_t = \eta_t = \tau_{tt} = 0 \tag{2.32}$$

After some more manipulations one finds that ξ , η and τ become

$$\begin{aligned} \xi &= c_0 + c_1 x + c_2 y, \\ \eta &= c_3 - c_2 x + c_1 y, \\ \tau &= c_4 + c_5 t. \end{aligned} \tag{2.33}$$

Then (2.27) is reduced to

$$\phi = (au+b)(2c_1 - c_5) \tag{2.34}$$

At this stage we construct the symmetry generators corresponding to each of the constant involved. These are total of six generators given by

$$
\begin{aligned}
&V_0 = \frac{\partial}{\partial x}, \quad V_1 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2(au+b)\frac{\partial}{\partial u}, \quad V_2 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}, \\
&V_3 = \frac{\partial}{\partial y}, \quad V_4 = \frac{\partial}{\partial t}, \qquad\qquad\qquad\qquad\quad V_5 = t\frac{\partial}{\partial t} - (au+b)\frac{\partial}{\partial u}.
\end{aligned}
\tag{2.35}
$$

3.2.2.2 Subcase II.b.2

In this subcase (2.29) and (2.30) are reduced to

$$\xi_t = \eta_t = 0 \tag{2.36}$$

By using (2.28), we get

$$\tau_{tt} = 0 \tag{2.37}$$

So ξ and η can be given as a solution of the following system

$$
\begin{aligned}
&\eta_x = -\xi_y, \\
&\xi_x = \eta_y, \\
&\eta_t = 0, \\
&\xi_t = 0.
\end{aligned}
\tag{2.38}
$$

If we restrict ξ , η to be polynomials of order two, then we get

$$
\begin{aligned}
&\xi = c_0 + c_1x + c_2y + c_3xy + c_4x^2 - c_4y^2, \\
&\eta = c_5 - c_2x + c_1y + 2c_4xy - \tfrac{1}{2}c_3x^2 + \tfrac{1}{2}c_3y^2, \\
&\tau = c_6 + c_7t.
\end{aligned}
\tag{2.39}
$$

Then (2.27) is reduced to

$$\phi = (-u+b)(2c_1+2c_3y+4c_4x-c_7) \tag{2.40}$$

At this stage we construct the symmetry generators corresponding to each of the constant involved. These are total of eight generators given by

$$\begin{aligned}
&V_0 = \tfrac{\partial}{\partial x}, && V_1 = x\tfrac{\partial}{\partial x} + y\tfrac{\partial}{\partial y} + 2(-u+b)\tfrac{\partial}{\partial u},\\
&V_2 = y\tfrac{\partial}{\partial x} - x\tfrac{\partial}{\partial y}, && V_3 = xy\tfrac{\partial}{\partial x} + \tfrac{1}{2}(y^2-x^2)\tfrac{\partial}{\partial y} + 2y(-u+b)\tfrac{\partial}{\partial u},\\
&V_4 = (x^2-y^2)\tfrac{\partial}{\partial x} + 2xy\tfrac{\partial}{\partial y} + 4x(-u+b)\tfrac{\partial}{\partial u}, && V_5 = \tfrac{\partial}{\partial y},\\
&V_6 = \tfrac{\partial}{\partial t}, && V_7 = t\tfrac{\partial}{\partial t} + (u-b)\tfrac{\partial}{\partial u}.
\end{aligned} \tag{2.41}$$

2.4 Classification of symmetries

In this section we give a classification of symmetries of the nonlinear heat equation (2.1) as a conclusion of the previous section

(1) $f(u) = \sigma e^{Au}$, where σ and A are nonzero constants.

In this case we have six generators which are given by

$$\begin{aligned}
&V_0 = \tfrac{\partial}{\partial x}, \quad V_1 = x\tfrac{\partial}{\partial x} + y\tfrac{\partial}{\partial y} + \tfrac{2}{A}\tfrac{\partial}{\partial u}, && V_2 = y\tfrac{\partial}{\partial x} - x\tfrac{\partial}{\partial y},\\
&V_3 = \tfrac{\partial}{\partial y}, \quad V_4 = \tfrac{\partial}{\partial t}, && V_5 = t\tfrac{\partial}{\partial t} - \tfrac{1}{A}\tfrac{\partial}{\partial u}.
\end{aligned} \tag{2.42}$$

It is easy to check that the symmetry generators found in (2.42) form a closed Lie algebra whose commutation relations are given in Table 2.1.

(2) $f(u) = \sigma \mid au+b \mid^{(1/a)}$, where σ, a are nonzero constants, b is a constant and $a \neq -1$.

Table 2.1: Case(1)

$[V_i,V_j]$	V_0	V_1	V_2	V_3	V_4	V_5
V_0	0	V_0	$-V_3$	0	0	0
V_1	$-V_0$	0	0	$-V_3$	0	0
V_2	V_3	0	0	$-V_0$	0	0
V_3	0	V_3	V_0	0	0	0
V_4	0	0	0	0	0	V_4
V_5	0	0	0	0	$-V_4$	0

In this case we have six generators which are given by

$$\begin{aligned} &V_0 = \tfrac{\partial}{\partial x}, \quad V_1 = x\tfrac{\partial}{\partial x} + y\tfrac{\partial}{\partial y} + 2(au+b)\tfrac{\partial}{\partial u}, \quad V_2 = y\tfrac{\partial}{\partial x} - x\tfrac{\partial}{\partial y}, \\ &V_3 = \tfrac{\partial}{\partial y}, \quad V_4 = \tfrac{\partial}{\partial t}, \qquad V_5 = t\tfrac{\partial}{\partial t} - (au+b)\tfrac{\partial}{\partial u}. \end{aligned} \tag{2.43}$$

It is easy to check that the symmetry generators found in (2.43) form a closed Lie algebra whose commutation relations are given in Table 2.2.

Table 2.2: Case(2)

$[V_i,V_j]$	V_0	V_1	V_2	V_3	V_4	V_5
V_0	0	V_0	$-V_3$	0	0	0
V_1	$-V_0$	0	0	$-V_3$	0	0
V_2	V_3	0	0	$-V_0$	0	0
V_3	0	V_3	V_0	0	0	0
V_4	0	0	0	0	0	V_4
V_5	0	0	0	0	$-V_4$	0

(3) $f(u) = \sigma / \mid b - u \mid$, where σ is a nonzero constant and b is a constant.

In this case we have eight generators which are given by

$$
\begin{aligned}
&V_0 = \frac{\partial}{\partial x}, && V_1 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2(-u+b)\frac{\partial}{\partial u},\\
&V_2 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}, && V_3 = xy\frac{\partial}{\partial x} + \frac{1}{2}(y^2-x^2)\frac{\partial}{\partial y} + 2y(-u+b)\frac{\partial}{\partial u},\\
&V_4 = (x^2-y^2)\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y} + 4x(-u+b)\frac{\partial}{\partial u}, && V_5 = \frac{\partial}{\partial y},\\
&V_6 = \frac{\partial}{\partial t}, && V_7 = t\frac{\partial}{\partial t} + (u-b)\frac{\partial}{\partial u}.
\end{aligned} \tag{2.44}
$$

It is easy to check that the symmetry generators found in (2.44) form a closed Lie algebra whose commutation relations are given in Table 2.3.

Table 2.3: Case(3)

$[V_i,V_j]$	V_0	V_1	V_2	V_3	V_4	V_5	V_6	V_7
V_0	0	V_0	$-V_5$	V_2	$2V_1$	0	0	0
V_1	$-V_0$	0	0	V_3	V_4	$-V_5$	0	0
V_2	V_5	0	0	$-1/2V_4$	$2V_3$	$-V_0$	0	0
V_3	$-V_2$	$-V_3$	$1/2V_4$	0	0	$-V_1$	0	0
V_4	$-2V_1$	$-V_4$	$-2V_3$	0	0	$2V_2$	0	0
V_5	0	V_5	V_0	V_1	$-2V_2$	0	0	0
V_6	0	0	0	0	0	0	0	V_6
V_7	0	0	0	0	0	0	$-V_6$	0

(4) For arbitrary $f(u)$, i.e., the *principle algebra* is five-dimensional, viz.,

$$\begin{aligned} &V_0=\frac{\partial}{\partial x},\quad V_1=x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+2t\frac{\partial}{\partial t},\quad V_2=y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}\\ &V_3=\frac{\partial}{\partial y},\quad V_4=\frac{\partial}{\partial t}. \end{aligned} \tag{2.45}$$

It is easy to check that the symmetry generators found in (2.45) form a closed Lie algebra whose commutation relations are given in Table 2.4.

Table 2.4: Case(4)

$[V_i,V_j]$	V_0	V_1	V_2	V_3	V_4
V_0	0	V_0	$-V_3$	0	0
V_1	$-V_0$	0	0	$-V_3$	$-2V_4$
V_2	V_3	0	0	$-V_0$	0
V_3	0	V_3	V_0	0	0
V_4	0	$2V_4$	0	0	0

2.5 Reductions

In this section we briefly show the steps involved in the reduction of the nonlinear heat equation to an ordinary differential equation, then find special invariant solutions. Since all reductions under all subalgebras cannot be given in this chapter, we restrict to giving reductions in two cases only. Reductions in the remaining cases with some special solutions are listed in the form of Appendices A,B,C and D.

2.5.1 From case (1) under V_1 and V_2

From Table 2.1, we find that the given generators commute $[V_1,V_2]= 0$. Thus either V_1 or V_2 can be used to start the reduction with. For our purpose we begin reduction with V_1. The characteristic equation associated with this generator is

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dt}{0} = \frac{A\,du}{2}. \tag{2.46}$$

Following standard procedure we integrate the characteristic equation to get three similarity variables.

$$s = \tfrac{y}{x}, \quad r = t, \quad w(r,s) = \tfrac{Au-2\ln(x)}{A} \tag{2.47}$$

Using these similarity variables, Eq. (2.1) can be recast in the form

$$Aw_r = \sigma e^{A\,w}(A^2s^2w_s^2 + A^2w_s^2 - 2sAw_s + As^2w_{ss} + A\,w_{ss} + 2) \tag{2.48}$$

At this stage we express V_2 in terms of the similarity variables defined in (2.47). It is straightforward to note that V_2 in the new variables takes the form

$$\tilde{V}_2 = -(s^2+1)\frac{\partial}{\partial s} - \frac{2s}{A}\frac{\partial}{\partial w} \tag{2.49}$$

The characteristic equation for $\tilde{V}_2$ is

$$\frac{-ds}{(s^2+1)} = \frac{dr}{0} = \frac{-A\ dw}{2s}. \tag{2.50}$$

Integrating this equation as before leads to new variables $\alpha = r$ and $\beta(\alpha) = \frac{A\ w - ln(s^2+1)}{A}$, which reduce (2.48) to a first-order differential equation

$$A\beta_\alpha = 4\sigma e^{\beta A} \tag{2.51}$$

Now by substitute $\alpha = t$ and $\beta = \frac{uA - \ln(x^2+y^2)}{A}$ in the solution of (2.51), we get the following special solution of (2.1)

$$u(x,y,t) = \ln\left(\frac{-x^2-y^2}{4\sigma(t+C_1)}\right)^{1/A} \tag{2.52}$$

2.5.2 From case (2) under V_1 and V_3

From Table 2.2, we find that the given generators satisfy the commutation relation $[V_1, V_3] = -V_3$. This suggests that reduction in this case should start with V_3. The characteristic equation associated with this generator is

$$\frac{dx}{0} = \frac{dy}{1} = \frac{dt}{0} = \frac{du}{0}. \tag{2.53}$$

Following standard procedure we integrate the characteristic equation to get three similarity variables

$$s = x, \quad r = t, \quad w(r,s) = u. \tag{2.54}$$

Using these similarity variables Eq. (2.1) can be recast in the form

$$w_r = \sigma(aw+b)^{(\frac{1-a}{a})}(aww_{ss} + bw_{ss} + w_s^2). \tag{2.55}$$

At this stage we express V_1 in terms of the similarity variables defined in (2.54). It is straightforward to note that V_1 in the new variables takes the form

$$\tilde{V}_1 = s\frac{\partial}{\partial s} + (2aw+2b)\frac{\partial}{\partial w}. \tag{2.56}$$

The characteristic equation for $\tilde{V}_1$ is

$$\frac{ds}{s} = \frac{dr}{0} = \frac{dw}{(2aw+2b)}. \tag{2.57}$$

Integrating this equation as before leads to new variables $\alpha = r$ and $\beta(\alpha) = \frac{aw+b}{as^{2a}}$, which reduce (2.55) to a first-order differential equation

$$\beta_\alpha = 2\sigma(1+2a)(a\beta)^{\frac{1+a}{a}} \tag{2.58}$$

Now by substitute $\alpha = t$ and $\beta = \frac{au+b}{ax^{2a}}$ in the solution of (2.58), we get the following special solution for (2.1)

$$u(x,y,t) = \frac{-b(-2\sigma a^{(\frac{1}{a})}t - 4\sigma a^{(\frac{1+a}{a})}t + C_1)^a + ax^{2a}}{a(-2\sigma a^{(\frac{1}{a})}t - 4\sigma a^{(\frac{1+a}{a})}t + C_1)^a}. \tag{2.59}$$

2.5.3 From case (4) under V_3 and V_4

From Table 2.4, we find that the given generators commute $[V_2,V_4]= 0$. Thus either V_2 or V_4 can be used to start the reduction with. For our purpose we begin reduction with V_2. The characteristic equation associated with this generator is

$$\frac{dx}{y} = \frac{-dy}{x} = \frac{dt}{0} = \frac{Adu}{0}. \tag{2.60}$$

Following standard procedure we integrate the characteristic equation to get three similarity variables.

$$s = x^2 + y^2, \quad r = t, \quad w(r,s) = u. \tag{2.61}$$

Using these similarity variables, Eq. (2.1) can be recast in the form

$$w_r = 4f(w)w_s + 4sf(w)w_{ss} + 4sf_w(w)w_s^2. \tag{2.62}$$

At this stage we express V_4 in terms of the similarity variables defined in (2.61). It is straightforward to note that V_4 in the new variables takes the form

$$\tilde{V}_2 = \frac{\partial}{\partial r}. \tag{2.63}$$

The characteristic equation for $\tilde{V}_4$ is

$$\frac{ds}{0} = \frac{dr}{1} = \frac{dw}{0}. \tag{2.64}$$

Integrating this equation as before leads to new variables $\alpha = s$ and $\beta(\alpha) = w$, which reduce (2.62) to a second-order differential equation

$$f(\beta)\beta_{\alpha} + \alpha f(\beta)\beta_{\alpha\alpha} + \alpha f'(\beta)\beta_{\alpha}^{2} = 0. \tag{2.65}$$

Now by substituting $\alpha = x^2 + y^2$ and $\beta = u$ in the solution of (2.65), we get the following special solution for (2.1)

$$\int f(u)\mathrm{d}u = \ln(c(x^2+y^2)^k), \tag{2.66}$$

where c, k are constants, $c > 0$.

Remark 2.5.1. Other reductions of (2.1) to odes under two-dimensional subalgebras of Lie symmetry generators are given in the Appendices.

2.6 Conclusion

In this chapter, we found the complete set of Lie point symmetry generators of a class of nonlinear diffusion type equations and reduced the equations to odes using two-dimensional Lie subalgebras. Some of these odes were solved whilst the remaining ones can be solved using other methods or reemploying the symmetry approach.

Chapter 3

Conservation Laws of a Nonlinear $(n+1)$ Wave Equation

This chapter* studies the conservation laws of the nonlinear $(n+1)$ wave equation $u_{tt} = \mathbf{div}(f(u)\mathbf{grad}u)$ involving an arbitrary function of the dependent variable. This equation is not derivable from a variational principle. By writing the equation, which admits a partial Lagrangian, in the partial Euler-Lagrange form, partial Noether operators associated with the partial Lagrangian are obtained for all possible cases of the arbitrary function. Partial Noether operators aid via a formula in the construction of the conservation laws of the wave equation. If $f(u)$ is an arbitrary function we show that there is a finite number of conservation laws for $n = 1$ and an infinite number of conservation laws for $n \geq 2$. None of the partial Noether symmetry operators is a Lie point symmetry of the equation. If f is constant, where all of the partial Noether operators are point symmetries of the equation, there is also an infinite number of conservation laws.

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3.1 Introduction

The relationship between symmetries and conservation laws of differential equations has been a topic of great interest (see, e.g. [31, 15, 6, 30, 16, 19, 25]. A systematic way for the determination of conservation laws associated with variational symmetries for systems of Euler-Lagrange equations is indeed the famous Noether theorem [29, 5] (see also [31, 15, 6, 30, 16, 25, 27]. This theorem requires a Lagrangian. There are approaches that do not require a Lagrangian or even assume the existence of a Lagrangian for differential equations (DEs), e.g. scalar evolution equations (see, e.g. [16] and the recent paper [28]). Direct construction methods for multipliers and hence the conservation laws [3], Lagrangian approach for evolution equations [21] and formula for relationship between symmetries and conservation laws, irrespective of the existence of a Lagrangian of the system [25] have been investigated. Also a basis of conservation laws was further investigated in [26] for DEs with and without Lagrangian formulation. Kara and Mahomed in [27] presented a new method to construct conservation laws of DEs via operators that are not necessarily symmetry generators of the underlying system. These partial Noether operators which are associated with partial Lagrangians aid via an explicit Noether-like formula in the construction of conservation laws of the system which need not be derivable from a variational principle. These systems are referred to as partial Euler-Lagrange equations with respect to partial Lagrangians. This approach provides a systematic way of obtaining conservation laws for systems which have partial Lagrangian formulations.

There has been much focus on the determination of conservation laws for various physical systems (see, e.g. [16, 28, 23]). In [28] a (2+1) evolution equation was considered for its conserved quantities using the direct method. Moreover there have been recent works on the (1+1) wave equation in [23] which contains two arbitrary functions. We provide a natural extension of [23] when one of the functions is zero to the case of n-space variables.

For $n = 1$ we recover the results of [23] in the general case.

The outline of this chapter is as follows. In Section 2, we present salient points of the necessary theory. Then in Section 3, investigation on the existence of conservation laws of the nonlinear (2+1) wave equation for all cases of $f(u)$ is carried out using the results of Section 2 and we derive new conservation laws for this equation. Then in Section 4, we generalize our work to the nonlinear $(n+1)$ wave equation for all possibilities of the function $f(u)$. Concluding remarks are given in the last section.

3.2 Operators and the Partial Noether's Theorem

Consider the kth-order system of partial differential equations (PDEs) of n independent variables $x = (x^1, x^2, ..., x^n)$ and m dependent variables $u = (u^1, u^2, ..., u^m)$

$$E^{\alpha}(x, u, u_{(1)}, ..., u_{(k)}) = 0, \quad \alpha = 1, ..., m \;, \tag{3.1}$$

where $u_{(1)}, u_{(2)}, ..., u_{(k)}$ denote the collections of all first, second,..., kth-order partial derivatives, i.e., $u_i^{\alpha} = D_i(u^{\alpha}), u_{ij}^{\alpha} = D_j D_i(u^{\alpha})$,...respectively, with the total differentiation operator with respect to x^i given by

$$D_i = \frac{\partial}{\partial x^i} + u_i^{\alpha}\frac{\partial}{\partial u^{\alpha}} + u_{ij}^{\alpha}\frac{\partial}{\partial u_j^{\alpha}} + ..., \quad i = 1, ..., n \;, \tag{3.2}$$

in which the summation convention is used.

The following definitions are well-known (see, e.g. [15, 27, 19]).

The Euler-Lagrange operator, for each α, is given by

$$\frac{\delta}{\delta u^{\alpha}} = \frac{\partial}{\partial u^{a}} + \sum_{s\geq 1}(-1)^s D_{i_1}...D_{i_s}\frac{\partial}{\partial u^{\alpha}_{i_1 i_2 \cdots i_s}}, \quad \alpha = 1, ..., m. \tag{3.3}$$

The Lie-Bäcklund operator is

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} \quad \xi^i, \eta^\alpha \in A\ , \tag{3.4}$$

where A is the space of *differentiable functions*. The operator (3.4) is an abbreviated form of the infinite formal sum

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} \zeta^\alpha_{i_1 i_2 \dots i_s} \frac{\partial}{\partial u^\alpha_{i_1 i_2 \dots i_s}}, \tag{3.5}$$

where the additional coefficients are determined uniquely by the prolongation formulae

$$\begin{aligned} \zeta^\alpha_i &= D_i(W^\alpha) + \xi^j u^\alpha_{ij} \\ \zeta^\alpha_{i_1 \dots i_s} &= D_{i_1} \dots D_{i_s}(W^\alpha) + \xi^j u^\alpha_{j i_1 \dots i_s}, \quad s > 1, \end{aligned} \tag{3.6}$$

in which W^α is the *Lie characteristic function*

$$W^\alpha = \eta^\alpha - \xi^j u^\alpha_j. \tag{3.7}$$

The *Noether operator associated* with the Lie-Bäcklund operator X is

$$N^i = \xi^i + W^\alpha \frac{\delta}{\delta u^\alpha_i} + \sum_{s \geq 1} D_{i_1} \dots D_{i_s}(W^\alpha) \frac{\delta}{\delta u^\alpha_{i i_1 i_2 \dots i_s}}, \quad i = 1, \dots, n\ , \tag{3.8}$$

where the Euler-Lagrange operators with respect to derivatives of u^α are obtained from (3.3) by replacing u^α by the corresponding derivatives.

The following are taken from [27].

Suppose that the system (3.1) is written as

$$E_\alpha = E_\alpha^0 + E_\alpha^1 = 0, \quad \alpha = 1,...,m. \tag{3.9}$$

If there exists a function $L = L(x,u,u_{(1)},...,u_{(l)}) \in A$, $l \leq k$ and nonzero functions $f_\alpha^\beta \in A$ ((f_α^β) is an invertible matrix) such that (3.9) can be written as $\delta L/\delta u^\alpha = f_\alpha^\beta E_\beta^1$, provided $E_\beta^1 \neq 0$ for some β, then L is called a *partial Lagrangian* of (3.9) and differential equations of the form

$$\tfrac{\delta L}{\delta u^\alpha} = f_\alpha^\beta E_\beta^1, \quad \alpha = 1,...,m, \tag{3.10}$$

are called *partial Euler-Lagrange equations*.

Note that if $E_\beta^1 = 0$ then the partial Lagrangian is the standard Lagrangian and we have the Euler-Lagrange equations $\delta L/\delta u^\alpha = 0$.

A Lie-Bäcklund or generalized operator X of the form (3.5) is called a *partial Noether operator* corresponding to a partial Lagrangian $L \in A$ if and only if there exists a vector $B = (B^1,...,B^n), B^i \in A$ such that

$$X(L) + LD_i(\xi^i) = W^\alpha \frac{\delta L}{\delta u^\alpha} + D_i(B^i), \tag{3.11}$$

where $W = (W^1,...,W^m), W^\alpha \in A$, is the characteristic of X.

Remark 3.2.1. Note that conditions (3.11) differs from the Noether determining equations [12, 2] as $\delta L/\delta u^\alpha \neq 0$ in general.

The adjoint of the fréchet derivative of G with $G = (G_1,...,G_m)$, for each α, is given by

$$(D_G^* W)_\alpha = \tfrac{\partial G^\beta}{\partial u^a} W^\beta + \sum_{s\geq 1} (-1)^s D_{i_1}...D_{i_s}(\tfrac{\partial G^\beta}{\partial u^\alpha_{i_1 i_2 \cdots i_s}} W^\beta), \quad \alpha = 1,...,m. \tag{3.12}$$

Theorem 3.2.2. (Partial Noether's Theorem). A Lie-Bäcklund operator X of the form (3.5) is a partial Noether operator of a partial Lagrangian L corresponding to the partial Euler-Lagrange system (3.10) if and only if the characteristic $W = (W^1, ..., W^m)$, $W^\alpha \in A$, of X is also the characteristic of the conservation law $D_i T^i = 0$, where

$$T^i = B^i - N^i(L), \quad i = 1, ..., n, \tag{3.13}$$

of the *partial Euler-Lagrange equations* (3.10).

Theorem 3.2.3. If X, of the form (3.5), is a partial Noether operator of a partial Lagrangian L, then X is the Lie-Bäcklund operator of the corresponding partial Euler-Lagrange equations $\delta L/\delta u^\alpha = f_\alpha^\beta E_\beta^1 = G_\alpha$ if and only if

$$XG_\alpha = \xi^i D_i G_\alpha + (D_G^* W)_\alpha, \tag{3.14}$$

for each $\alpha = 1, ..., m$.

3.3 Application to a nonlinear (2 + 1) wave equation

Our objective is to obtain all the first order conserved quantities of the nonlinear (2 + 1) wave equation

$$u_{tt} - (f(u)u_x)_x - (f(u)u_y)_y = 0. \tag{3.15}$$

It should be remarked that this equation is not derivable from a variational principle. Here, we investigate conservation laws of the equation for all possible forms of $f(u)$. We have looked at point type operators and we have restricted the gauge terms to be independent of derivatives. For simplicity we denote the derivative of u w.r.t. the independent variables (u_t, u_x, u_y) as (u_1, u_2, u_3).

The equation (3.15) has a partial Lagrangian $L = \frac{1}{2}{u_t}^2 - \frac{1}{2}f(u)\left({u_x}^2 + {u_y}^2\right)$ for which the partial Noether operators $X = \xi_1\partial_t + \xi_2\partial_x + \eta\partial_u$ satisfies (3.11), viz.

$$
\begin{aligned}
&\xi_1(u_1(-\tfrac{1}{2}u_2{}^2 f_u - \tfrac{1}{2}u_3{}^2 f_u) + u_{1,1}u_1 - u_{1,2}f(u)u_2 - u_{1,3}f(u)u_3) + \\
&\xi_2(u_2(-\tfrac{1}{2}u_2{}^2 f_u - \tfrac{1}{2}u_3{}^2 f_u) + u_{2,1}u_1 - u_{2,2}f(u)u_2 - u_{2,3}f(u)u_3) + \\
&\xi_3(u_3(-\tfrac{1}{2}u_2{}^2 f_u - \tfrac{1}{2}u_3{}^2 f_u) + u_{3,1}u_1 - u_{3,2}f(u)u_2 - u_{3,3}f(u)u_3) + \\
&(\eta - \xi_1u_1 - \xi_2u_2 - \xi_3u_3)(-\tfrac{1}{2}u_2{}^2 f_u - \tfrac{1}{2}u_3{}^2 f_u) + \\
&(\eta_t - \xi_{1,t}u_1 - \xi_{2,t}u_2 - \xi_{3,t}u_3 + u_1(\eta_u - \xi_{1,u}u_1 - \xi_{2,u}u_2 - \xi_{3,u}u_3) - \\
&u_{1,1}\xi_1 - u_{1,2}\xi_2 - u_{1,3}\xi_3)u_1 - (\eta_x - \xi_{1,x}u_1 - \xi_{2,x}u_2 - \xi_{3,x}u_3 + \\
&u_2(\eta_u - \xi_{1,u}u_1 - \xi_{2,u}u_2 - \xi_{3,u}u_3) - u_{2,1}\xi_1 - u_{2,2}\xi_2 - u_{2,3}\xi_3t)f(u)u_2 - \\
&(\eta_y - \xi_{1,y}u_1 - \xi_{2,y}u_2 - \xi_{3,y}u_3 + u_3(\eta_u - \xi_{1,u}u_1 - \xi_{2,u}u_2 - \xi_{3,u}u_3) - \\
&u_{3,1}\xi_1 - u_{3,2}\xi_2 - u_{3,3}\xi_3)f(u)u_3 + (-\tfrac{1}{2}f(u)u_2{}^2 - \tfrac{1}{2}f(u)u_3{}^2 + \tfrac{1}{2}u_1{}^2) \\
&(\xi_{1,t} + \xi_{1,u}u_1 + \xi_{2,x} + \xi_{2,u}u_2 + \xi_{3,y} + \xi_{3,u}u_3) = (\eta - \xi_1u_1 - \xi_2u_2 - \xi_3u_3) \\
&(-\tfrac{1}{2}u_2{}^2 f_u - \tfrac{1}{2}u_3{}^2 f_u) + B^1_t + u_1B^1_u + B^2_x + u_2B^2_u + B^3y + u_3B^3_u.
\end{aligned}
\tag{3.16}
$$

Separation by the derivatives of u yields the overdetermined linear system

$$
\begin{aligned}
&\xi_{1,u} = 0,\\
&\xi_{2,u} = 0,\\
&\xi_{3,u} = 0,\\
&\xi_1 f_u - f(u)\,\xi_{1,u} = 0,\\
&\xi_2 f_u - f(u)\,\xi_{2,u} = 0,\\
&\xi_3 f_u - f(u)\,\xi_{3,u} = 0,\\
&B^1_u - \eta_t = 0,\\
&B^2_u + f(u)\,\eta_x = 0,\\
&B^3_u + f(u)\,\eta_y = 0,\\
&f(u)\,(\xi_{3,x} + \xi_{2,y}) = 0,\\
&\xi_{2,t} - f(u)\,\xi_{1,x} = 0,\\
&\xi_{3,t} - f(u)\,\xi_{1,y} = 0,\\
&B^1_t + B^2_x + B^3_y = 0,\\
&f(u)\,(2\eta_u + \xi_{1,t} - \xi_{2,x} + \xi_{3,y}) = 0,\\
&f(u)\,(2\eta_u + \xi_{1,t} + \xi_{2,x} - \xi_{3,y}) = 0,\\
&2\eta_u - \xi_{1,t} + \xi_{2,x} + \xi_{3,y} = 0.
\end{aligned}
\tag{3.17}
$$

From (3.17)-(1, 2, 3, 16) we deduce

$$\eta\,(t,x,y,u) = \tfrac{1}{2}\left(\tfrac{\partial}{\partial t}\xi_1 - \tfrac{\partial}{\partial x}\xi_2 - \tfrac{\partial}{\partial y}\xi_3\right)u + \beta\,(t,x,y) \tag{3.18}$$

Moreover from (3.17)-(7, 8, 9) we obtain

$$B^1\,(t,x,y,u) = \left(\tfrac{\partial^2}{\partial t^2}\xi_1 - \tfrac{\partial^2}{\partial x \partial t}\xi_2 - \tfrac{\partial^2}{\partial y \partial t}\xi_3\right)\tfrac{u^2}{4} + \left(\tfrac{\partial}{\partial t}\beta\,(t,x,y)\right)u + f_I\,(t,x,y) \tag{3.19}$$

$$B^2(t,x,y,u) = -\int f(u)\left(\left(\frac{\partial^2}{\partial x\partial t}\xi_1 - \frac{\partial^2}{\partial x^2}\xi_2 - \frac{\partial^2}{\partial y\partial x}\xi_3\right)\frac{u}{2} + 2\frac{\partial}{\partial x}\beta(t,x,y)\right)du + f_2(t,x,y) \tag{3.20}$$

$$B^3(t,x,y,u) = -\int f(u)\left(\left(\frac{\partial^2}{\partial y\partial t}\xi_1 - \frac{\partial^2}{\partial y\partial x}\xi_2 - \frac{\partial^2}{\partial y^2}\xi_3\right)\frac{u}{2} + 2\frac{\partial}{\partial y}\beta(t,x,y)\right)du + f_3(t,x,y) \tag{3.21}$$

Also from (3.17)-(13), (3.19), (3.20) and (3.21) we find

$$\begin{gathered}
\frac{1}{4}\left(\frac{\partial^3}{\partial t^3}\xi_1 - \frac{\partial^3}{\partial x\partial t^2}\xi_2 - \frac{\partial^3}{\partial y\partial t^2}\xi_3\right)u^2 + \left(\frac{\partial^2}{\partial t^2}\beta(t,x,y)\right)u + \frac{\partial}{\partial t}f_1(t,x,y) - \\
\frac{1}{2}\int f(u)\left(\left(\frac{\partial^3}{\partial x^2\partial t}\xi_1 - \frac{\partial^3}{\partial x^3}\xi_2 - \frac{\partial^3}{\partial y\partial x^2}\xi_3\right)u + 2\frac{\partial^2}{\partial x^2}\beta(t,x,y)\right)du + \frac{\partial}{\partial x}f_2(t,x,y) - \\
\frac{1}{2}\int f(u)\left(\left(\frac{\partial^3}{\partial y^2\partial t}\xi_1 - \frac{\partial^3}{\partial y^2\partial x}\xi_2 - \frac{\partial^3}{\partial y^3}\xi_3\right)u + 2\frac{\partial^2}{\partial y^2}\beta(t,x,y)\right)du + \frac{\partial}{\partial y}f_3(t,x,y) = 0
\end{gathered} \tag{3.22}$$

We differentiate (3.22) w.r.t. u to get

$$\begin{gathered}
\frac{1}{2}\left(\frac{\partial^3}{\partial t^3}\xi_1 - \frac{\partial^3}{\partial x\partial t^2}\xi_2 - \frac{\partial^3}{\partial y\partial t^2}\xi_3\right)u + \frac{\partial^2}{\partial t^2}\beta(t,x,y) - \\
\frac{1}{2}f(u)\left(\left(\frac{\partial^3}{\partial x^2\partial t}\xi_1 - \frac{\partial^3}{\partial x^3}\xi_2 - \frac{\partial^3}{\partial y\partial x^2}\xi_3\right)u + 2\frac{\partial^2}{\partial x^2}\beta(t,x,y)\right) - \\
\frac{1}{2}f(u)\left(\left(\frac{\partial^3}{\partial y^2\partial t}\xi_1 - \frac{\partial^3}{\partial y^2\partial x}\xi_2 - \frac{\partial^3}{\partial y^3}\xi_3\right)u + 2\frac{\partial^2}{\partial y^2}\beta(t,x,y)\right) = 0
\end{gathered} \tag{3.23}$$

Now differentiate (3.22) w.r.t. u twice. This gives

$$\begin{aligned}
&\frac{1}{2}\frac{\partial^3}{\partial t^3}\xi_1 - \frac{1}{2}\frac{\partial^3}{\partial x\partial t^2}\xi_2 - \frac{1}{2}\frac{\partial^3}{\partial y\partial t^2}\xi_3 + \\
&f(u)\left(-\frac{1}{2}\frac{\partial^3}{\partial x^2\partial t}\xi_1 + \frac{1}{2}\frac{\partial^3}{\partial x^3}\xi_2 + \frac{1}{2}\frac{\partial^3}{\partial y\partial x^2}\xi_3\right) + \\
&f(u)\left(-\frac{1}{2}\frac{\partial^3}{\partial y^2\partial t}\xi_1 + \frac{1}{2}\frac{\partial^3}{\partial y^2\partial x}\xi_2 + \frac{1}{2}\frac{\partial^3}{\partial y^3}\xi_3\right) + \\
&\frac{d}{du}f(u)\left(-\frac{1}{2}\left(\frac{\partial^3}{\partial x^2\partial t}\xi_1 - \frac{\partial^3}{\partial x^3}\xi_2 - \frac{\partial^3}{\partial y\partial x^2}\xi_3\right)u - \frac{\partial^2}{\partial x^2}\beta(t,x,y)\right) + \\
&\frac{d}{du}f(u)\left(-\frac{1}{2}\left(\frac{\partial^3}{\partial y^2\partial t}\xi_1 - \frac{\partial^3}{\partial y^2\partial x}\xi_2 - \frac{\partial^3}{\partial y^3}\xi_3\right)u - \frac{\partial^2}{\partial y^2}\beta(t,x,y)\right) = 0
\end{aligned} \tag{3.24}$$

The analysis of (3.24) gives rise to two cases. The results are presented as follows.

(I) $\frac{d}{du}f(u) \neq 0$:

In this case we can obtain via (3.17)-(4, 5, 6), the equations

$$\begin{aligned}\xi_1(t,x,y) &= 0,\\ \xi_2(t,x,y) &= 0,\\ \xi_3(t,x,y) &= 0.\end{aligned} \tag{3.25}$$

By the substitution of (3.25) in both (3.23) and (3.24) we have

$$\eta(t,x,y) = \alpha(x,y)\,t + \gamma(x,y) \tag{3.26}$$

where $\alpha(x,y)$ and $\gamma(x,y)$ are the solutions of $\frac{\partial^2}{\partial x^2}u(x,y) + \frac{\partial^2}{\partial y^2}u(x,y) = 0$.

So we have an infinite number of nontrivial conservation laws that are given as follows:

$$\begin{aligned} T = \; & \Big(\alpha(x,y)\,u - u_1\left(\alpha(x,y)\,t + \gamma(x,y)\right),\\ & -\left(\frac{\partial}{\partial x}\alpha(x,y)\right) t\int f(u)\,du - \left(\frac{\partial}{\partial x}\gamma(x,y)\right)\int f(u)\,du + f(u)\,u_2\left(\alpha(x,y)\,t+\gamma(x,y)\right),\\ & -\left(\frac{\partial}{\partial y}\alpha(x,y)\right)t\int f(u)\,du - \left(\frac{\partial}{\partial y}\gamma(x,y)\right)\int f(u)\,du + f(u)\,u_3\left(\alpha(x,y)\,t+\gamma(x,y)\right)\Big)\end{aligned} \tag{3.27}$$

Now, in general the operator X is not a symmetry of the wave equation. Theorem 3.2.3 helps us to find the η's for which X is a symmetry generator of the wave equation. One can easily see that none of the partial Noether operators is a Lie point symmetry of the equation.

(II) $\frac{d}{du}f(u) = 0$: $f(u) = c$

This case is not of our interest, because we are interested in the nonlinear cases, but we will discuss it briefly in the next section.

3.4 Extension to nonlinear $(n+1)$ wave equation

Our aim here is to obtain all the first-order conserved quantities of the following nonlinear $(n+1)$ wave equation

$$u_{tt}-(f(u)u_{x_i})_{x_i}=0, \quad i=1,...,n. \tag{3.28}$$

It should be pointed out that this equation is not derivable from a variational principle. Here, we investigate conservation laws of the equation for all possible functions $f(u)$. We have looked at point type operators and we have restricted the gauge terms to be independent of derivatives. Again for simplicity we denote the variables $x_1,...,x_n$ as x_s and the derivative of u w.r.t. the independent variables (u_t,u_{x_i}) as $(u_1,u_{(i+1)})$, $i=1,...,n$.

The equation (3.28) has partial Lagrangian $L=\frac{1}{2}{u_t}^2-\frac{1}{2}f(u)\,{u_{x_i}}^2$, whose partial Noether operators of the form $X=\xi_1\partial_t+\xi_{(i+1)}\partial_{x_i}+\eta\partial_u$ which satisfy (3.11).

Similarly, by expansion then separation of the derivatives of u, we find the overdetermined linear system that consists of $5(n+1)+\frac{1}{2}n(n-1)$ equations:

$$\begin{aligned}
&\xi_{i,u}=0, && i=1,...,n+1\\
&\xi_i f_u-f(u)\,\xi_{i,u}=0, && i=1,...,n+1\\
&B^1_u-\eta_t=0, &&\\
&B^{(i+1)}_u+f(u)\,\eta_{x_i}=0, && i=1,...,n\\
&f(u)\,(\xi_{j+1,x_i}+\xi_{i+1,x_j})=0, && i\neq j,\ i=1,...,n;\ j=1,...,n\\
&\xi_{(i+1),t}-f(u)\,\xi_{1,x_i}=0, && i=1,...,n\\
&B^1_t+B^{(i+1)}_{x_i}=0, &&\\
&f(u)\,(2\eta_u+\xi_{1,t}+(-1)^{\delta_{ij}}\,\xi_{(i+1),x_i})=0, && j=1,...,n\\
&2\eta_u-\xi_{1,t}+\xi_{(i+1),x_i}=0. &&
\end{aligned} \tag{3.29}$$

From (3.29)-(1, 9) we have

$$\eta\,(t,x_s,u)=\tfrac{1}{2}(\tfrac{\partial}{\partial t}\xi_1-\tfrac{\partial}{\partial x_i}\xi_{(i+1)})u+\beta\,(t,x_s) \tag{3.30}$$

Also from (3.29)-(3, 4) we deduce

$$B^1\,(t,x_s,u)=\tfrac{1}{4}(\tfrac{\partial^2}{\partial t^2}\xi_1-\tfrac{\partial^2}{\partial x_i\partial t}\xi_{(i+1)})u^2+(\tfrac{\partial}{\partial t}\beta\,(t,x_s))u+f_1\,(t,x_s) \tag{3.31}$$

$$\begin{aligned}B^{(j+1)}\,(t,x_s,u)= &\ -\tfrac{1}{2}\textstyle\int f\,(u)\,((\tfrac{\partial^2}{\partial t\partial x_j}\xi_1-\tfrac{\partial^2}{\partial x_i\partial x_j}\xi_{(i+1)})u+2\,\tfrac{\partial}{\partial x_j}\beta\,(t,x_s)))du+\\ &f_{(j+1)}\,(t,x_s)\,,\,j=1,\ldots,n.\end{aligned} \tag{3.32}$$

Furthermore from (3.29)-(7), (3.31) and (3.32) we obtain

$$\begin{aligned}&\tfrac{1}{4}(\tfrac{\partial^3}{\partial t^3}\xi_1-\tfrac{\partial^2}{\partial x_i\partial t^2}\xi_{(i+1)})u^2+(\tfrac{\partial^2}{\partial t^2}\beta\,(t,x_s))u+\tfrac{\partial}{\partial t}f_1\,(t,x_s)-\\ &\tfrac{1}{2}\textstyle\int f\,(u)\,((\tfrac{\partial^3}{\partial t\partial x_j^2}\xi_1-\tfrac{\partial^3}{\partial x_i\partial x_j^2}\xi_{(i+1)})u+2\,\tfrac{\partial^2}{\partial x_j^2}\beta\,(t,x_s)))du+f_{(j+1)_{x_j}}\,(t,x_s)=0.\end{aligned} \tag{3.33}$$

We differentiate (3.33) w.r.t. u to arrive at

$$\begin{aligned}&\tfrac{1}{2}(\tfrac{\partial^3}{\partial t^3}\xi_1-\tfrac{\partial^2}{\partial x_i\partial t^2}\xi_{(i+1)})u+\tfrac{\partial^2}{\partial t^2}\beta\,(t,x_s)-\\ &\tfrac{1}{2}f\,(u)\,((\tfrac{\partial^3}{\partial t\partial x_j^2}\xi_1-\tfrac{\partial^3}{\partial x_i\partial x_j^2}\xi_{(i+1)})u+2\,\tfrac{\partial^2}{\partial x_j^2}\beta\,(t,x_s)))=0.\end{aligned} \tag{3.34}$$

Then differentiate (3.33) w.r.t. u two times to get

$$\begin{aligned}&\tfrac{1}{2}(\tfrac{\partial^3}{\partial t^3}\xi_1-\tfrac{\partial^2}{\partial x_i\partial t^2}\xi_{(i+1)})-\\ &\tfrac{1}{2}f\,(u)\,((\tfrac{\partial^3}{\partial t\partial x_j^2}\xi_1-\tfrac{\partial^3}{\partial x_i\partial x_j^2}\xi_{(i+1)}))-\\ &\tfrac{1}{2}\,\tfrac{d}{du}f\,(u)\,((\tfrac{\partial^3}{\partial t\partial x_j^2}\xi_1-\tfrac{\partial^3}{\partial x_i\partial x_j^2}\xi_{(i+1)})u+2\,\tfrac{\partial^2}{\partial x_j^2}\beta\,(t,x_s)))=0.\end{aligned} \tag{3.35}$$

The analysis of (3.35) provides the two cases (I) and (II). The results are presented as follows.

(I) $\frac{d}{du}f(u) \neq 0$:

In this case by using (3.29)-(2) we obtain

$$\xi_i(t, x_s) = 0, \quad i = 1, ..., n+1 \tag{3.36}$$

By substitution of (3.36) into (3.34) and (3.35), we find

$$\eta(t, x_s) = \alpha(x_s)t + \gamma(x_s) \tag{3.37}$$

where $\alpha(x_s)$ and $\gamma(x_s)$ are the solutions of the Laplace equation $\frac{\partial^2}{\partial x_j^2}u(x_s) = 0$.

This gives rise to the two subcases (I.a) and (I.b).

(I.a) $n = 1$:

There are four conservation laws (cf. [23]) that are given as:

$$\begin{aligned}
T_1 &= (u\,x_1 - u_1\,t\,x_1, -t\int f(u)\,du + f(u)\,u_2\,t\,x_1),\\
T_2 &= (u - u_1\,t, f(u)\,u_2\,t),\\
T_3 &= (-u_1\,x_1, -\int f(u)\,du + f(u)\;u_2\,x_1),\\
T_4 &= (-u_1, f(u)\,u_2).
\end{aligned} \tag{3.38}$$

(I.b) $n \geq 2$:

There is an infinite number of nontrivial conservation laws that are:

$$\begin{aligned}
T^1 = &\quad \alpha(x_s)u - u_1(\alpha(x_s)t + \gamma(x_s)),\\
T^{(i+1)} = &\; -\left(\frac{\partial}{\partial x_i}\alpha(x_s)\right)t\int f(u)\,du - \left(\frac{\partial}{\partial x_i}\gamma(x_s)\right)\int f(u)\,du +\\
&\quad f(u)\,u_{(i+1)}(\alpha(x_s)t + \gamma(x_s)), i = 1, ..., n
\end{aligned} \tag{3.39}$$

Now, in general X is not a symmetry of the wave equation. Theorem 3.2.3 helps us to find the η's for which X is a symmetry generator of the wave equation. One can easily see that none of the partial Noether operators is a Lie point symmetry of the equation.

(II) $\frac{d}{du}f(u)=0$: $(f(u)=c)$

Really this case is not of our interest, because we are interested in the nonlinear cases. So we summarized it here only when $c\neq 0$.

By the substitution of $f(u)=c$ into (3.33), we find the following two equations:

$$(\frac{\partial^2}{\partial t^2}-c\frac{\partial^2}{\partial x_j^2})(\frac{\partial}{\partial t}\xi_1-\frac{\partial}{\partial x_i}\xi_{(i+1)})=0 \tag{3.40}$$

$$(\frac{\partial^2}{\partial t^2}-c\frac{\partial^2}{\partial x_i^2})\beta(t,x_s)=0 \tag{3.41}$$

The analysis of (3.29)-(5, 6, 8), and (3.40) gives rise to the two subcases (II.a) and (II.b).

(II.a) $n=1$:

The equation (3.29)-(6, 8, 9), (3.40) and (3.41) reduce to the following system

$$\begin{aligned}&\frac{\partial}{\partial t}\xi_2=c\frac{\partial}{\partial x}\xi_1,\\&\frac{\partial}{\partial t}\xi_1=\frac{\partial}{\partial x}\xi_2,\\&(\frac{\partial^2}{\partial t^2}-c\frac{\partial^2}{\partial x^2})\eta(t,x)=0.\end{aligned} \tag{3.42}$$

Hence there are infinite number of nontrivial conservation laws. These are given as follows:

$$\begin{aligned}T=\;&(\eta_t u+\tfrac{1}{2}\xi_1 c{u_2}^2+\tfrac{1}{2}\xi_1{u_1}^2-u_1\eta+u_1\xi_2u_2,\\&-c\eta_{x_1}u-\tfrac{1}{2}\xi_2c{u_2}^2-\tfrac{1}{2}\xi_2{u_1}^2+cu_2\eta-cu_2\xi_1u_1)\end{aligned} \tag{3.43}$$

(II.b) $n\geq 2$:

The equation (3.29)-(5, 6, 8), (3.40) and (3.30) reduce to the following system

$$
\begin{aligned}
&\frac{\partial^3}{\partial t^3}\xi_1 = 0, && \\
&\xi_{j+1,x_i} + \xi_{i+1,x_j} = 0, && i \neq j, i = 1,...,n; j = 1,...,n \\
&\xi_{(j+1),t} - c\xi_{1,x_j} = 0, && j = 1,...,n \\
&\xi_{1,t} - \xi_{(j+1),x_j} = 0, && j = 1,...,n \\
&\eta\,(t, x_s, u) = \frac{(1-n)}{2}(\frac{\partial}{\partial t}\xi_1)u + \beta\,(t, x_s)\,. &&
\end{aligned}
\tag{3.44}
$$

From (3.44)-(1, 4, 5) we have

$$
\begin{aligned}
&\xi_1\,(t, x_s) = \tfrac{1}{2}\alpha\,(x_s)\,t^2 + \gamma(x_s)\,t + \sigma\,(x_s)\,, \\
&\xi_{j+1}\,(t, x_s) = \int(\alpha\,(x_s)\,t + \gamma(x_s))dx_j + Q_j\,(t, x_s)\,, \quad j = 1,...,n \\
&\eta\,(t, x_s, u) = \tfrac{(1-n)}{2}(\alpha\,(x_s)\,t + \gamma(x_s))u + \beta\,(t, x_s)\,.
\end{aligned}
\tag{3.45}
$$

such that $\frac{\partial}{\partial x_j}Q_j = 0, j = 1,...,n$. By substitution of (3.45) into (3.44)-(3) we find

$$
\begin{aligned}
Q_j\,(t, x_i) = \; & \tfrac{1}{6}\left(\tfrac{\partial}{\partial x_j}\alpha\,(x_s)\right)c\,t^3 + \tfrac{1}{2}\left(\tfrac{\partial}{\partial x_j}\gamma(x_s)\right)c\,t^2 - t\int\alpha\,(x_s)\,dx_j + ct\tfrac{\partial}{\partial x_j}\sigma\,(x_s) \\
& + F_j\,(x_s)\,, j = 1,...,n
\end{aligned}
\tag{3.46}
$$

such that $\frac{\partial}{\partial x_j}F_j = 0, j = 1,...,n$ By the inserting of (3.45) and (3.46) into (3.44)-(2) we obtain

$$
\begin{aligned}
&\frac{\partial^2}{\partial x_i \partial x_j}\alpha\,(x_s) = 0, \\
&\frac{\partial^2}{\partial x_i \partial x_j}\gamma(x_s) = 0, \\
&\frac{\partial^2}{\partial x_i \partial x_j}\sigma\,(x_s) = 0, \\
&\int\frac{\partial}{\partial x_i}\gamma(x_s)\,dx_j + \frac{\partial}{\partial x_i}F_j\,(x_s) + \int\frac{\partial}{\partial x_j}\gamma(x_s)\,dx_i + \frac{\partial}{\partial x_j}F_i\,(x_s) = 0. \\
&i \neq j,\; i = 1,...,n;\; j = 1,...,n
\end{aligned}
\tag{3.47}
$$

We differentiate (3.46) and (3.47)-(4) w.r.t. x_j to obtain

$$
\begin{aligned}
&\frac{\partial^2}{\partial x_j{}^2}\alpha(x_s) = 0, && j = 1, ..., n\\
&\frac{\partial^2}{\partial x_j{}^2}\gamma(x_s) = 0, && j = 1, ..., n\\
&\alpha(x_s) - c\frac{\partial^2}{\partial x_j{}^2}\sigma(x_s) = 0, && j = 1, ..., n\\
&\frac{\partial}{\partial x_i}\gamma(x_s) + \frac{\partial^2}{\partial x_j{}^2}F_i(x_s) = 0. && i \neq j,\ i = 1, ..., n;\ j = 1, ..., n
\end{aligned}
\tag{3.48}
$$

Next we differentiate (3.48)-(3, 4) w.r.t. x_j to find

$$
\begin{aligned}
&\frac{\partial^4}{\partial x_j{}^4}\sigma(x_s) = 0, \quad j = 1, ..., n\\
&\frac{\partial^3}{\partial x_j{}^3}F_i(x_s) = 0, \quad j = 1, ..., n
\end{aligned}
\tag{3.49}
$$

Then $\alpha(x_s), \gamma(x_s), \sigma(x_s)$, and $F_i(x_s)$ are determined by (3.47), (3.48) and (3.49) as polynomial functions.

Thus there is an infinite number of nontrivial conservation laws that are given as follows:

$$\begin{aligned}
T^1 = \quad & \tfrac{1}{4}u_1{}^2\alpha(x_s)t^2+\tfrac{1}{2}u_1{}^2\gamma(x_s)t+\tfrac{1}{2}u_1{}^2\sigma(x_s)+\\
& \tfrac{1}{4}cu_{j+1}{}^2\alpha(x_s)t^2+\tfrac{1}{2}cu_{j+1}{}^2\gamma(x_s)t+\tfrac{1}{2}cu_{j+1}{}^2\sigma(x_s)-\\
& \tfrac{1}{2}u_1u\alpha(x_s)t-\tfrac{1}{2}u_1u\gamma(x_s)+\tfrac{n}{2}u_1u\alpha(x_s)t+\tfrac{n}{2}u_1u\gamma(x_s)-\\
& u_1\beta(t,x_s)+u_1u_{j+1}\int\gamma(x_s)\,dx_j+\tfrac{1}{6}u_1u_{j+1}c\left(\tfrac{\partial}{\partial x_j}\alpha(x_s)\right)t^3+\\
& \tfrac{1}{2}u_1u_{j+1}c\left(\tfrac{\partial}{\partial x_j}\gamma(x_s)\right)t^2+u_1u_{j+1}tc\tfrac{\partial}{\partial x_j}\sigma(x_s)+u_1u_{j+1}F_j(x_i),\\
T^{(i+1)} = \quad & \tfrac{1}{2}cu_{j+1}{}^2\int\gamma(x_s)\,dx_i+\tfrac{1}{2}cu_{j+1}{}^2F_i(x_j)+cu_{i+1}\beta(t,x_s)+\\
& \tfrac{1}{2}c^2u_{j+1}{}^2t\tfrac{\partial}{\partial x_i}\sigma(x_s)+\tfrac{1}{4}c^2u_{j+1}{}^2\left(\tfrac{\partial}{\partial x_i}\gamma(x_s)\right)t^2-\tfrac{1}{4}u_1{}^2c\left(\tfrac{\partial}{\partial x_i}\gamma(x_s)\right)t^2-\\
& cu_{i+1}u_1\gamma(x_s)t-\tfrac{1}{6}c^2u_{i+1}u_{j+1}\left(\tfrac{\partial}{\partial x_j}\alpha(x_s)\right)t^3-\tfrac{1}{2}c^2u_{i+1}u_{j+1}\left(\tfrac{\partial}{\partial x_j}\gamma(x_s)\right)t^2-\\
& c^2u_{i+1}u_{j+1}t\tfrac{\partial}{\partial x_j}\sigma(x_s)-c\,u_{i+1}u_1\sigma(x_s)+\tfrac{1}{2}cu_{i+1}u\gamma(x_s)-\\
& \tfrac{1}{2}u_1{}^2tc\tfrac{\partial}{\partial x_i}\sigma(x_s)+\tfrac{1}{12}c^2u_{j+1}{}^2\left(\tfrac{\partial}{\partial x_i}\alpha(x_s)\right)t^3-cu_{i+1}u_{j+1}F_j(x_i)-\\
& \tfrac{1}{12}u_1{}^2c\left(\tfrac{\partial}{\partial x_i}\alpha(x_s)\right)t^3-\tfrac{1}{2}u_1{}^2F_i(x_j)-\tfrac{1}{2}u_1{}^2\int\gamma(x_s)\,dx_i-\\
& cu_{i+1}u_{j+1}\int\gamma(x_s)\,dx_j+\tfrac{1}{2}c\,u_{i+1}u\alpha(x_s)t-\tfrac{n}{2}c\,u_{i+1}u\alpha(x_s)t-\\
& \tfrac{n}{2}c\,u_{i+1}u\gamma(x_s)-\tfrac{1}{2}c\,u_{i+1}u_1\alpha(x_s)t^2.\ i=1,\dots,n
\end{aligned} \tag{3.50}$$

where $\beta(t,x_s)$ is a solution of (3.41).

One can verify that the conserved quantities (3.50) satisfy

$$(D_tT^1+D_{x_i}T^{(i+1)})\mid_{3.1}=0. \tag{3.51}$$

3.5 Conclusion

New conservation laws are constructed for the nonlinear $(n+1)$, $n\geq 1$ wave equation which is not derivable from a variational principle. We use the approach of [27]. For the equation containing an arbitrary function of the dependent variable, all possible cases are considered. When $f(u)$ is an arbitrary function we showed that there is a finite number of conservation laws for $n=1$ which concurs with [23] and an infinite number of conservation laws for

$n \geq 2$. None of the partial Noether symmetry operators is a Lie point symmetry of the equation in this case. If f is constant, where all of the partial Noether operators are also point symmetries of the equation, we found an infinite number of conservation laws.

It will be of interest to further study conservation laws for nonlinear wave equations with more than one arbitrary function for more space variables.

Chapter 4

Generalization of the double reduction theory

In a recent work [34, 35] Sjöberg remarked that generalization of the double reduction theory to partial differential equations of higher dimensions is still an open problem. This chapter* provides this generalization to find invariant solution for a nonlinear system of qth order partial differential equations with n independent and m dependent variables provided that the nonlinear system of partial differential equations admits a nontrivial conserved form which has at least one associated symmetry in every reduction.

4.1 Introduction

Applying a Lie point or Lie-Bäcklund symmetry generator to a conserved vector provide either (1) Conservation law associated with that symmetry or (2) Conservation law that may be trivial, known already or new. A pioneering work in this direction was published by Kara et. al [25, 26]. Sjöberg [34, 35] later showed that when the generated conserved vector is null, i.e. the symmetry is associated with the conserved vector (association defined as in [25]), a double reduction is possible for PDEs with two independent variables. In this dou-

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ble reduction the PDE of order q is reduced to an ODE of order $(q-1)$. Thus the use of one symmetry associated with a conservation law leads to two reductions, *the first being a reduction of the number of independent variables* and *the second being a reduction of the order of the DE.* Sjöberg also constructed the reduction formula for PDEs with two independent variables which transform the conserved form of the PDE to a reduced conserved form via an associated symmetry. Application of this method to the linear heat, the BBM and the sine-Gordon equation and a system of differential equations from one dimensional gas dynamics are given in [34]. The double reduction theory says that a PDE of order q with two independent and m dependent variables, which admits a nontrivial conserved form that has at least one associated symmetry can be reduced to an ODE of order $(q-1)$.
In his papers [34, 35] Sjöberg opines that generalizing the double reduction theory to PDEs of higher dimensions is still an open problem and it is not clear how to overcome the problem when not all derivatives of non-local variables are known explicitly. Further calculations for higher dimensions are quite tedious and cumbersome. There do not exist enough examples of potential symmetries and symmetries with associated conservation laws for higher dimensional PDEs so that the complexity of this problem can be demonstrated. Much work is needed to generalize (if possible) the theory to PDEs with more than two independent variables.

In this chapter we discuss *a generalization of the double reduction theory* showing that a nonlinear system of q^{th} order PDEs with n independent and m dependent variables can be reduced to a nonlinear system of $(q-1)^{th}$ order ODEs. It is shown that these reductions are possible provided the system admits a nontrivial conserved form with at least one associated symmetry in every reduction.

In order to solve this we use two main steps: (a) Generalize the reduction formula of

Sjöberg in [34] from two independent variable to n independent variables and (b) prove that the conserved form of PDEs with n independent variables can be transformed to a reduced conserved form via an associated symmetry.

4.2 The Fundamental Theorem of double reduction

Consider the qth-order system of partial differential equations (PDEs) of n independent variables $x = (x^1, x^2, \ldots, x^n)$ and m dependent variables $u = (u^1, u^2, \ldots, u^m)$

$$E^{\alpha}(x, u, u_{(1)}, \ldots, u_{(q)}) = 0, \quad \alpha = 1, \ldots, m \;, \tag{4.1}$$

Theorem 4.2.1. [4] Suppose $D_i T^i = 0$ is a conservation law of PDE system (4.1). Under the contact transformation, there exist functions $\tilde{T}^i$ such that $J\ D_i T^i = \tilde{D}_i \tilde{T}^i$ where $\tilde{T}^i$ is given explicitly in terms of the determinant obtained through replacing the i^{th} row of the Jacobian determinant by $[T^1, T^2, \ldots, T^n]$, where

$$J = \begin{vmatrix} \tilde{D}_1 x_1 & \tilde{D}_1 x_2 & \ldots & \tilde{D}_1 x_n \\ \tilde{D}_2 x_1 & \tilde{D}_2 x_2 & \ldots & \tilde{D}_2 x_n \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{D}_n x_1 & \tilde{D}_n x_2 & \ldots & \tilde{D}_n x_n \end{vmatrix} \tag{4.2}$$

Theorem 4.2.2. Suppose $D_i T^i = 0$ is a conservation law of PDE system (4.1). Under the contact transformation, there exist functions $\tilde{T}^i$ such that $J\ D_i T^i = \tilde{D}_i \tilde{T}^i$ where $\tilde{T}^i$ is given explicitly in terms of

$$\begin{pmatrix} \tilde{T}^1 \\ \tilde{T}^2 \\ \vdots \\ \tilde{T}^n \end{pmatrix} = J(A^{-1})^T \begin{pmatrix} T^1 \\ T^2 \\ \vdots \\ T^n \end{pmatrix}, \quad J \begin{pmatrix} T^1 \\ T^2 \\ \vdots \\ T^n \end{pmatrix} = A^T \begin{pmatrix} \tilde{T}^1 \\ \tilde{T}^2 \\ \vdots \\ \tilde{T}^n \end{pmatrix}, \tag{4.3}$$

where

$$A = \begin{pmatrix} \tilde{D}_1 x_1 & \tilde{D}_1 x_2 & \dots & \tilde{D}_1 x_n \\ \tilde{D}_2 x_1 & \tilde{D}_2 x_2 & \dots & \tilde{D}_2 x_n \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{D}_n x_1 & \tilde{D}_n x_2 & \dots & \tilde{D}_n x_n \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} D_1 \tilde{x}_1 & D_1 \tilde{x}_2 & \dots & D_1 \tilde{x}_n \\ D_2 \tilde{x}_1 & D_2 \tilde{x}_2 & \dots & D_2 \tilde{x}_n \\ \vdots & \vdots & \vdots & \vdots \\ D_n \tilde{x}_1 & D_n \tilde{x}_2 & \dots & D_n \tilde{x}_n \end{pmatrix} \tag{4.4}$$

and $J = det(A)$.

Proof. Using theorem 4.2.1 we can write

$$\tilde{T}^1 = \begin{vmatrix} T_1 & T_2 & \dots & T_n \\ \tilde{D}_2 x_1 & \tilde{D}_2 x_2 & \dots & \tilde{D}_2 x_n \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{D}_n x_1 & \tilde{D}_n x_2 & \dots & \tilde{D}_n x_n \end{vmatrix} = \frac{1}{J} \begin{vmatrix} J\,T_1 & \tilde{D}_2 x_1 & \dots & \tilde{D}_n x_1 \\ J\,T_2 & \tilde{D}_2 x_2 & \dots & \tilde{D}_n x_2 \\ \vdots & \vdots & \vdots & \vdots \\ J\,T_n & \tilde{D}_2 x_n & \dots & \tilde{D}_n x_n \end{vmatrix}, \tag{4.5}$$

$$\tilde{T}^2 = \begin{vmatrix} \tilde{D}_1 x_1 & \tilde{D}_1 x_2 & \dots & \tilde{D}_1 x_n \\ T_1 & T_2 & \dots & T_n \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{D}_n x_1 & \tilde{D}_n x_2 & \dots & \tilde{D}_n x_n \end{vmatrix} = \frac{1}{J} \begin{vmatrix} \tilde{D}_1 x_1 & J\,T_1 & \dots & \tilde{D}_n x_1 \\ \tilde{D}_1 x_2 & J\,T_2 & \dots & \tilde{D}_n x_2 \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{D}_1 x_n & J\,T_n & \dots & \tilde{D}_n x_n \end{vmatrix}, \tag{4.6}$$

$$\tilde{T}^n = \begin{vmatrix} \tilde{D}_1 x_1 & \tilde{D}_1 x_2 & \dots & \tilde{D}_1 x_n \\ \tilde{D}_2 x_1 & \tilde{D}_2 x_2 & \dots & \tilde{D}_2 x_n \\ \vdots & \vdots & \vdots & \vdots \\ T_1 & T_2 & \dots & T_n \end{vmatrix} = \frac{1}{J} \begin{vmatrix} \tilde{D}_1 x_1 & \tilde{D}_2 x_1 & \dots & J\,T_1 \\ \tilde{D}_1 x_2 & \tilde{D}_2 x_2 & \dots & J\,T_2 \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{D}_1 x_n & \tilde{D}_2 x_n & \dots & J\,T_n \end{vmatrix}. \tag{4.7}$$

Since

$$J = \begin{vmatrix} \tilde{D}_1 x_1 & \tilde{D}_1 x_2 & \dots & \tilde{D}_1 x_n \\ \tilde{D}_2 x_1 & \tilde{D}_2 x_2 & \dots & \tilde{D}_2 x_n \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{D}_n x_1 & \tilde{D}_n x_2 & \dots & \tilde{D}_n x_n \end{vmatrix} = \begin{vmatrix} \tilde{D}_1 x_1 & \tilde{D}_2 x_1 & \dots & \tilde{D}_n x_1 \\ \tilde{D}_1 x_2 & \tilde{D}_2 x_2 & \dots & \tilde{D}_n x_2 \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{D}_1 x_n & \tilde{D}_2 x_n & \dots & \tilde{D}_n x_n \end{vmatrix} = \left|A^T\right|, \tag{4.8}$$

one can use the Cramer's rule to find that $\tilde{T}^1, \tilde{T}^2, \ldots, \tilde{T}^n$ can be written as follows:

$$\begin{pmatrix} J\,T^1 \\ J\,T^2 \\ \vdots \\ J\,T^n \end{pmatrix} = A^T \begin{pmatrix} \tilde{T}^1 \\ \tilde{T}^2 \\ \vdots \\ \tilde{T}^n \end{pmatrix}. \tag{4.9}$$

Lastly, one can easily see that

$$AA^{-1} = I. \tag{4.10}$$

□

Lemma 4.2.3. Consider n independent variables $x = (x^1, x^2, ..., x^n)$, m dependent variables $u = (u^1, u^2, ..., u^m)$ and the change of independent variables $\tilde{x} = (\tilde{x}^1, \tilde{x}^2, ..., \tilde{x}^n)$, then any vector $f(x, u, u_1) = (f^1, f^2, ..., f^n)$ must satisfy the following identity

$$\begin{bmatrix} \tilde{D}_1 & \tilde{D}_1 & \ldots & \tilde{D}_1 \\ \tilde{D}_2 & \tilde{D}_2 & \ldots & \tilde{D}_2 \\ \vdots & \vdots & \ldots & \vdots \\ \tilde{D}_n & \tilde{D}_n & \ldots & \tilde{D}_n \end{bmatrix} \begin{pmatrix} f^1 & f^2 & \ldots & f^n \\ f^1 & f^2 & \ldots & f^n \\ \vdots & \vdots & \ldots & \vdots \\ f^1 & f^2 & \ldots & f^n \end{pmatrix} = A \begin{bmatrix} D_1 & D_1 & \ldots & D_1 \\ D_2 & D_2 & \ldots & D_2 \\ \vdots & \vdots & \ldots & \vdots \\ D_n & D_n & \ldots & D_n \end{bmatrix} \begin{pmatrix} f^1 & f^2 & \ldots & f^n \\ f^1 & f^2 & \ldots & f^n \\ \vdots & \vdots & \ldots & \vdots \\ f^1 & f^2 & \ldots & f^n \end{pmatrix}, \tag{4.11}$$

where

$$A = \begin{pmatrix} \tilde{D}_1 x_1 & \tilde{D}_1 x_2 & ... & \tilde{D}_1 x_n \\ \tilde{D}_2 x_1 & \tilde{D}_2 x_2 & ... & \tilde{D}_2 x_n \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{D}_n x_1 & \tilde{D}_n x_2 & ... & \tilde{D}_n x_n \end{pmatrix} \tag{4.12}$$

Proof.

Since

$$\tilde{D}_i f^j = \tilde{D}_i x_k D_k f^j, \quad i, j = 1, ..n, \tag{4.13}$$

then

$$\begin{pmatrix} \tilde{D}_1 f^1 & \tilde{D}_1 f^2 & \dots & \tilde{D}_1 f^n \\ \tilde{D}_2 f^1 & \tilde{D}_2 f^2 & \dots & \tilde{D}_2 f^n \\ \vdots & \vdots & \dots & \vdots \\ \tilde{D}_n f^1 & \tilde{D}_n f^2 & \dots & \tilde{D}_n f^n \end{pmatrix} = A \begin{pmatrix} D_1 f^1 & D_1 f^2 & \dots & D_1 f^n \\ D_2 f^1 & D_2 f^2 & \dots & D_2 f^n \\ \vdots & \vdots & \dots & \vdots \\ D_n f^1 & D_n f^2 & \dots & D_n f^n \end{pmatrix} \tag{4.14}$$

□

Theorem 4.2.4. (Fundamental Theorem of double reduction).

Suppose $D_i T^i = 0$ is a conservation law of PDE system (4.1). Under the similarity transformation of a symmetry X of the form (3.5) for the PDE, there exist functions $\tilde{T}^i$ such that X is still a symmetry for the PDE $\tilde{D}_i \tilde{T}^i = 0$ and

$$\begin{pmatrix} X\tilde{T}^1 \\ X\tilde{T}^2 \\ \vdots \\ X\tilde{T}^n \end{pmatrix} = J(A^{-1})^T \begin{pmatrix} [T^1, X] \\ [T^2, X] \\ \vdots \\ [T^n, X] \end{pmatrix}, \tag{4.15}$$

where

$$A = \begin{pmatrix} \tilde{D}_1 x_1 & \tilde{D}_1 x_2 & \dots & \tilde{D}_1 x_n \\ \tilde{D}_2 x_1 & \tilde{D}_2 x_2 & \dots & \tilde{D}_2 x_n \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{D}_n x_1 & \tilde{D}_n x_2 & \dots & \tilde{D}_n x_n \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} D_1 \tilde{x}_1 & D_1 \tilde{x}_2 & \dots & D_1 \tilde{x}_n \\ D_2 \tilde{x}_1 & D_2 \tilde{x}_2 & \dots & D_2 \tilde{x}_n \\ \vdots & \vdots & \vdots & \vdots \\ D_n \tilde{x}_1 & D_n \tilde{x}_2 & \dots & D_n \tilde{x}_n \end{pmatrix} \tag{4.16}$$

and $J = det(A)$.

Proof. By the above theorem there exist functions $\tilde{T}^i$ such that $J\ D_iT^i = \tilde{D}_i\tilde{T}^i$ and

$$\begin{pmatrix} \tilde{T}^1 \\ \tilde{T}^2 \\ \vdots \\ \tilde{T}^n \end{pmatrix} = J(A^{-1})^T \begin{pmatrix} T^1 \\ T^2 \\ \vdots \\ T^n \end{pmatrix}, \quad J \begin{pmatrix} T^1 \\ T^2 \\ \vdots \\ T^n \end{pmatrix} = A^T \begin{pmatrix} \tilde{T}^1 \\ \tilde{T}^2 \\ \vdots \\ \tilde{T}^n \end{pmatrix} \tag{4.17}$$

Then X is a symmetry for the PDE $\tilde{D}_i\tilde{T}^i = 0$, because $X(J)\ D_iT^i + J\ X(D_iT^i) = X(\tilde{D}_i\tilde{T}^i)$ and

$$\begin{pmatrix} X\tilde{T}^1 \\ X\tilde{T}^2 \\ \vdots \\ X\tilde{T}^n \end{pmatrix} = J(A^{-1})^T \begin{pmatrix} XT^1 \\ XT^2 \\ \vdots \\ XT^n \end{pmatrix} + J\,X((A^{-1})^T) \begin{pmatrix} T^1 \\ T^2 \\ \vdots \\ T^n \end{pmatrix} + X(J)(A^{-1})^T \begin{pmatrix} T^1 \\ T^2 \\ \vdots \\ T^n \end{pmatrix}. \tag{4.18}$$

Since $J = det(A)$, then

$$X(J) = \begin{vmatrix} \tilde{D}_1\xi^1 & \tilde{D}_1\xi^2 & \dots & \tilde{D}_1\xi^n \\ \tilde{D}_2x_1 & \tilde{D}_2x_2 & \dots & \tilde{D}_2x_n \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{D}_nx_1 & \tilde{D}_nx_2 & \dots & \tilde{D}_nx_n \end{vmatrix} + \begin{vmatrix} \tilde{D}_1x_1 & \tilde{D}_1x_2 & \dots & \tilde{D}_1x_n \\ \tilde{D}_2\xi^1 & \tilde{D}_2\xi^2 & \dots & \tilde{D}_2\xi^n \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{D}_nx_1 & \tilde{D}_nx_2 & \dots & \tilde{D}_nx_n \end{vmatrix} + \cdots + \begin{vmatrix} \tilde{D}_1x_1 & \tilde{D}_1x_2 & \dots & \tilde{D}_1x_n \\ \tilde{D}_2x_1 & \tilde{D}_2x_2 & \dots & \tilde{D}_2x_n \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{D}_n\xi^1 & \tilde{D}_n\xi^2 & \dots & \tilde{D}_n\xi^n \end{vmatrix} \tag{4.19}$$

Let ζ_{ij} denote the cofactor of $\tilde{D}_i\xi^j$, then it is the cofactor of $\tilde{D}_ix_j$ for the matrix A. Thus

$$X(J) = \tilde{D}_i\xi^j\ \ \zeta_{ij} = D_k\xi^j\ \ \tilde{D}_ix_k\ \ \zeta_{ij} = D_k\xi^j\ \ \delta_{jk}\ \ J. \tag{4.20}$$

Since $\tilde{D}_ix_k\ \ \zeta_{ij} = \delta_{jk}\ \ J$ for every fixed j and k, where δ_{jk} is the Kronecker delta, then

$$X(J) = J(D_1\xi^1 + D_2\xi^2 + \dots + D_n\xi^n) \tag{4.21}$$

Now using the previous lemma one gets,

$$\begin{bmatrix} \tilde{D}_1 & \tilde{D}_1 & \dots & \tilde{D}_1 \\ \tilde{D}_2 & \tilde{D}_2 & \dots & \tilde{D}_2 \\ \vdots & \vdots & \dots & \vdots \\ \tilde{D}_n & \tilde{D}_n & \dots & \tilde{D}_n \end{bmatrix} \begin{pmatrix} \xi^1 & \xi^2 & \dots & \xi^n \\ \xi^1 & \xi^2 & \dots & \xi^n \\ \vdots & \vdots & \dots & \vdots \\ \xi^1 & \xi^2 & \dots & \xi^n \end{pmatrix} = A \begin{bmatrix} D_1 & D_1 & \dots & D_1 \\ D_2 & D_2 & \dots & D_2 \\ \vdots & \vdots & \dots & \vdots \\ D_n & D_n & \dots & D_n \end{bmatrix} \begin{pmatrix} \xi^1 & \xi^2 & \dots & \xi^n \\ \xi^1 & \xi^2 & \dots & \xi^n \\ \vdots & \vdots & \dots & \vdots \\ \xi^1 & \xi^2 & \dots & \xi^n \end{pmatrix} \tag{4.22}$$

Now transposing both sides gives,

$$X(A^T) = \begin{pmatrix} D_1\xi^1 & D_2\xi^1 & \dots & D_n\xi^1 \\ D_1\xi^2 & D_2\xi^2 & \dots & D_n\xi^2 \\ \vdots & \vdots & \vdots & \vdots \\ D_1\xi^n & D_2\xi^n & \dots & D_n\xi^n \end{pmatrix} A^T \tag{4.23}$$

Since $A^T(A^{-1})^T = I$, then $X(A^T)(A^{-1})^T + A^T X((A^{-1})^T) = 0$, thus

$$X((A^{-1})^T) = -(A^T)^{-1}X(A^T)(A^{-1})^T = -(A^{-1})^T X(A^T)(A^T)^{-1}$$

$$= -(A^{-1})^T \begin{pmatrix} D_1\xi^1 & D_2\xi^1 & \dots & D_n\xi^1 \\ D_1\xi^2 & D_2\xi^2 & \dots & D_n\xi^2 \\ \vdots & \vdots & \vdots & \vdots \\ D_1\xi^n & D_2\xi^n & \dots & D_n\xi^n \end{pmatrix} . \tag{4.24}$$

Lastly we get the result

$$\begin{pmatrix} X\tilde{T}^1 \\ X\tilde{T}^2 \\ \vdots \\ X\tilde{T}^n \end{pmatrix} = J(A^{-1})^T \left(\begin{pmatrix} XT^1 \\ XT^2 \\ \vdots \\ XT^n \end{pmatrix} - \begin{pmatrix} D_1\xi^1 & D_2\xi^1 & \dots & D_n\xi^1 \\ D_1\xi^2 & D_2\xi^2 & \dots & D_n\xi^2 \\ \vdots & \vdots & \vdots & \vdots \\ D_1\xi^n & D_2\xi^n & \dots & D_n\xi^n \end{pmatrix} \begin{pmatrix} T^1 \\ T^2 \\ \vdots \\ T^n \end{pmatrix} + D_i\xi^i \begin{pmatrix} T^1 \\ T^2 \\ \vdots \\ T^n \end{pmatrix} \right) \tag{4.25}$$

□

Corollary 4.2.5. (The necessary and sufficient condition to get reduced conserved form) The conserved form $D_iT^i = 0$ of PDE system (4.1) can be reduced under the similarity transformation of a symmetry X to a reduced conserved form $\tilde{D}_i\tilde{T}^i = 0$ if and only if X is associated with the conservation law T, i.e. $[T,X]\mid_{(4.1)}= 0$.

Corollary 4.2.6. (The generalized double reduction theory)
A nonlinear system of q^{th} order PDEs with n independent and m dependent variables, which admits a nontrivial conserved form that has at least one associated symmetry in every reduction from the n reductions (the first step of double reduction) can be reduced to a nonlinear system $(q-1)^{th}$ order of ODEs .

Remark 4.2.7. According to the procedure of Sjöberg [34, 35], one can arrive from a PDE of order q to an ODE of order q provided there exists at least one associated symmetry in every reduction. This follows directly from the above theorem by the invariance of the fluxes and using canonical coordinates. Lastly, the q^{th} order ODE (written in the conserved form) is reduced to an ODE of order $(q-1)$.

Corollary 4.2.8. (The inherited symmetries)
Any symmetry Y for the conserved form $D_iT^i = 0$ of PDE system (4.1) can be transformed under the similarity transformation of a symmetry X for the PDE to the symmetry $\tilde{Y}$ for the PDE $\tilde{D}_i\tilde{T}^i = 0$.

Remark 4.2.9. There is a possibility to get an associated symmetry with a reduced conserved form by inhering of the non associated symmetry with the original conserved form. So there is an important use of the non associated symmetry also in Double reduction.
Finally, we conjecture that the reduction under a combination of an associated and a non associated symmetries will give us two PDE one of them is a reduced conserved form

and the second is a non reduced conserved form, we can separate them via the condition $X(\tilde{D}_i\tilde{T}^i) = 0$ such that the solution of a reduced conserved form is also a solution of the non reduced conserved form.

4.3 Conclusion

In order to find invariant solutions for a nonlinear system of q^{th} order PDEs with n independent and m dependent variables, a generalization of the double reduction theory due to Sjöberg [34, 35] is proposed. This generalization allows one to reduce the PDEs of order q to an ODE of $(q-1)$ from the association of the symmetry with its conserved form via the new generalized formula (4.3).

Chapter 5

The double reduction of a Nonlinear $(2+1)$ Wave Equation with different arbitrary functions

This chapter* provides the conservation laws of the nonlinear $(2+1)$ wave equation $u_{tt} = (f(u)u_x)_x + (g(u)u_y)_y$ involving arbitrary functions of the dependent variable, by writing the equation, which admits a partial Lagrangian, in the partial Euler-Lagrange form, partial Noether operators associated with the partial Lagrangian are obtained for all possible cases of the arbitrary functions. Partial Noether operators aid via a formula in the construction of the conservation laws of the wave equation. If $f(u)$ or $g(u)$ is an arbitrary nonconstant function we show that there is an infinite number of conservation laws. If both of $f(u)$ and $g(u)$ are arbitrary nonconstant functions we show that there is an infinite number of conservation laws for $f(u) = c_1 g(u) + c_2$ for some constants c_1 and c_2, otherwise there are eight conservation laws. Finally we applied The generalized double reduction theory for the Nonlinear $(2+1)$ wave equation by using seven conservation laws from these eight conservation laws to introduce new exact solutions.

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5.1 Introduction

The analysis of conservation laws of the nonlinear (2+1) wave equation with an arbitrary function was studied in chapter 3. Here we extend the study in the case of different arbitrary functions, then we apply The generalized double reduction theory for these conservation laws.

The outline of the chapter is as follows. In Section 2, investigation on the existence of conservation laws of the nonlinear (2+1) wave equation for all cases of $f(u)$ and $g(u)$ is carried out using the Partial Noether's Theorem and we derive new conservation laws for this equation. Finally, in Section 3, we apply the generalized double reduction theory for the nonlinear $(2+1)$ wave equation by seven conservation laws to introduce new exact solutions.

5.2 The Conservation laws of a nonlinear (2 + 1) wave equation with different arbitrary functions

Our objective is to obtain all the first order conserved quantities of the nonlinear (2 + 1) wave equation

$$u_{tt} - (f(u)u_x)_x - (g(u)u_y)_y = 0. \tag{5.1}$$

It should be remarked that this equation is not derivable from a variational principle. Here, we investigate conservation laws of the equation for all possible forms of $f(u)$. We have looked at point type operators and we have restricted the gauge terms to be independent of derivatives. For simplicity we denote the derivative of u w.r.t. the independent variables (u_t, u_x, u_y) as (u_1, u_2, u_3).

The equation (5.1) has partial Lagrangian $L = \frac{1}{2}{u_t}^2 - \frac{1}{2}f(u)u_x^2 - g(u)u_y^2$ for which the partial

Noether operators $X = \xi_1\partial_t + \xi_2\partial_x + \eta\partial_u$ satisfy (3.11), viz.

$$\begin{aligned}
&\xi_1(u_1(-\tfrac{1}{2}{u_2}^2 f_u - \tfrac{1}{2}{u_3}^2 g_u) + u_{1,1}u_1 - u_{1,2}f(u)u_2 - u_{1,3}g(u)u_3)+\\
&\xi_2(u_2(-\tfrac{1}{2}{u_2}^2 f_u - \tfrac{1}{2}{u_3}^2 g_u) + u_{2,1}u_1 - u_{2,2}f(u)u_2 - u_{2,3}g(u)u_3)+\\
&\xi_3(u_3(-\tfrac{1}{2}{u_2}^2 f_u - \tfrac{1}{2}{u_3}^2 g_u) + u_{3,1}u_1 - u_{3,2}f(u)u_2 - u_{3,3}g(u)u_3)+\\
&(\eta - \xi_1 u_1 - \xi_2 u_2 - \xi_3 u_3)(-\tfrac{1}{2}{u_2}^2 f_u - \tfrac{1}{2}{u_3}^2 g_u)+\\
&(\eta_t - \xi_{1,t}u_1 - \xi_{2,t}u_2 - \xi_{3,t}u_3 + u_1(\eta_u - \xi_{1,u}u_1 - \xi_{2,u}u_2 - \xi_{3,u}u_3)-\\
&u_{1,1}\xi_1 - u_{1,2}\xi_2 - u_{1,3}\xi_3)u_1 - (\eta_x - \xi_{1,x}u_1 - \xi_{2,x}u_2 - \xi_{3,x}u_3+ \qquad (5.2)\\
&u_2(\eta_u - \xi_{1,u}u_1 - \xi_{2,u}u_2 - \xi_{3,u}u_3) - u_{2,1}\xi_1 - u_{2,2}\xi_2 - u_{2,3}\xi_3 t)f(u)u_2-\\
&(\eta_y - \xi_{1,y}u_1 - \xi_{2,y}u_2 - \xi_{3,y}u_3 + u_3(\eta_u - \xi_{1,u}u_1 - \xi_{2,u}u_2 - \xi_{3,u}u_3)-\\
&u_{3,1}\xi_1 - u_{3,2}\xi_2 - u_{3,3}\xi_3)g(u)u_3 + (-\tfrac{1}{2}f(u){u_2}^2 - \tfrac{1}{2}g(u){u_3}^2 + \tfrac{1}{2}{u_1}^2)\\
&(\xi_{1,t} + \xi_{1,u}u_1 + \xi_{2,x} + \xi_{2,u}u_2 + \xi_{3,y} + \xi_{3,u}u_3) = (\eta - \xi_1 u_1 - \xi_2 u_2 - \xi_3 u_3)\\
&(-\tfrac{1}{2}{u_2}^2 f_u - \tfrac{1}{2}{u_3}^2 g_u) + B^1_t + u_1 B^1_u + B^2_x + u_2 B^2_u + B^3_y + u_3 B^3_u.
\end{aligned}$$

Separation by the derivatives of u yields the overdetermined linear system

$$
\begin{aligned}
&\xi_{1,u} = 0,\\
&\xi_{2,u} = 0,\\
&\xi_{3,u} = 0,\\
&\xi_1 f_u - f(u)\,\xi_{1,u} = 0,\\
&\xi_2 f_u - f(u)\,\xi_{2,u} = 0,\\
&\xi_3 f_u - f(u)\,\xi_{3,u} = 0,\\
&\xi_1 g_u - g(u)\,\xi_{1,u} = 0,\\
&\xi_2 g_u - g(u)\,\xi_{2,u} = 0,\\
&\xi_3 g_u - g(u)\,\xi_{3,u} = 0,\\
&B^1_u - \eta_t = 0,\\
&B^2_u + f(u)\,\eta_x = 0,\\
&B^3_u + g(u)\,\eta_y = 0,\\
&f(u)\,\xi_{3,x} + g(u)\,\xi_{2,y} = 0,\\
&\xi_{2,t} - f(u)\,\xi_{1,x} = 0,\\
&\xi_{3,t} - g(u)\,\xi_{1,y} = 0,\\
&B^1_t + B^2_x + B^3_y = 0,\\
&f(u)\,(2\eta_u + \xi_{1,t} - \xi_{2,x} + \xi_{3,y}) = 0,\\
&g(u)\,(2\eta_u + \xi_{1,t} + \xi_{2,x} - \xi_{3,y}) = 0,\\
&2\eta_u - \xi_{1,t} + \xi_{2,x} + \xi_{3,y} = 0.
\end{aligned}
\tag{5.3}
$$

From (5.3)-(1, 2, 3, 19) we deduce

$$
\eta\,(t,x,y,u) = \tfrac{1}{2}\left(\tfrac{\partial}{\partial t}\xi_1 - \tfrac{\partial}{\partial x}\xi_2 - \tfrac{\partial}{\partial y}\xi_3\right)u + \beta\,(t,x,y) \tag{5.4}
$$

Moreover from (5.3)-(10, 11, 12) we obtain

$$B^1(t,x,y,u) = (\frac{\partial^2}{\partial t^2}\xi_1 - \frac{\partial^2}{\partial x\partial t}\xi_2 - \frac{\partial^2}{\partial y\partial t}\xi_3)\frac{u^2}{4} + (\frac{\partial}{\partial t}\beta(t,x,y))u + f_1(t,x,y) \tag{5.5}$$

$$B^2(t,x,y,u) = -\frac{1}{2}\int f(u)((\frac{\partial^2}{\partial x\partial t}\xi_1 - \frac{\partial^2}{\partial x^2}\xi_2 - \frac{\partial^2}{\partial y\partial x}\xi_3)u + 2\frac{\partial}{\partial x}\beta(t,x,y)))du + f_2(t,x,y) \tag{5.6}$$

$$B^3(t,x,y,u) = -\frac{1}{2}\int g(u)((\frac{\partial^2}{\partial y\partial t}\xi_1 - \frac{\partial^2}{\partial y\partial x}\xi_2 - \frac{\partial^2}{\partial y^2}\xi_3)u + 2\frac{\partial}{\partial y}\beta(t,x,y)))du + f_3(t,x,y) \tag{5.7}$$

Also from (5.3)-(16), (5.5), (5.6) and (5.7) we find

$$\begin{gathered}\frac{1}{4}(\frac{\partial^3}{\partial t^3}\xi_1 - \frac{\partial^3}{\partial x\partial t^2}\xi_2 - \frac{\partial^3}{\partial y\partial t^2}\xi_3)u^2 + (\frac{\partial^2}{\partial t^2}\beta(t,x,y))u + \frac{\partial}{\partial t}f_1(t,x,y) - \\ \frac{1}{2}\int f(u)((\frac{\partial^3}{\partial x^2\partial t}\xi_1 - \frac{\partial^3}{\partial x^3}\xi_2 - \frac{\partial^3}{\partial y\partial x^2}\xi_3)u + 2\frac{\partial^2}{\partial x^2}\beta(t,x,y))\,du + \frac{\partial}{\partial x}f_2(t,x,y) - \\ \frac{1}{2}\int g(u)((\frac{\partial^3}{\partial y^2\partial t}\xi_1 - \frac{\partial^3}{\partial y^2\partial x}\xi_2 - \frac{\partial^3}{\partial y^3}\xi_3)u + 2\frac{\partial^2}{\partial y^2}\beta(t,x,y))\,du + \frac{\partial}{\partial y}f_3(t,x,y) = 0\end{gathered} \tag{5.8}$$

We differentiate (5.8) w.r.t. u to get

$$\begin{gathered}\frac{1}{2}(\frac{\partial^3}{\partial t^3}\xi_1 - \frac{\partial^3}{\partial x\partial t^2}\xi_2 - \frac{\partial^3}{\partial y\partial t^2}\xi_3)u + \frac{\partial^2}{\partial t^2}\beta(t,x,y) - \\ \frac{1}{2}f(u)((\frac{\partial^3}{\partial x^2\partial t}\xi_1 - \frac{\partial^3}{\partial x^3}\xi_2 - \frac{\partial^3}{\partial y\partial x^2}\xi_3)u + 2\frac{\partial^2}{\partial x^2}\beta(t,x,y)) - \\ \frac{1}{2}g(u)((\frac{\partial^3}{\partial y^2\partial t}\xi_1 - \frac{\partial^3}{\partial y^2\partial x}\xi_2 - \frac{\partial^3}{\partial y^3}\xi_3)u + 2\frac{\partial^2}{\partial y^2}\beta(t,x,y)) = 0\end{gathered} \tag{5.9}$$

Now differentiate (5.8) w.r.t. u twice. This gives

$$\begin{aligned}&\frac{1}{2}\frac{\partial^3}{\partial t^3}\xi_1 - \frac{1}{2}\frac{\partial^3}{\partial x\partial t^2}\xi_2 - \frac{1}{2}\frac{\partial^3}{\partial y\partial t^2}\xi_3 + \\ &f(u)(-\frac{1}{2}\frac{\partial^3}{\partial x^2\partial t}\xi_1 + \frac{1}{2}\frac{\partial^3}{\partial x^3}\xi_2 + \frac{1}{2}\frac{\partial^3}{\partial y\partial x^2}\xi_3) + \\ &g(u)(-\frac{1}{2}\frac{\partial^3}{\partial y^2\partial t}\xi_1 + \frac{1}{2}\frac{\partial^3}{\partial y^2\partial x}\xi_2 + \frac{1}{2}\frac{\partial^3}{\partial y^3}\xi_3) + \\ &\frac{d}{du}f(u)(-\frac{1}{2}(\frac{\partial^3}{\partial x^2\partial t}\xi_1 - \frac{\partial^3}{\partial x^3}\xi_2 - \frac{\partial^3}{\partial y\partial x^2}\xi_3)u - \frac{\partial^2}{\partial x^2}\beta(t,x,y)) + \\ &\frac{d}{du}g(u)(-\frac{1}{2}(\frac{\partial^3}{\partial y^2\partial t}\xi_1 - \frac{\partial^3}{\partial y^2\partial x}\xi_2 - \frac{\partial^3}{\partial y^3}\xi_3)u - \frac{\partial^2}{\partial y^2}\beta(t,x,y)) = 0\end{aligned} \tag{5.10}$$

Since the case when both $f(u)$ and $g(u)$ are constants is not of our interest, so the analysis of (5.10) gives rise to three cases. The results are presented as follows.

(I)$\frac{d}{du}g(u) \neq 0, f(u) = k_1, k_1$ is a constant:

In this case we can obtain via (5.3)-(7, 8, 9), the equations

$$\begin{aligned} \xi_1(t,x,y) &= 0, \\ \xi_2(t,x,y) &= 0, \\ \xi_3(t,x,y) &= 0. \end{aligned} \tag{5.11}$$

By the substitution of (5.11) in both (5.9) and (5.10) we have

$$\eta(t,x,y) = \alpha(t,x)\,y + \gamma(t,x) \tag{5.12}$$

where $\alpha(t,x)$ and $\gamma(t,x)$ are the solutions of $\frac{\partial^2}{\partial t^2}u(t,x) - k_1\frac{\partial^2}{\partial x^2}u(t,x) = 0$.

So we have an infinite number of nontrivial conservation laws that are given as follows:

$$\begin{aligned} T = \quad & ((\tfrac{\partial}{\partial t}\alpha(t,x)\,y + \tfrac{\partial}{\partial t}\gamma(t,x))u - u_1(\alpha(t,x)\,y + \gamma(t,x)), \\ & -k_1(\tfrac{\partial}{\partial x}\alpha(t,x)\,y + \tfrac{\partial}{\partial x}\gamma(t,x))u + k_1u_2(\alpha(t,x)\,y + \gamma(t,x)), \\ & -\alpha(t,x)\textstyle\int g(u)\,du + g(u)\,u_3(\alpha(t,x)\,y + \gamma(t,x))) \end{aligned} \tag{5.13}$$

(II)$\frac{d}{du}f(u) \neq 0, g(u) = k_2, k_2$ is a constant:

In this case we can obtain via (5.3)-(4, 5, 6), the equations

$$\begin{aligned} \xi_1(t,x,y) &= 0, \\ \xi_2(t,x,y) &= 0, \\ \xi_3(t,x,y) &= 0. \end{aligned} \tag{5.14}$$

By the substitution of (5.14) in both (5.9) and (5.10) we have

$$\eta(t,x,y) = \alpha(t,y)x + \gamma(t,y) \tag{5.15}$$

where $\alpha(t,y)$ and $\gamma(t,y)$ are the solutions of $\frac{\partial^2}{\partial t^2}u(t,y) - k_2\frac{\partial^2}{\partial y^2}u(t,y) = 0$.

So we have an infinite number of nontrivial conservation laws that are given as follows:

$$\begin{aligned} T = \quad & ((\tfrac{\partial}{\partial t}\alpha(t,y)x + \tfrac{\partial}{\partial t}\gamma(t,y))u - u_1(\alpha(t,y)x + \gamma(t,y)), \\ & -\alpha(t,y)\textstyle\int f(u)\,du + f(u)u_2(\alpha(t,y)x + \gamma(t,y)) \\ & -k_2(\tfrac{\partial}{\partial y}\alpha(t,y)x + \tfrac{\partial}{\partial y}\gamma(t,y))u + k_2u_3(\alpha(t,y)x + \gamma(t,y)),) \end{aligned} \tag{5.16}$$

(III)$\frac{d}{du}f(u) \neq 0, \frac{d}{du}g(u) \neq 0$:

In this case we can obtain via (5.3)-(4, 5, 6),or via (5.3)-(7, 8, 9) the equations

$$\begin{aligned} \xi_1(t,x,y) &= 0, \\ \xi_2(t,x,y) &= 0, \\ \xi_3(t,x,y) &= 0. \end{aligned} \tag{5.17}$$

here we have two subcases (III.a) and (III.b).

(III.a)$f(u) = c_1g(u) + c_2$:($f'(u)$ and $g'(u)$ are linearly dependent)

By the substitution of (5.17) in both (5.9) and (5.10) we have

$$\eta(t,x,y) = \beta(t,x,y) \tag{5.18}$$

where $\beta(t,x,y)$ is a solution of the system

$$\begin{gathered} c_1\frac{\partial^2}{\partial x^2}\beta + \frac{\partial^2}{\partial y^2}\beta = 0. \\ \frac{\partial^2}{\partial t^2}\beta = c_2\frac{\partial^2}{\partial x^2}\beta. \end{gathered} \tag{5.19}$$

So we have an infinite number of nontrivial conservation laws that are given as follows:

$$T=\left(\beta_t u-\beta u_1,-c_1\beta_x\int g(u)\,du-c_2\beta_x u+c_2\beta u_2+c_1 g(u)\beta u_2,-\beta_y\int g(u)\,du+g(u)\beta u_3\right) \tag{5.20}$$

(III.b)$f(u)\neq c_1 g(u)+c_2$:($f'(u)$ and $g'(u)$ are linearly independent)

By the substitution of (5.17) in both (5.9) and (5.10) we have

$$\eta(t,x,y)=C_1+C_2x+C_3y+C_4t+C_5xy+C_6ty+C_7tx+C_8txy \tag{5.21}$$

Therefore we obtain the following conservation laws.

$$\begin{aligned}
T_1=&\ (-u_1,f(u)u_2,g(u)u_3),\\
T_2=&\ (-u_1x,-\textstyle\int f(u)\,du+f(u)u_2x,g(u)u_3x),\\
T_3=&\ (-u_1y,f(u)u_2y,-\textstyle\int g(u)\,du+g(u)u_3y),\\
T_4=&\ (u-u_1t,f(u)u_2t,g(u)u_3t),\\
T_5=&\ (-u_1xy,-y\textstyle\int f(u)\,du+f(u)u_2xy,-x\int g(u)\,du+g(u)u_3xy),\\
T_6=&\ (uy-u_1ty,f(u)u_2ty,-t\textstyle\int g(u)\,du+g(u)u_3ty),\\
T_7=&\ (ux-u_1tx,-t\textstyle\int f(u)\,du+f(u)u_2tx,g(u)u_3tx),\\
T_8=&\ (uxy-u_1txy,-ty\textstyle\int f(u)\,du+f(u)u_2txy,-tx\int g(u)\,du+g(u)u_3txy).
\end{aligned} \tag{5.22}$$

5.3 Double reduction of a nonlinear $(2+1)$ wave equation

In this section we apply the generalized double reduction theory for the nonlinear $(2+1)$ wave equation

$$u_{tt}=(f(u)u_x)_x+(g(u)u_y)_y \tag{5.23}$$

that involves arbitrary different functions $f(u)$ and $g(u)$ by using seven conservation laws from the eight conservation laws (5.22) to introduce new exact solutions.

Really, this equation admits the following four symmetries:

$$\begin{aligned} &X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x} \\ &X_3 = \frac{\partial}{\partial y} \quad X_4 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}. \end{aligned} \tag{5.24}$$

In this chapter, we restrict to show reductions by T_1 and T_4 by detail, where the table that determine which of these four symmetries are associated with the eight conservation laws and the Reduction in the remaining cases with exact solutions are given in Table 5.1 and Table 5.2.

(1) The double reduction to the nonlinear $(2+1)$ ***wave equation by*** T_1***:***

We can get a reduced conserved form for the PDE by the associated symmetry which satisfies the following formula

$$X\begin{pmatrix} T^t \\ T^x \\ T^y \end{pmatrix} - \begin{pmatrix} D_t\xi^t & D_x\xi^t & D_y\xi^t \\ D_t\xi^x & D_x\xi^x & D_y\xi^x \\ D_t\xi^y & D_x\xi^y & D_y\xi^y \end{pmatrix}\begin{pmatrix} T^t \\ T^x \\ T^y \end{pmatrix} + (D_t\xi^t + D_x\xi^x + D_y\xi^y)\begin{pmatrix} T^t \\ T^x \\ T^y \end{pmatrix} = 0. \tag{5.25}$$

Then the only associated symmetries are X_1, X_2 and X_3, so we can get a reduced conserved form by the combination of them $X = \frac{\partial}{\partial t} + c_1\frac{\partial}{\partial x} + c_2\frac{\partial}{\partial y}$, where the generator X has a canonical form $X = \frac{\partial}{\partial q}$ when

$$\frac{dt}{1} = \frac{dx}{c_1} = \frac{dy}{c_2} = \frac{du}{0} = \frac{dr}{0} = \frac{ds}{0} = \frac{dq}{1} = \frac{dw}{0}, \tag{5.26}$$

or

$$r = y - c_2 t, \quad s = x - c_1 t, \quad q = t, \quad w(r,s) = u. \tag{5.27}$$

Using the following formula

$$\begin{pmatrix} T^r \\ T^s \\ T^q \end{pmatrix} = J(A^{-1})^T \begin{pmatrix} T^t \\ T^x \\ T^y \end{pmatrix}, \tag{5.28}$$

where

$$A^{-1} = \begin{pmatrix} D_t r & D_t s & D_t q \\ D_x r & D_x s & D_x q \\ D_y r & D_y s & D_y q \end{pmatrix}, \quad J = det(A). \tag{5.29}$$

We can get the reduced conserved form

$$D_r T^r + D_s T^s = 0, \tag{5.30}$$

where

$$\begin{aligned} T^r &= {c_2}^2 w_r + c_2 c_1 w_s - g(w) w_r, \\ T^s &= c_1 c_2 w_r + c_1^2 w_s - f(w) w_s, \\ T^q &= -c_2 w_r - c_1 w_s. \end{aligned} \tag{5.31}$$

The reduced conserved form admits the inherited symmetry:

$$\tilde{X}_4 = r\tfrac{\partial}{\partial r} + s\tfrac{\partial}{\partial s}, \tag{5.32}$$

Similarly we can get a reduced conserved form for the PDE by the associated symmetry which satisfies the following formula

$$X \begin{pmatrix} T^r \\ T^s \end{pmatrix} - \begin{pmatrix} D_r \xi^r & D_s \xi^r \\ D_r \xi^s & D_s \xi^s \end{pmatrix} \begin{pmatrix} T^r \\ T^s \end{pmatrix} + (D_r \xi^r + D_s \xi^s) \begin{pmatrix} T^r \\ T^s \end{pmatrix} = 0. \tag{5.33}$$

One can see that $\tilde{X}_4$ is an associated symmetry, so we can get a reduced conserved form by $Y = r\frac{\partial}{\partial r} + s\frac{\partial}{\partial s}$, where the generator Y has a canonical form $Y = \frac{\partial}{\partial m}$ when

$$\frac{dr}{r} = \frac{ds}{s} = \frac{dw}{0} = \frac{dn}{0} = \frac{dm}{1} = \frac{dv}{0}, \tag{5.34}$$

or

$$n = \tfrac{s}{r}, \quad m = ln\, r, \quad v(n) = w. \tag{5.35}$$

So by using the following formula, we can get the reduced conserved form

$$\begin{pmatrix} T^n \\ T^m \end{pmatrix} = J(A^{-1})^T \begin{pmatrix} T^r \\ T^s \end{pmatrix}, \tag{5.36}$$

where

$$A^{-1} = \begin{pmatrix} D_r n & D_r m \\ D_s n & D_s m \end{pmatrix}, \quad J = det(A). \tag{5.37}$$

Then the reduced conserved form is:

$$D_n T^n = 0, \tag{5.38}$$

where

$$\begin{aligned} T^n &= v_n(-{c_2}^2 n^2 + 2c_2 c_1 n + n^2 g(v) - {c_1}^2 + f(v)), \\ T^m &= -v_n(-c_2^2 n + c_2 c_1 + n g(v)). \end{aligned} \tag{5.39}$$

The second step of double reduction can be given as

$$v_n(-{c_2}^2 n^2 + 2c_2 c_1 n + n^2 g(v) - {c_1}^2 + f(v)) = C, \tag{5.40}$$

where C is a constant, $n = \frac{x - c_1 t}{y - c_2 t}$ and $v = u$.

(2)The double reduction to the nonlinear $(2+1)$ wave equation by T_4:

In this case the only associated symmetries are X_2 and X_3, so we can get a reduced conserved form by the combination of them $X = \frac{\partial}{\partial x} + c_1 \frac{\partial}{\partial y}$, where the generator X has a canonical form $X = \frac{\partial}{\partial q}$ when

$$\frac{dt}{0} = \frac{dx}{1} = \frac{dy}{c_1} = \frac{du}{0} = \frac{dr}{0} = \frac{ds}{0} = \frac{dq}{1} = \frac{dw}{0}, \tag{5.41}$$

or

$$r = y - c_1 x, \quad s = t, \quad q = x, \quad w(r,s) = u. \tag{5.42}$$

Then by using formula (5.28) we get the following 3-components of $\tilde{T}_4$

$$\begin{aligned} T^r &= sw_r(g(w) + c_1^2 f(w)), \\ T^s &= w - sw_s, \\ T^q &= -c_1 s f(w) w_r. \end{aligned} \tag{5.43}$$

So the conserved form will be reduced to

$$D_r T^r + D_s T^s = 0 \tag{5.44}$$

One can see that $\tilde{X}_1$ and $\tilde{X}_4$ are not associated symmetries with the reduced conservation law, however we can get a reduced conserved form by using the first part of Fundamental Theorem of double reduction by using the following procedure

First, take $Y = r\frac{\partial}{\partial r} + (1+s)\frac{\partial}{\partial s}$, where the generator Y has a canonical form $Y = \frac{\partial}{\partial m}$ when

$$\frac{dr}{r} = \frac{ds}{1+s} = \frac{dw}{0} = \frac{dn}{0} = \frac{dm}{1} = \frac{dv}{0}, \tag{5.45}$$

or

$$n = \frac{1+s}{r}, \quad m = ln\, r, \quad v(n) = w. \tag{5.46}$$

Then by using formula (5.36) we get the following 2-components of $\tilde{\tilde{T}}_4$

$$\begin{aligned} T^n &= -\left(n^3 v_n g\left(v\right) + c_1{}^2 n^3 v_n f\left(v\right) + v - n v_n\right) \mathrm{e}^m + v_n \left(n^2 g\left(v\right) + c_1{}^2 n^2 f\left(v\right) - 1\right), \\ T^m &= n^2 v_n (g(v) + c_1^2 f(v)) \mathrm{e}^m - n v_n (g(v) + c_1^2 f(v)). \end{aligned} \tag{5.47}$$

Now by using the first part of Fundamental Theorem we get that $D_n \frac{\partial T^n}{\partial m} + D_m \frac{\partial T^m}{\partial m} = 0$ so

$$D_n(-\left(n^3 v_n g\left(v\right) + c_1{}^2 n^3 v_n f\left(v\right) + v - n\, v_n\right) \mathrm{e}^m) + D_m(n^2 v_n (g(v) + c_1^2 f(v)) \mathrm{e}^m) = 0. \tag{5.48}$$

Thus by solving equation $D_n T^n + D_m T^m = 0$ with equation (5.48) we can reduce it to

$$v_n \left(n^2 g\left(v\right) + c_1{}^2 n^2 f\left(v\right) - 1\right) = C. \tag{5.49}$$

Where C is a constant, $n = \frac{1+t}{y - c_1 x}$ and $v = u$.

Note that equation (5.48) is an integral equation which has the same last solution (5.49).

Table 5.1: Conserved vector T^* generated by applying the symmetries to a conserved T

	X_1	X_2	X_3	X_4
T_1	0	0	0	T_1
T_2	0	T_1	0	$2T_2$
T_3	0	0	T_1	$2T_3$
T_4	T_1	0	0	$2T_4$
T_5	0	T_3	T_2	$3T_5$
T_6	T_3	0	T_4	$3T_6$
T_7	T_2	T_4	0	$3T_7$
T_8	T_5	T_6	T_7	$4T_8$

Table 5.2: The double reduction to the nonlinear$(2+1)$wave equation By $T_1 ... T_7$.

Conservation law	Reduction	n	$v(n)$
T_1	$v_n(-c_2{}^2n^2+2c_2c_1n+n^2g(v)-c_1{}^2+f(v))=C$	$\frac{x-c_1t}{y-c_2t}$	u
T_2	$v_n(-c_1{}^2n^2+n^2g(v)+f(v))=C$	$\frac{x+1}{y-c_1t}$	u
T_3	$v_n(-c_1{}^2+n^2g(v)+f(v))=C$	$\frac{x-c_1t}{y+1}$	u
T_4	$v_n(n^2g(v)+c_1{}^2n^2f(v)-1)=C$	$\frac{t+1}{y-c_1x}$	u
T_5	$v_n(n^2g(v)+f(v))=C$	$\frac{x+1}{y+1}$	u
T_6	$v_n(n^2g(v)-1)=C$	$\frac{t+1}{y+1}$	u
T_7	$v_n(n^2f(v)-1)=C$	$\frac{t+1}{x+1}$	u

5.4 Conclusion

New conservation laws are constructed for the nonlinear $(2+1)$ wave equation which is not derivable from a variational principle by using the results of [10]. For the equation containing an arbitrary function of the dependent variable, all possible cases are considered. When $f(u) \neq c_1g(u)+c_2$, we show that there are eight conservation laws. Finally we applied The generalized double reduction theory for the Nonlinear $(2+1)$ wave equation by using seven conservation laws from these eight conservation laws to introduce new exact solutions.

Chapter 6

Wave equation on spherically symmetric Lorentzian metrics

Introduction

The symmetry properties of most of the fundamental equations of mathematical physics, with flat background metric, have been well investigated [16, 17, 18]. In particular, an account of symmetry classification problem for a number of wave equations in flat space has been given in [2]-[9]. In this chapter we extend the earlier investigations by studying Noether symmetries of wave equation on a spherically symmetric metric. The equations determining Noether symmetries for this metric are solved up to explicit functions of θ and ϕ only. In order to solve these determining equations completely, we restrict to a specific spacetime metric, known as flat Friedmann metric. This metric represents an exact solution of the Einstein field equations of general Relativity and represents an expanding universe model [11]. The plan of this chapter is as follows: In the next section we discuss Noether symmetries of a $(3+1)$ wave equation on spherically symmetric metric. In section two, we solve the wave equation to find Noether symmetries admitted by it and using a conformal transformation re-cast it into a constant coefficient wave equation (with respect to derivatives). A brief discussion of results is given in the last section.

6.1 The Noether symmetries of a $(3+1)$ wave equation on spherically symmetric Lorentzian metrics

The spherically symmetric metric is given by the expression [11],

$$ds^2 = e^{\upsilon(t,r)}dt^2 - e^{\lambda(t,r)}dr^2 - e^{\mu(t,r)}d\theta^2 - e^{\mu(t,r)}\sin^2\theta d\phi^2 \tag{6.1}$$

The wave equation on this metric is written using the formula
$\square_g u = \frac{\partial}{\partial x_i}(\sqrt{|g|}g^{ik}\frac{\partial u}{\partial x_k}) = 0$, where g^{ik} is the inverse and $|g|$ is the determinant of the metric (6.1). Using (6.1) and simplifying $\square_g u = 0$, takes the form,

$$\frac{\partial}{\partial t}(e^{(\mu-\frac{\upsilon}{2}+\frac{\lambda}{2})}\sin\theta\frac{\partial u}{\partial t}) - \frac{\partial}{\partial r}(e^{(\mu+\frac{\upsilon}{2}-\frac{\lambda}{2})}\sin\theta\frac{\partial u}{\partial r}) - \frac{\partial}{\partial \theta}(e^{(\frac{\upsilon}{2}+\frac{\lambda}{2})}\sin\theta\frac{\partial u}{\partial \theta}) - \frac{\partial}{\partial \phi}(\frac{e^{(\frac{\upsilon}{2}+\frac{\lambda}{2})}\frac{\partial u}{\partial \phi}}{\sin\theta}) = 0. \tag{6.2}$$

The Lagrangian of the wave equation is given by the expression,

$$2\,L = e^{(\mu-\frac{\upsilon}{2}+\frac{\lambda}{2})}\sin\theta(\frac{\partial u}{\partial t})^2 - e^{(\mu+\frac{\upsilon}{2}-\frac{\lambda}{2})}\sin\theta(\frac{\partial u}{\partial r})^2 - e^{(\frac{\upsilon}{2}+\frac{\lambda}{2})}\sin(\theta)(\frac{\partial u}{\partial \theta})^2 - \frac{e^{(\frac{\upsilon}{2}+\frac{\lambda}{2})}(\frac{\partial u}{\partial \phi})^2}{\sin\theta}. \tag{6.3}$$

In order to investigate Noether symmetries for given $\upsilon(t,r), \lambda(t,r)$ and $\mu(t,r)$, we assume that the gauge term is independent of derivatives as discussed in chapter 3. Through out our calculations, we use a convention in which u_t,u_r,u_θ and u_ϕ are respectively written as u_1,u_2,u_3 and u_4.

The Noether symmetry generator $X = \xi_1\partial_t + \xi_2\partial_r + \xi_3\partial_\theta + \xi_4\partial_\phi + \eta\partial_u$ of the equation (6.2) satisfies (3.5), viz.

$$X\,L+L(\xi_{1,t}+u_1\xi_{1,u}+\xi_{2,r}+u_2\xi_{2,u}+\xi_{3,\theta}+u_3\xi_{3,u}+\xi_{4,\phi}+u_4\xi_{4,u})=B_{1,t}+u_1B_{1,u}+B_{2,r}+ u_2B_{2,u}+B_{3,\theta}+u_3B_{3,u}+B_{4,\phi}+u_4B_{4,u} \tag{6.4}$$

Separating equation (6.4) by the derivatives of u, yields the over determined linear system

$$
\begin{array}{lll}
u_1^3: & \xi_{1,u}=0, & (1)\\
u_2^3: & \xi_{2,u}=0, & (2)\\
u_3^3: & \xi_{3,u}=0, & (3)\\
u_4^3: & \xi_{4,u}=0, & (4)\\
u_1u_2: & \mathrm{e}^{\nu}\xi_{1,r}-\mathrm{e}^{\lambda}\xi_{2,t}=0, & (5)\\
u_1u_3: & \mathrm{e}^{\nu}\xi_{1,\theta}-\mathrm{e}^{\mu}\xi_{3,t}=0, & (6)\\
u_1u_4: & \mathrm{e}^{\nu}\xi_{1,\phi}-\mathrm{e}^{\mu}\sin^2\theta\xi_{4,t}=0, & (7)\\
u_3u_2: & \mathrm{e}^{\lambda}\xi_{2,\theta}+\mathrm{e}^{\mu}\xi_{3,r}=0, & (8)\\
u_2u_4: & \mathrm{e}^{\lambda}\xi_{2,\phi}+\mathrm{e}^{\mu}\sin^2\theta\xi_{4,r}=0, & (9)\\
u_3u_4: & \xi_{3,\phi}+\sin^2\theta\xi_{4,\theta}=0, & (10)\\
u_1: & B_{1,u}-e^{(\mu-\frac{\upsilon}{2}+\frac{\lambda}{2})}\sin\theta\ \eta_{1,t}=0, & (11)\\
u_2: & B_{2,u}+e^{(\mu+\frac{\upsilon}{2}-\frac{\lambda}{2})}\sin\theta\ \eta_{1,r}=0, & (12)\\
u_3: & B_{3,u}+e^{(\frac{\upsilon}{2}+\frac{\lambda}{2})}\ \eta_{1,\theta}\sin\theta=0, & (13)\\
u_4: & B_{4,u}+\frac{e^{(\frac{\upsilon}{2}+\frac{\lambda}{2})}\ \eta_{1,\phi}}{\sin\theta}=0, & (14)\\
c\ : & B_{1,t}+B_{2,r}+B_{3,\theta}+B_{4,\phi}=0, & (15)\\
u_3^2:eq_1: & ((\nu_t+\lambda_t)\xi_1+(\nu_r+\lambda_r)\xi_2+2\xi_{1,t}+2\xi_{2,r}-2\xi_{3,\theta}+2\xi_{4,\phi}+4\eta_{1,u})\sin\theta+2\xi_3\cos\theta=0, & (16)\\
u_4^2:eq_2: & ((\nu_t+\lambda_t)\xi_1+(\nu_r+\lambda_r)\xi_2+2\xi_{1,t}+2\xi_{2,r}+2\xi_{3,\theta}-2\xi_{4,\phi}+4\eta_{1,u})\sin\theta-2\xi_3\cos\theta=0, & (17)\\
u_2^2:eq_3: & ((\nu_t-\lambda_t+2\mu_t)\xi_1+(\nu_r-\lambda_r+2\mu_r)\xi_2+2\xi_{1,t}-2\xi_{2,r}+2\xi_{3,\theta}+2\xi_{4,\phi}+4\eta_{1,u})\sin\theta & \\
 & +2\xi_3\cos\theta=0, & (18)\\
u_1^2:eq_4: & ((\nu_t-\lambda_t-2\mu_t)\xi_1+(\nu_r-\lambda_r-2\mu_r)\xi_2+2\xi_{1,t}-2\xi_{2,r}-2\xi_{3,\theta}-2\xi_{4,\phi}-4\eta_{1,u})\sin\theta & \\
 & -2\xi_3\cos\theta=0. & (19)
\end{array}
\tag{6.5}$$

Equations (6.5)-(16, 17, 18, 19), eq_1, eq_2, eq_3 and eq_4, can be transformed to another equivalent equations e_1, e_2, e_3 and e_4 via the invertible matrix by using the formula

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} eq_1 \\ eq_2 \\ eq_3 \\ eq_4 \end{pmatrix} \tag{6.6}$$

where the equivalent new equations are:

$$\begin{aligned} e_1 :\quad & 2\,\eta_{1,u} + 2\,\xi_{1,t} + \xi_1 \nu_t + \xi_2 \nu_r = 0, \\ e_2 :\quad & 2\,\eta_{1,u} + 2\,\xi_{2,r} + \xi_1 \lambda_t + \xi_2 \lambda_r = 0, \\ e_3 :\quad & 2\,\eta_{1,u} + 2\,\xi_{3,\theta} + \xi_1 \mu_t + \xi_2 \mu_r = 0, \\ e_4 :\quad & 2\,\eta_{1,u} + 2\,\xi_{4,\phi} + \xi_1 \mu_t + \xi_2 \mu_r + 2\xi_3 \cot\theta = 0. \end{aligned} \tag{6.7}$$

From equations (6.7)-(1) and (6.5)-(1,2) we deduce that,

$$\eta_1 = \alpha(t,r,\theta,\phi)u + \beta(t,r,\theta,\phi) \tag{6.8}$$

By substituting η_1 in equations (6.7), it reduces to the following system:

$$\begin{aligned} & 2\,\alpha + 2\,\xi_{1,t} + \xi_1 \nu_t + \xi_2 \nu_r = 0, \\ & 2\,\alpha + 2\,\xi_{2,r} + \xi_1 \lambda_t + \xi_2 \lambda_r = 0, \\ & 2\,\alpha + 2\,\xi_{3,\theta} + \xi_1 \mu_t + \xi_2 \mu_r = 0, \\ & 2\,\alpha + 2\,\xi_{4,\phi} + \xi_1 \mu_t + \xi_2 \mu_r + 2\xi_3 \cot\theta = 0. \end{aligned} \tag{6.9}$$

Moreover from equations (6.5)-(11, 12, 13,14) and (6.8) we obtain,

$$B_1 = \frac{1}{2}\ \sin\theta\ e^{\mu-\frac{\nu}{2}+\frac{\lambda}{2}}(\alpha_t u^2 + 2\beta_t u) + f_1(t,r,\theta,\phi) \tag{6.10}$$

$$B_2 = -\frac{1}{2}\ \sin\theta\ e^{\mu+\frac{\nu}{2}-\frac{\lambda}{2}}(\alpha_r u^2 + 2\beta_r u) + f_2(t,r,\theta,\phi) \tag{6.11}$$

$$B_3 = -\frac{1}{2}\ \sin\theta e^{\frac{\nu}{2}+\frac{\lambda}{2}}(\alpha_\theta u^2 + 2\beta_\theta u) + f_3(t,r,\theta,\phi) \tag{6.12}$$

$$B_4 = -\frac{1}{2\sin\theta} e^{\frac{\nu}{2}+\frac{\lambda}{2}}(\alpha_\phi u^2 + 2\beta_\phi u) + f_4(t,r,\theta,\phi) \tag{6.13}$$

Also from (6.5)-(15) and (6.10)-(6.13), after comparing the different power of u, we find that α and β are solutions for the wave equation and f_1, f_2, f_3 and f_4 satisfy

$$f_{1,t} + f_{2,r} + f_{3,\theta} + f_{4,\phi} = 0, \tag{6.14}$$

Now, we start by evaluating ξ_1, ξ_2, ξ_3 and ξ_4 in terms of explicit functions of θ and ϕ. Taking the sum of the partial derivative of equations (6.5)-(6,7) with respect to ϕ and θ, respectively and using equation (6.5)-(10) then (6.5)-(7) again, we get,

$$D\,\xi_1 = 0, \tag{6.15}$$

where the operator D is defined as follows

$$D = \frac{\partial^2}{\partial\theta\partial\phi} - \cot\theta\frac{\partial}{\partial\phi} \tag{6.16}$$

Similarly,by taking the sum of the partial derivative of equations (6.5)-(8,9) with respect to ϕ and θ, respectively and using equation (6.5)-(10) then (6.5)-(9) again, we get

$$D\,\xi_2 = 0. \tag{6.17}$$

Applying the operator D on (6.9)-(1) and using (6.15) and (6.16), we get,

$$D\,\alpha = 0. \tag{6.18}$$

Also, applying the operator D on (6.9)-(3) and using (6.15), (6.17) and (6.18), we get,

$$D\,\xi_{3,\theta} = \xi_{3,\phi\theta\theta} - \cot\theta\,\xi_{3,\phi\theta} = T\,\xi_{3,\phi} = 0, \tag{6.19}$$

where the operator T is defined as follows

$$T = \frac{\partial^2}{\partial\theta^2} - \cot\theta\frac{\partial}{\partial\theta}. \tag{6.20}$$

Applying the operator T on (6.5)-(10) and using (6.19), we get,

$$T\,(\sin^2\theta\,\xi_{4,\theta}) = 0 \tag{6.21}$$

thus ξ_4 becomes,

$$\xi_4 = h_1(t,r,\phi)\csc\theta + h_2(t,r,\phi)\cot\theta + h_3(t,r,\phi) \tag{6.22}$$

By subtracting (6.9)-(3) from (6.9)-(4), then solving it give,

$$\xi_3 = -h_{1,\phi}\cos\theta - h_{3,\phi}\,\sin\theta\,\tanh^{-1}(\cos\theta) - h_{2,\phi} + h_4(t,r,\phi)\sin\theta \tag{6.23}$$

Substituting about the above values for ξ_3 and ξ_4 in equation (6.5)-(10) gives,

$$\begin{aligned}
h_1 &= k_1(t,r)\sin\phi + k_2(t,r)\cos\phi,\\
h_2 &= k_3(t,r)\sin\phi + k_4(t,r)\cos\phi,\\
h_3 &= k_5(t,r)\phi + k_6(t,r),\\
h_4 &= k_7(t,r).
\end{aligned} \tag{6.24}$$

Now, using equations (6.5)-(6,8) give us ξ_1 and ξ_2 in terms of explicit functions of θ:

$$\begin{aligned}
\xi_1 = \;& \mathrm{e}^{\mu-\nu}[\sin\theta(k_{2,t}\sin\phi - k_{1,t}\cos\phi) - k_{7,t}\cos\theta +\\
& k_{5,t}\cos\theta\ \tanh^{-1}(\cos\theta) + k_{5,t}\ln(\sin\theta) + \theta(k_{4,t}\sin\phi - k_{3,t}\cos\phi)] + h_5(t,r,\phi)\\
\xi_2 = \;& -\mathrm{e}^{\mu-\lambda}[\sin\theta(k_{2,r}\sin\phi - k_{1,r}\cos\phi) - k_{7,r}\cos\theta +\\
& k_{5,r}\cos\theta\ \tanh^{-1}(\cos\theta) + k_{5,r}\ln(\sin\theta) + \theta(k_{4,r}\sin\phi - k_{3,r}\cos\phi)] + h_6(t,r,\phi)
\end{aligned} \tag{6.25}$$

Now, substituting the above ξ_1 and ξ_4 in equation (6.5)-(7) yields

$h_5(t,r,\phi) = k_8(t,r), k_3(t,r) = w_3(r), k_4(t,r) = w_4(r), k_5(t,r) = w_5(r), k_6(t,r) = w_6(r)$.

Also, substituting the above ξ_2 and ξ_4 in equation (6.5)-(9) yields

$h_6(t,r,\phi) = k_9(t,r), w_3(r) = c_3, w_4(r) = c_4, w_5(r) = c_5, w_6(r) = c_6$.

Substituting the above ξ_1 and ξ_2 in equation (6.5)-(5) yields the following conditions

$$\begin{aligned}
& 2k_{1,rt} + (\mu_t - \lambda_t)k_{1,r} + (\mu_r - \nu_r)k_{1,t} = 0,\\
& 2k_{2,rt} + (\mu_t - \lambda_t)k_{2,r} + (\mu_r - \nu_r)k_{2,t} = 0,\\
& 2k_{7,rt} + (\mu_t - \lambda_t)k_{7,r} + (\mu_r - \nu_r)k_{7,t} = 0,\\
& \mathrm{e}^{\lambda}k_{9,t} - \mathrm{e}^{\nu}k_{8,r} = 0.
\end{aligned} \tag{6.26}$$

Now, equation (6.9)-(1) give α in terms of explicit functions of θ and ϕ as follows:

$$\begin{aligned}\alpha = \tfrac{1}{2}[\sin\theta\cos\phi \quad & [e^{\mu-\nu}(2k_{1,t,t} - k_{1,t}\nu_t + 2\mu_t k_{1,t}) - e^{\mu-\lambda}\nu_r k_{1,r}] \\ -\sin\theta\sin\phi \quad & [e^{\mu-\nu}(2k_{2,t,t} - \nu_t k_{2,t} + 2\mu_t k_{2,t}) - e^{\mu-\lambda}\nu_r k_{2,r}] \\ +\cos\theta \quad & [e^{\mu-\nu}(2k_{7,t,t} - k_{7,t}\nu_t + 2\mu_t k_{7,t}) - e^{\mu-\lambda}\nu_r k_{7,r}]] - \tfrac{1}{2}(\nu_t k_8 + \nu_r k_9 + 2k_{8,t}).\end{aligned} \tag{6.27}$$

Equation (6.9)-(2) give us the following conditions

$$\begin{aligned} & 2e^{\nu}k_{1,rr} + 2e^{\lambda}k_{1,tt} + e^{\nu}(2\mu_r - \nu_r - \lambda_r)k_{1,r} + e^{\lambda}(2\mu_t - \nu_t - \lambda_t)k_{1,t} = 0 \\ & 2e^{\nu}k_{2,rr} + 2e^{\lambda}k_{2,tt} + e^{\nu}(2\mu_r - \nu_r - \lambda_r)k_{2,r} + e^{\lambda}(2\mu_t - \nu_t - \lambda_t)k_{2,t} = 0 \\ & 2e^{\nu}k_{7,rr} + 2e^{\lambda}k_{7,tt} + e^{\nu}(2\mu_r - \nu_r - \lambda_r)k_{7,r} + e^{\lambda}(2\mu_t - \nu_t - \lambda_t)k_{7,t} = 0 \\ & \nu_r k_9 - \lambda_t k_8 - \lambda_r k_9 - 2k_{9,r} + \nu_t k_8 + 2k_{8,t} = 0 \end{aligned} \tag{6.28}$$

Equation (6.9)-(3) give us that $c_5 = 0$ with the following conditions

$$\begin{aligned} & 2e^{\mu}e^{\lambda}k_{1,tt} + e^{\mu}e^{\nu}(\mu_r - \nu_r)k_{1,r} + e^{\mu}e^{\lambda}(\mu_t - \nu_t)k_{1,t} + 2e^{\lambda}e^{\nu}k_1 = 0 \\ & 2e^{\mu}e^{\lambda}k_{2,tt} + e^{\mu}e^{\nu}(\mu_r - \nu_r)k_{2,r} + e^{\mu}e^{\lambda}(\mu_t - \nu_t)k_{2,t} + 2e^{\lambda}e^{\nu}k_2 = 0 \\ & 2e^{\mu}e^{\lambda}k_{7,tt} + e^{\mu}e^{\nu}(\mu_r - \nu_r)k_{7,r} + e^{\mu}e^{\lambda}(\mu_t - \nu_t)k_{7,t} + 2e^{\lambda}e^{\nu}k_7 = 0 \\ & \mu_t k_8 + \mu_r k_9 - \nu_t k_8 - \nu_r k_9 - 2k_{8,t} = 0. \end{aligned} \tag{6.29}$$

Finally, since α is a solution for the wave equation (6.2), the final solution is given as:

$$\begin{aligned} \xi_1 = \quad & e^{\mu-\nu}[\sin\theta(k_{2,t}\sin\phi - k_{1,t}\cos\phi) - k_{7,t}\cos\theta] + k_8 \\ \xi_2 = \quad & -e^{\mu-\lambda}[\sin\theta(k_{2,r}\sin\phi - k_{1,r}\cos\phi) - k_{7,r}\cos\theta] + k_9 \\ \xi_3 = \quad & (k_2\sin\phi - k_1\cos\phi)\cos\theta + k_7\sin\theta + c_4\sin\phi - c_3\cos\phi \\ \xi_4 = \quad & (k_1\sin\phi + k_2\cos\phi)\csc\theta + (c_3\sin\phi + c_4\cos\phi)\cot\theta + c_6 \end{aligned} \tag{6.30}$$

$$\begin{aligned}\alpha = \; & \tfrac{1}{2}[\sin\theta\cos\phi[\mathrm{e}^{\mu-\nu}(2k_{1,t,t}-k_{1,t}\nu_t+2\mu_t k_{1,t})-\mathrm{e}^{\mu-\lambda}\nu_r k_{1,r}] \\ & -\sin\theta\sin\phi[\mathrm{e}^{\mu-\nu}(2k_{2,t,t}-\nu_t k_{2,t}+2\mu_t k_{2,t})-\mathrm{e}^{\mu-\lambda}\nu_r k_{2,r}] \\ & +\cos\theta[\mathrm{e}^{\mu-\nu}(2k_{7,t,t}-k_{7,t}\nu_t+2\mu_t k_{7,t})-\mathrm{e}^{\mu-\lambda}\nu_r k_{7,r}]] \\ & -\tfrac{1}{2}(\nu_t k_8+\nu_r k_9+2k_{8,t}),\end{aligned} \tag{6.31}$$

subject to the following conditions:

$$\begin{aligned}& 2k_{1,rt}+(\mu_t-\lambda_t)k_{1,r}+(\mu_r-\nu_r)k_{1,t}=0 \\ & 2k_{2,rt}+(\mu_t-\lambda_t)k_{2,r}+(\mu_r-\nu_r)k_{2,t}=0 \\ & 2k_{7,rt}+(\mu_t-\lambda_t)k_{7,r}+(\mu_r-\nu_r)k_{7,t}=0 \\ & 2\mathrm{e}^{\nu}k_{1,rr}+2\mathrm{e}^{\lambda}k_{1,tt}+\mathrm{e}^{\nu}(2\mu_r-\nu_r-\lambda_r)k_{1,r}+\mathrm{e}^{\lambda}(2\mu_t-\nu_t-\lambda_t)k_{1,t}=0 \\ & 2\mathrm{e}^{\nu}k_{2,rr}+2\mathrm{e}^{\lambda}k_{2,tt}+\mathrm{e}^{\nu}(2\mu_r-\nu_r-\lambda_r)k_{2,r}+\mathrm{e}^{\lambda}(2\mu_t-\nu_t-\lambda_t)k_{2,t}=0 \\ & 2\mathrm{e}^{\nu}k_{7,rr}+2\mathrm{e}^{\lambda}k_{7,tt}+\mathrm{e}^{\nu}(2\mu_r-\nu_r-\lambda_r)k_{7,r}+\mathrm{e}^{\lambda}(2\mu_t-\nu_t-\lambda_t)k_{7,t}=0 \\ & 2\mathrm{e}^{\mu}\mathrm{e}^{\lambda}k_{1,tt}+\mathrm{e}^{\mu}\mathrm{e}^{\nu}(\mu_r-\nu_r)k_{1,r}+\mathrm{e}^{\mu}\mathrm{e}^{\lambda}(\mu_t-\nu_t)k_{1,t}+2\mathrm{e}^{\lambda}\mathrm{e}^{\nu}k_1=0 \\ & 2\mathrm{e}^{\mu}\mathrm{e}^{\lambda}k_{2,tt}+\mathrm{e}^{\mu}\mathrm{e}^{\nu}(\mu_r-\nu_r)k_{2,r}+\mathrm{e}^{\mu}\mathrm{e}^{\lambda}(\mu_t-\nu_t)k_{2,t}+2\mathrm{e}^{\lambda}\mathrm{e}^{\nu}k_2=0 \\ & 2\mathrm{e}^{\mu}\mathrm{e}^{\lambda}k_{7,tt}+\mathrm{e}^{\mu}\mathrm{e}^{\nu}(\mu_r-\nu_r)k_{7,r}+\mathrm{e}^{\mu}\mathrm{e}^{\lambda}(\mu_t-\nu_t)k_{7,t}+2\mathrm{e}^{\lambda}\mathrm{e}^{\nu}k_7=0 \\ & \nu_r k_9-\lambda_t k_8-\lambda_r k_9-2k_{9,r}+\nu_t k_8+2k_{8,t}=0 \\ & \mu_t k_8+\mu_r k_9-\nu_t k_8-\nu_r k_9-2k_{8,t}=0 \\ & \mathrm{e}^{\lambda}k_{9,t}-\mathrm{e}^{\nu}k_{8,r}=0\end{aligned} \tag{6.32}$$

$$\begin{aligned}& 2\mathrm{e}^{\mu+\nu}R_{1,rr}-2\mathrm{e}^{\mu+\lambda}R_{1,tt}+\mathrm{e}^{\mu+\nu}(2\mu_r+\nu_r-\lambda_r)R_{1,r}-\mathrm{e}^{\mu+\lambda}(2\mu_t-\nu_t+\lambda_t)R_{1,t}-4\,\mathrm{e}^{\nu+\lambda}R_1=0 \\ & 2\mathrm{e}^{\mu+\nu}R_{2,rr}-2\mathrm{e}^{\mu+\lambda}R_{2,tt}+\mathrm{e}^{\mu+\nu}(2\mu_r+\nu_r-\lambda_r)R_{2,r}-\mathrm{e}^{\mu+\lambda}(2\mu_t-\nu_t+\lambda_t)R_{2,t}-4\,\mathrm{e}^{\nu+\lambda}R_2=0 \\ & 2\mathrm{e}^{\mu+\nu}R_{3,rr}-2\mathrm{e}^{\mu+\lambda}R_{3,tt}+\mathrm{e}^{\mu+\nu}(2\mu_r+\nu_r-\lambda_r)R_{3,r}-\mathrm{e}^{\mu+\lambda}(2\mu_t-\nu_t+\lambda_t)R_{3,t}-4\,\mathrm{e}^{\nu+\lambda}R_3=0 \\ & 2\mathrm{e}^{\nu}R_{4,rr}-2\mathrm{e}^{\lambda}R_{4,tt}+\mathrm{e}^{\nu}(2\mu_r+\nu_r-\lambda_r)R_{4,r}-\mathrm{e}^{\lambda}(2\mu_t-\nu_t+\lambda_t)R_{4,t}=0,\end{aligned} \tag{6.33}$$

where

$$
\begin{aligned}
R_1(t,r) &= \mathrm{e}^{\mu-\nu}(2k_{1,t,t} - k_{1,t}\nu_t + 2\mu_t k_{1,t}) - \mathrm{e}^{\mu-\lambda}\nu_r k_{1,r}\\
R_2(t,r) &= \mathrm{e}^{\mu-\nu}(2k_{2,t,t} - \nu_t k_{2,t} + 2\mu_t k_{2,t}) - \mathrm{e}^{\mu-\lambda}\nu_r k_{2,r}\\
R_3(t,r) &= \mathrm{e}^{\mu-\nu}(2k_{7,t,t} - k_{7,t}\nu_t + 2\mu_t k_{7,t}) - \mathrm{e}^{\mu-\lambda}\nu_r k_{7,r}\\
R_4(t,r) &= \nu_t k_8 + \nu_r k_9 + 2k_{8,t}.
\end{aligned} \tag{6.34}
$$

Actually, the problem now is reduced to finding eight functions of two variables (t and r), namely, $k_1, k_2, k_7, k_8, k_9, \nu, \lambda$ and μ from a system of sixteen nonlinear partial differential equations. However, for given ν, λ and μ our system is reduced to a system of sixteen linear partial differential equations with five functions of two variables t and r.

Finally, from equations (6.10), (6.11), (6.12), and (6.13) we can estimate the gauge term to every Noether symmetry to construct the corresponding conservation laws by using the famous Noether theorem.

6.2 The wave equation on the Friedmann Robertson Walker universe

The Friedmann Robertson Walker universe is described by the line element:

$$ds^2 = dt^2 - a(t)^2\left(\tfrac{dr^2}{1-kr^2} + r^2 d\Omega^2\right), \tag{6.35}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$, $a(t)$ is the scale factor, and k is the curvature parameter with $k = -1, 0, 1$ corresponding to open, flat, and closed universes, respectively.

So the wave equation (6.2) on this metric when $\nu = 0$, $\lambda = 2\ln\left(\frac{a(t)}{\sqrt{1-kr^2}}\right)$, $\mu = \ln(r^2\, a(t)^2)$ will take the following form

$$\frac{\partial}{\partial t}\left(\frac{r^2\, a(t)^3 \sin\theta}{\sqrt{1-kr^2}}\frac{\partial u}{\partial t}\right) - \frac{\partial}{\partial r}\left(r^2\, a(t)\sin\theta\sqrt{1-kr^2}\frac{\partial u}{\partial r}\right) - \frac{\partial}{\partial \theta}\left(\frac{a(t)\sin\theta}{\sqrt{1-kr^2}}\frac{\partial u}{\partial \theta}\right) - \frac{\partial}{\partial \phi}\left(\frac{a(t)}{\sin\theta\sqrt{1-kr^2}}\frac{\partial u}{\partial \phi}\right) = 0. \tag{6.36}$$

Really, one can rewrite the wave equation (6.36) in the conserved form with respect to the cartesian coordinate by using Theorem 4.2.2 through the transformations $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$ and $z = r\cos\theta$, to get

$$\frac{\partial}{\partial t}\left(\frac{a(t)^3 u_t}{\sqrt{1-k(x^2+y^2+z^2)}}\right) - \frac{\partial}{\partial x}\left(\frac{a(t)(u_x - k\ x(x\ u_x + y\ u_y + z\ u_z))}{\sqrt{1-k(x^2+y^2+z^2)}}\right) - \frac{\partial}{\partial y}\left(\frac{a(t)(u_y - k\ y(x\ u_x + y\ u_y + z\ u_z))}{\sqrt{1-k(x^2+y^2+z^2)}}\right)$$
$$-\frac{\partial}{\partial z}\left(\frac{a(t)(u_z - k\ z(x\ u_x + y\ u_y + z\ u_z))}{\sqrt{1-k(x^2+y^2+z^2)}}\right) = 0. \tag{6.37}$$

6.2.1 Flat universe

The wave equation (6.36) when $k = 0$, will take the form

$$\frac{\partial}{\partial t}\left(r^2\ a(t)^3 \sin\theta \frac{\partial u}{\partial t}\right) - \frac{\partial}{\partial r}\left(r^2\ a(t)\sin\theta \frac{\partial u}{\partial r}\right) - \frac{\partial}{\partial \theta}\left(a(t)\sin\theta \frac{\partial u}{\partial \theta}\right) - \frac{\partial}{\partial \phi}\left(\frac{a(t)}{\sin\theta}\frac{\partial u}{\partial \phi}\right) = 0. \tag{6.38}$$

Now by solving the system (6.32) we will find the Noether symmetries for the above wave equation as follow.

From equation (6.32)-(1) we get

$$k_1 = F_1(r) + \frac{F_2(t)}{r} \tag{6.39}$$

From equations (6.32)-(4,7) we get

$$r^2 k_{1,rr} + r k_{1,r} - k_1 = 0 \tag{6.40}$$

Substituting equation (6.39) in equation (6.40) gives us the following Euler equation

$$r^2 F_{1,rr} + r F_{1,r} - F_1 = 0 \tag{6.41}$$

So F_1 is given as follows

$$F_1(r) = d_1 r + \frac{d_2}{r} \tag{6.42}$$

Substituting equation (6.39) and (6.42) in equation (6.32)-(7) gives us

$$a(t)^2 F_{2,tt} + a(t) a_t F_{2,t} + 2 d_1 = 0 \tag{6.43}$$

So F_2 is given as follows

$$F_2(t) = d_3\ b(t) - 2\ d_1\ \int \frac{b(t)}{a(t)} dt + C \tag{6.44}$$

where $b(t) = \int \frac{dt}{a(t)}$.

Substituting equation (6.42) and (6.44) in equation (6.39) gives us

$$k_1 = d_1 (r - \frac{2}{r} \int \frac{b(t)}{a(t)} dt) + \frac{d_2}{r} + d_3 \frac{b(t)}{r}. \tag{6.45}$$

Similarly we get that

$$\begin{aligned} k_2 &= d_4 (r - \tfrac{2}{r} \int \tfrac{b(t)}{a(t)} dt) + \tfrac{d_5}{r} + d_6 \tfrac{b(t)}{r}, \\ k_7 &= d_7 (r - \tfrac{2}{r} \int \tfrac{b(t)}{a(t)} dt) + \tfrac{d_8}{r} + d_9 \tfrac{b(t)}{r}, \end{aligned} \tag{6.46}$$

Also from equations (6.32)-(10,11) we get

$$k_9 - r\ k_{9,r} = 0. \tag{6.47}$$

So k_9 is given as follows

$$k_9 = W_1(t) r. \tag{6.48}$$

Substituting equation (6.48) in equation (6.32)-(12) gives us

$$k_8 = \frac{r^2}{2} a(t)^2 W_{1,t} + W_2(t). \tag{6.49}$$

Substituting equation (6.48) and (6.49) in equation (6.32)-(11) gives us the following two equations

$$\begin{aligned} &a(t) W_{1,tt} + a_t W_{1,t} = 0 \\ &a(t) W_{2,t} - a_t W_2(t) - a(t) W_1(t) = 0 \end{aligned} \tag{6.50}$$

So W_1 and W_2 are given as follows

$$\begin{aligned} &W_1 = d_{10} + 2 d_{11} b(t) \\ &W_2 = d_{10}\ a(t)\ b(t) + 2 d_{11}\ a(t)\ \int \tfrac{b(t)}{a(t)} dt + d_{12}\ a(t) \end{aligned} \tag{6.51}$$

Substituting equation (6.51) in equations (6.48) and (6.49) gives us

$$\begin{aligned} &k_8 = d_{10} a(t) b(t) + d_{11}(r^2 a(t) + 2a(t) \int \tfrac{b(t)}{a(t)} dt) + d_{12} a(t), \\ &k_9 = d_{10} r + 2 d_{11} r b(t), \end{aligned} \tag{6.52}$$

Finally, The solution of the system (6.32) for this metric is summarized as follows

$$\begin{aligned} &k_1 = d_1 (r - \tfrac{b(t)^2}{r}) + \tfrac{d_2}{r} + d_3 \tfrac{b(t)}{r}. \\ &k_2 = d_4 (r - \tfrac{b(t)^2}{r}) + \tfrac{d_5}{r} + d_6 \tfrac{b(t)}{r}, \\ &k_7 = d_7 (r - \tfrac{b(t)^2}{r}) + \tfrac{d_8}{r} + d_9 \tfrac{b(t)}{r}, \\ &k_8 = d_{10} a(t) b(t) + d_{11}(r^2 a(t) + a(t)\ b(t)^2) + d_{12} a(t), \\ &k_9 = d_{10}\ r + 2 d_{11}\ r\ b(t), \end{aligned} \tag{6.53}$$

where $b(t) = \int \frac{dt}{a(t)}$.

Now by using equations (6.33) and (6.34) we get that the maximal Noether symmetries are given when $a(t)\,\dot{a}(t)=c_1$ or $a(t)=\pm\sqrt{2\,c_1\,t+c_2}$ as follows

$$
\begin{aligned}
X_1 &= 2\,r\,ab\sin\theta\cos\phi\frac{\partial}{\partial t}+\sin\theta\cos\phi(b^2+r^2)\frac{\partial}{\partial r}+\frac{\cos\theta\cos\phi(b^2-r^2)}{r}\frac{\partial}{\partial\theta}-\frac{\sin\phi(b^2-r^2)}{r\sin\theta}\frac{\partial}{\partial\phi}\\
&\quad -2\,r\sin\theta\cos\phi(\dot{a}b+1)u\frac{\partial}{\partial u}\\
X_2 &= -\sin\theta\cos\phi\frac{\partial}{\partial r}-\frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta}+\frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\\
X_3 &= -r\,a\sin\theta\cos\phi\frac{\partial}{\partial t}-b\sin\theta\cos\phi\frac{\partial}{\partial r}-\frac{b\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta}+\frac{b\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}+r\dot{a}\sin\theta\cos\phi u\frac{\partial}{\partial u}\\
X_4 &= 2\,r\,ab\sin\theta\sin\phi\frac{\partial}{\partial t}+\sin\phi\sin\theta(b^2+r^2)\frac{\partial}{\partial r}+\frac{\cos\theta\sin\phi(b^2-r^2)}{r}\frac{\partial}{\partial\theta}+\frac{\cos\phi(b^2-r^2)}{r\sin\theta}\frac{\partial}{\partial\phi}\\
&\quad -2\,r\sin\theta\sin\phi(\dot{a}b+1)u\frac{\partial}{\partial u}\\
X_5 &= \sin\phi\sin\theta\frac{\partial}{\partial r}+\frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta}+\frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\\
X_6 &= r\,a\sin\theta\sin\phi\frac{\partial}{\partial t}+b\sin\theta\sin\phi\frac{\partial}{\partial r}+\frac{b\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta}+\frac{b\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}-r\dot{a}\sin\theta\sin\phi u\frac{\partial}{\partial u}\\
X_7 &= 2\,r\,ab\cos\theta\frac{\partial}{\partial t}+\cos\theta(b^2+r^2)\frac{\partial}{\partial r}-\frac{\sin\theta(b^2-r^2)}{r}\frac{\partial}{\partial\theta}-2\,r\cos\theta(\dot{a}b+1)u\frac{\partial}{\partial u}\\
X_8 &= -\cos\theta\frac{\partial}{\partial r}+\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\\
X_9 &= -ra\cos\theta\frac{\partial}{\partial t}-b\cos\theta\frac{\partial}{\partial r}+\frac{b\sin\theta}{r}\frac{\partial}{\partial\theta}+r\dot{a}\cos\theta u\frac{\partial}{\partial u}\\
X_{10} &= ab\frac{\partial}{\partial t}+r\frac{\partial}{\partial r}-(\dot{a}b+1)u\frac{\partial}{\partial u}\\
X_{11} &= a(b^2+r^2)\frac{\partial}{\partial t}+2\,rb\frac{\partial}{\partial r}-(\dot{a}(b^2+r^2)+2\,b)u\frac{\partial}{\partial u}\\
X_{12} &= a\frac{\partial}{\partial t}-\dot{a}u\frac{\partial}{\partial u}\\
X_{13} &= -\cos\phi\frac{\partial}{\partial\theta}+\frac{\cos\theta}{\sin\theta}\sin\phi\frac{\partial}{\partial\phi}\\
X_{14} &= \sin\phi\frac{\partial}{\partial\theta}+\frac{\cos\theta}{\sin\theta}\cos\phi\frac{\partial}{\partial\phi}\\
X_{15} &= \frac{\partial}{\partial\phi}\\
X_\beta &= \beta\frac{\partial}{\partial u}.
\end{aligned}
\tag{6.54}
$$

The commutation relations of the Lie algebra of the 15 Noether symmetries are shown in Table 6.1.

Table 6.1: Commutator table for the Lie algebra

$[X_i,X_j]$	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}	X_{11}	X_{12}	X_{13}	X_{14}	X_{15}
X_1	0	$2X_{10}$	X_{11}	0	$-2X_{15}$	0	0	$2X_{13}$	0	$-X_1$	0	$2X_3$	X_7	0	X_4
X_2	$-2X_{10}$	0	X_{12}	0	0	0	$2X_{13}$	0	0	X_2	$2X_3$	0	X_8	0	$-X_5$
X_3	$-X_{11}$	$-X_{12}$	0	0	0	$-X_{15}$	0	0	X_{13}	0	$-X_1$	$-X_2$	X_9	0	$-X_6$
X_4	0	0	0	0	$-2X_{10}$	$-X_{11}$	0	$-2X_{14}$	0	$-X_4$	0	$-2X_6$	0	$-X_7$	$-X_1$
X_5	$2X_{15}$	0	0	$2X_{10}$	0	X_{12}	$2X_{14}$	0	0	X_5	$2X_6$	0	0	X_8	X_2
X_6	0	0	X_{15}	X_{11}	$-X_{12}$	0	0	0	X_{14}	0	X_4	$-X_5$	0	X_9	X_3
X_7	0	$-2X_{13}$	0	0	$-2X_{14}$	0	0	$2X_{10}$	X_{11}	$-X_7$	0	$2X_9$	$-X_1$	X_4	0
X_8	$-2X_{13}$	0	0	$2X_{14}$	0	0	$-2X_{10}$	0	X_{12}	X_8	$2X_9$	0	$-X_2$	$-X_5$	0
X_9	0	0	$-X_{13}$	0	0	$-X_{14}$	$-X_{11}$	$-X_{12}$	0	0	$-X_7$	$-X_8$	$-X_3$	$-X_6$	0
X_{10}	X_1	$-X_2$	0	X_4	$-X_5$	0	X_7	$-X_8$	0	0	X_{11}	$-X_{12}$	0	0	0
X_{11}	0	$-2X_3$	X_1	0	$-2X_6$	$-X_4$	0	$-2X_9$	X_7	$-X_{11}$	0	$-2X_{10}$	0	0	0
X_{12}	$-2X_3$	0	X_2	$2X_6$	0	X_5	$-2X_9$	0	X_8	X_{12}	$2X_{10}$	0	0	0	0
X_{13}	$-X_7$	$-X_8$	$-X_9$	0	0	0	X_1	X_2	X_3	0	0	0	0	X_{15}	$-X_{14}$
X_{14}	0	0	0	X_7	$-X_8$	$-X_9$	$-X_4$	X_5	X_6	0	0	0	$-X_{15}$	0	X_{13}
X_{15}	$-X_4$	X_5	X_6	X_1	$-X_2$	$-X_3$	0	0	0	0	0	0	X_{14}	$-X_{13}$	0

Note that the number of Noether symmetries reduce to only seven, $X_2, X_5, X_8, X_{13}, X_{14}, X_{15}$ and X_β, when a(t) is taken an arbitrary function. Further, X_3, X_6, X_9 and X_{12} are Noether symmetries when $a(t)$ satisfies the differential constraint,

$$(a(t)\ \dot{a}(t))_{,tt} = 0,$$

whose solution is given by

$$a(t) = \pm\sqrt{c_1 t^2 + 2\ c_2\ t + c_3} \tag{6.55}$$

Similarly, $X_1, X_4, X_7$and X_{10} are Noether symmetries when $a(t)$ satisfies the constraint

$$b(t)a(t)(a(t)\ \dot{a}(t))_{,tt} + 2(a(t)\ \dot{a}(t))_{,t} = 0. \tag{6.56}$$

Finally, X_{11} is Noether symmetries when $a(t)\ \dot{a}(t) = c_1$ or $a(t) = \pm\sqrt{2\ c_1\ t + c_2}$.

6.2.2 Linearization of a $(3+1)$ wave equation on the flat universe

In this section we will transform the wave equation on the Friedmann flat metric (6.38), from linear PDE with variable coefficients to linear PDE with constant coefficients with respect to the derivative of the dependent variable.

The wave equation (6.37) at $k=0$ takes the form

$$\frac{\partial}{\partial t}(a(t)^3 u_t) = a(t)\ \nabla^2 u \tag{6.57}$$

Hence the Noether symmetries will transform under the cartesian transformations to:

(1)

$$Y_1 = X_1 = 2\ a\ b\ x\frac{\partial}{\partial t} + (x^2 - y^2 - z^2 + b^2)\frac{\partial}{\partial x} + 2\ x\ y\ \frac{\partial}{\partial y} + 2\ x\ z\frac{\partial}{\partial z} - 2x(\dot{a}b+1)u\frac{\partial}{\partial u} \tag{6.58}$$

(2)

$$Y_2 = X_4 = 2\ a\ b\ y\frac{\partial}{\partial t} + 2\ y\ x\ \frac{\partial}{\partial x} + (y^2 - x^2 - z^2 + b^2)\frac{\partial}{\partial y} + 2\ y\ z\frac{\partial}{\partial z} - 2y(\dot{a}b+1)u\frac{\partial}{\partial u} \tag{6.59}$$

(3)

$$Y_3 = X_7 = 2\ a\ b\ z\frac{\partial}{\partial t} + 2\ z\ x\ \frac{\partial}{\partial x} + 2\ z\ y\frac{\partial}{\partial y} + (z^2 - x^2 - y^2 + b^2)\frac{\partial}{\partial z} - 2z(\dot{a}b+1)u\frac{\partial}{\partial u} \tag{6.60}$$

(4)

$$\begin{aligned} Y_4 = X_{11} = \quad & a\ (x^2+y^2+z^2+b^2)\frac{\partial}{\partial t} + 2\ b\ x\ \frac{\partial}{\partial x} + 2\ b\ y\frac{\partial}{\partial y} + 2\ b\ z\frac{\partial}{\partial z} - \\ & (\dot{a}(x^2+y^2+z^2+b^2) + 2b)u\frac{\partial}{\partial u} \end{aligned} \tag{6.61}$$

(5)

$$Y_5 = X_3 = -a\,x\frac{\partial}{\partial t} - b\frac{\partial}{\partial x} + \dot{a}xu\frac{\partial}{\partial u} \tag{6.62}$$

(6)

$$Y_6 = X_6 = a\,y\frac{\partial}{\partial t} + b\frac{\partial}{\partial y} - \dot{a}yu\frac{\partial}{\partial u} \tag{6.63}$$

(7)

$$Y_7 = X_9 = -a\,z\frac{\partial}{\partial t} - b\frac{\partial}{\partial z} + \dot{a}zu\frac{\partial}{\partial u} \tag{6.64}$$

(8)

$$Y_8 = X_{10} = a\,b\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} - (\dot{a}b+1)u\frac{\partial}{\partial u} \tag{6.65}$$

(9) Invariance under translation in x:

$$Y_9 = X_2 = -\frac{\partial}{\partial x} \tag{6.66}$$

(10) Invariance under translation in y:

$$Y_{10} = X_5 = \frac{\partial}{\partial y} \tag{6.67}$$

(11) Invariance under translation in z:

$$Y_{11} = X_8 = -\frac{\partial}{\partial z} \tag{6.68}$$

(12)

$$Y_{12} = X_{12} \quad = a\frac{\partial}{\partial t} - \dot{a}u\frac{\partial}{\partial u} \tag{6.69}$$

(13) Rotation about the x-axis:

$$Y_{13} = X_{14} = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}, \tag{6.70}$$

(14) Rotation about the y-axis:

$$Y_{14} = X_{13} = -z\frac{\partial}{\partial x} + x\frac{\partial}{\partial z}, \tag{6.71}$$

(15) Rotation about the z-axis:

$$Y_{15} = X_{15} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}, \tag{6.72}$$

In addition to the symmetry which represent the invariance under addition An arbitrary solution of the wave equation, β to the solution $u(t,x,y,z)$:

$$X_\beta = \beta\frac{\partial}{\partial u}. \tag{6.73}$$

Now since the linear PDE with constant coefficients should be invariant under translation in x,y,z and t directions. Then if we define the invertible transformations $w = a(t)u$ and $\tau = -\int \frac{dt}{a(t)}$, the symmetry Y_{12} will transform to $Y_{12} = -\frac{\partial}{\partial \tau}$.

If we use the transformation $w = a(t)u$, the wave equation (6.57) will transform to

$$\frac{\partial}{\partial t}(a\, w_t) = \frac{1}{a}\,(\nabla^2 w + \frac{\partial}{\partial t}(a\, \dot{a})) \tag{6.74}$$

Then by using the new variable $\tau = -\int \frac{dt}{a(t)}$, the equation is transformed again to

$$w_{\tau\tau} = \nabla^2 w + w\frac{\partial}{\partial t}(a\, \dot{a}) \tag{6.75}$$

And the first eight symmetries will transform again under these transformations to:

(1)

$$Y_1 = X_1 = 2\, x\, \tau\frac{\partial}{\partial \tau} + (x^2 - y^2 - z^2 + \tau^2)\frac{\partial}{\partial x} + 2\, x\, y\, \frac{\partial}{\partial y} + 2\, x\, z\frac{\partial}{\partial z} - 2\, x\, w\frac{\partial}{\partial w} \tag{6.76}$$

(2)

$$Y_2 = X_4 = 2\, y\, \tau \frac{\partial}{\partial \tau} + 2\, y\, x\, \frac{\partial}{\partial x} + (y^2 - x^2 - z^2 + \tau^2) \frac{\partial}{\partial y} + 2\, y\, z \frac{\partial}{\partial z} - 2\, y\, w \frac{\partial}{\partial w} \tag{6.77}$$

(3)

$$Y_3 = X_7 = 2\, a\, b\, z \frac{\partial}{\partial t} + 2\, z\, x\, \frac{\partial}{\partial x} + 2\, z\, y \frac{\partial}{\partial y} + (z^2 - x^2 - y^2 + b^2) \frac{\partial}{\partial z} - 2\, z\, w \frac{\partial}{\partial w} \tag{6.78}$$

(4)

$$Y_4 = X_{11} = -(x^2 + y^2 + z^2 + \tau^2) \frac{\partial}{\partial \tau} - 2\, \tau\, x\, \frac{\partial}{\partial x} - 2\, \tau\, y \frac{\partial}{\partial y} - 2\, \tau\, z \frac{\partial}{\partial z} + 2\, \tau\, w \frac{\partial}{\partial w} \tag{6.79}$$

(5)

$$Y_5 = X_3 = x \frac{\partial}{\partial \tau} + \tau \frac{\partial}{\partial x} \tag{6.80}$$

(6)

$$Y_6 = X_6 = -(y \frac{\partial}{\partial \tau} + \tau \frac{\partial}{\partial y}) \tag{6.81}$$

(7)

$$Y_7 = X_9 = z \frac{\partial}{\partial \tau} + \tau \frac{\partial}{\partial z} \tag{6.82}$$

(8)The one-parameter dilation group of the equation

$$Y_8 = X_{10} = \tau \frac{\partial}{\partial \tau} + x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} - w\, \frac{\partial}{\partial w} \tag{6.83}$$

Now since the value of the scale factor for the flat universe is given as $a(t) = t^{\frac{2}{3}}$, then $\tau = -3\sqrt[3]{t}$ and $\frac{\partial}{\partial t}(a\, \dot{a}) = \frac{2}{\tau^2}$, so our equation will take this last form

$$w_{\tau\tau} = \nabla^2 w + \frac{2}{\tau^2} w. \tag{6.84}$$

The last equation can be dealt with by the method of separation of variable to obtain the ODE of second order and the Helmholtz PDE. The solution of such equations can be constructed if some specific initial and boundary conditions are given.

6.3 Conclusion

In this chapter we found the Noether symmetries of a $(3+1)$ wave equation on the general spherical metric explicitly in terms of the explicit functions of θ and ϕ. In order to solve the Noether symmetries in terms of known functions of all the spacetime variables we chose a specific flat Friedmann metric. In the case that $a(t) \cong t^{1/2}$ we get 15 Noether symmetries of which 3 are translations and 3 are rotations. In the case that $a(t) = t^{2/3}$ (a particular value for the flat Friedmann model) we only get ten Noether symmetries which are 3 translations, 3 rotations, Y_1, Y_2, Y_3 and Y_8. Lastly, we also have converted the wave equation with variable coefficients to the one with constant coefficients through a transformation $\tau = -3\sqrt[3]{t}$ and $w = t^{2/3}\ u$ which can solved by separation of variables method under certain initial/boundary conditions.

Chapter 7

Conclusion and Future Work

Partial differential equations play important role in all branches of all natural, social and engineering sciences and are studied from several perspectives, mostly concerned with their solutions. As for linear partial differential equations are concerned, there exist standard techniques that are used to solve them. Seeking solutions of these equations becomes a serious challenge when they are nonlinear. Whereas nonlinearity of these equations makes it difficult to solve, their beauty and predictions of true physical situations lies in their nonlinearity. An example of nonlinearity is the famous diffusion equation in which diffusion processes bring in nonlinearity in the equation and make the job of finding its solutions quite challenging. Sophus Lie in 1981 developed group theoretic methods to solve such equations. These methods rely on exploiting the symmetries of the partial differential equations (named after him as Lie symmetries) to find their exact solutions. Since the advent of these methods, tremendous amount of research is being carried out in this direction and solutions of partial differential equations representing interesting physical models have been successfully solved and analyzed. Following these techniques, results were published in which diffusion equation modeling variable thermal diffusivity of materials such as gases was considered. On the one hand the model was interesting in its own right, it was diluted somewhat by incorporating a strong assumption that the thermal diffusivity was allowed to be negligible. My first research work deals with the same diffusion equation while setting aside this assumption. A complete classification of solutions of this diffusion equation in two space and one time dimension is obtained. This is an interesting work as it not only gave a better understanding of the diffusion processes in higher dimensions, but also yielded new classes of solutions which were not given previously. This work is published in Journal of Nonlinear Dynamics, doi :10.1007/s11071-010-9704-8. As for symmetries are concerned, they have important relationship with existence of conservation laws admitted by the partial differential equations. In the light of this fact an important question arises as to how can one find such relationships between the Lie point symmetries and the conservation laws admitted by the partial differential equations. Such relationship was first introduced by Emmy Noether in her classical work in the 17^{th} century and is based on a requirement that the partial differential equations must possess a complete Lagrangian. Such symmetries were named as Noether symmetries. Existence of a Noether symmetry guarantees existence of a conservation law admitted by the partial differential equation. With this point in mind, I embarked on research in this direction by considering a wave equation having two different wave speeds. Following the procedure of Emmy Noether and procedures suggested in some of the recently published research, I obtained conservation laws of this particular wave equation possessing via its Lagrangian with respect to complete classification of an unknown variable. Interesting new insights resulted from this work regarding solutions and conservation laws admitted by the equation. This work is published in Nonlinear Analysis and Application-Real world problems, doi:10.1016/j.nonrwa.2009.10.009 Another direction of research that emerges from above studies is the one in which application of Lie point symmetry generators to a conserved vector is considered. A consequence of such consideration results in one obtaining a direct conservation law associated with that Lie point symmetry. Path breaking work in this direction was published by Sjoberg who showed that when the generated conserved vector is null, a double reduction is possible which reduces, in one step, both order as well as a variable of the partial differential equation under consideration. My recent research investigations include an extension of the theory of Sjoberg in which a partial differential equation of order n in m dependent variables can be dealt with. This is an interesting generalization. This work is also published in Nonlinear Analysis-Real world problems, doi:10.1016/j.nonrwa.2010.02.006. The theory is applied to a wave equation in (2+1) dimensions and is submitted for publication. Lastly, I have applied the idea of Lie point symmetries and conservation laws to a wave equation written on Lorentzian manifolds. I have tried to investigate effects of Lorentzian geometries on conservation laws of certain wave equations. A consequence of applying the techniques I learnt in my PhD research I have tried to solve a system of equations that emerged from a concept when Ricci inheritance symmetry was defined in general Relativity. My investigations have results in an interesting conjecture proving that all well known solutions of gravitational field equations do not admit any of the symmetries. This paper, though not directly related to my PhD research problem, has been published in Nuovo Cimento B, doi : 10.1393/ncb/i2010-10836-0.

From the work undertaken in this dissertation some new directions in research emerge as application of the symmetry analysis and conservation laws such as mapping linear partial differential equation with variable coefficient to constant coefficient equation, possibility of construction an invertible mapping of nonlinear PDE to linear PDE through admitted conservation laws and using complex Lie symmetries for linearization of systems of differential equations.

Chapter 8

Appendices

Appendix A

Table 8.1: Case (1)-Reductions

Algebra	Reduction	α	β
$[V_0, V_1] = V_0$	$A\beta_\alpha = 2\sigma e^{(\beta A)}$	t	$\frac{uA-2\ln y}{A}$
$[V_0, V_3] = 0$	$\beta_\alpha = 0$	t	u
$[V_0, V_4] = 0$	$\beta_{\alpha,\alpha} + A\beta_\alpha^2 = 0$	y	u
$[V_0, V_5] = 0$	$A\sigma e^{(\beta A)}(\beta_{\alpha,\alpha} + A\beta_\alpha^2) = -1$	y	$\frac{uA+\ln t}{A}$
$[V_1, V_2] = 0$	$A\beta_\alpha = 4\sigma e^{(\beta A)}$	t	$\frac{uA-\ln(x^2+y^2)}{A}$
$[V_1, V_3] = -V_3$	$A\beta_\alpha = 2\sigma e^{(\beta A)}$	t	$\frac{uA-2\ln x}{A}$
$[V_1, V_4] = 0$	$2 - 2A\alpha\beta_\alpha + A^2(\alpha^2+1)\beta_\alpha^2 + A(\alpha^2+1)\beta_{\alpha\alpha} = 0$	$\frac{y}{x}$	$\frac{uA-2\ln x}{A}$
$[V_1, V_5] = 0$	$\sigma e^{(\beta A)}(2 - 2A\alpha\beta_\alpha + A^2(\alpha^2+1)\beta_\alpha^2 + A(\alpha^2+1)\beta_{\alpha\alpha}) = -1$	$\frac{y}{x}$	$\frac{uA-\ln(\frac{x^2}{t})}{A}$
$[V_2, V_4] = 0$	$\beta_\alpha + A\alpha\beta_\alpha^2 + \alpha\beta_{\alpha\alpha} = 0$	x^2+y^2	u
$[V_2, V_5] = 0$	$4A\sigma e^{(\beta A)}(\beta_\alpha + A\alpha\beta_\alpha^2 + \alpha\beta_{\alpha\alpha}) = -1$	x^2+y^2	$\frac{uA+\ln t}{A}$
$[V_3, V_4] = 0$	$\beta_{\alpha,\alpha} + A\beta_\alpha^2 = 0$	x	u
$[V_3, V_5] = 0$	$A\sigma e^{(\beta A)}(\beta_{\alpha,\alpha} + A\beta_\alpha^2) = -1$	x	$\frac{uA+\ln t}{A}$
$[V_4, V_5] = V_4$	$W_{ss} + W_{rr} + AW_s^2 + AW_r^2 = 0$, such that $s = x, r = y$ and $W = u$		

Table 8.2: Case (1)-Solutions

Algebra	$u(x,y,t)$
$[V_0,V_1]=V_0$	$\ln(\frac{-y^2}{2\sigma(t+C_1)})^{1/A}$
$[V_0,V_3]=0$	C_1
$[V_0,V_4]=0$	$\ln(-C_1yA-C_1C_2A)^{1/A}$
$[V_0,V_5]=0$	$\ln(\frac{-y^2-2C_1\sigma Ay+2C_2\sigma A}{2\sigma t})^{1/A}$
$[V_1,V_2]=0$	$\ln(\frac{-x^2-y^2}{4\sigma(t+C_1)})^{1/A}$
$[V_1,V_3]=-V_3$	$\ln(\frac{-x^2}{2\sigma(t+C_1)})^{1/A}$
$[V_1,V_4]=0$	$\ln(C_2\cos(2\tan^{-1}(y/x))-C_1\sin(2\tan^{-1}(y/x)))^{1/A}+\ln(\frac{A(x^2+y^2)}{2})^{1/A}$
$[V_1,V_5]=0$	$\ln((-C_1\sigma A(x^2+y^2)-yx)\sin(2\tan^{-1}(y/x))+(C_2\sigma A(x^2+y^2)-x^2)\cos(2\tan^{-1}(y/x)))^{1/A}$ $-\ln(2t\sigma)^{1/A}$
$[V_2,V_4]=0$	$\ln(-C_1A\ln(x^2+y^2)-C_1C_2A)^{1/A}$
$[V_2,V_5]=0$	$\ln(\frac{-x^2-y^2+4C_1\sigma A-4C_2\sigma A\ln(x^2+y^2)}{4\sigma t})^{1/A}$
$[V_3,V_4]=0$	$\ln(-C_1Ax-C_1C_2A)^{1/A}$
$[V_3,V_5]=0$	$\ln(\frac{-x^2-2C_1\sigma Ax+2C_2\sigma A}{2\sigma t})^{1/A}$

Appendix B

Table 8.3: Case (2)-Reductions

Algebra	Reduction	α	$\beta(\alpha)$
$[V_0,V_1]=V_0$	$\beta_\alpha=2\sigma(1+2a)(a\beta)^{\frac{1+a}{a}}$	t	$\frac{au+b}{ay^{2a}}$
$[V_0,V_3]=0$	$\beta_{\alpha=0}$	t	u
$[V_0,V_4]=0$	$a\beta\beta_{\alpha\alpha}+b\beta_{\alpha\alpha}+\beta_\alpha^2=0$	y	u
$[V_0,V_5]=0$	$\sigma(a\beta\beta_{\alpha\alpha}+\beta_\alpha^2)+(a\beta)^{\frac{2a-1}{a}}=0$	y	$\frac{au+b}{at^{-a}}$
$[V_1,V_2]=0$	$\beta_\alpha=4\sigma(1+a)(a\beta)^{\frac{1+a}{a}}$	t	$\frac{au+b}{a(x^2+y^2)^a}$
$[V_1,V_3]=-V_3$	$\beta_\alpha=2\sigma(1+2a)(a\beta)^{\frac{1+a}{a}}$	t	$\frac{au+b}{ax^{2a}}$
$[V_1,V_4]=0$	$2a^2(1+2a)\beta^2+(1+\alpha^2)\beta_\alpha^2-2a\alpha(1+2a)\beta\beta_\alpha+a(1+\alpha^2)\beta\beta_{\alpha\alpha}=0$	$\frac{y}{x}$	$\frac{au+b}{ax^{2a}}$
$[V_1,V_5]=0$	$\sigma(2a^2(1+2a)\beta^2+(1+\alpha^2)\beta_\alpha^2$ $-2a\alpha(1+2a)\beta\beta_\alpha+a(1+\alpha^2)\beta\beta_{\alpha\alpha})+(a\beta)^{\frac{2a-1}{a}}=0$	$\frac{y}{x}$	$\frac{au+b}{ax^{2a}t^{-a}}$
$[V_2,V_4]=0$	$b\beta_\alpha+\alpha\beta_\alpha^2+a\beta\beta_\alpha+a\alpha\beta\beta_{\alpha\alpha}+b\alpha\beta_{\alpha\alpha}=0$	x^2+y^2	u
$[V_2,V_5]=0$	$4\sigma(a\beta\beta_\alpha+a\alpha\beta\beta_{\alpha\alpha}+\alpha\beta_\alpha^2)+(a\beta)^{\frac{2a-1}{a}}=0$	x^2+y^2	$\frac{au+b}{at^{-a}}$
$[V_3,V_4]=0$	$a\beta\beta_{\alpha\alpha}+b\beta_{\alpha\alpha}+\beta_\alpha^2=0$	x	u
$[V_3,V_5]=0$	$\sigma(a\beta\beta_{\alpha\alpha}+\beta_\alpha^2)+(a\beta)^{\frac{2a-1}{a}}=0$	x	$\frac{au+b}{at^{-a}}$
$[V_4,V_5]=V_4$	$aWW_{ss}+bW_{s,s}+aWW_{r,r}+bW_{rr}+W_s^2+W_r^2=0$, such that $s=x,r=y$ and $W=u$		

Table 8.4: Case (2)-Solutions

Algebra	$u(x,y,t)$
$[V_0,V_1]=V_0$	$\frac{-b(-2\sigma a^{(\frac{1}{a})}t-4\sigma a^{(\frac{1+a}{a})}t+C_1)^a+ay^{2a}}{a(-2\sigma a^{(\frac{1}{a})}t-4\sigma a^{(\frac{1+a}{a})}t+C_1)^a}$
$[V_0,V_3]=0$	C_1
$[V_0,V_4]=0$	$\frac{(-C_1(y+C_2)(1+a))^{(\frac{a}{1+a})}-b}{a}$
$[V_1,V_2]=0$	$\frac{-b(-4\sigma a^{(\frac{1}{a})}t-4\sigma a^{(\frac{1+a}{a})}t+C_1)^a+a(x^2+y^2)^a}{a(-4\sigma a^{(\frac{1}{a})}t-4\sigma a^{(\frac{1+a}{a})}t+C_1)^a}$
$[V_1,V_3]=-V_3$	$\frac{-b(-2\sigma a^{(\frac{1}{a})}t-4\sigma a^{(\frac{1+a}{a})}t+C_1)^a+ax^{2a}}{a(-2\sigma a^{(\frac{1}{a})}t-4\sigma a^{(\frac{1+a}{a})}t+C_1)^a}$
$[V_2,V_4]=0$	$\frac{(-C_1(\ln(x^2+y^2)+C_2)(1+a))^{(\frac{a}{1+a})}-b}{a}$
$[V_3,V_4]=0$	$\frac{(-C_1(x+C_2)(1+a))^{(\frac{a}{1+a})}-b}{a}$

Appendix C

Table 8.5: Case (3)-Reductions

Algebra	Reduction	α	$\beta(\alpha)$
$[V_0,V_1]=V_0$	$\beta_\alpha+2\sigma=0$	t	$(u-b)y^2$
$[V_0,V_5]=0$	$\beta_\alpha=0$	t	u
$[V_0,V_6]=0$	$\beta_\alpha^2-\beta\beta_{\alpha\alpha}+b\beta_{\alpha\alpha}=0$	y	u
$[V_0,V_7]=0$	$\beta^3+\sigma(\beta\beta_{\alpha\alpha}-\beta_\alpha^2)=0$	y	$\frac{u-b}{t}$
$[V_1,V_2]=0$	$\beta_\alpha=0$	t	$(u-b)(x^2+y^2)$
$[V_1,V_3]=V_3$	$\beta_\alpha+2\sigma=0$	t	$(u-b)x^2$
$[V_1,V_4]=V_4$	$\beta_\alpha+2\sigma=0$	t	$(u-b)y^2$
$[V_1,V_5]=-V_5$	$\beta_\alpha+2\sigma=0$	t	$(u-b)x^2$
$[V_1,V_6]=0$	$2\beta^2-(1+\alpha^2)\beta_\alpha^2+2\alpha\beta\beta_\alpha+(1+\alpha^2)\beta\beta_{\alpha\alpha}=0$	$\frac{y}{x}$	$(u-b)x^2$
$[V_1,V_7]=0$	$\beta^3+\sigma(2\beta^2-(1+\alpha^2)\beta_\alpha^2+2\alpha\beta\beta_\alpha+(1+\alpha^2)\beta\beta_{\alpha\alpha})=0$	$\frac{y}{x}$	$\frac{(u-b)x^2}{t}$
$[V_2,V_6]=0$	$b\beta_\alpha+\alpha\beta_\alpha^2-\beta\beta_\alpha-\alpha\beta\beta_{\alpha\alpha}+b\alpha\beta_{\alpha\alpha}=0$	x^2+y^2	u
$[V_2,V_7]=0$	$\beta^3+4\sigma(-\alpha\beta_\alpha^2+\beta\beta_\alpha+\alpha\beta\beta_{\alpha\alpha})=0$	x^2+y^2	$\frac{u-b}{t}$
$[V_3,V_4]=0$	$\beta_\alpha=0$	t	$(u-b)(x^2+y^2)^2$
$[V_3,V_6]=0$	$2\beta^2-\alpha^2\beta_\alpha^2+2\alpha\beta\beta_\alpha+\alpha^2\beta\beta_{\alpha\alpha}=0$	$\frac{x^2+y^2}{x}$	$(u-b)x^2$
$[V_3,V_7]=0$	$\beta^3+\sigma(2\beta^2-\alpha^2\beta_\alpha^2+2\alpha\beta\beta_\alpha+\alpha^2\beta\beta_{\alpha\alpha})=0$	$\frac{x^2+y^2}{x}$	$\frac{(u-b)x^2}{t}$
$[V_4,V_6]=0$	$2\beta^2-\alpha^2\beta_\alpha^2+2\alpha\beta\beta_\alpha+\alpha^2\beta\beta_{\alpha\alpha}=0$	$\frac{x^2+y^2}{y}$	$(u-b)y^2$
$[V_4,V_7]=0$	$\beta^3+\sigma(2\beta^2-\alpha^2\beta_\alpha^2+2\alpha\beta\beta_\alpha+\alpha^2\beta\beta_{\alpha\alpha})=0$	$\frac{x^2+y^2}{y}$	$\frac{(u-b)y^2}{t}$
$[V_5,V_6]=0$	$\beta_\alpha^2-\beta\beta_{\alpha\alpha}+b\beta_{\alpha\alpha}=0$	x	u
$[V_5,V_7]=0$	$\beta^3+\sigma(\beta\beta_{\alpha\alpha}-\beta_\alpha^2)=0$	x	$\frac{u-b}{t}$
$[V_6,V_7]=V_6$	$-WW_{ss}+bW_{s,s}-WW_{r,r}+bW_{rr}+W_s^2+W_r^2=0$, such that $s=x,r=y$ and $W=u$		

Table 8.6: Case(3)-Solutions

Algebra	$u(x,y,t)$
$[V_0,V_1]=V_0$	$\frac{by^2-2\sigma t+C_1}{y^2}$
$[V_0,V_5]=0$	C_1
$[V_0,V_6]=0$	$\frac{-1+C_2be^{C_1y}}{C_2e^{C_1y}}$
$[V_0,V_7]=0$	$\frac{2C_1b+t-t\tanh(\frac{y+C_2}{2\sqrt{C_1\sigma}})^2}{2C_1}$
$[V_1,V_2]=0$	$\frac{by^2+bx^2+C_1}{y^2+x^2}$
$[V_1,V_3]=V_3$	$\frac{bx^2-2\sigma t+C_1}{x^2}$
$[V_1,V_4]=V_4$	$\frac{by^2-2\sigma t+C_1}{y^2}$
$[V_1,V_5]=-V_5$	$\frac{bx^2-2\sigma t+C_1}{x^2}$
$[V_1,V_6]=0$	$\frac{b(x^2+y^2)e^{(-C_2\tan^{-1}(y/x))}+C_1}{(x^2+y^2)e^{(-C_2\tan^{-1}(y/x))}}$
$[V_1,V_7]=0$	$\frac{2C_1b(x^2+y^2)+t-t\tanh(\frac{C_2-\tan^{-1}(y/x)}{\sqrt{4C_1\sigma}})^2}{2C_1(x^2+y^2)}$
$[V_2,V_6]=0$	$\frac{-1+C_2b(x^2+y^2)^{C_1}}{C_2(x^2+y^2)^{C_1}}$
$[V_2,V_7]=0$	$\frac{C_1b(x^2+y^2)+t-t\tanh(\frac{\ln(x^2+y^2)+C_2}{\sqrt{8C_1\sigma}})^2}{C_1(x^2+y^2)}$
$[V_3,V_4]=0$	$\frac{b(x^2+y^2)^2+C_1}{(x^2+y^2)^2}$
$[V_3,V_6]=0$	$\frac{e^2e^{(\frac{C_2x}{x^2+y^2})}+C_1b(x^2+y^2)^2}{C_1(x^2+y^2)^2}$
$[V_3,V_7]=0$	$\frac{2C_1b(x^2+y^2)^2+t-t\tanh(\frac{C_2(x^2+y^2)+x}{\sqrt{4C_1\sigma}(x^2+y^2)})^2}{2C_1(x^2+y^2)^2}$
$[V_4,V_6]=0$	$\frac{e^2e^{(\frac{C_2y}{x^2+y^2})}+C_1b(x^2+y^2)^2}{C_1(x^2+y^2)^2}$
$[V_4,V_7]=0$	$\frac{2C_1b(x^2+y^2)^2+t-t\tanh(\frac{C_2(x^2+y^2)+y}{\sqrt{4C_1\sigma}(x^2+y^2)})^2}{2C_1(x^2+y^2)^2}$
$[V_5,V_6]=0$	$\frac{-1+C_2be^{C_1x}}{C_2e^{C_1x}}$
$[V_5,V_7]=0$	$\frac{2C_1b+t-t\tanh(\frac{x+C_2}{2\sqrt{C_1\sigma}})^2}{2C_1}$

Appendix D

Table 8.7: Case (4)-Reductions

Algebra	Reduction	α	$\beta(\alpha)$
$[V_0,V_1]=V_0$	$\beta_\alpha=2\alpha(3f(\beta)\beta_\alpha+2\alpha f(\beta)\beta_{\alpha\alpha}+2\alpha f'(\beta)\beta_\alpha^2)$	$\frac{t}{y^2}$	u
$[V_0,V_3]=0$	$\beta_{\alpha=0}$	t	u
$[V_0,V_4]=0$	$f(\beta)\beta_{\alpha\alpha}+f'(\beta)\beta_\alpha^2=0$	y	u
$[V_1,V_2]=0$	$\beta_\alpha=4\alpha(f(\beta)\beta_\alpha+\alpha f(\beta)\beta_{\alpha\alpha}+\alpha f'(\beta)\beta_\alpha^2)$	$\frac{t}{x^2+y^2}$	u
$[V_1,V_3]=-V_3$	$\beta_\alpha=2\alpha(3f(\beta)\beta_\alpha+2\alpha f(\beta)\beta_{\alpha\alpha}+2\alpha f'(\beta)\beta_\alpha^2)$	$\frac{t}{x^2}$	u
$[V_1,V_4]=-2V_4$	$2\alpha f(\beta)\beta_\alpha+(1+\alpha^2)f(\beta)\beta_{\alpha\alpha}+(1+\alpha^2)f'(\beta)\beta_\alpha^2=0$	$\frac{y}{x}$	u
$[V_2,V_4]=0$	$f(\beta)\beta_\alpha+\alpha f(\beta)\beta_{\alpha\alpha}+\alpha f'(\beta)\beta_\alpha^2=0$	x^2+y^2	u
$[V_3,V_4]=0$	$f(\beta)\beta_{\alpha\alpha}+f'(\beta)\beta_\alpha^2=0$	x	u

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