ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF SOME VISCOELASTIC PROBLEMS

BY

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DEDICATION

I solely dedicate this thesis to Allah (*Subhanahu Wa Ta'ala*), the Lord of the Worlds. May He accept it from me as path to seeking knowledge. "Truly, my prayer and my service of sacrifice, my life and my death are (all) for Allah, the Cherisher of the Worlds".

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THESIS ABSTRACT

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In this thesis we investigate asymptotic behavior of solutions of some viscoelastic problems in bounded domains and establish some stability results for the problems. In this regard, we establish exponential, polynomial and general decay rate results. The decay results are established in the absence, as well as in the presence of a source term.

ملخص الرسالة

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في هذه الرسالة نفحص السلوك التقاربي لحلول بعض مسائل المرونة اللزجة في مجالات محدودة ونثبت بعض النتائج المتعلقة باستقرار هذه المسائل.

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CHAPTER 1

INTRODUCTION

1.1 Viscoelastic Materials

Elasticity is the material deformation behavior described by Hooke's law which states that displacement is linearly proportional to the applied load. An elastic material returns to the undeformed state once the loads are removed and the effects of multiple load systems can be computed by simple linear superposition. Moreover, the work done by the forces is calculated by multiplying the loads by the displacements. On the other hand, viscosity is an internal property of a fluid that offers resistance to flow. Viscous liquid has no definite shape, and it flows irreversibly under the action of external forces. However, there are materials with properties that are intermediate between elasticity and viscosity.

Viscoelasticity, as its name suggests, incorporates aspects of both time dependent fluid behavior (viscous) and time independent solid behavior (elastic). Viscoelastic materials share some properties with elastic solids and some others with Newtonian viscous fluid. They exhibit an instantaneous elasticity effect and creep characteristics at the same time. In fact they can display all the intermediate range of properties. For instance, at low temperatures, or high frequencies of measurement, a polymer may be glass-like and it will break or flow at great strains. On the other hand, at high temperatures, permanent deformation occurs under load, and polymer behaves like a highly viscous liquid. However, in an intermediate temperature or frequency range, commonly called the glass transition range, the polymer is neither glassy nor rubber-like. Hence, polymers are usually described as viscoelastic materials and may dissipate a considerable amount of energy on being strained. In the rubber-like state, a polymer may be subjected to large deformation and still shows a complete recovery. To a good approximation, this is an elastic behavior at large strain [22], [34].

The importance of the viscoelastic properties of materials has been realized because of the rapid developments in rubber and plastics industry. Many advances in the studies of constitutive relations, failure theories and life prediction of viscoelastic materials and structures were reported and reviewed in the last two decades [18]. Time dependence of mechanical behavior of viscoelastic materials reveals the existence of inner clock or intrinsic time, which can be influenced by many factors such as temperature [3], physical aging [11], [55], [58], damage, pressure and solvent concentration [31], [38], strain and stress level [7], [33], [53], etc.

Depending on the change of strain rate versus stress inside a material the viscosity can be categorized as having a linear, non-linear, or plastic response. When a material exhibits a linear response it is categorized as a Newtonian material. In this case the stress is linearly proportional to the strain rate. If the material exhibits a non-linear response to the strain rate, it is categorized as Non-Newtonian fluid. There is also an interesting case where the viscosity decreases as the shear/strain rate remains constant. A material which exhibits this type of behavior is known as thixotropic. In addition, when the stress is independent of this strain rate, the material exhibits plastic deformation [35].

Viscoelastic behavior can be represented by combinations of springs and dashpots (pistons that move inside a viscous fluid). While linear springs instantaneously produce deformation proportional to the load, a dashpot produces a velocity proportional to the load at each instant. If a spring and a dashpot are placed in parallel one obtains Maxwell's viscoelastic model. If they are arranged in series, one has Voigt's model. Finally, a series/parallel arrangement yields Kelvin's model [35]

Two commonly observed viscoelastic behaviors are stress relaxation and (low temperature, viscoelastic) creep. Stress relaxation describes the time dependent change in stress following the application of an instantaneous strain. Alternatively, creep is the time dependent change of strain following the application of an instantaneous stress. Hence, creep is in some sense the inverse of stress relaxation, and refers to the general characteristic of viscoelastic materials to undergo increased deformation under a constant stress, until an asymptotic level of strain is reached. Any materials that exhibit hysteresis, creep or stress relaxation can be considered viscoelastic materials. In comparison, elastic materials do not exhibit energy dissipation or hysteresis as their loading and unloading curve is the same. Indeed, the fact that all energy due to deformation is stored is a characteristic of elastic materials. Furthermore, under fixed stress, elastic materials will reach a fixed strain and stay at that level. However, under fixed strain, elastic materials will reach a fixed stress and stay at that level with no relaxation [35].

Boltzmann (1844-1906) first proposed to use superposition to compute the stress-strain response of a viscoelastic solid subjected to an arbitrary loading history. He assumed that creep at any time is a function of the entire prior loading history and that each loading step makes an independent contribution to the deformation. Hence for an applied stress $\sigma(t)$, the strain is

$$\epsilon(t) = \int_{-\infty}^{t} J(t-\tau) d\sigma(t)$$

where J is time dependent creep compliance

Likewise, if a strain $\epsilon(t)$ is applied

$$\sigma(t) = \int_{-\infty}^{t} G(t-\tau) d\epsilon(t)$$

where G time dependent stress relaxation modulus [35].

We consider viscoelasticity in the isothermal approximation, which means that the temperature does not enter the model (state and constitutive relation). So the state involves the deformation gradient only while the constitutive equation is in fact a stress-strain relation. We obtain [10], [35]

$$\sigma(t) = G(t)\epsilon(0) + \int_{0}^{t} G(t-\tau) \frac{\partial \epsilon(\tau)}{\partial \tau} d\tau = G * \epsilon$$

The integrating functions $G(t)$ are mechanical properties of the material and are called "relaxation functions". It can be considered to be the formulation of Boltzmann's superposition principle such that the current stress is determined by the superposition of the responses to the complete spectrum of increments of strain. More so, the right hand side is called the convolution of G and ϵ [35].

The relaxation function G brings about damping effect of the solutions to the problem. This viscous damping ensures global existence of smooth solutions decaying uniformly under constant density as time goes to infinity. This is true for sufficiently smooth and/or small data and history. We shall mainly be concerned with this phenomenon in our problems

1.2 Literature Review

In [26], Giorgi C. *et al* considered the following semilinear hyperbolic equation with linear memory in a bounded domain $\Omega \in \mathbb{R}^3$

$$u_{tt} - k(0)\Delta u - \int_0^\infty k^{'}(s)\Delta u(t-s)ds + g(u) = f \quad \text{in } \Omega \times \mathbb{R}^+ \tag{1.1}$$

with $k(0),\ k(\infty) > 0$ and $k^{'}(s) \leq 0, \forall s \in \mathbb{R}^+$ and established longtime behavior of solutions. In particular, in the autonomous case, they established existence of global attractors for the solutions. Later, Monica Conti and Vittorino Pata [19] considered

$$u_{tt} + \alpha u_t - k(0)\Delta u - \int_0^\infty k^{'}(s)\Delta u(t-s)ds + g(u) = f, \quad \text{in } \Omega \times \mathbb{R}^+ \tag{1.2}$$

where $g : \mathbb{R} \to \mathbb{R}$ is a nonlinear term of (at most) cubic growth satisfying some dissipativity conditions and the memory kernel k is a convex decreasing smooth function such that $k(0) > k(\infty) > 0$ and established the existence of a regular global attractor, thereby improving the result of [26].

In [4], Appleby J. A. D *et al* investigated the linear integro-differential equation

$$u^{''}(t) + Au(t) + \int_{-\infty}^{t} k(t-s)Au(s)ds = 0, \quad t > 0 \tag{1.3}$$

and established results concerning the exponential decay of strong solutions in Hilbert space. Recently, Vittorino Pata in [54] discussed the decay properties of the semigroup generated by a linear integro-differential equation in a Hilbert space, which is an abstract version of the equation:

$$\partial_{tt}u(t) - \Delta u(t) + \int_0^\infty \mu(s)\Delta u(t-s)ds = 0, \tag{1.4}$$

describing the dynamics of linearly viscoelastic bodies and established the necessary as well as the sufficient conditions for the exponential stability.

For the finite history case, Cavalcanti *et al.* in [15] investigated the following viscoelastic problem:

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau = 0, & \text{in } \Omega \times (0,\infty) \\ u = 0, & \text{on } \Gamma_0 \times (0,\infty) \\ \dfrac{\partial u}{\partial v} - \int_0^t g(t-\tau)\dfrac{\partial u}{\partial v}(\tau)d\tau + h(u_t) = 0, & \text{on } \Gamma_1 \times (0,\infty) \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in \Omega, \end{cases} \tag{1.5}$$

where Ω is a bounded domain of R^n with a smooth boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1$. Here, Γ_0 and Γ_1 are closed and disjoint, with *meas* $(\Gamma_0) > 0$, ν is the unit outward normal to $\partial\Omega$, and g and h are specific functions. They established a global existence result for strong and weak solutions. In addition, some uniform decay rate results were proved under quite restrictive assumptions on both the damping function h and the kernel g. In fact, the function g had to behave exactly like $e^{-mt}, m > 0$, and the function h had a polynomial behavior near zero. Later, Cavalcanti *et al.* [14] considered (1.1) without imposing a growth assumption on h and under weaker conditions on g. They improved the result of [15] and established uniform stability depending on the behavior of h near the origin and on the behavior of g at infinity provided that $\|g\|_{L^1(0,\infty)}$ is small enough. In particular, they obtained explicit decay rate results for some special cases. This result has been recently improved by Messaoudi and Mustapha [46]. All these results are in the spirit of the work of Fabrizio *et al.* [22] in which they considered

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau + u_t = 0, \quad \text{in } \Omega \times (0,\infty) \tag{1.6}$$

and showed that the exponential decay of the relaxation function is a necessary condition for the exponential decay of the solution energy. In other words, the presence of the memory term may prevent the exponential decay due to the linear frictional damping term. They also obtained a similar result for the polynomial decay case.

In [50], Muñoz Rivera considered equations for linear isotropic homogeneous viscoelastic solids of integral type which occupy a bounded domain or the whole space $\mathbb{R}^n$, with zero boundary and history data, and in the absence of body forces. In the bounded domain case, an exponential decay result was proved for exponentially decaying memory kernels. For the whole space case, a polynomial decay result was established and the rate of the decay was given. This result was later generalized to a situation, where the kernel is decaying algebraically but not exponentially by Cabanillas and Rivera [12]. In this paper, the authors showed that the decay of solutions is also algebraic, at a rate which can be determined by the rate of the decay of the relaxation function. Also, the authors considered both cases the bounded domains and that of a material occupying the entire space. This result was later improved by Baretto *et al*. in [6], where equations related for linear viscoelastic plates were treated. Precisely, they showed that the solution energy decays at the same decay rate of the relaxation function. For partially viscoelastic materials, Rivera *et al.* [52] showed that solutions decay exponentially to zero, provided the relaxation function decays in similar fashion, regardless to the size of the viscoelastic part of the material. In [16], Cavalcanti *et al*. considered

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau + a(x)u_t + |u|^\gamma u = 0, \text{ in } \Omega \times (0,\infty), \tag{1.7}$$

for $a : \Omega \to \mathbb{R}^+$, a function which may vanish on a part $\omega \subset \Omega$ of positive measure. Under some geometry restrictions on ω and for

$$a(x) \geq a_0 > 0, \qquad \forall x \in \omega$$

$$-\xi_1 g(t) \leq g'(t) \leq -\xi_2 g(t), \quad t \geq 0,$$

the authors established an exponential rate of decay. Berrimi and Messaoudi [8] improved Cavalcanti's result by introducing a different functional which allowed them to weaken the conditions on both a and g. Furthermore, Berrimi and Messaoudi [9] considered

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau = [u]^{p-2}u \tag{1.8}$$

in a bounded domain and $p > 2$. They established a local existence and showed, under weaker conditions than those in [16], that the local solution is global and decays uniformly if the initial data are small enough. In particular, the function a can vanish on the whole domain Ω and consequently the geometry condition is no longer needed. In [17], Cavalcanti *et al* considered

$$u_{tt} - k_0 \Delta u + \int_0^t div[a(x) g(t-\tau) \nabla u(\tau)]\, d\tau + b(x) h(u_t) + f(u) = 0 \tag{1.9}$$

under similar conditions on the relaxation function g and $a(x) + b(x) \geq \rho > 0$, for all $x \in \Omega$. They improved the result of [16] by establishing exponential stability for g decaying exponentially and h linear and polynomial stability for g decaying polynomially and h nonlinear. For quasilinear viscoelastic equations, Cavalcanti *et al* [13] studied, in a bounded domain, the following equation:

$$|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-\tau) \Delta u(\tau) d\tau - \gamma \Delta u_t = 0, \quad \rho > 0 \tag{1.10}$$

and proved a global existence result for $\gamma \geq 0$ and an exponential decay for $\gamma > 0$. This latter result has been extended to a situation, where a source term is competing with the strong mechanism damping and the one induced by the viscosity, by Messaoudi and Tatar [47]. Furthermore, Messaoudi and Tatar [49], [48] established, for $\gamma = 0$, exponential and polynomial decay results in the absence, as well as in the presence, of a source term. Messaoudi [42] considered the equation

$$u_{tt} - \Delta u + \int_0^t g(t-\tau) \Delta u(\tau) d\tau + a u_t |u_t|^m = b |u|^\gamma u, \quad \text{in } \Omega \times (0, \ \infty) \tag{1.11}$$

and showed, under suitable conditions on g, that solutions with negative energy blow up in finite time if $\gamma > m$, and continue to exist if $m \geq \gamma$.

In the absence of the viscoelastic term $(g = 0)$, the problem has been extensively studied and many results concerning global existence and nonexistence have been proved. For instance, for the problem

$$u_{tt} - \Delta u + a u_t |u_t|^m = b|u|^\gamma u, \quad \text{in } \Omega \times (0, \infty) \qquad (1.12)$$

with $m, \gamma \geq 0$, it is well known that, for $a = 0$, the source term $b|u|^\gamma u$ $(\gamma > 0)$ causes finite time blow up of solutions with negative initial energy [5], [28] and for $b = 0$, the damping term $a u_t |u_t|^m$ assures global existence for arbitrary initial data [27], [32]. The interaction between the damping and the source terms was first considered by Levine [37], [36] in the linear damping case $(m = 0)$. He showed that solutions with negative initial energy blow up in finite time. Georgiev and Todorova [25] extended Levine's result to the nonlinear damping case $(m > 0)$. In their work, the authors introduced a different method and determined suitable relations between m and γ and for which there is global existence or alternatively finite time blow up. Precisely; they showed that solutions with negative energy continue to exist globally 'in time' if $m \geq \gamma$, and blow up in finite time if $\gamma > m$ and the initial energy is sufficiently negative. Without imposing the condition that the initial energy is sufficiently negative, Messaoudi [41] extended the blow up result of [25] to solutions with negative initial energy only.

Recently, Messaoudi [44], [45] considered

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau = b|u|^\gamma u, \text{ in } \Omega \times (0, \infty) \qquad (1.13)$$

for $b = 0$ and $b = 1$ and for a wider class of relaxation functions. He established a more general decay result, from which the usual exponential and polynomial decay results are only special cases.

A related result is the work of Kawashima [29], in which he considered a one-dimensional model equation for viscoelastic materials of integral type where the memory function is allowed to have an integrable singularity. For small initial data, Muñoz Rivera and Baretto [51] proved that the first and the second–order energies of the solution to a viscoelastic plate, decay exponentially provided that the kernel of the memory decays exponentially. Kirane and Tatar [30] considered a mildly damped wave equation and proved that any small integral dissipation is sufficient to uniformly stabilize the solution by means of a nonlinear feedback of memory type acting on a part of the boundary. This result was established without any restriction on the space dimension or geometrical conditions on the domain or its boundary. For more about the subject see [1], [24], [40], [43], [57] and [56].

1.3 Results Description

The aim of this thesis is to investigate the asymptotic behavior of solutions of some viscoelastic problems in bounded domains. In this regard, we study several problems and establish exponential, polynomial and general decay rate results. The decay results are established in the absence, as well as in the presence of a source term. This thesis contains five chapters. In Chapter 1, we discuss in much details the properties and significance of viscoelastic materials and we end the chapter by reviewing some literatures related to our problems. In Chapter 2, we present some principal concepts, some theorems on Sobolev embeddings and some lemmas which are of essential use in the proofs of our results. We devote the other chapters to the discussion of our problems which is to study the asymptotic behaviors of solutions of some viscoelastic problems. In Chapter 3, we study the case when the relaxation function is decaying exponentially in the presence of a source term. In this regard, we establish a decay result which depends on the behavior of the external force. The case when the relaxation function is decaying polynomially in the absence of a source term is treated in Chapter 4. In the last chapter, we study a general decay result for the finite history case in the presence of an external force.

CHAPTER 2

PRINCIPAL CONCEPTS

The main objective of this chapter is to present without proof brief discussion of some concepts and properties related to our problems. Reader should consult [2], [21] and [39] for proofs and more details.

2.1 Preliminaries

Definition 2.1.1. Let Ω be a domain in $\mathbb{R}^n$ and let m be a non-negative integer. We define by $C^m(\Omega)$ the linear space of continuous functions on Ω whose partial derivatives $D^\alpha u$, $|\alpha| \leq m$, exist and continuous, where

$$D^\alpha u(x) = \frac{\partial^{|\alpha|} u(x)}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots\ldots \partial x_n^{\alpha_n}} \tag{2.1}$$

$\alpha = (\alpha_1, \ldots\ldots \alpha_n)$ is called a multi-index of dimension n and

$$|\alpha| = \sum_{i=1}^{n} \alpha_i$$

Definition 2.1.2. The support of a continuous function f defined on $\mathbb{R}^n$ is the closure of the set of point where $f(x)$ is nonzero;

$$supp\, f = \overline{\{x \in \mathbb{R}^n : f(x) \neq 0\}}.$$

The closed and bounded sets in $\mathbb{R}^n$ are precisely the compact sets, so if $supp\, f$ is bounded, we say f has a compact support and denote the set of such functions by

$C_0(\mathbb{R}^n)$. Similarly, $C_0(\Omega)$ denotes the set of continuous functions on Ω whose supports are compact subsets of Ω.

In addition, $C_0^\infty(\Omega)$ denotes the class of the functions $u\ in\ \Omega$ such that

$a)$ u is infinity differentiable, which means that $D^\alpha u$ is uniformly continuous in $\overline{\Omega}$, for any α;

$b)$ u is compactly supported: *supp* u is a compact subset of Ω.

Corollary 2.1.1. $C^m(\Omega)$ is a Banach space with respect to the norm

$$\|u\|_{C^m(\Omega)} := \max_{|\alpha|\le m} \sup_{x\in\Omega} |D^\alpha u(x)| \tag{2.2}$$

Remark 2.1.1. If $m = 0$, we denote $C^0(\Omega) = C(\Omega)$

2.2 Lebesgue Spaces

Definition 2.2.1. [21] Let Ω be a domain in $\mathbb{R}^n$; for $1 \le p < \infty$, $L^p(\Omega)$ denote the measurable real-valued functions u on Ω for which

$$\int_\Omega |u(x)|^p dx < \infty.$$

In addition, $L^\infty(\Omega)$ denotes the measurable real valued functions that are essentially bounded (bounded except on a set of measure zero). For $u \in L^p(\Omega)$, we define the norms

$$\|u\|_p = \left(\int_\Omega |u(x)|^p dx \right)^{\frac{1}{p}}, \qquad for\ 1 \le p < \infty, \tag{2.3}$$

$$\|u\|_\infty = ess\ sup|u(x)| = inf\{M:\ \mu\{x: u(x) > M\} = 0\} \tag{2.4}$$

Lemma 2.2.1. [2] If $1 \le p < \infty$, and $a, b \ge 0$, then

$$(a+b)^p \le 2^{p-1}(a^p + b^p) \tag{2.5}$$

Theorem 2.2.1. (**Hölder's inequality**) [2] Let $1 < p < \infty$, and let q denote the conjugate exponent defined by

$$q = \frac{p}{p-1}, \quad \text{that is} \quad \frac{1}{p} + \frac{1}{q} = 1$$

which also satisfies $1 < q < \infty$. If $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$, then $uv \in L^1(\Omega)$ and

$$\int_\Omega |u(x)v(x)|dx \le \|u\|_p \|u\|_q \tag{2.6}$$

Equality holds if and only if for some constants α and β, not both zero,

$$\alpha|u(x)|^p = \beta|v(x)|^q \;\; a.e \text{ in } \Omega$$

Corollary 2.2.1. By taking $p = q = 2$, we obtain the Cauchy-Schwarz inequality

$$\int_\Omega |u(x)v(x)|dx \le \|u\|_2 \|u\|_2 \tag{2.7}$$

Theorem 2.2.2. (**Young's inequality**) [21] Let $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and $a, b \ge 0$. Then for any $\eta > 0$,

$$ab \le \eta a^p + C_\eta b^q \tag{2.8}$$

where $C_\eta = \dfrac{1}{q(\eta p)^{\frac{q}{p}}}$

For $p = q = 2$, the inequality takes the form

$$ab \le \eta a^2 + \frac{b^2}{4\eta} \tag{2.9}$$

Theorem 2.2.3. [2] $L^p(\Omega)$ equipped with the norm (2.3) is a Banach space if $1 \le p \le \infty$,

Corollary 2.2.2. [2] $L^2(\Omega)$ is a Hilbert space with respect to the inner product

$$\langle u, v \rangle = \int_{\Omega} u(x)\overline{v(x)}dx$$

The associated norm is then

$$\|u\|_2^2 = \langle u, u \rangle$$

Theorem 2.2.4. (**Density Theorem**) [21] If $f \in L^p(\Omega)$, $1 \le p < \infty$, then there exists a sequence $(f_n) \subset C_0^\infty(\Omega)$ which converges to f with respect to the norm $\|.\|_p$. This implies that $C_0^\infty(\Omega)$ is dense in $L^p(\Omega)$.

2.3 Sobolev Spaces

Definition 2.3.1. (**Weak derivative**) [2] If $u, v \in L^p(\Omega)$, v is called a weak derivative of order α of u if

$$\int_{\Omega} u(x)D^\alpha \Phi(x)dx = (-1)^{|\alpha|}\int_{\Omega} v(x)\Phi(x)dx, \quad \forall \Phi \in C_0^\infty(\Omega). \tag{2.10}$$

For the definition of α and $D^\alpha \Phi(x)$ we refer to (2.1).

Definition 2.3.2. (**Sobolev spaces**) [2] Let Ω be an open set of $\mathbb{R}^n$, then the Sobolev space $W^{k,p}(\Omega)$, $1 \le p \le \infty$, $k \in \mathbb{N}^*$(positive integer number), is the set of all functions $u \in L^p(\Omega)$ such that the weak derivatives $D^\alpha u$ of order α, $|\alpha| \le k$, exist and lie in $L^p(\Omega)$. That is

$$W^{k,p}(\Omega) := \{u \in L^p(\Omega) | D^\alpha u \in L^p(\Omega), |\alpha| \le k \}$$

$W^{k,p}(\Omega)$ is equipped with the following norm:

$$\|u\|_{k,p} = \left(\sum_{|\alpha|\le k} \|D^\alpha u\|_p^p \right)^{\frac{1}{p}}, \quad 1 \le p < \infty$$
$$\|u\|_{k,\infty} = \max_{|\alpha|\le k} \|D^\alpha u\|_\infty \tag{2.11}$$

Remark 2.3.1. If $u \in C^m(\Omega)$, then all weak derivatives are classical.

Theorem 2.3.1. [2] $W^{k,p}(\Omega)$ is a Banach space with respect to the norm (2.11)

Remark 2.3.2. If $p = 2$, we denote $W^{k,2}(\Omega)$ by $H^k(\Omega)$ and it is a Hilbert space with respect to the inner product

$$\langle u, v\rangle_k = \int_\Omega \sum_{|\alpha|\le k} D^\alpha u(x) D^\alpha v(x)\, dx, \qquad \forall u, v \in H^k(\Omega). \tag{2.12}$$

Definition 2.3.3. (**Sobolev spaces of order one in $\mathbb{R}^n$**)

Let Ω be an open domain of $\mathbb{R}^n$ and $1 \le p \le \infty$. Then

$$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega) \,\middle|\, \exists v_i \in L^p(\Omega), \int_\Omega u \frac{\partial \phi}{\partial x_i} = -\int_\Omega v_i \phi\, , i = 1, 2, \dots, n, \forall \phi \in C_0^\infty(\Omega) \right\}$$

is called the Sobolev space of order one and it is equipped with the norm

$$\|u\|_{1,p} = \|u\|_p + \sum_{i=1}^n \left\| \frac{\partial u}{\partial x_i} \right\|_p \tag{2.13}$$

or equivalently with

$$\|u\|_{1,p} = \left(\|u\|_p^p + \sum_{i=1}^n \left\| \frac{\partial u}{\partial x_i} \right\|_p^p \right)^{\frac{1}{p}}, \quad 1 < p < \infty \tag{2.14}$$

Remark 2.3.3. $W^{1,2}(\Omega) = H^1(\Omega)$ is a Hilbert space with respect to the inner product

$$\langle u, v\rangle = \int_\Omega uv dx + \sum_{i=1}^{n} \int_\Omega \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} dx$$

Definition 2.3.4. $\left(\textbf{The space } \mathbf{W_0^{1,p}(\Omega)}\right)$ Let Ω be a domain of $\mathbb{R}^n$ and $1 \leq p < \infty$, we define the space $W_0^{1,p}(\Omega)$ to be the closure of $C_0^1(\Omega)$ with respect to the norm of $W^{1,P}(\Omega)$.

Theorem 2.3.2. If $u \in W_0^{1,p}(\Omega) \cap C(\overline{\Omega})$, then $u(x) = 0$ for every $x \in \partial\Omega$.

Theorem 2.3.3. $\left(\textbf{Poincaré's inequality}\right)$ [39] Assume that Ω is bounded in one direction and $1 \leq p < \infty$. Then there is a positive constant $C = C(\Omega, p)$ such that

$$\|u\|_p \leq C_p \|\nabla u\|_p, \quad \forall u \in W_0^{1,p}(\Omega)$$

Definition 2.3.5. Let V and W be two Banach spaces. We say that V is continuously embedded in W and we write $V \hookrightarrow W$, if we have, for some $C > 0$,

$$\|v\|_W \leq C\|v\|_V, \quad \forall v \in V$$

Theorem 2.3.4. (**An embedding theorem for L^p spaces**) [2] Suppose that

$$vol\ (\Omega) = \int_\Omega dx < \infty$$

and $1 \leq p \leq q \leq \infty$. If $u \in L^q(\Omega)$, then $u \in L^p(\Omega)$, and

$$\|u\|_p \leq \big(vol\,(\Omega)\big)^{\frac{1}{p}-\frac{1}{q}}\|u\|_q$$

hence

$$L^q(\Omega) \hookrightarrow L^p(\Omega)$$

Theorem 2.3.5. (**Sobolev embedding theorems**) [2] Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain, $m \geq 1$ and $1 \leq p \leq \infty$. Then, the following mappings represent continuous embeddings

$$W^{m,p}(\Omega) \hookrightarrow L^{p^*}(\Omega), \quad \frac{1}{p^*} = \frac{1}{p} - \frac{m}{n}, \; if\; m < \frac{n}{p},$$

$$W^{m,p}(\Omega) \hookrightarrow L^q(\Omega), \quad 1 \leq q < \infty, \; if\; m = \frac{n}{p},$$

$$W^{m,p}(\Omega) \hookrightarrow C^{0,m-\frac{n}{p}}(\overline{\Omega}), \quad if\; \frac{n}{p} < m < \frac{n}{p} + 1, \tag{2.15}$$

$$W^{m,p}(\Omega) \hookrightarrow C^{0,\alpha}(\overline{\Omega}), \quad 0 < \alpha < 1, \; if\; m = \frac{n}{p} + 1,$$

$$W^{m,p}(\Omega) \hookrightarrow C^{0,1}(\overline{\Omega}), \quad if\; m > \frac{n}{p} + 1.$$

Theorem 2.3.6. (**Sobolev**, **Gagliardo**, **Nirenberg**) [39] If $1 \leq p < n$, then

$$W^{1,p}(\mathbb{R}^n) \hookrightarrow L^{p^*}(\mathbb{R}^n), \quad \frac{1}{p^*} = \frac{1}{p} - \frac{1}{n} \tag{2.16}$$

and there exists a constant $C = C(n,\ p)$ such that

$$\|u\|_{p^*} \leq C\|\nabla u\|_p, \quad \forall u \in W^{1,p}(\mathbb{R}^n)$$

Corollary 2.3.1 If $1 \leq p < n$, then

$$W^{1,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n), \quad 1 \leq q \leq p^* \tag{2.17}$$

Theorem 2.3.7. [39] If $p = n$, then

$$W^{1,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n), \quad n \leq q < \infty \tag{2.18}$$

Theorem 2.3.8. (**Morrey**) [38] If $p > n$, then

$$W^{1,p}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$$

Moreover, if $u \in W^{1,P}(\mathbb{R}^n)$, then u is a continuous function.

Remark 2.3.4. The above theorems remain valid if we substitute $\mathbb{R}^n$ by a domain $\Omega \subset \mathbb{R}^n$ with a smooth boundary $\partial\Omega$.

We conclude this chapter by introducing a very important formula we use very often to estimate some integrals and to prove many results in our problems.

2.4 Green's Formula

Let Ω be a bounded domain of $\mathbb{R}^n$ with a smooth boundary, then $\forall u \in H^1,\ \forall v \in H^2$, we have

$$\int_\Omega u\Delta v dx = -\int_\Omega \nabla u \cdot \nabla v dx + \int_{\partial\Omega} u\nabla v \cdot \zeta ds \tag{2.20}$$

where ζ is the outer unit normal to $\partial\Omega$ [39].

Remark 2.4.1. If $u \in H_0^1(\Omega)$, then Green's formula is reduced to

$$\int_\Omega u\Delta v dx = -\int_\Omega \nabla u \cdot \nabla v dx$$

CHAPTER 3

EXPONENTIAL DECAY OF SOLUTION OF A VISCOELASTIC PROBLEM

3.1 Introduction

In this chapter, we consider the viscoelastic problem

$$\begin{cases} u_{tt} - \Delta u(x,t) + \int_0^\infty g(s)\Delta u(t-s)ds = f(x,t), \ \text{in } \Omega \times \mathbb{R}^+ \\ u(x,t) = 0, \qquad x \in \partial\Omega, \qquad t \in \mathbb{R}^+ \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \qquad x \in \Omega \end{cases} \tag{3.1}$$

where Ω is a bounded domain of $\mathbb{R}^n (n \geq 1)$ with a smooth boundary $\partial\Omega$, $f \in L^2(\Omega \times (0, +\infty))$ and g is a positive non-increasing function satisfying the following conditions:

(G_1) $\quad g: \mathbb{R}^+ \to \mathbb{R}^+$ is a differentiable function such that

$$g(0) > 0, \qquad 1 - \int_0^\infty g(s)ds = l > 0$$

(G_2) $\quad$ There exists a positive constant ξ such that

$$g'(t) \leq -\xi g(t), \quad \forall t \geq 0.$$

Following the idea of Dafermos [20], we introduce

$$\eta^t(x,s) = u(x,t) - u(x,t-s), \qquad s \geq 0. \tag{3.2}$$

Consequently, by adding and subtracting the term Δu, (3.1) transforms into

$$\begin{cases} u_{tt} - l\Delta u(x,t) - \int_0^\infty g(s)\Delta\eta^t(x,s)ds = f(x,t), \text{ in } \Omega \times \mathbb{R}^+ \\ \eta_t^t(x,s) + \eta_s^t(x,s) = u_t(x,t) \\ u(x,t) = \eta^t(x,s) = 0, \qquad x \in \partial\Omega, \qquad \forall\, t,\ s \geq 0 \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ \forall x \in \Omega \\ \eta^0(x,s) = \eta_0(x,s) = u_0(x) - v_0(x,-s), \eta^t(x,0) = 0, \forall x \in \Omega,\ s \geq 0 \end{cases} \tag{3.3}$$

Theorem 3.1.1. Let $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ and $\eta_0 \in L_g^2\big(\mathbb{R}^+, H_0^1(\Omega)\big)$ be given. Assume that $f \in L^2\big(\Omega \times (0,+\infty)\big)$ and g satisfies (G_1) and (G_2), then problem (3.3) has a unique global weak solution

$$u \in C\left([0,\infty); H_0^1(\Omega)\right),\ u_t \in C\big([0,\infty); L^2(\Omega)\big),\ \eta^t \in L_g^2\left(\mathbb{R}^+ \times \mathbb{R}^+, H_0^1(\Omega)\right), \tag{3.4}$$

where

$$L_g^2\big(\mathbb{R}^+, H_0^1(\Omega)\big) = \left\{u : \mathbb{R}^+ \to H_0^1(\Omega) \,/ \int_0^\infty g(s)\|\nabla u(x,s)\|_2^2 ds < \infty\right\}.$$

Definition 3.1. By a weak solution, we mean $(u,\ \eta^t)$ which satisfies (3.4) and

$$\frac{d}{dt}\int_\Omega u_t(x,t)\Phi(x)dx + l\int_\Omega \nabla u(x,t)\cdot\nabla\Phi(x)dx + \int_\Omega\int_0^\infty g(s)\nabla\eta^t(x,s)\cdot\nabla\Phi(x)dsdx$$

$$= \int_\Omega f(x,t)\Phi(x)dx\,, \quad \text{for } a.e\ t > 0$$

Proof of Theorem $\mathbf{3.1.1.}$ This theorem can be established by using the Galerkin method or the linear semigroup theory as in [54] or by repeating the steps of [23] with the necessary modification.

3.2 Modified Energy Functional

Multiply $(3.3)_1$ by u_t and integrate over Ω, we obtain

$$\int_{\Omega} u_t u_{tt} dx - l \int_{\Omega} u_t \Delta u(t) dx - \int_{\Omega} \int_{0}^{\infty} g(s) u_t \Delta \eta^t(s) ds dx = \int_{\Omega} u_t f(x,t) dx \tag{3.5}$$

The terms in (3.5) are estimated as follows:

First Term

$$\int_{\Omega} u_t u_{tt} dx = \frac{1}{2}\frac{d}{dt} \|u_t\|_2^2. \tag{3.6}$$

Second Term

Using Green's formula and the boundary conditions, we obtain

$$-l \int_{\Omega} u_t \Delta u(t) dx = l \int_{\Omega} \nabla u_t \cdot \nabla u(t) dx = \frac{l}{2}\frac{d}{dt} \|\nabla u(t)\|_2^2. \tag{3.7}$$

Third Term

Using Green's formula and the boundary conditions, we get

$$-\int_{\Omega} \int_{0}^{\infty} g(s) u_t \Delta \eta^t(s) ds dx = \int_{0}^{\infty} g(s) \int_{\Omega} \nabla u_t \cdot \nabla \eta^t(s) dx ds.$$

Using $(3.3)_2$, we have

$$-\int_\Omega \int_0^\infty g(s)u_t\Delta\eta^t(s)dsdx = \int_0^\infty g(s)\int_\Omega \{\nabla\eta_t^t(s)+\nabla\eta_s^t(s)\}\cdot\nabla\eta^t(s)dxds$$

$$= \int_0^\infty g(s)\int_\Omega \nabla\eta_t^t(s)\cdot\nabla\eta^t(s)dxds + \int_0^\infty g(s)\int_\Omega \nabla\eta_s^t(s)\cdot\nabla\eta^t(s)\,dxds$$

$$= \frac{1}{2}\int_0^\infty g(s)\frac{d}{dt}\int_\Omega |\nabla\eta^t(s)|^2dxds + \frac{1}{2}\int_0^\infty g(s)\frac{d}{ds}\int_\Omega |\nabla\eta^t(s)|^2dxds$$

$$= \frac{1}{2}\frac{d}{dt}\left(\int_0^\infty g(s)\|\nabla\eta^t(s)\|_2^2ds\right) + \frac{1}{2}\left[g(s)\|\nabla\eta^t(s)\|_2^2\Big|_0^\infty - \int_0^\infty g'(s)\|\nabla\eta^t(s)\|_2^2ds\right]$$

$$= \frac{1}{2}\frac{d}{dt}\left(\int_0^\infty g(s)\|\nabla\eta^t(s)\|_2^2ds\right) - \frac{1}{2}\int_0^\infty g'(s)\|\nabla\eta^t(s)\|_2^2ds. \qquad (3.8)$$

Fourth term

Using Young's inequality, we obtain for any $\delta_2 > 0$

$$\int_\Omega u_t f(x,t)\,dx \le \delta_2\|u_t\|_2^2 + \frac{1}{4\delta_2}\|f(x,t)\|_2^2. \qquad (3.9)$$

By substituting $(3.6)-(3.9)$ in (3.5), we obtain

$$\frac{d}{dt}\left(\frac{1}{2}\|u_t\|_2^2 + \frac{l}{2}\|\nabla u(t)\|_2^2 + \frac{1}{2}\int_0^\infty g(s)\|\nabla\eta^t(s)\|_2^2ds\right)$$

$$\le \frac{1}{2}\int_0^\infty g'(s)\,\|\nabla\eta^t(s)\|_2^2ds + \delta_2\|u_t\|_2^2 + \frac{1}{4\delta_2}\|f(x,t)\|_2^2. \qquad (3.10)$$

Set

$$E(t) = \frac{1}{2}\|u_t\|_2^2 + \frac{l}{2}\|\nabla u(t)\|_2^2 + \frac{1}{2}(go\nabla\eta^t)(t), \tag{3.11}$$

where

$$(go\nabla\eta^t)(t) = \int_0^\infty g(s)\|\nabla\eta^t(s)\|_2^2 ds.$$

E(t) is called the Modified Energy Functional

Hence (3.10) becomes

$$E'(t) \le \frac{1}{2}(g' o\nabla\eta^t)(t) + \delta_2\|u_t\|_2^2 + \frac{1}{4\delta_2}\|f(.,t)\|_2^2. \tag{3.12}$$

3.3 Decay of Solution

In this section we state and prove the main result in this chapter. For this purpose, we set

$$F(t) = E(t) + \epsilon_1\psi(t) + \epsilon_2\chi(t), \tag{3.13}$$

where ϵ_1 and ϵ_2 are positive constants to be chosen properly later and

$$\psi(t) = \int_\Omega u u_t dx, \qquad \chi(t) = -\int_\Omega u_t \int_0^\infty g(s)\eta^t(s)\, ds dx. \tag{3.14}$$

Lemma 3.3.1. For ϵ_1 and ϵ_2 small enough, the inequality

$$\alpha_1 F(t) \le E(t) \le \alpha_2 F(t) \tag{3.15}$$

holds for two positive constants α_1 and α_2.

Proof.

$$F(t) = E(t) + \epsilon_1 \int_{\Omega} u u_t dx - \epsilon_2 \int_{\Omega} u_t \int_0^{\infty} g(s)\eta^t(s)\, ds dx. \tag{3.16}$$

Using Young's inequality, we have

$$F(t) \le E(t) + \frac{\epsilon_1}{2}\|u(t)\|_2^2 + \frac{\epsilon_1 + \epsilon_2}{2}\|u_t\|_2^2 + \frac{\epsilon_2}{2}\int_{\Omega}\left(\int_0^{\infty} g(s)\eta^t(s)\, ds\right)^2 dx. \tag{3.17}$$

We estimate the fourth term in the right-hand side of (3.17) as follows:

$$\left(\int_0^{\infty} g(s)\eta^t(s)\, ds\right)^2 = \left(\int_0^{\infty} \sqrt{g(s)}\,\sqrt{g(s)}\eta^t(s)\, ds\right)^2.$$

Using Cauchy-Schwarz inequality, we have

$$\left(\int_0^{\infty} g(s)\eta^t(s)\, ds\right)^2 \le \left(\left(\int_0^{\infty} g(s)\, ds\right)^{\frac{1}{2}}\left(\int_0^{\infty} g(s)|\eta^t(s)|^2\, ds\right)^{\frac{1}{2}}\right)^2$$

$$\le \left(\int_0^{\infty} g(s)\, ds\right)\left(\int_0^{\infty} g(s)|\eta^t(s)|^2\, ds\right)$$

$$\left(\int_0^{\infty} g(s)\eta^t(s)\, ds\right)^2 \le (1-l)\left(\int_0^{\infty} g(s)|\eta^t(s)|^2\, ds\right). \tag{3.18}$$

We substitute (3.18) in (3.17), then use Poincaré's inequality, to obtain

$$F(t) \le E(t) + \frac{\epsilon_1 + \epsilon_2}{2}\|u_t\|_2^2 + \frac{\epsilon_1}{2}C_P\|\nabla u(t)\|_2^2 + \frac{\epsilon_2(1-l)}{2}C_P(go\nabla\eta^t)(t). \tag{3.19}$$

Note the following:

$$E(t) \geq \frac{1}{2}(go\nabla\eta^t)(t).$$

Thus,

$$\epsilon_2(1-l)C_P E(t) \geq \frac{\epsilon_2(1-l)}{2} C_P (go\nabla\eta^t)(t). \tag{3.20}$$

Similarly,

$$\frac{\epsilon_1}{l} C_P E(t) \geq \frac{\epsilon_1}{2} C_P \|\nabla u(t)\|_2^2 \tag{3.21}$$

and

$$(\epsilon_1+\epsilon_2)E(t) \geq \frac{(\epsilon_1+\epsilon_2)}{2}\|u_t\|_2^2. \tag{3.22}$$

Hence, equation (3.19) becomes

$$F(t) \leq E(t) + \frac{\epsilon_1 C_P}{l} E(t) + (\epsilon_1+\epsilon_2)E(t) + \epsilon_2(1-l)C_P E(t)$$

$$\leq \left(1 + \frac{\epsilon_1 C_P}{l} + (\epsilon_1+\epsilon_2) + \epsilon_2(1-l)C_P\right) E(t) \leq \beta_1 E(t),$$

where $\beta_1 = 1 + \frac{\epsilon_1 C_P}{l} + (\epsilon_1+\epsilon_2) + \epsilon_2(1-l)C_P$,

which implies

$$E(t) \geq \alpha_1 F(t), \quad \alpha_1 = \frac{1}{\beta_1}. \tag{3.23}$$

Similarly, from (3.16) using Young's inequality, we obtain

$$F(t) \geq E(t) - \frac{\epsilon_1}{2}\|u(t)\|_2^2 - \frac{\epsilon_1 + \epsilon_2}{2}\|u_t\|_2^2 - \frac{\epsilon_2}{2}\int_\Omega \left(\int_0^\infty g(s)\eta^t(s)\,ds\right)^2 dx.$$

Using (3.18), we have

$$F(t) \geq E(t) - \frac{\epsilon_1}{2}\|u(t)\|_2^2 - \frac{\epsilon_1 + \epsilon_2}{2}\|u_t\|_2^2 - \frac{\epsilon_2(1-l)}{2}\left(\int_0^\infty g(s)|\eta^t(s)|^2\,ds\right).$$

Using Poincaré's inequality, we get

$$F(t) \geq E(t) - \frac{\epsilon_1 + \epsilon_2}{2}\|u_t\|_2^2 - \frac{\epsilon_1}{2}C_P\|\nabla u(t)\|_2^2 - \frac{\epsilon_2(1-l)}{2}C_P(go\nabla\eta^t)(t). \qquad (3.24)$$

Substituting (3.20) − (3.22) in (3.24), we obtain

$$F(t) \geq \left(1 - \frac{\epsilon_1 C_P}{l} - (\epsilon_1 + \epsilon_2) - \epsilon_2(1-l)C_P\right)E(t). \qquad (3.25)$$

By choosing ϵ_1 and ϵ_2 small enough so that

$$\alpha = \frac{\epsilon_1 C_P}{l} + (\epsilon_1 + \epsilon_2) + \epsilon_2(1-l)C_P < 1,$$

we arrive at

$$F(t) \geq (1-\alpha)E(t).$$

Hence,

$$E(t) \leq \alpha_2 F(t), \quad \alpha_2 = \frac{1}{1-\alpha}. \qquad (3.26)$$

Combination of (3.23) and (3.26) gives

$$\alpha_1 F(t) \leq E(t) \leq \alpha_2 F(t),$$

which completes the proof.

Lemma 3.3.2 Let (u, u_t, η^t) be the solution of (3.3). Then under the assumptions (G_1) and (G_2), the functional

$$\psi(t) = \int_\Omega u u_t dx$$

satisfies, for any $\delta_1 > 0$,

$$\psi'(t) \le \|u_t\|_2^2 - \left\{\frac{l}{2} - \delta_1 C_P\right\} \|\nabla u(t)\|_2^2 + \frac{1}{2}\left(\frac{1-l}{l}\right)(go\nabla\eta^t)(t) + \frac{1}{4\delta_1}\|f(.,t)\|_2^2. \tag{3.27}$$

Proof.

By taking the derivative of ψ and using $(3.3)_1$, we get

$$\psi'(t) = \|u_t\|_2^2 + l\int_\Omega u(t)\Delta u(t)dx + \int_\Omega u(t)\int_0^\infty g(s)\,\Delta\eta^t(s)dsdx + \int_\Omega u(t)f(x,t)dx$$

Using Green's formula, the boundary conditions and Young's inequality, we obtain, for any $\delta_1 > 0$

$$\psi'(t) \le \|u_t\|_2^2 - l\|\nabla u(t)\|_2^2 - \int_\Omega \nabla u(t)\cdot\int_0^\infty g(s)\,\nabla\eta^t(s)dsdx + \delta_1\|u(t)\|_2^2$$

$$+\frac{1}{4\delta_1}\|f(.,t)\|_2^2.$$

By Poincaré's inequality, we have

$$\psi'(t) \le \|u_t\|_2^2 - \left(l - \delta_1 C_p\right)\|\nabla u(t)\|_2^2 - \int_\Omega \nabla u(t) \cdot \int_0^\infty g(s)\, \nabla\eta^t(s) ds dx + \frac{1}{4\delta_1}\|f(.,t)\|_2^2.$$

(3.28)

By estimating the third term in (3.28), using Young's inequality, we obtain for any $\delta_3 > 0$

$$-\int_\Omega \nabla u(t) \cdot \int_0^\infty g(s)\, \nabla\eta^t(s) ds dx \le \delta_3 \|\nabla u(t)\|_2^2 + \frac{1}{4\delta_3}\int_\Omega \left(\int_0^\infty g(s)|\nabla\eta^t(s)| ds\right)^2 dx.$$

Using (3.19), we get

$$\int_\Omega \nabla u(t) \cdot \int_0^\infty g(s)\, \nabla\eta^t(s) ds dx \le \delta_3 \|\nabla u(t)\|_2^2 + \frac{(1-l)}{4\delta_3}\int_\Omega \int_0^\infty g(s)|\nabla\eta^t(s)|^2 \, ds dx.$$

(3.29)

By substituting (3.29) in (3.28), we obtain

$$\psi'(t) \le \|u_t\|_2^2 - \left\{(l - \delta_3) - \delta_1 C_p\right\}\|\nabla u(t)\|_2^2 + \frac{(1-l)}{4\delta_3}(go\nabla\eta^t)(t) + \frac{1}{4\delta_1}\|f(.,t)\|_2^2.$$

By choosing $\delta_3 = \dfrac{l}{2}$, we get

$$\psi'(t) \le \|u_t\|_2^2 - \left\{\frac{l}{2} - \delta_1 C_p\right\}\|\nabla u(t)\|_2^2 + \frac{1-l}{2l}(go\nabla\eta^t)(t) + \frac{1}{4\delta_1}\|f(.,t)\|_2^2,$$

(3.30)

which complete the proof.

Lemma 3.3.3 Let (u, u_t, η^t) be the solution of (3.3). Then under the assumptions (G_1) and (G_2), the functional

$$\chi(t) = -\int_{\Omega} u_t \int_0^{\infty} g(s)\eta^t(s)\, ds dx$$

satisfies, for any $\delta,\ \delta_4 > 0$,

$$\chi'(t) \leq \{\delta - (1-l)\}\|u_t\|_2^2 + l^2\delta_4\|\nabla u(t)\|_2^2 + \left\{\left(1 + \frac{1}{4\delta_4} + \frac{C_p}{4\delta}\right)(1-l)\right\}(go\nabla\eta^t)(t)$$

$$-\frac{g(0)}{4\delta} C_P\left(g' o\nabla\eta^t\right)(t) + \delta\|f(.,t)\|_2^2 \tag{3.31}$$

Proof.

By taking the derivative of χ and using $(3.3)_1$, we get

$$\chi'(t) = -l\int_{\Omega} \Delta u(t) \int_0^{\infty} g(s)\,\eta^t(s) ds dx$$

$$-\int_{\Omega}\left(\int_0^{\infty} g(s)\,\Delta\eta^t(s) ds\right)\left(\int_0^{\infty} g(s)\,\eta^t(s) ds\right) dx$$

$$-\int_{\Omega} u_t \int_0^{\infty} g(s)\eta_t^t(s)\, ds dx - \int_{\Omega} f(x,t)\int_0^{\infty} g(s)\eta^t(s)\, ds dx.$$

Using Green's formula and the boundary conditions, we obtain

$$\chi'(t) = l\int_{\Omega} \nabla u(t)\cdot \int_0^{\infty} g(s)\,\nabla\eta^t(s) ds dx + \int_{\Omega}\left(\int_0^{\infty} g(s)\,\nabla\eta^t(s) ds\right)^2 dx$$

$$-\int_{\Omega} u_t \int_0^{\infty} g(s)\eta_t^t(s)\, dsdx - \int_{\Omega} f(x,t) \int_0^{\infty} g(s)\eta^t(s)\, dsdx. \tag{3.32}$$

The terms in (3.32) are estimated below.

The first Term

Using Young's inequality and Cauchy-Schwarz inequality, we obtain, for any $\delta_4 > 0$,

$$l \int_{\Omega} \nabla u(t) \cdot \int_0^{\infty} g(s)\, \nabla\eta^t(s) dsdx \leq l^2 \delta_4 \|\nabla u(t)\|_2^2 + \frac{(1-l)}{4\delta_4}(go\nabla\eta^t)(t). \tag{3.33}$$

The second Term

Using Cauchy-Schwarz inequality, we obtain

$$\int_{\Omega} \left(\int_0^{\infty} g(s)\, \nabla\eta^t(s) ds \right)^2 dx \leq (1-l)(go\nabla\eta^t)(t). \tag{3.34}$$

The third Term

Using $(3.3)_2$, integration by parts and the initial conditions, we get

$$-\int_{\Omega} u_t \int_0^{\infty} g(s)\eta_t^t(s)\, dsdx = -(1-l)\|u_t\|_2^2 + \int_{\Omega} u_t \left[g(s)\eta^t(s)|_0^{\infty} - \int_0^{\infty} g'(s)\eta^t(s)\, ds \right] dx$$

$$= -(1-l)\|u_t\|_2^2 - \int_{\Omega} u_t \int_0^{\infty} g'(s)\eta^t(s)\, dsdx.$$

Using Young's inequality, we obtain for any $\delta > 0$,

$$-\int_{\Omega} u_t \int_0^{\infty} g(s)\eta_t^t(s)\, dsdx \leq \{\delta - (1-l)\}\|u_t\|_2^2 + \frac{1}{4\delta} \int_{\Omega} \left(\int_0^{\infty} g'(s)\, \eta^t(s) ds \right)^2 dx$$

$$\leq \{\delta - (1-l)\}\|u_t\|_2^2 + \frac{1}{4\delta}\int_\Omega \left(\int_0^\infty \sqrt{-g'(s)}\sqrt{-g'(s)}\eta^t(s)ds\right)^2 dx.$$

Using Cauchy-Schwarz inequality, we obtain

$$-\int_\Omega u_t \int_0^\infty g(s)\eta_t^t(s)\,dsdx \leq \{\delta - (1-l)\}\|u_t\|_2^2 + \frac{g(0)}{4\delta}\int_\Omega \int_0^\infty -g'(s)\,|\eta^t(s)|^2 dsdx.$$

By Poincaré's inequality, we have

$$-\int_\Omega u_t \int_0^\infty g(s)\eta_t^t(s)\,dsdx \leq \{\delta - (1-l)\}\|u_t\|_2^2 - \frac{g(0)}{4\delta}C_P(g'o\nabla\eta^t)(t). \qquad (3.35)$$

The fourth Term

Using Young's inequality, we obtain, for all $\delta > 0$,

$$-\int_\Omega f(x,t)\int_0^\infty g(s)\eta^t(s)\,dsdx \leq \delta\|f(.,t)\|_2^2 + \frac{1}{4\delta}\int_\Omega \left(\int_0^\infty g(s)\eta^t(s)\,dsdx\right)^2 dx.$$

Using Cauchy-Schwarz inequality and Poincaré's inequality, we obtain

$$-\int_\Omega f(x,t)\int_0^\infty g(s)\eta^t(s)\,dsdx \leq \delta\|f(.,t)\|_2^2 + \frac{1-l}{4\delta}C_P(go\nabla\eta^t)(t). \qquad (3.36)$$

By substituting (3.33), (3.34), (3.35) and (3.36), in (3.32), we obtain

$$\chi'(t) \leq \{\delta - (1-l)\}\|u_t\|_2^2 + l^2\delta_4\|\nabla u(t)\|_2^2$$

$$+\left\{\left(1+\frac{1}{4\delta_4}+\frac{C_p}{4\delta}\right)(1-l)\right\}(go\nabla\eta^t)(t) - \frac{g(0)}{4\delta}C_P(g'o\nabla\eta^t)(t) + \delta\|f(.,t)\|_2^2,$$

which completes the proof.

THEOREM 3.3.1. Let $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ and $\eta_0 \in L_g^2\left(\mathbb{R}^+, H_0^1(\Omega)\right)$ be given. Assume that $f \in L^2\left(\Omega \times (0, +\infty)\right)$ and g satisfies (G_1) and(G_2). Then there exist strictly positive constants K_1 and K_2 such that, for all $t \geq 0$,

$$E(t) \leq K_1 e^{-kt} + K_2 e^{-kt} \int_0^t \|f(.,s)\|_2^2 \, e^{ks} ds, \quad \forall t \geq 0. \tag{3.37}$$

Proof

Taking the derivative of F, then substituting (3.12), (3.27), (3.31) and , we obtain

$$F'(t) \leq -\{\varepsilon_2[(1-l) - \delta] - \varepsilon_1 - \delta_2\}\|u_t\|_2^2 - \left\{\varepsilon_1 \left(\frac{l}{2} - \delta_1 C_P\right) - \varepsilon_2 l^2 \delta_4\right\} \|\nabla \mathrm{u}(t)\|_2^2$$

$$+ \left\{\left(\frac{\varepsilon_1}{2l} + \varepsilon_2 \left(1 + \frac{1}{4\delta_4} + \frac{C_P}{4\delta}\right)\right)(1-l)\right\}(go\nabla\eta^t)(t)$$

$$+ \left\{\left(\frac{1}{2} - \varepsilon_2 \frac{g(0)}{4\delta} C_P\right)\right\}(g' o\nabla\eta^t)(s) + \left\{\frac{1}{4\delta_2} + \frac{\varepsilon_1}{4\delta_1} + \varepsilon_2\delta\right\}\|f(.,t)\|_2^2.$$

Using (G_2), we get

$$F'(t) \leq -\{\varepsilon_2[(1-l) - \delta] - \varepsilon_1 - \delta_2\}\|u_t\|_2^2 - \left\{\varepsilon_1 \left(\frac{l}{2} - \delta_1 C_P\right) - \varepsilon_2 l^2 \delta_4\right\} \|\nabla \mathrm{u}(t)\|_2^2$$

$$- \left\{\xi\left(\frac{1}{2} - \varepsilon_2 \frac{g(0)}{4\delta} C_P\right) - \left[\frac{\varepsilon_1}{2l} + \varepsilon_2 \left(1 + \frac{1}{4\delta_4} + \frac{C_P}{4\delta}\right)\right](1-l)\right\}(go\nabla\eta^t)(t)$$

$$+ \left\{\frac{1}{4\delta_2} + \frac{\varepsilon_1}{4\delta_1} + \varepsilon_2\delta\right\}\|f(.,t)\|_2^2.$$

By choosing $\delta_1 = \dfrac{l}{4C_P}$, $\delta_4 = \dfrac{\delta}{l^2}$, we obtain

$$F'(t) \le -\{\varepsilon_2[(1-l)-\delta]-\varepsilon_1-\delta_2\}\|u_t\|_2^2-\left\{\frac{\varepsilon_1 l}{4}-\varepsilon_2\delta\right\}\|\nabla \mathrm{u}(t)\|_2^2$$

$$-\left\{\xi\left(\frac{1}{2}-\varepsilon_2\frac{g(0)}{4\delta}C_P\right)-\left[\frac{\varepsilon_1}{2l}+\varepsilon_2\left(1+\frac{l^2}{4\delta}+\frac{C_P}{4\delta}\right)\right](1-l)\right\}(go\nabla\eta^t)(t)$$

$$+\left\{\frac{1}{4\delta_2}+\frac{\varepsilon_1 C_P}{l}+\varepsilon_2\delta\right\}\|f(.,t)\|_2^2. \qquad (3.38)$$

Now choose δ so small that,

$$(1-l)-\delta > \frac{1}{2}(1-l),$$

$$\frac{4}{l}\delta < \frac{1}{4}(1-l).$$

Whence δ is fixed, any choice of ε_1 and ε_2, such that

$$\frac{(1-l)}{4}\varepsilon_2 < \varepsilon_1 < \frac{(1-l)}{2}\varepsilon_2 \qquad (3.39)$$

will make

$$\varepsilon_2\big((1-l)-\delta\big) > \frac{\varepsilon_2(1-l)}{2} > \varepsilon_1,$$

$$\frac{4}{l}\varepsilon_2\delta < \frac{\varepsilon_2(1-l)}{4} < \varepsilon_1.$$

Thus, we have

$$K_0 = \varepsilon_2[(1-l)-\delta]-\varepsilon_1 > 0,$$

$$K_1 = \frac{\varepsilon_1 l}{4}-\varepsilon_2\delta > 0.$$

We then pick ε_1 and ε_2 so small that (3.15) and (3.39) remain valid and

$$K_3 = \left\{\xi\left(\frac{1}{2} - \varepsilon_2 \frac{g(0)}{4\delta} C_P\right) - \left[\frac{\varepsilon_1}{2l} + \varepsilon_2\left(1 + \frac{l^2}{4\delta} + \frac{C_P}{4\delta}\right)\right](1-l)\right\} > 0$$

and take δ_2 so small that

$$K_2 = K_0 - \delta_2 > 0.$$

Finally, we choose $\beta = \min\{K_1, K_2, K_3\}$ to obtain

$$F'(t) \leq -\beta\{\|u_t\|_2^2 + \|\nabla \mathrm{u}(t)\|_2^2 + (go\nabla\eta^t)(t)\} + C\|f(.,t)\|_2^2, \tag{3.40}$$

where $C = \dfrac{1}{4\delta_2} + \dfrac{\varepsilon_1 C_P}{l} + \varepsilon_2\delta.$

A combination of (3.15) and (3.40) yields

$$F'(t) \leq -kF(t) + C\|f(.,t)\|_2^2, \qquad k = \beta\alpha_1 \quad \forall t \geq 0. \tag{3.41}$$

We introduce the following functional:

$$H(t) = F(t) - Ce^{-kt}\int_0^t \|f(.,s)\|_2^2\, e^{ks} ds \tag{3.42}$$

Taking the derivative of (3.42), we obtain

$$H'(t) = F'(t) + kCe^{-kt}\int_0^t \|f(.,s)\|_2^2\, e^{ks} ds - Ce^{-kt}\|f(.,t)\|_2^2 e^{kt},$$

which implies that,

$$H'(t) = F'(t) + kCe^{-kt}\int_0^t \|f(.,s)\|_2^2\, e^{ks} ds - C\|f(.,t)\|_2^2 \tag{3.43}$$

By substituting (3.42) and (3.43) in (3.41), we obtain

$$H'(t) \leq -kH(t). \tag{3.44}$$

A simple integration of (3.44) over $(0,\ t)$ gives

$$H(t) \leq H(0)e^{-kt}. \tag{3.45}$$

Using (3.42), we obtain

$$F(t) \leq \left\{ F(0) + C \int_0^t \|f(.,s)\|_2^2 \, e^{ks} ds \right\} e^{-kt}. \tag{3.46}$$

A combination of (3.15) and (3.46) gives

$$E(t) \leq K_1 e^{-kt} + K_2 e^{-kt} \int_0^t \|f(.,s)\|_2^2 \, e^{ks} ds, \quad \forall t \geq 0, \tag{3.47}$$

where

$$K_1 = \alpha_2 F(0), \qquad K_2 = C\alpha_2.$$

Thus the estimate (3.37) is proved.

Remark 3.3.1

1. If $f \equiv 0$, then $E(t) \leq K_1 e^{-kt}, \quad \forall t \geq 0$
2. If $\|f\|_2^2 \leq M$, then $E(t) \leq K_1 e^{-kt} + \lambda, \quad \forall t \geq 0$ where $\lambda = \frac{K_2 M}{k}$
3. If $\|f\|_2^2 \leq M e^{-\gamma t}$, then

$$E(t) \leq K_1 e^{-kt} + \lambda_1 e^{-\gamma t}, \quad \forall t \geq 0, \text{where } \lambda_1 = \frac{K_2 M}{k - \gamma} \text{ if } K \neq \gamma$$

$$* \quad E(t) \leq (K_1 + \lambda_2 t) e^{-\gamma t}, \quad \forall t \geq 0, \text{where } \lambda_2 = K_2 M \text{ if } K = \gamma$$

In $*$, $E(t)$ does not necessarily converge to zero as $e^{-\gamma t}$ when t goes to $+\infty$.

CHAPTER 4

POLYNOMIAL DECAY OF SOLUTION OF A VISCOELASTIC PROBLEM

4.1 Introduction

In this chapter, we consider the viscoelastic problem

$$\begin{cases} u_{tt} - \Delta u(x,t) + \int_0^{\infty} g(s)\Delta u(t-s)ds = 0 \quad \text{in } \Omega \times \mathbb{R}^+ \\ u(x,t) = 0, \qquad x \in \partial\Omega, \qquad t \in \mathbb{R}^+ \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \quad x \in \Omega \end{cases} \tag{4.1}$$

where Ω is a bounded domain of $\mathbb{R}^n (n \geq 1)$ with a smooth boundary $\partial\Omega$ and g is a positive non increasing function satisfying the following conditions

(G_1) $g: \mathbb{R}^+ \to \mathbb{R}^+$ is a differentiable function such that

$$g(0) > 0, \qquad 1 - \int_0^{\infty} g(s)ds = l > 0$$

(G_2) There exist a positive constant ξ and $1 < p < \frac{3}{2}$ such that

$$g'(t) \leq -\xi g^p(t), \quad \forall t \geq 0.$$

Following the idea of Dafermos [20], we introduce

$$\eta^t(x,s) = u(x,t) - u(x,t-s), \qquad s \geq 0. \tag{4.2}$$

Consequently, by adding and subtracting the term Δu, (4.1) transforms into

$$\begin{cases} u_{tt} - l\Delta u(x,t) - \int_0^\infty g(s)\Delta\eta^t(x,s)ds = f(x,t), \text{ in } \Omega \times \mathbb{R}^+ \\ \eta_t^t(x,s) + \eta_s^t(x,s) = u_t(x,t) \\ u(x,t) = \eta^t(x,s) = 0, \qquad x \in \partial\Omega, \qquad \forall\, t,\ \ s \geq 0 \\ u(x,0) = u_0(x),\ \ u_t(x,0) = u_1(x),\ \ \forall x \in \Omega \\ \eta^0(x,s) = \eta_0(x,s) = u_0(x) - v_0(x,-s), \eta^t(x,0) = 0, \forall x \in \Omega,\ \ s \geq 0 \end{cases} \tag{4.3}$$

Theorem 4.1.1. Let $(u_0, u_1) \in \left(H_0^1(\Omega) \times L^2(\Omega)\right)$ and $\eta_0 \in L_g^2\left(\mathbb{R}^+,\ H_0^1(\Omega)\right)$ be given. Assume that g satisfies (G_1) and (G_2), then problem (4.3) has a unique global weak solution

$$u \in C\left([0,\infty); H_0^1(\Omega)\right),\ \ u_t \in C\left([0,\infty); L^2(\Omega)\right),\ \ \eta^t \in L_g^2\left(\mathbb{R}^+ \times \mathbb{R}^+, H_0^1(\Omega)\right), \tag{4.4}$$

where

$$L_g^2\left(\mathbb{R}^+, H_0^1(\Omega)\right) = \left\{u : \mathbb{R}^+ \to H_0^1(\Omega)\ /\int_0^\infty g(s)\|\nabla u(x,s)\|_2^2 ds < \infty\right\}$$

Proof. This result can be established by using the Galerkin method or the linear semigroup theory as in [54] or by repeating the steps of [23] with the necessary modification.

4.2 Modified Energy Functional

The modified energy functional $E(t)$ is already obtained in chapter 3. Thus we have

$$E(t) = \frac{1}{2}\|u_t\|_2^2 + \frac{l}{2}\|\nabla \mathrm{u}(t)\|_2^2 + \frac{1}{2}(go\nabla\eta^t)(t). \qquad (4.5)$$

Remark 4.2.1. The modified energy functional $E(t)$ satisfies, along the solution of (4.3),

$$E'(t) = \frac{1}{2}(g'o\nabla\eta^t)(t) \leq 0. \qquad (4.6)$$

Lemma 4.2.1. Let (u, u_t, η^t) be a solution of (4.3). Then, there exists a constant $C > 0$ such that, for $1 < p < \frac{3}{2}$ we have,

$$(go\nabla\eta^t)(t) \leq C\{(g^p o\nabla\eta^t)(s)\}^{\frac{1}{2p-1}}.$$

Proof:

$$(go\nabla\eta^t)(t) = \int_0^\infty g(t)\,\|\nabla\eta^t(s)\|_2^2 ds$$

$$= \int_0^\infty g^{\frac{1}{2r}}(s)\,\|\nabla\eta^t(s)\|_2^{\frac{2}{r}} g^{\frac{2r-1}{2r}}(s)\|\nabla\eta^t(s)\|_2^{\frac{2r-2}{r}} ds.$$

Using Hölder's inequality, we obtain, for any $r > 1$,

$$(go\nabla\eta^t)(t) \leq \left(\int_0^\infty g^{\frac{1}{2}}(s)\,\|\nabla\eta^t(s)\|_2^2 ds\right)^{\frac{1}{r}} \left(\int_0^\infty g^{\frac{2r-1}{2r-2}}(s)\,\|\nabla\eta^t(s)\|_2^2 ds\right)^{\frac{r-1}{r}}.$$

By taking $r = \frac{2p-1}{2p-2}$, we obtain

$$(go\nabla\eta^t)(t) \le \left(\int_0^\infty g^{\frac{1}{2}}(s)\,\|\nabla\eta^t(s)\|_2^2 ds\right)^{\frac{2p-2}{2p-1}} \left(\int_0^\infty g^p(s)\,\|\nabla\eta^t(s)\|_2^2 ds\right)^{\frac{1}{2p-1}}. \quad (4.7)$$

We estimate the first term in the right-hand side of (4.7) as follows:

$$\int_0^\infty g^{\frac{1}{2}}(s)\,\|\nabla\eta^t(s)\|_2^2 ds = \int_0^\infty g^{\frac{1}{2}}(s)\,\|\nabla u(x,t) - \nabla u(x,t-s)\|_2^2 ds.$$

Using the fact that

$$\|\nabla u(x,t)\|_2^2 \le \frac{2}{l}E(t) \le \frac{2}{l}E(0),$$

$$\|\nabla u(x,t-s)\|_2^2 \le \frac{2}{l}E(t-s) \le \frac{2}{l}E(0)$$

and

$$|\mathrm{a} - b|^2 \le 2(|a|^2 + |b|^2),$$

we obtain

$$\int_0^\infty g^{\frac{1}{2}}(s)\,\|\nabla\eta^t(s)\|_2^2 ds \le \frac{8E(0)}{l}\int_0^\infty g^{\frac{1}{2}}(s)\,ds.$$

Note that,

$$\int_0^\infty g^{\frac{1}{2}}(s)\,ds \le \int_0^\infty g^{\frac{1}{2}-p}(s) g^p(s)\,ds.$$

Hence using G_2, we obtain

$$\int_0^\infty g^{\frac{1}{2}}(s)\,ds \le -\frac{1}{\xi}\int_0^\infty g^{\frac{1}{2}-p}(s)g'(s)\,ds \le -\frac{2}{\xi(3-2p)}g^{\frac{3-2p}{2}}(s)\Big|_0^\infty < +\infty.$$

Consequently, for a constant $C > 0$, (4.7) becomes

$$(go\nabla\eta^t)(t) \le C\{(g^p o\nabla\eta^t)(t)\}^{\frac{1}{2p-1}},$$

which completes the proof.

4.3 Decay of Solution

In this section we state and prove the main result. For this purpose, we set

$$F(t) = E(t) + \varepsilon_1\psi(t) + \varepsilon_2\chi(t), \tag{4.8}$$

where ε_1 and ε_2 are positive constants to be chosen properly later and

$$\psi(t) = \int_\Omega uu_t dx, \qquad \chi(t) = -\int_\Omega u_t \int_0^\infty g(s)\eta^t(s)\,dsdx.$$

Lemma 4.3.1. For ϵ_1 and ϵ_2 small enough, the inequality

$$\alpha_1 F(t) \le E(t) \le \alpha_2 F(t). \tag{4.9}$$

Proof. For the proof of this Lemma, see the proof of Lemma 3.3.1

Lemma 4.3.2. Let (u, u_t, η^t) be the solution of (4.3). Then under the assumptions (G_1) and (G_2), the functional

$$\psi(t) = \int_\Omega uu_t dx$$

satisfies

$$\psi'(t) \le \|u_t\|_2^2 - \frac{l}{2}\|\nabla \mathrm{u}(t)\|_2^2 + \frac{1}{2l}\left(\int_0^\infty g^{2-p}(s)\,ds\right)(g^p o \nabla\eta^t)(t). \qquad (4.10)$$

Proof.

By taking the derivative of ψ and using $(4.3)_1$, we obtain

$$\psi'(t) = \|u_t\|_2^2 + l\int_\Omega u(t)\Delta u(t)dx + \int_\Omega u(t)\int_0^\infty g(s)\,\Delta\eta^t(s)dsdx.$$

Using Green's formula, the boundary conditions and Young's inequality, we obtain, for any $\delta_1 > 0$,

$$\psi'(t) \le \|u_t\|_2^2 - (l-\delta_1)\|\nabla u(t)\|_2^2 + \frac{1}{4\delta_1}\int_\Omega\left(\int_0^\infty g(s)|\nabla\eta^t(s)|ds\right)^2 dx. \quad (4.11)$$

Using Cauchy-Schwarz inequality, the estimation of third term in right-hand side of (4.11) becomes

$$\int_\Omega\left(\int_0^\infty g(s)|\nabla\eta^t(s)|ds\right)^2 dx$$

$$\le \int_\Omega\left\{\left(\int_0^\infty\left(g^{1-\frac{p}{2}}(s)\right)^2 ds\right)^{\frac{1}{2}}\left(\int_0^\infty\left(g^{\frac{p}{2}}(s)\right)^2|\nabla\eta^t(s)|^2ds\right)^{\frac{1}{2}}\right\}^2 dx$$

$$\int_{\Omega}\left(\int_0^{\infty} g(s)|\nabla\eta^t(s)|ds\right)^2 dx \le \left(\int_0^{\infty} g^{2-p}(s)ds\right)(g^p o\nabla\eta^t)(t), \tag{4.12}$$

where

$$(g^p o\nabla\eta^t)(t) = \int_0^{\infty} g^p(s)\|\nabla\eta^t(s)\|_2^2 ds.$$

Substituting (4.12) in (4.11), we obtain

$$\psi'(t) \le \|u_t\|_2^2 - (l-\delta_1)\|\nabla u(t)\|_2^2 + \frac{1}{4\delta_1}\left(\int_0^{\infty} g^{2-p}(s)ds\right)(g^p o\nabla\eta^t)(t).$$

By choosing $\delta_1 = \frac{l}{2}$, we obtain

$$\psi'(t) \le \|u_t\|_2^2 - \frac{l}{2}\|\nabla u(t)\|_2^2 + \frac{1}{2l}\left(\int_0^{\infty} g^{2-p}(s)ds\right)(g^p o\nabla\eta^t)(t),$$

which completes the proof.

Lemma 4.3.3 Let (u, u_t, η^t) be the solution of (4.3). Then under the assumptions (G_1) and (G_2), the functional

$$\chi(t) = -\int_{\Omega} u_t \int_0^{\infty} g(s)\eta^t(s)\, ds dx$$

satisfies, for any $\delta,\ \delta_2 > 0$

$$\chi'(t) \le \{\delta - (1-l)\}\|u_t\|_2^2 + l^2\delta_2\|\nabla u(t)\|_2^2$$

$$+\left\{\left(1+\frac{1}{4\delta_2}\right)\left(\int_0^{\infty} g^{2-p}(s)ds\right)\right\}(g^p o\nabla\eta^t)(t) - \frac{g(0)}{4\delta}C_P(g' o\nabla\eta^t)(t). \tag{4.13}$$

Proof.

By taking the derivative of χ and using $(4.3)_1$, we obtain

$$\chi'(t) = -l\int_{\Omega} \Delta u(t)\int_0^{\infty} g(s)\,\eta^t(s)dsdx$$

$$-\int_{\Omega}\left(\int_0^{\infty} g(s)\,\Delta\eta^t(s)ds\right)\left(\int_0^{\infty} g(s)\,\eta^t(s)ds\right)dx$$

$$-\int_{\Omega} u_t\int_0^{\infty} g(s)\eta_t^t(s)\,dsdx.$$

Using Green's formula and the boundary conditions, we obtain

$$\chi'(t) = l\int_{\Omega} \nabla u(t)\cdot\int_0^{\infty} g(s)\,\nabla\eta^t(s)dsdx + \int_{\Omega}\left(\int_0^{\infty} g(s)\,\nabla\eta^t(s)ds\right)^2 dx$$

$$-\int_{\Omega} u_t\int_0^{\infty} g(s)\eta_t^t(s)\,dsdx. \quad (4.14)$$

The terms in (4.14) are estimated below.

The first Term

Using Young's inequality and Cauchy-Schwarz inequality, we obtain for any $\delta_2 > 0$

$$l\int_{\Omega} \nabla u(t)\cdot\int_0^{\infty} g(s)\,\nabla\eta^t(s)dsdx \le l^2\delta_2\|\nabla u(t)\|_2^2$$

$$+\frac{1}{4\delta_2}\left(\int_0^{\infty} g^{2-p}(s)\,ds\right)(g^p o\nabla\eta^t)(t). \quad (4.15)$$

The second Term

This is the same as (4.12), hence we have

$$\int_{\Omega}\left(\int_{0}^{\infty} g(s)\,\nabla\eta^{t}(s)ds\right)^{2} dx \leq \left(\int_{0}^{\infty} g^{2-p}(s)ds\right)(g^{p}o\nabla\eta^{t})(t). \tag{4.16}$$

The third Term

This is the same as (3.35), hence we have,

$$-\int_{\Omega} u_t \int_{0}^{\infty} g(s)\eta_t^t(s)\,dsdx \leq \{\delta-(1-l)\}\|u_t\|_2^2 - \frac{g(0)}{4\delta}C_P(g^{'}o\nabla\eta^{t})(t). \tag{4.17}$$

By substituting $(4.15)-(4.17)$ in (4.14), we obtain

$$\chi'(t) \leq \{\delta-(1-l)\}\|u_t\|_2^2 + l^2\delta_2\|\nabla u(t)\|_2^2$$

$$+\left\{\left(1+\frac{1}{4\delta_2}\right)\left(\int_{0}^{\infty} g^{2-p}(s)ds\right)\right\}(g^{p}o\nabla\eta^{t})(t) - \frac{g(0)}{4\delta}C_P(g'o\nabla\eta^{t})(t),$$

which completes the proof.

Theorem 4.3.1 Let $(u_0, u_1) \in H_0^1(\Omega)\times L^2(\Omega)$ and $\eta_0 \in L_g^2\big(\mathbb{R}^+, H_0^1(\Omega)\big)$ be given. Assume that g satisfies (G_1) and(G_2). Then there exist strictly positive constants K such that, for all $t\geq 0$,

$$E(t) \leq K(1+t)^{\frac{-1}{2(p-1)}}, \qquad \forall t \geq 0 \tag{4.18}$$

Proof.

We take the derivative of F and substitute (4.6), (4.10) and (4.13), to obtain

$$F'(t) \le -\{\epsilon_2[(1-l)-\delta]-\epsilon_1\}\|u_t\|_2^2 - \left\{\frac{\epsilon_1 l}{2} - \epsilon_2 l^2 \delta_2\right\}\|\nabla u(t)\|_2^2$$

$$+\left\{\left(\frac{\epsilon_1}{2l}+\epsilon_2\left(1+\frac{1}{4\delta_2}\right)\right)\left(\int_0^\infty g^{2-p}(s)ds\right)\right\}(g^p o \nabla\eta^t)(t)$$

$$+\left(\frac{1}{2}-\epsilon_2\frac{g(0)C_P}{4\delta}\right)(g' o \nabla\eta^t)(t).$$

By using (G_2) and letting $\delta_2 = \dfrac{\delta}{l^2}$, we get

$$F'(t) \le -\{\epsilon_2[(1-l)-\delta]-\epsilon_1\}\|u_t\|_2^2 - \left\{\frac{\epsilon_1 l}{2} - \epsilon_2 \delta\right\}\|\nabla u(t)\|_2^2$$

$$-\left\{\xi\left(\frac{1}{2}-\epsilon_2\frac{g(0)C_P}{4\delta}\right)-\left(\frac{\epsilon_1}{2l}+\epsilon_2\left(1+\frac{1}{4\delta_2}\right)\right)\left(\int_0^\infty g^{2-p}(s)ds\right)\right\}(g^p o \nabla\eta^t)(s). \tag{4.19}$$

Note that $g^{2-p}(s) = g^{2(1-p)}(s)g^p(s)$.

So,

$$g^p(s) \le -\frac{1}{\xi}\, g'(s).$$

Consequently, we have

$$g^{2-p}(s) \le -\frac{1}{\xi} g^{2(1-p)}(s) g'(s). \tag{4.20}$$

A simple integration of (4.20) over $(-\infty,\ t)$ yields

$$\int_0^\infty g^{2-p}(s)ds \le \frac{1}{\xi}\frac{g^{3-2p}(0)}{3-2p} < +\infty, \quad \forall p < \frac{3}{2}. \tag{4.21}$$

We now choose δ so small that

$$(1-l)-\delta > \frac{1}{2}(1-l),$$

$$\frac{4}{l}\delta < \frac{1}{4}(1-l).$$

Whence δ is fixed, any choice of ϵ_1 and ϵ_2, such that

$$\frac{\epsilon_2(1-l)}{4} < \epsilon_1 < \frac{\epsilon_2(1-l)}{2}, \qquad (4.22)$$

makes

$$\frac{4}{l}\varepsilon_2\delta < \frac{\varepsilon_2(1-l)}{4} < \varepsilon_1,$$

$$\frac{4}{l}\varepsilon_2\delta < \frac{\varepsilon_2(1-l)}{4} < \varepsilon_1.$$

Consequently, we have

$$K_1 = \epsilon_2[(1-l)-\delta]-\epsilon_1 > 0,$$

$$K_1 = \frac{\varepsilon_1 l}{4} - \varepsilon_2\delta > 0.$$

We now choose ϵ_1 and ϵ_2 so small that (4.9) and (4.22) remain valid and

$$K_3 = \xi\left(\frac{1}{2} - \epsilon_2\frac{g(0)C_P}{4\delta}\right) - \left[\frac{\epsilon_1}{2l} + \epsilon_2\left(1 + \frac{1}{4\delta_2}\right)\right]\left(\int_0^\infty g^{2-p}(s)ds\right) > 0.$$

Finally, we choose $\beta = \min\{K_1, K_2, K_3\}$, to obtain

$$F^{'}(t) \le -\beta[\|u_t\|_2^2 + \|\nabla u(t)\|_2^2 + (g^p o\nabla\eta^t)(t)], \quad \forall t \ge 0. \quad (4.23)$$

Since $E(t) \le \|u_t\|_2^2 + \|\nabla u(t)\|_2^2 + (go\nabla\eta^t)(t)$

So,

$$E^{\sigma}(t) \leq [\|u_t\|_2^2 + \|\nabla u(t)\|_2^2 + (go\nabla\eta^t)(t)]^{\sigma}, \quad \text{for } \sigma > 1.$$

Using Lemma 2.2.1, we obtain

$$E^{\sigma}(t) \leq C[\|u_t\|_2^{2\sigma} + \|\nabla u(t)\|_2^{2\sigma} + \{(go\nabla\eta^t)(t)\}^{\sigma}]$$

$$E^{\sigma}(t) \leq C\left[\|u_t\|_2^2\|u_t\|_2^{2(\sigma-1)} + \|\nabla u(t)\|_2^2\|\nabla u(t)\|_2^{2(\sigma-1)}\right] + C\{(go\nabla\eta^t)(t)\}^{\sigma}.$$

Since, for all $C > 1$, we have

$$\|u_t\|_2^2 \leq CE(t) \leq CE(0).$$

Hence,

$$\|u_t\|_2^{2(\sigma-1)} \leq C^{\sigma-1}E^{\sigma-1}(0).$$

Similarly,

$$\|\nabla u(t)\|_2^{2(\sigma-1)} \leq C^{\sigma-1}E^{\sigma-1}(0).$$

Thus,

$$E^{\sigma}(t) \leq C[\|u_t\|_2^2 C^{\sigma-1}E^{\sigma-1}(0) + \|\nabla u(t)\|_2^2 C^{\sigma-1}E^{\sigma-1}(0)] + C\{(go\nabla\eta^t)(t)\}^{\sigma}$$

$$E^{\sigma}(t) \leq C^{\sigma}E^{\sigma-1}(0)[\|u_t\|_2^2 + \|\nabla u(t)\|_2^2] + C\{(go\nabla\eta^t)(t)\}^{\sigma}. \qquad (4.24)$$

Using Lemma 4.2.1, we obtain

$$E^{\sigma}(t) \leq CE^{\sigma-1}(0)[\|u_t\|_2^2 + \|\nabla u(t)\|_2^2] + C\left\{C\{(g^p o\nabla\eta^t)(t)\}^{\frac{1}{2p-1}}\right\}^{\sigma}$$

$$E^{\sigma}(t) \leq CE^{\sigma-1}(0)[\|u_t\|_2^2 + \|\nabla u(t)\|_2^2] + C\{(g^p o\nabla\eta^t)(t)\}^{\frac{\sigma}{2p-1}}.$$

Let $\sigma = 2p - 1$, we have

$$E^{2p-1}(t) \leq C\big[[\|u_t\|_2^2 + \|\nabla u(t)\|_2^2] + (g^p o\nabla\eta^t)(t)\big]. \qquad (4.25)$$

By using (4.23), we obtain

$$F'(t) \leq -\beta_2 E^{2p-1}(t), \qquad \text{for some } \beta_2 > 0. \tag{4.26}$$

A combination of (4.9) and (4.26), yields

$$F'(t) \leq -\beta_2 {\alpha_1}^{2p-1} F^{2p-1}(t), \qquad \forall t \geq 0. \tag{4.27}$$

From (4.9), we know that $F(t) > 0$

Since $1 < p < \frac{3}{2}$ implies $1 < 2p - 1 < 2$. Thus, we have $F^{2p-1}(t) > 0$

Multiply (4.27) by $F^{(1-2p)}(t)$ and integrate over $(0,\ t)$, we obtain

$$\int_0^t F^{(1-2p)}(s)\, F'(s) ds \leq -\int_0^t \beta_2 {\alpha_1}^{2p-1} ds$$

$$\left. \frac{F^{2(1-p)}(s)}{2(1-p)} \right|_0^t \leq -\beta_2 {\alpha_1}^{2p-1} t$$

$$\frac{F^{2(1-p)}(t) - F^{2(1-p)}(0)}{2(1-p)} \leq -\beta_2 {\alpha_1}^{2p-1} t.$$

Since $p > 1$ implies $2(1-p) < 0$

Therefore,

$$\begin{aligned} F^{2(1-p)}(t) &\geq -2\beta_2 (1-p) {\alpha_1}^{2p-1} t + F^{2(1-p)}(0) \\ &\geq 2\beta_2 (p-1) {\alpha_1}^{2p-1} t + F^{2(1-p)}(0). \end{aligned}$$

Also, $1 < p < \dfrac{3}{2}$ implies $0 < 2(\mathrm{p} - 1) < 1$.

Thus, $2\beta_2 (p-1) {\alpha_1}^{2p-1} t > 0, \ \ \forall t > 0$

Hence,

$$F^{2(p-1)}(t) \le \frac{1}{2\beta_2(p-1){\alpha_1}^{2p-1}t + F^{2(1-p)}(0)}.$$

By choosing $\lambda = \min\left\{2\beta_2{\alpha_1}^{2p-1}(p-1),\ \ F^{2(1-p)}(0)\right\} > 0,$

we obtain, $F^{2(p-1)}(t) \le \dfrac{1}{\lambda(1+t)}.$

which implies that $F(t) \le \left(\dfrac{1}{\lambda(1+t)}\right)^{\frac{1}{2(p-1)}}$

Thus,

$$F(t) \le C(1+t)^{\frac{-1}{2(p-1)}}, \quad C = (\lambda)^{\frac{1}{2(p-1)}}, \quad \forall t \ge 0. \tag{4.28}$$

Combination of (4.9) and (4.28) gives

$$E(t) \le K(1+t)^{\frac{-1}{2(p-1)}}, \quad K = \alpha_2 C, \quad \forall t \ge 0,$$

which completes the proof.

CONCLUSION:

The above proof shows that in past history case, the polynomial decay is slower compared to the finite history case which gives

$$E(t) \le K(1+t)^{\frac{-1}{p-1}}, \ \ \forall t \ge 0$$

See [9] and [49].

CHAPTER 5

GENERAL DECAY OF SOLUTION OF A VISCOELASTIC PROBLEM

5.1 Introduction

In this chapter, we consider the viscoelastic problem

$$\begin{cases} u_{tt} - \Delta u(x,t) + \int_0^t g(t-s)\Delta u(s)ds = f(x,t), \ \ in\ \Omega \times (0,\infty) \\ u(x,t) = 0, \qquad x \in \partial\Omega, \qquad t \geq 0 \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \qquad x \in \Omega \end{cases} \tag{5.1}$$

where Ω is a bounded domain of $\mathbb{R}^n (n \geq 1)$ with a smooth boundary $\partial\Omega$, $f \in L^2(\Omega \times (0,+\infty))$ and g is a positive non increasing function satisfying the following conditions

(G_1) $g: \mathbb{R}^+ \to \mathbb{R}^+$ is a differentiable function such that

$$g(0) > 0, \qquad 1 - \int_0^\infty g(s)ds = l > 0$$

(G_2) There exists a differentiable function $\xi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$g^{'}(t) \leq -\xi(t)g(t), \quad \forall t \geq 0, \quad \xi^{'}(t) \leq 0.$$

Theorem 5.1.1. Let $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ be given.

Assume that $f \in L^2\big(\Omega \times (0, +\infty)\big)$and g satisfies (G_1) $and(G_2)$, then the problem has a unique global weak solution

$$u \in C\big([0, \infty); H_0^1(\Omega)\big), \qquad u_t \in C\big([0, \infty); L^2(\Omega)\big) \tag{5.2}$$

Proof. This result can be established by using the Galerkin method or the linear semigroup theory as in [54] or by repeating the steps of [23] with the necessary modification.

5.2 Modified Energy Functional

Multiply$(5.1)_1$ by u_t and integrate over Ω, we obtain

$$\int_\Omega u_t u_{tt} dx - \int_\Omega u_t \Delta u(x,t) dx + \int_\Omega \int_0^t g(t-s) u_t \Delta u(s) ds dx = \int_\Omega u_t f(x,t)\, dx. \tag{5.3}$$

The terms in (5.3) are estimated as follows:

First Term

$$\int_\Omega u_t u_{tt} dx = \frac{1}{2}\frac{d}{dt}\|u_t\|_2^2. \tag{5.4}$$

Second Term

Using Green's formula and the boundary conditions, we obtain

$$-\int_\Omega u_t \Delta u(t) dx = \int_\Omega \nabla u_t \cdot \nabla u(t) dx = \frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|_2^2. \tag{5.5}$$

Third Term

$$\int_{\Omega}\int_{0}^{t} g(t-s)u_t\Delta u(s)d\tau dx = \int_{0}^{t} g(t-s)\int_{\Omega} u_t\Delta u(s)dxds.$$

Using Green's formula and the boundary conditions, we obtain

$$\int_{\Omega}\int_{0}^{t} g(t-s)u_t\Delta u(s)dsdx = -\int_{0}^{t} g(t-s)\int_{\Omega} \nabla u_t\cdot\nabla u(s)dxds$$

$$= -\int_{0}^{t} g(t-s)\int_{\Omega} \nabla u_t\cdot[\nabla u(s)-\nabla u(t)]dxds - \int_{0}^{t} g(t-s)\int_{\Omega} \nabla u_t\cdot\nabla u(t)dxds$$

$$= \frac{1}{2}\int_{0}^{t} g(t-\mathrm{s})\frac{d}{dt}\int_{\Omega} |\nabla u(s)-\nabla u(t)|^2dxds - \frac{1}{2}\int_{0}^{t} g(t-\mathrm{s})\frac{d}{dt}\int_{\Omega} |\nabla u(t)|^2dxds$$

$$= \frac{1}{2}\frac{d}{dt}\left(\int_{0}^{t} g(t-\mathrm{s})\int_{\Omega} |\nabla u(s)-\nabla u(t)|^2dxds\right)$$

$$-\frac{1}{2}\int_{0}^{t} g^{'}(t-\mathrm{s})\int_{\Omega} |\nabla u(s)-\nabla u(t)|^2dxds - \frac{1}{2}\frac{d}{dt}\left(\int_{0}^{t} g(t-\mathrm{s})ds\int_{\Omega} |\nabla u(t)|^2dx\right)$$

$$+\frac{1}{2}\int_{0}^{t} g^{'}(t-s)ds\int_{\Omega} |\nabla u(t)|^2dx + \frac{1}{2}g(0)\int_{\Omega} |\nabla u(t)|^2dx$$

$$= \frac{1}{2}\frac{d}{dt}\left(\int_{0}^{t} g(t-\mathrm{s})\|\nabla u(s)-\nabla u(t)\|_2^2\mathrm{ds}\right) - \frac{1}{2}\int_{0}^{t} g'(t-\mathrm{s})\|\nabla u(s)-\nabla u(t)\|_2^2ds$$

$$-\frac{1}{2}\frac{d}{dt}\left(\int_{0}^{t} g(t-\mathrm{s})\mathrm{ds}\right)\|\nabla u(t)\|_2^2 + \frac{1}{2}g(t)\|\nabla u(t)\|_2^2. \tag{5.6}$$

Fourth term

Using Young's inequality, we obtain, for any $\delta_2 > 0$,

$$\int_\Omega u_t f(x,t)\, dx \leq \delta_2 \|u_t\|_2^2 + \frac{1}{4\delta_2}\|f(x,t)\|_2^2. \tag{5.7}$$

By substituting $(5.4) - (5.7)$ in (5.3), we obtain

$$\frac{1}{2}\frac{d}{dt}||u_t||_2^2 + \frac{1}{2}\frac{d}{dt}||\nabla u(t)||_2^2 + \frac{1}{2}\frac{d}{dt}\left(\int_0^t g(t-s)\|\nabla u(s) - \nabla u(t)\|_2^2 ds\right)$$

$$-\frac{1}{2}\int_0^t g'(t-s)\|\nabla u(s) - \nabla u(t)\|_2^2 ds - \frac{1}{2}\frac{d}{dt}\left(\int_0^t g(t-\mathrm{s})\mathrm{ds}\right)\|\nabla u(t)\|_2^2$$

$$+\frac{1}{2}g(t)\|\nabla u(t)\|_2^2 \leq \delta_2\|u_t\|_2^2 + \frac{1}{4\delta_2}\|f(x,t)\|_2^2.$$

So,

$$\frac{d}{dt}\left\{\frac{1}{2}\left||u_t|\right|_2^2 + \frac{l}{2}\left||\nabla u(t)|\right|_2^2 + \frac{1}{2}\int_0^t g(t-s)\|\nabla u(s) - \nabla u(t)\|_2^2 ds\right\} + \frac{1}{2}g(t)\|\nabla u(t)\|_2^2$$

$$-\frac{1}{2}\int_0^t g'(t-s)||\nabla u(s) - \nabla u(t)||_2^2 ds \leq \delta_2\|u_t\|_2^2 + \frac{1}{4\delta_2}||f(x,t)||_2^2. \tag{5.8}$$

Set

$$E(t) = \frac{1}{2}||u_t||_2^2 + \frac{l}{2}||\nabla u(t)||_2^2 + \frac{1}{2}(go\nabla u)(t), \tag{5.9}$$

where

$$(go\nabla u)(t) = \int_0^t g(t-s)\|\nabla u(s) - \nabla u(t)\|_2^2 ds.$$

E(t) is called the Modified Energy Functional.

Hence (5.8) becomes

$$E'(t) \leq \frac{1}{2}(g'o\nabla u)(t) - \frac{1}{2}g(t)\|\nabla u(t)\|_2^2 + \delta_2\|u_t\|_2^2 + \frac{1}{4\delta_2}\|f(x,t)\|_2^2,$$

which implies that

$$E'(t) \leq \frac{1}{2}(g'o\nabla u)(t) + \delta_2\|u_t\|_2^2 + \frac{1}{4\delta_2}\|f(.,t)\|_2^2. \tag{5.10}$$

5.3 Decay of Solution

In this section we state and prove the main result in this chapter. For this purpose, we set

$$F(t) = E(t) + \epsilon_1\psi(t) + \epsilon_2\chi(t), \tag{5.11}$$

where ϵ_1 and ϵ_2 are positive constants, to be chosen properly later and

$$\psi(t) = \int_\Omega u u_t dx, \qquad \chi(t) = -\int_\Omega u_t \int_0^t g(t-s)[u(t) - u(s)]dsdx. \tag{5.12}$$

Lemma 5.3.1. For ϵ_1 and ϵ_2 small enough, the inequality

$$\alpha_1 F(t) \leq E(t) \leq \alpha_2 F(t), \tag{5.13}$$

holds for two positive constants α_1 and α_2.

Proof.

$$F(t) = E(t) + \epsilon_1 \int_\Omega u u_t dx - \epsilon_2 \int_\Omega u_t \int_0^t g(t-s)[u(t) - u(s)] ds dx. \qquad (5.14)$$

Using Young's inequality, we have

$$F(t) \leq E(t) + \frac{\epsilon_1}{2} \left||u(t)|\right|_2^2 + \frac{\epsilon_1 + \epsilon_2}{2} \left||u_t|\right|_2^2$$

$$+ \frac{\epsilon_2}{2} \int_\Omega \left(\int_0^t g(t-s)[u(t) - u(s)] ds \right)^2 dx. \qquad (5.15)$$

We estimate the fourth term in the right-hand side of (5.15) as follows

$$\left(\int_0^t g(t-s)[u(t) - u(s)] ds \right)^2 dx = \left(\int_0^t \sqrt{g(t-s)} \sqrt{g(t-s)} [u(t) - u(s)] ds \right)^2.$$

Using Cauchy-Schwarz inequality, we have

$$\left(\int_0^t g(t-s)[u(t) - u(s)] ds \right)^2 dx$$

$$\leq \left(\left(\int_0^t g(t-s)\, ds \right)^{\frac{1}{2}} \left(\int_0^t g(t-s) |u(t) - u(s)|^2\, ds \right)^{\frac{1}{2}} \right)^2$$

$$\leq \left(\int_0^t g(t-s)\, ds \right) \left(\int_0^t g(t-s) |u(t) - u(s)|^2\, ds \right)$$

$$\leq (1-l) \left(\int_0^t g(t-s) |u(t) - u(s)|^2\, ds \right). \qquad (5.16)$$

We substitute (5.16) in (5.15), then using Poincaré's inequality, to obtain

$$F(t) \leq E(t) + \frac{\epsilon_1 + \epsilon_2}{2} ||u_t||_2^2 + \frac{\epsilon_1}{2} C_P ||\nabla u(t)||_2^2 + \frac{\epsilon_2(1-l)}{2} C_P (go\nabla u)(t). \tag{5.17}$$

Since $\frac{1}{2}||u_t||_2^2 \leq E(t)$, then

$$\frac{\epsilon_1 + \epsilon_2}{2} ||u_t||_2^2 \leq (\epsilon_1 + \epsilon_2) E(t) \tag{5.18}$$

In the same way, we have

$$\frac{\epsilon_1}{2} C_P ||\nabla u(t)||_2^2 \leq \frac{\epsilon_1}{l} C_P E(t), \tag{5.19}$$

$$\frac{\epsilon_2(1-l)}{2} C_P (go\nabla u)(t) \leq \epsilon_2 (1-l) C_P E(t). \tag{5.20}$$

Hence, equation (5.17) becomes

$$F(t) \leq \left(1 + \frac{\epsilon_1 C_P}{l} + (\epsilon_1 + \epsilon_2) + \epsilon_2(1-l)C_P\right) E(t) \leq \beta_1 E(t),$$

where $\beta_1 = 1 + \frac{\epsilon_1 C_P}{l} + (\epsilon_1 + \epsilon_2) + \epsilon_2(1-l)C_P$,

which implies

$$E(t) \geq \alpha_1 F(t), \qquad \alpha_1 = \frac{1}{\beta_1}. \tag{5.21}$$

Similarly, from (5.14) using Young's inequality, we obtain

$$F(t) \geq E(t) - \frac{\epsilon_1}{2} ||u(t)||_2^2 - \frac{\epsilon_1 + \epsilon_2}{2} ||u_t||_2^2 - \frac{\epsilon_2}{2} \int_\Omega \left(\int_0^t g(t-s)[u(t) - u(s)] ds \right)^2 dx.$$

From (5.16), we have

$$F(t) \geq E(t) - \frac{\epsilon_1}{2}||u(t)||_2^2 - \frac{\epsilon_1 + \epsilon_2}{2}||u_t||_2^2 - \frac{\epsilon_2(1-l)}{2}\int_0^t g(t-s)\,\|u(t)-u(s)\|_2^2 ds.$$

Using Poincarè's inequality, we have

$$F(t) \geq E(t) - \frac{\epsilon_1 + \epsilon_2}{2}||u_t||_2^2 - \frac{\epsilon_1}{2}C_P||\nabla u(t)||_2^2 - \frac{\epsilon_2(1-l)}{2}C_P(go\nabla u)(t).$$

Using $(5.18) - (5.20)$, we obtain

$$F(t) \geq \left\{1 - \left(\frac{\epsilon_1 C_P}{l} + (\epsilon_1 + \epsilon_2) + \epsilon_2(1-l)C_P\right)\right\}E(t). \tag{5.22}$$

By choosing ϵ_1 and ϵ_2 small enough so that

$$\alpha = \frac{\epsilon_1 C_P}{l} + (\epsilon_1 + \epsilon_2) + \epsilon_2(1-l)C_P < 1,$$

we obtain

$$F(t) \geq (1-\alpha)E(t).$$

Hence,

$$E(t) \leq \alpha_2 F(t), \quad \alpha_2 = \frac{1}{1-\alpha}. \tag{5.23}$$

Combination of (5.21) and (5.23) gives

$$\alpha_1 F(t) \leq E(t) \leq \alpha_2 F(t),$$

which completes the proof.

Lemma 5.3.2 Under the assumptions (G_1) and (G_2) the functional

$$\psi(t) = \int_{\Omega} u(t)u_t dx$$

satisfies, along the solution (5.1),

$$\psi'(t) \leq \|u_t\|_2^2 - \left\{\frac{l}{2} - \delta_1 C_P\right\} \|\nabla u(t)\|_2^2 + \frac{1}{2}\left(\frac{1-l}{l}\right)(go\nabla u)(t) + \frac{1}{4\delta_1}\|f(.,t)\|_2^2. \tag{5.24}$$

Proof. By taking the derivative of ψ and using $(5.1)_1$, we get

$$\psi'(t) = \|u_t\|_2^2 + \int_{\Omega} u(t)\Delta u(t)dx - \int_{\Omega} u(t)\int_0^t g(t-s)\Delta u(s)dsdx$$

$$+ \int_{\Omega} u(t)f(x,t)dx.$$

Using Green's formula and the boundary conditions, we obtain

$$\psi'(t) = \|u_t\|_2^2 - \|\nabla u(t)\|_2^2 + \int_{\Omega} \nabla \mathrm{u}(\mathrm{t}) \cdot \int_0^t g(t-s)\nabla \mathrm{u}(s)dsdx + \int_{\Omega} u(t)f(x,t)dx. \tag{5.25}$$

Using Young's inequality and Poincarè's inequality, we have, for all $\delta_1 > 0$

$$\psi'(t) \leq \|u_t\|_2^2 - \|\nabla u(t)\|_2^2 + \int_{\Omega} \nabla \mathrm{u}(\mathrm{t}) \cdot \int_0^t g(t-s)\nabla \mathrm{u}(s)dsdx$$

$$+ \delta_1 C_P\|\nabla u(t)\|_2^2 + \frac{1}{4\delta_1}\|f(.,t)\|_2^2. \tag{5.26}$$

By estimating the third term in (5.26), using Young's inequality, we obtain,

$$\int_\Omega \nabla \mathrm{u}(\mathrm{t}) \cdot \int_0^t g(t-s)\nabla \mathrm{u}(s)dsdx \le \frac{1}{2}\|\nabla u(t)\|_2^2 + \frac{1}{2}\int_\Omega \left(\int_0^t g(t-s)|\nabla \mathrm{u}(s)|ds\right)^2 dx$$

$$\le \frac{1}{2}\|\nabla u(t)\|_2^2 + \frac{1}{2}\int_\Omega \left(\int_0^t g(t-s)[|\nabla \mathrm{u}(s) - \nabla \mathrm{u}(t)| + |\nabla \mathrm{u}(t)|]\,ds\right)^2 dx. \qquad (5.27)$$

We estimate the second term in (5.27) as follows:

$$\int_\Omega \left(\int_0^t g(t-s)[|\nabla \mathrm{u}(s) - \nabla \mathrm{u}(t)| + |\nabla \mathrm{u}(t)|]\,ds\right)^2 dx$$

$$= \int_\Omega \left(\int_0^t g(t-s)|\nabla \mathrm{u}(s) - \nabla \mathrm{u}(t)|ds\right)^2 dx + \int_\Omega \left(\int_0^t g(t-s)|\nabla \mathrm{u}(t)|ds\right)^2 dx$$

$$+2\int_\Omega \left(\int_0^t g(t-s)|\nabla \mathrm{u}(s) - \nabla \mathrm{u}(t)|ds\right)\left(\int_0^t g(t-s)|\nabla \mathrm{u}(t)|ds\right)dx.$$

Using Young's inequality, we obtain, for all $\eta > 0$,

$$\int_\Omega \left(\int_0^t g(t-s)[|\nabla \mathrm{u}(s) - \nabla \mathrm{u}(t)| + |\nabla \mathrm{u}(t)|]\,ds\right)^2 dx$$

$$\le (1+\eta)\int_\Omega \left(\int_0^t g(t-s)|\nabla \mathrm{u}(t)|ds\right)^2 dx$$

$$+\left(1+\frac{1}{\eta}\right)\int_\Omega \left(\int_0^t g(t-s)\,|\nabla \mathrm{u}(s) - \nabla \mathrm{u}(t)|ds\right)^2 dx$$

$$\leq (1+\eta)(1-l)^2\|\nabla \mathrm{u}(t)\|_2^2$$

$$+\left(1+\frac{1}{\eta}\right)\int_\Omega \left(\int_0^t \sqrt{g(t-s)}\sqrt{g(t-s)}|\nabla \mathrm{u}(s)-\nabla \mathrm{u}(t)|ds\right)^2 dx.$$

Using Cauchy-Schwarz inequality, we obtain

$$\int_\Omega \left(\int_0^t g(t-\tau)\,[|\nabla \mathrm{u}(s)-\nabla \mathrm{u}(t)|+|\nabla \mathrm{u}(t)|]ds\right)^2 dx \leq (1+\eta)(1-l)^2\|\nabla \mathrm{u}(t)\|_2^2$$

$$+\left(1+\frac{1}{\eta}\right)\int_\Omega \left(\left(\int_0^t g(t-s)ds\right)^{\frac{1}{2}}\left(\int_0^t g(t-s)|\nabla \mathrm{u}(s)-\nabla \mathrm{u}(t)|^2 ds\right)^{\frac{1}{2}}\right)^2 dx$$

$$\leq (1+\eta)(1-l)^2\|\nabla \mathrm{u}(t)\|_2^2+\left(1+\frac{1}{\eta}\right)(1-l)(go\nabla u)(t). \tag{5.28}$$

By substituting (5.28) in (5.27), we obtain

$$\int_\Omega \left(\int_{-\infty}^t g(t-\tau)\,[|\nabla \mathrm{u}(\tau)-\nabla \mathrm{u}(t)|+|\nabla \mathrm{u}(t)|]d\tau\right)^2 dx$$

$$\leq \frac{1}{2}\{1+(1+\eta)(1-l)^2\}\|\nabla \mathrm{u}(t)\|_2^2+\frac{1}{2}\left(1+\frac{1}{\eta}\right)(1-l)(go\nabla u)(t). \tag{5.29}$$

By substituting (5.29) in (5.26), we obtain

$$\psi'(t) \leq \|\mathrm{u_t}\|_2^2-\left\{\frac{1}{2}[1-(1+\eta)(1-l)^2]-\delta_1 C_P\right\}\|\nabla \mathrm{u}(t)\|_2^2$$

$$+\frac{1}{2}\left(1+\frac{1}{\eta}\right)(1-l)(go\nabla u)(t)+\frac{1}{4\delta_1}\|f(.,t)\|_2^2.$$

By choosing $\eta=\dfrac{l}{1-l}$, we obtain

$$\psi'(t) \le \|u_t\|_2^2 - \left\{\frac{l}{2} - \delta_1 C_P\right\} \|\nabla u(t)\|_2^2 + \frac{1}{2}\left(\frac{1-l}{l}\right)(go\nabla u)(t) + \frac{1}{4\delta_1}\|f(.,t)\|_2^2,$$

which completes the proof.

Lemma 4.3.3 Under the assumptions (G_1) and (G_2) the functional

$$\chi(t) = -\int_\Omega u_t \int_0^t g(t-s)[u(t)-u(s)]dsdx$$

satisfies, along the solution of (5.1) and for any $\delta > 0$,

$$\chi'(t) \le \left\{\delta - \int_0^t g(t-s)\,ds\right\}\|u_t\|_2^2 + \delta\{1+2(1-l)^2\}\|\nabla u(t)\|_2^2$$

$$+\left(2\delta + \frac{1}{2\delta} + \frac{C_P}{4\delta}\right)(1-l)(go\nabla u)(t) + \frac{g(0)}{4\delta}C_P[-(g'o\nabla u)(t)] + \delta\|f(.,t)\|_2^2. \tag{5.30}$$

Proof.

By taking the derivative of χ and using $(5.1)_1$, we get

$$\chi'(t) = -\int_\Omega \Delta u(t) \int_0^t g(t-s)[u(t)-u(s)]dsdx$$

$$+\int_\Omega \left(\int_0^t g(t-s)\,\Delta u(s)ds\right)\left(\int_0^t g(t-s)[u(t)-u(s)]ds\right)dx$$

$$-\int_\Omega u_t \int_0^t g'(t-s)[u(t)-u(s)]dsdx - \left\{\int_0^t g(t-s)\,ds\right\}\|u_t\|_2^2$$

$$-\int_\Omega f(x,t)\int_0^t g(t-s)[u(t)-u(s)]ds\,dx.$$

Using Green's formula and the boundary conditions, we obtain

$$\chi^{'}(t) = \int_{\Omega} \nabla u(t) \cdot \int_{0}^{t} g(t-s)[\nabla u(t) - \nabla u(s)]dsdx - \left(\int_{0}^{t} g(t-s)\,ds\right)\|u_t\|_2^2$$

$$-\int_{\Omega} \left(\int_{0}^{t} g(t-s)\,\nabla u(s)ds\right)\cdot\left(\int_{0}^{t} g(t-s)[\nabla u(t) - \nabla u(s)]ds\right)dx$$

$$-\int_{\Omega} u_t \int_{0}^{t} g^{'}(t-s)[u(t) - u(s)]dsdx$$

$$-\int_{\Omega} f(x,t) \int_{0}^{t} g(t-s)[u(t) - u(s)]ds\,dx.$$

(5.31)

The terms in (5.31) are estimated below.

The first Term

By repeating the steps $(5.27) - (5.29)$, we obtain

$$\int_{\Omega} \nabla u(t) \cdot \int_{0}^{t} g(t-s)[\nabla u(t) - \nabla u(s)]dsdx \le \delta||\nabla u(t)||_2^2 + \frac{1-l}{4\delta}(go\nabla u)(t).$$

(5.32)

The second Term

Using Young's inequality, we have, for all $\delta > 0$

$$-\int_{\Omega}\left(\int_0^t g(t-s)\,\nabla u(s)ds\right)\cdot\left(\int_0^t g(t-s)[\nabla u(t)-\nabla u(s)]ds\right)dx$$

$$\leq \delta\int_{\Omega}\left(\int_0^t g(t-s)\,\nabla u(s)ds\right)^2 dx+\frac{1}{4\delta}\int_{\Omega}\left(\int_0^t g(t-s)[\nabla u(t)-\nabla u(s)]ds\right)^2 dx$$

$$\leq \delta\int_{\Omega}\left|\int_0^t g(t-s)\,\nabla u(s)ds\right|^2 dx+\frac{1}{4\delta}\int_{\Omega}\left|\int_0^t g(t-s)[\nabla u(t)-\nabla u(s)]ds\right|^2 dx$$

$$\leq \delta\int_{\Omega}\left|\int_0^t g(t-s)\,[\nabla u(s)-\nabla u(t)]ds+\int_0^t g(t-s)\,[\nabla u(t)]ds\right|^2 dx$$

$$+\frac{1}{4\delta}\int_{\Omega}\left|\int_0^t g(t-s)[\nabla u(t)-\nabla u(s)]ds\right|^2 dx.$$

Using Lemma 2.3.1, we obtain

$$-\int_{\Omega}\left(\int_0^t g(t-s)\,\nabla u(s)ds\right)\cdot\left(\int_0^t g(t-s)[\nabla u(t)-\nabla u(s)]ds\right)dx$$

$$\leq\left(2\delta+\frac{1}{4\delta}\right)\int_{\Omega}\left(\int_0^t g(t-s)\,|\nabla u(s)-\nabla u(t)|ds\right)^2 dx$$

$$+2\delta\int_{\Omega}\left(\int_0^t g(t-s)\,|\nabla u(t)|ds\right)^2 dx.$$

By applying Cauchy-Schwarz inequality, we obtain

$$-\int_{\Omega}\left(\int_0^t g(t-s)\,\nabla u(s)ds\right)\cdot\left(\int_0^t g(t-s)[\nabla u(t)-\nabla u(s)]ds\right)dx$$

$$\leq\left(2\delta+\frac{1}{4\delta}\right)(1-l)(go\nabla u)(t)+2\delta(1-l)^2\|\nabla u(t)\|_2^2. \tag{5.33}$$

The third Term

Using Young's inequality, we obtain, for all $\delta > 0$

$$-\int_{\Omega} u_t\int_0^t g'(t-s)[u(t)-u(s)]dsdx$$

$$\leq \delta\|u_t\|_2^2+\frac{1}{4\delta}\int_{\Omega}\left(\int_0^t g'(t-s)|u(t)-u(s)\,|ds\right)^2 dx$$

$$\leq \delta\|u_t\|_2^2+\frac{1}{4\delta}\int_{\Omega}\left(\int_0^t \sqrt{-g'(t-s)}\sqrt{-g'(t-s)}|u(t)-u(s)\,|ds\right)^2 dx.$$

Using Cauchy-Schwarz inequality, we obtain

$$-\int_{\Omega} u_t\int_0^t g'(t-s)[u(t)-u(s)]dsdx \leq \delta\|u_t\|_2^2$$

$$+\frac{1}{4\delta}\int_{\Omega}\left(\left\{\int_0^t -g'(t-s)ds\right\}^{\frac{1}{2}}\left\{\int_0^t -g'(t-s)|u(t)-u(s)|^2\,ds\right\}^{\frac{1}{2}}\right)^2 dx$$

$$\leq \delta\|u_t\|_2^2+\frac{1}{4\delta}\left(\int_0^t -g'(t-s)ds\right)\int_{\Omega}\left(\int_0^t -g'(t-s)|u(t)-u(s)|^2\,ds\right)dx$$

$$\leq \delta\|u_t\|_2^2+\frac{1}{4\delta}[g(0)-g(t)]\int_{\Omega}\int_0^t -g'(t-s)|u(t)-u(s)|^2\,ds\,dx.$$

By Poincaré's inequality, we obtain

$$-\int_{\Omega} u_t \int_{-\infty}^{t} g^{'}(t-\tau)[u(t)-u(\tau)]d\tau dx$$

$$\leq \delta\|u_t\|_2^2 + \frac{1}{4\delta}C_p[g(0)-g(t)][-(g^{'}o\nabla u)(t)]$$

$$\leq \delta\|u_t\|_2^2 + \frac{g(0)}{4\delta}C_P[-(g'o\nabla u)(t)]. \tag{5.34}$$

The fourth Term

Using Young's inequality, we obtain, for all $\delta > 0$

$$-\int_{\Omega} f(x,t)\int_{0}^{t} g(t-s)[u(t)-u(s)]ds\, dx$$

$$\leq \delta\|f(.,t)\|_2^2 + \frac{1}{4\delta}\int_{\Omega}\left(\int_{0}^{t} g(t-s)[u(t)-u(s)]ds\right)^2 dx.$$

Using Cauchy-Schwarz inequality and Poincaré's inequality, we obtain

$$-\int_{\Omega} f(x,t)\int_{0}^{t} g(t-s)[u(t)-u(s)]ds\, dx \leq \delta\|f(.,t)\|_2^2 + \frac{1-l}{4\delta}C_P(go\nabla u)(t). \tag{5.35}$$

By substituting $(5.32)-(5.35)$ in (5.31), we obtain

$$\chi'(t) \leq \left(\delta - \int_{0}^{t} g(t-s)\,ds\right)\|u_t\|_2^2 + \delta\{1+2(1-l)^2\}\|\nabla u(t)\|_2^2$$

$$+\left(2\delta + \frac{1}{2\delta} + \frac{C_P}{4\delta}\right)(1-l)(go\nabla u)(t) + \frac{g(0)}{4\delta}C_P[-(g^{'}o\nabla u)(t)] + \delta\|f(.,t)\|_2^2,$$

which completes the proof.

THEOREM 5.3.1 Let $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ be given. Assume that $f \in L^2(\Omega \times (0, +\infty))$ and g satisfies (G_1) and (G_2). Then, for any $t \geq 0$

$$E(t) \leq C_2 E(0) e^{-\lambda \int_0^t \xi(s) ds} \text{ if } f \equiv 0. \tag{5.36}$$

Otherwise,

$$E(t) \leq C_2 \left\{ E(0) + \int_0^t \left(\frac{1}{\xi(s)} + \xi(s) \right) \|f(.,s)\|_2^2 e^{\lambda \int_0^s \xi(\varsigma) d\varsigma} ds \right\} e^{-\lambda \int_0^t \xi(s) ds}, \tag{5.37}$$

where C_2 and λ are positive constants.

Proof

By taking the derivative of F and substituting (5.10), (5.24), (5.30) we obtain

$$\begin{aligned}
F'(t) \leq \; & \frac{1}{2}(g' o \nabla u)(t) + \delta_2 \|u_t\|_2^2 + \frac{1}{4\delta_2} \|f(.,t)\|_2^2 \\
& + \epsilon_1 \left\{ \|u_t\|_2^2 - \left(\frac{l}{2} - \delta_1 C_P \right) \|\nabla u(t)\|_2^2 + \frac{1}{2}\left(\frac{1-l}{l} \right) (g o \nabla u)(t) + \frac{1}{4\delta_1} \|f(.,t)\|_2^2 \right\} \\
& + \epsilon_2 \left\{ \left(\delta - \int_0^t g(t-s)\, ds \right) \|u_t\|_2^2 + \delta [1 + 2(1-l)^2] \|\nabla u(t)\|_2^2 \right. \\
& \qquad + \left(2\delta + \frac{1}{2\delta} + \frac{C_P}{4\delta} \right) (1-l)(g o \nabla u)(t) + \frac{g(0)}{4\delta} C_P [-(g' o \nabla u)(t)] \\
& \qquad \left. + \delta \|f(.,t)\|_2^2 \right\}.
\end{aligned} \tag{5.38}$$

Since g is continuous, positive and $g(0) > 0$, then for any $t_0 > 0$, we have

$$\int_0^t g(s)ds \geq \int_0^{t_0} g(s)ds = g_0, \quad \forall t \geq t_0.$$

Hence (5.38) becomes

$$F'(t) \leq -\{\epsilon_2(g_0 - \delta) - \epsilon_1 - \delta_2\}\|u_t\|_2^2$$

$$-\left\{\epsilon_1\left(\frac{l}{2} - \delta_1 C_P\right) - \epsilon_2\delta[1 + 2(1-l)^2]\right\}\|\nabla \mathrm{u}(t)\|_2^2$$

$$+\left\{\left(\frac{\epsilon_1}{2l} + \epsilon_2\left(2\delta + \frac{1}{2\delta} + \frac{C_P}{4\delta}\right)\right)(1-l)\right\}(go\nabla u)(t) + \left\{\frac{1}{2} - \epsilon_2\frac{g(0)}{4\delta}C_P\right\}(g'o\nabla u)(t)$$

$$+\left(\frac{1}{4\delta_2} + \frac{\epsilon_1}{4\delta_1}C_P + \epsilon_2\delta\right)\|f(.,t)\|_2^2, \quad \forall t \geq t_0. \tag{5.39}$$

By choosing $\delta_1 = \dfrac{l}{4C_P}$, we obtain

$$F'(t) \leq -\{\epsilon_2(g_0 - \delta) - \epsilon_1 - \delta_2\}\|u_t\|_2^2 - \left\{\frac{\epsilon_1 l}{4} - \epsilon_2\delta[1 + 2(1-l)^2]\right\}\|\nabla \mathrm{u}(t)\|_2^2$$

$$+\left\{\left(\frac{\epsilon_1}{2l} + \epsilon_2\left(2\delta + \frac{1}{2\delta} + \frac{C_P}{4\delta}\right)\right)(1-l)\right\}(go\nabla u)(t)$$

$$+\left\{\frac{1}{2} - \epsilon_2\frac{g(0)}{4\delta}C_P\right\}(g'o\nabla u)(t) + \left(\frac{1}{4\delta_2} + \frac{\epsilon_1}{l}C_P + \epsilon_2\delta\right)\|f(.,t)\|_2^2, \forall t \geq t_0. \tag{5.40}$$

Now choose δ so small that

$$g_0 - \delta > \frac{1}{2}g_0,$$

$$\frac{4}{l}\delta[1 + 2(1-l)^2] < \frac{g_0}{4}.$$

Whence δ is fixed, any choice of ϵ_1 and ϵ_2, such that

$$\frac{g_0}{4}\epsilon_2 < \epsilon_1 < \frac{g_0}{2}\epsilon_2, \tag{5.41}$$

will make

$$\epsilon_2(g_0 - \delta) > \frac{g_0}{2}\epsilon_2 > \epsilon_1,$$

$$\frac{4}{l}\epsilon_2\delta[1 + 2(1 - l)^2] < \frac{g_0}{4}\epsilon_2 < \epsilon_1.$$

Thus, we have

$$K_0 = \epsilon_2(g_0 - \delta) - \epsilon_1 > 0,$$

$$K_1 = \frac{4\epsilon_1}{l} - \epsilon_2\delta[1 + 2(1 - l)^2] > 0.$$

We then pick ϵ_2 so small that (5.13) and (5.41) remain valid and

$$\delta > \epsilon_2 \frac{g(0)}{2} C_P$$

and δ_2 so small that

$$K_2 = K_0 - \delta_2 > 0.$$

Thus

$$K_3 = \frac{1}{2} - \epsilon_2 \frac{g(0)}{4\delta} C_P > 0.$$

Hence from (5.40), we have $\forall t \geq t_0$

$$F^{'}(t) \leq -\{K_2\|u_t\|_2^2 + K_1\|\nabla \mathrm{u}(t)\|_2^2\} + K_4(go\nabla u)(t) + K_3\big(g^{'}o\nabla u\big)(t) + K_5\|f(.,t)\|_2^2,$$

(5.42)

where

$$K_4 = \left(\frac{\epsilon_1}{2l} + \epsilon_2\left(2\delta + \frac{1}{2\delta} + \frac{C_P}{4\delta}\right)\right)(1-l),$$

$$K_5 = \frac{1}{4\delta} + \frac{\epsilon_1}{l}C_P + \epsilon_2\delta.$$

Therefore,

$$F^{'}(t) \leq -\{K_2\|u_t\|_2^2 + K_1\|\nabla \mathrm{u}(t)\|_2^2 + K_6(go\nabla u)(t)\} + K_7(go\nabla u)(t)$$

$$+K_3(g^{'}o\nabla u)(t) + K_5\|f(.,t)\|_2^2, \;\; \forall t \geq t_0,$$

where $K_7 = K_4 + K_6$.

Choose $\beta = \min\{K_1, K_2, K_6\}$ to obtain

$$F^{'}(t) \leq -\beta\{\|u_t\|_2^2 + \|\nabla \mathrm{u}(t)\|_2^2 + (go\nabla u)(t)\} + K_7(go\nabla u)(t) + K_3(g^{'}o\nabla u)(t)$$

$$+K_5\|f(.,t)\|_2^2, \;\; \forall t \geq t_0,$$

which implies that

$$F^{'}(t) \leq -\beta E(t) + K_7(go\nabla u)(t) + K_5\|f(.,t)\|_2^2, \;\; \forall t \geq t_0. \tag{5.43}$$

Multiply (5.43) by $\xi(t)$ and use (G_2), to obtain

$$\xi(t)F'(t) \leq -\beta\xi(t)E(t) + K_7[-(g'o\nabla u)(t)] + K_5\xi(t)\|f(.,t)\|_2^2, \forall t \geq t_0. \tag{5.44}$$

Using (5.10) and the fact that $E(t) \geq \frac{1}{2}\|u_t\|_2^2$, we obtain, for all $\mu > 0$,

$$-(g'o\nabla u)(t) \leq -2E'(t) + 4\mu E(t) + \frac{1}{2\mu}\|f(.,t)\|_2^2. \tag{5.45}$$

Substitute (5.45) in (5.44), we obtain, for all $\mu > 0$,

$$\xi(t)F'(t) \leq -(\beta\xi(t) - 4\mu K_7)E(t) - 2K_7E'(t) + \left(\frac{K_7}{2\mu} + K_5\xi(t)\right)\|f(.,t)\|_2^2, \quad \forall t \geq t_0.$$

Take $\mu = \frac{\varepsilon}{2}\xi(t)$, ε is a positive constant. Then, for all $\varepsilon > 0$, we have

$$\xi(t)F'(t) \leq -(\beta - C\varepsilon)\xi(t)E(t) + \left(\frac{K_8}{\xi(t)} + K_5\xi(t)\right)\|f(.,t)\|_2^2 - CE'(t), \quad \forall t \geq t_0.$$

where $C = 2K_7$ *and* $K_8 = \dfrac{K_7}{\varepsilon}$.

Let $\varepsilon = \dfrac{\beta}{2C}$, we obtain

$$\xi(t)F'(t) \leq -\beta_1\xi(t)E(t) + \left(\frac{K_8}{\xi(t)} + K_5\xi(t)\right)\|f(.,t)\|_2^2 - CE'(t), \quad \forall t \geq t_0,$$

where $\beta_1 = \dfrac{\beta}{2}$.

$$\frac{d}{dt}[\xi(t)F(t) + CE(t)] - \xi'(t)F(t) \leq -\beta_1\xi(t)E(t) + \left(\frac{K_8}{\xi(t)} + K_5\xi(t)\right)\|f(.,t)\|_2^2, \forall t \geq t_0.$$

Since $\xi'(t) \leq 0$. Hence,

$$\frac{d}{dt}[\xi(t)F(t) + CE(t)] \leq -\beta_1\xi(t)E(t) + \left(\frac{K_8}{\xi(t)} + K_5\xi(t)\right)\|f(.,t)\|_2^2, \forall t \geq t_0 \tag{5.46}$$

Using the fact that $\xi(t)$ is non-increasing and setting $R(t) = \xi(t)F(t) + CE(t)$.

Hence,

$$CE(t) \leq R(t) \leq C_1 E(t), \tag{5.47}$$

holds for two positive numbers $C \; and \; C_1$.

Using (5.47), we obtain, $\forall t \geq t_0$

$$R'(t) \leq -\lambda\xi(t)R(t) + \left(\frac{K_8}{\xi(t)} + K_5\xi(t)\right)\|f(.,t)\|_2^2, \;\; \lambda = \frac{\beta_1}{C_1}. \tag{5.48}$$

CASE 1 If $f \equiv 0$, then

$$R'(t) \leq -\lambda\xi(t)R(t), \quad \forall t \geq t_0.$$

Simple integration over (t_0, t), gives,

$$R(t) \leq R(t_0)e^{-\lambda\int_{t_0}^{t}\xi(s)ds}, \qquad \forall t \geq t_0.$$

Using (5.47), we obtain, for some positive constant $\tilde{C}_2$,

$$E(t) \leq \tilde{C}_2 E(t_0)e^{-\lambda\int_{t_0}^{t}\xi(s)ds}, \quad \forall t \geq t_0.$$

Since $E(t) \leq E(t_0) \leq E(0), \;\; \forall t \geq t_0$. Thus, we get

$$E(t) \leq \tilde{C}_2 E(0)e^{\lambda\int_0^{t_0}\xi(s)ds}e^{-\lambda\int_0^{t}\xi(s)ds}, \quad \forall t \geq t_0.$$

Thus the estimate (5.36) is proved with $C_2 = \tilde{C}_2 e^{\lambda\int_0^{t_0}\xi(s)ds}$.

CASE 2 If $f(x,t) \neq 0$,

Let's set

$$H(t) = R(t) - e^{-\lambda\int_{t_0}^{t}\xi(s)ds}\int_{t_0}^{t}\left(\frac{K_8}{\xi(s)} + K_5\xi(s)\right)\|f(.,s)\|_2^2\, e^{\lambda\int_{t_0}^{s}\xi(\varsigma)d\varsigma}ds,$$

which implies

$$R(t) = H(t) + e^{-\lambda \int_{t_0}^{t} \xi(s)ds} \int_{t_0}^{t} \left(\frac{K_8}{\xi(s)} + K_5 \xi(s) \right) \|f(.,s)\|_2^2 \, e^{\lambda \int_{t_0}^{s} \xi(\varsigma)d\varsigma} ds. \tag{5.49}$$

Simple differentiation of (5.49) gives

$$R'(t) = H'(t) - \lambda \xi(t) e^{-\lambda \int_{t_0}^{t} \xi(s)ds} \int_{t_0}^{t} \left(\frac{K_8}{\xi(s)} + K_5 \xi(s) \right) \|f(.,s)\|_2^2 \, e^{\lambda \int_{t_0}^{s} \xi(\varsigma)d\varsigma} ds$$

$$+ \left(\frac{K_8}{\xi(t)} + K_5 \xi(t) \right) \|f(.,t)\|_2^2, \qquad \forall t \geq t_0. \tag{5.50}$$

Using (5.49), we obtain

$$R'(t) = H'(t) - \lambda \xi(t) R(t) + \lambda \xi(t) H(t) + \left(\frac{K_8}{\xi(t)} + K_5 \xi(t) \right) \|f(.,t)\|_2^2, \qquad \forall t \geq t_0. \tag{5.51}$$

Substitute (5.51) in (5.48), we obtain

$$H'(t) \leq -\lambda \xi(t) H(t), \quad \forall t \geq t_0. \tag{5.52}$$

Simple integration of (5.52) over (t_0, t) gives

$$H(t) \leq H(t_0) e^{-\lambda \int_{t_0}^{t} \xi(s)ds}, \qquad \forall t \geq t_0,$$

which implies

$$R(t) \leq \left\{ R(t_0) + \int_{t_0}^{t} \left(\frac{K_8}{\xi(s)} + K_5 \xi(s) \right) \|f(.,s)\|_2^2 e^{\lambda \int_{t_0}^{s} \xi(\varsigma)d\varsigma} ds \right\} e^{-\lambda \int_{t_0}^{t} \xi(s)ds}, \forall t \geq t_0. \tag{5.53}$$

Thus, using (5.47), we obtain, for some positive constant $\tilde{C}_2$,

$$E(t) \le \left\{ \widetilde{C}_2 E(t_0) + \int_{t_0}^{t} \left(\frac{K_8}{\xi(s)} + K_5 \xi(s) \right) \|f(.,s)\|_2^2 e^{\lambda \int_{t_0}^{s} \xi(\varsigma) d\varsigma} ds \right\} e^{-\lambda \int_{t_0}^{t} \xi(s) ds}, \forall t \ge t_0.$$

Thus the estimate (5.37) is proved with $C_2 = \left(max\{\tilde{C}_2, K_8, K_5\}\right) e^{\lambda \int_0^{t_0} \xi(s)}$

Remark 5.3.1.

1 If $\xi(t) \equiv a > 0$, then

$$E(t) \le C_2 E(0) e^{-\lambda a t} + b e^{-\lambda a t} \int_0^t \|f(.,s)\|_2^2 e^{\lambda a s} ds, \qquad b = \frac{1}{a} + a, \quad \forall t \ge 0$$

2 If $\xi(t) \equiv a > 0$ and $\|f(x,t)\|_2^2 \le c e^{-\gamma t}$, then, for all $t \ge 0$,

$$E(t) \le C_2 E(0) e^{-\lambda a t} + b_1 e^{-\gamma t}, \quad \forall t \ge t_0$$

where $b_1 = \dfrac{bcC_2}{\lambda a - \gamma}$ if $\lambda a \ne \gamma$

$*$ $E(t) \le (C_2 E(0) + b_2 t) e^{-\lambda a t}$ if $\lambda a = \gamma$

where $b_1 = bcC_2$

In $*$, $E(t)$ does not necessarily converge to zero as $e^{-\lambda a t}$ when t goes to $+\infty$.

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