Tilting and Star Module Theory in Commutative Rings

BY

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A Dissertation Presented to the

DEANSHIP OF GRADUATE STUDIES

KING FAHD UNIVERSITY OF PETROLEUM & MINERALS

DHAHRAN, SAUDI ARABIA

In Partial Fulfillment of the

Requirements for the Degree of

DOCTOR OF PHILOSOPHY

In

MATHEMATICS

May, 2009

KING FAHD UNIVERSITY OF PETROLEUM & MINIRALS
DHAHRAN, SAUDI ARABIA

DEANSHIP OF GRADUATE STUDIES

This dissertation, written by MOHAMMAD SAMEEH MAHMOUD JARRAR under the direction of his thesis advisors and approved by his thesis committee, has been presented to and accepted by the Dean of Graduate Studies, in partial fulfillment of the requirements for the degree of DOCTOR OF PHIILOSOPHY IN MATHEMATICS.

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To my mother, brothers and sister

Acknowledgements

All praise and glory to ALLAH almightily who taught man through the pen. And peace and blessing of ALLAH be upon his last messenger Mohammad (Sallallah-Alihe-Wasallam), who brought the absolute truth and wisdom to mankind.

First and the foremost acknowledgment are due to the wonderful university that gave me the opportunity to work on my Ph.D program; King Fahd University of Petroleum & Minerals.

My deep appreciation and heartfelt gratitude goes to my thesis advisors Dr. Jawad Abuhulial and Prof. Salah-eddine Qabbaj for their guidance and encouragement throughout this research work. This work could not be accomplished without their valuable suggestions.

I extend my deepest gratitude to the committee members for their helpful suggestions.

Big thanks are due to my beloved mother and rest of the family members who always pray for me to success.

I also owe big thanks to my faithful friends Fadi, Ahmed, Basel and Abdulaziz for their support and encouragement.

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DISSERTATION ABSTRACT

Name: Mohammad Sameeh Mahmoud Jarrar

Title of study: Tilting and Star Module Theory

in Commutative Rings

Major Field: Mathematics

Date of degree: May, 2009

This Ph.D. thesis traverses two chapters which contribute to the study of tilting modules and $\star$-modules over commutative rings. Chapter 1 provides a complete classification of all tilting modules and tilting classes over almost perfect domains, which generalizes the classifications of tilting modules and tilting classes over Dedekind and 1-Gorenstein domains. Assuming the almost perfect domain is Noetherian, a complete classification of all cotilting modules is obtained (via duality). Chapter 2 studies the multiplicative ideal structure of commutative rings in which every finitely generated ideal is a $\star$-module, by investigating the correlation of this class of rings with well-known Prüfer conditions; namely, we prove that this class stands strictly between the two classes of arithmetical rings and Gaussian rings. This allows us to generalize Osofsky's theorem on the weak global dimension of arithmetical rings and partially resolve Bazzoni-Glaz's related conjecture on Gaussian rings. We also explore various contexts of trivial ring extensions in order to build original examples of rings subject to the $\star$-property, marking its distinction from related Prüfer conditions. We close this chapter with a study of n-star modules over Dedekind domains.

Mathematics Subject Classification: 13C05, 13D07, 13H99, 16L99, 13F05, 13B05, 13C13, 16D40, 16B50, 16D90, 13F05, 13F30, 13G05, 13DXX, 13H10, 13C15, 13A15.

Key Words and Phrases: Tilting module, Fuchs-Salce tilting module, perfect ring, almost perfect domain, coprimely packed ring, Dedekind domain, 1-Gorenstein domain, h-local

domain, Matlis domain, Prüfer domain, arithmetical ring, Gaussian ring, Prüfer ring, semi-hereditary ring, $\star$-module, quasi-projective module, weak global dimension.

ملخص بحث

درجة الدكتوراة في الفلسفة

الاســــــــــم: محمـــد سميــح محمــــود جــــرار

عنوان الرسالة: نظـريـــة الـمقــاييـــس المائلـــة والنجمـــة في الحلقـــات التبديليـــة

التخصـــــص: الريـــاضيـــات

تاريخ التخرج: مايو ٢٠٠٩

قسمت رسالة الدكتوراه هذه إلى ثلاث وحدات: الوحدة الأولى عبارة عن مقدمة في المقاييس المائلة ومقاييس النجمة بشكل عام والعلاقة بينهما, اما الوحدة الثانية فتختص بالمقاييس المائلة على الحلقات التبديلية, فيما تختص الوحدة الثالثة بدراسة مقاييس النجمة على الحلقات التيديلية.

ففي الوحدة الثانية أعطينا وصفا كاملا لكل المقاييس المائلة و الصفوف المائلة على المجال التام غالبا و الذي هو تعميم للمقاييس و الصفوف المائلة على مجال "ديديكند" و مجال "جورنيشتاين" الاول. و بفرضية ان المجال التام غالبا هو "نوثيرين" اعطينا وصفا كاملا للمقاييس و الصفوف "الكومائلة".

اما في الوحدة الثالثة فقد درسنا بناء المثالية الضربية لحلقات تبديلية تكون فيها كل مثالية محدودة التوليد عبارة عن مقياس نجمة, حيث اثبتنا ان هذه الحلقات تقع بشكل قوي بين الحلقات الرياضية و حلقات "جاوسين". و بهذه النظرية نكون قد عممنا نظرية "اوسوفيسكي" على الاتجاه العام الضعيف على الحلقات الرياضية و حللنا جزئيا مسالة "باتزوني و جلاس" على حلقات "جاوسين", و اعطينا امثلة عامة على هذه الحلقات الجديدة بواسطة التمديد الطبيعي للحلقات التبديلية. اخيرا, ختمنا هذه الوحدة بدراسة مقاييس النجمة النونية على مجال "ديديكند".

Introduction

An important result in the theory of classical tilting modules is Brenner-Butler's famous theorem which asserts that a finitely generated tilting module induces some equivalences between certain subcategories [18]. In this vein, $\star$-modules investigated by Menini and Orsatti [66] as well as n-star modules considered in [94] are generalizations of tilting modules. Indeed, the classical 1-tilting modules are just the $\star$-modules which generate all injective modules [23] and n-tilting modules coincide with those n-star modules which n-generate all injective modules [94].

An R-module T is said to be tilting if $\mathrm{Gen}_n(T) = T^{\perp_\infty}$ for some positive integer n, where $\mathrm{Gen}_n(T)$ denotes the class of T-n-generated R-modules and $T^{\perp_\infty} := \bigcap_{i\geq 1} \mathrm{Ker}(\mathrm{Ext}^i_R(T,-))$. Classical tilting modules of projective dimension at most 1 were introduced by Brenner and Butler [18] and then generalized and developed by Happel and Ringel [55]. In this thesis, tilting modules are allowed to be infinitely generated. This is essential for any useful application (since a finitely generated tilting module over a commutative ring is necessarily progenerator [68]). It is however worth recalling that infinitely generated tilting modules are somehow close to the finitely generated setting to the effect that all tilting modules are of finite type [13], i.e., the tilting class $T^{\perp_\infty}$ of T is the Ext-orthogonal of a set of modules having a projective resolution consisting of finitely generated projective modules. One of the first examples of infinitely generated 1-tilting modules is the Fuchs divisible module ∂, which was introduced over an arbitrary integral domain in [39] and was proved to have the tilting property by Facchini in [35, 36].

The study of tilting modules over special classes of commutative rings and domains was initiated by Göbel and Trlifaj [49], who classified tilting Abelian groups by assuming Gödel's axiom of constructibility (this condition was removed later in [7]). Tilting modules were classified too over Dedekind domains by Bazzoni et al. [7], over valuation and Prüfer domains by Salce in [81, 77], and recently over arbitrary 1-Gorenstein rings by Trlifaj and Pospíšil [88].

The $\star$-modules were introduced by Menini and Orsatti [66] and, then, were proved to be always finitely generated by Trlifaj [87]. This notion was extended to the so-called $\star^n$-module by Wei et al. in [96] for $n > 1$; noting that $\star^1$-modules coincide with $\star$-modules (and hence are finitely generated), while for $n \geq 2$, they provided an example of a $\star^2$-module which is not finitely generated. Further, the self-smallness assumption on $\star^n$-modules was dropped in [94] and the new modules were called n-star modules.

Over commutative rings, Trlifaj showed that the class of $\star$-modules and the class of quasi-progenerators coincide [87]. Zanardo proved that every $\star$-module over a valuation ring R is isomorphic to $(R/I)^n$ for some positive integer n and ideal I of R [102]. An open problem in [49, Page 254] is to "characterize all tilting modules and classes over Matlis domains." Recall that a domain is Matlis if the projective dimension of its quotient field is at most 1. Recalling that almost perfect domains are Matlis [80, Proposition 2.5], a natural problem in this connection was stated by Salce: "Characterize all tilting modules and classes over almost perfect domains." One of the main results of this thesis (in Chapter 1) resolves completely this problem.

This thesis consists of two chapters. Chapter 1 provides a complete classification of all tilting modules and tilting classes over an almost perfect domain (APD, for short), which generalizes the classifications of tilting modules and tilting classes over Dedekind domains and 1-Gorenstein domains. Assuming the APD to be Noetherian, a complete classification of all cotilting modules is also obtained (via duality). For this purpose, we first characterize injective modules, torsion-free modules, and divisible modules over almost perfect domains. Moreover, we show that, over such integral domains,

$$\mathscr{I} = \mathscr{I}_1 \text{ and } \mathscr{F} = \mathscr{F}_1 = \mathscr{P}_1 = \mathscr{P}$$

(see the notation in Section 1.1). Thereby, we establish the main results of this chapter (Theorem 1.4.15) which deals with Salce's problem mentioned above; namely, for a nontrivial APD R, the following assertions hold:

(1) All tilting R-modules are 1-tilting and represented by

$$\{T(X) := \bigcap_{\mathfrak{m}\in X} R_{\mathfrak{m}} \bigoplus \frac{\bigcap_{\mathfrak{m}\in X} R_{\mathfrak{m}}}{R} \mid X \subseteq \mathrm{Max}(R)\}.$$

(2) $\{X\text{-Div} \mid X \subseteq \mathrm{Max}(R)\}$ is the class of all tilting classes, where

$$X\text{-Div} := \{M_R \mid \mathfrak{m}M = M \text{ for every } \mathfrak{m} \in X\}.$$

(3) If R is coprimely packed, then the following set of Fuchs-Salce tilting modules classifies all tilting R-modules

$$\{\partial_S \mid S \subseteq R^{\times} \text{ is a multiplicative subset}\}.$$

Chapter 2 studies the multiplicative ideal structure of commutative rings in which every finitely generated ideal is a $\star$-module, by investigating the correlation of this class of rings with well-known Prüfer conditions. This allows us to generalize Osofsky's theorem on the weak global dimension of arithmetical rings and partially resolve Bazzoni-Glaz's related conjecture on Gaussian rings. We then explore various contexts of trivial ring extensions in order to build original examples of rings subject to the $\star$-property. As a prelude to this, we show that the notion of $\star$-module coincides with quasi-projectivity for finitely generated modules over commutative rings. We introduce the class of fqp-rings in which consist of rings in which every finitely generated ideal s quasi-projective equivalently a $\star$-module. Then, the first main result (Theorem 2.3.1) asserts that the class of fqp-rings stands strictly between the two classes of arithmetical rings and Gaussian rings; that is,

"R is an arithmetical ring $\Rightarrow$ R is an fqp-ring $\Rightarrow$ R is a Gaussian ring."

The second main result (Theorem 2.3.10) extends Osofsky's theorem on the weak global dimension of arithmetical rings and partially resolves Bazzoni-Glaz's related conjecture on Gaussian rings; we prove that:

"The weak global dimension of an fqp-ring is equal to 0, 1, or ∞."

The third main result (Theorem 2.3.13) establishes the transfer of the concept of fqp-ring between a local ring and its total ring of quotients; namely:

"A local ring R is an fqp-ring if and only if R is Prüfer and $Q(R)$ is an fqp-ring."

Finally, the main result (Theorem 2.4.4) about trivial ring extensions states that

"If $(A,\mathfrak{m})$ is a local ring, E a nonzero $\frac{A}{\mathfrak{m}}$-vector space, and $R := A \propto E$ the trivial ring extension of A by E, then R is an fqp-ring if and only if $\mathfrak{m}^2 = 0$."

This result generates new and original examples of fqp-rings, marking the distinction between the $\star$-property and related Prüfer conditions. We close this chapter with a brief study of n-star modules over Dedekind domains.

Chapter 1

Tilting modules over commutative rings

This chapter* provides a complete classification of all tilting modules and tilting classes over almost perfect domains, which generalizes the classifications of tilting modules and tilting classes over Dedekind and 1-Gorenstein domains. Assuming the APD is Noetherian, a complete classification of all cotilting modules is obtained (as duals of the tilting ones).

1.1 Introduction

Throughout, R is a commutative ring with $1_R \neq 0_R$ and all R-modules are unital. By $Z(R)$ we denoted the set of zero-divisors of R and set $R^{\times} := R \backslash Z(R)$. By $Q = (R^{\times})^{-1}R$ we denote the total ring of quotients of R (the field of quotients, if R is an integral domain). With R-Mod we denoted the category of R-modules.

Let M be an R-module. The character module of M is $M^c := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$. With

***Part of this chapter is submitted for publication under the title "Tilting modules over almost perfect domains" (in collaboration with Dr. J. Abuhlail).**

$\mathrm{Max}(M)$ we denote the (possibly empty) spectrum of maximal R-submodules and define

$$\mathrm{rad}(M_R) := \bigcap_{L \in \mathrm{Max}(M)} L \ \ (= M, \text{ if } \mathrm{Max}(M) = \varnothing).$$

In particular, $\mathrm{Max}(R)$ is the spectrum of maximal R-ideals and $J(R) := \mathrm{rad}(R_R)$ is the Jacobson radical of R. We denote with $\mathrm{p.d.}_R(M)$ (resp., $\mathrm{i.d.}_R(M)$, $\mathrm{w.d.}_R(M)$) the projective (resp., injective, weak or flat) dimension of M_R. Moreover, we set

$$\begin{array}{rclcrcl}
\mathscr{P}_n & := & \{M_R \mid \mathrm{p.d.}_R(M) \leq n\}; & & \mathscr{P} & := & \bigcup_{n=0}^{\infty} \mathscr{P}_n; \\
\mathscr{I}_n & := & \{M_R \mid \mathrm{i.d.}_R(M) \leq n\}; & & \mathscr{I} & := & \bigcup_{n=0}^{\infty} \mathscr{I}_n; \\
\mathscr{F}_n & := & \{M_R \mid \mathrm{w.d.}_R(M) \leq n\}; & & \mathscr{F} & := & \bigcup_{n=0}^{\infty} \mathscr{F}_n.
\end{array}$$

In particular, $\mathscr{P}\mathscr{R} := \mathscr{P}_0$ is the class of projective R-modules, $\mathscr{I}\mathscr{N} := \mathscr{I}_0$ is the class of injective R-modules, and $\mathscr{F}\mathscr{L} := \mathscr{F}_0$ is the class of flat R-modules. The class of torsion-free R-modules will be denoted with $\mathscr{T}\mathscr{F}$. For a multiplicative subset $S \subseteq R^{\times}$, the class of S-divisible R-modules is

$$\mathscr{D}_S := \{{}_R M \mid sM = M \text{ for every } s \in S\}.$$

In particular, $\mathscr{D}\mathscr{I} := \mathscr{D}_{R^{\times}}$ is the class of divisible R-modules. For any unreferenced material on domains and their modules we refer the reader to [42].

It is well known that every module over any ring has an injective envelope as shown by Eckmann and Schopf [29] (see [98, 17.9]). The dual result does not hold for the categorical dual notion of projective covers. Rings over which every (finitely generated) module has a projective cover were considered first by H. Bass [5] and called (semi-)perfect rings. At the beginning of the current century, Bican, Bashir, and Enochs [15] solved the so-called flat cover conjecture proving that every module has a flat cover. Recalling that the class of strongly flat modules $\mathscr{SFL}$ lies strictly between $\mathscr{FL}$ and $\mathscr{PR}$, rings over which every (finitely generated torsion) module has a strongly flat cover were studied by Bazzoni and Salce [11]; such rings were characterized as being almost (semi-)perfect, in the sense that every proper homomorphic image of such rings is (semi-)perfect (see also [12]). Since almost perfect rings that are not domains are perfect, and since perfect domains are fields, the interest is restricted to almost perfect domains (APD's). Although local APD's were studied earlier by Smith [85] under the name "local domains with topologically T-nilpotent radical" (local TTN-domains), the interest in them resurfaced only recently in connection with the revival of the theory of cotorsion pairs introduced by Salce [76]. Our main reference on APD's and their modules is the survey by Salce [80] (see also [11, 103, 12, 82, 78, 79, 104, 40]).

Tilting modules were introduced by Brenner and Butler [18] and then generalized by several authors (e.g. [55, 67, 27, 100, 2]). Cotilting modules appeared as vector space duals of tilting modules over finite dimensional (Artin) algebras (e.g. [54, IV.7.8.]) and then generalized in a number of papers (e.g. [25, 2, 101, 6]). A classification of (co)tilting modules over special classes of commutative rings and domains was initiated by Göbel and Trlifaj [49], who classified (co)tilting Abelian groups (assuming Gödel's axiom of constructibility; a condition removed later in [7]). (Co)tilting modules were classified also over Dedekind domains by Bazzoni et al. [7] (removing set theoretical assumptions in [89]), over valuation and Prüfer domains by Salce in [81, 77], and recently over arbitrary 1-Gorenstein rings by Trlifaj and Pospíšil [88].

An open problem in [49, Page 254] is "Characterize all tilting modules and classes over Matlis domains" (R is Matlis if $\text{p.d.}_R(Q) \leq 1$). Recalling that APD's are Matlis domains by [80, Proposition 2.5], a natural question in this connection was raised by L. Salce: "Characterize all tilting modules and classes over APD's". Our main result (Theorem 1.4.15) provides a complete answer:

MAIN THEOREM. Let R be an APD that is not a field.

(1) All tilting R-modules are 1-tilting and represented (up to equivalence) by

$$\{T(X) := \bigcap_{\mathfrak{m}\in X} R_{\mathfrak{m}} \bigoplus \frac{\bigcap_{\mathfrak{m}\in X} R_{\mathfrak{m}}}{R} \mid X \subseteq \mathrm{Max}(R)\}.$$

(2) $\{X\text{-Div} \mid X \subseteq \mathrm{Max}(R)\}$ is the class of all tilting classes, where

$$X\text{-Div} := \{{}_R M \mid \mathfrak{m}M = M \text{ for every } \mathfrak{m} \in X\}.$$

(3) If R is coprimely packed, then the set of Fuchs-Salce tilting modules

$$\{\partial_S \mid S \subseteq R^{\times} \text{ is a multiplicative subset}\}$$

classifies all tilting R-modules (up to equivalence).

This provides a partial solution to the above mentioned open problem on Matlis domains and generalizes the classification of tilting modules over 1-Gorenstein domains (which are properly contained in the class of APD's) and Dedekind domains.

The chapter is organized as follows. After this introductory section, we collect in the second section some preliminaries on (semi)perfect rings and almost (semi)perfect domains. In the third section, we characterize some classes of modules over APD's, proving the following:

$$\mathscr{I} = \mathscr{I}_1,\ \mathscr{F} = \mathscr{F}_1 = \mathscr{P}_1 = \mathscr{P},\ \mathscr{I}\mathscr{N} = \mathscr{D}\mathscr{I} \cap \mathscr{I}_1,\ \mathscr{F}\mathscr{L} = \mathscr{T}\mathscr{F} \cap \mathscr{P}_1,$$

$$\mathscr{D}\mathscr{I} = \{M \mid \mathrm{rad}(M_R) = M\}.$$

Although these results are meant to serve in proving the main result (Theorem 1.4.15), we include them in a separate section since we believe they are interesting on their own. In the fourth section, we present our main results. Since $\mathscr{I} = \mathscr{I}_1$ and $\mathscr{P} = \mathscr{P}_1$, we notice first that all (co)tilting modules over APD's are 1-(co)tilting. Moreover, we conclude (analogously to the case of Prüfer domains) that all torsion-free tilting modules over APD's are projective. In the local case, we prove that every tilting module over a local APD is either divisible or projective (see Theorem 1.4.11). Finally, we present in Theorem 1.4.15 a complete classification of all tilting modules over APD's that are not fields. Assuming moreover that the APD R is coprimely packed (e.g., R is semilocal), we show that any tilting R-module is equivalent to a Fuchs-Salce tilting R-module ∂_S for some suitable multiplicative subset $S \subseteq R^{\times}$. If R is a coherent APD, then the cotilting R-modules are precisely the (dual) character modules of the tilting ones (see Corollary 1.4.17). The fifth section sheds light on more interesting properties of modules over APD's. We close this chapter with a brief study, in the sixth section, of tilting modules over (n,d)-rings. In particular, we notice that for a ring R, "every finitely generated ideal is 1-tilting if and only if R is a Prüfer domain" (Corollary 1.6.6).

1.2 Perfect rings and almost perfect domains

In this section, we collect some preliminaries on (semi)perfect rings and almost (semi)perfect domains.

Definition 1.2.1 ([5])**.** The ring R is said to be (semi)perfect if every (finitely generated) R-module has a projective cover.

For the reader's convenience, we collect in the following lemma some of the characterizations of perfect commutative rings (e.g. [1, Section 28], [98, Section 43], [61, Chapter 8], [12, Theorem 1.1]):

Lemma 1.2.2. *The following are equivalent:*

(1) *R is perfect;*

(2) *every semisimple R-module has a projective cover;*

(3) *every flat R-module is (self-)projective;*

(4) *direct limits of projective R-modules are (self-)projective;*

(5) *R is semilocal and every nonzero R-module has a maximal submodule;*

(6) *R is semilocal and every nonzero R-module contains a simple submodule;*

(7) *R contains no infinite set of orthogonal idempotents and every nonzero R-module contains a simple submodule;*

(8) *$R/J(R)$ is semisimple and $J(R)$ is T-nilpotent;*

(9) *$R/J(R)$ is semisimple and R is semiartinian;*

(10) *R satisfies the DCC for principal (finitely generated) ideals;*

(11) *Any R-module satisfies the DCC on its cyclic (finitely generated) R-submodules;*

(12) *Any R-module satisfies the ACC on its cyclic R-submodules;*

(13) *R is a finite direct product of local rings with T-nilpotent maximal ideals;*

(14) *R is semilocal and $R_{\mathfrak{m}}$ is a perfect ring for every $\mathfrak{m} \in \mathrm{Max}(R)$;*

(15) *R is semilocal and semiartinian.* □

Definition 1.2.3 ([11, 12])**.** R is an almost (semi)perfect ring if R/I is (semi)perfect for every nonzero ideal $I \trianglelefteq R$.

Remark 1.2.4. An almost perfect ring that is not a domain is necessarily perfect by [12, Proposition 1.3]. On the other hand, any perfect domain is a field (e.g. [80, Corollary 1.3]). This restricts the interest to almost perfect domains (APD's).

Lemma 1.2.5 ([11, Theorem 4.9] and [42, Theorem IV.3.7])**.** *The following are equivalent for an integral domain R :*

(1) *R is almost semiperfect;*

(2) *every finitely generated torsion R-module has a strongly flat cover;*

(3) $Q/R \simeq \bigoplus_{\mathfrak{m}\in \mathrm{Max}(R)} (Q/R)_{\mathfrak{m}}$ *canonically;*

(4) *R is h-local (i.e., R/I is semilocal for every nonzero ideal $0 \neq I \trianglelefteq R$ and R/P is local for every nonzero prime ideal $0 \neq P \in \mathrm{Spec}(R)$).* □

In the following lemma we collect several characterizations of APD's (see [80, Main Theorem], [11], and [12]):

Lemma 1.2.6. *For an integral domain R with $Q \neq R$ the following are equivalent:*

(1) *R is an APD;*

(2) *R is almost semi-perfect and $R_{\mathfrak{m}}$ is an APD for every* $\mathfrak{m} \in \mathrm{Max}(R)$;

(3) *R is h-local and $R_{\mathfrak{m}}$ is an APD for every* $\mathfrak{m} \in \mathrm{Max}(R)$;

(4) *R is h-local and Q/R is semiartinian;*

(5) *R is h-local and for every proper nonzero ideal I of R the R-module R/I contains a simple R-submodule;*

(6) *every flat R-module is strongly flat;*

(7) *every R-module has a strongly flat cover;*

(8) *every weakly cotorsion R-module is cotorsion ;*

(9) *every R-module with weak dimension at most* 1 *has projective dimension at most* 1 *(i.e.,* $\mathscr{F}_1 = \mathscr{P}_1$*);*

(10) *every divisible R-module is weak-injective.* □

Remarks 1.2.7. Let R be an integral domain.

(1) R is a coherent APD if and only if R is Noetherian and 1-dimensional (see [11, Propositions 4.5, 4.6]). Whence, Dedekind domains are precisely the Prüfer APD's.

(2) A valuation domain R is an APD if and only if R is a DVR (e.g. [80, Example 2.2]).

(3) We have the following implications (e.g. [42], [80]): R is Dedekind $\Rightarrow$ R is 1-Gorenstein $\Rightarrow$ R is 1-dimensional and Noetherian $\Rightarrow$ R is an APD $\Rightarrow$ R is a 1-dimensional h-local $\Rightarrow$ R is a Matlis domain.

The following examples illustrate that the implications above are not reversible:

Examples 1.2.8. (1) Let d be a square-free integer such that $d \equiv 1 \pmod 4$ and consider the commutative Noetherian subring

$$R := \{\frac{m}{2n+1} + \frac{m'}{2n'+1}\sqrt{d} \mid m, m', n, n' \in \mathbb{Z}\} \subseteq \mathbb{Q}[\sqrt{d}].$$

By [86, Corollary 4.5], R is a 1-Gorenstein domain that is not Dedekind.

(2) Let K be a field. Then $R = K[|t^3, t^5, t^7|]$ is a Noetherian 1-dimensional domain which is not 1-Gorenstein (e.g. [65, Ex. 18.8]).

(3) Let K be a field and $V = (K[[x]], M)$ the local domain of power series in the indeterminate x with coefficients in K and with maximal ideal $M := xK[[x]]$. Let $(D, \mathfrak{m})$ be a local subring of K and consider the local integral domain $R := (D+M, \mathfrak{m}+M)$. By [12, Lemma 3.1], R is an APD if and only if D is a field. Moreover, by [12, Example 3.3], if $D = F$ is a field and $K = F(X)$, then R is Noetherian if and only if $[K : F] < \infty$. So, if $[K : F] = \infty$, then R is a non-Noetherian APD whence not 1-Gorenstein.

(4) Any rank-one non-discrete valuation domain is a 1-dimensional local Matlis domain that is not an APD (a concrete example is [104, Example 1.3]).

(5) Any almost Dedekind domain which is not Dedekind is a 1-dimensional Matlis domain that is not of finite character, whence not h-local (for a concrete example see [42, Example III.5.5]).

Generalizing the so-called Prime Avoidance Theorem (e.g. [83, 3.61]) by allowing infinite unions of prime ideals led to the following notions.

1.2.9 ([74, 33])**.** An ideal I of a commutative ring R is said to be coprimely packed (resp., compactly packed) if for any set of maximal (resp., prime) R-ideals $\{P_\lambda\}_\Lambda$ we have

$$I \subseteq \bigcup_{\lambda\in\Lambda} P_\lambda \Rightarrow I \subseteq P_{\lambda_0} \text{ for some } \lambda_0 \in \Lambda. \tag{1.1}$$

A class of R-ideals $\mathscr{E}$ said to be coprimely packed (resp., compactly packed) if every ideal in $\mathscr{E}$ is so. The ring R is said to be coprimely packed (resp., compactly packed) if every ideal of R is coprimely packed (resp., compactly packed).

Remark 1.2.10. By [34, Lemma 2] (resp., [14, Theorem 2.3]), a ring R is coprimely packed (resp., compactly packed) if and only if Spec(R) is coprimely packed (resp., compactly packed). Indeed, 1-dimensional rings (e.g. APD's) are coprimely packed if and only if they are compactly packed. By [74] a Dedekind domain is compactly packed (equivalently coprimely packed) if and only if its ideal class group is torsion (see also [33, Theorem 1.4]). Semilocal rings are obviously coprimely packed (by the Prime Avoidance Theorem). A coprimely packed domain R is h-local if, for example, R is 1-dimensional by [33, Proposition 1.3] and [64, Theorem 3.22] (see also [42, Theorem 3.7, EX. IV.3.3]) or if Q/R is injective by [17, Theorem 9]. While clearly all compactly packed rings are coprimely packed, it had been shown in [74] that a Noetherian compactly packed ring has Krull dimension at most one; thus any semilocal Noetherian ring with Krull dimension at least 2 is coprimely packed but not compactly packed.

Example 1.2.11. Let K be an algebraically closed field and F a proper subfield such that $[K:F]=\infty$ and X an indeterminate. By [80, Example 5.5], $R := F + XK[X]$ is a non-coherent APD with $\mathrm{Max}(R) = \{XK[X]\} \cup \{(1-aX)R \mid a \in K^{\times}\}$. Clearly, R is a coprimely packed (compactly packed) APD that is not semilocal.

1.3 Special modules over almost perfect domains

In this section, we characterize the injective modules, the torsion-free modules, and the divisible modules over almost perfect domains. Moreover, we show that over such integral domains $\mathscr{I} = \mathscr{I}_1$, $\mathscr{F} = \mathscr{F}_1 = \mathscr{P}_1 = \mathscr{P}$. Throughout in this section, R is an almost perfect domain with $Q \neq R$.

Dedekind domains are characterized by the fact that every divisible module is injective (e.g. [75, Theorem 4.24], [98, 40.5]). This inspires:

Proposition 1.3.1. *An R-module M is injective if and only if M is divisible and* $\mathrm{i.d.}_R(M) \leq 1$, *i.e.,*

$$\mathscr{I}\mathscr{N} = \mathscr{D}\mathscr{I} \cap \mathscr{I}_1. \tag{1.2}$$

Proof. ($\Rightarrow$) Injective modules over any ring are divisible (e.g. [98, 16.6]).

($\Leftarrow$) Assume that M_R is divisible and $\text{i.d.}_R(M) \leq 1$.

Case 1. $(R, \mathfrak{m})$ is local. Let $0 \neq r \in R$. By Lemma 1.2.6 (5), the R-module R/Rr contains a simple R-submodule J/Rr ($\simeq R/\mathfrak{m}$, since $\text{Max}(R) = \{\mathfrak{m}\}$). So, we have a short exact sequence of R-modules

$$0 \to J/Rr \to R/Rr \to R/J \to 0.$$

Applying the contravariant functor $\text{Hom}_R(-, M)$, we get a long exact sequence

$$\cdots \to \text{Ext}^1_R(R/Rr, M) \to \text{Ext}^1_R(J/Rr, M) \to \text{Ext}^2_R(R/J, M) \to \cdots$$

Since M_R is divisible, we have $\text{Ext}^1_R(R/Rr, M) = 0$ by [42, Lemma I.7.2]; and since $\text{i.d.}_R(M) \leq 1$, we have $\text{Ext}^2_R(R/J, M) = 0$. It follows that $\text{Ext}^1_R(R/\mathfrak{m}, M) \simeq \text{Ext}^1_R(J/Rr, M) = 0$, whence M_R is injective by [80, Proposition 8.1. (1)].

Case 2. R is arbitrary. Let $\mathfrak{m} \in \text{Max}(R)$ be arbitrary. Since R is h-local, it follows by [42, Theorem IX.7.6] that localizing any injective coresolution of R-modules at $\mathfrak{m}$ yields an injective coresolution of $R_\mathfrak{m}$-modules, hence $\text{i.d.}_{R_\mathfrak{m}}(M_\mathfrak{m}) \leq 1$. Since ${}_{R_\mathfrak{m}}M_\mathfrak{m}$ is also divisible, we conclude that ${}_{R_m}M_\mathfrak{m}$ is injective by the proof of Case 1. Since R is h-local, we have (e.g. [64], [42, Theorem IX.7.6])

$$\text{i.d.}_R(M) = \sup\{\text{i.d.}_{R_\mathfrak{m}}(M_\mathfrak{m}) \mid \mathfrak{m} \in \text{Max}(R)\} = 0.$$

□

It is well-known that for 1-Gorenstein domains (and general 1-Gorenstein rings), we have $\mathscr{I} = \mathscr{I}_1 = \mathscr{F} = \mathscr{F}_1 = \mathscr{P} = \mathscr{P}_1$ (e.g. [32, 9.1.10], [49, 7.1.12]). For the strictly larger class of APD's (see Example 1 (3)), these hold partially.

Proposition 1.3.2. *We have*

$$\mathscr{I} = \mathscr{I}_1, \ \mathscr{F} = \mathscr{F}_1 = \mathscr{P}_1 = \mathscr{P}. \tag{1.3}$$

Proof. Let R be an APD.

- We prove, by induction, that any R-module M with finite injective dimension at most n has injective dimension at most 1. If $n = 0$, we are done. Let $n \geq 1$ and assume the statement is true for $n-1$. Let

$$0 \to M \xrightarrow{f_0} E_0 \xrightarrow{f_1} E_1 \to \cdots \longrightarrow E_{n-2} \xrightarrow{f_{n-1}} E_{n-1} \xrightarrow{f_n} E_n \longrightarrow 0$$

  be an injective coresolution of M_R and $L := \mathrm{Im}(f_{n-1}) = \mathrm{Ker}(f_n)$. Being a homomorphic image of a divisible R-module, L is divisible and obviously $\mathrm{i.d.}_R(L) \leq 1$ whence L_R is injective by Proposition 1.3.1. It follows that $\mathrm{i.d.}_R(M) \leq n-1$, whence $\mathrm{i.d.}_R(M) \leq 1$ by the induction hypothesis.

- Let M be with finite weak (flat) dimension at most n. By [42, Proposition IX. 7.7] we have for any injective cogenerator $\mathbf{E}_R$:

$$\mathrm{i.d.}_R(\mathrm{Hom}_R(M, \mathbf{E})) = \mathrm{w.d.}_R(M) \tag{1.4}$$

  and we conclude that $\mathrm{w.d.}_R(M) \leq 1$ by the first part of the proof.

- Let M_R be with finite projective dimension at most n. Since $\mathrm{w.d.}_R(M) \leq \mathrm{p.d.}_R(M) \leq n$, we have $M \in \mathscr{F}_1 = \mathscr{P}_1$ by Lemma 1.2.6 (9). □

Using Proposition 1.3.2 we conclude that an APD is either Dedekind or has (weak) global dimension ∞. This provides new characterizations of Dedekind domains and recovers the fact that Dedekind domains are precisely the Prüfer APD's.

Corollary 1.3.3. *An integral domain R (not necessarily APD) is Dedekind if and only if R is an APD with finite (weak) global dimension if and only if R is an APD with (weak) global dimension at most one if and only if R is a Prüfer APD.*

Proposition 1.3.4. *An R-module M is flat if and only if M is torsion-free and* $\mathrm{p.d.}_R(M) \le 1$, *i.e.,*

$$\mathscr{F}\mathscr{L} = \mathscr{T}\mathscr{F} \cap \mathscr{P}_1 = \mathscr{T}\mathscr{F} \cap \mathscr{F}_1. \tag{1.5}$$

Proof. ($\Rightarrow$) Follows by the well-known fact that flat modules over domains are torsion-free (e.g. [98, 36.7]). So, we are done by $\mathscr{F}_1 = \mathscr{P}_1$ (Lemma 1.2.6 (9)).

($\Leftarrow$) Since M_R is torsion-free, it embeds in a vector space over Q (e.g. [75, Lemma 4.33]). So, we have a short exact sequence of R-modules

$$0 \to M \to Q^{(\Lambda)} \to Q^{(\Lambda)}/M \to 0.$$

Since $Q_R^{(\Lambda)}$ is flat, $\mathrm{p.d.}_R(Q^{(\Lambda)}) \le 1$ by Lemma 1.2.6 (9). It follows by [42, Lemma VI.2.4] that $\mathrm{p.d.}_R(Q^{(\Lambda)}/M) < \infty$, whence $Q^{(\Lambda)}/M \in \mathscr{P}_1 = \mathscr{F}_1$ by Proposition 1.3.2. Consequently, M_R is flat. □

1.3.5 ([49])**.** An R-module over an (arbitrary ring) R is said to be strongly finitely presented if it possesses a projective resolution consisting of finitely generated R-modules. With R-mod we denote the class of strongly finitely presented modules. In case R is coherent, R-mod coincides with the class of finitely presented R-modules.

Proposition 1.3.6. *The following are equivalent for an R-module M :*

(1) *M_R is divisible;*

(2) $\mathrm{rad}(M_R) = M$ *(i.e., M has no maximal R-submodules);*

(3) $\mathfrak{m}M = M$ *for every* $\mathfrak{m} \in \mathrm{Max}(R)$.

Proof. The result is obvious for $M = 0$. So, assume $M \neq 0$. The equivalence $(1) \Leftrightarrow (3)$ is already known for APD's (e.g. L. Salce [80, Proposition 8.1]).

$(1) \Rightarrow (2)$ Suppose that M contains a maximal R-submodule L. Then $M/L \simeq R/\mathfrak{m}$ for some maximal ideal $\mathfrak{m} \trianglelefteq R$. Since M_R is divisible by assumption, it follows that $R/\mathfrak{m}$ is also a divisible R-module (a contradiction).

$(2) \Rightarrow (1)$ Suppose M_R is not divisible. Then there exists $0 \neq r \in R$ such that $rM \neq M$. By Lemma 1.2.2 (5), the nonzero R/rR-module M/rM contains a maximal submodule N/rM. Then there exists $\mathfrak{m} \in \mathrm{Max}(R)$, such that

$$R/\mathfrak{m} \simeq (R/rR)/(\mathfrak{m}/rR) \simeq (M/rM)/(N/rM) \simeq M/N.$$

This implies that $N \in \mathrm{Max}(M)_R$ (a contradiction). □

Definition 1.3.7. A non-empty set $\mathscr{L}$ of R-ideals is said to be a localizing system (or a Gabriel topology) if for any ideals $I, J \trianglelefteq R$ we have:

(LS1) If $I \in \mathscr{L}$ and $I \subseteq J$, then $J \in \mathscr{L}$;

(LS2) If $I \in \mathscr{L}$ and $(J :_R r) \in \mathscr{L}$ for every $r \in I$, then $J \in \mathscr{L}$.

Definition 1.3.8. Let R be an integral domain and $\mathscr{E}$ be a class of R-ideals. We say an R-module M is $\mathscr{E}$-divisible if $IM = M$ for every $I \in \mathscr{E}$.

For any classes $\mathscr{M}$ of R-modules and $\mathscr{E}$ of R-ideals we set

$$\begin{aligned} \mathscr{D}(\mathscr{M}) &:= \{I \trianglelefteq R \mid IM = M \text{ for every } M \in \mathscr{M}\}; \\ \mathscr{E}\text{-Div} &:= \{M_R \mid IM = M \text{ for every } I \in \mathscr{E}\}. \end{aligned}$$

If R is a domain, then $\mathscr{D}(M_R)$ is a localizing system by [78, Lemma 1.1].

Lemma 1.3.9. *Let R be an APD and $\mathfrak{F}$ a localizing system. An R-module M is $\mathfrak{F}$-divisible if and only if $\mathfrak{m}M = M$ for all maximal ideals $\mathfrak{m}$ in $\mathfrak{F}$, i.e.,*

$$\mathfrak{F}\text{-Div} = (\mathfrak{F} \cap \operatorname{Max}(R))\text{-Div}. \tag{1.6}$$

Proof. Let $M \in (\mathfrak{F} \cap \operatorname{Max}(R))$-Div. Let $I \in \mathfrak{F}$ and set $\mathscr{M}(I) := \{\mathfrak{m} \in \operatorname{Max}(R) \mid I \subseteq \mathfrak{m}\} \subseteq \mathfrak{F}$ by (LS1). Let $\mathfrak{m} \in \operatorname{Max}(R)$. If $\mathfrak{m} \in \mathscr{M}(I)$, then $\mathfrak{m}_\mathfrak{m} M_\mathfrak{m} = (\mathfrak{m}M)_\mathfrak{m} = M_\mathfrak{m}$ whence the $R_\mathfrak{m}$-module $M_\mathfrak{m}$ is divisible by Proposition 1.3.6, and it follows that $(IM)_\mathfrak{m} = I_\mathfrak{m} M_\mathfrak{m} = M_\mathfrak{m}$. On the other hand, if $\mathfrak{m} \notin \mathscr{M}(I)$, then $I_\mathfrak{m} = R_\mathfrak{m}$ and so $(IM)_\mathfrak{m} = R_\mathfrak{m} M_\mathfrak{m} = M_\mathfrak{m}$. Since $(IM)_\mathfrak{m} = M_\mathfrak{m}$ for every $\mathfrak{m} \in \operatorname{Max}(R)$, we conclude that $IM = M$ (i.e., $M \in \mathfrak{F}$-Div). □

1.4 Classification of tilting and cotilting modules over APD's

This section is devoted to the classification of (co)tilting modules over APD's. For any unreferenced material we refer the reader to [49]. The next general definition of a tilting module is due to Angeleri Hügel and Coelho in [2].

Definition 1.4.1. An R-module T is tilting, provided that

(1) $\text{p.d.}_R(T) < \infty$,

(2) $\operatorname{Ext}^i_R(T, T^{(\Lambda)}) = 0$ for every index set Λ and all i ≥ 1,

(3) There exist $T_0, ..., T_n \in \operatorname{Add}({}_RT)$ fitting in an exact sequence

$$0 \to R \to T_0 \to ... \to T_n \to 0$$

where $\operatorname{Add}({}_RT)$ denotes the class of all R-modules which are isomorphic to summands of direct sums of T.

For any class of R-modules $\mathscr{M}$ we set

$$\begin{aligned}\mathscr{M}^{\perp_\infty} &:= \{N_R \mid \operatorname{Ext}^i_R(M,N)=0 \text{ for all } i\geq 1 \text{ and every } M\in\mathscr{M}\};\\ {}^{\perp_\infty}\mathscr{M} &:= \{N_R \mid \operatorname{Ext}^i_R(N,M)=0 \text{ for all } i\geq 1 \text{ and every } M\in\mathscr{M}\}.\end{aligned}$$

Moreover, we set

$$\mathscr{M}^{\perp} := \bigcap_{M\in\mathscr{M}} \operatorname{Ker}(\operatorname{Ext}^1_R(M,-)) \text{ and } {}^{\perp}\mathscr{M} := \bigcap_{M\in\mathscr{M}} \operatorname{Ker}(\operatorname{Ext}^R_1(-,M)).$$

1.4.2. For an R-module X_R, let $\operatorname{Gen}_n(X_R)$ be the class of R-modules M possessing an exact sequence of R-modules $X^{(\Lambda_n)} \to \cdots \to X^{(\Lambda_1)} \to M \to 0$ (for index sets $\Lambda_1,\cdots,\Lambda_n$). Dually, let $\operatorname{Cogen}_n(X_R)$ be the class of R-modules M possessing an exact sequence of R-modules $0\to M\to X^{\Lambda_1}\to\cdots\to X^{\Lambda_n}$ (for index sets $\Lambda_1,\cdots,\Lambda_n$). In particular, $\operatorname{Gen}(X_R) := \operatorname{Gen}_1(X_R)$ is the class of X-generated R-modules and $\operatorname{Cogen}(X_R) := \operatorname{Cogen}_1(X_R)$ is the class of X-cogenerated R-modules.

1.4.3. Let $\mathscr{A}$ and $\mathscr{B}$ be two classes of R-modules. Then $(\mathscr{A},\mathscr{B})$ is said to be a cotorsion pair if $\mathscr{A} = {}^{\perp}\mathscr{B}$ and $\mathscr{B} = \mathscr{A}^{\perp}$. If, moreover, $\operatorname{Ext}^i_R(A,B)=0$ for all $i\geq 1$ and $A\in\mathscr{A}$, $B\in\mathscr{B}$ we say $(\mathscr{A},\mathscr{B})$ is hereditary. Each class $\mathscr{M}$ of R-modules generates a cotorsion pair $({}^{\perp}(\mathscr{M}^{\perp}),\mathscr{M}^{\perp})$ and cogenerates a cotorsion pair $({}^{\perp}\mathscr{M},({}^{\perp}\mathscr{M})^{\perp})$. For two cotorsion pairs $(\mathscr{A},\mathscr{B})$, $(\mathscr{A}',\mathscr{B}')$, we have $\mathscr{A}=\mathscr{A}'$ if and only if $\mathscr{B}=\mathscr{B}'$.

1.4.4. An R-module T is said to be n-tilting if $\operatorname{Gen}_n(T_R) = T^{\perp_\infty}$; the induced n-tilting class $T^{\perp_\infty}$ cogenerates a hereditary cotorsion pair $({}^{\perp}(T^{\perp_\infty}),T^{\perp_\infty})$ with $\mathscr{A} := {}^{\perp}(T^{\perp_\infty}) \subseteq \mathscr{P}_n$ by [50, Lemma 5.1.8] (in particular, $\text{p.d.}_R(T)\leq n$). By [50, Lemma 6.1.2] (see also [6, Theorem 3.11]), T_R is 1-tilting if $\operatorname{Gen}(T_R) = T^{\perp}$. An R-module T is tilting if and only if T is n-tilting for some $n\geq 0$. Two tilting R-modules T_1, T_2 are said to be equivalent ($T_1\sim T_2$) if and only if $T_1^{\perp_\infty} = T_2^{\perp_\infty}$.

1.4.5. An R-module C is said to be n-cotilting if $\mathrm{Cogen}_n(C_R) = {}^{\perp_\infty}C$; the induced n-cotilting class ${}^{\perp_\infty}C$ generates a hereditary cotorsion pair $({}^{\perp_\infty}C, ({}^{\perp_\infty}C)^{\perp})$ with $\mathscr{B} := ({}^{\perp_\infty}C)^{\perp} \subseteq \mathscr{I}_n$ by [50, Lemma 8.1.4] (in particular, $\mathrm{i.d.}_R(C) \leq n$). By [50, Lemma 8.2.2] (see also [6, Theorem 3.11]), C_R is 1-cotilting if and only if $\mathrm{Cogen}(C_R) = {}^{\perp}C$. An R-module C is said to be cotilting if C is n-cotilting for some $n \geq 0$. Two cotilting R-modules C_1, C_2 are said to be equivalent ($C_1 \sim C_2$) if ${}^{\perp_\infty}C_1 = {}^{\perp_\infty}C_2$.

Remark 1.4.6. Obviously, the 0-tilting modules are precisely the projective generators, while the 0-cotilting modules are precisely the injective cogenerators.

Example 1.4.7. Let R be an integral domain, $S \subseteq R^{\times}$ a multiplicative subset, and $\omega = ()$ be the empty sequence. Let F be the free R-module with basis

$$\beta := \{(s_0, \cdots, s_n) \mid n \geq 0 \text{ and } s_j \in S \text{ for } 0 \leq j \leq n\} \cup \{\omega\}$$

and G the R-submodule of F (which is in fact free) generated by

$$\{(s_0, \cdots, s_n)s_n - (s_0, \cdots, s_{n-1}) \mid n > 0 \text{ and } s_j \in S \text{ for } 0 \leq j \leq n\} \cup \{(s)s - \omega\}.$$

The R-module $\partial_S := F/G$ is a 1-tilting R-module with $\partial_S^{\perp} = \mathrm{Gen}(\partial_S) = \mathscr{D}_S$ as shown in [41] and we call it the Fuchs-Salce module. It generalizes the Fuchs module $\partial := \partial_{R^{\times}}$ (introduced in [39]), which was studied and shown to be 1-tilting with $\partial^{\perp} = \mathrm{Gen}(\partial_R) = \mathscr{D}\mathscr{I}$ by A. Facchini in [35] and [36].

Definition 1.4.8 ([50])**.** A Matlis localization of the commutative ring R is $S^{-1}R$, where $S \subseteq R^{\times}$ is a multiplicative subset and $\mathrm{p.d.}_R(S^{-1}R) \leq 1$.

Lemma 1.4.9 ([50, Proposition 5.2.24] and [3, Theorem 1.1])**.** *Let R be a commutative ring and $S \subseteq R^{\times}$ a multiplicative subset.*

(1) *Let T be an n-tilting R-module, $\mathscr{T} := T^{\perp_\infty}$ the induced n-tilting class and*

$$\mathscr{T}_S := \{{}_{S^{-1}R}N \mid N \simeq S^{-1}M \text{ for some } M \in \mathscr{T}\}.$$

Then $S^{-1}T$ is an n-tilting $S^{-1}R$-module and its induced n-tilting class is

$$(S^{-1}T)^{\perp_\infty} := \bigcap_{i\geq 1} \mathrm{Ker}(\mathrm{Ext}^i_{S^{-1}R}(S^{-1}T, -)) = \mathscr{T}_S = T^{\perp_\infty} \cap S^{-1}R\text{-Mod}.$$

Moreover, $M_R \in \mathscr{T}$ if and only if $M_\mathfrak{m} \in \mathscr{T}_\mathfrak{m}$ for every $\mathfrak{m} \in \mathrm{Max}(R)$. If T' is another n-tilting R-module, then

$$T \sim T' \Leftrightarrow T_\mathfrak{m} \sim T'_\mathfrak{m} \text{ for all maximal ideals } \mathfrak{m} \in \mathrm{Max}(R). \tag{1.7}$$

(2) *The following are equivalent:*

(a) $\mathrm{p.d.}_R(S^{-1}R) \leq 1$ *(i.e., $S^{-1}R$ is a Matlis localization);*

(b) $T(S) := S^{-1}R \oplus \frac{S^{-1}R}{R}$ *is a 1-tilting R-module;*

(c) $\mathrm{Gen}(S^{-1}R_R) = \mathscr{D}_S$.

Moreover, in this case $T(S)^{\perp_\infty} = \mathrm{Gen}(T(S)) = \mathscr{D}_S$. □

We prove now some fundamental properties of (co)tilting modules over APD's, some of which are analogous to the case of Prüfer domains:

Proposition 1.4.10. *Let R be an APD with $R \neq Q$.*

(1) *All tilting R-modules are 1-tilting.*

(2) *The torsion-free tilting R-modules are precisely the projective generators (i.e., the 0-tilting R-modules) and are all equivalent to R.*

(3) *Every divisible tilting R-modules generates* $\mathscr{DI}$, *whence is equivalent to* ∂.

(4) *All localizations of R are Matlis localizations. For every multiplicative subset* $S \subseteq R^{\times}$ *we have a tilting R-module* $T(S) := S^{-1}R \oplus S^{-1}R/R \sim \partial_S$ *and a cotilting R-module* $T(S)^c \sim \partial_S^c$.

(5) *All cotilting R-modules are* 1*-cotilting.*

(6) *The divisible cotilting R-modules are precisely the injective cogenerators (i.e., the* 0*-cotilting R-modules) and are equivalent to* $R^c := \mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})$.

Proof.

(1) Follows directly from $\mathscr{P} = \mathscr{P}_1$ (1.3).

(2) If T_R is a torsion-free tilting R-module, then by "1": $T \in \mathscr{TF} \cap \mathscr{P}_1 \overset{(1.5)}{=} \mathscr{FL}$, whence T_R is projective (since flat 1-tilting modules over arbitrary rings are projective by [10, Corollary 2.8]). In this case, $\mathrm{Gen}(T_R) = T^{\perp} = R\text{-Mod} = R^{\perp}$; consequently, T_R is a projective generator and $T \sim R$.

(3) Recall that $\mathscr{F}_1$ generates a cotorsion pair $(\mathscr{F}_1, \mathscr{WI})$, where (by definition) $\mathscr{WI} := \mathscr{F}_1^{\perp}$ is the class of weak-injective R-modules. Notice that conditions (8) and (9) of Lemma 1.2.6 can be expressed as $(\mathscr{F}_1, \mathscr{WI}) = (\mathscr{P}_1, \mathscr{DI})$. Let T be a tilting R-module and consider the induced cotorsion pair $({}^{\perp}(T^{\perp}), T^{\perp})$. If T_R is divisible, then $T^{\perp} = \mathrm{Gen}(T_R) \subseteq \mathscr{DI}$, whence $\mathscr{P}_1 = {}^{\perp}\mathscr{DI} \subseteq {}^{\perp}(T^{\perp}) \subseteq \mathscr{P}_1$. So, $\partial^{\perp} = \mathscr{DI} = \mathscr{P}_1^{\perp} = T^{\perp} = \mathrm{Gen}(T_R)$, i.e., T generates $\mathscr{DI}$ and $T \sim \partial$.

(4) For every multiplicative subset $S \subseteq R^{\times}$, the localization $S^{-1}R$ is a flat R-module whence $\mathrm{p.d.}_R(S^{-1}R) \leq 1$ by Lemma 1.2.6 (9). It follows by Lemma 1.4.9 (2) that $T(S) := S^{-1}R \oplus \frac{S^{-1}R}{R}$ is a tilting R-module with $T(S)^{\perp} = \mathscr{D}_S = \partial_S^{\perp}$, whence $T(S) \sim \partial_S$. The character module of any tilting R-module is cotilting by [50, Theorem 8.1.2],

whence $T(S)^c$ is a cotilting R-module which is equivalent to ∂_S^c (e.g. [50, Theorem 8.1.13]).

(5) Follows directly from $\mathscr{I} = \mathscr{I}_1$ (1.3).

(6) If C_R is a divisible cotilting R-module, then by "6": $C \in \mathscr{D}\mathscr{I} \cap \mathscr{I}_1 \overset{(1.2)}{=} \mathscr{I}\mathscr{N}$. In this case, $\mathrm{Cogen}(C_R) = {}^{\perp}C = R\text{-Mod} = {}^{\perp}R^c$; consequently, C_R is an injective cogenerator and $C \sim R^c$. □

The following is a key result that will be used frequently in the sequel.

Theorem 1.4.11. *Let $(R,\mathfrak{m})$ be a local APD with $R \neq Q$. Any tilting R-module is either projective or divisible. Hence, R has exactly two tilting modules $\{R, \partial\}$ (up to equivalence) and exactly two tilting classes $\{R\text{-Mod}, \mathscr{D}\mathscr{I}\}$.*

Proof. Let T be a tilting R-module and assume that T_R is not divisible. Then $T \neq 0$ and contains by Proposition 1.3.6 a maximal R-submodule N such that $T/N \simeq R/\mathfrak{m}$. By [13] all tilting modules (over arbitrary rings) are of finite type. So, there exists $\mathscr{S} \subseteq \mathscr{P}_1 \cap R\text{-mod}$ such that $R/\mathfrak{m} \in \mathrm{Gen}(T_R) = T^{\perp} = \mathscr{S}^{\perp}$. Let $M \in \mathscr{S}$, so that $\mathrm{Ext}_R^1(M, R/\mathfrak{m}) = 0$. Since the field $R/\mathfrak{m}$ is indeed injective as a module over itself, it follows (e.g. [42, Page 34 (6)]) that

$$\begin{aligned} \mathrm{Tor}_1^R(R/\mathfrak{m}, M) &\simeq \mathrm{Tor}_1^R(\mathrm{Hom}_{R/\mathfrak{m}}(R/\mathfrak{m}, R/\mathfrak{m}), M) \\ &\simeq \mathrm{Hom}_{R/\mathfrak{m}}(\mathrm{Ext}_R^1(M, R/\mathfrak{m}), R/\mathfrak{m}) = 0. \end{aligned}$$

By [16, II.3.2.Corollary 2], M_R is projective (being finitely presented and flat). So, $\mathscr{S} \subseteq \mathscr{P}\mathscr{R}$, whence T_R is projective. □

Recall (from [53]) that an R-submodule M of an R-module N is said to be a restriction submodule if $M_{\mathfrak{m}} = N_{\mathfrak{m}}$ or $M_{\mathfrak{m}} = 0$ for every $\mathfrak{m} \in \mathrm{Max}(R)$. For any subset $X \subseteq \mathrm{Max}(R)$, we set

$$R_{(X)} := \bigcap_{\mathfrak{m} \in X} R_{\mathfrak{m}} \ (:= Q, \text{ if } X = \varnothing) \,.$$

Lemma 1.4.12. *Let $R \neq Q$, $X \subseteq \mathrm{Max}(R)$, $X' := \mathrm{Max}(R)\backslash X$ and consider*

$$M_1 := \frac{R_{(X)}}{R} \text{ and } M_2 := \frac{R_{(X')}}{R}.$$

(1) *If R is an h-local domain, then $M_1, M_2 \subseteq \frac{Q}{R}$ are restriction R-submodules and*

$$\frac{Q}{R} = M_1 \oplus M_2 = \frac{R_{(X)}}{R} \oplus \frac{R_{(X')}}{R}. \tag{1.8}$$

(2) *If R is a 1-dimensional h-local domain, then*

$$T(X) := R_{(X)} \bigoplus \frac{R_{(X)}}{R} \ \ (= Q \oplus \frac{Q}{R}, \text{ if } X = \varnothing)$$

is a 1-tilting R-module.

Proof. Recall first that if $\mathfrak{m}, \mathfrak{m}' \in \mathrm{Max}(R)$ are such that $\mathfrak{m} \neq \mathfrak{m}'$, then we have by [64, Theorem 3.19] (see also [42, IV.3.2]):

$$R_{\mathfrak{m}} \otimes_R R_{\mathfrak{m}'} \simeq (R_{\mathfrak{m}})_{\mathfrak{m}'} = Q. \tag{1.9}$$

Moreover, if $\{R_\lambda\}_\Lambda$ is a class of R-submodules of Q with $\bigcap_{\lambda \in \Lambda} R_\lambda \neq 0$, then it follows from [42, IV.3.10] that

$$(\bigcap_{\lambda \in \Lambda} R_\lambda)_{\mathfrak{m}} = \bigcap_{\lambda \in \Lambda} (R_\lambda)_{\mathfrak{m}} \text{ for every } \mathfrak{m} \in \mathrm{Max}(R). \tag{1.10}$$

(1) Clearly $M_1 \cap M_2 = 0$. Let $\mathfrak{m}' \in \operatorname{Max}(R)$. Then

$$(M_1)_{\mathfrak{m}'} = \frac{(R_{(X)})_{\mathfrak{m}'}}{R_{\mathfrak{m}'}} \overset{(1.10)}{=} \frac{\bigcap_{\mathfrak{m}\in X} (R_{\mathfrak{m}})_{\mathfrak{m}'}}{R_{\mathfrak{m}'}} \overset{(1.9)}{=} \begin{cases} 0, & \mathfrak{m}' \in X \\ \\ \frac{Q}{R_{\mathfrak{m}'}}, & \mathfrak{m}' \notin X \end{cases}.$$

Similarly,

$$(M_2)_{\mathfrak{m}'} = \begin{cases} \frac{Q}{R_{\mathfrak{m}'}}, & \mathfrak{m}' \in X \\ \\ 0, & \mathfrak{m}' \notin X \end{cases}.$$

So, $M_1, M_2 \subseteq \frac{Q}{R}$ are restriction R-submodules. Moreover, we have $(M_1 \oplus M_2)_{\mathfrak{m}'} = (M_1)_{\mathfrak{m}'} \oplus (M_2)_{\mathfrak{m}'} = \frac{Q}{R_{\mathfrak{m}'}} = (\frac{Q}{R})_{\mathfrak{m}'}$ for all $\mathfrak{m}' \in \operatorname{Max}(R)$, and so $\frac{Q}{R} = M_1 \oplus M_2$.

(2) Notice first that a 1-dimensional h-local domain is a Matlis domain (in fact $\text{p.d.}_R(Q) = \text{p.d.}_R(\frac{Q}{R}) = 1$ as shown in [80, Lemma 2.4]). For any $X \subseteq \operatorname{Max}(R)$, we have $\frac{Q}{R} \overset{(1.8)}{=} \frac{R_{(X)}}{R} \oplus \frac{R_{(X')}}{R}$ and so $T(X)$ is a 1-tilting R-module by [3, Theorem 8.2]. □

Remark 1.4.13. Although we proved (1.8) for general h-local domains, we point out here that it can be obtained for an APD R by applying [3, Theorem 3.10] to $M_1 := \frac{R_{(X)}}{R}$. Then $X_1 := \operatorname{Supp}(M_1) = \operatorname{Max}(R) \backslash X$ and $X_2 := \operatorname{Supp}(Q/R) \backslash X_1 = X$. Consider the embedding $\varphi : \frac{Q}{R} \to \prod_{\mathfrak{m}\in\operatorname{Max}(R)} (\frac{Q}{R})_{\mathfrak{m}}$. Since R is h-local, it follows by [42, Theorem IV.3.7] (3) that $M_1 \simeq \bigoplus_{\mathfrak{m}\notin\operatorname{Max}(R)} (M_1)_{\mathfrak{m}} = \bigoplus_{\mathfrak{m}\in X} \frac{Q}{R_{\mathfrak{m}}}$. So, $M_2 := \varphi^{-1}(\prod_{\mathfrak{m}\in X} (\frac{Q}{R})_{\mathfrak{m}}) = \frac{R_{(X')}}{R}$. Notice that $\text{w.d.}_R(\frac{Q}{R_{(X)}}) \leq 1$ and so $\text{p.d.}_R(\frac{Q}{R_{(X)}}) \leq 1$ by Lemma 1.2.6 (9). The equality (1.8) follows now by [3, Theorem 3.10].

Lemma 1.4.14. *Let R be an APD with $R \neq Q$. If T is a tilting R-module, then*

$$T^{\perp_\infty} = \operatorname{Gen}(T_R) = \mathscr{D}(T_R)\text{-Div}. \tag{1.11}$$

Proof. Clearly $\text{Gen}(T_R) \subseteq \mathscr{D}(T)\text{-Div}$. Let $M \in \mathscr{D}(T)\text{-Div}$, $\mathfrak{m} \in \text{Max}(R)$ and consider the tilting $R_\mathfrak{m}$-module $T_\mathfrak{m}$. By Theorem 1.4.11, ${}_{R_\mathfrak{m}}T_\mathfrak{m}$ is either divisible or projective. If $\mathfrak{m} \in \mathscr{D}(T)$, then $T_\mathfrak{m}$ is divisible and generates all divisible $R_\mathfrak{m}$-modules by Proposition 1.4.10 (3). Moreover, $\mathfrak{m}_\mathfrak{m} M_\mathfrak{m} = (\mathfrak{m}M)_\mathfrak{m} = M_\mathfrak{m}$ and it follows by Proposition 1.3.6 that $M_\mathfrak{m}$ is a divisible $R_\mathfrak{m}$-module, whence $M_\mathfrak{m} \in \text{Gen}({}_{R_\mathfrak{m}}T_\mathfrak{m})$. On the other hand, if $\mathfrak{m} \notin \mathscr{D}(T)$ then $T_\mathfrak{m}$ is a projective $R_\mathfrak{m}$-module whence a generator in $R_\mathfrak{m}$-Mod by Proposition 1.4.10 (2). In either cases $M_\mathfrak{m} \in \text{Gen}({}_{R_\mathfrak{m}}T_\mathfrak{m}) = T_\mathfrak{m}^{\perp_\infty}$ for every $\mathfrak{m} \in \text{Max}(R)$, whence $M \in T^{\perp_\infty} = \text{Gen}(T_R)$ by Lemma 1.4.9 (1). □

Theorem 1.4.15. *Let R be an APD with $R \neq Q$.*

(1) *The set*

$$\{T(X) \mid X \subseteq \text{Max}(R)\}$$

is a representative set (up to equivalence) of all tilting R-modules.

(2) *There is a bijective correspondence between the set of all tilting torsion classes of R-modules and the power set of the maximal spectrum $\mathfrak{B}(\text{Max}(R))$. The correspondence is given by the mutually inverse assignments:*

$$\mathscr{T} \mapsto \quad \mathscr{D}\mathscr{M}(\mathscr{T}) \quad := \quad \{\mathfrak{m} \in \text{Max}(R) \mid \mathfrak{m}M = M \textit{ for every } M \in \mathscr{T}\};$$

and

$$X \mapsto \quad X\text{-Div} \quad := \quad \{M_R \mid \mathfrak{m}M = M \textit{ for every } \mathfrak{m} \in X\}.$$

(3) *If R is coprimely packed, then the class of Fuchs-Salce tilting modules*

$$\{\partial_S \mid S \subseteq R^\times \textit{ is a multiplicative subset}\}$$

classifies all tilting R-modules (up to equivalence).

Proof.

(1) Let T be a tilting R-module and set

$$\begin{aligned}\Omega_1 &:= \{\mathfrak{m} \in \mathrm{Max}(R) \mid T_\mathfrak{m} \text{ is a divisible } R_\mathfrak{m}\text{-module}\};\\ \Omega_2 &:= \{\mathfrak{m} \in \mathrm{Max}(R) \mid T_\mathfrak{m} \text{ is a projective } R_\mathfrak{m}\text{-module}\}.\end{aligned}$$

Notice first that $\mathrm{Max}(R) = \Omega_1 \cup \Omega_2$ by Theorem 1.4.11 (a disjoint union by applying Proposition 1.4.10 (2) & (3) to the ring $R_\mathfrak{m}$).

Claim: $T \sim T(\Omega_2)$. One can show (as in the proof of Lemma 1.4.12), that if $\mathfrak{m} \in \mathrm{Max}(R)$ then

$$T(\Omega_2)_\mathfrak{m} = \begin{cases} Q \oplus \frac{Q}{R_\mathfrak{m}}, & \mathfrak{m} \in \Omega_1 \\ \\ R_\mathfrak{m}, & \mathfrak{m} \in \Omega_2 \end{cases}.$$

So, $T_\mathfrak{m} \sim T(\Omega_2)_\mathfrak{m}$ for every $\mathfrak{m} \in \mathrm{Max}(R)$ whence $T \sim T(\Omega_2)$ by (1.7).

(2) Let $\mathscr{T} = T^{\perp_\infty}$ be a tilting torsion class for some tilting R-module T. Then

$$\mathscr{DM}(\mathscr{T})\text{-Div} = \mathscr{DM}(T)\text{-Div} \overset{(1.6)}{=} \mathscr{D}(T)\text{-Div} \overset{(1.11)}{=} \mathrm{Gen}(T_R) = T^{\perp_\infty} = \mathscr{T}.$$

On the other hand, let $X \subseteq \mathrm{Max}(R)$, $\overline{X} := \mathrm{Max}(R) \backslash X$, and $T' := T(\overline{X})$. Then clearly $\mathscr{DM}(T') = X$ and so

$$\mathscr{DM}(X\text{-Div}) = \mathscr{DM}(\mathscr{DM}(T')\text{-Div}) = \mathscr{DM}(T') = X.$$

(3) Let R be compactly packed. Let Ω_1 and Ω_2 be as in "1".

Case 1. $\mathrm{Max}(R) = \Omega_1$ (i.e., $T_\mathfrak{m}$ is a divisible $R_\mathfrak{m}$-module for all $\mathfrak{m} \in \mathrm{Max}(R)$). In this case, T_R is divisible whence $T \sim Q \oplus Q/R$ and we can take $S = R^\times$.

Case 2. $\mathrm{Max}(R) = \Omega_2$ (i.e., $T_\mathfrak{m}$ is a projective $R_\mathfrak{m}$-module for all $\mathfrak{m} \in \mathrm{Max}(R)$). In this case, T_R is projective whence $T \sim R$ and we can take $S = \{1\}$.

Case 3. $\mathrm{Max}(R) \neq \Omega_1$ and $\mathrm{Max}(R) \neq \Omega_2$. Let

$$S := R \setminus \bigcup_{\mathfrak{m}\in\Omega_2} \mathfrak{m} \text{ and } T(S) := S^{-1}R \oplus S^{-1}R/R.$$

Let $\mathfrak{m} \in \Omega_2$, so that $T_\mathfrak{m}$ is projective and $S \subseteq R \setminus \mathfrak{m}$. Then $(S^{-1}R)_\mathfrak{m} = R_\mathfrak{m}$. Therefore $(T(S))_\mathfrak{m} = (S^{-1}R)_\mathfrak{m} \oplus (S^{-1}R/R)_\mathfrak{m} = R_\mathfrak{m}$ is equivalent to the projective $R_\mathfrak{m}$-module $T_\mathfrak{m}$. On the other hand, let $\mathfrak{m} \in \Omega_1$ so that $T_\mathfrak{m}$ is a divisible $R_\mathfrak{m}$-module. Then $\mathfrak{m} \cap S \neq \varnothing$ (otherwise $\mathfrak{m} \subseteq \bigcup_{\mathfrak{m}\in\Omega_2} \mathfrak{m}$ and so $\mathfrak{m} \in \Omega_2$ since R is coprimely packed; a contradiction since $\Omega_1 \cap \Omega_2 = \varnothing$). Let $\widetilde{s} \in S \cap \mathfrak{m}$. Clearly $\widetilde{s}(S^{-1}R)_\mathfrak{m} = (S^{-1}R)_\mathfrak{m}$, whence $(S^{-1}R)_\mathfrak{m}$ is a divisible $R_\mathfrak{m}$-module by Proposition 1.3.6. It follows that $(T(S))_\mathfrak{m} = (S^{-1}R)_\mathfrak{m} \oplus (S^{-1}R)_\mathfrak{m}/R_\mathfrak{m}$ is a divisible $R_\mathfrak{m}$-module, whence $T(S)_\mathfrak{m} \sim T_\mathfrak{m}$ as $R_\mathfrak{m}$-modules by Proposition 1.4.10 (3) (applied to the ring $R_\mathfrak{m}$). Since $T_\mathfrak{m} \sim T(S)_\mathfrak{m}$ for all $\mathfrak{m} \in \mathrm{Max}(R)$, we conclude that $T \sim T(S)$ by (1.7). □

Remark 1.4.16. Let R be a 1-Gorenstein ring and T_R be a tilting R-module. By [88] there exists $X \subseteq \mathbf{P}_1$ (the set of prime ideals of height 1) and some (unique) R-module R_X, satisfying $R \subseteq R_X \subseteq Q$ and fitting in an exact sequence

$$0 \to R \to R_X \to \bigoplus_{\mathfrak{m}\in X} \mathrm{E}(R/\mathfrak{m}) \to 0,$$

such that T is equivalent to the so-called Bass tilting module $B(X) := R_X \oplus \bigoplus_{\mathfrak{m}\in X} \mathrm{E}(R/\mathfrak{m})$. Let $\mathfrak{m} \in \mathrm{Max}(R)$. By the proof of [88, Theorem 0.1], the $R_\mathfrak{m}$-module $B(X)_\mathfrak{m}$ is injective, whence divisible, if $\mathfrak{m} \in X$ and projective if $\mathfrak{m} \notin X$. If R is a 1-Gorenstein domain (whence an APD), the same holds for the $R_\mathfrak{m}$-module $T(X')_\mathfrak{m}$, where $X' := \mathrm{Max}(R) \setminus X$. It follows that, in this case, $B(X) \sim T(X')$ by (1.7) and so $T \sim T(X')$.

A direct application of Theorem 1.4.15, and [50, Theorem 8.2.8] yields

Corollary 1.4.17. *Let R be a coherent (Noetherian) APD.*

(1) *All cotilting R-modules are of cofinite type and $\{T(X)^c \mid X \subseteq \mathrm{Max}(R)\}$ is a representative set (up to equivalence) of all cotilting R-modules.*

(2) *If R is coprimely packed, then $\{\partial_S^c \mid S \subseteq R^\times$ is a multiplicative subset$\}$ classifies all cotilting R-modules (up to equivalence).* □

1.5 Divisibility and injectivity over APD's

This section studies the intertwined correlation between (Gorenstein) injectivity and divisibility (and related objects) over almost perfect domains. We first notice that the notion of almost perfect domain can be defined via principal ideals, i.e., *R is an APD if and only if $\frac{R}{rR}$ is a perfect ring, for every nonzero $r \in R$.* Indeed, if I is a nonzero ideal of R and $r \in I$, then $R/I \cong (\frac{R}{rR})/(\frac{I}{rR})$ is perfect by [1, Corollary 28.7].

It is known that over Dedekind domains the class of divisible modules and the class of injective modules coincide. Next, we study the relation between these two classes over APD's. Recall that an R-module M is RD-injective (RD stands for relatively divisible) if every homomorphism $A \to M$ can be extended to a homomorphism $B \to M$ for every RD-submodule A of B (i.e., $A \cap rB = rA$ for all $r \in R$). Clearly, an injective module is RD-injective.

Corollary 1.5.1. *Let R be an APD. Then M is injective if and only if M is divisible and RD-injective.*

Proof. $(\Rightarrow)$ Trivial.

($\Leftarrow$) Assume M to be divisible and RD-injective. Then $\text{i.d.}_R(M) \leq 1$ by [42, Theorem XII.1.1]. Hence M is injective by Proposition 1.3.1. $\square$

The next result prvides a context where divisible modules over APD's have the (relative) injective property.

Corollary 1.5.2. *Let* $0 \to A \to B \to C \to 0$ *be an exact sequence of R-modules over an APD such that* $\text{p.d.}_R(C) \leq 1$*. Every divisible R-module has the injective property relative to this sequence.*

Proof. Let D be a divisible R-module. Applying the functor $\text{Hom}_R(-,D)$ to the exact sequence $0 \to A \to B \to C \to 0$ induces an exact sequence

$$\text{Hom}_R(B,D) \to \text{Hom}_R(A,D) \to \text{Ext}^1_R(C,D).$$

As in the proof of Proposition 1.4.10 (3), $\text{Ext}^1_R(C,D)$ vanishes, as desired. $\square$

The next two results relate the divisibility over APD's to the Fuchs divisible module ∂ [39].

Proposition 1.5.3. *An R-module M over an APD with projective dimension 1 is divisible if and only if M is a summand of a direct sum of copies of* ∂.

Proof. ($\Rightarrow$) The pair $(\mathscr{P}_1, \mathscr{D}\mathscr{I})$ over the *APD R* is the cotorsion pair induced by ∂. So, any divisible R-module with projective dimension 1 belongs to $\mathscr{P}_1 \cap \mathscr{D}\mathscr{I} = \text{Add}(\partial)$ (see [49, Proposition 5.1.8(c)]).

($\Leftarrow$) Let M be a summand of $\partial^{(\Lambda)}$ for some index set Λ. Since $\text{p.d.}_R(\partial) = 1$, $\text{p.d.}_R(\partial^{(\Lambda)}) = 1$ and so $\text{p.d.}_R(M) \leq 1$. Recall that any direct sum of divisible R-modules is divisible and every epimorphism image of a divisible module is divisible. Therefore M is divisible. $\square$

Corollary 1.5.4. *Over an APD, any $M \in \mathscr{P}_1$ can be embedded in a module which belongs to* $\mathrm{Add}(\partial)$

Proof. Let R be an APD and $M \in \mathscr{P}_1$. Then, by [40, Lemma 3.1], we have an exact sequence

$$0 \to M \to W(M) \to W(M)/M \to 0$$

such that $W(M)$ is weak-injective and $\mathrm{w.d.}_R(W(M)/M) \leq 1$. By Theorem 1.2.6, $\mathrm{p.d.}_R(W(M)/M) \leq 1$ and $W(M)$ is divisible. Hence $\mathrm{p.d.}_R(W(M)) \leq 1$ by [42, Lemma VI.2.9]. So, by Proposition 1.5.3, $W(M) \in \mathrm{Add}(\partial)$ and we are done. □

Corollary 1.5.5. *R is an APD and $K := Q/R$ is an injective R-module if and only if R is a* 1*-Gorenstein domain.*

Proof. Assume R is an APD and K is injective. Then $\mathrm{i.d.}_R(R) = 1$, so R is Noetherian by [12, Proposition 2.6]. Consequently, R is 1-Gorenstein. □

An element of a ring is of finite character if it belongs only to a finite number of maximal ideals. A ring R is said to be of finite character if every nonzero element of R is of finite character. Recall also that a localizing system (Definition 1.3.7) $\mathfrak{F}$ is said to be finitely generated provided that, for every ideal $I \in \mathfrak{F}$, there exists a finitely generated ideal $J \subseteq I$ with $J \in \mathscr{F}$. Finally set $\mathscr{M}(I) := \{\mathfrak{m} \in \mathrm{Max}(R) \mid I \subseteq \mathfrak{m}\}$ for a given ideal I of R.

Salce proved in [81] that for any tilting module T over a Prüfer domain R, $\mathscr{D}(T) := \{I$ ideal of $R \mid IT = T\}$ is a finitely generated localizing system. Next, we prove a more general result over APD's.

Lemma 1.5.6 ([42, Lemma III.2.7])**.** *Let I be an ideal of a domain R which contains an element a_0 of finite character. Then I contains a finitely generated ideal J such that $a_0 \in J$ and $\mathscr{M}(J) = \mathscr{M}(I)$.*

Proposition 1.5.7. *Let R be an APD and M an R-module. Then $\mathscr{D}(M)$ is a finitely generated localizing system.*

Proof. Recall that $\mathscr{D}(M)$ is a localizing system for any arbitrary domain R [78, Lemma 1.1]. Now, we claim that if A is an ideal of R with $\mathscr{M}(A) \subseteq \mathscr{D}(M)$, then $A \in \mathscr{D}(M)$. Indeed, suppose $\mathscr{M}(A) \subseteq \mathscr{D}(M)$.

Let $\mathfrak{m} \in \mathrm{Max}(R)$. If $\mathfrak{m} \in \mathscr{M}(A)$, then $\mathfrak{m}_{\mathfrak{m}} M_{\mathfrak{m}} = (\mathfrak{m}M)_{\mathfrak{m}} = M_{\mathfrak{m}}$ whence the $R_{\mathfrak{m}}$-module $M_{\mathfrak{m}}$ is divisible by Proposition 1.3.6, and it follows that $(AM)_{\mathfrak{m}} = A_{\mathfrak{m}} M_{\mathfrak{m}} = M_{\mathfrak{m}}$. On the other hand, if $\mathfrak{m} \notin \mathscr{M}(A)$, then $A_{\mathfrak{m}} = R_{\mathfrak{m}}$ and so $(AM)_{\mathfrak{m}} = R_{\mathfrak{m}} M_{\mathfrak{m}} = M_{\mathfrak{m}}$. Since $(AM)_{\mathfrak{m}} = M_{\mathfrak{m}}$ for every $\mathfrak{m} \in \mathrm{Max}(R)$, we conclude that $AM = M$. This proves the claim. Since R is an APD, R is of finite character. Now, let $I \in \mathscr{D}(M)$. By Lemma 1.5.6, there exists a finitely generated ideal J with $\mathscr{M}(J) = \mathscr{M}(I) \subseteq \mathscr{D}(M)$. The above claim gives that $J \in \mathscr{D}(M)$, as desired. □

We close this section with a result on tilting modules over perfect rings.

Corollary 1.5.8. *Every tilting module over a perfect ring is projective.*

Proof. Let T be a tilting module over a perfect ring R and let $P \in \mathrm{Max}(R)$. By Lemma 1.4.9, T_P is a tilting R_P-module and contains a maximal submodule. As in the proof of Proposition 1.4.11, T_P is a projective R_P-module, hence flat. We are done by the fact that over perfect rings flat R-modules are projective. □

1.6 Tilting modules over (n,d)-rings

Given positive integers n and d, we say that a ring R is an (n,d)-ring if $\mathrm{p.d.}_R(M) \leq d$ for each n-presented R-module M [28]. A ring (resp., domain) R is semihereditary (resp., Prüfer) if and only if R is a (1,1)-ring (resp., (1,1)-domain); and R is von Neumann regular if and only if R is a (1,0)-ring. Finally, recall that $(n,0)$-rings are termed n-von Neumann regular rings. For more details about the (n,d)-theory, we refer the reader to [28, 58, 63].

In 2004, Bazzoni proved in [6] that "*a tilting module T over a Prüfer domain has projective dimension at most 1 (i.e., T is 1-tilting).*" One aim of this section is to generalize this result to (n,d)-rings (Theorem 1.6.2). Before proceeding with this, we record below some notation and facts from the literature.

Let $\mathscr{X}$ be a class of modules. A module M is said to be $\mathscr{X}$-filtered provided there is an increasing chain $\{M_\alpha \mid \alpha \leq \kappa\}$ (where α and κ are cardinal numbers) of submodules of M such that $M_0 = 0, M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ (where β is a cardinal number) for each $\alpha \leq \kappa$, $\frac{M_{\alpha+1}}{M_\alpha} \cong X_\alpha$ for some $X_\alpha \in \mathscr{X}$ for each $\alpha < \kappa$, and $M_\kappa = M$. An R-module M is said to be strongly finitely presented if it projective resolution consist of finitely generated modules. For a given class C of R-modules, $C^{<\omega}$ denotes the intersection of C with the class of all strongly finitely presented R-modules.

Theorem 1.6.1 ([50, Theorem 5.2.20])**.** *Let R be a ring. Then every tilting R-module T is equivalent to a tilting R-module T_1 such that T_1 is $\mathscr{A}^{<\omega}$-filtered, where $\mathscr{A}$ denotes ${}^\perp(T^{\perp_\infty})$.*

Next, we generalize Bazzoni's result mentioned above.

Theorem 1.6.2. *A tilting module T over an (n,d)-ring has projective dimension at most d (i.e., T is d-tilting).*

Proof. Let R be an (n,d)-ring and T be a tilting R-module with the induced cotorsion pair $(\mathscr{A},\mathscr{B})$. There exists a tilting R-module T_1 such that $T \sim T_1$ and T_1 is $\mathscr{A}^{<\omega}$-filtered by Theorem 1.6.1. Since R is an (n,d)-ring, $\mathscr{A}^{<\omega} \subseteq \mathscr{P}_d := \{M_R \text{— p.d}_R(M) \leq d\ d\}$. Whence T_1 is $\mathscr{P}_d$-filtered. So, p.d.$_R(T_1) \leq d$ by [50, Lemma 3.1.4]. Therefore $\mathscr{A} \subseteq \mathscr{P}_d$ by [50, Lemma 5.1.8]. Consequently, p.d.$_R(T) \leq d$, as desired. □

We deduce the following result.

Corollary 1.6.3. (1) *If R is an n-von Neumann regular ring, then any tilting R-module is a projective generator.*

(2) *If R is an $(n,1)$-ring, then any tilting module is 1-tilting.*

Next, we handle the finitely generated case. First, recall that if M is a finitely generated faithful R-module, then M generates all simple R-modules [98, Proposition 18.9].

Proposition 1.6.4. *Let R be a (commutative) ring and n a positive integer. If T is a finitely generated n-tilting R-module, then T is projective.*

We weren't able to locate a proof of this result in the literature. Miyashita mentioned this fact in [68] without proof. Next, we provide a proof, due to Silvana Bazzoni, that we got through private correspondence. Another proof was provided to us by Trilifaj.

Proof. By induction on n. The result was proved for $n = 1$ by Colpi in [23]. Suppose the statement is true for $n-1$. Let T be finitely generated and n-tilting. Then by [13, Lemma 3.4] T is strongly finitely presented. We envisage two cases.

CASE 1. Assume that (R,P) is a local ring. Then T_R is faithful and finitely generated. By [98, Proposition 18.9], we have an exact sequence of R-modules $0 \to J \to T \to R/P \to 0$. By applying the functor $\mathrm{Hom}_R(T,-)$ to get the exact sequence $0 = \mathrm{Ext}^n_R(T,T) \to \mathrm{Ext}^n_R(T,R/P) \to \mathrm{Ext}^{n+1}_R(T,J) = 0$ and so $\mathrm{Ext}^n_R(T,R/P) = 0$. The following sequence is an exact sequence of R-modules $0 \to H \to R^{(n_1)} \to T \to 0$ where H is strongly finitely presented and $\mathrm{p.d.}_R(H) \leq n-1$. Apply the functor $\mathrm{Hom}_R(-,R/P)$ to get the exact sequence $0 = \mathrm{Ext}^{n-1}_R(R^{(n_1)},R/P) \to \mathrm{Ext}^{n-1}(H,R/P) \to \mathrm{Ext}^n_R(T,R/P) = 0$, so $\mathrm{Ext}^{n-1}_R(H,R/P) = \mathrm{Ext}^1_R(K_{n-3},R/P) = 0$, where $\{K_n\}_n$ denote the syzygies of H. Then we have

$$\begin{aligned}\mathrm{Tor}^R_1(R/P,K_{n-3}) &\cong \mathrm{Tor}^R_1(\mathrm{Hom}_{R/P}(R/P,R/P),K_{n-3}) \\ &\cong \mathrm{Hom}_{R/P}(\mathrm{Ext}^1_R(K_{n-3},R/P),R/P) = 0.\end{aligned}$$

By [16, II.3.2.Corollary 2], K_{n-3} is projective (being finitely presented and flat). This implies that $\mathrm{p.d.}_R(H) \leq n-2$. So, $\mathrm{p.d.}_R(T) \leq n-1$ and, by the induction hypothesis T is projective.

CASE 2. R is not necessarily local. By Case 1, T_P is flat (projective) for each maximal ideal P of R . Since T is finitely presented, we conclude that T is projective, completing the proof. □

As applications of Proposition 1.6.4, we offer the next two corollaries.

Corollary 1.6.5. *Let M be a strongly finitely presented R-module. If $M, R \in M^{\perp_\infty}$ and* $\text{p.d.}_R(M) < \infty$*, then M is projective.*

Proof. We claim that $L := R \oplus M$ is a tilting R-module. Indeed, since M is strongly finitely presented, so is L. First of all, L has finite projective dimension, i.e., the first condition in Definition 1.4.1 is satisfied. By assumption M, $R \in M^{\perp_\infty}$, whence $\text{Ext}^i_R(L, L) = 0$ for all i. The second condition also holds (see [50] page 189). The sequence $0 \to R \to L \to M \to 0$ illustrates the third condition of Definition 1.4.1. By Proposition 1.6.4, L is projective. Consequently, M is projective. □

Corollary 1.6.6. *Let R be a ring. The following statements are equivalent:*

(1) *Every finitely generated ideal is 1-tilting;*

(2) *Every finitely generated ideal is n-tilting for some positive integer n;*

(3) *R is a Prüfer domain.*

Proof. (1) $\Rightarrow$ (2) Trivial.

(2) $\Rightarrow$ (3) Since each tilting R-module generates all injective R-modules, it is faithful. Let $0 \neq r \in R$. Then rR is faithful and so r is not a zero-divisor. Consequently, R is a domain. Proposition 1.6.4 ensures that every finitely generated tilting ideal is projective. By (2), R is a Prüfer domain.

(3) $\Rightarrow$ (1) Any faithful finitely generated projective R-module is a progenerator, hence 0-tilting, whence 1-tilting, completing the proof. □

Chapter 2

Star modules over commutative rings

This chapter* studies the multiplicative ideal structure of commutative rings in which every finitely generated ideal is a $\star$-module. An R-module M is a $\star$-module if the functor $\mathrm{Hom}(M,-)$ induces an equivalence between the two categories $\mathrm{Gen}(M)$ and $\mathrm{Cogen}(M^{\star}_{S})$ where $S := \mathrm{End}(M)$ and $M^{\star} := \mathrm{Hom}_R(M,E)$ for an injective cogenerator E_R. We investigate the correlation with well-known Prüfer conditions; namely, we prove that this class of rings stands strictly between the two classes of arithmetical rings and Gaussian rings. Thereby, we generalize Osofsky's theorem on the weak global dimension of arithmetical rings and partially resolve Bazzoni-Glaz's related conjecture on Gaussian rings. We also establish an analogue of Bazzoni-Glaz results on the transfer of Prüfer conditions between a ring and its total ring of quotients. We also explore various contexts of trivial ring extensions in order to generate new and original examples of rings subject to the $\star$-property, marking its distinction from related Prüfer conditions. We close with a study of n-star modules over Dedekind domains.

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*Part of this chapter is submitted for publication under the title "Commutative rings in which every finitely generated ideal is a $\star$-module" (in collaboration with Dr. J. Abuhlail and Dr. S. Kabbaj).**

2.1 Introduction

All rings considered in this chapter, unless otherwise specified, are commutative with identity element and all modules are unital. There are five well-known extensions of the notion of Prüfer domain [60, 72] to arbitrary rings. Namely:

(1) Semihereditary ring, i.e., every finitely generated ideal is projective [20];

(2) Ring with weak global dimension ≤ 1 [46, 47];

(3) Arithmetical ring, i.e., every finitely generated ideal is locally principal [38];

(4) Gaussian ring, i.e., $c(fg) = c(f)c(g)$ for any polynomials f, g with coefficients in the ring, where $c(f)$ denotes the content of f [90];

(5) Prüfer ring, i.e., every finitely generated regular ideal is projective [19, 52].

In the domain context, all these forms coincide with the original definition of a Prüfer domain [48], that is, every nonzero finitely generated ideal is invertible [72]. Prüfer domains occur naturally in several areas of commutative algebra, including valuation theory, star and semistar operations, dimension theory, representations of overrings, trace properties, in addition to several homological extensions.

In 1970 Koehler [59] studied associative rings for which every cyclic module is quasi-projective. She noticed that any commutative ring satisfies this property. Later, rings in which every left ideal is quasi-projective were studied by Jain and others [57, 51] and called left qp-rings. Several characterizations of (semi-)perfect qp-rings were obtained. Moreover, Mohammad [69] and Singh-Mohammad [84] studied local or semi-perfect rings in which every finitely generated ideal is quasi-projective. A ring is said to be an fqp-ring (or to satisfy the $\star$-property) if every finitely generated ideal is quasi-projective or, equivalently, a $\star$-module.

This chapter studies the multiplicative ideal structure of fqp-rings. Section 2 provides ample details on $\star$-modules and quasi-projectivity and shows that these two concepts coincide for finitely generated modules over commutative rings. Section 3 investigates the correlation between the $\star$-property and well-known Prüfer conditions. In this vein, the first main result (Theorem 2.3.1) asserts that the class of fqp-rings stands strictly between the two classes of arithmetical rings and Gaussian rings; that is, "arithmetical ring $\Rightarrow$ fqp-ring $\Rightarrow$ Gaussian ring." Further, the second main result (Theorem 2.3.10) extends Osofsky's theorem on the weak global dimension of arithmetical rings and partially resolves Bazzoni-Glaz's related conjecture on Gaussian rings; we prove that "the weak global dimension of an fqp-ring is equal to 0, 1, or ∞." The third main result (Theorem 2.3.13) establishes the transfer of the concept of fqp-ring between a local ring and its total ring of quotients; namely, "a local ring R is an fqp-ring if and only if R is Prüfer and Q is an fqp-ring." Section 4 studies the possible transfer of the $\star$-property to various contexts of trivial ring extensions. The main result (Theorem 2.4.4) states that "if $(A,\mathfrak{m})$ is a local ring, E a nonzero $\frac{A}{\mathfrak{m}}$-vector space, and $R := A \propto E$ the trivial ring extension of A by E, then R is an fqp-ring if and only if $\mathfrak{m}^2 = 0$." This result generates new and original examples of fqp-rings, marking the distinction between the $\star$-property and related Prüfer conditions.

The following diagram of implications puts the notion of fqp-ring in perspective within the family of Prüfer-like rings [8, 9], where the third and fourth implications are established by Theorem 2.3.1:

Semihereditary ring

$\Downarrow$

Ring with weak global dimension ≤ 1

$\Downarrow$

Arithmetical ring

$\Downarrow$

fqp-Ring

$\Downarrow$

Gaussian ring

$\Downarrow$

Prüfer ring

2.2 Quasi-projectivity and the $\star$-property

All rings considered in this section are associative with identity element and, unless otherwise specified, are not assumed to be commutative. This section traces categorical developments that gave rise to $\star$-modules and shows how this concept coincides with quasi-projectivity for finitely generated modules over commutative rings. Thereby, we define the notion of a commutative fqp-ring by subjecting all finitely generated ideals to any one of the $\star$-property or quasi-projectivity. The results stated or recalled in this section will be in use in the next section which deals with commutative settings.

Throughout let R be a ring and fix an injective cogenerator E_R in the category of right R-modules. Let V be a right R-module, $\mathrm{Ann}(V)$ the annihilator of V in R, and $V^* := \mathrm{Hom}_R(V,E)$ considered as a right module over $S := \mathrm{End}(V)$ (the endomorphism ring of V).

Also let $\mathbb{M}_R$ be the category of right R-modules and $\mathbb{M}_S$ the category of right S-modules. Let $\mathrm{Gen}(V) \subseteq \mathbb{M}_R$ denote the full subcategory of V-generated right R-modules (i.e., epimorphic images of direct sums of V) and $\mathrm{Cogen}(V_S^*) \subseteq \mathbb{M}_S$ denote the full subcategory of V^*-cogenerated right S-modules (i.e., S-submodules of direct products of V_S^*). Finally, let $\sigma[V] := \overline{\mathrm{Gen}(V)}$ denote Wisbauer's category of V-subgenerated right R-modules [98], that is, the smallest subclass of $\mathbb{M}_R$ that contains $\mathrm{Gen}(V)$ and is closed under submodules.

Definition 2.2.1. (1) Let M be a right R-module. The right R-module V is M-projective if $\mathrm{Hom}_R(V,M) \to \mathrm{Hom}_R(V,M/N)$ is surjective for every submodule N of M.

(2) V is quasi-projective (or self-projective) if V is V-projective, that is, $\mathrm{Hom}_R(V,V) \to \mathrm{Hom}_R(V,V/U)$ is surjective for every submodule U of V.

Recall that V is a generator (resp., self-generator) if $\mathrm{Gen}(V) = \mathbb{M}_R$ (resp., V generates each of its submodules). Also V is a progenerator (resp., quasi-progenerator) if V is finitely generated, projective, and generator (resp., quasi-projective and self-generator).

Next we examine these notions in the commutative setting.

Lemma 2.2.2. *Let R be a commutative ring and V a finitely generated R-module. Then the following assertions hold:*

(1) *V is a progenerator if and only if V is projective and faithful.*

(2) *V is a quasi-progenerator if and only if V is quasi-projective if and only if V is projective over $\frac{R}{\mathrm{Ann}(V)}$.*

Proof. (1) This is [98, 18.11].

(2) Set $\overline{R} := \frac{R}{\mathrm{Ann}(V)}$ and let $\{v_1, \dots, v_k\} \subseteq V$ be a finite generating system of V. Then we have a monomorphism

$$\rho : \overline{R} \hookrightarrow V^k,\ \overline{r} \mapsto r(v_1, \dots, v_k).$$

In particular, $\overline{R} \in \sigma[V_R]$, whence $\sigma[V] = \mathbb{M}_{\overline{R}}$ (since $\sigma[V_R]$ is closed under epimorphic images and direct sums). By [98, 18.3], V is quasi-projective if and only if V is projective in $\sigma[V_R] = \mathbb{M}_{\overline{R}}$, as desired. On the other hand, if V is projective over $\overline{R}$, then V is a progenerator in $\mathbb{M}_{\overline{R}} = \sigma[V_R]$ by (1) (since V is faithful over $\overline{R}$). By [98, 18.5], V_R is a quasi-progenerator, completing the proof of the lemma. □

Now, let R be a ring and $V \in \mathbb{M}_R$. Consider the following functors

$$\mathbb{M}_R \underset{(-\otimes_S V)}{\overset{\mathrm{Hom}_R(V,-)}{\rightleftarrows}} \mathbb{M}_S.$$

One observes that

$$\mathrm{Hom}_R(V, \mathbb{M}_R) \subseteq \mathrm{Cogen}(V_S^*)$$

and

$$\mathbb{M}_S \otimes_S V \subseteq \mathrm{Gen}(V_R).$$

This led Menini and Orsatti [66] to introduce and study modules satisfying the property $(\star)$ below.

Definition 2.2.3. An R-module V is called a $\star$-module if the following equivalence of categories holds

$$\mathrm{Gen}(V_R) \underset{(-\otimes_S V)}{\overset{\mathrm{Hom}_R(V,-)}{\approx}} \mathrm{Cogen}(V_S^*). \qquad (*)$$

Several homological characterizations for such modules were given by Colpi [22, 23] who termed them $\star$-modules. He showed that $\star$-modules generalize progenerators which are characterized by $\mathbb{M}_R \approx \mathbb{M}_S$ (Morita [70]) and quasi-progenerators which are characterized by $\sigma[V] \approx \mathbb{M}_S$ (Fuller [43]). Also it is worth recalling that a $\star$-module is necessarily finitely generated (Trlifaj [87]). Moreover, over commutative rings we have:

Lemma 2.2.4. *Let R be a commutative ring and V a finitely generated R-module. Then the* *following assertions hold:*

(1) *V is a progenerator if and only if V is a faithful quasi-progenerator if and only if V is a faithful $\star$-module.*

(2) *V is a quasi-progenerator if and only if V is a $\star$-module if and only if V is quasi-projective.*

Proof. Since R is commutative, V is a $\star$-module if and only if V is a quasi-progenerator [21, Theorem 2.4.5] and [26, Theorem 2.4].

(1) Every progenerator is a faithful quasi-progenerator, hence a faithful $\star$-module. On the other hand, since R is commutative, if V is a faithful $\star$-module, then V is a progenerator in $\sigma[V_R] = \mathbb{M}_{\overline{R}} = \mathbb{M}_R$ as shown in the proof of Lemma 2.2.2.

(2) Conclude by Lemma 2.2.2(2). □

Next, we provide -via Lemmas 2.2.2 and 2.2.4- a complete description of $\star$-modules over arbitrary commutative rings. For the special case of local rings, it recovers the description of $\star$-modules over valuation rings (i.e., chained rings) obtained by Zanardo in [102].

Theorem 2.2.5. *Let R be a commutative ring. An R-module V is a $\star$-module if and only if V is a direct summand of $(R/I)^n$ for some ideal I of R and integer $n \geq 0$. If, moreover, R is local, then V is a $\star$-module if and only if $V \cong (R/I)^n$ for some ideal I of R and integer $n \geq 0$.*

Proof. Let V_R be an R-module, $J := \mathrm{Ann}_R(V)$ and $\overline{R} := R/J$. Assume that V_R is a $\star$-module. Then V_R is finitely generated by [87]. So, $V_{\overline{R}}$ is finitely generated, and projective by Lemmas 2.2.2 and 2.2.4. It follows that $V_{\overline{R}}$, whence V_R, is a direct summand of $(R/J)^n$ for some $n \geq 0$. Conversely, let V be a direct summand of $(R/I)^n$ for some ideal I of R and integer $n \geq 0$. Then $V_{R/I}$, whence $V_{\overline{R}}$ is finitely generated and projective (notice that $I \subseteq J$). Consequently, V_R is a $\star$-module by Lemma 2.2.4.

Now assume that R is local. If V_R is a $\star$-module, then $V_{\overline{R}}$ is finitely generated and projective as shown above, whence free since $\overline{R}$ is local. It follows that $V_R \cong (R/J)^n$ for some $n \geq 0$. The converse was shown to be true for arbitrary commutative rings. □

As a consequence of Theorem 2.2.5, we obtain the next result which generalizes Fuller's well-known result on ring extensions [44, Theorem 2.2] in the commutative context.

Corollary 2.2.6. *Let $\xi : A \to R$ be a morphism of commutative rings. If U_A is a $\star$-module, then $V_R := R \otimes_A U$ is a $\star$-module.*

Proof. Let U be a $\star$-module over A. Then $U \oplus X = (A/I)^n$ for some ideal I of A, an integer $n \geq 0$, and an A-module X. It follows that $(R \otimes_A U) \oplus (R \otimes_A X) \cong R \otimes_A (A/I)^n \cong (R/RI)^n$, whence $V_R := R \otimes_A U$ is a $*$-module by Theorem 2.2.5. □

Remark 2.2.7. Let $J := \mathrm{Ann}_A(U)$ and $\overline{A} := A/J$. If U_A is a $\star$-module, then U_A is a quasi-progenerator and so the faithful module $U_{\overline{A}}$ is a progenerator. In particular, $U_{\overline{A}}$ generates $V_{\overline{A}}$, hence U_A generates V_A (notice that $J \subseteq \mathrm{Ann}_A(V)$). This shows that the assumption "U_A generates V_A" in [44, Theorem 2.2] is automatically satisfied for $\star$-modules over commutative rings.

In view of the above study, we set the following definition.

Definition 2.2.8. A commutative ring R is said to be an fqp-ring (or, equivalently, satisfy the $\star$-property) if every finitely generated ideal of R is quasi-projective (or, equivalently, a $\star$-module).

The following two examples show that the two notions of $\star$-module and quasi-projective module do not coincide, in general, beyond the commutative context.

Example 2.2.9 ([43])**.** Consider

$$M := \begin{bmatrix} \mathbb{Z} \\ \mathbb{Q} \end{bmatrix}, \; L := \begin{bmatrix} 0 \\ \mathbb{Q} \end{bmatrix} \text{ and } R := \mathrm{End}_{\mathbb{Z}}(M) \cong \begin{bmatrix} \mathbb{Z} & 0 \\ \mathbb{Q} & \mathbb{Q} \end{bmatrix}.$$

Then M is a finitely generated faithful projective left R-module and L is a left R-submodule of M such that $L \notin \mathrm{Gen}({}_R M)$. Hence M is not a self-generator. By [22, Theorem 4.7.], a quasi-projective $\star$-module is a quasi-progenerator, whence M is not a $\star$-module.

Example 2.2.10. Let A be a ring with a non-projective tilting module ${}_A T$ (e.g., A is a finite dimensional hereditary non-semisimple algebra). Let $P_A := A_A^{(\mathbb{N})}$ and $R := \mathrm{End}(P_A)$ which is isomorphic to the ring of all $\mathbb{N} \times \mathbb{N}$ column-finite matrices with entries in A. The left R-module $V := P \otimes_A T$ is a faithful $\star$-module that is not a quasi-progenerator [24, Corollary 9]. Once again we appeal to [22, Theorem 4.7.] to conclude that V is not quasi-projective.

The $\star$-property and the notion of fqp-ring are distinct, in general, in non-commutative rings, as shown by the next example.

Example 2.2.11. Let $\mathbb{F}$ be a field and consider

$$R := \begin{bmatrix} \mathbb{F} & \mathbb{F} \\ 0 & \mathbb{F} \end{bmatrix}, \; I := \begin{bmatrix} \mathbb{F} & \mathbb{F} \\ 0 & 0 \end{bmatrix} \text{ and } J := \begin{bmatrix} 0 & \mathbb{F} \\ 0 & 0 \end{bmatrix}.$$

By [98, Page 399], R is (left and right) hereditary. Further, I is an ideal of R and J is a subideal of I. By [105], $\mathrm{Hom}_R(I,J) = 0$, whence $J_R \notin \mathrm{Gen}(I_R)$. Consequently, I_R is a finitely generated projective right R-module that is not a self-generator (whence not quasi-progenerator). Every quasi-projective $\star$-module is a quasi-progenerator [22, Theorem 4.7.]. So I_R is not a $\star$-module. Consequently, R is a hereditary ring (hence an fqp-ring) which does not satisfy the $\star$-property.

2.3 fqp-Rings

This section investigates the correlation between (commutative) fqp-rings and the Prüfer-like rings mentioned in the introduction. The first result of this section (Theorem 2.3.1) states that the class of fqp-rings contains strictly the class of arithmetical rings and is contained strictly in the class of Gaussian rings. Its proof provides specific examples proving that the respective containments are strict. Consequently, fqp-rings stand as a new class of Prüfer-like rings (to the effect that, in the domain context, the fqp-notion coincides with the definition of a Prüfer domain).

In 1969, Osofsky proved that the weak global dimension of an arithmetical ring is either less than or equal to one or infinite [71]. Recently, Bazzoni and Glaz studied the homological aspects of Gaussian rings, showing, among others, that Osofsky's result is valid in the context of coherent Gaussian rings (resp., coherent Prüfer rings) [47, Theorem 3.3] (resp., [9, Theorem 6.1]). They closed with a conjecture sustaining that "the weak global dimension of a Gaussian ring is 0, 1, or ∞" [9]. In this vein, Theorem 2.3.10 generalizes Osofsky's theorem as well as validates Bazzoni-Glaz conjecture in the class of fqp-rings.

We close this section with a satisfactory analogue (for fqp-rings) to Bazzoni-Glaz results on the transfer of Prüfer conditions between a ring and its total ring of quotients [9, Theorems 3.3 , 3.6 , 3.7 & 3.12].

Next we announce the first result of this section.

Theorem 2.3.1. *For a ring R, we have*

$$R \text{ arithmetical} \Rightarrow R \text{ fqp-ring} \Rightarrow R \text{ Gaussian}$$

where the implications are irreversible in general.

The proof of this theorem involves the following lemmas which are of independent interest.

Lemma 2.3.2 ([91, Lemma 2]). *Let R be a ring and M a quasi-projective R-module. Assume $M = M_1 + \ldots + M_n$, where M_i is a submodule of M for $i = 1, \ldots, n$. Then there are endomorphisms f_i of M such that $f_1 + \ldots + f_n = 1_M$ and $f_i(M) \subseteq M_i$ for $i = 1, \ldots, n$.* □

Lemma 2.3.3 ([59]). *Every cyclic module over a commutative ring is quasi-projective.*

Proof. Here is a direct proof. Let R be a (commutative) ring and M a cyclic R-module. Then $M \cong \frac{R}{\mathrm{Ann}(M)}$. So M is a projective $\frac{R}{\mathrm{Ann}(M)}$-module, hence quasi-projective by Lemma 2.2.2. □

Lemma 2.3.4 ([45, Corollary 1.2]). *Let $\{M_i\}_{1 \le i \le n}$ be a finite family of R-modules. Then $\bigoplus_{i=1}^{n} M_i$ is quasi-projective if and only if M_i is M_j-projective for all $i, j \in \{1, \ldots, n\}$.* □

Lemma 2.3.5. *If R is an fqp-ring, then $S^{-1}R$ is an fqp-ring, for any multiplicatively closed subset S of R.*

Proof. Let J be a finitely generated ideal of $S^{-1}R$ and let I be a finitely generated ideal of R such that $J := S^{-1}I$. Then I is a $\star$-module that is faithful over $\frac{R}{\mathrm{Ann}(I)}$. By a combination of Lemma 2.2.4 and Lemma 2.2.2, I is projective over $\frac{R}{\mathrm{Ann}(I)}$. So that $J := S^{-1}I$ is projective over $\frac{S^{-1}R}{S^{-1}\mathrm{Ann}(I)} = \frac{S^{-1}R}{\mathrm{Ann}(S^{-1}I)}$. By Lemma 2.2.2, J is quasi-projective, as desired. □

Lemma 2.3.6 ([99, 19.2] and [97]). *Let R be a (commutative) ring and M a finitely generated R-module. Then M is quasi-projective if and only if $M_{\mathfrak{m}}$ is quasi-projective over $R_{\mathfrak{m}}$ and $(\mathrm{End}(M))_{\mathfrak{m}} \cong \mathrm{End}_{R_{\mathfrak{m}}}(M_{\mathfrak{m}})$, for every maximal ideal $\mathfrak{m}$ of R.* □

Lemma 2.3.7. *Let R be a local ring and a, b two nonzero elements of R such that (a) and (b) are incomparable. If (a, b) is quasi-projective (in particular, if R is an fqp-ring), then:*

(1) $(a) \cap (b) = 0$,

(2) $a^2 = b^2 = ab = 0$,

(3) $\text{Ann}(a) = \text{Ann}(b)$.

Proof. (1) $I := (a,b)$ is quasi-projective; so by Lemma 2.3.2, there exist f_1, f_2 in $\text{End}_R(I)$ with $f_1(I) \subseteq (a)$, $f_2(I) \subseteq (b)$, and $f_1 + f_2 = 1_I$. So

$$a = f_1(a) + f_2(a) \; ; \; b = f_1(b) + f_2(b).$$

Let $f_1(a) = x_1 a$, $f_2(a) = y_1 b$, $f_1(b) = x_2 a$, and $f_2(b) = y_2 b$. We obtain

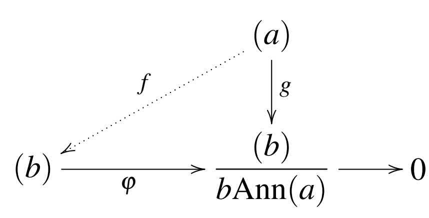
$$a = x_1 a + y_1 b \; ; \; b = x_2 a + y_2 b.$$

This forces x_1 to be a unit and $1 - y_2$ to not be a unit. Let $z \in (a) \cap (b)$; say, $z = c_1 a = c_2 b$ for some $c_1, c_2 \in R$. We get

$$z = f_1(c_1 a) + f_2(c_2 b) = x_1 z + y_2 z.$$

Therefore $(x_1 - (1 - y_2))z = 0$, hence $z = 0$ (since $x_1 - (1 - y_2)$ is necessarily a unit), as desired.

(2) We have $I = (a) \oplus (b)$. So (a) is (b)-projective by Lemma 2.3.4. Consider the following diagram of R-maps



where φ denotes the canonical map and g is (well) defined by $g(ra) = \overline{rb}$. Since (a) is (b)-projective, there exists an R-map $f : (a) \to (b)$ with $\varphi \circ f = g$. In particular, $\overline{f(a)} = \overline{b}$ (mod $\frac{(b)}{bAnn(a)}$). Therefore $f(a) = cb$ for some $c \in R$ hence $cb - b = bd$ for some $d \in \text{Ann}(a)$.

Further, since $ab = 0$ (recall $(a) \cap (b) = 0$), we have $0 = f(ab) = bf(a) = cb^2$. Multiplying the above equality by b, we get $(d+1)b^2 = 0$. It follows that $b^2 = 0$ as $d+1$ is a unit (since d is a zero-divisor and R is local). Likewise, $a^2 = 0$. Thus $I^2 = 0$, as claimed.

(3) The above equality $cb - b = bd$ yields $(d+1-c)b = 0$. Hence the fact that $d+1$ is a unit forces c to be a unit too (since $b \neq 0$). Now, let $x \in \mathrm{Ann}(a)$. Then $0 = f(xa) = xf(a) = cxb$, whence $x \in \mathrm{Ann}(b)$. So $\mathrm{Ann}(a) \subseteq \mathrm{Ann}(b)$. Likewise, $\mathrm{Ann}(b) \subseteq \mathrm{Ann}(a)$, completing the proof of the lemma. □

It is worthw noting that Lemma 2.3.7 sharpens and recovers [69, Lemma 3] and [84, Lemma 3] where the authors require the hypothesis that "every finitely generated ideal is quasi-projective" (i.e., R is an fqp-ring).

Proof of Theorem 2.3.1. Assume R to be an arithmetical ring. Let I be a nonzero finitely generated ideal of R and J a subideal of I (possibly equal to 0). Let P be any prime ideal of R. Then $I_P := IR_P$ is a principal ideal of R_P (possibly equal to R_P) and hence a $\star$-module by Lemma 2.3.3. Moreover, we claim that

$$(\mathrm{Hom}_R(I,I))_P \cong \mathrm{Hom}_{R_P}(I_P, I_P).$$

We only need to prove

$$(\mathrm{Hom}_R(I,I))_P \cong \mathrm{Hom}_R(I, I_P).$$

Consider the function

$$\begin{array}{rrcl} \phi: & (\mathrm{Hom}_R(I,I))_P & \longrightarrow & \mathrm{Hom}_R(I, I_P) \\ & \frac{f}{s} & \longrightarrow & \phi(\frac{f}{s}) : I \to I_P \ ; \ x \mapsto \frac{f(x)}{s} \end{array}$$

Obviously, ϕ is a well-defined R-map. Moreover, one can easily check that ϕ is injective since I is finitely generated. It remains to prove the surjectivity. Let $g \in \mathrm{Hom}_R(I, I_P)$.

Clearly, the R_P-module I_P is cyclic and so $I_P = aR_P$ for some $a \in I$. Therefore there exists $\lambda \in R$ such that $g(a) = \frac{\lambda a}{s}$. Let $f : I \to I$ be defined by $f(x) = \lambda x$. Then $f \in \mathrm{Hom}_R(I, I)$. Let $x \in I$, so $\frac{x}{1} = \frac{ra}{u}$ and $u \in R \backslash P$, whence $tux = tra$ for some $t \in R \backslash P$. We have

$$\phi(\frac{f}{s})(x) = \frac{f(x)}{s} = \frac{\lambda}{s}\frac{x}{1} = \frac{\lambda}{s}\frac{ra}{u} = \frac{r}{u}g(a) = \frac{1}{tu}g(tra) = \frac{1}{tu}g(tux) = g(x).$$

This proves the claim. By Lemma 2.3.6, I is a $\star$-module and hence R is an fqp-ring, proving the first implication.

Next assume R to be an fqp-ring. The Gaussian notion is a local property, that is, R is Gaussian if and only if R_P is Gaussian for every $P \in \mathrm{Spec}(R)$ [9]. This fact combined with Lemma 2.3.5 reduces the proof to the local case. Now, R is a local fqp-ring. Let a, b be any two elements of R. We envisage two cases. Case 1: Suppose $(a, b) = (a)$ or (b), say, (a). Then $(a, b)^2 = (a^2)$ and if in addition $ab = 0$, then $b \in (a)$ implies that $b^2 = 0$. Case 2: Suppose $I := (a, b)$ with $I \neq (a)$ and $I \neq (b)$. Obviously, $a \neq 0$ and $b \neq 0$. By Lemma 2.3.7, $I^2 = 0$. Consequently, both cases satisfy the conditions of [9, Theorem 2.2(d)] and thus R is a Gaussian ring, proving the second implication.

It remains to show that both implications are, in general, irreversible. This is handled by the next two examples. $\square$

Example 2.3.8. There is an example of an fqp-ring that is not arithmetical.

Proof. From [48], consider the local ring $R := \frac{\mathbb{F}_2[x,y]}{(x,y)^2} \cong \mathbb{F}_2[\bar{x}, \bar{y}]$ with maximal ideal $\mathfrak{m} := (\bar{x}, \bar{y})$. The proper ideals of R are exactly (0), $(\bar{x})$, $(\bar{y})$, $(\bar{x} + \bar{y})$, and $\mathfrak{m}$. By Lemma 2.3.3, $(\bar{x})$, $(\bar{y})$, and $(\bar{x} + \bar{y})$ are $\star$-modules. Further $\mathfrak{m} := (\bar{x}) \oplus (\bar{y})$ implies that $\mathfrak{m}$ is quasi-projective by Lemma 2.3.4. Hence R is an fqp-ring. Clearly, R is not an arithmetical ring since $\mathfrak{m}$ is not principal. $\square$

Example 2.3.9. There is an example of a Gaussian ring that is not an fqp-ring.

Proof. Let $\mathbb{K}$ be a field and consider the local Noetherian ring $R := \frac{\mathbb{K}[x,y]}{(x^2,,xy,y^3)} \cong \mathbb{K}[\bar{x},\bar{y}]$ with maximal ideal $\mathfrak{m} := (\bar{x},\bar{y})$. One can easily verify that $\mathrm{Ann}(\mathfrak{m}) = (\bar{x},\bar{y}^2)$ and then $\frac{R}{\mathrm{Ann}(\mathfrak{m})} \cong \frac{\mathbb{K}[y]}{(y^2)}$. Therefore $\frac{R}{\mathrm{Ann}(\mathfrak{m})}$ is a principal ring and hence an arithmetical ring. It follows that R is a Gaussian ring (see first paragraph right after Theorem 2.2 in [9]). Finally, we claim that $\mathfrak{m}$ is not quasi-projective. Deny. Since $\mathfrak{m} = (\bar{x},\bar{y})$ with $\mathfrak{m} \neq (\bar{x})$ and $\mathfrak{m} \neq (\bar{y})$, then Lemma 2.3.7 yields $\mathfrak{m}^2 = 0$, absurd. Thus R is a not an fqp-ring, as desired. □

Next, in view of Theorem 2.3.1 and Example 2.3.8, we extend Osofsky's theorem on the weak global dimension of arithmetical rings to the class of fqp-rings.

Theorem 2.3.10. *The weak global dimension of an fqp-ring is equal to 0, 1, or ∞.*

The proof uses the following result.

Lemma 2.3.11 ([84, Theorem 2])**.** *Let R be a local fqp-ring. Then either* $\mathrm{Nil}(R)^2 = 0$ *or R is a chained ring* (*i.e., its ideals are linearly ordered with respect to inclusion*). □

Proof of Theorem 2.3.10. Let R be an fqp-ring. We envisage two cases. Case 1: Suppose R is reduced. Then Theorem 2.3.1 combined with [47, Theorem 2.2] forces the weak global dimension of R to be less than or equal to one, as desired. Case 2: Suppose R is not reduced. We wish to show that $\mathrm{w.dim}(R) = \infty$. Indeed, since $\mathrm{Nil}(R)$ is not null, there is $P \in \mathrm{Spec}(R)$ such that $\mathrm{Nil}(R)R_P \neq 0$. The ring R_P is now a non-reduced local fqp-ring (Lemma 2.3.5) with $\mathrm{Nil}(R_P) = \mathrm{Nil}(R)R_P$. By Lemmas 2.3.5 and 2.3.11, $(\mathrm{Nil}(R_P))^2 = 0$ or R_P is a chained ring. By Theorem 2.3.1, R_P is Gaussian, so the statement "$(\mathrm{Nil}(R_P))^2 = 0$" yields $\mathrm{w.dim}(R_P) = \infty$ by [9, Theorem 6.4]. On the other hand, the statement "R_P is a chained ring" implies that R_P is a local arithmetical ring with zero-divisors (since $\mathrm{Nil}(R_P) \neq 0$)), hence R_P has an infinite weak global dimension (Osofsky [71]). Finally, the known fact "$\mathrm{w.dim}(R_P) \leq \mathrm{w.dim}(R)$, $\forall\, P \in \mathrm{Spec}(R)$" leads to the conclusion, completing the proof of the theorem. □

In 2005, Glaz proved that Osofsky's result is valid in the class of coherent Gaussian rings [47, Theorem 3.3]. In 2007, Bazzoni and Glaz conjectured that the same must hold in the whole class of Gaussian rings [9]. Theorem 2.3.10 widens the scope of validity of this conjecture beyond the class of coherent Gaussian rings, as shown by next example:

Example 2.3.12. There is an example of an fqp-ring that is not coherent.

Proof. Let $\mathbb{K}$ be field and $\{x_1, x_2, ...\}$ an infinite set of indeterminates over $\mathbb{K}$. Let $R := \frac{\mathbb{K}[x_1,x_2,...]}{\mathfrak{m}^2} = \mathbb{K}[\overline{x_1}, \overline{x_2}, ...]$, where $\mathfrak{m} := (x_1, x_2, ...)$. One can easily check that R has the following features:

(1) $R = \mathbb{K} + \frac{\mathfrak{m}}{\mathfrak{m}^2}$ is local with maximal ideal $\frac{\mathfrak{m}}{\mathfrak{m}^2}$.

(2) $\forall\, f \in \frac{\mathfrak{m}}{\mathfrak{m}^2}$, $\mathrm{Ann}(f) = \frac{\mathfrak{m}}{\mathfrak{m}^2}$.

(3) $\forall\, i \neq j$, $(\overline{x_i}) \cap (\overline{x_j}) = 0$.

(4) $\forall\, f \in \frac{\mathfrak{m}}{\mathfrak{m}^2}$ and $\forall\, i \geq 1$, $(f) \cong (\overline{x_i})$.

(5) For every finitely generated ideal I of R, we have $I \cong \bigoplus_{k=1}^{n} (\overline{x_{i_k}})$ for some indeterminates $x_{i_1}, ..., x_{i_n}$ in $\{x_1, x_2, ...\}$.

Let I be a finitely generated ideal of R. By (4), $(\overline{x_i})$ is $(\overline{x_j})$-projective for all $i, j \geq 1$. So (5) forces I to be quasi-projective by Lemma 2.3.4. Therefore R is an fqp-ring. Moreover, by (2), the following sequence of natural homomorphisms

$$0 \rightarrow \frac{\mathfrak{m}}{\mathfrak{m}^2} \rightarrow R \rightarrow R\overline{x_1} \rightarrow 0$$

is exact. So $R\overline{x_1}$ is not finitely presented and hence R is not coherent, as desired. □

In [9], Bazzoni and Glaz proved that a Prüfer ring R satisfies any of the other four Prüfer conditions (mentioned in the introduction) if and only if its total ring of quotients Q satisfies that same condition. This fact narrows the scope of study of the Prüfer conditions to the class of total rings of quotients; specifically, "a Prüfer ring is Gaussian (resp., is arithmetical, has $\mathrm{w.dim}(R) \leq 1$, is semihereditary) if and only if so is Q" [9, Theorems 3.3 , 3.6 , 3.7 & 3.12]. Next, we establish an analogue for the $\star$-property in the local case.

Theorem 2.3.13. *Let R be a local ring. Then R is Prüfer and Q is an fqp-ring if and only if R is an fqp-ring.*

Proof. A Gaussian ring is Prüfer [48, Theorem 3.4.1] and [62, Theorem 6]. So in view of Theorem 2.3.1 and Lemma 2.3.5 only the necessity has to be proved. Assume R is Prüfer and Q is an fqp-ring. Notice first that R is a (local) Gaussian ring by [9, Theorem 3.3] and hence the lattice of its prime ideals is linearly ordered [90]. Therefore the set of zero-divisors $\mathrm{Z}(R)$ of R is a prime ideal and hence $Q = R_{\mathrm{Z}(R)}$ is local. Next, let S denote the set of all regular elements of R and let I be a finitely generated ideal of R with a minimal generating set $\{x_1, \ldots, x_n\}$. If I is regular, then I is projective (since R is Prüfer). Suppose I is not regular, that is, $I \cap S = \emptyset$. We wish to show that I is quasi-projective. We first claim that

$$\left(\frac{x_i}{1}\right)Q \cap \left(\frac{x_j}{1}\right)Q = 0, \ \forall\, i \neq j \in \{1, \ldots, n\}.$$

Indeed, for any $i \neq j$, the ideals $(\frac{x_i}{1})$ and $(\frac{x_j}{1})$ are incomparable in Q: Otherwise if, say, $\frac{x_i}{1} \in (\frac{x_j}{1})$, then $sx_i = ax_j$ for some $a \in R$ and $s \in S$. Since s is regular, the ideal (a, s) is projective in R (which is Prüfer). Moreover, by Lemma 2.3.7, we obtain $(a, s) = (s)$ or $(a, s) = (a)$ and, in this case, necessarily $a \in S$. It follows that x_i and x_j are linearly dependent which contradicts minimality. Therefore, by Lemma 2.3.7 applied to the ideal $(\frac{x_i}{1}, \frac{x_j}{1})$ in the local fqp-ring $Q(R)$, we get $(\frac{x_i}{1}) \cap (\frac{x_j}{1}) = 0$, proving the claim. Since S consists of regular elements, $x_iR \cap x_jR = 0$, for each $i \neq j$, whence $I = \bigoplus_{i=1}^{n} x_iR$. Further, by Lemma 2.3.7, we have

$$\mathrm{Ann}_{Q(R)}\left(\frac{x_i}{1}\right) = \mathrm{Ann}_{Q}\left(\frac{x_j}{1}\right), \ \forall\, i \neq j \in \{1, \ldots, n\}.$$

Therefore, we obtain

$$\mathrm{Ann}(x_i) = \mathrm{Ann}(x_j),\ \forall\, i \neq j \in \{1, \ldots, n\}.$$

Consequently, $x_iR \cong x_jR$ and hence x_iR is x_jR-projective for all i, j. Once again, we appeal to Lemma 2.3.4 to conclude that I is quasi-projective, as desired. □

The global case holds for coherent rings as shown next.

Corollary 2.3.14. *Let R be a coherent ring. Then R is Prüfer and Q is an fqp-ring if and only if R is an fqp-ring.*

Proof. Here too only necessity has to be proved. Assume R is Prüfer and Q is an fqp-ring and let I be a finitely generated ideal of R. By [9, Theorem 3.3], R is Gaussian. Let P be a prime ideal of R. Then R_P is a local Prüfer ring (since Gaussian). Moreover, by [9, Theorem 3.4], the total ring of quotients of R_P is a localization of Q (with respect to a multiplicative subset of R) and hence an fqp-ring by Lemma 2.3.5. By Theorem 2.3.13, R_P is an fqp-ring. Consequently, I is locally quasi-projective. Since I is finitely presented, I is quasi-projective [37, Theorem 2], as desired. □

We close this section with a discussion of the global case. Recall first that the Gaussian and arithmetical properties are local, i.e., R is Gaussian (resp., arithmetical) if and only if $R_{\mathfrak{m}}$ is Gaussian (resp., arithmetical) for every maximal ideal $\mathfrak{m}$ of R. The same holds for rings with weak global dimension ≤ 1. We were not able to prove or disprove this fact for the $\star$-property. Notice however that the $\star$-module notion is not local. Indeed, from [73], let $\mathbb{K}$ be a field and R the algebra over $\mathbb{K}$ generated by $\{x_1, x_2, ...\}$ such that $x_n x_{n+1} = x_n$ for all $n \geq 1$. The ring R has a finitely generated flat module that is not projective. So there is a non-finitely generated R-module M that is locally finitely generated and projective [92, 93]. Hence M is locally a $\star$-module but not a $\star$-module (since M is not finitely generated). But in order to build counterexamples for the fqp-ring notion, one needs first to investigate non-local settings which yield Gaussian rings that are not fqp-rings.

Moreover, the transfer result for the semihereditary notion (which is not a local property) was made possible by Endo's result that "a total ring of quotients is semihereditary if and only if it is von Neumann regular" [30]. No similar phenomenon occurs for the $\star$-property; namely, a total ring of quotients that is an fqp-ring is not necessarily arithmetical (see Example 2.3.8).

Based on the above discussion, we conjecture that Theorem 2.3.13 is not true, in general, beyond the local and coherent cases. That is, a Prüfer ring, with a total ring of quotients which is an fqp-ring, is not necessarily an fqp-ring.

2.4 Examples via trivial ring extensions

This section studies the $\star$-property in various trivial ring extensions. Our objective is to generate new and original examples to enrich the current literature with new families of fqp-rings. It is worthw noticing that trivial extensions have been thoroughly investigated in [4] for the other five Prüfer conditions (mentioned in the introduction).

Let A be a ring and E an A-module. The trivial (ring) extension of A by E (also called the idealization of E over A) is the ring $R := A \propto E$ whose underlying group is $A \times E$ with multiplication given by $(a_1, e_1)(a_2, e_2) = (a_1a_2, a_1e_2 + a_2e_1)$. For the reader's convenience, recall that if I is an ideal of A and E' is a submodule of E such that $IE \subseteq E'$, then $J := I \propto E'$ is an ideal of R; ideals of R need not be of this form [58, Example 2.5]. However, prime (resp., maximal) ideals of R have the form $p \propto E$, where p is a prime (resp., maximal) ideal of A [56, Theorem 25.1(3)]. Also an ideal of R of the form $I \propto IE$, where I is an ideal of A, is finitely generated if and only if I is finitely generated [46, p. 141]. A Suitable background on commutative trivial ring extensions is [46, 56].

We first state a necessary condition for the inheritance of the $\star$-property in a general context of trivial extensions.

Proposition 2.4.1. *Let A be a ring, E an A-module, and $R := A \propto E$ the trivial ring extension of A by E. If R is an fqp-ring, then so is A.*

Proof. Assume that R is an fqp-ring. Let I be a finitely generated ideal of A, J a subideal of I, and $f \in \mathrm{Hom}_A(I, I/J)$. We wish to prove the existence of $h \in \mathrm{Hom}_A(I, I)$ such that $f(x) = \overline{h(x)}$ (mod J), for every $x \in I$. Clearly, $I \propto IE$ is a finitely generated ideal of R and $J \propto IE$ a subideal of $I \propto IE$. Let $F : I \propto IE \longrightarrow \dfrac{I \propto IE}{J \propto IE}$ be defined by $F(x,e) = \overline{(a,0)}$ (mod $J \propto IE$) where $a \in I$ with $f(x) = \overline{a}$ (mod J). It is easily seen that F is a well-defined R-map. By assumption, $I \propto IE$ is a $\star$-module. So there exists $H \in \mathrm{Hom}_R(I \propto IE, I \propto IE)$ such that $F(x,e) = \overline{H(x,e)}$ (mod $J \propto IE$), for every $(x,e) \in I \propto IE$. Now, for each $x \in I$, let $h(x)$ denote the first coordinate of $H(x,0)$; that is, $H(x,0) = (h(x), e_x)$ for some $e_x \in IE$. One can easily check that $h : I \longrightarrow I$ is an A-map. Moreover, let $x \in I$ and $a \in I$ with $f(x) = \overline{a}$. We have $\overline{(a,0)} = F(x,0) = \overline{H(x,0)} = \overline{(h(x), e_x)}$ (mod $J \propto IE$). It follows that $f(x) = \overline{a} = \overline{h(x)}$ (mod J), as desired. □

Remark 2.4.2. One can also prove Proposition 2.4.1 via Corollary 2.2.6. Indeed, assume $R := A \propto E$ is an fqp-ring and let I be a finitely generated ideal of A. Then $U_R := I \propto IE$ is a finitely generated ideal of R and hence a $\star$-module. Now consider the ring homomorphism $\varphi : R \longrightarrow A$ defined by $\varphi(a,e) = a$. Clearly, the fact $A \cong \frac{R}{0 \propto E}$ leads to the conclusion (to the effect that $A \otimes_R U \cong \frac{R}{0 \propto E} \otimes_R I \propto IE \cong \frac{I \propto IE}{0 \propto IE} \cong I$).

Example 2.4.7 below provides a counter-example for the converse of Proposition 2.4.1. The next two theorems establish necessary and sufficient conditions for the transfer of of the $\star$-property in special contexts of trivial extensions. The next result examines the case of trivial extensions of integral domains.

Theorem 2.4.3. *Let $A \subseteq B$ be an extension of domains and $K := \mathrm{qf}(A)$. Let $R := A \propto B$ be the trivial ring extension of A by B. Then the following statements are equivalent:*

(1) *A is a Prüfer domain with $K \subseteq B$;*

(2) *R is a Prüfer ring;*

(3) *R is a Gaussian ring;*

(4) *R is an fqp-ring.*

Proof. The implications (1) $\iff$ (2) $\iff$ (3) and (4) $\Longrightarrow$ (3) are handled by [4, Theorem 2.1] and Theorem 2.3.1, respectively. It remains to prove (3) $\Longrightarrow$ (4). Notice first that $(a,b) \in R$ is regular if and only if $a \neq 0$. Assume that R is Gaussian and let I be a (nonzero) finitely generated ideal of R. If I contains a regular element, then I is projective (since R is a Prüfer ring). If $I \subseteq 0 \propto B$, then I is a torsion free A-module and hence projective (since A is a Prüfer domain). But $A \cong \frac{R}{0 \propto B}$ with $\mathrm{Ann(I)} = 0 \propto \mathrm{B}$, hence I is quasi-projective by Lemma 2.2.2. Therefore R is an fqp-ring. □

Next we study the transfer of the $\star$-property in trivial extensions of local rings by vector spaces over the residue fields.

Theorem 2.4.4. *Let $(A,\mathfrak{m})$ be a local ring and E a nonzero $\dfrac{A}{\mathfrak{m}}$-vector space. Let $R := A \propto E$ be the trivial ring extension of A by E. Then R is an fqp-ring if and only if $\mathfrak{m}^2 = 0$.*

The proof relies on the next preliminary results.

Lemma 2.4.5. *Let R be a local fqp-ring which is not a chained ring. Then $\mathrm{Z}(R) = \mathrm{Nil}(R)$.*

Proof. Let $s \in \mathrm{Z}(R)$. Assume by way of contradiction that $s \notin \mathrm{Nil}(R)$. Let x, y be nonzero elements of R such that (x) and (y) are incomparable (since R is not a chained ring). Lemma 2.3.7 forces (x) and (s) to be comparable and a fortiori $x \in (s)$. Likewise $y \in (s)$;

say, $x = sx'$ and $y = sy'$ for some $x', y' \in R$. Necessarily, (x') and (y') are incomparable and hence $(x') \cap (y') = 0$ (by the same lemma). Now let $0 \neq t \in R$ be such that $st = 0$. Next let's consider three cases. If (x') and (t) are incomparable, then $\mathrm{Ann}(x') = \mathrm{Ann}(t)$ by Lemma 2.3.7(3). It follows that $x = sx' = 0$, absurd. If $(t) \subseteq (x')$, then $(t) \cap (y') \subseteq (x') \cap (y') = 0$. So (y') and (t) are incomparable, whence similar arguments yield $y = sy' = 0$, absurd. If $(x') \subseteq (t)$; say, $x' = rt$ for some $r \in R$, then $x = sx' = str = 0$, absurd. All possible cases end up with an absurdity, the desired contradiction. Therefore $s \in \mathrm{Nil}(R)$ and thus $\mathrm{Z}(R) = \mathrm{Nil}(R)$. □

Lemma 2.4.6. *Let $(R, \mathfrak{m})$ be a local ring. If $\mathfrak{m}^2 = 0$, then R is an fqp-ring.*

Proof. Let I be a nonzero proper finitely generated ideal of R. Then $\mathrm{Ann}(I) = \mathfrak{m}$. So $\dfrac{R}{\mathrm{Ann}(I)} \cong \dfrac{R}{\mathfrak{m}}$. Hence I is a free $\frac{R}{\mathrm{Ann}(I)}$-module, whence I is quasi-projective (Lemma 2.2.2). Consequently, R is an fqp-ring. □

Proof of Theorem 2.4.4. Recall first that R is local with maximal ideal $\mathfrak{m} \propto E$ as well as a total ring of quotients (i.e., $Q = R$). Now suppose that R is an fqp-ring. Without loss of generality, we may assume A not to be a field. Notice that R is not a chained ring since, for $e := (1, 0, 0, \ldots) \in E$ and $0 \neq a \in \mathfrak{m}$, $\big((a,0)\big)$ and $\big((0,e)\big)$ are incomparable. Therefore Lemma 2.4.5 yields $\mathfrak{m} \propto E = \mathrm{Z}(R) = \mathrm{Nil}(R)$. By Lemma 2.3.11, $(\mathfrak{m} \propto E)^2 = 0$, hence $\mathfrak{m}^2 = 0$, as desired.

Conversely, suppose $\mathfrak{m}^2 = 0$. Then $(\mathfrak{m} \propto E)^2 = 0$ which leads to the conclusion via Lemma 2.4.6, completing the proof of the theorem. □

[4, Theorem 3.1] states that "$R := A \propto E$ is Gaussian if and only if so is A" and "R is arithmetical if and only if $A := K$ is a field and $\dim_K E = 1$." Theorem 2.4.4 generates new and original examples of rings with zerodivisors subject to Prüfer conditions as shown below.

Example 2.4.7. $R := \frac{\mathbb{Z}}{8\mathbb{Z}} \propto \frac{\mathbb{Z}}{2\mathbb{Z}}$ is a Gaussian total ring of quotients which is not an fqp-ring.

Example 2.4.8. $R := \frac{\mathbb{Z}}{4\mathbb{Z}} \propto \frac{\mathbb{Z}}{2\mathbb{Z}}$ is an fqp total ring of quotients which is not arithmetical.

2.5 *n*-Star modules over Dedekind domains

Throughout n is a positive integer. This section deals with n-star modules (initially considered in [94]) where the aim is to examine this notion over Dedekind domains.

Definition 2.5.1. An R-module T is n-$\sum$-quasi-projective if $\mathrm{Hom}_R(T,-)$ is exact with respect to all exact sequences

$$0 \to N \to P \to M \to 0$$

with $P \in \mathrm{Add}(T)$ and $N \in \mathrm{Gen}_{n-1}(T)$.

Note that the class of finitely generated quasi-projective modules coincide with the class of finitely generated 1-Σ-quasi-projective modules [1, Proposition 16.12].

Definition 2.5.2. An R-module T is said to be an n-star module if T is (n+1)-$\sum$-quasi-projective and $\mathrm{Gen}_{n+1}(T) = \mathrm{G}en_n(T)$.

Recall that an R-module M is self-small if the covariant functor $\mathrm{Hom}(M,-)$ commutes with the direct sums of copies of M. An R-module T is an $\star^n$-module if T is a self-small n-star module. By [22, Proposition 4.3], the class of $\star$-modules coincides with the class of $\star^1$-modules. Later, the self-smallness assumption was dropped in [94] and the new modules were called n-star modules. Accordingly, n-star modules are generalizations of $\star$-modules.

Wei proved in [95] that $\mathbb{Q}$ is 2-star $\mathbb{Z}$-module. Next, we generalize this fact as follows.

Proposition 2.5.3. *Let R be a domain and Q its quotient field. Let B be a ring containing Q. Then B is a 2-star R-module.*

Proof. First of all it is clear that B_R is torsion-free and divisible (being a vector space over Q). Let $0 \to N \to B_0 \to M \to 0$ be an exact sequence with $B_0 \in \mathrm{Add}(B)$ and $N \in \mathrm{Gen}_2(B)$. Note that N is a torsion-free module, since N_R is a submodule of a torsion-free module. Since N_R is a homomorphic image of a divisible R-module, N_R is divisible. Since every torsion free divisible is injective by [42, Theorem IX.1.1], N is an injective R-module. The following sequence

$$0 \to \mathrm{Hom}_R(B,N) \to \mathrm{Hom}_R(B,B_0) \to \mathrm{Hom}_R(B,M) \to 0$$

is exact and hence B_R is $3-\Sigma-quasi-projective$. It is clear that $\mathrm{Gen}_3(B_R) \subseteq \mathrm{Gen}_2(B_R)$. Conversely, let $X_R \in \mathrm{Gen}_2(B_R)$; so we have the following exact sequence

$$0 \to K \to B^{(I)} \to X \to 0$$

with $K \in \mathrm{Gen}(B_R)$ and one can easily check that K is injective. Hence $X \in \mathrm{Add}(B_R)$ and thus $X \in \mathrm{Gen}_3(B_R)$, as desired. □

Corollary 2.5.4. *Over a Dedekind domain, any injective module is a 2-star module.*

Proof. Every divisible module over a Dedekind domain is injective (e.g., [75, Theorem 4.27]). Use this argument in the second part of the proof of Proposition 2.5.3 to prove the corollary. □

Next, our main result, characterizes Dedekind domains via the notion of n-star module. For the proof, we make use of some known results in the literature, which we record below for the reader's convenience.

Proposition 2.5.5 ([98, Proposition 18.9])**.** *Let R be a commutative ring and M a finitely generated R-module. If M is faithful, then M generates all simple R-modules.*

Lemma 2.5.6 ([94, Lemma 3.2])**.** *Let T be an n-star module. Assume that* $0 \to L \to M \to N \to 0$ *is exact with* $L, M \in \mathrm{Gen}_n(T)$*. Then* $N \in \mathrm{Gen}_n(T)$.

Proposition 2.5.7 ([94, Proposition 3.3])**.** *Let T be an n-star module. Assume that the exact sequence* $0 \to L \to M \to N \to 0$ *stays exact under the functor* $\mathrm{Hom}(T, -)$*. If two of the three modules* L, M, N *are in* $\mathrm{Gen}_n(T)$*, then so is the third one.*

Theorem 2.5.8 ([94, Theorem 3.4])**.** *Let T be an n-star module. Then the functor* $\mathrm{Hom}_R(T, -)$ *preserves short exact sequences in* $\mathrm{Gen}_n(T)$.

Corollary 2.5.9 ([28, Corollary 4.3])**.** *Let R be a ring with flat maximal ideals. If R is Noetherian, then R is hereditary.*

Lemma 2.5.10. *Let R be a domain and let $\mathfrak{m}$ be a finitely generated maximal ideal of R. If* $\mathrm{Gen}(\mathfrak{m})$ *is closed under extensions, then $\mathfrak{m}$ is a projective module.*

Proof. Let $\mathfrak{m}$ be a finitely generated maximal ideal of R and consider the following exact sequence $0 \to \mathfrak{m} \to R \to R/\mathfrak{m} \to 0$. Since R is an integral domain, $\mathfrak{m}$ is a finitely generated faithful R-module and hence $R/\mathfrak{m} \in \mathrm{Gen}(\mathfrak{m})$ by Proposition 2.5.5. Since $\mathrm{Gen}(\mathfrak{m})$ is closed under extensions, $R \in \mathrm{Gen}(\mathfrak{m})$. By [86, Theorem 3.1], $\mathfrak{m}$ is projective. □

Theorem 2.5.11. *Let R be a Noetherian domain and n a positive integer. The following statements are equivalent:*

(1) *R is a Dedekind domain;*

(2) $\mathrm{Gen}_n(\mathfrak{m})$ *contains all simple R-modules and $\mathfrak{m}$ is an n-star R-module, for each maximal ideal $\mathfrak{m}$ of R;*

(3) $\mathrm{Gen}_n(\mathfrak{m})$ *is closed under maximal submodules and $\mathfrak{m}$ is an n-star R-module, for each maximal ideal $\mathfrak{m}$ of R;*

(4) $\mathrm{Gen}(\mathfrak{m})$ *is closed under extensions, for each maximal ideal* $\mathfrak{m}$ *of* R.

Proof. (2) $\Rightarrow$ (1) Let $\mathfrak{m}$ be a maximal ideal of R. For each $0 \neq x \in \mathfrak{m}$, the R-module $xR/x\mathfrak{m} \simeq R/\mathfrak{m}$ is a simple R-module. Now, we construct the following exact commutative diagram:

$$\begin{array}{ccccccccc} & & & & & & 0 & & \\ & & & & & & \downarrow & & \\ & & & & & & xR/x\mathfrak{m} & & \\ & & & & & & \downarrow & & \\ 0 & \rightarrow & x\mathfrak{m} & \rightarrow & \mathfrak{m} & \rightarrow & \mathfrak{m}/x\mathfrak{m} & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & xR & \rightarrow & \mathfrak{m} & \rightarrow & \mathfrak{m}/xR & \rightarrow & 0 \\ & & & & & & \downarrow & & \\ & & & & & & 0 & & \end{array}$$

By applying the functor $\mathrm{Hom}_R(\mathfrak{m}, -)$, we get the following exact commutative diagram

$$\begin{array}{ccccccccc} & & & & & & 0 & & \\ & & & & & & \downarrow & & \\ & & & & & & \mathrm{Hom}_R(\mathfrak{m}, xR/x\mathfrak{m}) & & \\ & & & & & & \downarrow & & \\ 0 & \rightarrow & \mathrm{Hom}_R(\mathfrak{m}, x\mathfrak{m}) & \rightarrow & \mathrm{Hom}_R(\mathfrak{m}, \mathfrak{m}) & \rightarrow & \mathrm{Hom}_R(\mathfrak{m}, \mathfrak{m}/x\mathfrak{m}) & \dashrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & \mathrm{Hom}_R(\mathfrak{m}, xR) & \rightarrow & \mathrm{Hom}_R(\mathfrak{m}, \mathfrak{m}) & \rightarrow & \mathrm{Hom}_R(\mathfrak{m}, \mathfrak{m}/xR) & \dashrightarrow & 0 \\ & & & & & & \downarrow & & \\ & & & & & & 0 & & \end{array}$$

Note that $x\mathfrak{m} \simeq \mathfrak{m}$, and $\mathfrak{m} \in \mathrm{Gen}_n(\mathfrak{m})$; so $x\mathfrak{m} \in \mathrm{Gen}_n(\mathfrak{m})$. By Lemma 2.5.6, we obtain that $\mathfrak{m}/x\mathfrak{m} \in \mathrm{Gen}_n(\mathfrak{m})$. Therefore, the first row is exact by Theorem 2.5.8. Moreover, $xR/x\mathfrak{m} \simeq R/\mathfrak{m}$ is a simple R-module, and so $xR/x\mathfrak{m} \in \mathrm{Gen}_n(\mathfrak{m})$ and $\mathfrak{m}/x\mathfrak{m} \in \mathrm{Gen}_n(\mathfrak{m})$ as shown above, it follows that $\mathfrak{m}/xR \in \mathrm{Gen}_n(\mathfrak{m})$. Then by Theorem 2.5.8, the column is exact. By commutativity of the diagram, we conclude that the second row is also exact. As shown above, $\mathfrak{m}$ and $\mathfrak{m}/xR \in \mathrm{Gen}_n(\mathfrak{m})$, and so by Proposition 2.5.7 $xR \in \mathrm{Gen}_n(\mathfrak{m})$ and hence $xR \cong R$. Then $\mathfrak{m}$ is a progenerator R-module since $\mathfrak{m}$ is an n-star module. Therefore R is a Dedekind domain by Corollary 2.5.9.

(1) $\Rightarrow$ (2) Clear.

(3) $\Rightarrow$ (2) Let $\mathfrak{m}$ be a maximal ideal of R. Let U be a simple R-module. We have an exact sequence of R-modules $0 \to J \to \mathfrak{m} \to U \to 0$. Applying Lemma 2.5.6, we get that $U \in \mathrm{Gen}_n(\mathfrak{m})$.

(1) $\Rightarrow$ (3) Clear.

(1) $\Rightarrow$ (4) Clear.

(4) $\Rightarrow$ (1) By Lemma 2.5.10. $\square$

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Glossary

$J(R)$: The Jacobson radical of R.

Max(R):The spectrum of maximal R-ideals.

$R^{\times}$: The set of all non zero- divisors.

$Q = (R^{\times})^{-1}R$: The total ring of quotients of R

Ann(M): $\{r \in R : rM = 0\}$

$\mathscr{DI}$: The class of all divisible R-modules.

$\mathscr{F}_n : \{M_R \mid \text{w.d.}_R(M) \leq n\}$.

$$\mathscr{F} : \bigcup_{n=0}^{\infty} \mathscr{F}_n.$$

$\mathscr{FL}$: The class of flat R-modules

$\mathscr{I}_n : \{M_R \mid \text{i.d.}_R(M) \leq n\}$

$\mathscr{I}$: $\bigcup_{n=0}^{\infty} \mathscr{I}_n$

$\mathscr{I}\mathscr{N}$: The class of injective R-modules.

i.d.$_R(M)$: The injective dimension of M

$\mathscr{P}$: $\bigcup_{n=0}^{\infty} \mathscr{P}_n$

$\mathscr{P}_n$: $\{M_R \mid \text{p.d.}_R(M) \leq n\}$

$\mathscr{P}\mathscr{R}$: The class of projective R-modules

p.d.$_R(M)$: The projective dimension of M

R-Mod: The category of R-modules.

S-divisible: $\mathscr{D}_S := \{_R M \mid sM = M \text{ for every } s \in S\}$ where $S \subseteq R$.

$\mathscr{T}\mathscr{F}$.: The class of torsion-free R-modules

w.d.$_R(M))$: The weak dimension of M

$Z(R)$: The set of zero-divisors of R

Almost Dedekind domain : A domain R such that R_P is a DVR for each maximal ideal

P of R.

Cotorsion module : An R- module M is called cotorsion module if $\text{Ext}^1_R(N,M) = 0$ for every R-module $N \in \mathscr{F}_0$.

Faithful module : An R- module M is called faithful module if $\text{Ann}_R(M) = 0$.

h**-local domain** : A domain R where every is h element is contained in a finite number of maximal ideals and every prime ideal is contained in one maximal ideal.

Matlis domain : A domain R with projective dimension of $Q(R)$ is at most 1.

n**-Gorenstein ring** : A Noetherian ring R withe injective dimension of R at most n.

Semiartinian module : A module M such that if $0 \neq M/N$ contains a simple submodule for every proper submodule N of M.

Semilocal ring: A ring with finite number of maximal ideals .

Weakly cotorsion module : A M such that $\text{Ext}^1_R(Q,M) = 0$.

Strongly flat module M : A module M with $\text{Ext}^i_R(M,N) = 0$ for every weakly cotorsion module N.

T**-nilpotent ideal** : An ideal I is called T-nilpotent if for any sequence $\{a_1, a_2, ...\} \subseteq I$ there exists n such that $a_1 ... a_n = 0$.

Von Neumann regular ring : A ring R such that for every element $a \in R$ there exists an element $x \in R$ such that $a = a^2x$.

Weak-injective module : A module M such that $\mathrm{Ext}^1_R(N,M) = 0$ for every $N \in \mathscr{F}_1$.

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VITA

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