STUDY OF INTERACTION BETWEEN VISCOELASTIC AND FRICTIONAL DAMPINGS IN WAVE EQUATIONS

BY

SAEED MOHAMMED SALMAN ALI

A Thesis Presented to the
DEANSHIP OF GRADUATE STUDIES

KING FAHD UNIVERSITY OF PETROLEUM & MINERALS

DHAHRAN, SAUDI ARABIA

In Partial Fulfillment of the
Requirements for the Degree of

MASTER OF SCIENCE

In

MATHEMATICS

APRIL, 2009

KING FAHD UNIVERSITY OF PETROLEUM & MINERALS
DHAHRAN 31261, SAUDI ARABIA

DEANSHIP OF GRADUATE STUDIES

This thesis, written by **SAEED MOHAMMED SALMAN ALI** under the direction of his thesis advisor and approved by his thesis committee, has been presented to and accepted by the Dean of Graduate Studies, in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE IN MATHEMATICS

Thesis Committee

Prof. Salim A. Messaoudi (Advisor)

Dr. Nasser-eddine Tatar(Member)

Dr. Khled Furati (Member)

Dr. Suliman S. Al-Homidan
Department Chairman

Dr. Salam A. Zummo
Dean of Graduate Studies

2/6/09

الإهــداء

أهدي هذا العمل

إلى والدي الكريمين وعمى حسن العزيز ... ثمرة من ثمار تربيتهم وغرسهم

إلى زوجتى العزيزه وبناتى العزيزات.... وفاءً لهنّ ولصبرهنّ وتضحيتهـنّ

إلى إخوتـــــــــــــى وأخــــــــــــواتى ... رداً لبعض معروفهم وإحسانهـــم

DEDICATED

TO

MY PARENTS AND FAMILY

ACKNOWLEDGMENT

First and foremost thanks to Allah who gave me strength, patience and ability to accomplish this research.

Acknowledgment is due to the King Fahd University of Petroleum & Minerals for supporting this research.

I wish to express my appreciation to Professor. Salim Messaoudi who served as my major advisor, for his guidance and patience through the thesis. His continuous support and encouragement can never be forgotten. I would like also to thank Dr. Nasser-eddine Tatar and Dr. Khaled Furati for their suggestions and valuable comments. Thanks are also due to the Chairman of Mathematics Department Dr. Suliman for providing all the available facilities. I am also grateful to all faculty members for their encouragement and their direct or indirect help. Finally, I thank my family for their patience and encouragement during my years of study.

TABLE OF CONTENTS

THESIS ABSTRACT

Name: SAEED MOHAMMED SALMAN

Title: STUDY OF INTERACTION BETWEEN VISCOELASTIC DAMPING AND FRICTIONAL DAMPING IN WAVE EQUATIONS

Major Field: MATHEMATICS

Date of Degree: April, 20009

In this thesis we study the interaction between viscoelastic damping and frictional damping in wave equations. In this regard, we study several problems and establish exponential and polynomial decay rate results. Our results are obtained for relaxation functions which decay exponentially or polynomially. These results extend several earlier results in the literature.

ملخص الرسالة

الاســــــــم: **سعيد محمد سلمان علي**

عنوان الرسالة: **دراسة التفاعل بين التخميد المرن اللزج والاحتكاكي فى معادلة الموجه**

التخصـــــص: **رياضيات**

تاريخ التخــرج: **جماد الاول 1430 هجرية**

ندرس فى هذه الرساله التفاعل بين التخميد المرن اللزج والاحتكاكي فى معادلة الموجه. لهذا الغرض نقوم بدراسة عدد من المسائل ونتوصل الى الصيغه الرياضيه التى تعطى معدل اضمحلال أسى اوكثيري حدود فى كل حاله. وهذه النتائج التي تم التوصل اليها لدوال استرخاء تضمحل اسيا اوكثيري حدود. ان هذه النتائج التي حصلنا عليها امتداد لمجموعه من النتائج السابقه المتعلقه بمسائل المرونه اللزجه.

Chapter 1

INTRODUCTION

1.1 Viscoelastic Materials

An elastic solid has usually a definite shape. When it is subjected to an external force during a certain period, the body is deformed to another equilibrium shape and it stores all the energy obtained from the work created by the external force during the deformation. When the external force is removed, the body returns to its original shape. On the other hand, a viscous liquid has no definite shape, and it flows irreversibly under the action of external forces.

There are materials with properties that are intermediate between those of an elastic solid and a viscous liquid. For instance, the most interesting features of high polymers is that a given polymer can display all the intermediate range of properties, depending on the temperature and the chosen time scale during the experiment [30,31,32]. A polymer can also exhibit all the properties of a glassy, brittle solid, an elastic rubber, or a viscous liquid, depending on the temperature and time scale of the experiment. Polymers are then regarded as viscoelastic materials, a generic term which is used to emphasize their intermediate position between viscous liquids and elastic solids. At low temperatures, or high frequencies of measurement, a polymer may be glass-like, and it will break or flow at great strains. At high temperatures, a permanent deformation occurs under load. In this situation, the polymer behaves like a highly viscous liquid. In an intermediate temperature or frequency range, the polymer is neither glassy nor rubber-like. The viscoelastic property shows an intermediate modulus, and it may dissipate a considerable amount of energy on being strained. This state is commonly called the glass transition range.

In the rubber-like state, a polymer may be subjected to large deformations and still shows a complete recovery. To a good approximation, this is an elastic behavior at large strains [42,43].

If a viscoelastic fluid is subjected to a fixed simple shear state of stress, it responds, after transient effects have died out, with a steady state flow. Also, when it is subjected to a fixed simple shear state of deformation, it produces a stress state which will eventually decay to zero. This is contrary to an isotropic viscoelastic solid which, when subjected to a simple shear state of deformation, will have a corresponding component of stress that remains nonzero as long as the state of deformation is maintained. In simple words, a viscoelastic fluid has an unlimited number of undeformed configurations, while a viscoelastic solid may have only one [33,34,35].

Assume that a viscoelastic material, with the instantaneous elasticity and creep characteristics described above, undergoes two nonsimultaneously applied sudden changes in uniform stress, superimposed upon each other. During the period between the two applications, the material responds in some time- dependent manner which depends on the magnitude of the first stress state. However, if we consider the situation at an arbitrarily small interval of time after the sudden application of the second stress state, the material not only experiences the instantaneous response to the second change in surface tractions, but it also experiences a continuing time-dependent response due to the first applied level of stress. Thus, this is a more general type of material, possessing a characteristic which can be referred to as a memory effect. That is, the material response depends not only on the current state of stress, but also on all past states of stress, and in a general sense the material has a memory keeping all past states of stress. Though several early contributors have existed, such as Maxwell, Kelvin, and Voigt, Boltzmann in 1874 apparently supplied the first formulation of a three-dimensional theory of isotropic viscoelasticity, while Volterra obtained comparable forms for anisotropic solids in 1909. His models of (linear) viscoelastic solid were elaborated on the basis of the following assumptions:

At any (fixed) point of the body, the stress at any time t depends upon the strain at all the preceding times. If the strain at all preceding times is in the same direction, then the effect is to reduce the corresponding stress. The influence of a previous strain on the stress depends on the time elapsed since that strain occurred, and it is weaker for those strains that occurred long ago. Such properties make the model of solid elaborated by Boltzmann a material with fading memory. In addition, Boltzmann made the assumption that a superposition of the influence of previous strains holds, which means that the stress-strain relation is linear.

The one-dimensional viscoelastic response is often interpreted as a combination of elastic and viscous fluid responses. It is natural to try representing viscoelastic properties by a combination of mechanical analogs for these simpler responses with more complicated mechanisms. In fact, viscoelastic mechanical response is usually simulated by different spring-viscous damper combinations. The aim of these models is to stimulate only "macroscopic" behavior, as might be observed in a finite-sized specimen. They do not necessarily provide insight into the molecular basis for viscoelastic response, and they cannot be thought of as providing a rigorous mathematical foundation for the study of the response of viscoelastic materials.

These mechanical analogs lead to a stress- history strain- history relation for the one-dimensional response under certain conditions. It turns out that the combination of springs and viscous dampers which may stimulate a specific viscoelastic response is not unique. Different combinations of springs and viscous dampers can provide the same simulation.

We consider viscoelasticity in the isothermal approximation, in which the model (state and constitutive relation) is independent of the temperature. So the state involves the deformation gradient only, and consequently the constitutive equation is a stress-strain relation only. We obtain [30,44]

$$\sigma(t) = G(t)\varepsilon(0) + \int_0^t G(t-\tau)\frac{d\varepsilon(\tau)}{d\tau}d\tau.$$

The integrating functions $G(t)$ represent mechanical properties of the material and are called "relaxation functions". This can be considered to be the formulation of Boltzmann's superposition principle such that the current stress is determined by the superposition of the responses to the complete spectrum of increments of strain.

Mathematically, this is interpreted by the time convolution of a "relaxation function" with the Laplacian transform of the solution. As a consequence, a subtle damping effect is produced. For a constant density, it has been observed that, for sufficiently smooth and/or small data and history, this viscous damping should ensure the global existence of smooth solutions decaying uniformly as time goes to infinity. We shall be concerned with this phenomenon for our problem.

1.2 Literature Review

In this present thesis we are concerned with the following equation :

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\,\Delta u(\tau)\,d\tau + a(x)\,h(u_t) = f(u), \qquad \text{in } \Omega \times (0,\infty) \quad (1.1)$$

with a localized damping $a(x)\,h(u_t)$, where $a(x) : \Omega \longrightarrow \mathbb{R}_+$ is a function that may vanish on a part ω of the domain. The first work that dealt with uniform decay was by Dassios et al. [14] in which a viscoelastic problem was studied and a polynomial decay result was proved for exponentially decaying kernels. Also, the uniform stability, for some problems in linear viscoelasticity, was established by Fabrizio and Morro [15] in 1992.

In 1994, Rivera [22] considered equations for linear isotropic homogeneous viscoelastic solids of integral type which occupy a bounded domain or the whole space, with zero boundary and history data and in the absence of body forces. In the bounded domain case, an exponential decay result was proved for exponentially decaying memory kernels. For the whole-space case, a polynomial decay result was established, and the rate of the decay was given.

This result was later generalized by Rivera et al. [5] to a situation where the kernel is decaying algebraically but not exponentially. These authors showed that the decay of solutions is also algebraic, at a rate which can be determined by the rate of the decay of the relaxation function. Also, the authors considered both cases: the bounded domain case and that of a material occupying the entire space.

This result was later improved by Baretto et al. in [1], to viscoelastic plates. Precisely, they showed that the solution energy decays at the same decay rate as the one of the relaxation function. For partially viscoelastic materials, Rivera et al. [24] showed that solutions decay exponentially to zero, provided the relaxation function decays in similar fashion, regardless of the size of the viscoelastic part of the material.

In [25], a class of abstract viscoelastic systems of the form

$$\begin{aligned} &u_{tt} + Au\,(t) + \beta u\,(t) + (g * A^{\alpha}u)\,(t) = 0 \\ &u\,(0) = u_0, \quad u_t\,(0) = u_1 \end{aligned} \tag{1.2}$$

for $0 \leq \alpha \leq 1$, $\beta \geq 0$, were investigated. The main focus was on the case when $0 < \alpha < 1$ and the main result was that the solution for (1.2) decay polynomially even when the kernel decay exponentially. It is a sharp result (see Theorem 12 [25]).

This result was generalized by Rivera et al. [27], to a more general abstract problem than (1.2) and established a necessary and sufficient condition to obtain an exponential decay. In the case of lack of exponential decay, a polynomial decay was proved. In the latter case, the authors showed that the rate of decay can be improved by taking more regular initial data.

For system with a localized frictional damping cooperating with the dissipation induced by the viscoelastic term, we mention the work of Cavalcanti et al. [7], where (1.1), with $h\,(u_t) = u_{t,}$ was considered under the condition

$$-\zeta_1 g\,(t) \leq g'\,(t) \leq -\zeta_2 g\,(t)\,, \qquad t \geq 0$$

with $\|g\|_{L^1(0,\infty)}$ small enough. These authors obtained an exponential rate of decay.

Berrimi et al. [3] improved the result of Cavalcanti et al. by showing that the viscoelastic dissipation alone is enough to stabilize the system. To achieve their goal, Berrimi et al. [3] introduced a different functional, which allowed them to weaken the conditions on the kernel as well as on the localized damping. This result was later extended to a situation where a source is competing with the viscoelastic dissipation, by Berrimi et al. [4]. Also, Cavalcanti et al. [8] considered

$$u_{tt} - k_0 \Delta u(t) + \int_0^t div\left[a(x)\, g(t-\tau)\, \nabla u(\tau)\right] d\tau + b(x)\, h(u_t) + f(u) = 0 \qquad (1.3)$$

under similar conditions as in [7] on the relaxation function g and $a(x)+b(x) \geq \delta > 0$ and improved the result in [7]. They established an exponential stability result when g is decaying exponentially and h is linear and a polynomial stability result when g is decaying polynomially and h is nonlinear. A related problem, in a bounded domain, of the form

$$|u_t|^{\rho} u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-\tau)\, \Delta u(\tau)\, d\tau - \gamma \Delta u_t = 0, \qquad (1.4)$$

for $\rho > 0$, was also studied by Cavalcanti et al. [6]. Results on global existence for $\gamma \geq 0$, and exponential decay for $\gamma \geq 0$, have been established.

This latter result has been extended to a situation where $\gamma = 0$, by Messaoudi and Tatar [20], [21]. Exponential and polynomial decay results have been established in the absence, as well as in the presence, of a source term.

For viscoelastic systems with oscillating kernels, Rivera et al. [26] showed that, if the kernel satisfies $g(0) > 0$ and decays exponentially to zero, then the solution decays exponentially to zero. On the other hand, if the kernel decays polynomially, then the corresponding solutions also decay polynomially to zero with the same rate of decay.

In all previous works, the rate of decay was either exponential or polynomial. Messaoudi [40,41] considered

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\,\Delta u(\tau)\,d\tau = bu|u|^{\gamma}, \qquad \text{in } \Omega \times (0,\infty) \tag{1.5}$$

for $b \geq 0$, $\gamma > 0$, and for certain relaxation functions which are not necessarily of exponential or polynomial type, and he showed that the rate of decay of the solution energy is exactly equal to that of the relaxation function.

Recently, Messaoudi [18] considered

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\,\Delta u(\tau)\,d\tau + au_t|u_t|^{m} = bu|u|^{\gamma}, \qquad \text{in } \Omega \times (0,\infty) \tag{1.6}$$

and showed, under suitable conditions on g, that local solutions with negative energy blow up in finite time if $\gamma > m$ and continue to exist globally if $m \geq \gamma$. This blow-up result has been extended to some situations where the initial energy is positive, by Messaoudi [19]. A similar result has been also obtained by Wu [29] using a different method.

For more early and recent results related to stability and asymptotic behavior, we refer the reader to the books by Renardy et al. [28], Liu and Zheng [16], and to the works by Liu and Liu [17], Rivera and Oquendo [23], Baretto et al. [2], and Alabau-Boussouira [39].

1.3 Results description

The aim of this thesis is to investigate the interaction between the viscoelastic damping and frictional damping in wave equations. We study several problems and establish exponential and polynomial decay rate results, for relaxation functions that decay exponentially and polynomially in the presence of linear and nonlinear frictional damping.

In addition to chapter 1, the thesis contains five chapters. In chapter 2, we present some principal concepts, some theorems on the Sobolev embeddings and some lemmas which are of essential use in the proofs of our results.

In other chapters, we study the interaction between the viscoelastic damping and the frictional damping. This study is divided into four cases. In chapter 3, we study the case when the relaxation function is decaying exponentially and the frictional is linear. This is the exponential-linear case. The other cases (exponential-nonlinear, polynomial-linear and polynomial-nonlinear) are investigated in chapters 4,5 and 6 respectively. In each case, we notice that the rate of decay of the solution energy is exactly equal to that of the relaxation function whether the frictional damping is linear or superlinear.

Chapter 2

Principal Concepts

2.1 Introduction

The smooth functions play an important role in functional analysis and in differential equations. The space of these functions is considered one of the most important and famous spaces to study such equations. In the last century, it was extended by other spaces that allow us to find solutions for various partial differential equations. In this chapter, we discuss a brief review of some concepts and the properties related to these spaces in particular, Sobolev spaces, since they are the spaces where the solution of our problem lies.

2.2 Lebesgue Spaces

Definition 2.2.1. [37] Let Ω be a domain in $\mathbb{R}^n$; for $1 \leq p < \infty$, $L^p(\Omega)$ denote the measurable real-valued functions u on Ω for which $\int_\Omega |u(x)|^p \, dx < \infty$. In addition, $L^\infty(\Omega)$ denotes the measurable real valued functions that are essentially bounded (bounded except on a set of measure zero). For $u \in L^p(\Omega)$ we define the norms

$$\begin{aligned} \|u\|_p &= \left(\int_\Omega |u(x)|^p \, dx \right)^{\frac{1}{p}}, \qquad \text{for } 1 \leq p < \infty \\ \|u\|_\infty &= ess \sup_{x \in \Omega} |u(x)| = \inf \{ M : \mu \{ x : u(x) > M \} = 0 \} \end{aligned} \tag{2.1}$$

Lemma 2.2.1. [36] If $1 \leq p < \infty$ and $a, b \geq 0$, then

$$(a+b)^p \leq 2^{p-1} (a^p + b^p) \tag{2.2}$$

Theorem 2.2.1. (Hölder's inequality) [36] Let $1 < p < \infty$ and let q denote the conjugate exponent defined by

$$q = \frac{p}{p-1}, \qquad \text{that is} \qquad \frac{1}{p} + \frac{1}{q} = 1$$

which also satisfies $1 < q < \infty$. If $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$, then $uv \in L^1(\Omega)$ and

$$\int_{\Omega} |u(x)\, v(x)|\, dx \leq \|u\|_p \|v\|_q \tag{2.3}$$

The equality holds if and only if $|u(x)|^p$ and $|v(x)|^q$ are proportional a.e. in Ω.

Corollary 2.2.1. By taking $p = q = 2$, we obtain the Cauchy-Schwarz inequality

$$\int_{\Omega} |u(x)\, v(x)|\, dx \leq \|u\|_2 \|v\|_2$$

Theorem 2.2.2. (Young's inequality) [37] Let $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and $a, b \geq 0$. Then for any $\eta > 0$,

$$ab \leq \eta a^p + C_\eta b^q \tag{2.4}$$

where $C_\eta = \frac{1}{q(\eta p)^{\frac{q}{p}}}$. For $p = q = 2$, the inequality takes the form

$$ab \leq \eta a^2 + \frac{1}{4\eta} b^2 \tag{2.5}$$

Theorem 2.2.3. [36] $L^p(\Omega)$ equipped with the norms (2.1) is a Banach space if $1 \leq p \leq \infty$.

Corollary 2.2.2. [36] $L^2(\Omega)$ is a Hilbert space with respect to the inner product

$$\langle u, v\rangle = \int_{\Omega} u(x)\overline{v(x)}dx$$

The associated norm is then

$$\|u\|_2^2 = \langle u, u\rangle$$

Definition 2.2.2. The support of a continuous function $f(x)$ defined on $\mathbb{R}^n$ is the closure of the set of point where $f(x)$ is nonzero:

$$\text{supp } f = \overline{\{x \in \mathbb{R}^n : f(x) \neq 0\}}.$$

The closed and bounded sets in $\mathbb{R}^n$ are precisely the compact sets, so if supp f is bounded, we say f has a compact support and denote the set of such functions by $C_0(\mathbb{R}^n)$. Similiarly, $C_0(\Omega)$ denotes the set of continuous functions on Ω whose support are compct subsets of Ω.

Theorem 2.2.4. (Density Theorem) [37] If $f \in L^p(\Omega)$, $1 \leq p < \infty$. Then there exists a sequence $(f_n) \subset C_0^\infty(\Omega)$ which converges to f with respect to the norm $\|.\|_p$. That is $C_0^\infty(\Omega)$ is dense in $L^p(\Omega)$.

2.3 Sobolev Spaces

Definition 2.3.1. If $u, v \in L^p(\Omega)$, v is called a weak derivative of order α of u if

$$\int_{\Omega} u(x)\, D^{\alpha}\Phi(x)\, dx = (-1)^{|\alpha|} \int_{\Omega} v(x)\, \Phi(x)\, dx, \qquad \forall \Phi \in C_0^{\infty}(\Omega),$$

where

$$D^{\alpha}\Phi(x) = \frac{\partial^{|\alpha|}\Phi(x)}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_n^{\alpha_n}}$$

and $\alpha = (\alpha_1, \dots, \alpha_n)$ is called a multi-index of dimension n and $|\alpha| = \sum_{i=1}^{n} \alpha_i$ is the length of the multi-index α.

Definition 2.3.2. Let Ω be an open set of $\mathbb{R}^n$ then the Sobolev space $W^{k,p}(\Omega)$, $1 \leq p \leq \infty$, $k \geq 1$, is the set of all functions $u \in L^p(\Omega)$ such that the weak derivatives $D^{\alpha}u$ of order α, $|\alpha| \leq k$, exist and lie in $L^p(\Omega)$. $W^{k,p}$ is equipped with the following norm

$$\begin{aligned} \|u\|_{k,p} &= \left(\sum_{|\alpha| \leq k} \|D^{\alpha}u\|_p^p \right)^{\frac{1}{p}}, \qquad 1 \leq p < \infty \\ \|u\|_{k,\infty} &= \max_{0 \leq |\alpha| \leq k} \|D^{\alpha}u\|_{\infty} \end{aligned} \tag{2.6}$$

Remark 2.3.1. If $u \in C^m(\Omega)$, then all weak derivatives are classical.

Theorem 2.3.1. [36] $W^{k,p}(\Omega)$ is a Banach space with respect to the norm (2.6).

Remark 2.3.2. If $p = 2$, we denote $W^{k,2}(\Omega)$ by $H^k(\Omega)$ and it is a Hilbert space with respect to the inner product

$$\langle u, v \rangle_k = \int_\Omega \sum_{|\alpha| \le k} D^\alpha u(x) \, D^\alpha v(x), \qquad \forall u, v \in H^k(\Omega).$$

Question: why are we interested in working on these spaces? To answer this question, we introduce the following example. This example shows that we are forced to look for weak solutions when we cannot find classical solutions for some problems. These weak solutions reduce the strong requirement of classical solutions.

Example 2.3.1 We consider the following problem

$$\begin{aligned} -u''(x) + u(x) &= f(x), \qquad 0 < x < 1 \\ u(0) = u(1) &= 0 \end{aligned} \tag{2.7}$$

If $f \in C([0,1])$, then the solution $u \in C^2([0,1])$ is called a classical solution. However, if $f \in L^2((0,1))$ then we can transform the previous equation to the weak formulation by multiplying the equation by $\phi \in C_0^1((0,1))$ and integrating over the interval $(0,1)$, that is

$$\int_0^1 (-u''(x) + u(x)) \, \phi(x) \, dx = \int_0^1 f(x) \, \phi(x) \, dx.$$

Using integration by parts, we obtain

$$\int_0^1 (u'\phi' + u\phi) \, dx = \int_0^1 f\phi dx, \qquad \forall \phi \in C_0^1((0,1)) \tag{2.8}$$

This is called the weak formulation to problem 2.7. A function $u \in L^2((0,1))$, with a weak derivative $u' \in L^2((0,1))$, is called a weak solution of problem 2.7 if it satisfies

2.8.

Definition 2.3.3. Let Ω be a domain of $\mathbb{R}^n$, then

$$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega) / \exists v_i \in L^p(\Omega), \int_\Omega u \frac{\partial \phi}{\partial x_i} = -\int_\Omega v_i \phi,\ i = 1, 2, \ldots, n,\ \ \forall \phi \in C_0^\infty(\Omega) \right\}$$

is called Sobolev space of order one, and it is equipped with the norm

$$\|u\|_{1,p} = \|u\|_p + \sum_{i=1}^{n} \left\| \frac{\partial u}{\partial x_i} \right\|_p \tag{2.9}$$

or the equivalent norm

$$\|u\|_{1,p} = \left(\|u\|_p^p + \sum_{i=1}^{n} \left\| \frac{\partial u}{\partial x_i} \right\|_p^p \right)^{\frac{1}{p}}. \tag{2.10}$$

Remark 2.3.4. $W^{1,2}(\Omega) = H^1(\Omega)$ is a Hilbert space with respect to the inner product

$$\langle u, v \rangle = \int_\Omega uv dx + \sum_{i=1}^{n} \int_\Omega \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx, \tag{2.11}$$

Definition 2.3.4. Let Ω be a domain of $\mathbb{R}^n$, we define the Sobolev space $W_0^{1,p}(\Omega)$ to be the closure of $C_0^1(\Omega)$ in the norm of $W^{1,p}(\Omega)$ space.

Theorem 2.3.2. If $u \in W_0^{1,p}(\Omega) \cap C\left(\bar{\Omega}\right)$, then $u(x) = 0$, for every $x \in \partial\Omega$.

Theorem 2.3.3. (Poincaré's inequality) [38] Assume that Ω is bounded in one direction and $1 \leq p < \infty$. Then there is a positive constant $C = C(\Omega, p)$ such that

$$\|u\|_p \leq C \|\nabla u\|_p, \qquad \forall u \in W_0^{1,p}(\Omega).$$

Now, we outline some basic facts and theorems on the embedding property which is considered to be the most important property of Sobolev spaces.

Definition 2.3.5. Let V and W be two Banach spaces. We say that V is embedded in W and we write $V \hookrightarrow W$, if we have

$$\|v\|_W \leq C \|v\|_V , \quad \forall v \in V.$$

Theorem 2.3.4. (**An embedding theorem for L^p spaces**) [**36**]

Suppose that $vol(\Omega) = \int\limits_{\Omega} dx < \infty$, and $1 \leq p \leq q \leq \infty$. If $u \in L^q(\Omega)$, then $u \in L^p(\Omega)$ and

$$\|u\|_p \leq (vol(\Omega))^{\left(\frac{1}{p}\right)-\left(\frac{1}{q}\right)} \|u\|_q$$

hence

$$L^q(\Omega) \hookrightarrow L^p(\Omega).$$

Now, we introduce the embedding relations between different Sobolev spaces. In addition to the obvious embedding, we have, for $vol(\Omega) < \infty$,

$$W^{1,p}(\Omega) \hookrightarrow W^{1,q}(\Omega), \qquad p \geq q. \tag{2.12}$$

Also, there are several other embeddings in the case of bounded domains Ω. We mention some of them in the following theorems.

♦ **One-dimensional case**

Theorem 2.3.5. [38] If I is an open interval of $\mathbb{R}$, then

$$W^{1,p}(I) \hookrightarrow L^q(I), \qquad p \leq q \leq \infty \tag{2.13}$$

and we have

$$\|u\|_q \leq C\|u\|_{1,p}$$

where $C = C(|I|, p)$. Moreover, if I is bounded then (2.13) hold for $1 \leq q \leq +\infty$.

♦ **Higher-dimensional case**

Theorem 2.3.6. (Sobolev, Gagliardo, Nirenberg) [38] If $1 \leq p < n$, then

$$W^{1,p}(\mathbb{R}^n) \hookrightarrow L^{p^*}(\mathbb{R}^n) \qquad \frac{1}{p^*} = \frac{1}{p} - \frac{1}{n} \tag{2.14}$$

and there exists a constant $C = C(n, p)$ such that

$$\|u\|_{p^*} \leq C\|\nabla u\|_p, \qquad \forall u \in W^{1,p}(\mathbb{R}^n).$$

Corollary 2.3.1. If $1 \leq p < n$, then

$$W^{1,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n), \qquad \forall q \in [p, p^*] \tag{2.15}$$

Theorem 2.3.7. [38] If $p = n$, then

$$W^{1,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n), \qquad \forall q \in [n, \infty) \tag{2.16}$$

Theorem 2.3.8. **(Morrey)** [38] If $p > n$, then

$$W^{1,p}(\mathbb{R}^n) \hookrightarrow L^{\infty}(\mathbb{R}^n).$$

Moreover, if $u \in W^{1,p}(\mathbb{R}^n)$ then u is a continuous function.

Remark 2.3.5. The above theorems remain valid if we substitute $\mathbb{R}^n$ by a domain $\Omega \subset \mathbb{R}^n$ with a smooth boundary $\partial\Omega$.

Remark 2.3.6. $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$ for $2 \leq q \leq \frac{2n}{n-2}$ if $n \geq 3$ or $q \geq 2$ if $n = 1, 2$ and we have

$$\|u\|_q \leq C \|\nabla u\|_2$$

Now, we introduce a very important formula that we often use to estimate some integrals and to prove many results.

2.4 Green's Formula [38]

Let Ω be a bounded domain of $\mathbb{R}^n$ with a smooth boundary, then $\forall u \in H^1$, $v \in H^2$, we have

$$\int_{\Omega} u\Delta v dx = -\int_{\Omega} \nabla u.\nabla v dx + \int_{\partial\Omega} u\nabla v.\zeta ds, \tag{2.17}$$

where ζ is the outer unit normal to $\partial\Omega$. If $u \in H_0^1(\Omega)$, then Green's formula is reduced to

$$\int_{\Omega} u\Delta v dx = -\int_{\Omega} \nabla u.\nabla v dx.$$

Chapter 3

Exponential Decay in a Linear Viscoelastic Problem

3.1 Introduction

In this chapter, we study the interaction between viscoelastic damping and linear frictional damping in a wave equation. We consider the following problem:

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) + \int_0^t g(t-\tau)\Delta u(x,\tau)\,d\tau + a u_t(x,t) = 0, & \text{in } \Omega \times (0,\infty) \\ u(x,t) = 0, \quad x \in \partial\Omega \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \end{cases} \tag{3.1}$$

where Ω is a bounded domain of $\mathbb{R}^n$ $(n \geq 1)$ with a smooth boundary $\partial\Omega$, a is a positive constant and g is a positive nonincreasing function satisfying the following conditions

(G$_1$) $g : \mathbb{R}_+ \to \mathbb{R}_+$ is differentiable function such that

$$g(0) > 0, \quad 1 - \int_0^\infty g(s)\,ds = l > 0$$

(G$_2$) There exists a positive constant ξ such that

$$g'(t) \leq -\xi g(t), \quad \forall t \geq 0$$

Theorem 3.1.1. Let $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ be given. Assume that g satisfies

(G_1) and (G_2) then problem (3.1) has a unique global solution

$$u \in C\left([0,\infty);H_0^1(\Omega)\right), \qquad u_t \in C\left([0,\infty);L^2(\Omega)\right). \tag{3.2}$$

Proof. For the proof of this theorem, we refer the reader to [6], [7].

3.2 Modified Energy Functional

By multiplying equation 3.1 by u_t and integrating over Ω, we obtain

$$\begin{aligned}&\int_\Omega u_{tt}(x,t)\,u_t(x,t)\,dx-\int_\Omega \Delta u(x,t)\,u_t(x,t)\,dx+\int_\Omega\int_0^t g(x,t)\,\Delta u(x,\tau)\,u_t(x,t)\,d\tau dx\\&+a\int_\Omega u_t^2(x,t)\,dx=0.\end{aligned} \tag{3.3}$$

The terms in 3.3 are estimated below. All the calculations are done for regular solutions but they remain valid for weak solutions by density arguments.

First term

$$\int_\Omega u_t(x,t)\,u_{tt}(x,t)\,dx=\frac{1}{2}\int_\Omega \frac{d}{dt}u_t^2(x,t)\,dx=\frac{1}{2}\frac{d}{dt}\int_\Omega u_t^2(x,t)\,dx. \tag{3.4}$$

Second term

By using Green's formula and the boundary conditions, we obtain

$$-\int_\Omega \Delta u(x,t)\,u_t(x,t)\,dx=\int_\Omega \nabla u(x,t)\,.\nabla u_t(x,t)\,dx=\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u(x,t)|^2\,dx. \tag{3.5}$$

Third term

$$
\begin{aligned}
&\int_\Omega\int_0^t g\left(t-\tau\right)\Delta u\left(x,t\right)u_t\left(x,t\right)d\tau dx \\
&=\int_0^t g\left(t-\tau\right)\int_\Omega \Delta u\left(x,t\right)u_t\left(x,t\right)dxd\tau \\
&=-\int_0^t g\left(t-\tau\right)\int_\Omega \nabla u_t\left(x,t\right).\nabla u\left(x,\tau\right)dxd\tau \\
&=-\int_0^t g\left(t-\tau\right)\int_\Omega \nabla u_t\left(x,t\right).[\nabla u\left(x,\tau\right)-\nabla u\left(x,t\right)]dxd\tau-\int_0^t g\left(t-\tau\right)\int_\Omega \nabla u_t\left(x,t\right).\nabla u\left(x,t\right)dxd\tau \\
&=\frac{1}{2}\int_0^t g\left(t-\tau\right)\frac{d}{dt}\int_\Omega \left|\nabla u\left(x,\tau\right)-\nabla u\left(x,t\right)\right|^2 dxd\tau-\frac{1}{2}\left(\frac{d}{dt}\int_\Omega \left|\nabla u\left(x,t\right)\right|^2 dx\right)\int_0^t g\left(\tau\right)d\tau \\
&=\frac{1}{2}\frac{d}{dt}\left[\int_0^t g\left(t-\tau\right)\int_\Omega \left|\nabla u\left(x,\tau\right)-\nabla u\left(x,t\right)\right|^2 dxd\tau\right]-\frac{1}{2}\frac{d}{dt}\left[\int_0^t g\left(\tau\right)\int_\Omega \left|\nabla u\left(x,t\right)\right|^2 dxd\tau\right] \\
&-\frac{1}{2}\int_0^t g'\left(t-\tau\right)\int_\Omega \left|\nabla u\left(x,\tau\right)-\nabla u\left(x,t\right)\right|^2 dxd\tau+\frac{1}{2}g\left(t\right)\int_\Omega \left|\nabla u\left(x,t\right)\right|^2 dx.
\end{aligned}
\tag{3.6}
$$

By substituting 3.4, 3.5, and 3.6 in 3.3, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\frac{1}{2}\left[\int_\Omega \left|u_t\left(x,t\right)\right|^2 dx+\frac{1}{2}\left(1-\int_0^t g\left(s\right)ds\right)\int_\Omega \left|\nabla u\left(x,t\right)\right|^2 dx\right. \\
&\left.+\frac{1}{2}\int_\Omega\int_0^t g\left(t-\tau\right)\left|\nabla u\left(x,\tau\right)-\nabla u\left(x,t\right)\right|^2 d\tau dx\right]+\frac{1}{2}g\left(t\right)\int_\Omega \left|\nabla u\left(x,t\right)\right|^2 dx \\
&-\frac{1}{2}\int_\Omega\int_0^t g'\left(t-\tau\right)\left|\nabla u\left(x,\tau\right)-\nabla u\left(x,t\right)\right|^2 dxd\tau+a\int_\Omega u_t^2\left(x,t\right)dx=0.
\end{aligned}
\tag{3.7}
$$

We then set

$$
E\left(t\right)=\frac{1}{2}\int_\Omega \left|u_t\left(x,t\right)\right|^2 dx+\frac{1}{2}\left(1-\int_0^t g\left(s\right)ds\right)\int_\Omega \left|\nabla u\left(x,t\right)\right|^2 dx+\frac{1}{2}\left(g\circ\nabla u\right)\left(t\right),
\tag{3.8}
$$

where

$$
\left(g\circ\nabla u\right)\left(t\right)=\int_\Omega\int_0^t g\left(t-\tau\right)\left|\nabla u\left(x,\tau\right)-\nabla u\left(x,t\right)\right|^2 dxd\tau.
$$

$E(t)$ is called the modified energy functional. As a result of 3.7, we have:

Lemma 3.2.1. The modified energy satisfies, in accordance with the solution of equation 3.1

$$\begin{aligned} E'(t) &= \tfrac{1}{2}\left(g' \circ \nabla u\right)(t) - \tfrac{1}{2} g(t) \left\|\nabla u(x,t)\right\|_2^2 - a \left\|u_t(x,t)\right\|_2^2 \\ &\leq \tfrac{1}{2}\left(g' \circ \nabla u\right)(t) - a \left\|u_t(x,t)\right\|_2^2 \leq 0. \end{aligned} \tag{3.9}$$

3.3 Decay of Solutions

In this section we state and prove the main result of this chapter. For this purpose, we set

$$F(t) := E(t) + \varepsilon \Psi(t), \tag{3.10}$$

where ε is a positive constant to be chosen properly later and

$$\Psi(t) := \int_{\Omega} u(x,t)\, u_t(x,t)\, dx + \frac{1}{2} \int_{\Omega} a u^2(x,t)\, dx \tag{3.11}$$

Lemma 3.3.1. For ε small enough, the inequality

$$\alpha_1 F(t) \leq E(t) \leq \alpha_2 F(t) \tag{3.12}$$

holds for two positive constants α_1 and α_2.

Proof. By direct substitution of 3.11 in 3.10, using Young's inequality and Poincarè's inequality, we obtain

$$
\begin{aligned}
F(t) &\le E(t) + \tfrac{\varepsilon}{2}\int_{\Omega} u^2(x,t)\,dx + \tfrac{\varepsilon}{2}\int_{\Omega} u_t^2(x,t)\,dx + \tfrac{\varepsilon}{2}a\int_{\Omega} u^2 dx \\
&\le E(t) + \tfrac{\varepsilon}{2}C_p\int_{\Omega} |\nabla u(x,t)|^2\,dx + \tfrac{\varepsilon}{2}\int_{\Omega} |u_t^2(x,t)|\,dx + \tfrac{\varepsilon}{2}aC_p\int_{\Omega} |\nabla u(x,t)|^2\,dx \\
&\le E(t) + \alpha E(t) = (1+\alpha)E(t)
\end{aligned}
\tag{3.13}
$$

Therefore,

$$
F(t) \le \alpha_2 E(t). \tag{3.14}
$$

Similiarly, we have

$$
\begin{aligned}
F(t) &\ge E(t) - \tfrac{\varepsilon}{2}C_p\int_{\Omega} |\nabla u(x,t)|^2\,dx - \tfrac{\varepsilon}{2}\int_{\Omega} |u_t^2(x,t)|\,dx - \tfrac{\varepsilon}{2}aC_p\int_{\Omega} |\nabla u(x,t)|^2\,dx \\
&\ge \tfrac{l}{2}\int_{\Omega} |\nabla u(x,t)|^2\,dx + \tfrac{1}{2}\int_{\Omega} |u_t^2(x,t)|\,dx + \tfrac{1}{2}(g \circ \nabla u)(t) \\
&\quad -\tfrac{\varepsilon}{2}C_p\int_{\Omega} |\nabla u(x,t)|^2\,dx - \tfrac{\varepsilon}{2}\int_{\Omega} |u_t^2(x,t)|\,dx - \tfrac{\varepsilon}{2}aC_p\int_{\Omega} |\nabla u(x,t)|^2\,dx \\
&\ge \tfrac{1}{2}(l - \varepsilon C_p(1+a))\|\nabla u(x,t)\|_2^2 + \tfrac{1}{2}(1-\varepsilon)\|u_t(x,t)\|_2^2 + \tfrac{1}{2}(g \circ \nabla u)(t).
\end{aligned}
$$

By choosing ε small such that

$l - \varepsilon C_p(1+a) > 0$, $1-\varepsilon > 0$ and taking $\alpha_1 = \min\{l - \varepsilon C_p(1+a), 1-\varepsilon, 1\}$, we get

$$
F(t) \ge \alpha_1 E(t). \tag{3.15}
$$

This completes the proof.

Lemma 3.3.2. Under the assumptions on g, the functional 3.11 satisfies, in accordance with the solution of equation 3.1, the estimate

$$\Psi'(t) \leq \|u_t(x,t)\|_2^2 + \frac{1}{2}\frac{(1-l)}{l}(g \circ \nabla u)(t) - \frac{l}{2}\|\nabla u(x,t)\|_2^2 \tag{3.16}$$

Proof. By differentiating 3.11 and using 3.1, we easily see that

$$\begin{aligned}
\Psi'(t) &= \int_\Omega (uu_{tt} + u_t^2)\,dx + a\int_\Omega uu_t dx \\
&= \int_\Omega u_t^2 dx + \int_\Omega u\Delta u(t)\,dx - \int_\Omega u\int_0^t g(t-\tau)\,\Delta u(\tau)\,d\tau dx - a\int_\Omega uu_t dx + a\int_\Omega uu_t dx \\
&= \int_\Omega u_t^2 dx - \int_\Omega |\nabla u|^2\,dx + \int_\Omega \nabla u(t)\,.\int_0^t g(t-\tau)\,\nabla u(\tau)\,d\tau dx
\end{aligned} \tag{3.17}$$

We now estimate the third term in the right-hand side of equation 3.17 as follows (keeping in mind that all the estimates are valid for weak solutions by a simple density argument)

$$\begin{aligned}
&\int_\Omega \nabla u(t)\,.\int_0^t g(t-\tau)\,\nabla u(\tau)\,d\tau dx \\
.\ &\leq \tfrac{1}{2}\int_\Omega |\nabla u(t)|^2\,dx + \tfrac{1}{2}\int_\Omega \left(\int_0^t g(t-\tau)\,|\nabla u(\tau)|\,d\tau\right)^2 dx \\
&\leq \tfrac{1}{2}\int_\Omega |\nabla u(t)|^2\,dx + \tfrac{1}{2}\int_\Omega \left(\int_0^t g(t-\tau)\,(|\nabla u(\tau) - \nabla u(t)| + |\nabla u(t)|)\,d\tau\right)^2 dx
\end{aligned} \tag{3.18}$$

We then use the Cauchy-Schwarz inequality, Young's inequality and the fact that

$\int\limits_0^t g(\tau)\,d\tau \le \int\limits_0^\infty g(\tau)\,d\tau = 1-l$ to obtain, for any $\eta > 0$,

$$
\begin{aligned}
&\int\limits_\Omega \left(\int\limits_0^t g(t-\tau)\left(|\nabla u(\tau)-\nabla u(t)| + |\nabla u(t)|\right) d\tau\right)^2 dx \\
&\le \int\limits_\Omega \left(\int\limits_0^t g(t-\tau)\left(|\nabla u(\tau)-\nabla u(t)|\right) d\tau\right)^2 dx + \int\limits_\Omega \left(\int\limits_0^t g(t-\tau)\,|\nabla u(t)|\,d\tau\right)^2 dx \\
&+2\int\limits_\Omega \left(\int\limits_0^t g(t-\tau)\left(|\nabla u(\tau)-\nabla u(t)|\right) d\tau\right)\left(\int\limits_0^t g(t-\tau)\,|\nabla u(t)|\,d\tau\right) dx \\
&\le \int\limits_\Omega \left(\int\limits_0^t g(t-\tau)\left(|\nabla u(\tau)-\nabla u(t)|\right) d\tau\right)^2 dx + \int\limits_\Omega \left(\int\limits_0^t g(t-\tau)\,|\nabla u(t)|\,d\tau\right)^2 dx \\
&+2\left[\tfrac{\eta}{2}\int\limits_\Omega \left(\int\limits_0^t g(t-\tau)\,|\nabla u(t)|\,d\tau\right)^2 dx + \tfrac{1}{2\eta}\int\limits_\Omega \left(\int\limits_0^t g(t-\tau)\left(|\nabla u(\tau)-\nabla u(t)|\right) d\tau\right)^2\right] \\
&\le (1+\eta)\int\limits_\Omega \left(\int\limits_0^t g(t-\tau)\,|\nabla u(t)|\,d\tau\right)^2 dx + \left(1+\tfrac{1}{\eta}\right)\int\limits_\Omega \left(\int\limits_0^t g(t-\tau)\left(|\nabla u(\tau)-\nabla u(t)|\right) d\tau\right)^2 \\
&\le (1+\eta)(1-l)^2\,\|\nabla u(t)\|_2^2 + \left(1+\tfrac{1}{\eta}\right)\int\limits_\Omega \left(\int\limits_0^t \sqrt{g(t-\tau)}\sqrt{g(t-\tau)}\left(|\nabla u(\tau)-\nabla u(t)|\right) d\tau\right)^2 \\
&\le (1+\eta)(1-l)^2\,\|\nabla u(t)\|_2^2 + \left(1+\tfrac{1}{\eta}\right)\int\limits_\Omega \left(\int\limits_0^t g(t-\tau)\,d\tau\right)\left(\int\limits_0^t g(t-\tau)\,|\nabla u(\tau)-\nabla u(t)|^2\,d\tau\right) \\
&\le (1+\eta)(1-l)^2\,\|\nabla u(t)\|_2^2 + \left(1+\tfrac{1}{\eta}\right)(1-l)\,(g\circ\nabla u)(t).
\end{aligned}
\tag{3.19}
$$

We insert 3.19 in 3.18, to obtain

$$
\begin{aligned}
&\int\limits_\Omega \nabla u(t)\,.\int\limits_0^t g(t-\tau)\,\nabla u(\tau)\,d\tau dx \\
&\le \tfrac{1}{2}\left[1+(1+\eta)(1-l)^2\right]\|\nabla u(t)\|_2^2 + \tfrac{1}{2}\left(1+\tfrac{1}{\eta}\right)(1-l)\,(g\circ\nabla u)(t)
\end{aligned}
\tag{3.20}
$$

By substituting 3.20 in 3.17, we get

$$
\begin{aligned}
\Psi'(t) &\le \|u_t(x,t)\|_2^2 + \tfrac{1}{2}\left(1+\tfrac{1}{\eta}\right)(1-l)\,(g\circ\nabla u)(t) \\
&\quad +\tfrac{1}{2}\left[(1+\eta)(1-l)^2-1\right]\|\nabla u(x,t)\|_2^2.
\end{aligned}
$$

We choose $\eta = \frac{l}{1-l}$ to arrive at

$$\Psi'(t) \le \|u_t(x,t)\|_2^2 + \frac{1}{2}\frac{(1-l)}{l}(g \circ \nabla u)(t) - \frac{l}{2}\|\nabla u(x,t)\|_2^2 .$$

This completes the proof.

Theorem 3.3.1. Let $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ be given. Assume that g satisfies (G_1) and (G_2). Then there exist strictly positive constants K and k such that the solution given by 3.2 satisfies, for all $t \ge 0$,

$$E(t) \le Ke^{-kt}.$$

Proof. By using $3.9, 3.10$ and 3.16, we obtain

$$\begin{aligned} F'(t) \le &-[a-\varepsilon]\|u_t(x,t)\|_2^2 - \tfrac{\varepsilon}{2} l \|\nabla u(x,t)\|_2^2 \\ &-\tfrac{1}{2}\left[\xi - \varepsilon\tfrac{(1-l)}{l}\right](g \circ \nabla u)(t). \end{aligned}$$

So, by choosing ε so small that $k_1 = a - \varepsilon > 0$, $k_2 = \frac{\xi}{2} - \frac{\varepsilon}{2}\frac{(1-l)}{l} > 0$, and 3.12 remains valid, we get, for $\beta = \min\left\{k_1, k_2, \frac{\varepsilon}{2} l\right\}$,

$$F'(t) \le -\beta\left[\|u_t\|_2^2 + \|\nabla u\|_2^2 + (g \circ \nabla u)(t)\right]. \tag{3.21}$$

As a result of 3.8, we have

$$E(t) \le \|u_t\|_2^2 + \|\nabla u\|_2^2 + (go\nabla u)(t). \tag{3.22}$$

A combination of 3.12, 3.21 and 3.22 yields

$$F'(t) \leq -\beta_1 E(t) \leq -\beta_1 \alpha_1 F(t), \qquad \forall t \geq 0 \tag{3.23}$$

A simple integration of 3.23 leads to

$$F(t) \leq F(0)\, e^{-\beta_1 \alpha_1 t}, \qquad \forall t \geq 0. \tag{3.24}$$

Thus, 3.12 and 3.24 yield

$$E(t) \leq \alpha_2 F(0)\, e^{-\beta_1 \alpha_1 t} = K e^{-kt}, \qquad \forall t \geq 0$$

where $K = \alpha_2 F(0)$ and $k = \beta_1 \alpha_1$. This completes the proof.

Chapter 4

Exponential Decay in a Nonlinear Viscoelastic Problem

4.1 Introduction

In this chapter, we study the interaction between a viscoelastic damping and a nonlinear frictional damping in a wave equation. We consider the following problem

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) + \int_0^t g(t-\tau)\,\Delta u(x,\tau)\,d\tau + a u_t(x,t)\,|u_t(x,t)|^{m-2} = 0 \\ \text{in } \ \Omega \times (0,\infty) \\ u(x,t) = 0, \qquad x \in \partial\Omega, \qquad t \geq 0 \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \qquad x \in \Omega, \end{cases} \tag{4.1}$$

where Ω is a bounded domain of $\mathbb{R}^n$ $(n \geq 1)$, $(m > 2)$, with a smooth boundary $\partial\Omega$, a is a positive constant and g is a positive nonincreasing function satisfying the following conditions

(G_1) $\quad g : \mathbb{R}_+ \to \mathbb{R}_+$ is differentiable such that

$$g(0) > 0, \qquad 1 - \int_0^\infty g(s)\,ds = l > 0$$

(G_2) There exists a positive constant ζ such that

$$g'(t) \leq -\zeta g(t), \qquad \forall t \geq 0$$

Theorem 4.1.1. Let $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ be given. Assume that g satisfies

(G_1) and (G_2). Assume further that

$$\begin{aligned} & 2 \leq m \leq \tfrac{2n}{n-2}, \quad n \geq 3 \\ & m \geq 2, \quad\quad n = 1, 2. \end{aligned} \tag{*}$$

Then problem 4.1 has a unique global solution

$$u \in C\left([0,\infty); H_0^1(\Omega)\right), \qquad u_t \in C\left([0,\infty); L^2(\Omega)\right). \tag{4.2}$$

Proof. For the proof of this theorem, we refer the reader to [6], [7].

Before stating the main result, we find the modified energy functional in a similar way to the previous chapter. Thus, we have

$$E(t) = \frac{1}{2}\|u_t(x,t)\|_2^2 + \frac{1}{2}\left(1 - \int_0^t g(s)\,ds\right)\|\nabla u(x,t)\|_2^2 + \frac{1}{2}(g \circ \nabla u)(t), \tag{4.3}$$

where

$$(g \circ \nabla u)(t) = \int_0^t g(t-\tau)\|\nabla u(t) - \nabla u(\tau)\|_2^2\,d\tau.$$

Remark 4.1.1. The modified energy functional satisfies, in accordance with the solution of equation 4.1,

$$\begin{aligned} E'(t) &= \tfrac{1}{2}(g' \circ \nabla u)(t) - \tfrac{1}{2}g(t)\|\nabla u(x,t)\|_2^2 - a\|u_t(x,t)\|_m^m \\ &\leq \tfrac{1}{2}(g' \circ \nabla u)(t) - a\|u_t(x,t)\|_m^m \leq 0. \end{aligned} \tag{4.4}$$

4.2 Decay of solution

In this section we state and prove the main result in this chapter. For this purpose, we set

$$F(t) := E(t) + \varepsilon_1 \Psi(t) + \varepsilon_2 \chi(t), \tag{4.5}$$

where ε_1 and ε_2 are positive constants to be chosen properly later and

$$\begin{aligned} \Psi(t) &:= \int_\Omega u u_t dx \\ \chi(t) &:= -\int_\Omega u_t \int_0^t g(t-\tau)(u(t)-u(\tau))\, d\tau dx. \end{aligned} \tag{4.6}$$

In order to prove the main result, we establish several lemmas.

Lemma 4.2.1. For ε_1 and ε_2 small enough, the inequality

$$\alpha_1 F(t) \le E(t) \le \alpha_2 F(t) \tag{4.7}$$

holds for two positive constants α_1 and α_2.

Proof. By direct substitution of 4.6 in 4.5, using Young's inequality and Poincarè's inequality, we get

$$\begin{aligned} F(t) &\le E(t) + \left(\tfrac{\varepsilon_1+\varepsilon_2}{2}\right) \int_\Omega |u_t(x,t)|^2 dx + \tfrac{\varepsilon_1}{2} C_p \int_\Omega |\nabla u(x,t)|^2 dx \\ &+ \tfrac{\varepsilon_2}{2} \int_\Omega \left(\int_0^t g(t-\tau)(u(t)-u(\tau))\, d\tau \right)^2 dx \end{aligned} \tag{4.8}$$

We estimate the third term in the right-hand side of 4.8 by using Hölder's inequality and the fact that $\int\limits_0^t g(s)\,ds \leq \int\limits_0^\infty g(s)\,ds = 1-l$ as follows

$$\begin{aligned}
&\left(\int\limits_0^t g(t-\tau)(u(t)-u(\tau))\,d\tau\right)^2 = \left(\int\limits_0^t \sqrt{g(t-\tau)}\sqrt{g(t-\tau)}(u(t)-u(\tau))\,d\tau\right)^2 \\
&\leq \left(\int\limits_0^t g(t-\tau)\,d\tau\right)\left(\int\limits_0^t g(t-\tau)\left|u(t)-u(\tau)\right|^2 d\tau\right) \\
&\leq (1-l)\int\limits_0^t g(t-\tau)\left|u(t)-u(\tau)\right|^2 d\tau
\end{aligned} \tag{4.9}$$

We substitute 4.9 in 4.8, to obtain

$$\begin{aligned}
F(t) &\leq E(t) + \left(\tfrac{\varepsilon_1+\varepsilon_2}{2}\right)\int\limits_\Omega \left|u_t(x,t)\right|^2 dx + \tfrac{\varepsilon_1}{2}C_p\int\limits_\Omega \left|\nabla u(x,t)\right|^2 dx \\
&+\tfrac{\varepsilon_2}{2}C_p(1-l)\int\limits_\Omega\int\limits_0^t g(t-\tau)\left|\nabla u(t)-\nabla u(\tau)\right|^2 d\tau dx
\end{aligned} \tag{4.10}$$

Since

$$1-\int\limits_0^t g(s)\,ds \geq l,$$

Then

$$\frac{1}{1-\int\limits_0^t g(s)\,ds} \leq \frac{1}{l}$$

Thus,

$$\frac{1}{2}\varepsilon_1 C_p \leq \frac{\varepsilon_1}{2l}C_p\left(1-\int\limits_0^t g(s)\,ds\right) \tag{4.11}$$

By substituting 4.11 in 4.10, we obtain

$$F(t) \leq E(t) + \left(\tfrac{\varepsilon_1+\varepsilon_2}{2}\right) \int_\Omega |u_t(x,t)|^2 \, dx + \tfrac{\varepsilon_1}{2l} C_p \left(1 - \int_0^t g(s)\, ds\right) \int_\Omega |\nabla u(x,t)|^2 \, dx$$
$$+ \tfrac{\varepsilon_2}{2} C_p (1-l) (g \circ \nabla u)(t) \tag{4.12}$$

We choose $\beta = \max\left\{\varepsilon_1 + \varepsilon_2, \tfrac{\varepsilon_1}{l} C_p,\ \varepsilon_2 C_p (1-l)\right\}$, then 4.12 takes the form

$$F(t) \leq (1+\beta) E(t) \leq \alpha_2 E(t) \tag{4.13}$$

Similarly, we have

$$F(t) \geq E(t) - \tfrac{\varepsilon_1}{2} \int_\Omega |u_t(x,t)|^2 \, dx - \tfrac{\varepsilon_1}{2} \int_\Omega |u(x,t)|^2 \, dx - \tfrac{\varepsilon_2}{2} \int_\Omega |u_t(x,t)|^2 \, dx$$
$$- \tfrac{\varepsilon_2}{2} C_p (1-l) (g \circ \nabla u)(t)$$
$$\geq E(t) - \left(\tfrac{\varepsilon_1+\varepsilon_2}{2}\right) \int_\Omega |u_t(x,t)|^2 \, dx - \tfrac{\varepsilon_1}{2} C_p \int_\Omega |\nabla u(x,t)|^2 \, dx - \tfrac{\varepsilon_2}{2} C_p (1-l) (g \circ \nabla u)(t) \tag{4.14}$$

Since $1 - \int_0^t g(s)\, ds \geq l$, then $\frac{1}{2}\left(1 - \int_0^t g(s)\, ds\right) \geq \frac{l}{2}$; therefore, 4.14 becomes

$$F(t) \geq \tfrac{l}{2} \int_\Omega |\nabla u(x,t)|^2 \, dx + \tfrac{1}{2} \int_\Omega |u_t(x,t)|^2 \, dx + \tfrac{1}{2} (g \circ \nabla u)(t)$$
$$- \left(\tfrac{\varepsilon_1+\varepsilon_2}{2}\right) \int_\Omega |u_t(x,t)|^2 \, dx - \tfrac{\varepsilon_1}{2} C_p \int_\Omega |\nabla u(x,t)|^2 \, dx - \tfrac{\varepsilon_2}{2} C_p (1-l) (g \circ \nabla u)(t)$$
$$\geq \tfrac{1}{2} \left[1 - (\varepsilon_1 + \varepsilon_2)\right] \|u_t(x,t)\|_2^2 + \tfrac{1}{2} (l - \varepsilon_1 C_p) \|\nabla u(x,t)\|_2^2$$
$$+ \tfrac{1}{2} \left[1 - \varepsilon_2 C_p (1-l)\right] (g \circ \nabla u)(t) \tag{4.15}$$

By choosing ε_1 and ε_2 small enough so that

$$1 - (\varepsilon_1 + \varepsilon_2) > 0,\ l - \varepsilon_1 C_p > 0,\ 1 - \varepsilon_2 C_p (1 - l) > 0$$

and taking

$$\alpha = \min \{1 - (\varepsilon_1 + \varepsilon_2), l - \varepsilon_1 C_p, 1 - \varepsilon_2 C_p (1 - l)\},$$

we obtain

$$F(t) \geq \frac{\alpha}{2} \left[\|u_t(x,t)\|_2^2 + \|\nabla u(x,t)\|_2^2 + (g \circ \nabla u)(t) \right]$$

But, from 4.3, we have

$$E(t) \leq \|u_t(x,t)\|_2^2 + \|\nabla u(x,t)\|_2^2 + (g \circ \nabla u)(t)$$

Therefore,

$$F(t) \geq \alpha_1 E(t), \qquad \text{where} \quad \alpha_1 = \frac{\alpha}{2}$$

This completes the proof.

Lemma 4.2.2. Under the assumptions (G_1) and (G_2) the functional

$$\Psi(t) := \int_\Omega u u_t dx$$

satisfies, in accordance with the solution of equation 4.1,

$$\begin{aligned}\Psi'(t) &\leq \|u_t(x,t)\|_2^2 - \tfrac{1}{2}\left[1-(1+\eta)(1-l)^2 - 2a\eta C\right]\|\nabla u(x,t)\|_2^2 \\ &+ac(\eta)\|u_t(x,t)\|_m^m + \tfrac{1}{2}\left(1+\tfrac{1}{\eta}\right)(1-l)(g\circ\nabla u)(t)\end{aligned} \tag{4.16}$$

where C is a positive constant depending on $E(0)$ only.

Proof. By using equation 4.1, we see that

$$\begin{aligned}\Psi'(t) &= \int_\Omega \left(uu_{tt}+u_t^2\right)dx = \int_\Omega u_t^2 dx - \int_\Omega |\nabla u(t)|^2 dx \\ &+\int_\Omega \nabla u(t) . \int_0^t g(t-\tau)\nabla u(\tau)\,d\tau dx - a\int_\Omega uu_t|u_t|^{m-2}dx\end{aligned} \tag{4.17}$$

The estimation of the third term, as we did in the previous chapter, is

$$\begin{aligned}&\int_\Omega \nabla u(t) . \int_0^t g(t-\tau)\nabla u(\tau)\,d\tau dx \\ &\leq \tfrac{1}{2}\left[1+(1+\eta)(1-l)^2\right]\|\nabla u(t)\|_2^2 + \tfrac{1}{2}\left(1+\tfrac{1}{\eta}\right)(1-l)(g\circ\nabla u)(t)\end{aligned} \tag{4.18}$$

We now estimate the fourth term in the right-hand side of 4.17. By using Young's inequality, we get

$$-\int_\Omega uu_t|u_t|^{m-2}dx \leq \int_\Omega |u|\,|u_t|^{m-1}dx \leq \eta\|u\|_m^m + c(\eta)\|u_t\|_m^m \tag{4.19}$$

Using the embedding $H_0^1(\Omega)\hookrightarrow L^m(\Omega)$ for $2\leq m\leq \frac{2n}{n-2}$ if $n\geq 3$ or $m\geq 2$ if $n=1,2$ we have

$$\|u\|_m^m \leq C_1\|\nabla u\|_2^m = C_1\|\nabla u\|_2^2\,\|\nabla u\|_2^{m-2} \tag{4.20}$$

By recalling 4.3 and the fact that $E(t)$ is decreasing, we have

$$\|\nabla u\|_2^2 \leq CE(t) \leq CE(0),$$

hence

$$\|\nabla u\|_2^{m-2} = \left(\|\nabla u\|_2^2\right)^{\frac{m-2}{2}} \leq (CE(0))^{\frac{m-2}{2}}$$

Therefore,

$$\|u\|_m^m \leq C\|\nabla u\|_2^2, \quad \text{where } C = (CE(0))^{\frac{m-2}{2}} C_1 \tag{4.21}$$

By substituting 4.21 in 4.19, we obtain

$$-\int_\Omega u u_t |u_t|^{m-2} dx \leq \eta C \|\nabla u\|_2^2 + c(\eta)\|u_t\|_m^m \tag{4.22}$$

We substitute 4.18 and 4.22 in 4.17, to arrive at

$$\begin{aligned}\Psi'(t) \leq \|u_t(x,t)\|_2^2 - \tfrac{1}{2}\left[1-(1+\eta)(1-l)^2 - 2a\eta C\right]\|\nabla u(x,t)\|_2^2 \\ +ac(\eta)\|u_t(x,t)\|_m^m + \tfrac{1}{2}\left(1+\tfrac{1}{\eta}\right)(1-l)(g\circ\nabla u)(t)\end{aligned} \tag{4.23}$$

This completes the proof.

Remark 4.2.1. If we choose η such that

$$0 < \eta < \frac{l(2-l)}{(1-l)^2 + 2aC}$$

then

$$k_1 = 1-(1+\eta)(1-l)^2 - 2a\eta C > 0$$

Lemma 4.2.3. Under the assumptions (G_1) and (G_2) the functional

$$\chi(t) := -\int_{\Omega} u_t \int_0^t g(t-\tau)(u(t)-u(\tau))\, d\tau dx$$

satisfies, in accordance with the solution of equation 4.1 and for any $\delta > 0$,

$$\begin{aligned}\chi'(t) \leq \left[\delta + 2\delta(1-l)^2\right] \|\nabla u(x,t)\|_2^2 + \left[\delta - \int_0^t g(s)\, ds\right] \|u_t(x,t)\|_2^2 + a\delta \|u_t(x,t)\|_m^m \\ + \left[2\delta + \tfrac{1}{2\delta} + ac(\delta)(1-l)^{m-2}\right](1-l)(g \circ \nabla u)(t) + \tfrac{g(0)}{4\delta} C_p \left[-(g' \circ \nabla u)(t)\right].\end{aligned} \tag{4.24}$$

Proof. By using equation 4.1, we see that

$$\begin{aligned}\chi'(t) = & -\int_{\Omega} \nabla u(t) . \int_0^t g(t-\tau)(\nabla u(t) - \nabla u(\tau))\, d\tau dx \\ & -\int_{\Omega} \left(\int_0^t g(t-\tau) \nabla u(\tau)\, d\tau\right) \left(\int_0^t g(t-\tau)(\nabla u(t) - \nabla u(\tau))\, d\tau\right) dx \\ & -\left(\int_0^t g(s)\, ds\right) \|u_t\|_2^2 + a \int_{\Omega} u_t |u_t|^{m-2} \int_0^t g(t-\tau)(u(t)-u(\tau))\, d\tau dx \\ & -\int_{\Omega} u_t \int_0^t g'(t-\tau)(u(t)-u(\tau))\, d\tau dx\end{aligned} \tag{4.25}$$

Using Hölder's inequality, Young's inequality, and the fact that $\int_0^t g(s)\,ds \le 1-l$, we estimate the first term in the right hand side of 4.25 as follows

$$
\begin{aligned}
&-\int_\Omega \nabla u(t).\int_0^t g(t-\tau)(\nabla u(t)-\nabla u(\tau))\,d\tau dx \\
&\quad\le \delta\int_\Omega |\nabla u(t)|^2\,dx+\tfrac{1}{4\delta}\int_\Omega\left(\int_0^t g(t-\tau)|\nabla u(t)-\nabla u(\tau)|\,d\tau\right)^2 dx \\
&\quad\le \delta\int_\Omega |\nabla u(t)|^2\,dx+\tfrac{1}{4\delta}\int_\Omega\left(\int_0^t \sqrt{g(t-\tau)}\sqrt{g(t-\tau)}\,(|\nabla u(\tau)-\nabla u(t)|)\,d\tau\right)^2 dx \\
&\quad\le \delta\int_\Omega |\nabla u(t)|^2\,dx+\tfrac{1}{4\delta}\int_\Omega\left(\int_0^t g(t-\tau)\,d\tau\right)\left(\int_0^t g(t-\tau)|\nabla u(\tau)-\nabla u(t)|^2\,d\tau\right) dx \\
&\quad\le \delta\|\nabla u(t)\|_2^2+\tfrac{1}{4\delta}(1-l)(g\circ\nabla u)(t)
\end{aligned}
\tag{4.26}
$$

The second term

$$
\begin{aligned}
&\int_\Omega\left(\int_0^t g(t-\tau)\nabla u(\tau)\,d\tau\right)\left(\int_0^t g(t-\tau)(\nabla u(t)-\nabla u(\tau))\,d\tau\right)dx \\
&\quad\le \delta\int_\Omega\left|\int_0^t g(t-\tau)\nabla u(\tau)\,d\tau\right|^2 dx+\tfrac{1}{4\delta}\int_\Omega\left|\int_0^t g(t-\tau)(\nabla u(t)-\nabla u(\tau))\,d\tau\right|^2 dx \\
&\quad\le \delta\int_\Omega\left|\int_0^t g(t-\tau)(|\nabla u(\tau)-\nabla u(t)|+|\nabla u(t)|)\,d\tau\right|^2 dx \\
&\quad+\tfrac{1}{4\delta}\int_\Omega\left|\int_0^t g(t-\tau)(\nabla u(t)-\nabla u(\tau))\,d\tau\right|^2 dx
\end{aligned}
\tag{4.27}
$$

We estimate the first term in the RHS of 4.27 by using Lemma 2.2.1 as follows

$$
\begin{aligned}
&\int_\Omega\left(\int_0^t g(t-\tau)(|\nabla u(\tau)-\nabla u(t)|+|\nabla u(t)|)\,d\tau\right)^2 dx \\
&\quad\le 2\int_\Omega\left(\int_0^t g(t-\tau)(|\nabla u(\tau)-\nabla u(t)|)\,d\tau\right)^2 dx+2\int_\Omega\left(\int_0^t g(t-\tau)(|\nabla u(t)|)\,d\tau\right)^2 dx \\
&\quad\le 2\int_\Omega\left(\int_0^t g(t-\tau)(|\nabla u(\tau)-\nabla u(t)|)\,d\tau\right)^2 dx+2\int_\Omega\left(\int_0^t g(t-\tau)(|\nabla u(t)|)\,d\tau\right)^2 dx
\end{aligned}
\tag{4.28}
$$

By substiuting 4.28 in 4.27, we obtain

$$\begin{aligned}
&\int_{\Omega}\left(\int_0^t g(t-\tau)\nabla u(\tau)\,d\tau\right)\left(\int_0^t g(t-\tau)(\nabla u(t)-\nabla u(\tau))\,d\tau\right)dx\\
&\quad\le \left(2\delta+\tfrac{1}{4\delta}\right)\int_{\Omega}\left(\int_0^t g(t-\tau)(|\nabla u(\tau)-\nabla u(t)|)\,d\tau\right)^2 dx+2\delta(1-l)^2\|\nabla u(t)\|_2^2\\
&\quad\le \left(2\delta+\tfrac{1}{4\delta}\right)(1-l)(g\circ\nabla u)(t)+2\delta(1-l)^2\|\nabla u(t)\|_2^2
\end{aligned} \tag{4.29}$$

By exploiting Young's inequality, the fourth term of 4.25 is estimated as follows:

$$\begin{aligned}
&\int_{\Omega}u_t|u_t|^{m-2}\int_0^t g(t-\tau)(u(t)-u(\tau))\,d\tau dx\le\int_{\Omega}|u_t|^{m-1}\int_0^t g(t-\tau)|(u(t)-u(\tau))|\,d\tau dx\\
&\le\delta\|u_t\|_m^m+c(\delta)\int_{\Omega}\left(\int_0^t g(t-\tau)|(u(t)-u(\tau))|\,d\tau\right)^m dx
\end{aligned} \tag{4.30}$$

We estimate the second term of 4.30 by using Hölder's inequality with $p=\frac{m}{m-1}$, $q=m$, the Sobolev embedding, and the fact that $\int_0^t g(s)\,ds\le 1-l$. Thus, we get

$$\begin{aligned}
&\int_{\Omega}\left(\int_0^t g(t-\tau)|(u(t)-u(\tau))|\,d\tau\right)^m dx\\
&\le\int_{\Omega}\left(\int_0^t g^{1-\frac{1}{m}}(t-\tau)\,g^{\frac{1}{m}}(t-\tau)|(u(t)-u(\tau))|\,d\tau\right)^m dx\\
&\le\int_{\Omega}\left\{\left(\int_0^t\left(g^{1-\frac{1}{m}}(t-\tau)\right)^{\frac{m}{m-1}}d\tau\right)^{\frac{m-1}{m}}\left(\int_0^t\left(g^{\frac{1}{m}}(t-\tau)|(u(t)-u(\tau))|\right)^m d\tau\right)^{\frac{1}{m}}\right\}^m dx\\
&\le(1-l)^{m-1}\int_{\Omega}\int_0^t g(t-\tau)|(u(t)-u(\tau))|^m\,d\tau dx\\
&\le(1-l)^{m-1}C_1\int_0^t g(t-\tau)\|\nabla u(t)-\nabla u(\tau)\|_2^m\,d\tau\\
&\le(1-l)^{m-1}C_1\int_0^t g(t-\tau)\|\nabla u(t)-\nabla u(\tau)\|_2^2\|\nabla u(t)-\nabla u(\tau)\|_2^{m-2}\,d\tau
\end{aligned} \tag{4.31}$$

Since $m > 2$, then

$$\|\nabla u(t) - \nabla u(\tau)\|_2^{m-2} = \left(\|\nabla u(t) - \nabla u(\tau)\|_2^2\right)^{\frac{m-2}{2}} \leq \left[\left(\|\nabla u(t)\|_2^2 + \|\nabla u(\tau)\|_2^2\right)\right]^{\frac{m-2}{2}}$$
$$\leq [CE(t) + CE(\tau)]^{\frac{m-2}{2}} \leq [CE(0) + CE(0)]^{\frac{m-2}{2}} \leq [2CE(0)]^{\frac{m-2}{2}} = C \tag{4.32}$$

A combination of 4.32 and 4.31 gives

$$\int_\Omega \left(\int_0^t g(t-\tau)\left|(u(t) - u(\tau))\right| d\tau\right)^m dx \leq C_2 (g \circ \nabla u)(t) \tag{4.33}$$

where $C_2 = CC_1 (1-l)^{m-1}$ is depending on $E(0)$.

Thus, the estimation of the fourth term of the right-hand side of 4.25 takes the form

$$\int_\Omega u_t |u_t|^{m-2} \int_0^t g(t-\tau)(u(t) - u(\tau))\, d\tau dx \leq \delta \|u_t\|_m^m + c_2(\delta)(g \circ \nabla u)(t) \tag{4.34}$$

where $c_2(\delta) = C_2 c(\delta)$.

Now, we estimate the fifth term in the right-hand side of 4.25. By using Young's inequality we obtain, for $\delta > 0$,

$$-\int_\Omega u_t \int_0^t g'(t-\tau)(u(t) - u(\tau))\, d\tau dx \leq \delta \|u_t\|_2^2 + \frac{1}{4\delta} \int_\Omega \left(\int_0^t g'(t-\tau)\left|u(t) - u(\tau)\right| d\tau\right)^2 dx \tag{4.35}$$

By using Hölder's inequality and Poincaré's inequality, the second term in the right-hand side of 4.35 is handled as follows:

$$\left(\int_0^t g'(t-\tau)(u(t)-u(\tau))\,d\tau\right)^2 = \left(\int_0^t \sqrt{-g'(t-\tau)}\sqrt{-g'(t-\tau)}\,|u(t)-u(\tau)|\,d\tau\right)^2$$
$$\leq \left(\int_0^t -g'(t-\tau)\,d\tau\right)\left(\int_0^t -g'(t-\tau)\,|u(t)-u(\tau)|^2\,d\tau\right)$$
$$\leq [g(0)-g(t)]\,C_p\left[-(g'\circ\nabla u)(t)\right] \leq g(0)\,C_p\left[-(g'\circ\nabla u)(t)\right]. \tag{4.36}$$

We combine 4.36 and 4.35 to get

$$-\int_\Omega u_t \int_0^t g'(t-\tau)(u(t)-u(\tau))\,d\tau dx \leq \delta\,\|u_t\|_2^2 + \frac{g(0)}{4\delta}C_p\left[-(g'\circ\nabla u)(t)\right]. \tag{4.37}$$

By substituting 4.26, 4.29, 4.34, and 4.37 in 4.25, the assertion of Lemma 4.2.3 is established.

Theorem 4.2.1. Let $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ be given. Assume that $(*)$ of Theorem 4.1.1 holds and g satisfies (G_1) and (G_2). Then, for each $t_0 > 0$, there exist strictly positive constants K and k such that the solution given by 4.2 satisfies, for all $t \geq t_{0,}$

$$E(t) \leq Ke^{-kt}.$$

Proof. Since g is continuous, positive and $g(0) > 0$, then for any $t_0 > 0$, we have

$$\int_0^t g(s)\,ds \geq \int_0^{t_0} g(s)\,ds = g_0 > 0, \quad \forall t \geq t_0 \tag{4.38}$$

By using 4.5, 4.4, 4.16, 4.24 and 4.38, we get

$$\begin{aligned} F'(t) \leq & -\left[\varepsilon_2\left(g_0-\delta\right)-\varepsilon_1\right]\left\|u_t\right\|_2^2-a\left[1-\left(\varepsilon_1 c(\eta)+\varepsilon_2 \delta\right)\right]\left\|u_t\right\|_m^m \\ & -\left[\varepsilon_1\left\{\frac{1-(1+\eta)(1-l)^2-2 a \eta C}{2}\right\}-\varepsilon_2 \delta\left\{1+2(1-l)^2\right\}\right]\|\nabla u\|_2^2 \\ & -\zeta\left[\frac{1}{2}-\varepsilon_2 \frac{g(0)}{4 \delta} C_p-\left\{\frac{\varepsilon_1}{2}\left(1+\frac{1}{\eta}\right)\right.\right. \\ & \left.\left.+\varepsilon_2\left(2 \delta+\frac{1}{2 \delta}+a c_2(\delta)(1-l)^{m-2}\right)\right\}(1-l)\right](g \circ \nabla u)(t) \end{aligned} \tag{4.39}$$

By recalling Remark 4.2.1, we can choose δ so small that

$$g_0-\delta>\frac{1}{2} g_0$$

$$\frac{2}{1-(1+\eta)(1-l)^2-2 a \eta C} \delta\left[1+2(1-l)^2\right]<\frac{1}{4} g_0$$

Whence δ is fixed, any choice of ε_1 and ε_2, such that

$$\frac{1}{4} g_0 \varepsilon_2<\varepsilon_1<\frac{1}{2} g_0 \varepsilon_2, \tag{4.40}$$

makes

$$\varepsilon_2\left(g_0-\delta\right)>\frac{1}{2} g_0 \varepsilon_2>\varepsilon_1 \Longrightarrow k_2=\varepsilon_2\left(g_0-\delta\right)-\varepsilon_1>0,$$

$$\frac{2 \varepsilon_2 \delta}{1-(1+\eta)(1-l)^2-2 a \eta C}\left[1+2(1-l)^2\right]<\frac{1}{4} g_0 \varepsilon_2<\varepsilon_1$$

This implies that

$$k_3=\varepsilon_1\left[\frac{1-(1+\eta)(1-l)^2-2 a \eta C}{2}\right]-\varepsilon_2 \delta\left\{1+2(1-l)^2\right\}>0$$

We then pick ε_1 and ε_2 so small that 4.7 and 4.40 remain valid,

$$k_4 = \frac{1}{2} - \varepsilon_2 \frac{g(0)}{4\delta} C_p - \left\{ \frac{\varepsilon_1}{2} \left(1 + \frac{1}{\eta}\right) + \varepsilon_2 \left(2\delta + \frac{1}{2\delta} + ac_2(\delta)(1-l)^{m-2}\right) \right\} (1-l) > 0$$

and

$$k_5 = 1 - (\varepsilon_1 c(\eta) + \varepsilon_2 \delta) > 0$$

Finally, we choose $\beta = \min\{k_i,\ i = 2, 3, 4\}$ to obtain

$$\begin{aligned} F'(t) &\leq -\beta \left[\|u_t\|_2^2 + \|\nabla u\|_2^2 + (g \circ \nabla u)(t)\right] - ak_5 \|u_t\|_m^m \\ &\leq -\beta \left[\|u_t\|_2^2 + \|\nabla u\|_2^2 + (g \circ \nabla u)(t)\right], \qquad \forall t \geq t_0 \end{aligned} \tag{4.41}$$

We combine 4.7 and 4.41 to get

$$F'(t) \leq -\beta E(t) \leq -\beta \alpha_1 F(t) \qquad \forall t \geq t_0 \tag{4.42}$$

By integrating 4.42 over the interval (t_0, t) and due to the fact $F(t) > 0$, $\forall t \geq 0$, we obtain

$$\ln \frac{F(t)}{F(t_0)} \leq -\beta \alpha_1 (t - t_0)$$

which implies that

$$F(t) \leq \left(F(t_0) e^{\beta \alpha_1 t_0}\right) e^{-\beta \alpha_1 t} \tag{4.43}$$

Thus, 4.7 and 4.43 yield

$$E(t) \leq \left(\alpha_2 F(t_0)\, e^{\beta\alpha_1 t_0}\right) e^{-\beta\alpha_1 t} = Ke^{-kt}, \qquad \forall t \geq t_0 \tag{4.44}$$

This completes the proof.

Remark 4.2.2. Estimate 4.44 is also true for $0 \leq t \leq t_0$ by boundedness and continuity of $E(t)$.

Chapter 5

Polynomial Decay in a Linear Viscoelastic Problem

5.1 Introduction

In this chapter, we study the interaction between viscoelastic damping and linear frictional damping in a wave equation. We consider the following problem:

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) + \int_0^t g(t-\tau)\Delta u(x,\tau)\,d\tau + au_t(x,t) = 0, & \text{in } \Omega \times (0,\infty) \\ u(x,t) = 0, \qquad x \in \partial\Omega \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \qquad x \in \Omega, \end{cases} \tag{5.1}$$

where Ω is a bounded domain of $\mathbb{R}^n$ $(n \geq 1)$ with a smooth boundary $\partial\Omega$, a is a positive constant and g is a positive nonincreasing function satisfying the following conditions

(G$_1$) $g : \mathbb{R}_+ \to \mathbb{R}_+$ is differentiable such that

$$g(0) > 0, \quad 1 - \int_0^\infty g(s)\,ds = l > 0$$

(G$_2$) There exists a positive constant ξ such that

$$g'(t) \leq -\xi g^p(t), \quad \forall t \geq 0, \quad 1 < p < \frac{3}{2}$$

Theorem 5.1.1. Let $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ be given. Assume that g satisfies (G_1) and (G_2) then problem 5.1 has a unique global solution

$$u \in C\left([0,\infty); H_0^1(\Omega)\right), \qquad u_t \in C\left([0,\infty); L^2(\Omega)\right). \tag{5.2}$$

Proof. For the proof of this theorem, we refer the reader to [6], [7].

5.2 Modified Energy Functional

In chapter 3, we saw that the modified energy functional is

$$E(t) = \frac{1}{2}\|u_t(x,t)\|_2^2 + \frac{1}{2}\left(1 - \int_0^t g(s)\,ds\right)\|\nabla u(x,t)\|_2^2 + \frac{1}{2}(g \circ \nabla u)(t). \tag{5.3}$$

Remark 5.2.1. The modified energy functional $E(t)$ satisfies, in accordance with the solution of equation 5.1,

$$E'(t) \leq \frac{1}{2}(g' \circ \nabla u)(t) - a\|u_t(x,t)\|_2^2 \leq 0. \tag{5.4}$$

Remark 5.2.2. Using (G_2), one can easily see that

$$\int_0^\infty g^{\frac{1}{2}}(s)\,ds < +\infty.$$

Lemma 5.2.1 [8]. Assume that $u \in L^\infty(0,T;H^1(\Omega))$ and g is a continuous function. Then, there exists $C > 0$ such that

$$(g \circ \nabla u)(t) \leq C\left[\int_0^t \|u(\tau)\|_{H^1}^2\,d\tau + t\,\|u(t)\|_{H^1}^2\right]^{\frac{p-1}{p}} \{(g^p \circ \nabla u)(t)\}^{\frac{1}{p}}$$

and

$$(g \circ \nabla u)(t) \leq C\{(g^p \circ \nabla u)(t)\}^{\frac{1}{2p-1}}$$

5.3 Decay of Solution

In this section we state and prove the main result. For this purpose, we set

$$F(t) := E(t) + \varepsilon \Psi(t), \tag{5.5}$$

where ε is a positive constant and

$$\Psi(t) := \int_\Omega u u_t dx + \frac{1}{2} \int_\Omega a u^2 dx \tag{5.6}$$

Lemma 5.3.1. For ε small enough, the inequality

$$\alpha_1 F(t) \leq E(t) \leq \alpha_2 F(t) \tag{5.7}$$

holds for two positive constants α_1 and α_2.

Proof. See Lemma 3.3.1.

Lemma 5.3.2. Under the assumptions (G_1) and (G_2), the functional (5.6)

$$\Psi(t) := \int_\Omega u u_t dx + \frac{1}{2} \int_\Omega a u^2 dx$$

satisfies, in accordance with the solution of equation 5.1, the estimate

$$\Psi'(t) \leq \|u_t\|_2^2 - \frac{l}{2} \|\nabla u\|_2^2 + \frac{1}{2l} \left[\int_0^t g^{2-p}(s)\, ds \right] (g^p \circ \nabla u)(t) \tag{5.8}$$

Proof. By differentiating 5.6 and using 5.1, we obtain, similar to 3.17,

$$\Psi'(t) = \|u_t\|_2^2 - \|\nabla u\|_2^2 + \int_\Omega \nabla u(t) . \int_0^t g(t-\tau) \nabla u(\tau) \, d\tau dx \tag{5.9}$$

By using Young's inequality, the estimation of the third term in the right-hand side of equation 5.9 becomes

$$\begin{aligned}
&\int_\Omega \nabla u(t) . \int_0^t g(t-\tau) \nabla u(\tau) \, d\tau dx \\
&\leq \frac{1}{2} \int_\Omega |\nabla u(t)|^2 \, dx + \frac{1}{2} \int_\Omega \left(\int_0^t g(t-\tau) |\nabla u(\tau)| \, d\tau \right)^2 dx \\
&\leq \frac{1}{2} \int_\Omega |\nabla u(t)|^2 \, dx + \frac{1}{2} \int_\Omega \left(\int_0^t g(t-\tau) (|\nabla u(\tau) - \nabla u(t)| + |\nabla u(t)|) \, d\tau \right)^2 dx
\end{aligned} \tag{5.10}$$

Similarly to equation 3.18 in Chapter 3, the estimation of the second term in right-hand side of equation 5.10 becomes

$$\begin{aligned}
&\int_\Omega \left(\int_0^t g(t-\tau) (|\nabla u(\tau) - \nabla u(t)| + |\nabla u(t)|) \, d\tau \right)^2 dx \\
&\leq \int_\Omega \left(\int_0^t g(t-\tau) (|\nabla u(\tau) - \nabla u(t)|) \, d\tau \right)^2 dx + \int_\Omega \left(\int_0^t g(t-\tau) |\nabla u(t)| \, d\tau \right)^2 dx \\
&+2 \int_\Omega \left(\int_0^t g(t-\tau) (|\nabla u(\tau) - \nabla u(t)|) \, d\tau \right) \left(\int_0^t g(t-\tau) |\nabla u(t)| \, d\tau \right) dx \\
&\leq \int_\Omega \left(\int_0^t g(t-\tau) (|\nabla u(\tau) - \nabla u(t)|) \, d\tau \right)^2 dx + \int_\Omega \left(\int_0^t g(t-\tau) |\nabla u(t)| \, d\tau \right)^2 dx \\
&+2 \left[\frac{\eta}{2} \int_\Omega \left(\int_0^t g(t-\tau) |\nabla u(t)| \, d\tau \right)^2 dx + \frac{1}{2\eta} \int_\Omega \left(\int_0^t g(t-\tau) (|\nabla u(\tau) - \nabla u(t)|) \, d\tau \right)^2 \right] \\
&\leq (1+\eta) \int_\Omega \left(\int_0^t g(t-\tau) |\nabla u(t)| \, d\tau \right)^2 dx + \left(1 + \frac{1}{\eta}\right) \int_\Omega \left(\int_0^t g(t-\tau) (|\nabla u(\tau) - \nabla u(t)|) \, d\tau \right)^2
\end{aligned} \tag{5.11}$$

By using the Cauchy-Schwarz inequality, the estimation of the second term in right-hand side of 5.11 is

$$\begin{aligned}
&\int_{\Omega}\left(\int_0^t g(t-\tau)\left(|\nabla u(\tau)-\nabla u(t)|\right)d\tau\right)^2 dx \\
&=\int_{\Omega}\left(\int_0^t g^{1-\frac{p}{2}}(t-\tau)\,g^{\frac{p}{2}}(t-\tau)\left(|\nabla u(\tau)-\nabla u(t)|\right)d\tau\right)^2 dx \\
&\leq\int_{\Omega}\left\{\left[\int_0^t\left(g^{1-\frac{p}{2}}(t-\tau)\right)^2 d\tau\right]^{\frac{1}{2}}\left[\int_0^t\left(g^{\frac{p}{2}}(t-\tau)\left(|\nabla u(\tau)-\nabla u(t)|\right)\right)^2 d\tau\right]^{\frac{1}{2}}\right\}^2 dx \\
&\leq\int_{\Omega}\left(\int_0^t g^{2-p}(t-\tau)\,d\tau\right)\left(\int_0^t g^{p}(t-\tau)\,|\nabla u(\tau)-\nabla u(t)|^2 d\tau\right)dx \\
&\leq\left[\int_0^t g^{2-p}(t-\tau)\,d\tau\right]\int_{\Omega}\int_0^t g^{p}(t-\tau)\,|\nabla u(\tau)-\nabla u(t)|^2 d\tau dx \\
&\leq\left[\int_0^t g^{2-p}(t-\tau)\,d\tau\right](g^p\circ\nabla u)(t)
\end{aligned} \tag{5.12}$$

By substituting 5.12 in 5.11 and using the fact that $\int_0^t g(s)\,ds\leq\int_0^\infty g(s)\,ds=1-l$, we get

$$\begin{aligned}
&\int_{\Omega}\left(\int_0^t g(t-\tau)\left(|\nabla u(\tau)-\nabla u(t)|+|\nabla u(t)|\right)d\tau\right)^2 dx \\
&\quad\leq(1+\eta)(1-l)^2\,\|\nabla u\|_2^2+\left(1+\tfrac{1}{\eta}\right)\left[\int_0^t g^{2-p}(t-\tau)\,d\tau\right](g^p\circ\nabla u)(t)
\end{aligned} \tag{5.13}$$

Inserting 5.13 in 5.10 leads to

$$\begin{aligned}
&\int_{\Omega}\nabla u(t)\,.\int_0^t g(t-\tau)\,\nabla u(\tau)\,d\tau dx \\
&\leq\tfrac{1}{2}\left[1+(1+\eta)(1-l)^2\right]\|\nabla u\|_2^2+\tfrac{1}{2}\left(1+\tfrac{1}{\eta}\right)\left[\int_0^t g^{2-p}(t-\tau)\,d\tau\right](g^p\circ\nabla u)(t)
\end{aligned} \tag{5.14}$$

Now, we substitute 5.14 in 5.9, and we obtain

$$\begin{aligned}\Psi'(t) &\leq \|u_t\|_2^2 + \tfrac{1}{2}\left[(1+\eta)(1-l)^2 - 1\right]\|\nabla u\|_2^2 \\ &+\tfrac{1}{2}\left(1+\tfrac{1}{\eta}\right)\left[\int_0^t g^{2-p}(t-\tau)\,d\tau\right](g^p \circ \nabla u)(t)\end{aligned}$$

By choosing $\eta = \frac{l}{1-l}$, we arrive at

$$\Psi'(t) \leq \|u_t\|_2^2 - \frac{l}{2}\|\nabla u\|_2^2 + \frac{1}{2l}\left[\int_0^t g^{2-p}(s)\,ds\right](g^p \circ \nabla u)(t)$$

This completes the proof.

Theorem 5.3.1. Let $(u_{0,}u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ be given. Assume that g satisfies (G_1) and (G_2). Then there exist a strictly positive constant K such that the solution given by 5.2 satisfies, for all $t \geq 0$

$$E(t) \leq K(1+t)^{\frac{-1}{p-1}}.$$

Proof. By using 5.4, 5.5 and 5.8, we obtain

$$\begin{aligned}F'(t) &= E'(t) + \varepsilon\Psi'(t) \\ &\leq \tfrac{1}{2}(g' \circ \nabla u)(t) - a\|u_t\|_2^2 + \varepsilon\|u_t\|_2^2 - \tfrac{\varepsilon l}{2}\|\nabla u\|_2^2 + \tfrac{\varepsilon}{2l}\left[\int_0^t g^{2-p}(s)\,ds\right](g^p \circ \nabla u)(t) \\ &\leq -(a-\varepsilon)\|u_t\|_2^2 - \tfrac{\varepsilon l}{2}\|\nabla u\|_2^2 - \tfrac{1}{2}\left\{\zeta - \left(\tfrac{\varepsilon}{2l}\right)\left[\int_0^t g^{2-p}(s)\,ds\right]\right\}(g^p \circ \nabla u)(t)\end{aligned}$$

Since $g^{2-p}(t) = g^{2-2p}(t)\,g^p(t)$, then by using (G_1) and (G_2) we get

$$g^{2-p}(t) \leq -\frac{1}{\zeta} g^{2-2p}(t)\, g'(t),$$

Consequently,

$$\int_0^\infty g^{2-p}(s)\, ds \leq -\frac{1}{\zeta}\int_0^\infty g^{2-2p}(s)\, g'(s)\, ds \leq \frac{-1}{\zeta(3-2p)}\left[g^{3-2p}(s)\right|_{s=o}^{s=\infty}.$$

Since g is nonincreasing, then the above integral is finite if $p < \frac{3}{2}$. Thus, we choose ε small enough for 5.7 to remain valid and

$$k_1 = a - \varepsilon > 0. \tag{5.15}$$

$$k_2 = \zeta - \left(\frac{\varepsilon}{2l}\right)\left[\int_0^\infty g^{2-p}(s)\, ds\right] > 0$$

Therefore,

$$F'(t) \leq -\beta\left[\|u_t\|_2^2 + \|\nabla u\|_2^2 + (g^p \circ \nabla u)(t)\right], \tag{5.16}$$

where

$$\beta = \min\left\{k_1, k_2, \frac{\varepsilon}{2} l\right\}$$

Since

$$E(t) = \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\left(1 - \int_0^t g(s)\, ds\right)\|\nabla u\|_2^2 + \frac{1}{2}(g \circ \nabla u)(t)$$

then

$$E(t) \leq \|u_t\|_2^2 + \|\nabla u\|_2^2 + (g \circ \nabla u)(t)$$

Therefore, for $\sigma > 1$, using Lemma 2.2.1, we obtain

$$\begin{aligned} E^{\sigma}(t) &\leq \left[\|u_t\|_2^2 + \|\nabla u\|_2^2 + (g \circ \nabla u)(t)\right]^{\sigma} \\ &\leq C\left[\|u_t\|_2^{2\sigma} + \|\nabla u\|_2^{2\sigma} + \{(g \circ \nabla u)(t)\}^{\sigma}\right] \\ &\leq C\left[\|u_t\|_2^2 \|u_t\|_2^{2(\sigma-1)} + \|\nabla u\|_2^2 \|\nabla u\|_2^{2(\sigma-1)}\right] + C\{(g \circ \nabla u)(t)\}^{\sigma} \end{aligned}$$

Since $\|u_t\|_2^2 \leq CE(t) \leq CE(0)$, then $\|u_t\|_2^{2(\sigma-1)} \leq (C)^{(\sigma-1)} E^{\sigma-1}(0)$.

Similarly, $\|\nabla u\|_2^{2(\sigma-1)} \leq (C)^{(\sigma-1)} E^{\sigma-1}(0)$

Thus,

$$E^{\sigma}(t) \leq CE^{\sigma-1}(0)\left[\|u_t\|_2^2 + \|\nabla u\|_2^2\right] + C\{(g \circ \nabla u)(t)\}^{\sigma} \tag{5.17}$$

By using Lemma 5.2.1, the inequality 5.17 takes the form

$$\begin{aligned} E^{\sigma}(t) &\leq CE^{\sigma-1}(0)\left[\|u_t\|_2^2 + \|\nabla u\|_2^2\right] \\ &+C\left\{\left(\int_0^{\infty} g^{\frac{1}{2}}(s)\,ds\right)E(0)\right\}^{\frac{\sigma(2p-2)}{2p-1}} \{(g^p o \nabla u)(t)\}^{\frac{\sigma}{2p-1}} \end{aligned} \tag{5.18}$$

By choosing $\sigma = 2p-1$, the inequality 5.18 becomes

$$E^{\sigma}(t) \leq C\left[\|u_t\|_2^2 + \|\nabla u\|_2^2 + (g^p \circ \nabla u)(t)\right] \tag{5.19}$$

From 5.16 and 5.19, we infer

$$F'(t) \leq -\beta_2 E^{\sigma}(t) \tag{5.20}$$

for some constant $\beta_2 > 0$. By combining 5.7 and 5.20, we obtain

$$F'(t) \leq -\beta_2 (\alpha_1)^{\sigma} F^{\sigma}(t) \tag{5.21}$$

By integrating 5.21 over $(0, t)$, we get

$$\int_0^t F^{-\sigma}(s) F'(s)\, ds \leq -\beta_2 (\alpha_1)^{\sigma} \int_0^t ds$$

which implies that

$$\frac{F^{-\sigma+1}(t) - F^{-\sigma+1}(0)}{-\sigma + 1} \leq -\beta_2 (\alpha_1)^{\sigma} t$$

Since $1 - \sigma < 0$, then

$$\begin{aligned} F^{-\sigma+1}(t) &\geq (1-\sigma)(-\beta_2)(\alpha_1)^{\sigma} t + F^{-\sigma+1}(0) \\ &\geq \beta_2 (\sigma - 1)(\alpha_1)^{\sigma} t + F^{-\sigma+1}(0) \end{aligned}$$

Also, since $1 < p < \frac{3}{2}$ and $\sigma = 2p - 1$, then $1 < \sigma < 2$, and hence $0 < \sigma - 1 < 1$. Therefore,

$$\beta_2 (\sigma - 1)(\alpha_1)^{\sigma} t > 0, \qquad \forall t > 0$$

Thus,

$$F^{\sigma-1}(t) \leq \frac{1}{\beta_2 (\sigma - 1)(\alpha_1)^{\sigma} t + F^{-\sigma+1}(0)}$$

By choosing $\lambda = \min\{\beta_2 (\sigma - 1)(\alpha_1)^{\sigma},\ F^{-\sigma+1}(0)\} > 0$, we obtain

$$F^{\sigma-1}(t) \leq \frac{1}{\lambda (t+1)}$$

That is,

$$F(t) \le C(1+t)^{\frac{-1}{\sigma-1}}, \quad \forall t \ge 0 \quad \text{where } C = \left(\frac{1}{\lambda}\right)^{\frac{1}{\sigma-1}} \tag{5.22}$$

From 5.22, we conclude that

$$\int_0^\infty F(t)\,dt \le C\int_0^\infty (1+t)^{\frac{-1}{\sigma-1}}\,dt = \frac{C(\sigma-1)}{\sigma-2}\left[(1+t)^{1-\frac{1}{\sigma-1}}\right|_0^\infty$$

Since $\sigma = 2p-1$, $1 < p < \frac{3}{2}$, then $1 - \frac{1}{\sigma-1} < 0$. So that $\frac{1}{\sigma-1} - 1 \ge \delta > 0$

Therefore,

$$\int_0^\infty F(t)\,dt \le -\frac{C}{\delta}\left[\frac{1}{(1+t)^\delta}\right|_0^\infty = -\frac{C}{\delta}[0-1] = \frac{C}{\delta} < \infty$$

Similiarly,

$$tF(t) \le Ct(1+t)^{\frac{-1}{\sigma-1}} \le C(1+t)(1+t)^{\frac{-1}{\sigma-1}}$$
$$\le C(1+t)^{1-\frac{1}{\sigma-1}} \le \tfrac{C}{(1+t)^\delta} \le C_1, \qquad \forall\ t \ge 0$$

That is, $tF(t) \le C_1$ and hence sup $tF(t) < \infty$.

We deduce that

$$\int_0^\infty F(t)\,dt + \sup_{t\ge 0}\ tF(t) < \infty \tag{5.23}$$

Therefore, by using Lemma 5.2.1, we have

$$\begin{aligned}
(go\nabla u)(t) &\le C\left[\int_0^t \|u(\tau)\|_{H^1}^2\,d\tau + t\|u(t)\|_{H^1}^2\right]^{\frac{p-1}{p}} \{(g^p \circ \nabla u)(t)\}^{\frac{1}{p}} \\
&\le C_1\left[\int_0^t E(\tau)\,d\tau + tE(t)\right]^{\frac{p-1}{p}} \{(g^p \circ \nabla u)(t)\}^{\frac{1}{p}} \\
&\le C_2\left[\int_0^t F(\tau)\,d\tau + tF(t)\right]^{\frac{p-1}{p}} \{(g^p \circ \nabla u)(t)\}^{\frac{1}{p}} \\
&\le C_3 (g^p \circ \nabla u)^{\frac{1}{p}}(t)
\end{aligned} \tag{5.24}$$

or

$$g^p \circ \nabla u \geq C_4 \left(g \circ \nabla u\right)^p \tag{5.25}$$

By combining 5.25 and 5.16, we get

$$F'\left(t\right) \leq -C_5 \left[\|u_t\|_2^2 + \|\nabla u\|_2^2 + \left(g \circ \nabla u\right)^p \left(t\right)\right], \qquad \forall t \geq 0 \tag{5.26}$$

Similar to 5.17, we have

$$E^p \left(t\right) \leq C_6 \left[\|u_t\|_2^2 + \|\nabla u\|_2^2 + \left(g \circ \nabla u\right)^p \left(t\right)\right], \quad \forall t \geq 0 \tag{5.27}$$

A combination of 5.26, 5.27 and 5.7 yields

$$F'\left(t\right) \leq -C_6 F^p \left(t\right), \quad \forall t \geq 0 \tag{5.28}$$

We integrate 5.28 over $(0, t)$ to give

$$F\left(t\right) \leq K \left(1 + t\right)^{\frac{-1}{p-1}}, \quad \forall t \geq 0 \tag{5.29}$$

Consequently,

$$E\left(t\right) \leq K \left(1 + t\right)^{\frac{-1}{p-1}}, \quad \forall t \geq 0$$

This completes the proof.

Chapter 6

Polynomial Decay in a Nonlinear Viscoelastic Problem

6.1 Introduction

In this chapter, we study the interaction between a viscoelastic damping and a nonlinear frictional damping in a wave equation. We consider the following problem

$$\begin{cases} u_{tt}(x,t)\,u - \Delta u(x,t) + \int\limits_0^t g(t-\tau)\,\Delta u(x,\tau)\,d\tau + a u_t(x,t)\,|u_t(x,t)|^{m-2} = 0 \\ \text{in } \ \Omega \times (0,\infty) \\ u(x,t) = 0, \qquad x \in \partial\Omega, \qquad t \geq 0 \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \qquad x \in \Omega, \end{cases} \tag{6.1}$$

where Ω is a bounded domain of $\mathbb{R}^n$ $(n \geq 1)$, $(m > 2)$, with a smooth boundary $\partial\Omega$, a is a positive constant and g is a positive nonincreasing function satisfying the following condition

(G_1) $\qquad g : \mathbb{R}_+ \to \mathbb{R}_+$ is a differentiable function such that

$$g(0) > 0, \qquad 1 - \int\limits_0^\infty g(s)\,ds = l > 0$$

(G_2) There exists a positive constant ζ such that

$$g'(t) \leq -\zeta g^p(t), \qquad \forall t \geq 0, \quad 1 < p < \frac{3}{2}$$

Theorem 6.1.1. Let $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ be given. Assume that g satisfies

(G_1) and (G_2). Assume further that

$$\begin{aligned} &2 \leq m \leq \tfrac{2n}{n-2}, \quad n \geq 3 \\ &m \geq 2, \quad n = 1, 2. \end{aligned} \tag{6.2}$$

Then problem 6.1 has a unique global solution

$$u \in C\left([0, \infty) ; H_0^1(\Omega)\right), \qquad u_t \in C\left([0, \infty) ; L^2(\Omega)\right). \tag{6.3}$$

Proof. For the proof of this theorem, we refer the reader to [6], [7].

The modified energy functional is obtained in the previous chapters. Thus we have

$$E(t) = \frac{1}{2}\|u_t(x,t)\|_2^2 + \frac{1}{2}\left(1 - \int_0^t g(s)\,ds\right)\|\nabla u(x,t)\|_2^2 + \frac{1}{2}(go\nabla u)(t), \tag{6.4}$$

where

$$(g \circ \nabla u)(t) = \int_0^t g(t-\tau)\,\|\nabla u(t) - \nabla u(\tau)\|_2^2\,d\tau$$

Remark 6.1.1. The modified energy functional $E(t)$ satisfies, in accordance with the solution of equation 6.1,

$$\begin{aligned} E'(t) &= \tfrac{1}{2}(g' \circ \nabla u)(t) - \tfrac{1}{2}g(t)\|\nabla u\|_2^2 - a\|u_t\|_m^m \\ &\leq \tfrac{1}{2}(g' \circ \nabla u)(t) - a\|u_t\|_m^m \leq 0 \end{aligned} \tag{6.5}$$

6.2 Decay of solution

In this section we state and prove the main result. For this purpose, we set

$$F(t) := E(t) + \varepsilon_1 \Psi(t) + \varepsilon_2 \chi(t), \tag{6.6}$$

where ε_1 and ε_2 are positive constants to be chosen properly later and

$$\begin{aligned} \Psi(t) &:= \int_\Omega u u_t dx \\ \chi(t) &:= -\int_\Omega u_t \int_0^t g(t-\tau)(u(t)-u(\tau))\, d\tau dx \end{aligned} \tag{6.7}$$

Lemma 6.2.1. For ε_1 and ε_2 small enough, the inequality

$$\alpha_1 F(t) \leq E(t) \leq \alpha_2 F(t) \tag{6.8}$$

holds for two positive constants α_1 and α_2.

Proof. The proof of this Lemma is similar to the proof of lemma 4.2.1.

Lemma 6.2.2. Under the assumptions (G_1) and (G_2) the functional

$$\Psi(t) := \int_\Omega u u_t dx$$

satisfies, in accordance with the solution of 6.1,

$$\begin{aligned} \Psi'(t) \leq \|u_t\|_2^2 - \tfrac{1}{2}\left[1-(1+\eta)(1-l)^2 - 2a\eta K\right]\|\nabla u\|_2^2 \\ + ac(\eta)\|u_t\|_m^m + \tfrac{1}{2}\left(1+\tfrac{1}{\eta}\right)\left[\int_0^t g^{2-p}(\tau)\, d\tau\right](g^p \circ \nabla u)(t) \end{aligned} \tag{6.9}$$

where K is a positive constant depending on $E(0)$ only.

Proof. By using equation 6.1, we easily see that

$$\begin{aligned}\Psi'(t) &= \int_\Omega \left(uu_{tt} + u_t^2\right) dx \\ &= \int_\Omega u_t^2 dx - \int_\Omega |\nabla u(t)|^2 dx + \int_\Omega \nabla u(t) . \int_0^t g(t-\tau)\nabla u(\tau)\, d\tau dx - a\int_\Omega uu_t |u_t|^{m-2} dx\end{aligned} \tag{6.10}$$

By repeating the steps 5.10-5.14, the third term of the right-hand side of 6.10 is estimated as follows:

$$\begin{aligned}&\int_\Omega \nabla u(t) . \int_0^t g(t-\tau)\nabla u(\tau)\, d\tau dx \\ &\leq \tfrac{1}{2}\left[1 + (1+\eta)(1-l)^2\right] \|\nabla u\|_2^2 + \tfrac{1}{2}\left(1 + \tfrac{1}{\eta}\right)\left[\int_0^t g^{2-p}(t-\tau)\, d\tau\right] (g^p o \nabla u)(t)\end{aligned} \tag{6.11}$$

The fourth term in the right-hand side of 6.10 is estimated in chapter 4, as follows

$$-\int_\Omega uu_t |u_t|^{m-2} dx \leq \eta K \|\nabla u\|_2^2 + c(\eta)\|u_t\|_m^m \tag{6.12}$$

We substitute 6.12 and 6.13 in 6.10, to obtain

$$\begin{aligned}\Psi'(t) \leq& \|u_t\|_2^2 - \tfrac{1}{2}\left[1 - (1+\eta)(1-l)^2 - 2a\eta K\right] \|\nabla u\|_2^2 \\ &+ac(\eta)\|u_t\|_m^m + \tfrac{1}{2}\left(1 + \tfrac{1}{\eta}\right)\left[\int_0^t g^{2-p}(\tau)\, d\tau\right](g^p o \nabla u)(t)\end{aligned} \tag{6.13}$$

This completes the proof.

Remark 6.2.1. If we choose η such that

$$0 < \eta < \frac{l(2-l)}{(1-l)^2 + 2aK}$$

then

$$k_1 = 1 - (1+\eta)(1-l)^2 - 2a\eta K > 0$$

Lemma 6.2.3. Under the assumptions (G_1) and (G_2) the functional

$$\chi(t) := -\int_\Omega u_t \int_0^t g(t-\tau)(u(t) - u(\tau))\, d\tau dx$$

satisfies, in accordance with the solution of equation 6.1 and for any $\delta > 0$,

$$\begin{aligned}\chi'(t) \leq & \left[\delta + 2\delta(1-l)^2\right] \|\nabla u\|_2^2 + \left[\delta - \int_0^t g(s)\, ds\right] \|u_t\|_2^2 + a\delta \|u_t\|_m^m \\ & \left\{ \left(2\delta + \tfrac{1}{2\delta}\right) \left[\int_0^t g^{2-p}(\tau)\, d\tau\right] + ac(\delta) \left[\int_0^t g^{\frac{m-p}{m-1}}(\tau)\, d\tau\right]^{m-1} \right\} (g^p \circ \nabla u)(t) \\ & + \tfrac{g(0)}{4\delta} C_p \left[-(g' \circ \nabla u)(t)\right].\end{aligned} \tag{6.14}$$

Proof. By using equation 6.1, we see that

$$\begin{aligned}\chi'(t) = & -\int_\Omega \nabla u(t) . \int_0^t g(t-\tau)(\nabla u(t) - \nabla u(\tau))\, d\tau dx - \int_\Omega u_t \int_0^t g'(t-\tau)(u(t) - u(\tau))\, d\tau dx \\ & -\int_\Omega \left(\int_0^t g(t-\tau) \nabla u(\tau)\, d\tau\right) \left(\int_0^t g(t-\tau)(\nabla u(t) - \nabla u(\tau))\, d\tau\right) dx \\ & - \left(\int_0^t g(s)\, ds\right) \|u_t\|_2^2 + a\int_\Omega u_t |u_t|^{m-2} \int_0^t g(t-\tau)(u(t) - u(\tau))\, d\tau dx\end{aligned} \tag{6.15}$$

By using Hölder's inequality, Young's inequality, and the fact that $\int\limits_0^t g(s)\,ds \leq 1 - l$, we estimate the first term in the right-hand side of 6.15 as follows

$$\begin{aligned}
&-\int\limits_\Omega \nabla u(t) . \int\limits_0^t g(t-\tau)(\nabla u(t) - \nabla u(\tau))\,d\tau dx \\
&\quad\leq \delta \int\limits_\Omega |\nabla u(t)|^2 dx + \tfrac{1}{4\delta}\int\limits_\Omega \left(\int\limits_0^t g(t-\tau)|\nabla u(t) - \nabla u(\tau)|\,d\tau\right)^2 dx \\
&\quad\leq \delta \int\limits_\Omega |\nabla u(t)|^2 dx + \tfrac{1}{4\delta}\int\limits_\Omega \left(\int\limits_0^t g^{1-\frac{p}{2}}(t-\tau)\, g^{\frac{p}{2}}(t-\tau)\left(|\nabla u(\tau) - \nabla u(t)|\right) d\tau\right)^2 dx \\
&\quad\leq \delta \int\limits_\Omega |\nabla u(t)|^2 dx + \tfrac{1}{4\delta}\int\limits_\Omega \left[\int\limits_0^t g^{2-p}(\tau)\,d\tau\right](g^p \circ \nabla u)(t).
\end{aligned} \tag{6.16}$$

The third term

$$\begin{aligned}
&\int\limits_\Omega \left(\int\limits_0^t g(t-\tau)\nabla u(\tau)\,d\tau\right)\left(\int\limits_0^t g(t-\tau)(\nabla u(t) - \nabla u(\tau))\,d\tau\right) dx \\
&\quad\leq \delta \int\limits_\Omega \left|\int\limits_0^t g(t-\tau)\nabla u(\tau)\,d\tau\right|^2 dx + \tfrac{1}{4\delta}\int\limits_\Omega \left|\int\limits_0^t g(t-\tau)(\nabla u(t) - \nabla u(\tau))\,d\tau\right|^2 dx \\
&\quad\leq \delta \int\limits_\Omega \left|\int\limits_0^t g(t-\tau)\left(|\nabla u(\tau) - \nabla u(t)| + |\nabla u(t)|\right) d\tau\right|^2 dx \\
&\qquad + \tfrac{1}{4\delta}\int\limits_\Omega \left|\int\limits_0^t g(t-\tau)(\nabla u(t) - \nabla u(\tau))\,d\tau\right|^2 dx
\end{aligned} \tag{6.17}$$

We estimate the first term in the right-hand side of 6.17 by using Lemma 2.2.1 as follows

$$\begin{aligned}
&\int\limits_\Omega \left(\int\limits_0^t g(t-\tau)\left(|\nabla u(\tau) - \nabla u(t)| + |\nabla u(t)|\right) d\tau\right)^2 dx \\
&\quad\leq 2\int\limits_\Omega \left(\int\limits_0^t g(t-\tau)\left(|\nabla u(\tau) - \nabla u(t)|\right) d\tau\right)^2 dx + 2\int\limits_\Omega \left(\int\limits_0^t g(t-\tau)\left(|\nabla u(t)|\right) d\tau\right)^2 dx
\end{aligned} \tag{6.18}$$

We combine 6.17 and 6.18 to get

$$
\begin{aligned}
&\int_\Omega \left(\int_0^t g(t-\tau)\nabla u(\tau)\,d\tau\right)\left(\int_0^t g(t-\tau)(\nabla u(t)-\nabla u(\tau))\,d\tau\right)dx \\
&\quad\le \left(2\delta+\tfrac{1}{4\delta}\right)\int_\Omega \left(\int_0^t g(t-\tau)(|\nabla u(\tau)-\nabla u(t)|)\,d\tau\right)^2 dx + 2\delta(1-l)^2\|\nabla u(t)\|_2^2 \\
&\quad\le \left(2\delta+\tfrac{1}{4\delta}\right)\left[\int_0^t g^{2-p}(\tau)\,d\tau\right](g^p\circ\nabla u)(t) + 2\delta(1-l)^2\|\nabla u(t)\|_2^2
\end{aligned} \tag{6.19}
$$

The fifth term in the right-hand side of 6.15 is estimated as follows:

$$
\begin{aligned}
&\int_\Omega u_t|u_t|^{m-2}\int_0^t g(t-\tau)(u(t)-u(\tau))\,d\tau dx \le \int_\Omega |u_t|^{m-1}\int_0^t g(t-\tau)|(u(t)-u(\tau))|\,d\tau dx \\
&\le \delta\|u_t\|_m^m + c(\delta)\int_\Omega\left(\int_0^t g(t-\tau)|(u(t)-u(\tau))|\,d\tau\right)^m dx
\end{aligned} \tag{6.20}
$$

We estimate the second term of 6.20 by using Hölder's inequality with $p=\frac{m}{m-1}$, $q=m$, the Sobolev embedding, and the fact that $\int_0^t g(s)\,ds \le 1-l$. Thus, we obtain

$$
\begin{aligned}
&\int_\Omega\left(\int_0^t g(t-\tau)|(u(t)-u(\tau))|\,d\tau\right)^m dx \le \int_\Omega\left(\int_0^t g^{1-\frac{p}{m}}(t-\tau)\,g^{\frac{p}{m}}(t-\tau)|(u(t)-u(\tau))|\,d\tau\right)^m dx \\
&\quad\le \int_\Omega\left\{\left(\int_0^t \left(g^{1-\frac{p}{m}}(\tau)\right)^{\frac{m}{m-1}}d\tau\right)^{\frac{m-1}{m}}\left(\int_0^t \left(g^{\frac{p}{m}}(t-\tau)|(u(t)-u(\tau))|\right)^m d\tau\right)^{\frac{1}{m}}\right\}^m dx \\
&\quad\le \left(\int_0^t g^{\frac{m-p}{m-1}}(\tau)\,d\tau\right)^{m-1}\int_\Omega\int_0^t g^p(t-\tau)|(u(t)-u(\tau))|^m\,d\tau dx \\
&\quad\le C_1\left(\int_0^t g^{\frac{m-p}{m-1}}(\tau)\,d\tau\right)^{m-1}\int_0^t g^p(t-\tau)\|\nabla u(t)-\nabla u(\tau)\|_2^m\,d\tau \\
&\quad\le C_1\left(\int_0^t g^{\frac{m-p}{m-1}}(\tau)\,d\tau\right)^{m-1}\int_0^t g^p(t-\tau)\|\nabla u(t)-\nabla u(\tau)\|_2^2\|\nabla u(t)-\nabla u(\tau)\|_2^{m-2}\,d\tau
\end{aligned} \tag{6.21}
$$

Since $m>2$, then

$$\|\nabla u(t)-\nabla u(\tau)\|_2^{m-2}=\left(\|\nabla u(t)-\nabla u(\tau)\|_2^2\right)^{\frac{m-2}{2}}\leq\left[\|\nabla u(t)\|_2^2+\|\nabla u(\tau)\|_2^2\right]^{\frac{m-2}{2}}$$
$$\leq\left[CE(t)+CE(\tau)\right]^{\frac{m-2}{2}}\leq\left[CE(0)+CE(0)\right]^{\frac{m-2}{2}}\leq\left[2CE(0)\right]^{\frac{m-2}{2}} \tag{6.22}$$

By substituting 6.22 in 6.21, we obtain

$$\int_\Omega\left(\int_0^t g(t-\tau)\left|(u(t)-u(\tau))\right|d\tau\right)^m dx\leq C_2\left(\int_0^t g^{\frac{m-p}{m-1}}(\tau)\,d\tau\right)^{m-1}(g^p\circ\nabla u)(t) \tag{6.23}$$

where $C_2=\left[2CE(0)\right]^{\frac{m-2}{2}}C_1$.

Thus, the estimation of the fifth term of the right-hand side of 6.15 takes the form

$$\int_\Omega u_t\left|u_t\right|^{m-2}\int_0^t g(t-\tau)(u(t)-u(\tau))\,d\tau dx\leq\delta\|u_t\|_m^m+c_2(\delta)\left(\int_0^t g^{\frac{m-p}{m-1}}(\tau)\,d\tau\right)^{m-1}(g^p\circ\nabla u)(t) \tag{6.24}$$

where $c_2(\delta)=C_2c(\delta)$.

Similarly to 4.35, the second term of the right-hand side of 6.15 is estimated as follows:

$$-\int_\Omega u_t\int_0^t g'(t-\tau)(u(t)-u(\tau))\,d\tau dx\leq\delta\|u_t\|_2^2+\frac{g(0)}{4\delta}C_p\left[-(g'\circ\nabla u)(t)\right]. \tag{6.25}$$

By substituting 6.16, 6.19, 6.24 and 6.25 in 6.15 the assertion of Lemma 6.2.3. is established.

Theorem 6.2.1. Let $(u_0,u_1)\in H_0^1(\Omega)\times L^2(\Omega)$ be given and assume that 6.2 holds and g satisfies (G_1) and (G_2). Then, for each $t_0>0$, there exists a strictly positive constant K such that the solution given by 6.3 satisfies, for all $t \geq t_0$,

$$E(t) \leq K(1+t)^{\frac{-1}{p-1}}$$

Proof. Since g is continuous, positive and $g(0) > 0$, then for any $t_0 > 0$, we have

$$\int_0^t g(s)\,ds \geq \int_0^{t_0} g(s)\,ds = g_0 > 0, \quad \forall t \geq t_0 \tag{6.26}$$

By using 6.5, 6.6, 6.9, 6.14 and 6.26, we get

$$\begin{aligned} F'(t) \leq & -[\varepsilon_2(g_0-\delta)-\varepsilon_1]\|u_t\|_2^2 - a[1-(\varepsilon_1 c(\eta)+\varepsilon_2\delta)]\|u_t\|_m^m \\ & -\left\{\varepsilon_1\left[\frac{1-(1+\eta)(1-l)^2-2a\eta K}{2}\right]-\varepsilon_2\delta\left[1+2(1-l)^2\right]\right\}\|\nabla u\|_2^2 \\ & -\zeta\left\{\frac{1}{2}-\varepsilon_2\frac{g(0)}{4\delta}C_p-\left(\left\{\frac{\varepsilon_1}{2}\left(1+\frac{1}{\eta}\right)+\varepsilon_2\left(2\delta+\frac{1}{2\delta}\right)\right\}\left[\int_0^t g^{2-p}(\tau)\,d\tau\right]\right.\right. \\ & \left.\left.+\varepsilon_2 C\left[\int_0^t g^{\frac{m-p}{m-1}}(\tau)\,d\tau\right]^{m-1}\right)\right\}(g^p\circ\nabla u)(t) \end{aligned} \tag{6.27}$$

By recalling Remark 6.2.1, we can choose δ so small that

$$g_0-\delta > \frac{1}{2}g_0$$

$$\frac{2}{1-(1+\eta)(1-l)^2-2a\eta K}\delta\left[1+2(1-l)^2\right] < \frac{1}{4}g_0$$

Whence δ is fixed, any choice of ε_1 and ε_2, such that

$$\frac{1}{4}g_0\varepsilon_2 < \varepsilon_1 < \frac{1}{2}g_0\varepsilon_2, \tag{6.28}$$

makes

$$\varepsilon_2(g_0-\delta) > \frac{1}{2}g_0\varepsilon_2 > \varepsilon_1 \Longrightarrow k_2 = \varepsilon_2(g_0-\delta)-\varepsilon_1 > 0,$$

$$\frac{2\varepsilon_2\delta}{1-(1+\eta)(1-l)^2-2a\eta K}\left[1+2(1-l)^2\right] < \frac{1}{4}g_0\varepsilon_2 < \varepsilon_1$$

which implies that

$$k_3 = \varepsilon_1\left[\frac{1-(1+\eta)(1-l)^2-2a\eta K}{2}\right] - \varepsilon_2\delta\left\{1+2(1-l)^2\right\} > 0$$

We have

$$g^{\frac{m-p}{m-1}}(s) = g^{\frac{m(1-p)}{m-1}}(s)\,g^p(s)$$

Therefore,

$$\begin{aligned}\int_0^\infty g^{\frac{m(1-p)}{m-1}}(s)\,g^p(s)\,ds &\le \frac{-1}{\zeta}\int_0^\infty g^{\frac{m(1-p)}{m-1}}(s)\,g'(s)\,ds \le \frac{-1}{\zeta(2m-p-1)}\left[g^{\frac{m(1-p)}{m-1}+1}\right]_0^\infty \\ &\le C\left[g^{\frac{2m-pm-1}{m-1}}\right]_0^\infty\end{aligned}$$

Since g is nonincreasing, then the above integral is finite if $m > 2$ and $p < \frac{3}{2}$. Also, we showed in the previous chapter that the integral $\int_0^\infty g^{2-p}(\tau)\,d\tau$ is finite.

Consequently, we pick ε_1 and ε_2 so small that 6.8 and 6.28 remain valid and further

$$k_4 = \tfrac{1}{2} - \varepsilon_2 \tfrac{g(0)}{4\delta} C_p$$
$$- \left[\left\{ \tfrac{\varepsilon_1}{2} \left(1 + \tfrac{1}{\eta}\right) + \varepsilon_2 \left(2\delta + \tfrac{1}{2\delta}\right) \right\} \left[\int_0^\infty g^{2-p} (\tau)\, d\tau \right] + \varepsilon_2 a c_2(\delta) \left[\int_0^\infty g^{\frac{m-p}{m-1}} (\tau)\, d\tau \right]^{m-1} \right] > 0$$

and

$$k_5 = 1 - (\varepsilon_1 c(\eta) + \varepsilon_2 \delta) > 0$$

Finally, we choose $\beta = \min \{k_i, i = 2, 3, 4\}$, to obtain

$$\begin{aligned} F'(t) &\leq -\beta \left[\|u_t\|_2^2 + \|\nabla u\|_2^2 + (g^p \circ \nabla u)(t) \right] - a k_5 \|u_t\|_m^m \\ &\leq -\beta \left[\|u_t\|_2^2 + \|\nabla u\|_2^2 + (g^p \circ \nabla u)(t) \right], \qquad \forall t \geq t_0 \end{aligned} \tag{6.29}$$

By repeating the steps 5.16-5.29, we arrive at

$$E(t) \leq K (1 + t)^{\frac{-1}{p-1}}, \quad \forall t \geq t_0 \tag{6.30}$$

This completes the proof.

Remark 6.2.2. Estimate 6.30 is also true for $0 \leq t \leq t_0$ by boundedness and continuity of $E(t)$.

REFERENCES

1. Barreto R., Lapa E. C. and Muñoz Rivera J.E., Decay rates for viscoelastic plateswith memory, J. Elasticity 44 # 1 (1996), 61-87.

2. Barreto R. and Muñoz Rivera J.E., Uniform rates of decay in nonlinear viscoelasticity for polynomial decaying kernels, Appl. Anal. 60 (1996), 263-283.

3. Berrimi S. and Messaoudi S.A., Exponential decay of solutions to a viscoelasticequation with nonlinear localized damping, Elect. J. Diff. Egns. Vol. 2004 # 88 (2004), 1-10.

4. Berrimi S. and Messaoudi S.A., Existence and decay of solutions of a viscoelasticequation with a nonlinearsource, Nonlinear Analysis 64 (2006), 2314-2331.

5. Cabanillas E.L. and Muñoz Rivera J.E., decay rates of solutions of an anisotropic inhomogeneousn-dimensional viscelastic equation with polynomial decaying kernels, Comm. Math. Phys. 177 (1996), 583-602.

6. Cavalcanti M.M., Domingos V.N. and Ferreira J., Existence and uniform decay for nonlinear viscoelastic equations with strong damping, Math. Meth. Appl. Sci. 24 (2001), 1043-1053.

7. Cavalcanti M.M., Domingos Cavalcanti V.N. and Soriano J.A., Exponential decay for the solution of semilinear viscoelastic wave equation with localized damping, Elect. J. Diff. Eqs. Vol. 2002 # 44 (2002), 1-14.

8. Cavalcanti M.M. and Oquendo H.P., Frictional versus viscoelastic damping in a semilinear wave equation, SIAM J. Control Optim. Vol. 42 # 4 (2003), 1310-1324.

9. Coleman B.D. and Noll W., Foundations of linear viscoelasticity, Rev. Modern Phys. 33 (1961), 239-249.

10. Coleman B.D. and Mizel V.J., On the general theory of fading memory. Arch. Rational Mech. Anal.29 (1968) 18-31.

11. Dafermos C.M., Asymptotic stability in viscoelasticity, Arch. Rational Mech. Anal. 37 (1970), 297-308.

12. Dafermos C.M., On abstract Volterra equations with applications to linear viscoelasticity, J. Differential Equations 7 (1970), 554-569.

13. Dafermos C.M., Development of singularities in the motion of material with fading memory, Arch. Rational Mech. Anal. 91 # (1985), 193-205.

14. Dassios G. and Zafiropoulos F., Equipartition of energy in linearized 3-d viscoelasticity, Quart. Appl. Math. 48# 4 (1990), 715-730.

15. Fabrizio M. and Morro A., Mathematical Problems in Linear Viscoelasticity, SIAM Studies in Appl. Math., Vol. 12, Philadelphia 1992.

16. Liu Z. and Zheng S., Semigroups associated with dissipative systems, Chapman & Hall / CRC Research Notes in Math., Vol. 398, Chapman & Hall /CRC, Boca Raton, FL 1999.

17. Liu K. and Liu Z., On the type of semigroups associated with the abstract linear viscoelastic system, Z. Angew Math. Phys. Vol. 47 # 1 (1996), 1-15.

18. Messaoudi S.A., Blow-up and global existence in a nonlinear viscoelastic wave equation, Mathematische Nachrichten Vol. 260 (2003), 58-66.

19. Messaoudi S.A., Blow-up of positive-initial-energy solutions of a nonlinear viscoelastic hyperbolic equation, J. Math. Anal. Appl. 320 (2006), 902-915.

20. Messaoudi S.A., and Tatar N.E., Exponential and Polynomial Decay for a Quasilinear Viscoelastic Equation, Nonl. Anal.

21. Messaoudi S.A., and Tatar N.E., Global existence and uniform stabilityof solutions for a quasilinear viscoelastic problem, Math. Meth. Appl. Sci.Vol 30 (2007), 665-680.

22. Muñoz Rivera J.E., Asymptotic behaviour in linear viscoelastcity, Quart. Appl. Math. 25 # 4 (1994),628-648.

23. Muñoz Rivera J.E. and Oquendo Portillo H., Exponential stability to a contact problem of partiallyviscoelastic materials, J. Elsticity 63 # 2 (2001), 87-111.

24. Muñoz Rivera J.E. and Press Salvatierra A., Asymptotic behavior of the energy in partially viscoelasticmaterials, Quart. Appl. Math. 59 # 3 (2001), 557-578.

25. Muñoz Rivera J.E., Naso M.G. and Vegni F.M., Asymptotic behaviour of the energy fora class of weakly dissipative second-order systems with memory, J. Math.

Anal. Appl. 286 # 2(2003), 692-704.

26. Muñoz Rivera J.E. , Naso M.G., On the decay of the energy for systems with memory and indefinite dissipation, Asympt. Anal. 49 # 3-4 (2006), 189-204.

27. Muñoz Rivera J.E., Naso M.G., Asymptotic stability of semigroups associated to weak dissipative systems with memory, J. Math. Anal. Appl. 326 # 1 (2007), 691-707.

28. Renardy M., Hrusa W.J., and Nohel J.A., Mathematical Problems in Viscoelasticity, Pitman Monographs and Surveys in Pure and Appl. Math., Vol. 35, Longman Scientific & Technical, Harlow, 1987.

29. Wu S.T., Blow-up of solutions for an integro-differential equation with a nonlinear source, Elect. J. Diff. Eqns. Vol. 2006 # 45 (2006), 1-9.

30. Bland D.R., The Theory of Linear Viscoelasticity, Pergamon Press, Oxford 1960.

31. Christensen R.M., Theory of Viscoelasticity: An Introduction, New York and London: Academic Press 1971.

32. Day W .A., The Thermodynamics of Materials with Memory, in Material with Memory, D. Graffi, ed., Liguori, Naples 1979.

33. Dill E.H., Simple Material with Fading Memory, Continuum Physics II, A.C.

Eringen, ed. Academic, New York 1975.

34. Ferry J.D., Viscoelastic Properties of Polymers, New-York: Wiley 1980.

35. Gross B., Mathematical Structures of the Theories of Viscoelasticity, Pris: Hermann 1953.

36. Adams R. A. and Fournier J.F., Sobolev Spaces, Second Edition, Academic Press, (2003).

37. Evans L. C., Partial Differential Equations, Graduate Studies in Mathematics 19, AMS. (1998).

38. Mcowen, R., Partial Differential Equations Methods & Applications, Second Edition, Prentice Hall (2003).

39. Alabau-Boussouira F., Decay estimate for second order evolution equations with memory, Functional Analysis 254 (2008) 1342-1372.

40. Messaoudi S. A., General decay of the solution energy in a viscoelastic equation with a nonlinear source, Nonlinear Analysis 69 (2008) 2589-2598.

41. Messaoudi S. A., General decay of solution of a viscoelastic equation, Math. Appl. 341 (2008) 1457-1467.

42. Laun H.M., Description of the non-linear shear behavior of a low density polyethylene melt by means of an experimentally determined strain-dependent memory function, Rheol. Acta 17 (1978), 1-15.

43. Fabrizio M., Polidord S., Asympototic decay for some differential systems with fading memory, Appl. Analysis 81 # 6(2002) 1245-1264.

Curriculum Vitae

Name: Saeed Mohammed Salman Ali

Birth date & Place: January, 1, 1970, Dhamar, Yemen

Nationality: Yemeni

E-mail saeedmohammed199@hotmail.com

Major Field: Applied Mathematics

Area of Specialization: Partial Differential Equations

Education

University	Year	Degree
King Fahd University of Petroleum & Minerals (KSA)	2004-2009	MS in Applied Mathematics
University of Hodideh (Yemen)	1991-1994	BS in Math