CHAPTER 1

HISTORICAL BACKGROUND

1.1 Introduction

During the 1960s, a method to solve an important class of non-linear partial differential equation was introduced in the field of mathematical physics [1,2,5,6,7.34]. This method is now commonly known as the inverse scattering transform method. This transformation is capable of producing an exact solution for non-linear differential equations. One of the main successes of this transformation is the relationship found between the Schrödinger equation:

$$-\frac{\partial^2 w(x,t)}{\partial x^2}+u_0(x)w(x,t)=\lambda(t)w(x,t), \tag{1.1}$$

and the Korteweg-de Vries equation:

$$\begin{aligned} &u_t-6uu_x+u_{xxx}=0\\ &u(x,0)=u_0(x). \end{aligned} \tag{1.2}$$

In fact, the basic idea behind this transformation is to view the initial solution of the non-linear partial differential equation as the potential $u_0(x)$ of the Schrödinger equation for a fixed time. Then, we study the scattering data (Reflection coefficient, Transmission coefficient and the Bound States) and, as the time evolves, the potential

$u(x,t)$ and the scattering data both evolve with time. These evolution equations are independent of the potential, and as a result of that, we study the scattering data instead of studying the potential itself. These evolution equations are given by separated linear differential equations. However, by solving a linear integral equation which is known as Gel'fand and Levitan equation, we can recover the potential $u_0(x)$ from the scattering data at a later time. This is exactly the inverse scattering problem. Moreover, for some kinds of potentials, the inverse scattering problem reduces to solve separable integral equations [5,6,32].

1.2 The Discovery of the Solitary Wave

The discovery of the solitary wave has passed through different stages and in order to introduce some of these stages, we need the following technical definition about the dispersive waves.

Definition 1.1

If waves of different wavelengths propagate at different speeds, then we say that the waves are dispersive. Figure 1-1 below shows the propagation of dispersive waves.

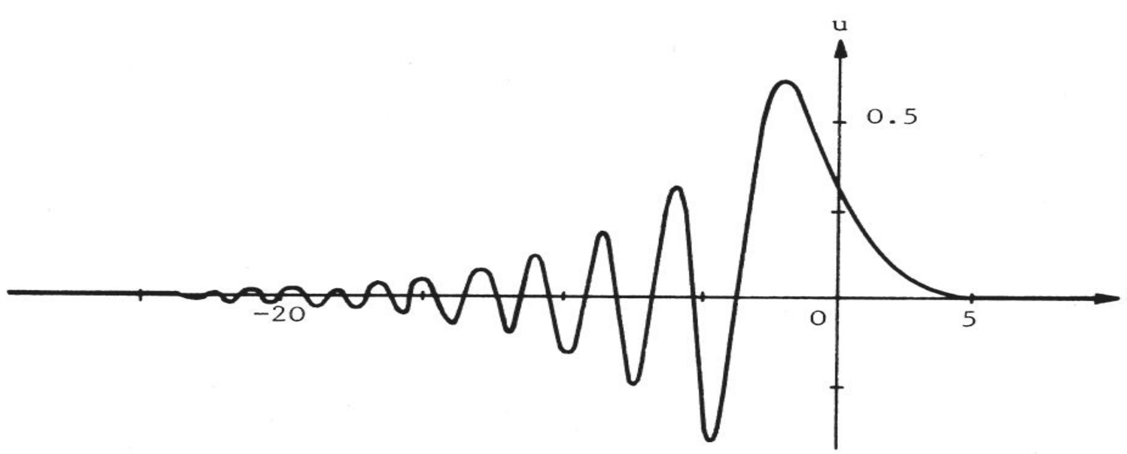


Figure 1-1 Propagation of Dispersive Waves.

Moreover, dispersive waves depend not only on the wave maker but also the media in which they propagate. In fact, we can send dispersive waves through most materials in nature. Water is an excellent example of a dispersive medium [2,5,32].

Example 1.1 Consider the Klein-Gorden equation

$$\theta_{tt}(x,t) - \theta_{xx}(x,t) + \theta(x,t) = 0, \tag{1.3}$$

and assume the above equation has the following wave solution

$$\theta(x,t) = A\cos k\left(x - \frac{w}{k}t\right), \tag{1.4}$$

where w is called the frequency, k is called the wave number and $\lambda = \frac{1}{k}$ is called the wavelength. By substituting the above solution into the differential equation we obtain

$$-Aw^2\cos(kx - wt) + Ak^2\cos(kx - wt) + A\cos(kx - wt) = 0$$

$$\left(-w^2 + k^2 + 1\right)\cos(kx - wt) = 0. \tag{1.5}$$

Hence, the differential equation has a nontrivial solution if and only if

$$-w^2 + k^2 + 1 = 0$$

$$\Rightarrow w^2 = k^2 + 1. \tag{1.6}$$

Equation (1.6) is called a dispersion relation and the phase shift (the observable velocity) is $\frac{w}{k}$ which is given by:

$$\frac{w}{k} = \pm\sqrt{1 + \frac{1}{k^2}} = \pm\sqrt{1 + \lambda^2}, \tag{1.7}$$

and from this relation (1.7), we can conclude the following points:

(i) Waves of different wavelengths propagate at different speeds.

(ii) Dispersive waves with a fixed wavelength propagate in two directions since the phase shift has positive and negative signs.

(iii) The larger the wave length, the higher the propagation speed.

Example 1.2 Consider the linearized Korteweg-de Vries Equation

$$\theta_t(x,t)+\frac{3}{2}\theta_x(x,t)+\frac{1}{6}\theta_{xxx}(x,t)=0, \tag{1.8}$$

with the following wave solution

$$\theta(x,t)=A\cos\left(kx-wt\right). \tag{1.9}$$

By direct substitution, we can easily obtain the following dispersive relation

$$w=\frac{3}{2}k-\frac{1}{6}k^3, \tag{1.10}$$

and the observable velocity is

$$\frac{w}{k}=\frac{3}{2}-\frac{1}{6}k^2. \tag{1.11}$$

By careful analysis for the above relation, we can conclude that the wave is dispersive and the waves with a fixed wave lengths propagate only in one direction since there is only one given sign for a given k.

John Scott Russell discovered the solitary wave during 1838 in front of a boat on Edinburgh - Glasgow canal and he summarized his observations in the following statement:

"A large solitary elevation, a rounded, smooth and well defined heap of water, which continued its course along the channel apparently without change of form or diminution of spead....

Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance of interview with that singular and beautiful phenomenon" [5,6].

Some time later, he conducted extensive experiments in the laboratory and as a result of that he managed to find the relationship [5]:

$$c = \sqrt{g\left(h + a\right)}, \tag{1.12}$$

between the velocity "c", amplitude "a", depth of the canal "h" and the earth acceleration due to the gravity "g". Figure 1-2 is a simple sketch for a solitary wave.

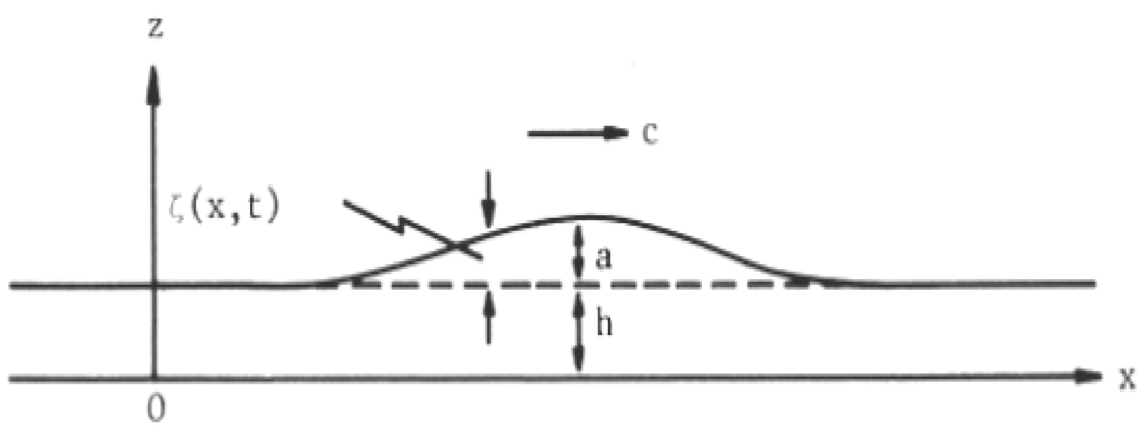


Figure 1-2 A Solitary Wave.

From those experiments, he made several observations

- Taller solitary wave travels faster than the smaller one (see figure 1-3a and figure 1-3b).

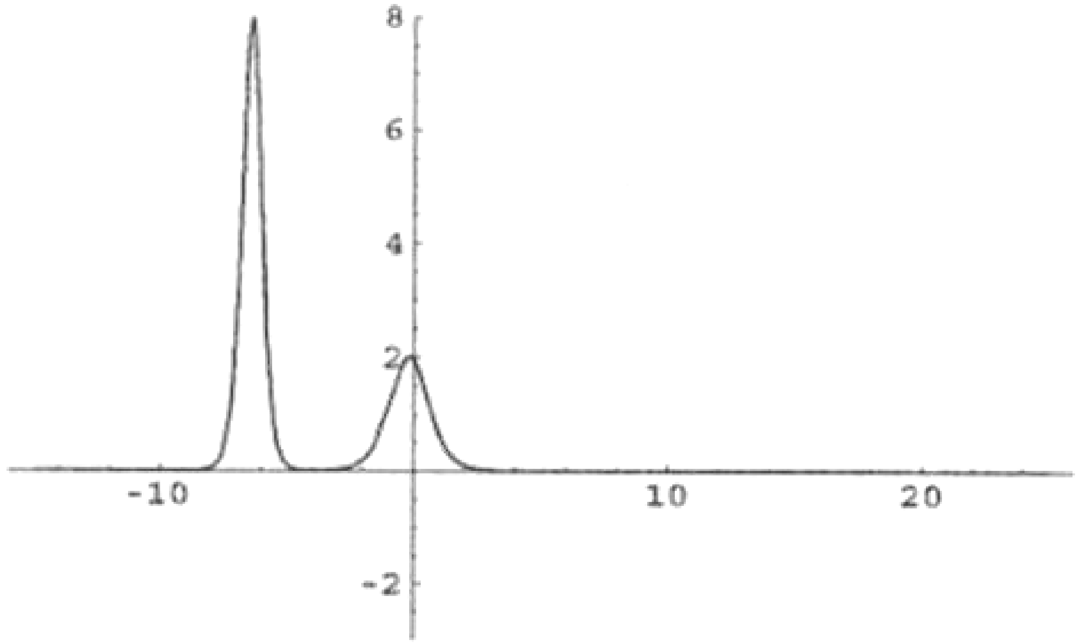


Figure 1-3a Two Solitons Before the Collision.

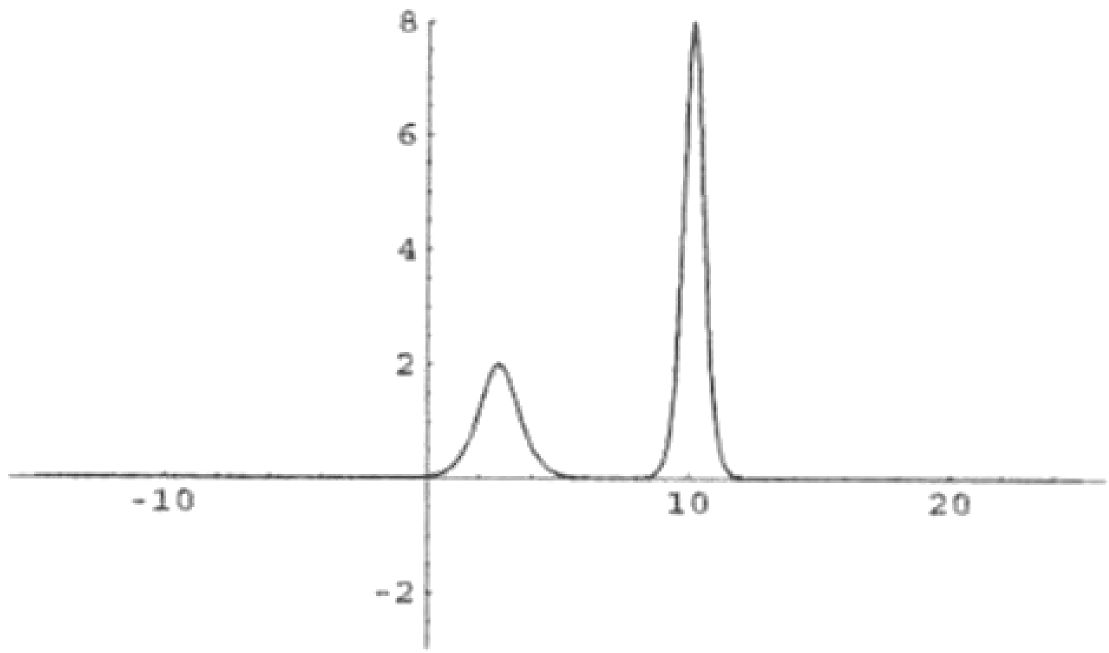


Figure 1-3b Two Solitons After Collision.

- Raising the weight from the bottom of the water, dispersive waves were generated [5,6] (see figure 1-4).

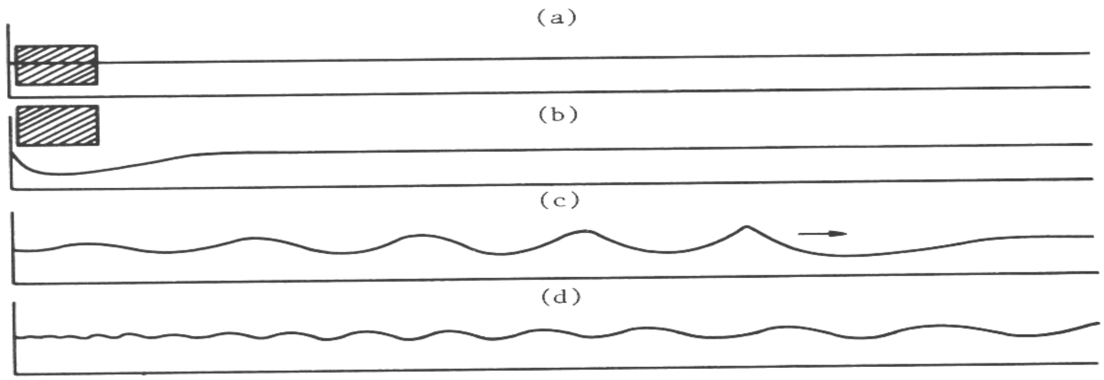


Figure 1-4 Dispersive Waves.

- Initial depression becomes a train of oscillatory waves whose length increases and amplitude decreases with time (see figure 1-4 and figure 1-5).

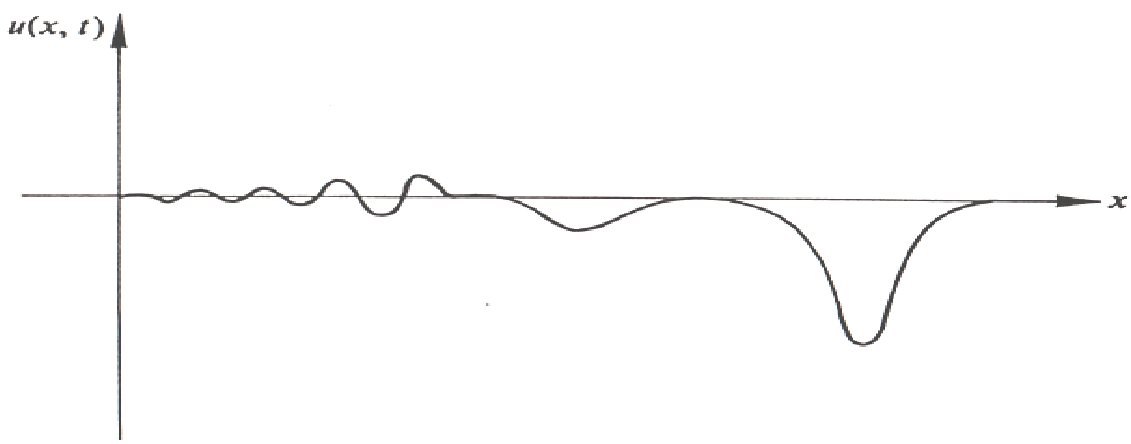


Figure 1-5 A Train of Oscillatory Waves.

Later, both Boussinssq and Lord Rayleigh made some other observations and one of these observations is that a solitary wave has a length much greater than the depth of the water [2]. They independently showed that the solitary wave can be described by the sech^2 profile and the wave length above the mean level "h" is given by

$$u(x,t) = a\ \mathrm{sech}^2\left\{\frac{x-ct}{b}\right\}, \qquad (1.13)$$

where

$$b^2 = \frac{4h^2(h+a)}{3a}. \qquad (1.14)$$

This phenomenon took almost another 50 years before both Korteweg and de-Veries developed the theory and derived a partial differential equation which governs the two-dimensional motion of weakly nonlinear long waves:

$$u_t - 6uu_x + u_{xxx} = 0, \qquad (1.15)$$

which has sech^2 shaped solitary wave. The equation (1.15) is known as the Korteweg-de Vries or KdV equation [2,5,6].

1.3 Solitary Wave

For certain initial conditions (potentials), the KdV equation admits some solutions that are termed as solitons [7,8,23,34,37]. In fact, a soliton solution is the simplest type of solution for the KdV equation which can be obtained by looking for a solution of the form $u(x-vt)$ and this method is given in appendix (I). Moreover, both Kruskal and Zabusky [36,37] found that the KdV equation has special permanent wave solution which is the solitary wave

$$\zeta(x,t) = 2k^2\mathrm{sech}^2 k(x-4k^2t-x_0), \qquad (1.16)$$

where k and x_0 are two constants. Also, it is obvious, that the velocity is related to the amplitude and proportional by a factor of 2. This solution is a disturbance (see figure 1-2) that moves with a constant speed in the direction of positive x-axis and exhibits some features of nonlinear waves [5,14]. One of these features is that the velocity and the amplitude for the disturbance are related. Later we will show that the disturbance for the KdV equation is given by the relationship [1,2]

$$u(x,t) = 2k^2 \operatorname{sech}^2 k(x - 4k^2 t - x_0). \tag{1.17}$$

Moreover, by looking closely at this solution, we notice that the larger amplitude pulse moves faster and is narrower in width.

1.4 Some General Properties of Solitons

In this section, we present some features of the solitons. In order to do so, we start by the following definition:

Definition 1.2

An evolution equation is a partial differential equation for an unknown function $u(x,t)$ of the form

$$\frac{\partial u(x,t)}{\partial t} = f(u), \tag{1.18}$$

where $f(u)$ is an expression involving only u and its derivatives with respect to x. If the expression $f(u)$ is nonlinear, equation (1.18) is called a nonlinear evolution equation.

So far, there is no precise definition for solitons in the literature but they are solutions that satisfy nonlinear equations and usually they have the following characteristics:

i. Solitons are waves dying out at infinity and they have profiles, which are unaltered after colliding with other solitons [1,2].

ii. They evolve with time and therefore they satisfy certain evolution equations [1].

iii. They are stable solutions and they do not disperse apart when they collide with other solitons [5].

iv. In collision with other solitons, there is nonlinear interaction. However, they retain their original shape shortly afterwards, only slightly displaced [3, 32].

v. Soliton with larger amplitude pulse moves faster and is narrower in width than the smaller soliton.

To demonstrate the interaction of these waves, Zabusky and Kruskal [36,37] suggested the following: assume that two waves are given and well separated from each other at time $t = 0$ and the smaller wave is to the right. After sometime, the two waves will overlap and interact and sometime later they will retain there original shapes and velocities with a phase shift (see figure 1-6a, Figure1-6b and Figure.1.7).

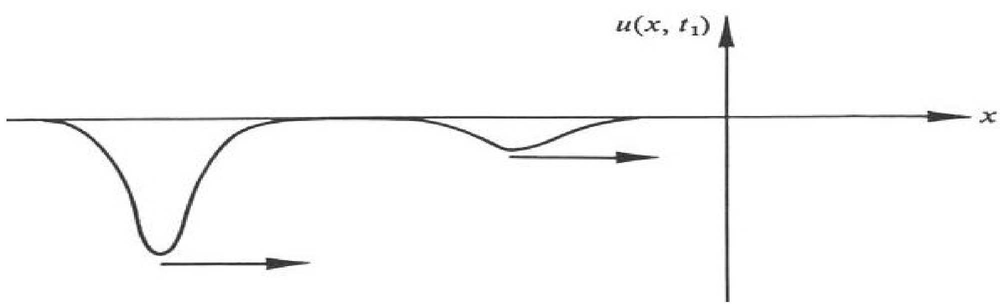


Figure1-6a Before Interaction.

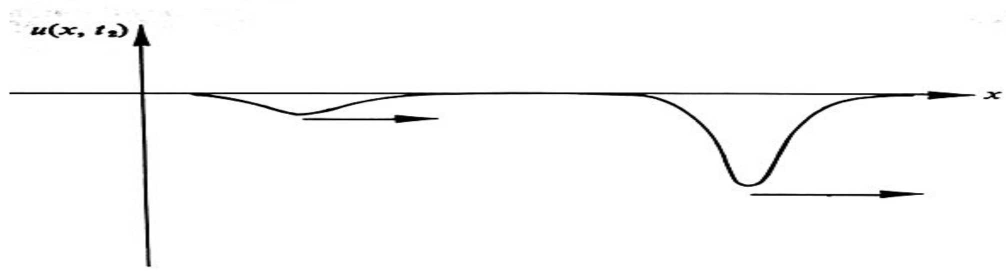


Figure 1-6b After Interaction.

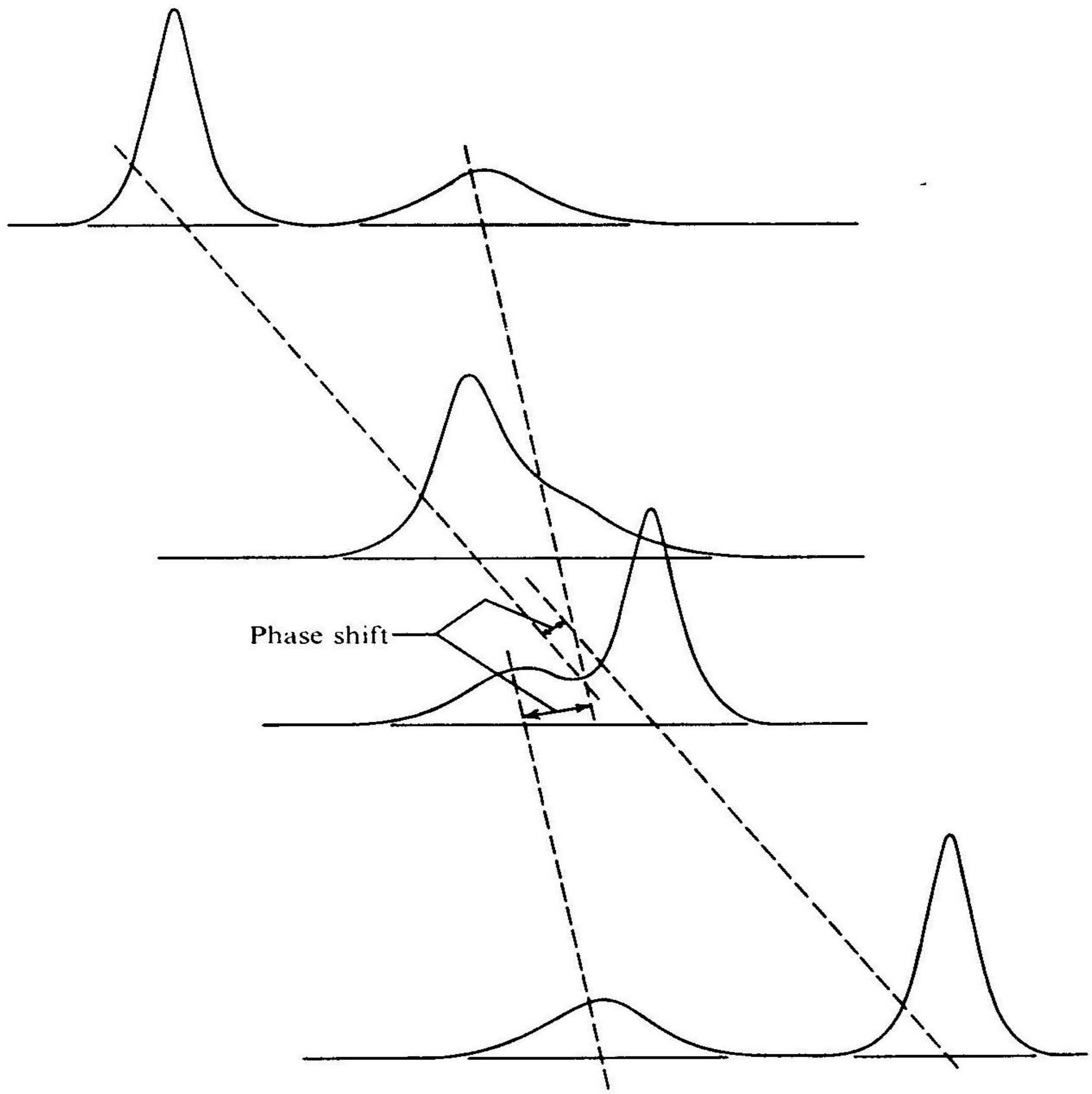


Figure1-7 Interaction Between Two Solitons.

CHAPTER 2

SCHRODINGER AND KDV EQUATIONS

2.1 Inverse Scattering Transform

A remarkable method, which was discovered by Gardner, Greene, Kruskal and Miura [8,9] (abbr. GGKM) to solve nonlinear partial differential equations, has been developed quite rapidly during the past thirty years or so. This method is known as the inverse scattering transform method, which can be used to produce stable solutions that are known as solitons [18,35].

The philosophy behind the inverse scattering transform comes from associating the initial solution of the nonlinear partial differential equation to a linear eigenvalue equation whose eigenvalues are constants. Moreover, the initial solution for the nonlinear partial differential equation is known as the potential in the eigenvalue equation. The idea simply, is to map the potential to the scattering data of the eigenvalue equation and the evolution of these data can be easily computed by using certain evolution equations [18,19,25]. Finally, the inverse scattering techniques are applied to get the solution for the nonlinear differential equation at any time. Moreover, we will see that the nonlinear differential equation has a unique solution [13,33] for potential that decays sufficiently rapidly as $|x| \to \infty$.

Definition 2.1

A C^∞ function $u(x,t)$ on $\Re$ (where t is regarded as a smoothly varying parameter) is said to decay sufficiently rapidly if $u(x,t)$ and all its x-derivatives go to zero as $|x| \to \infty$.

2.2 Sturm-Liouville Problem

The differential equation

$$\frac{d^2y}{dx^2} + \left[\lambda - u(x)\right]y = 0 \qquad a < x < b, \tag{2.1}$$

with the boundary conditions imposed at $x = a$ and $x = b$ (either or both of which may be at infinity) appears quite often in applied mathematics [19]. Equation (2.1) has been extensively studied during the last three decades and it is known as Schrödinger equation, in the context of quantum theory. Moreover, the function $u(x)$ is known as the potential for the Schrödinger equation. For a given potential $u(x)$ the above equation (2.1) gives specific solutions $y(x)$ (eigenfunctions) depending on λ. The dependence of the solution upon the parameter λ, and the dependence of the parameter upon the boundary conditions, is known (SL) problem [1,19].

Example 2.1

One of the simplest examples of (SL) problem is to set $u(x) = 0$.

$$\frac{d^2 y}{dx^2} + \lambda y = 0,$$

with the following boundary conditions:

$$y(0) = y(b) = 0.$$

Solution:

It is clear that $\lambda \leq 0$ is not an eigenvalue. However, when $\lambda > 0$, then the solution of the (SL) problem is given by

$$y(x) = A\cos\left(\sqrt{\lambda}\right)x + B\sin\left(\sqrt{\lambda}\right)x.$$

First applying the boundary condition $y(0) = 0$ we set $A = 0$. Thus

$$y(x) = B\sin\left(\sqrt{\lambda}\right)x.$$

Applying the second condition $y(b) = 0$ the above equation yields,

$$y(b) = B\sin\left(\sqrt{\lambda}\right)b = 0$$

Since $B = 0$ gives a trivial solution, we set $B \neq 0$ and choose

$$\left(\sqrt{\lambda}\right)b = n\pi \quad \Rightarrow \lambda = \left[\frac{n\pi}{b}\right]^2 \qquad where\ n = 1, 2, 3, \ldots$$

In the light of above, the corresponding eigenfunctions are

$$y_n(x) = B\sin\left(\frac{n\pi}{b}\right)x \qquad n = 1, 2, 3, \ldots$$

In fact, there are relatively few functions $u(x)$ for which the (SL) problem may be solved in terms of standard transcendental functions [5,6,19, 35].

Example 2.2

Another example is to set the potential $u(x) = -2\text{sech}^2(x)$.

Solution:

The corresponding (SL) problem is

$$\frac{d^2y}{dx^2} + \left[\lambda + 2\text{sech}^2(x)\right]y = 0, \qquad -\infty < x < +\infty.$$

Let $s = \tanh(x)$. This transformation map $(-\infty, \infty)$ for x to $[-1,1]$ for s

$$y_x = y_s(1-s^2),$$
$$y_{xx} = y_{ss}(1-s^2)^2 - 2s(1-s^2)y_s.$$

In the transformed variables, the (SL) problem takes the form

$$\frac{d}{ds}\left[\left(1-s^2\right)\frac{dy}{ds}\right] + \left[2 + \frac{\lambda}{1-s^2}\right]y = 0, \qquad -1 < s < 1.$$

Comparing the above equation with the generalized Legender equation

$$\frac{d}{d\xi}\left[\left(1-\xi^2\right)\frac{d\psi}{d\xi}\right] + \left[\ell(\ell+1) - \frac{m^2}{1-\xi^2}\right]\psi = 0.$$

It is easy to find that

$$\ell(\ell+1) = 2; \qquad \lambda = -m^2; \qquad \ell \geq 0, \qquad 0 < |m| \leq \ell.$$

$$\therefore \ell = 1 \quad \Rightarrow \quad \lambda_1 = -1.$$

It is clear that this is the only eigenvalue and the corresponding eigenfunction can be found from the Legender polynomials as follows:

$$y(x) = (s^2-1)^{\frac{1}{2}}\frac{d}{ds}p_1(s) = (s^2-1)^{\frac{1}{2}}\frac{d}{ds}(s) = s^2-1)^{\frac{1}{2}} = \text{sech}(x).$$

$$\therefore \lambda_1 = -1; \qquad y(x) = \text{sech}(x).$$

Example 2.3

If we chose our initial distribution $u(x) = -6\text{sech}^2(x)$, we get two eigenvalues for the (SL) problem

Solution:

In this case, the corresponding (SL) problem is

$$\frac{d^2y}{dx^2} + \left[\lambda + 6\text{sech}^2(x)\right]y = 0; \qquad -\infty < x < +\infty.$$

Using the transformation, $s = \tanh(x)$ we map $(-\infty, \infty)$ for x to $[-1,1]$ for s

$$y_x = y_s(1-s^2),$$
$$y_{xx} = y_{ss}(1-s^2)^2 - 2s(1-s^2)y_s.$$

In the light of above, (SL) problem takes the form

$$\frac{d}{ds}\left[\left(1-s^2\right)\frac{dy}{ds}\right] + \left[6 + \frac{\lambda}{1-s^2}\right]y = 0; \qquad -1 < s < 1.$$

Comparing the above equation with the generalized Legender equation

$$\frac{d}{d\xi}\left[\left(1-\xi^2\right)\frac{d\psi}{d\xi}\right] + \left[\ell(\ell+1) - \frac{m^2}{1-\xi^2}\right]\psi = 0,$$

we get:

$$\ell(\ell+1) = 6; \qquad \lambda = -m^2; \qquad \ell \geq 0, \qquad 0 < |m| \leq \ell.$$

$$\ell = 2 \;\Rightarrow \lambda_1 = -1 \quad \& \quad \lambda_2 = -2^2 \Rightarrow \lambda_2 = -4$$

In order to find the corresponding eigenfunctions, we use the associated Legender polynomials

$$y_1(x) = P_2^1(s)$$
$$y_1(x) = (s^2-1)^{\frac{1}{2}} \frac{d}{ds} p_1(s)$$
$$= (s^2-1)^{\frac{1}{2}} \frac{d}{ds}(\frac{1}{2}(3s^3-1))$$
$$= 3s(s^2-1)^{\frac{1}{2}} = 3i(1-s^2)^{\frac{1}{2}}(s)$$
$$= 3i\text{sech}(x)\tanh(x).$$

$$y_2(x) = P_2^2(s)$$
$$y_2(x) = (s^2-1)\frac{d^2}{ds^2} p_2(s)$$
$$= (s^2-1)\frac{d}{ds}(3s)$$
$$= 3(s^2-1)$$
$$= -3\text{sech}^2(x).$$

$$\therefore \ \lambda_1 = -1, \quad y_1(x) = 3i\,\text{sech}(x)\tanh(x) \quad \& \quad \lambda_2 = -4, \quad y_2(x) = 3\,\text{sech}^2(x).$$

Remark: As a result of these examples, we can conclude that the (SL) problem will generally has N-eigenvalues and N-eigenfunctions if the initial potential would be of this form $$u(x) = -n(n+1)\text{sech}^2(x).$$

2.3 Analysis for the Scattering Data

In this section, we discuss the scattering data in some details. We start with eigenvalues and we will show that the eigenvalues are constants. Then, we talk about the normalizing constants, reflection coefficient and transmission coefficient.

2.3.1 The Eigenvalues

We now turn attention to developing our working definitions and analysis by introducing the following theorem. This theorem will give us a remarkable result.

Theorem 2.1

If $u(x,t)$ is a solution of $u_t - 6uu_x + u_{xxx} = 0$, $-\infty < x < \infty$ $t > 0$; $u(x,t=0) = u_0(x)$ and if λ is an eigenvalue of $-\frac{\partial^2 w(x,t)}{\partial x^2} + u_0(x)w(x,t) = \lambda(t)w(x,t)$, $w_0(\pm\infty) = 0$, $w_x(\pm\infty) = 0$, then λ is a constant independent of time.

Proof:

Define $\psi = w_t + u_x w - 2(u+2\lambda)w_x$, then

$$w\psi_x - w_x\psi = w^2 \frac{\partial}{\partial x}(\frac{\psi}{w})$$

$$= w^2\left(\frac{w(w_{tx} + u_{xx}w + u_x w - 2u_x w_x - 2(u+2\lambda)w_{xx}) - w_x(w_t + u_x w - 2(u+2\lambda)w_x)}{w^2}\right)$$

$$= ww_{tx} + u_{xx}w^2 + u_x w^2 - 2u_x w_x w - 2(u+2\lambda)w_{xx}w - w_x(w_t + u_x w - 2(u+2\lambda)w_x)$$

$$= ww_{tx} + u_{xx}w^2 + u_x w^2 - 2u_x w_x w - 2(u+2\lambda)w_{xx}w - w_x w_t - u_x w_x w + 2(u+2\lambda)w_x^2$$

$$= ww_{tx} + u_{xx}w^2 + u_x w^2 - 2u_x w_x w - w_x w_t + u_x w_x w - 2(u+2\lambda)(w_{xx}w - w_x^2).$$

$$\therefore w\psi_x - w_x\psi$$

$$= ww_{tx} + u_{xx}w^2 + u_x w^2 - 2u_x w_x w - w_x w_t + u_x w_x w - 2(u+2\lambda)(w_{xx}w - w_x^2).$$

$$\frac{\partial}{\partial x}(w\psi_x - w_x\psi)$$

$$= \frac{\partial}{\partial x}\left(ww_{tx} + u_{xx}w^2 + u_x w^2 - 2u_x w_x w - w_x w_t + u_x w_x w - 2(u+2\lambda)(w_{xx}w - w_x^2)\right)$$

$$= ww_{xxt} + w_x w_{xt} - w_{xx}w_t - w_x w_{xt} + u_{xxx}w^2 + 2ww_x u_{xx} - 2u_x w_x^2 - 2u_x ww_{xx}$$

$$+2u_{xx}ww_x - 2u_x(w_{xx}w - w_x^2) - 2(u+2\lambda)(w_x w_{xx} + ww_{xxx} - 2w_x w_{xx})$$

$$= w^2\frac{\partial}{\partial t}\left(\frac{w_{xx}}{w}\right) + w^2 u_{xxx} - 4u_x ww_{xx} - 2(u+2\lambda)(ww_{xxx} - w_x w_{xx})$$

$$= w^2\frac{\partial}{\partial t}(u-\lambda) + w^2 u_{xxx} - 4u_x w^2(u-\lambda) - 2(u+2\lambda)w^2\frac{\partial}{\partial x}\left(\frac{w_{xx}}{w}\right)$$

$$= w^2(u_t - \lambda_t + u_{xxx} - 6uu_x).$$

Since

$$u_t + u_{xxx} - 6uu_x = 0,$$

then

$$w^2\lambda_t + \frac{\partial}{\partial x}\left(w\psi_x - w_x\psi\right) = 0.$$

Integrating both sides over the real line we get

$$\lambda_t \int_{-\infty}^{\infty} w^2 dx = -\left[w\,\psi_x - w_x\psi\right]_{-\infty}^{\infty} = 0.$$

Since the eigenfunctions are square integrable, then

$$\lambda_t \int_{-\infty}^{\infty} w^2 dx = 0 \Rightarrow \lambda_t = 0,$$

$$\therefore\ \frac{d\lambda}{dt} = 0. \qquad (2.2)$$

Therefore, the eigenvalues λ are independent of time and this completes the proof.

2.3.2 Normalization Constant

Second, we study the evolution of the normalization constant $c_n(t)$ for the eigenfunction $\phi_n(x)$. By definition; the normalizing constant is defined as:

$$c_n(t) = \left\{ \int_{-\infty}^{\infty} \phi_n^2(x,t)\, dx \right\}^{-1}.$$

In order to study the evolution of the normalizing constant, we differentiate both sides with respect to the time

$$\frac{dc^{-1}(t)}{dt} = \frac{d}{dt}\left(\int_{-\infty}^{\infty} \phi^2(x)\, dx \right) = \int_{-\infty}^{\infty} 2\phi\phi_t dx = 2\int_{-\infty}^{\infty} \phi\phi_t dx\,.$$

By substituting the evolution equation into the integrand, we get

$$\frac{dc^{-1}(t)}{dt} = 2\int_{-\infty}^{\infty} \phi\phi_t dx$$

$$=2\int_{-\infty}^{\infty}\phi\left(u_x\phi-2\left(u+2\lambda\right)\phi_x+4k^3\phi\right)\,dx$$

$$=2\int_{-\infty}^{\infty}\left(2\left(u+2\lambda\right)\phi_x\phi-u_x\phi^2-4k^3\phi^2\right)dx$$

$$=2\int_{-\infty}^{\infty}\left(4\left(\phi_{xx}-\lambda\phi\right)\phi_x+4\lambda\phi\phi_x-\left(-u\phi^2\right)_x\right)dx-8k^3\int_{-\infty}^{\infty}\phi^2dx$$

$$=2\left[2\phi_x^2+2\lambda\phi^2+2\lambda\phi^2-u\phi^2\right]_{-\infty}^{\infty}-8k^3c^{-1}(t)$$

$$=-8k^3c^{-1}(t).$$

The solution for the above equation gives us the equation of evolution for the normalizing constant.

$$c_n(t)=Ae^{8k_n^3t}\Rightarrow c_n(0)=A,$$

$$c_n(t)=c_n(0)e^{8k_n^3t}. \qquad (2.3)$$

2.3.3 Transmission and Reflection Coefficients

Since $\lambda>0$ gives rise to unbound state [5,6,35,37], then we have

$$\frac{\psi}{w}=\text{ Function of time only}$$

$$=\left[\frac{w_t+u_xw-2(u+2\lambda)w_x}{w}\right].$$

$$\therefore\lim_{x\to\infty}\left(\frac{\psi}{w}\right)=\left[\frac{b_te^{jkx}+0-2(0+2\lambda)ik(-1e^{-jkx}+be^{jkx})}{e^{-jkx}+be^{jkx}}\right]_{x=\infty}.$$

Whence, in order that these exponential functions to vanish, we must equate the coefficients of e^{ikx} and e^{-ikx}. For the seek of simplicity, we may first rewrite the above expression as

$$= \left[\frac{(b_t - 4\lambda ikb)e^{ikx} + (4\lambda ik)e^{-ikx})}{e^{-kx} + be^{ikx}} \right]_{x=\infty}$$

$$= \left[\frac{Ae^{ikx} + Be^{-ikx}}{Ce^{ikx} + De^{ikx}} \right] = \alpha \Rightarrow (A - \alpha C)e^{ikx} + (B - \alpha D)e^{-ikx} = 0.$$

Since e^{ikx} and e^{-ikx} are linearly independent then

$$(B - \alpha D) = 0 \Rightarrow 4ik\lambda - \alpha = 0 \Rightarrow \alpha = 4ik\lambda .$$

In order that the second term to vanish, the coefficient must equal to zero

$$(A - \alpha C) = 0 \Rightarrow (b_t - 4\lambda ikb) - [4ik\lambda](b) = 0 \Rightarrow b_t = 8ik^3 b.$$

$$\therefore b(k,t) = b(k,0)e^{(8ik)t} \qquad \text{for all} \quad t > 0 . \tag{2.4}$$

Similarly

$$\frac{\psi}{w} = \text{Function of time only}$$

$$= \left[\frac{w_t + u_x w - 2(u + 2\lambda)w_x}{w} \right]_{x=-\infty}$$

$$= \left[\frac{a_t e^{-ikx} + 0 - 4\lambda(-ikae^{-ikx})}{ae^{-ikx}} \right]_{x=-\infty}$$

$$= \frac{a_t}{a} + 4ik^3 = 4ik^3.$$

$$\therefore a_t = 0.$$

$$\therefore a(k,t) = a(k,0), \qquad \text{for all } t > 0 . \tag{2.5}$$

and the relationship between the reflection coefficient and the transmission coefficient is given by the relation

$$|T|^2 + |R|^2 = 1 . \tag{2.6}$$

Finally, from the above theorem, we can state the following theorem which summarizes our analysis for the scattering data.

Theorem 2.2

If $C_m(0)$, $R(k,0)$, $T(k,0)$ are given as above then

$$\begin{cases} C_m(t) = C_m(0)\, exp(4k_m^3 t) \\ R(k,t) = R(k,0)\, exp(8ik^3 t) \\ T(k,t) = T(k,0) \end{cases}$$

where $C_m(0) = \lim_{x \to \infty} \phi_m(x,0)\, exp(4k_m x)$, $R(k,0)$ and $T(k,0)$ are obtained from the initial data for the KdV equation $u(x,t=0) = u(x)$.

CHAPTER 3

INVERSE SCATTERING TRANSFORM

3.1 Inverse Scattering Transform for KdV Equation

The scattering data $w(x,t)$ can be found by solving the scattering problem [19]. However, the inverse scattering transform is used to find the potential $u(x,t)$ from the scattering data [5]. In fact, during 1955 a certain procedure was established by both Gel'fand and Levitan to recover the potential $u(x,t)$ by solving a linear integral equation [12,28,34]:

$$u(x,t) = -2\frac{\partial}{\partial x}K(x,x,t), \tag{3.1}$$

where $K(x,x,t)$ is the solution for the Gel'fand and Levitan equation

$$K(x,y,t) + B(x+y,t) + \int_x^\infty B(x+y,t)K(x,z,t)dz = 0, \tag{3.2}$$

where $B(x+y,t)$ is called the integral kernel defined by

$$B(\xi,t)=\sum_{m=1}^{N}c_m^2(t)\exp(-k_m\xi)+\frac{1}{2\pi}\int_{-\infty}^{\infty}R(k,t)\exp(ik\xi))dk. \qquad (3.3)$$

Using the theorem (2.2) from the previous chapter, we can rewrite $B(\xi,t)$ as:

$$B(\xi,t)=\sum_{m=1}^{N}c_m^2(0)\exp(8k_m^2t-k_m\xi)+\frac{1}{2\pi}\int_{-\infty}^{\infty}R(k,0)\exp(i(8k^3t+k\xi))dk\ , \qquad (3.4)$$

where the terms inside the kernel are defined as [35] :

- $k_m^2=-\lambda_m$ where $m=1,2,3,...,N.$
- $C_m(t)=\lim_{x\to\infty}\psi_m(x,t)\exp(k_m x).$
- $|T|^2+|R|^2=1.$

The beauty of the inverse scattering transform is to solve a nonlinear partial differential equation by solving two linear equations. In fact, this mentioned point is a remarkable discovery in the field of mathematics and the nonlinear wave research. To be more precise, with the above equations, the KdV equation can be easily solved by solving two linear equations. One of these equations is the eigenvalue problem for a time independent Schrödinger equation (1.1) while the second is the above Gel'fand and Levitan equation (3.2). To end this section, we will summarize what we are trying to do in the coming three sections. In section 3.2, we will give a simple sketch for the method of inverse scattering transform. In section 3.3, we will derive the kernel for the n-bounded states with vanishing reflection coefficient. Finally, some examples will be provided to show how this method is used to solve the KdV equation.

3.2 Sketch for the Inverse Scattering Transform

- Sketch for the Inverse Scattering Transform (IST)

Gel'fand and Levitan equation

$$K(x,y,t)+B(x+y,t)+\int_{x}^{\infty}B(x+y,t)K(x,z,t)\,dz=0.$$

$B(\zeta,t)$

Solve $\psi_{xx}+\left(-u_0(x)+\lambda\right)\psi=0$ using special functions. Find $k_m^2=-\lambda_m,\psi_m(x),R(k,0),$ $C_m(0),C_m(t),R(k,t)$

Solve the Gel'fand-Lavitan equation by seperation of varibles to get $K(x,y,t)$

$u_0(x)$ → → $u(x,t)$

Standard KdV equation

$u_t-6uu_x+u_{xxx}=0$

3.3 The Kernel for N-bound States with Vanishing Reflection Coefficient

When the initial solution is reflectionless, then the integral part inside the Gel'fand-Levitan equation will disappear from the equation. As a result of that, we try to look for a solution which is separable

$$K(x:y,t)=\sum_{n=1}^{N}k_n(x,t)g_n(y). \tag{3.5}$$

By substituting the above assumption into Gel'fand-Levitan equation, we get

$$K(x:y,t)+B(x+y;t)+\int_x^\infty K(x:s,t)B(s+y;t)ds=0,$$

$$\sum_{n=1}^{N}k_n(x,t)g_n(y)+\sum_{n=1}^{N}f_n(x,t)g_n(y)+\int_x^\infty\left\{\sum_{n=1}^{N}k_n(x,t)g_n(s)\sum_{n=1}^{N}f_n(s,t)g_n(y)\right\}ds=0. \tag{3.6}$$

By comparing the coefficients of $g_n(y)$, we obtain the following system

$$k_1(x,t)+f_n(x,t)+\int_x^\infty\sum_{n=1}^{N}k_n(x,t)g_n(s)f_1(s,t)ds=0$$

$$k_1(x,t)+f_n(x,t)+\int_x^\infty c_1^2(t)\sum_{n=1}^{N}k_n(x,t)e^{-(k_1+k_n)s}ds=0$$

$$k_1(x,t)+f_n(x,t)+c_1^2(t)\sum_{n=1}^{N}k_n(x,t)\int_x^\infty e^{-(k_1+k_n)s}ds=0$$

$$k_1(x,t)+f_n(x,t)+c_1^2(t)\sum_{n=1}^{N}k_n(x,t)\left[\frac{e^{-(k_1+k_n)s}}{k_1+k_n}\right]_\infty^x=0$$

$$k_1(x,t)+f_n(x,t)+c_1^2(t)\sum_{n=1}^{N}k_n(x,t)\frac{e^{-(k_1+k_n)x}}{k_1+k_n}=0. \tag{3.7}$$

whence, we can in general write the general formula

$$k_m(x,t)+f_n(x,t)+c_m^2(t)\sum_{n=1}^{N}k_n(x,t)\frac{e^{-(k_m+k_n)x}}{k_m+k_n}=0 \qquad \text{where } 1\le \text{m} \le \text{N}. \tag{3.8}$$

The above equation can be written in matrix notation as

$$Mk + f = 0 \tag{3.9}$$

where k and f are column vectors with entries given by

$$k = \begin{pmatrix} k_1 \\ k_2 \\ : \\ \\ k_n \end{pmatrix}, \qquad f = \begin{pmatrix} f_1 \\ f_2 \\ : \\ \\ f_n \end{pmatrix},$$

where M is an $N \times N$ matrix with entries given by

$$M_{ij} = \delta_{ij} + c_i^2(t) \frac{e^{-(k_i + k_j)x}}{k_i + k_j},$$

where δ_{ij} is the Kronecker delta which is given by

$$\delta_{ij} = \begin{cases} 1 & if \;\; i = j \\ 0 & if \;\; i \neq j \end{cases}.$$

Now solving the matrix equation (3.9), we obtain

$$k = -M^{-1} f, \tag{3.10}$$

whence, the kernel can be written as

$$K(x : x, t) = \sum_{n=1}^{N} k_n(x, t) g_n(x) = -M^{-1} f g = -g^T M^{-1} f. \tag{3.11}$$

Since

$$\frac{\partial}{\partial x}(M_{ij}) = \frac{\partial}{\partial x}(\delta_{ij} + c_i^2(t) \frac{e^{-(k_i + k_j)x}}{k_i + k_j}) = 0 - c_i^2(t) e^{-(k_i + k_j)x} = -c_i^2(t) e^{-k_i x} e^{-k_j x}.$$

$$\frac{\partial}{\partial x}(M_{ij}) = -f_i(x, t) g_j(x). \tag{3.12}$$

then equation (3.12) and equation (3.11) give us the final form of the kernel

$$K(x : x, t) = \sum_{n=1}^{N} k_n(x, t) g_n(x) = -M^{-1} f g = M^{-1} \frac{\partial}{\partial x}(M_{ij}). \tag{3.13}$$

3.4 Applications with Vanishing Reflection Coefficient

In this section, we are going to apply the inverse scattering transformation which was described in section 3.2 to show how it works.

3.4.1 One Soliton Solution.

Consider the initial value problem

$$u_t - 6uu_x + u_{xxx} = 0\,,$$

with the initial solution

$$u(x,0) = u_0(x) = -2\text{sech}^2(\text{x}).$$

Solution:

<u>**Step 1** Finding the eigenvalues and eigenfunctions</u>

$$\frac{d^2\psi}{dx^2} + \left[\lambda + 2sech^2(x)\right]\psi = 0 \qquad \psi(\pm\infty) = 0.$$

Let $s = \tan(x)$ this transformation map $(-\infty,\infty)$ for x to $[-1,1]$ for s .

$$\psi_x = \psi_s(1-s^2),$$

$$\psi_{xx} = \psi_{ss}(1-s^2)^2 - 2s(1-s^2)\psi_s\,.$$

Therefore, the (SL) problem becomes

$$\frac{d}{ds}\left[\left(1-s^2\right)\frac{d\psi}{ds}\right] + \left[2 + \frac{\lambda}{1-s^2}\right]\psi = 0 \qquad -1 < s < 1.$$

Comparing this equation with the generalized Legender equation

$$\frac{d}{d\xi}\left[\left(1-\xi^2\right)\frac{d\psi}{d\xi}\right] + \left[\ell(\ell+1) - \frac{m^2}{1-\xi^2}\right]\psi = 0\,,$$

we get

$$\ell(\ell+1) = 2; \qquad \lambda = -m^2; \qquad \ell \ge 0, \qquad 0 < |\text{m}| \le \ell.$$

$$\therefore \ell = 1 \quad \Rightarrow \quad \lambda_1 = -1. \tag{1}$$

It is clear that this is the only eigenvalue and the corresponding eigenfunction can be found from the Legender polynomials as follow:

$$\psi(x) = (s^2 - 1)^{\frac{1}{2}} \frac{d}{ds} p_1(s)$$

$$= (s^2 - 1)^{\frac{1}{2}} \frac{d}{ds}(s) = (s^2 - 1)^{\frac{1}{2}} sech(x).$$

$$\therefore \; \lambda_1 = -1 \quad \psi(x) = sech(x). \tag{2}$$

<u>**Step 2** Normalization of the eigenfunction</u>

$$\int_{-\infty}^{\infty} |\psi_1(x)|^2 \, dx = \int_{-\infty}^{\infty} |sech(x)|^2 \, dx = \int_{-\infty}^{\infty} sech^2(x) dx = tanh(x)\Big|_{-\infty}^{\infty} = 2.$$

Therefore, the normalized eigenfunction is

$$\overline{\psi}_1(x) = \frac{\sqrt{2}}{2} sech(x). \tag{3}$$

<u>**Step 3** Determination of c(0) and c(t)</u>

By using the definition

$$c_1(0) = \lim_{x \to \infty} \left(\overline{\psi}_1(x) \exp(k_1 x) \right)$$

$$= \lim_{x \to \infty} \left(\frac{\sqrt{2}}{2} sech(x) \exp(x) \right)$$

$$= \sqrt{2}. \tag{4}$$

Therefore, the evolution equation for the normalization constant is given by

$$c_1(t) = c_1(0) \exp(4k_1^2 t) = \sqrt{2} \exp(4t). \tag{5}$$

Step 4 Determination of integration kernel

$$B(\xi,t)=\sum_{m=1}^{N}c_m^2(0)\exp(8k_m^2t-k_m\xi)+\frac{1}{2\pi}\int_{-\infty}^{\infty}R(k,0)\exp(i(8k^3t+k\xi))dk$$

$$B(\xi,t)=\sum_{m=1}^{1}c_m^2(0)\exp(8k_m^2t-k_m\xi)$$

$$=2\exp(8t-\xi)$$

$$=2\exp(8t)\exp(-\xi). \tag{6}$$

Step 5 Writing Gel'fand and Levitan equation

$$K(x,y,t)+B(x+y,t)+\int_x^{\infty}B(y+z,t)K(x,z,t)dz=0$$

$$K(x,y;t)+2\exp(8t)\exp(-x-y)+\int_x^{\infty}\left[2\exp(8t)\exp(-z-y)\right]K(x,z,t)dz=0$$

$$K(x,y;t)+2\exp(8t)\exp(-x)\exp(-y)+ \int_x^{\infty}\left[2\exp(8t)\exp(-z)\exp(-y)\right]K(x,z,t)dz=0. \tag{7}$$

We solve the above equation by separation of variables by assuming

$$K(x,y;t)=f(x,t)\exp(-y). \tag{8}$$

Putting (8) into equation (7), we get

$$f(x,t)\exp(-y)+2\exp(8t)\exp(-x)\exp(-y) +\int_x^{\infty}\left[2\exp(8t)\exp(-z)\exp(-y)\left[f(x,t)\exp(-z)\right]dz=0. \tag{9}$$

Comparing coefficients of $\exp(-y)$ in equation (9) gives

$$f(x,t)+2\exp(8t)\exp(-x)+\int_x^{\infty}\left[2\exp(8t)\exp(-z)\left[f(x,t)\exp(-z)\right]dz=0$$

$$\Rightarrow f(x,t)+2\exp(8t-x)+2f(x,t)\exp(8t)\int_x^{\infty}\left[\exp(-2z)dz=0\right.$$

$$\Rightarrow f(x,t)+2\exp(8t-x)+f(x,t)\exp(8t-2x)=0$$

$$\Rightarrow f(x,t) = \frac{-2\exp(8t-x)}{1+\exp(8t-2x)}$$

$$\Rightarrow f(x,t) = \frac{-2\exp(4t-x)\exp(4t)}{1+\exp(8t-2x)}\left(\frac{\exp(x-4t)}{\exp(x-4t)}\right)$$

$$\Rightarrow f(x,t) = -\exp(4t)\frac{2}{\exp(x-4t)+\exp(4t-x)}$$

$$f(x,t) = -\mathrm{e}^{4t}\,\frac{2}{\mathrm{e}^{(x-4t)}+\mathrm{e}^{-(x-4t)}} = -\mathrm{e}^{4t}\,sech(x-4t).$$

$$f(x,t) = -\mathrm{e}^{4t}\,sech(x-4t). \tag{10}$$

By substituting equation (10) into equation (8) we obtain the kernel

$$K(x,y;t) = f(x,t)\exp(-y) = -\mathrm{e}^{4t-y}\,sech(x-4t).$$

Therefore

$$K(x,x;t) = -\mathrm{e}^{4t-x}\,sech(x-4t). \tag{11}$$

<u>Step 6</u> <u>Solution</u>

$$u(x,t) = -2\frac{\partial}{\partial x}K(x,x,t)$$

$$u(x,t) = -2\frac{\partial}{\partial x}\left(-\mathrm{e}^{4t-x}\,sech(x-4t)\right)$$

$$u(x,t) = -2\frac{\partial}{\partial x}\left(\frac{-2e^{4t-x}}{\mathrm{e}^{x-4t}+e^{4t-x}}\right) = -2\frac{\partial}{\partial x}\left(\frac{-2}{\mathrm{e}^{2x-8t}+1}\right)$$

$$u(x,t) = -2\left(\frac{4e^{2x-8t}}{\left(\mathrm{e}^{2x-8t}+1\right)^2}\right) = -2\left(\frac{2}{\mathrm{e}^{(x-4t)}+\mathrm{e}^{-(x-4t)}}\right)^2.$$

We get the exact solution

$$u(x,t) = -2sech^2(x-4t).$$

3.4.2 Two Soliton Solution

Consider the initial value problem

$$u_t - 6uu_x + u_{xxx} = 0,$$

with the following initial solution

$$u(x,0) = u_0(x) = -6sech^2(x).$$

Solution:

<u>**Step 1** Finding the eigenvalues and eigenfunctions</u>

$$\frac{d^2\psi}{dx^2} + \left[\lambda + 6sech^2(x)\right]\psi = 0; \qquad \psi(\pm\infty) = 0.$$

Using the transformation $s = \tan(x)$, this transformation map $(-\infty,\infty)$ for x to $[-1,1]$ for s.

$$\psi_x = \psi_s(1-s^2),$$

$$\psi_{xx} = \psi_{ss}(1-s^2)^2 - 2s(1-s^2)\psi_s.$$

Therefore, the (SL) problem becomes

$$\frac{d}{ds}\left[\left(1-s^2\right)\frac{d\psi}{ds}\right] + \left[6 + \frac{\lambda}{1-s^2}\right]\psi = 0; \qquad -1 < s < 1.$$

Comparing this equation with the generalized Legender equation

$$\frac{d}{d\xi}\left[\left(1-\xi^2\right)\frac{d\psi}{d\xi}\right] + \left[\ell(\ell+1) - \frac{m^2}{1-\xi^2}\right]\psi = 0.$$

We get

$$\ell(\ell+1) = 6; \quad \lambda = -m^2; \quad \ell \geq 0, \quad 0 < |\mathrm{m}| \leq \ell.$$

$$\therefore \ell = 2 \;\Rightarrow\; \lambda_1 = -1 \;\&\; \lambda_2 = -2^2 \Rightarrow \lambda_2 = -4.$$

These are the only two eigenvalues for the (SL) problem and in order to find the corresponding eigenfunctions, we use the associated Legender polynomials:

$$\psi_1(x) = P_2^1(s)$$
$$\psi_1(x) = (s^2-1)^{\frac{1}{2}} \frac{d}{ds} p_1(s)$$
$$= (s^2-1)^{\frac{1}{2}} \frac{d}{ds}(\frac{1}{2}(3s^3-1))$$
$$= 3s(s^2-1)^{\frac{1}{2}} = 3i(1-s^2)^{\frac{1}{2}}(s)$$
$$= 3isech(x)\tanh(x).$$

$$\psi_2(x) = P_2^2(s)$$
$$\psi_2(x) = (s^2-1)\frac{d^2}{ds^2} p_2(s)$$
$$= (s^2-1)\frac{d}{ds}(3s)$$
$$= 3(s^2-1)$$
$$= -3sech^2(x).$$

$$\therefore\ \lambda_1 = -1, \quad \psi_1(x) = 3isech(x)\tanh(x) \quad \& \quad \lambda_2 = -4, \quad \psi_2(x) = 3sech^2(x).$$

Step 2 Normalization of the eigenfunctions

$$\int_{-\infty}^{\infty} |\psi_1(x)|^2 dx = \int_{-\infty}^{\infty} |3isech(x)\tanh(x)|^2 dx = 9\int_{-\infty}^{\infty} sech^2(x)\tanh^2(x)dx = 6.$$

$$\int_{-\infty}^{\infty} |\psi_2(x)|^2 dx = \int_{-\infty}^{\infty} |-3sech^2(x)|^2 dx = 9\int_{-\infty}^{\infty} sech^4(x)dx = 12.$$

Therefore, the normalized eigenfunctions are

$$\overline{\psi}_1(x) = \sqrt{\frac{3}{2}} sech(x)\tanh(x).$$

$$\overline{\psi}_2(x) = \frac{\sqrt{3}}{2} sech^2(x).$$

Step 3 Determination of c(0) and c(t)

By using the definition $\quad c_1(0) = \lim_{x\to\infty}\left(\overline{\psi}_1(x)\exp(k_1 x)\right)$

$$= \lim_{x\to\infty}\left(\sqrt{\frac{3}{2}} sech(x)\tanh(x)\exp(x)\right) = \sqrt{6}.$$

$$c_2(0) = \lim_{x\to\infty}\left(\overline{\psi}_2(x)\exp(k_2 x)\right)$$

$$= \lim_{x\to\infty}\left(\frac{\sqrt{3}}{2} sech^2(x)\exp(2x)\right) = 2\sqrt{3}.$$

Whence, the evolution equation is given by

$$c_1(t) = c_1(0)\exp(4k_1^3 t)$$

$$= \sqrt{6}\exp(4t).$$

$$c_2(t) = c_2(0)\exp(4k_2^3 t)$$

$$= 2\sqrt{3}\exp(32t).$$

<u>**Step 4** Determination of integration</u>

$$B(\xi,t) = \sum_{m=1}^{N} c_m^2(0)\exp(8k_m^2 t - k_m\xi) + \frac{1}{2\pi}\int_{-\infty}^{\infty} R(k,0)\exp(i(8k^3 t + k\xi))dk$$

$$B(\xi,t) = \sum_{m=1}^{2} c_m^2(0)\exp(8k_m^2 t - k_m\xi)$$

$$= 6\exp(8t-\xi) + 12\exp(64t - 2\xi)$$

$$= 6\exp(8t)\exp(-\xi) + 12\exp(64t)\exp(-2\xi).$$

<u>**Step 5** Writing Gel'fand and Levitan equation</u>

$$K(x,y,t) + B(x+y,t) + \int_x^{\infty} B(y+z,t)K(x,z,t)dz = 0$$

$$K(x,y;t) + 6\exp(8t)\exp(-x-y) + 12\exp(64t)\exp(-2x-2y) +$$

$$+\int_x^{\infty}\left[6\exp(8t)\exp(-z-y) + 12\exp(64t)\exp(-2z-2y)\right]K(x,z,t)dz = 0$$

$$K(x,y;t) + 6\exp(8t)\exp(-x)\exp(-y) + 12\exp(64t)\exp(-2x)\exp(-2y)$$

$$+\int_x^{\infty}\left[6\exp(8t)\exp(-z)\exp(-y) + 12\exp(64t)\exp(-2z)\exp(-2y)\right]K(x,z,t)dz = 0.$$

We solve the above equation by separation of variables by assuming

$$K(x,y;t)=f_1(x,t)\exp(-y)+f_2(x,t)\exp(-2y). \tag{16}$$

Putting the assumption into Gel'fand and Levitan equation, we get

$$\begin{aligned}&f_1(x,t)\exp(-y)+f_2(x,t)\exp(-2y)+6\exp(8t)\exp(-x)\exp(-y)\\&+12\exp(64t)\exp(-2x)\exp(-2y)+\int_x^\infty\left[6\exp(8t)\exp(-z)\exp(-y)+\right.\\&\left.12\exp(64t)\exp(-2z)\exp(-2y)\right]\left[f_1(x,t)\exp(-z)+f_2(x,t)\exp(-2z)\right]dz=0.\end{aligned} \tag{17}$$

Comparing coefficients of $\exp(-y)$ and $\exp(-2y)$ in equation (17) gives

$$\Rightarrow f_1(x,t)+6\exp(8t)\exp(-x)+\int_x^\infty\left[6\exp(8t)\exp(-z)\right]\left[f_1(x,t)\exp(-z)+f_2(x,t)\exp(-2z)\right]dz=0$$

$$\Rightarrow f_1(x,t)+\exp(8t)\exp(-x)+6\exp(8t)f_1(x,t)\int_x^\infty\exp(-2z)dz+6\exp(8t)f_2(x,t)\int_x^\infty\exp(-3z)dz=0$$

$$\Rightarrow f_1(x,t)+\exp(8t-x)+3\exp(8t+2x)f_1(x,t)+2\exp(8t+3x)f_2(x,t)=0$$

$$\left[1+3\exp(8t-2x)\right]f_1(x,t)+\left[2\exp(8t-3x)\right]f_2(x,t)=-6\exp(8t-x). \tag{a}$$

The second equation is

$$f_2(x,t)+12\exp(64t)\exp(-2x)+\int_x^\infty\left[12\exp(64t)\exp(-2z)\right]\left[f_1(x,t)\exp(-z)+f_2(x,t)\exp(-2z)\right]dz=0$$

$$f_2(x,t)+12\exp(64t-2x)+12f_1(x,t)\exp(64t)\int_x^\infty\exp(-3z)dz+12f_2(x,t)\exp(64t)\int_x^\infty\exp(-4z)dz=0$$

$$f_2(x,t)+12\exp(64t-2x)+4f_1(x,t)\exp(64t-3x)+3f_2(x,t)\exp(64t-4x)=0$$

$$\left[4\exp(64t-3x)\right]f_1(x,t)+\left[1+3\exp(64t-4x)\right]f_2(x,t)=-12\exp(64t-2x). \tag{b}$$

from (a) and (b) , we have following the system:

$$\begin{cases}\left[1+3e^{8t-2x}\right]f_1(x,t)+\left[2e^{8t-3x}\right]f_2(x,t)=-6e^{8t-x}\\\left[4e^{64t-3x}\right]f_1(x,t)+\left[1+3e^{64t-4x}\right]f_2(x,t)=-12e^{64t-2x}\end{cases}$$

The above system of algebraic equations can be solved by using Cramer's rule

$$f_1(x,t)=\frac{D_{f_1}}{D}, \qquad f_2(x,t)=\frac{D_{f_2}}{D}.$$

where

$$D=\begin{vmatrix}1+3e^{8t-2x} & 2e^{8t-3x}\\ 4e^{64t-3x} & 1+3e^{64t-4x}\end{vmatrix}=\left(1+3e^{8t-2x}\right)\left(1+3e^{64t-4x}\right)-\left(4e^{64t-3x}\right)\left(2e^{8t-3x}\right)$$
$$=1+3e^{64t-4x}+3e^{8t-2x}+9e^{72t-6x}-8e^{72t-6x}$$
$$\therefore D=1+3e^{64t-4x}+3e^{8t-2x}+e^{72t-6x}.$$

$$D_{f_1}=\begin{vmatrix}-6e^{8t-x} & 2e^{8t-3x}\\ -12e^{64t-2x} & 1+3e^{64t-4x}\end{vmatrix}=\left(-6e^{8t-x}\right)\left(1+3e^{64t-4x}\right)-\left(-12e^{64t-2x}\right)\left(2e^{8t-3x}\right)$$
$$=-6e^{8t-x}-18e^{72t-5x}+24e^{72t-5x}$$
$$\therefore D_{f_1}=-6e^{8t-x}+6e^{72t-5x}.$$

$$D_{f_2}=\begin{vmatrix}1+3e^{8t-2x} & -6e^{8t-x}\\ 4e^{64t-3x} & -12e^{64t-2x}\end{vmatrix}=\left(1+3e^{8t-2x}\right)\left(-12e^{64t-2x}\right)-\left(4e^{64t-3x}\right)\left(-6e^{8t-x}\right)$$
$$=-12e^{64t-2x}-36e^{72t-4x}+24e^{72t-4x}$$
$$\therefore D_{f_2}=-12e^{64t-2x}-12e^{72t-4x}.$$

Substituting the above results into equation (16) we obtain

$$K(x,y;t)=f_1(x,t)\exp(-y)+f_2(x,t)\exp(-2y)$$

$$K(x,y;t)=\frac{\left[-6e^{8t-x}+6e^{72t-5x}\right]e^{-y}+\left[-12e^{64t-2x}-12e^{72t-4x}\right]e^{-2y}}{1+3e^{64t-4x}+3e^{8t-2x}+e^{72t-6x}}.$$

Therefore

$$K(x,x;t)=\frac{\left[-6e^{8t-x}+6e^{72t-5x}\right]e^{-x}+\left[-12e^{64t-2x}-12e^{72t-4x}\right]e^{-2x}}{1+3e^{64t-4x}+3e^{8t-2x}+e^{72t-6x}}$$

$$=\frac{\left[-6e^{8t-2x}+6e^{72t-6x}\right]+\left[-12e^{64t-4x}-12e^{72t-6x}\right]}{1+3e^{64t-4x}+3e^{8t-2x}+e^{72t-6x}}$$

$$K(x,x;t)=\frac{-6\left(e^{8t-2x}+e^{72t-6x}+2e^{64t-4x}\right)}{1+3e^{64t-4x}+3e^{8t-2x}+e^{72t-6x}}$$

Step 6 Solution

$$u(x,t) = -2\frac{\partial}{\partial x}K(x,x,t)$$

$$u(x,t) = -2\frac{\partial}{\partial x}\left(\frac{-6\left(e^{8t-2x}+e^{72t-6x}+2e^{64t-4x}\right)}{1+3e^{64t-4x}+3e^{8t-2x}+e^{72t-6x}}\right)$$

$$u(x,t) = -12\frac{N}{\left(1+3e^{64t-4x}+3e^{8t-2x}+e^{72t-6x}\right)^2}$$

$$\begin{aligned} N = &\left(-2e^{8t-2x}-6e^{72t-6x}-8e^{64t-4x}\right)\left(1+3e^{64t-4x}+3e^{8t-2x}+e^{72t-6x}\right) \\ &-\left(e^{8t-2x}+e^{72t-6x}+2e^{64t-4x}\right)\left(-12e^{64t-4x}-6e^{8t-2x}-6e^{72t-6x}\right). \end{aligned}$$

The above expression can be simplified and written as[35]

$$u(x,t) = -12\frac{3+4\cosh(2x-8t)+\cosh(4x-64t)}{\left(3\cosh(x-28t)+\cosh(3x-36t)\right)^2}.$$

Figure 3-1 below shows the movement of the two soliton solution through different values of time.

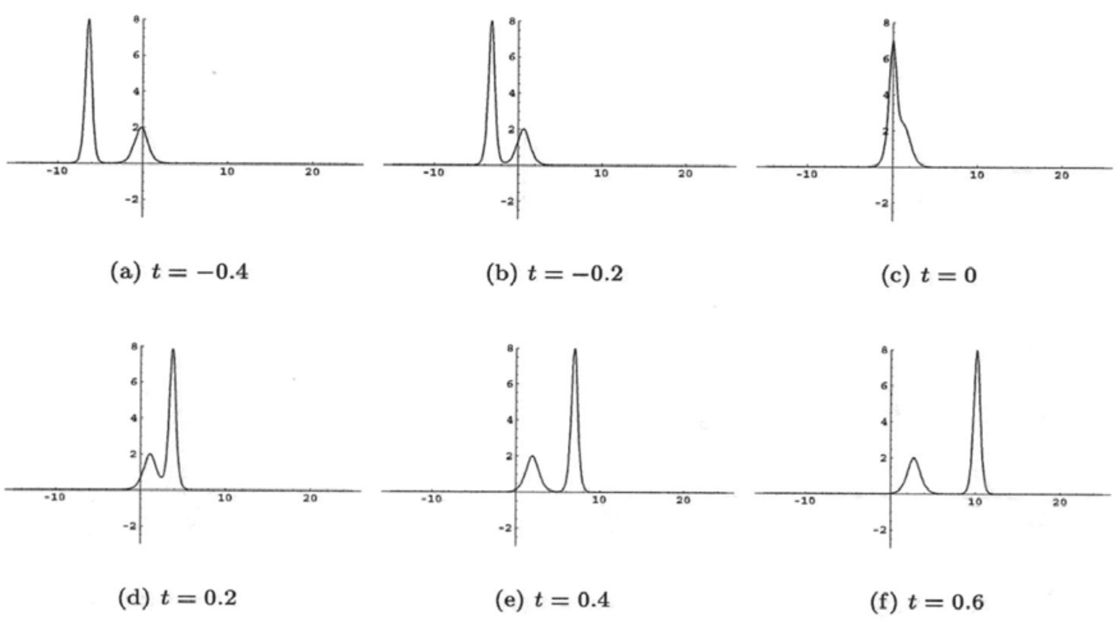


Figure 3-1 Two Soliton Solution for the KdV Equation at Different Values of Time.

3.4.3 Three Soliton Solution

Consider the initial value problem

$$u_t - 6uu_x + u_{xxx} = 0,$$

with the following initial solution

$$u(x,0) = u_0(x) = -12sech^2(x).$$

Solution:

<u>**Step 1** Finding eigenvalues and eigenfunctions.</u>

$$\frac{d^2\psi}{dx^2} + \left[\lambda + 12sech^2(x)\right]\psi = 0 \qquad \psi(\pm\infty) = 0$$

Using the transformation, $s = \tan(x)$, the transformation map $(-\infty,\infty)$ for x to $[-1,1]$

for s, $\psi_x = \psi_s(1-s^2)$,

$$\psi_{xx} = \psi_{ss}(1-s^2)^2 - 2s(1-s^2)\psi_s.$$

Thus, the (SL) problem becomes

$$\frac{d}{ds}\left[\left(1-s^2\right)\frac{d\psi}{ds}\right] + \left[12 + \frac{\lambda}{1-s^2}\right]\psi = 0; \qquad -1 < s < 1.$$

Comparing this equation with the generalized Legender equation

$$\frac{d}{d\xi}\left[\left(1-\xi^2\right)\frac{d\psi}{d\xi}\right] + \left[\ell(\ell+1) - \frac{m^2}{1-\xi^2}\right]\psi = 0.$$

We get

$$\ell(\ell+1) = 12; \quad \lambda = -m^2; \quad \ell \geq 0, \quad 0 < |\mathrm{m}| \leq \ell.$$

$$\therefore \ell = 3 \;\Rightarrow\; \lambda_1 = -1, \;\&\; \lambda_2 = -2^2 \Rightarrow \lambda_2 = -4, \;\&\; \lambda_3 = -3^2 \Rightarrow \lambda_3 = -9.$$

These are the only three eigenvalues for the (SL) problem and in order to find the corresponding eigenfunctions, we use the associated Legender polynomials:

$$\psi_1(x) = P_3^1(s)$$

$$\psi_1(x) = (s^2-1)^{\frac{1}{2}} \frac{d}{ds} p_3(s) = (s^2-1)^{\frac{1}{2}} \frac{d}{ds}(\frac{1}{2}(5s^3-3z))$$

$$= \frac{3}{2}(s^2-1)^{\frac{1}{2}}(5s^2-1) = \frac{3}{2} sech(x)(4-5sech^2(x)).$$

$$\psi_2(x) = P_3^2(s)$$

$$\psi_2(x) = (s^2-1)\frac{d^2}{ds^2} p_3(s) = (s^2-1)\frac{d}{ds}(\frac{5s^2-1}{2})$$

$$= 3(s^2-1)(5s) = 15sech^2(x)tanh(x).$$

$$\psi_3(x) = P_3^3(s)$$

$$\psi_3(x) = (s^2-1)^{\frac{3}{2}} \frac{d^3}{ds^3} p_3(s) = (s^2-1)\frac{d}{ds}(5s)$$

$$= 15(s^2-1)^{\frac{3}{2}} = 15sech^3(x).$$

Thus, the three eigenfunctions:

$$\psi_1(x) = \frac{3}{2} sech(x)(4-5sech^2(x)).$$

$$\psi_2(x) = 15sech^2(x)tanh(x).$$

$$\psi_3(x) = 15sech^3(x).$$

<u>Step 2</u> <u>Normalization of eigenfunctions</u>

$$\int_{-\infty}^{\infty} |\psi_1(x)|^2 dx = \int_{-\infty}^{\infty} \left| \frac{3}{2} sech(x)(4-5sech^2(x)) \right|^2 dx = 12.$$

$$\int_{-\infty}^{\infty} |\psi_2(x)|^2 dx = \int_{-\infty}^{\infty} \left| 15\sec h^2(x)\tanh(x) \right|^2 dx = 60.$$

$$\int_{-\infty}^{\infty} |\psi_2(x)|^2 dx = \int_{-\infty}^{\infty} \left| 15sech^3(x) \right|^2 dx = 240.$$

Therefore, the normalized eigenfunctions are

$$\overline{\psi}_1(x) = \frac{\sqrt{3}}{4} sech(x)(4 - 5sech^2(x)).$$

$$\overline{\psi}_2(x) = \frac{\sqrt{15}}{2} sech^2(x)tanh(x).$$

$$\overline{\psi}_3(x) = \frac{\sqrt{15}}{4} sech^3(x).$$

<u>**Step 3** Determination of c(0) and c(t)</u>

$$c_1(0) = \lim_{x\to\infty}\left(\overline{\psi}_1(x)\exp(k_1 x)\right)$$

$$= \lim_{x\to\infty}\left(\frac{\sqrt{3}}{4} sech(x)(4 - 5sech^2(x))\exp(x)\right) = 2\sqrt{3}.$$

$$c_2(0) = \lim_{x\to\infty}\left(\overline{\psi}_2(x)\exp(k_2 x)\right)$$

$$= \lim_{x\to\infty}\left(\frac{\sqrt{15}}{2} sech^2(x)tanh(x)\exp(2x)\right) = 2\sqrt{15}.$$

$$c_3(0) = \lim_{x\to\infty}\left(\overline{\psi}_3(x)\exp(k_3 x)\right)$$

$$= \lim_{x\to\infty}\left(\frac{\sqrt{15}}{4} sech^3(x)\exp(3x)\right) = 4\sqrt{15}.$$

Using the definition, the evolution equations for the normalization constants are

$$c_1(t) = c_1(0)\exp(4k_1^3 t) = 2\sqrt{3}\exp(4t).$$

$$c_2(t) = c_2(0)\exp(4k_2^3 t) = 2\sqrt{15}\exp(32t).$$

$$c_3(t) = c_3(0)\exp(4k_3^3 t) = 4\sqrt{15}\exp(108t).$$

Step 4 Determination of integration kernel

$$B(\xi,t)=\sum_{m=1}^{N}c_m^2(0)\exp(8k_m^2t-k_m\xi)+\frac{1}{2\pi}\int_{-\infty}^{\infty}R(k,0)\exp(i(8k^3t+k\xi))dk$$

$$B(\xi,t)=\sum_{m=1}^{3}c_m^2(0)\exp(8k_m^3t-k_m\xi)$$

$$=12\exp(8t-\xi)+60\exp(64t-2\xi)+240\exp(216t-3\xi)$$

$$=12e^{8t}e^{-\xi}+60e^{64t}e^{-2\xi}+240e^{210t}e^{-3\xi}.$$

Step 5 Finding Gel'fand and Levitan equation

$$K(x,y,t)+B(x+y,t)+\int_{x}^{\infty}B(y+z,t)K(x,z,t)dz=0$$

$$K(x,y;t)+12e^{8t}e^{-x}e^{-y}+60e^{64t}e^{-2x}e^{-2y}+240e^{210t}e^{-3x}e^{-3y}+$$
$$+\int_{x}^{\infty}\left[12e^{8t}e^{-z}e^{-y}+60e^{64t}e^{-2z}e^{-2y}+240e^{210t}e^{-3z}e^{-3y}\right]K(x,z,t)dz=0$$

We solve the above equation by separation of variables by assuming

$$K(x,y;t)=f_1(x,t)e^{-y}+f_2(x,t)e^{-2y}+f_3(x,t)e^{-3y}. \tag{18}$$

Using the above assumption into Gel'fand and Levitan equation, we get

$$f_1(x,t)e^{-y}+f_2(x,t)e^{-2y}+f_3(x,t)e^{-3y}+12e^{8t}e^{-x}e^{-y}+60e^{64t}e^{-2x}e^{-2y}$$
$$+240e^{210t}e^{-3x}e^{-3y}+\int_{x}^{\infty}\left[12e^{8t}e^{-z}e^{-y}+60e^{64t}e^{-2z}e^{-2y}++240e^{210t}e^{-3z}e^{-3y}\right]$$
$$\left[f_1(x,t)e^{-z}+f_2(x,t)e^{-2z}+f_3(x,t)e^{-3z}\right]dz=0.$$

First comparing coefficients of e^{-y}, e^{-2y} and e^{-3y}, and then integrating by part we obtain the following system

$$f_m(x,t)+\sum_{n=1}^{3}c_n^2(t)f_n(x,t)\frac{\exp\left[-(k_m+k_n)x\right]}{(k_m+k_n)}=\exp(-k_mx)\quad \text{where}\quad m=1,2,3.$$

Using Cramer's rule the solution for the above system becomes,

$$f_1(x,t)=\frac{\Delta_1}{\Delta} \tag{19}$$

$$f_2(x,t)=\frac{\Delta_2}{\Delta} \tag{20}$$

$$f_3(x,t)=\frac{\Delta_3}{\Delta} \tag{21}$$

where

$$\Delta_1=\exp(5x)-5\exp(216t-x)+\exp(280t-5x)-5\exp(64t+x).$$

$$\Delta_2=\exp(4x)-2\exp(216t-2x)+2\exp(8t+2x)-\exp(224t-4x).$$

$$\Delta_3=\exp(3x)+3\exp(64t-x)+3\exp(8t+x)-5\exp(72t+3x).$$

$$\Delta=\exp(6x)+10\exp(216t)+15\exp(64t+2x)+6\exp(280t-4x)$$
$$+6\exp(8t+4x)+15\exp(224t-2x)+10\exp(72t+2x)+\exp(288t-6x).$$

Substituting equations (19), (20) and (21) into equation (18) and then using equation (3.1), we obtain the following solution

$$u(x,t)=-\frac{U_1(x,t)}{U_2(x,t)}$$

where $U_1(x,t)$ and $U_2(x,t)$ are given by,

$$U_1(x,t)=3024+24\cosh(10x-280t)+240\cosh(8x-224t)+360\cosh(6x-72t)$$
$$+720\cosh(6x-216t)+960\cosh(4x-208t)+600\cosh(x-152t)$$
$$+1200\cosh(2x-8t)+3240\cosh(2x-56t).$$

$$U_2(x,t)=\cosh(6x-144t)+6\cosh(4x-136t)+15\cosh(2x-80t)+10\cosh(72t).$$

CHAPTER 4

VARIABLE COEFFICIENT KDV EQUATION

4.1 Introduction

As we know from the last three chapters, the KdV equation is used to study the propagation of non-linear disperse waves in homogenous media. However, the variable-coefficient KdV equation may arise to describe the disperse waves if the boundaries are non-uniform or the medium are inhomogeneous [4,10,15,16,17]. In this chapter, we will study the shallow water nonlinear waves moving over an uneven bottom. In fact, this physical problem can be modeled by variable- coefficient KdV equation [16,17]. In fact, a study was conducted by Johnson [17] to study some numerical solutions for variable-coefficient KdV equation. However, the inverse scattering transform can be constructed only for certain types of variable coefficient KdV equations [16,17]. We find that if the following compatibility condition

$$\begin{aligned} \alpha(t)\left[2f_1'(t)-f_2'(t)\right]-\alpha(t)\frac{a_3'(t)}{a_3(t)}\left[2f_1(t)-f_2(t)\right] \\ -\left[2f_1(t)-f_2(t)\right]\left[f_1(t)-2f_2(t)\right]=0 \end{aligned} \tag{4.1}$$

is satisfied. Then, the variable coefficient KdV equation

$$\alpha(t)u_t-\frac{3}{2}a_3(t)uu_x+\frac{1}{4}a_3(t)u_{xxx}+f_1(t)xu_x+f_2(t)u=0 \tag{4.2}$$

is integrable by the method of inverse scattering transform [4].

4.2 Inverse Scattering Transform for Variable-Coefficient KdV Equation

In order to study and construct the inverse scattering transform for the variable coefficient KdV equation in shallow water waves, we first need to find the equation that describes the shallow water waves and then the compatibility condition which is given by equation (4.1). In fact, there is more than one form that describes the shallow water waves. However, the variable –coefficient KdV equation for the shallow water disperse waves over an uneven bottom has been derived by Johnson [16,17] in the form

$$u_X + \frac{3}{2} d^{-7/4}(\sigma X) u u_\xi + \frac{1}{6} d^{1/2}(\sigma X) u_{\xi\xi\xi} = 0, \tag{4.3}$$

where

X is the far field horizontal space variable with $X = \varepsilon x$,

x is a dimensionless horizontal space variable,

$d(\sigma X)$ is the local depth with $d(0) = 1$. Also, whereas

ξ is a variable related to X and time through the equation

$$\xi = \int_0^x d^{1/2} dx - t, \text{ then}$$

$d^{\frac{-1}{4}} u$:is the free surface displacement.

In order to find the compatibility condition, we have to first transform the variable coefficient KdV equation into the standard form. For this purpose, we introduce the transformation

$$U = -u \text{ and } y = \left(\frac{3}{2}\right)^{\frac{1}{2}} d^{\frac{-9}{8}} (\sigma X)\xi.$$

to obtain

$$u_X = -U_X\,,\quad \frac{dy}{d\xi} = \left(\frac{3}{2}\right)^{\frac{1}{2}} d^{\frac{-9}{8}}(\sigma X)\,,\quad \frac{\partial u}{\partial \xi} = -\frac{\partial U}{\partial \xi} = -\frac{\partial U}{\partial y}\frac{\partial y}{\partial \xi} = -\left(\frac{3}{2}\right)^{\frac{1}{2}} d^{\frac{-9}{8}}(\sigma X)U_y\,,$$

First, rewriting equation (4.2) in the form

$$-U_X + \frac{3}{2}d^{-7/4}(\sigma X)\left(\frac{3}{2}\right)^{\frac{1}{2}} d^{-9/8}(\sigma X)UU_y -$$

$$\frac{1}{6}d^{1/2}(\sigma X)\left(\frac{3}{2}\right)^{\frac{3}{2}} d^{-27/8}(\sigma X)U_{yyy} - \frac{9d_X(\sigma X)}{8d(\sigma X)}yU_y = 0.$$

and then simplifying terms we obtain

$$U_X - \frac{3}{2}\left(\frac{3}{2}\right)^{\frac{1}{2}} d^{-23/8}(\sigma X)UU_y + \frac{1}{4}\left(\frac{3}{2}\right)^{\frac{1}{2}} d^{-23/8}(\sigma X)U_{yyy} - \frac{9d_X(\sigma X)}{8d(\sigma X)}yU_y = 0\,. \tag{4.4}$$

At this stage we compare equations (4.4) and (4.2) to easily identify that

$$\alpha(X) = 1, \qquad f_1(X) = -\frac{9d_X(\sigma X)}{8d(\sigma X)},$$

$$f_2(t) = 0, \qquad a_3(t) = \left(\frac{3}{2}\right)^{\frac{1}{2}} d^{-23/8}(\sigma X).$$

Now substituting the above functions into equation (4.1), the compatibility condition takes the form

$$\frac{H(X)}{dX} + 4H^2(X) = 0, \tag{4.5}$$

where

$$H(X) = \frac{d_X(\sigma X)}{d(\sigma X)}\,. \tag{4.5}$$

Thus the solution of equation (4.5) becomes

$$H(X) = \frac{1}{4X + c_1}\,. \tag{4.6}$$

In the light of above, we first write equation (4.5) as,

$$H(X) = \frac{1}{4X + c_1}$$

$$\frac{d_X(\sigma X)}{d(\sigma X)} = \frac{1}{4}\frac{4}{4X + c_1}$$

and then integrate to get

$$d(\sigma X) = c_2\left(4X + c_1\right)^{\frac{1}{4}}. \qquad (4.7)$$

to get the local depth, since $d(\sigma X) > 0$, therefore it gives

$$c_2 > 0$$

Further, since c_1 and c_2 are two integration constants, we can choose

$$c_2 = \left(\frac{\sigma c}{4}\right)^{\frac{1}{4}}$$

Thus, the equation for the local depth becomes

$$d(\sigma X) = c_2\left(4X + c_1\right)^{\frac{1}{4}} = \left(\frac{\sigma c}{4}\right)^{\frac{1}{4}}\left(4X + c_1\right)^{\frac{1}{4}} = \left(\sigma c X + \frac{\sigma c c_1}{4}\right)^{\frac{1}{4}}.$$

with equation (4.7) taking the form

$$d(\sigma X) = \left(\sigma c X + 1\right)^{\frac{1}{4}}.$$

Moreover, the inverse scattering transform can be applied to equation (4.2) if the water bottom takes the following form

$$d(\sigma X) = \left(\sigma c X + 1\right)^{\frac{1}{4}}. \qquad (4.8)$$

Substituting the above equation into equation (4.4) gives,

$$U_X - \frac{3}{2}\left(\frac{3}{2}\right)^{\frac{1}{2}}\left(\sigma c X + 1\right)^{\frac{-23}{32}} U U_y + \frac{1}{4}\left(\frac{3}{2}\right)^{\frac{1}{2}}\left(\sigma c X + 1\right)^{\frac{-23}{32}} U_{yyy} - \frac{9\sigma c}{32\left(\sigma c X + 1\right)} y U_y = 0. \qquad (4.9)$$

The inverse scattering transform for the above equation (4.9) has been provided by Dai [4] and these equations take the following forms:

$$K_X - \frac{9}{32}\frac{k\sigma}{1+k\sigma X}\left(K + yK_y + zK_z\right) + \left(\frac{3}{2}\right)^{\frac{1}{2}}\left(1+k\sigma X\right)^{-23/32}\left(K_{yyy} + K_{zzz}\right)$$
$$- \frac{3}{4}\left(\frac{3}{2}\right)^{\frac{1}{2}}\left(\sigma c X + 1\right)^{\frac{-23}{32}}\left[v\,\frac{\partial K}{\partial y} + \frac{\partial\left(K_v\right)}{\partial y}\right] = 0, \tag{4.10}$$

$$K_{yy} - K_{zz} = \left[v + \frac{3k\sigma}{8}\left(\frac{3}{2}\right)^{-\frac{1}{2}}\left(1+k\sigma X\right)^{\frac{-9}{32}}\left(y - z\right)\right]K, \tag{4.11}$$

$$F_X - \frac{9}{32}\frac{k\sigma}{1+k\sigma X}\left(F + yF_y + zF_z\right) + \left(\frac{3}{2}\right)^{\frac{1}{2}}\left(1+k\sigma X\right)^{-23/32} \times \left(F_{yyy} + F_{zzz}\right) = 0, \tag{4.12}$$

and

$$F_{yy} - F_{zz} = \frac{3k\sigma}{8}\left(\frac{3}{2}\right)^{-\frac{1}{2}}\left(1+k\sigma X\right)^{\frac{-9}{32}}\left(y - z\right)K, \tag{4.13}$$

where $K(y,z,X)$ and $F(y,z,X)$ are related through the Gel'fand Levitan equation

$$F(y,z,X) + K(y,x,X) + \int_{y}^{+\infty} K(y,s,X)F(s,z,X)\,ds = 0. \tag{4.14}$$

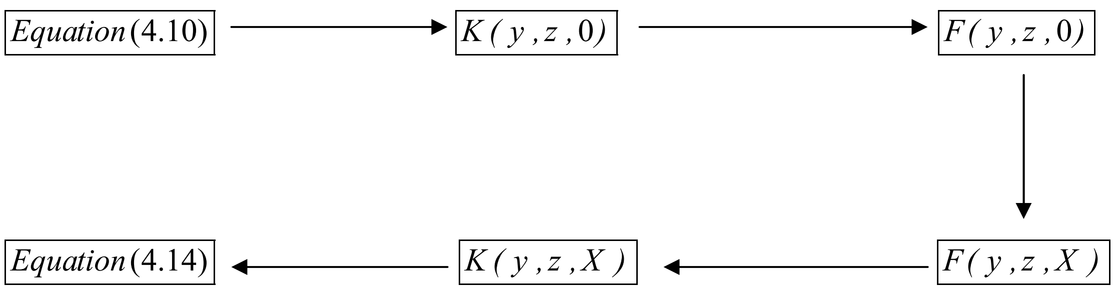
After solving the Gel'fand Levitan equation and finding the kernel, the solution for the variable KdV equation becomes

$$v(y,X) = -2\frac{d}{dy}K(y,y,X). \tag{4.15}$$

In the coming two sections, we will illustrate how to apply the inverse scattering transform for the variable-coefficient KdV equation (4.9) we obtained in this chapter. In fact, we will consider two cases. The first case of these deals with the solution for an increasing local depth, while the second will be the solution for the decreasing local depth.

4.3 Solution for the Increasing Local Depth

At this stage, we solve the initial value problem by solving a series of linear equations. This series can be illustrated through the following diagram:



To find the solution for equation (4.11), we assume

$$\xi_1 = h_1(X)\,y - h_2(X)\,,$$

$$\xi_2 = h_1(X)\,z - h_2(X)\,,$$

and

$$F(y,z,X) = h_3(X)\,G(\xi_1,\xi_2)\,, \tag{4.16}$$

where $h_1(X)$, $h_2(X)$ and $h_3(X)$ are arbitrary functions to be determined. By substituting the above equations into equation (4.11), we obtain,

$$h_1^3(X)\left(G_{\xi_1\xi_1} - G_{\xi_2\xi_2}\right) = \frac{3k\sigma}{8}\left(\frac{2}{3}\right)^{\frac{1}{2}}\left(1+k\sigma X\right)^{\frac{-9}{32}}\left(\xi_1-\xi_2\right)G. \tag{4.17}$$

The above equation can be compared to give,

$$h_1(X) = \frac{|k|}{k}\left(\frac{3|k|\sigma}{8}\right)^{\frac{1}{3}}\left(\frac{2}{3}\right)^{\frac{1}{6}}\left(1+k\sigma X\right)^{\frac{-3}{32}}. \tag{4.18}$$

$$G_{\xi_1\xi_1} - G_{\xi_2\xi_2} = \left(\xi_1 - \xi_2\right)G. \tag{4.19}$$

To solve equation (4.19), we use the method of separation of variables by substituting

$$G(\xi_1, \xi_2) = g_1(\xi_1)\, g_2(\xi_2). \tag{4.20}$$

With this substitution, equation (4.19) gives

$$g_1''(\xi_1) - \left(\xi_1 - \lambda\right) g_1(\xi_1) = 0, \tag{4.21}$$

$$g_2''(\xi_2) - \left(\xi_2 - \lambda\right) g_2(\xi_2) = 0, \tag{4.22}$$

where λ is a separation constant and can be chosen $\lambda = 0$. Solving above equations we set the general solution [4]

$$g_1(\xi_1) = a_{11} Ai(\xi_1) + a_{12} Bi(\xi_1) \tag{4.23}$$

$$g_2(\xi_2) = a_{21} Ai(\xi_2) + a_{22} Bi(\xi_2) \tag{4.24}$$

where a_{11}, a_{12}, a_{21} and a_{22} are arbitrary constants. Also, the $Ai(*)$ and $Bi(*)$ are the Airy functions of the first and second kinds respectively. In order to find the solution for equation (4.16), we will consider two cases.

Case (i) If we let $a_{12} = a_{22} = 0$ and $a_{11} = a_{21} = 1$ then, the solution takes the following form

$$F(y, z, X) = h_3(X) G(\xi_1, \xi_2) = h_3(X) Ai(\xi_1) Ai(\xi_2). \tag{4.25}$$

By substituting this solution into equation (8) we get

$$\begin{aligned} R_1(X) Ai(\xi_1) Ai(\xi_2) + R_2(X) Ai'(\xi_1) Ai(\xi_2) + R_2(X) Ai(\xi_1) Ai'(\xi_2) \\ + R_3(X) Ai'(\xi_1) Ai(\xi_2) + R_3(X) Ai(\xi_1) Ai'(\xi_2) = 0, \end{aligned} \tag{4.26}$$

where

$$R_1(X) = h_3'(X) - \frac{9}{32}\frac{k\sigma}{\left(1+k\sigma X\right)} h_3(X) + 2\left(\frac{3}{2}\right)^{\frac{1}{2}}\left(1+k\sigma X\right)^{-23/32} h_3(X) h_1^3(X), \tag{4.27}$$

$$R_2(X) = -h_3(X)\left[h_2'(X) + 2\left(\frac{3}{2}\right)^{\frac{1}{2}} \left(1+k\sigma X\right)^{-23/32} h_2(X) h_1^3(X) \right], \qquad (4.28)$$

and

$$R_3(X) = h_3(X)\left[h_1'(X) - \frac{9}{32}\frac{k\sigma}{\left(1+k\sigma X\right)} h_1(X) + \left(\frac{3}{2}\right)^{\frac{1}{2}} \left(1+k\sigma X\right)^{-23/32} h_1^4(X) \right], (4.29)$$

In driving the above equation, use has been made of

$$Ai'''(\xi_\vartheta) = Ai'(\xi_\vartheta) + Ai(\xi_\vartheta), \qquad \vartheta = 1, 2.$$

In fact, it can be shown that [4]

$$R_1(X) = 0 \;,\; R_2(X) = 0 \text{ and } R_3(X) = 0$$

$$h_2(X) = a_2\left(1+k\sigma X\right)^{-3/8} = a_2 h_4(X), \qquad h_3(X) = a_1\left(1+k\sigma X\right)^{-15/32} = a_1 h_5(X)$$

$$F(y,z,X) = a_1 h_5(X) Ai(h_1(X)y - a_2 h_4(X)) Ai(h_1(X)z - a_2 h_4(X)). \qquad (4.30)$$

To obtain the kernel for the variable KdV equation we substitute equation (4.30) into the Gel'fand Levitan equation (4.14)

$$K(y,z,X) = \frac{-a_1 h_5(X) Ai(h_1(X)y - a_2 h_4(X)) Ai(h_1(X)z - a_2 h_4(X))}{1 + a_1 h_5(X)\int_y^\infty Ai^2(h_1(X)s - a_2 h_4(X))ds}. \qquad (4.31)$$

In this situation, we have to examine the convergence of the above integral. In fact, the Airy function has the following asymptotic properties [4]

$$Ai(\xi) = \frac{1}{2}\pi\xi^{-\frac{1}{4}} e^{-\frac{2}{3}\xi^{\frac{3}{2}}}\left[1 + O(\xi^{-\frac{3}{2}})\right] \qquad (|\arg\xi| < \pi) \text{ as } \xi \to \infty.$$

$$Ai(-\xi) = -\pi^{\frac{-1}{2}}\xi^{-\frac{1}{4}}\left[sin(\frac{2}{3}\xi^{\frac{3}{2}} + \frac{\pi}{4}) + O(\xi^{-\frac{3}{2}})\right] \qquad (|\arg\xi| < \frac{2}{3}\pi) \text{ as } \xi \to \infty.$$

With these asymptotic properties, it can be shown [4] that the above integral is zero at $-\infty$. Therefore, the solution becomes,

$$v(y,X) = -2\frac{d^2}{dy^2} ln\left\{1+\frac{a_1h_5(X)}{h_1(X)}\left[\left(Ai'(h_1(X)y-a_2h_4(X))\right)^2 -\left(h_1(X)y-a_2h_4(X)\right)Ai^2(h_1(X)y-a_2h_4(X))\right]\right\}. \tag{4.32}$$

Also, the bounded solution for the free surface displacement is given by [4]

$$d^{-\frac{1}{4}}(\sigma X)u = -\frac{4}{3}d^2(\sigma X)\frac{d^2}{d\xi^2} ln\left\{1+c_1d^{-\frac{3}{2}}(\sigma X)\left[\left(Ai'(\eta(\xi,X))\right)^2 -\eta(\xi,X)Ai^2(\eta(\xi,X))\right]\right\} \tag{4.33}$$

where

$$\eta(\xi,X) = \left(\frac{9k\sigma}{16}\right)^{\frac{-1}{3}} d^{-\frac{3}{2}}(\sigma X)\xi - a_2d^{\frac{3}{2}}(\sigma X),$$

$$c_1 = a_1\left(\frac{3k\sigma}{8}\right)^{\frac{-1}{3}}\left(\frac{2}{3}\right)^{\frac{-1}{6}}.$$

To generate another solution, we let

$$F(y,z,X) = \sum_{i=1}^{N} a_ih_5(X)Ai(h_1(X)y-a_ih_4(X))Ai(h_1(X)z-a_ih_4(X)).$$

$$v(y,X) = -2\frac{d^2}{dy^2} ln\,\Delta, \tag{4.34}$$

where

$$\Delta = det\left|\delta_{ij} + a_{1i}h_5(X)\int_y^{\infty} Ai(h_1(X)s-a_{2i}h_4(X))Ai(h_1(X)s-a_{2i}h_4(X))ds\right|.$$

4.4 Solution for the Decreasing Local Dept

Here, we will to find the solution for the variable KdV equation for decreasing local depth. In fact, we will to move parallel to the case of increasing local depth.

Case (ii) By taking the following choice $a_{12} = a_{21} = 0$ and $a_{11} = a_{22} = 1$ we obtain the following solution

$$F(y,z,X) = h_3(X)G(\xi_1,\xi_2) = h_3(X)Ai(\xi_1)Bi(\xi_2). \qquad (4.35)$$

Substituting the above solution into equation (4.9) we notice that $h_3(X)$ and $h_2(X)$ are given by,

$$h_3(X) = a_1\left(1+k\sigma X\right)^{-15/32} = a_1 h_5(X). \qquad (4.36)$$

$$h_2(X) = a_2\left(1+k\sigma X\right)^{-3/8} = a_2 h_4(X). \qquad (4.37)$$

Using equations (4.36), (4.37) give the following solution,

$$F(y,z,X) = a_1 h_5(X)Ai(h_1(X)y - a_2h_4(X))Bi(h_1(X)z - a_2h_4(X)),$$

Substituting into the Gel'fand Levitan equation (4.14) gives the kernel as,

$$K(y,z,X) = \frac{-a_1h_5(X)Ai(h_1(X)y - a_2h_4(X))Bi(h_1(X)z - a_2h_4(X))}{1 + a_1h_5(X)\int_y^\infty Ai(h_1(X)s - a_2h_4(X))Bi(h_1(X)s - a_2h_4(X))ds}. \qquad (4.38)$$

As before we examine the convergence of the above integral. In fact, the Airy function has the asymptotic properties [4],

$$Bi(\xi) = \pi^{\frac{-1}{2}}\xi^{-\frac{1}{4}}e^{-\frac{2}{3}\xi^{\frac{3}{2}}}\left[1 + O(\xi^{-\frac{3}{2}})\right] \qquad (|\arg\xi| < \frac{1}{13}\pi) \text{ as } \xi \to \infty,$$

$$Bi(-\xi) = \pi^{\frac{-1}{2}}\xi^{-\frac{1}{4}}\left[cos(\frac{2}{3}\xi^{\frac{3}{2}} + \frac{\pi}{4}) + O(\xi^{-\frac{3}{2}})\right] \qquad (|\arg\xi| < \frac{2}{3}\pi) \text{ as } \xi \to \infty.$$

These asymptotic equations show that the above integral is divergent if $h_1(X) > 0$ and convergent if $h_1(X) < 0$. Since, the Gel'fand Levitan equation (4.38) is valid only for a decreasing local depth and one solution is given by,

$$v(y,X) = -2\frac{d^2}{dy^2}ln\left[1 + a_1 h_5(X)\int_y^{\infty} Ai(h_1(X)s - a_2 h_4(X))Bi(h_1(X)s - a_2 h_4(X))ds\right]$$

$$v(y,X) = -2\frac{d^2}{dy^2}ln\left[1 - \frac{a_1 h_5(X)}{h_1(X)}\int_{-\infty}^{h_1(X)y - a_2 h_4(X)} Ai(\omega)Bi(\omega)d\omega\right]. \qquad (4.39)$$

In order to evaluate the integral inside the solution, we use integration by parts and the following identities:

$Ai''(\omega) = \omega Ai(\omega)$ and $Bi''(\omega) = \omega Bi(\omega)$,

$\int Ai(\omega)Bi(\omega)d\omega = \omega Ai(\omega)Bi(\omega) - Ai'(\omega)Bi'(\omega)$.

Keeping in mind the asymptotic properties of Airy functions,

$$Ai'(-\xi) = -\pi^{\frac{-1}{2}}\xi^{\frac{1}{4}}\left[cos(\frac{2}{3}\xi^{\frac{3}{2}} + \frac{\pi}{4}) + O(\xi^{-\frac{3}{2}})\right] \qquad (|\arg\xi| < \frac{2}{3}\pi) \text{ as } \xi \to \infty .$$

$$Bi'(-\xi) = -\pi^{\frac{-1}{2}}\xi^{\frac{1}{4}}\left[sin(\frac{2}{3}\xi^{\frac{3}{2}} + \frac{\pi}{4}) + O(\xi^{-\frac{3}{2}})\right] \qquad (|\arg\xi| < \frac{2}{3}\pi) \text{ as } \xi \to \infty .$$

It can be shown [4] that the above integral is zero at $-\infty$ and therefore, the solution takes the following form

$$v(y,X) = 2\frac{d^2}{dy^2}ln\left\{1 + \frac{a_1 h_5(X)}{h_1(X)}\left[Ai'(h_1(X)y - a_2 h_4(X))Bi'(h_1(X)y - a_2 h_4(X))\right.\right.$$
$$\left.\left. -\left(h_1(X)y - a_2 h_4(X)\right)Ai(h_1(X)y - a_2 h_4(X))Bi(h_1(X)y - a_2 h_4(X))\right]\right\}.$$

By using the original variables, the bounded solution for the free surface displacement is

$$d^{-\frac{1}{4}}(\sigma X)u = -\frac{4}{3}d^{2}(\sigma X)\frac{d^{2}}{d\xi^{2}} ln\left\{1 + c_2 d^{-\frac{3}{2}}(\sigma X)\left[Ai'(\eta_1(\xi, X))Bi'(\eta_1(\xi, X)) - \eta_1(\xi, X)Ai(\eta_1(\xi, X))Bi(\eta_1(\xi, X))\right]\right\}, \tag{4.40}$$

where

$$\eta_1(\xi, X) = \left(-\frac{9k\sigma}{16}\right)^{\frac{1}{3}} d^{-\frac{3}{2}}(\sigma X)\xi - a_2 d^{\frac{3}{2}}(\sigma X),$$

and

$$c_2 = a_1\left(-\frac{3k\sigma}{8}\right)^{\frac{-1}{3}}\left(\frac{2}{3}\right)^{\frac{-1}{6}}.$$

To generate another, solution, we let

$$F(y, z, X) = \sum_{i=1}^{N} a_i h_5(X) Ai(h_1(X)y - a_{2i}h_4(X))Bi(h_1(X)z - a_{2i}h_4(X)).$$

where a_{1i} and a_{2i} are arbitrary constants. In fact, it can be shown [4] that another bounded solution for the variable coefficient KdV equation can obtained by substituting the above equation into the Gel'fand Levitan equation

$$v(y, X) = \frac{d^2}{dy^2} ln\,\Delta, \tag{4.41}$$

where

$$\Delta = det\left|\delta_{ij} + a_{1i}h_5(X)\int_{y}^{\infty} Ai(h_1(X)s - a_{2i}h_4(X))Bi(h_1(X)s - a_{2i}h_4(X))ds\right|.$$

Finally, there is another choice for the constants $a_{11} = a_{22} = 0$, to get a bounded solution for the variable KdV equation. In fact, the above mentioned choice leads to the same solution. However, other choices for the constants lead to a divergent integral and as a result of that we do not get a bounded solution.

4.5 Transforming Variable-Coefficient KdV into Standard KdV Equation

We find that variable-coefficient KdV equation is directly intergable by the method of inverse scattering transform if the compactable condition is satisfied. In fact, we find that the local depth has the following function

$$d(\sigma X) = \left(\sigma c X + 1\right)^{\frac{1}{4}}. \tag{4.42}$$

According to Dai [4], we can transform the variable coefficient KdV equation into standard KdV equation by introducing the following transformations:

$$U = \frac{3}{2} d^{\frac{3}{4}}(\sigma X) - \frac{2}{3} d^{-\frac{9}{4}}(\sigma X)\theta = (1 + k\sigma X)^{\frac{-9}{16}} \left(\frac{3}{8} k\sigma\xi - \frac{2}{3}\theta\right), \tag{4.43}$$

$$\psi = \frac{1}{6}\int d^{\frac{-25}{4}}(\sigma s)ds = \frac{4}{27k\sigma}(1 + k\sigma X)^{\frac{-9}{16}}, \tag{4.44}$$

$$\phi = \xi d^{\frac{-9}{4}}(\sigma X) = \xi(1 + k\sigma X)^{\frac{-9}{16}}. \tag{4.45}$$

Substituting these equations (4.43) - (4.45) into the variable- coefficient KdV equation (4.3), we obtain the standard KdV equation

$$\theta_{\psi} - 6\theta\theta_{\phi} + \theta_{\phi\phi\phi} = 0, \tag{4.46}$$

which has the solitary solution given by

$$\theta = -\frac{1}{2} c \ \operatorname{sech}\left(\frac{1}{2}\sqrt{c}\,(\phi - c\psi - \phi_0)\right). \tag{4.47}$$

Now using the original variables in equations (4.43) - (4.45), we can write the solution as:

$$U = \frac{3}{8}(1+k\,\sigma X)^{\frac{-9}{16}} k\,\sigma\xi + \frac{1}{3}(1+k\,\sigma X)^{\frac{-9}{16}} \mathrm{c}$$
$$\mathrm{sech}^2\left(\frac{1}{2}\sqrt{c}\left[\xi(1+k\,\sigma X)^{\frac{-9}{16}} + \frac{4}{27}\frac{c}{k\,\sigma}(1+k\,\sigma X)^{\frac{-9}{16}} - \phi_0)\right]\right). \tag{4.48}$$

As a remark, it is important to indicate that whereas the solution does not represent one soliton solution, it becomes unbounded as $\xi \to \infty$.

CHAPTER 5

INTERACTION OF TWO SOLITONS

5.1 Introduction

In this chapter, we will study and discuss the relation and interaction between the solitons which are obtained as the solution for the KdV equation in the previous chapters. In fact, the main idea is to decompose the N-solitons solution into a linear sum even though the KdV equation is nonlinear and the superposition principle fails to hold. In practice it can be done by looking through some decomposition from literature [24,26,27]. Once the decomposition is introduced, we shall find out that the solitons exchange identities [24] and emit dual ghost particles [27,29] during the interaction. We shall conclude this chapter by an example in support of work.

5.2 Decay Eigenvalues

As we know from chapter two that for certain choice of the initial condition, the KdV equation has a discrete energy spectrum. With the above mentioned statement, consider the non-linear partial differential equation KdV

$$u_t - 6uu_x + u_{xxx} = 0\,, \qquad -\infty < x < \infty \quad t > 0 \tag{5.1}$$

subject to the initial condition

$$u(x, t=0) = u_0(x) \tag{5.2}$$

and assume that the solution is reflection-less .Therefore, if the solution is reflection-less then the initial data has only a discrete energy spectrum $\{\lambda_1 < \lambda_2 < ... < \lambda_n < 0\}$ which can be obtained by solving the time-independent Schrödinger equation

$$\frac{d^2\psi}{dx^2} + \left[\lambda - u_0(x)\right]\psi = 0. \tag{5.3}$$

As a result of that, we obtain the eigenfunctions $\{\psi_1, \psi_2, ..., \psi_n\}$ corresponding to the discrete eigenvalues. The eigenfunctions can be normalized and the normalizing constants can be calculated through the definition

$$\int_{-\infty}^{\infty} \psi_n(x,t)dx = 1, \qquad \mathrm{c_n} = \lim_{x\to\infty} e^{k_n x}\psi_n(x,t). \tag{5.4}$$

Thus, the initial data is used to produce N-soliton solution [7,12,31,34] for the KdV equation through the determinant formula, which was obtained before by the inverse scattering method

$$u(x,t) = -2\frac{\partial^2}{\partial x^2}\Big(\log\big(\det\left(I + A\right)\big)\Big), \tag{5.5}$$

where, A is a square matrix of dimension N and with entries defined as:

$$a_{mn} = \frac{c_m c_n}{k_m + k_n} e^{(k_m + k_n)x - 4\left(k_m^2 + k_n^2\right)t}, \tag{5.6}$$

and the parameter $k_n > 0$ is defined through the relation $\lambda_n = -k_n^2$. Since $A = (a_{mn})$ is symmetric and positive definite, these properties allow us to diagonalize the matrix $A = (a_{mn})$ so that:

$$P^{-1}AP = D = \begin{pmatrix} \mu_1(x,t) & 0 & .. & 0 \\ 0 & \mu_2(x,t) & & \vdots \\ \vdots & & & 0 \\ 0 & & & \mu_n(x,t) \end{pmatrix}, \tag{5.7}$$

where

1. $\{\mu_1(x,t), \mu_2(x,t), ..., \mu_n(x,t)\}$ is the ordered set of real positive eigenvalues of A.

and

2. B is the orthogonal matrix which consists of the orthonormal basis of the eigenvlues of A.

By using the above representation, we can write the solution for the KdV equation in term of the eigenvalues,

$$\det(I + A) = \det\left(P(I + A)P^{-1}\right)$$

$$= \det\left(I + PAP^{-1}\right) = \det(I + D) = \prod_{n=1}^{N}\left(1 + \mu_n(x,t)\right). \tag{5.8}$$

Since the above derivation allows us to decompose the solution into sum of separated solitons, the solution for the KdV equation becomes,

$$u(x,t) = -2\frac{\partial^2}{\partial x^2}\left(\log\left(\det(I + A)\right)\right)$$

$$= -2\frac{\partial^2}{\partial x^2}\left(\log\left(\prod_{n=1}^{N}\left(1 + \mu_n(x,t)\right)\right)\right) = -2\frac{\partial^2}{\partial x^2}\left(\sum_{n=1}^{N}\log\left[1 + \mu_n(x,t)\right]\right)$$

$$= \sum_{n=1}^{N} -2\frac{\partial^2}{\partial x^2}\log\left[1 + \mu_n(x,t)\right] = \sum_{n=1}^{N} u_n(x,t),$$

therefor

$$u(x,t) = \sum_{n=1}^{N} u_n(x,t). \tag{5.9}$$

5.3 Decay of Two Solitons

The easiest way to see the interaction of N-solitons for KdV equation is to look for the interaction of two solitons. In this section, we define and discuss the decay functions for two solitons in terms of u_1 and u_2 that can be then generalized for N solitons.

Definition 5.2.1

Let $\{\mu_1(x,t), \mu_2(x,t), ..., \mu_n(x,t)\}$ be the set of real positive eigenvalues of A, then we define $u(x,t) = \sum_{n=1}^{N} u_n(x,t)$ as decay decomposition, where

$$u_n(x,t) = -2\frac{\partial^2}{\partial x^2}\log\left[1+\mu_n(x,t)\right] \quad \text{as decay function.}$$

Definition 5.2.2

We define the n-th soliton particle of $u(x,t)$ as

$$s_n(v_n) = -2\sec h^2(k_n v_n) \qquad \text{n=1,2,3,...,N.}$$

where $v_n = x - 4k_n$ is the nth moving frame.

Since we are going to talk about the decay function of two solitons, the matrix A associated with two solitons becomes,

$$A = \begin{pmatrix} \frac{c_1^2}{2k_1}e^{2k_1v_1} & \frac{c_1c_2}{k_1+k_2}e^{k_1v_1+k_2v_2} \\ \frac{c_1c_2}{k_1+k_2}e^{k_1v_1+k_2v_2} & \frac{c_2^2}{2k_2}e^{2k_2v_2} \end{pmatrix}, \tag{5.10}$$

The two eigenvalues associated with A are

$$\mu_1(x,t) = \frac{1}{2}\left(Tr(A) + \sqrt{\left[Tr(A)\right]^2 - 4\det(A)}\right), \tag{5.11}$$

$$\mu_2(x,t) = \frac{1}{2}\left(Tr(A) - \sqrt{\left[Tr(A)\right]^2 - 4\det(A)}\right). \tag{5.12}$$

Definition 5.2.3

The ghost matrix of A_g is defined by

$$A_g = \begin{pmatrix} \frac{c_1^2}{2k_1} e^{2k_1 v_g} & \frac{c_1 c_2}{k_1 + k_2} e^{(k_1+k_2) v_g} \\ \frac{c_1 c_2}{k_1 + k_2} e^{(k_1+k_2) v_g} & \frac{c_2^2}{2k_2} e^{2k_2 v_g} \end{pmatrix},$$

where

$$v_g = x - 4k_g^2 t \text{ and } k_g = \sqrt{k_1^2 + k_1 k_2 + k_2^2}.$$

Definition 5.2.4

Let γ_1 and γ_2 to be the eigenvalues of A_g corresponding to u_1 and u_2 respectively. Then, the ghost and anti-ghost particles are defined respectively as,

$$g(v_g) = -2 \frac{\partial^2}{\partial v_g^2} \log\left[\gamma_1(v_g)\right]$$

$$\overline{g}(v_g) = -2 \frac{\partial^2}{\partial v_g^2} \log\left[\gamma_2(v_g)\right].$$

Remarks

- v_g represents the moving frames for ghost particles g and $\overline{g}$.
- The velocity $4k_g^2$ for the ghost particles exceeds the velocity of the solitons particles.

In fact, the relationship we presented above between the two matrices A and A_g and their corresponding eigenvalues is well defined and the following lemma assure that relationship.

Lemma 6.2.1

Let $\overline{k}^2 = k_1^2 k_2 + k_1 k_2^2$ then

$$i)\ A = e^{8\overline{k}^2 t} A_g .$$

$$ii)\ \mu_n = e^{8\overline{k}^2 t} \gamma_n \text{ for } n = 1,2 .$$

Proof:

To prove the first part, it suffices to show that every coefficient of the matrix A has $e^{8\overline{k}^2 t}$ as a common factor when rewritten in term of v_g

$$\begin{aligned} e^{k_n v_n} &= e^{k_n (x - 4k_n^2 t)} \\ &= e^{k_n (v_g + 4k_g^2 t - 4k_n^2 t)} = e^{k_n v_g} e^{4K_n (k_g^2 - k_n^2)t} \\ &= e^{k_n v_g} e^{4K_n (k_1^2 + k_1 k_2 + k_2^2 - k_n^2)t} = e^{k_n v_g} e^{4\overline{k}^2 t} . \end{aligned}$$

To prove the second part, we use the result we obtained in the first part.

$$ii)\ \mu_n = e^{8\overline{k}^2 t} \gamma_n \text{ for } n = 1,2.$$

5.4 Asymptotic Relations for the Decay of Two Solitons

In this section, we present the main theorem that describes the asymptotic behaviors for two solitons. We try to support the theorem by presenting some graphs that clarify the idea behind the theorem.

Theorem 6.3.1

The following asymptotic relations hold for u_1 and u_2:

$$\text{i)} \begin{array}{ll} u_1 \sim s_1(v_1 + \delta_1) & as\ t \to -\infty \\ u_1 \sim s_2(v_2 + \delta_2) + g(v_g) & as\ t \to +\infty \end{array}$$

in the sense that

$$\lim_{\substack{v_1 fixed \\ t\to-\infty}} u_1 = s_1(v_1+\delta_1), \qquad \lim_{\substack{v_2 fixed \\ t\to\infty}} u_1 = s_2(v_2+\delta_2), \text{ and } \lim_{\substack{v_g fixed \\ t\to\infty}} u_1 = g(v_g),$$

with the relative phase shifts δ_1 and δ_2 defined as,

$$e^{2k_1\delta_1} = \frac{c_1^2}{2k_1}, \qquad e^{2k_2\delta_2} = \frac{c_2^2}{2k_2}.$$

ii) $$\begin{aligned} &u_2 \sim s_2(v_2+\delta_2+\Delta) && as\ t\to-\infty. \\ &u_2 \sim s_1(v_1+\delta_1+\Delta)+\overline{g}(v_g) && as\ t\to+\infty. \end{aligned}$$

in the sense that

$$\lim_{\substack{v_2 fixed \\ t\to-\infty}} u_2 = s_2(v_2+\delta_2+\Delta), \qquad \lim_{\substack{v_1 fixed \\ t\to\infty}} u_2 = s_1(v_1+\delta_1+\Delta), \text{ and } \lim_{\substack{v_g fixed \\ t\to\infty}} u_2 = \overline{g}(v_g),$$

with Δ defined as,

$$e^{2k_2\Delta} = \frac{(k_1-k_2)^2}{(k_1+k_2)^2}.$$

Proof

In order to prove the theorem, we are going to look for the particles through three different frames (i.e. v_1, v_2 and v_g).

<u>The first frame</u> v_1.

Here we move with the velocity of v_1 to see what is going on with the particles. Since we have

$$Tr(A) = \frac{c_1^2}{2k_1} e^{2k_1 v_1} + \frac{c_2^2}{2k_2} e^{2k_2 v_2},$$

$$\det(A) = \left(\frac{c_1^2 c_2^2}{4k_1 k_2} - \frac{c_1^2 c_2^2}{[k_1+k_2]^2} \right) e^{2k_1 v_1 + 2k_2 v_2}$$

$$=c_1^2c_2^2\left(\frac{1}{4k_1k_2}-\frac{1}{\left[k_1+k_2\right]^2}\right)e^{2k_1v_1+2k_2v_2}=\frac{c_1^2c_2^2}{4k_1k_2}\left(\frac{k_1-k_2}{k_1+k_2}\right)^2 e^{2(k_1v_1+k_2v_2)}.$$

then

$$\lim_{\substack{v_1 \text{ fixed} \\ t\to-\infty}} Tr(A)=\frac{c_1^2}{2k_1}e^{2k_1v_1}, \qquad \lim_{\substack{v_1 \text{ fixed} \\ t\to-\infty}} \det(A)=0. \tag{5.13}$$

$$\lim_{\substack{v_1 \text{ fixed} \\ t\to-\infty}} (1+\mu_1)=\lim_{\substack{v_1 \text{ fixed} \\ t\to-\infty}}\left[1+\frac{1}{2}\left(Tr(A)+\sqrt{\left[Tr(A)\right]^2-4\det(A)}\right)\right]=1+\frac{c_1^2}{2k_1}e^{2k_1v_1}. \tag{5.14}$$

The decay function is

$$\lim_{\substack{v_1 \text{ fixed} \\ t\to-\infty}} u_1(x,t)=\lim_{\substack{v_1 \text{ fixed} \\ t\to-\infty}}\left\{-2\frac{\partial^2}{\partial x^2}\log\left[1+\mu_1(x,t)\right]\right\}$$

$$=-2\frac{\partial^2}{\partial x^2}\log\left[1+\frac{c_1^2}{2k_1}e^{2k_1v_1}\right]$$

$$=\frac{-8k_1c_1^2e^{2k_1v_1}}{\left(1+\frac{c_1^2}{2k_1}e^{2k_1v_1}\right)^2}=s_1(v_1+\delta_1). \tag{5.15}$$

where

$$e^{2k_1\delta_1}=\frac{c_1^2}{2k_1} \quad \text{i.e.} \quad \frac{c_1^2}{2k_1}e^{2k_1v_1}=e^{2k_1(\delta_1+v_1)}.$$

The second frame v_2.

Here we move the velocity of v_2 instead of v_1 to see what is going on with the particles.

Since we have

$$\lim_{\substack{v_2 \text{ fixed} \\ t\to\infty}} Tr(A)=\frac{c_2^2}{2k_2}e^{2k_2v_2}, \qquad \lim_{\substack{v_2 \text{ fixed} \\ t\to\infty}} \det(A)=0,$$

then

$$\lim_{\substack{v_2 \text{ fixed} \\ t\to\infty}} \left(1+\mu_1\right) = \lim_{\substack{v_2 \text{ fixed} \\ t\to\infty}} \left[1+\frac{1}{2}\left(Tr(A)+\sqrt{\left[Tr(A)\right]^2-4\det(A)}\right)\right] = 1+\frac{c_2^2}{2k_2}e^{2k_2v_2}. \tag{5.16}$$

The decay function is

$$\lim_{\substack{v_2 \text{ fixed} \\ t\to\infty}} u_1(x,t) = \lim_{\substack{v_2 \text{ fixed} \\ t\to\infty}} \left\{-2\frac{\partial^2}{\partial x^2}\log\left[1+\mu_1(x,t)\right]\right\}$$

$$= -2\frac{\partial^2}{\partial x^2}\log\left[1+\frac{c_2^2}{2k_2}e^{2k_2v_2}\right]$$

$$= \frac{-8k_2c_2^2e^{2k_2v_2}}{\left(1+\frac{c_2^2}{2k_2}e^{2k_2v_2}\right)^2} = s_2(v_2+\delta_2), \tag{5.17}$$

where

$$e^{2k_2\delta_2} = \frac{c_2^2}{2k_2} \quad \text{i.e.} \quad \frac{c_2^2}{2k_2}e^{2k_2v_2} = e^{2k_2(\delta_2+v_2)}.$$

<u>The third frame</u> v_g.

Here we move with the velocity of v_g to see what is going on with the particles.

$$\lim_{\substack{v_g \text{ fixed} \\ t\to-\infty}} u_1(x,t) = \lim_{\substack{v_g \text{ fixed} \\ t\to-\infty}} \left\{-2\frac{\partial^2}{\partial x^2}\log\left[1+\mu_1(x,t)\right]\right\} = -2\frac{\partial^2}{\partial x^2}\log\left[1+e^{8\bar{k}^2t}\gamma_1\right] = 0.$$

$$\lim_{\substack{v_g \text{ fixed} \\ t\to\infty}} u_1(x,t) = \lim_{\substack{v_g \text{ fixed} \\ t\to\infty}} \left\{-2\frac{\partial^2}{\partial x^2}\log\left[1+\mu_1(x,t)\right]\right\} = \lim_{\substack{v_g \text{ fixed} \\ t\to\infty}} \left\{-2\frac{\partial^2}{\partial x^2}\log\left[1+e^{8\bar{k}^2t}\gamma_1\right]\right\}$$

$$= \lim_{\substack{v_g \text{ fixed} \\ t\to\infty}} \left\{-2\frac{\partial^2}{\partial x^2}\left[\log\left(e^{4\bar{k}^2t}\right)+\log\left(e^{-4\bar{k}^2t}+\gamma_1\right)\right]\right\}$$

$$= \lim_{\substack{v_g \text{ fixed} \\ t\to\infty}} \left\{-2\frac{\partial^2}{\partial x^2}\log\left(\gamma_1\right)\right\}$$

$$= g.$$

(ii) To prove the second part, we are going to consider again the three different frames (i.e. v_1, v_2 and v_g).

<u>The first frame</u> v_2.

Here we to move with the velocity of the v_2 to see what is going on with the particles.

Since we have $\quad 2k_1 v_1 = 2k_2 v_2 - 8(k_2^2 - k_1^2)t$,

then, we can rewrite $Tr(A)$ and $\det(A)$ as:

$$Tr(A) = \frac{c_1^2}{2k_1} e^{2k_1 v_1} + \frac{c_2^2}{2k_2} e^{2k_2 v_2}$$

$$= e^{8k_1(k_2^2 - k_1^2)t} \left(\frac{c_1^2}{2k_1} e^{2k_1 v_2} + \frac{c_2^2}{2k_2} e^{2k_2 v_2 - 8k_1(k_2^2 - k_1^2)t} \right). \tag{5.18}$$

$$\det(A) = \left(\frac{c_1^2 c_2^2}{4k_1 k_2} - \frac{c_1^2 c_2^2}{[k_1 + k_2]^2} \right) e^{2k_1 v_1 + 2k_2 v_2}$$

$$= e^{8k_1(k_2^2 - k_1^2)t} \frac{c_1^2 c_2^2}{4k_1 k_2} \left(\frac{k_1 - k_2}{k_1 + k_2} \right)^2 e^{2(k_1 + k_2)v_2}. \tag{5.19}$$

$$\lim_{\substack{v_2 \text{ fixed} \\ t \to -\infty}} \frac{\det(A)}{Tr(A)} = \frac{c_2^2}{2k_2} \left(\frac{k_1 - k_2}{k_1 + k_2} \right)^2 e^{2k_2 v_2} \qquad \& \qquad \lim_{\substack{v_2 \text{ fixed} \\ t \to -\infty}} \frac{\det(A)}{[Tr(A)]^2} = 0. \tag{5.20}$$

Whence, the second eigenvalue of A behaves as

$$\lim_{\substack{v_2 \text{ fixed} \\ t \to -\infty}} (1 + \mu_1) = \lim_{\substack{v_2 \text{ fixed} \\ t \to -\infty}} \left[1 + \frac{1}{2} \left(Tr(A) - \sqrt{[Tr(A)]^2 - 4\det(A)} \right) \right]$$

$$= \lim_{\substack{v_2 \text{ fixed} \\ t \to -\infty}} \left[1 + \frac{1}{2} \left(\frac{\frac{4\det(A)}{Tr(A)}}{1 + \sqrt{1 - \frac{4\det(A)}{[Tr(A)]^2}}} \right) \right]$$

$$= \lim_{\substack{v_2 \text{ fixed} \\ t \to -\infty}} \left[1 + \frac{\frac{4\det(A)}{Tr(A)}}{1 + \sqrt{1 - \frac{4\det(A)}{[Tr(A)]^2}}} \right] = \frac{c_2^2}{2k_2}\left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2 e^{2k_2 v_2}. \tag{5.21}$$

and the decay function is

$$\lim_{\substack{v_2 \text{ fixed} \\ t \to -\infty}} u_2(x,t) = \lim_{\substack{v_2 \text{ fixed} \\ t \to -\infty}} \left\{ -2\frac{\partial^2}{\partial x^2} \log\left[1 + \mu_2(x,t)\right] \right\}$$

$$= -2\frac{\partial^2}{\partial x^2} \log\left[1 + \frac{c_2^2}{2k_2}\left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2 e^{2k_2 v_2} \right]$$

$$= s_2(v_2 + \delta_2 + \Delta).$$

where

$$e^{2k_2\Delta} = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2 \qquad \& \qquad e^{2k_2\delta_2} = \frac{c_2^2}{2k_2}.$$

<u>The second frame</u> v_1.

Here we move with the velocity of the v_1 instead of v_2 to see what is going on with the particles but this is exactly similar to the proof of the second part in the first frame which is given by,

$$\lim_{\substack{v_2 \text{ fixed} \\ t \to \infty}} u_2(x,t) = s_1(v_1 + \delta_1 + \Delta),$$

where $$e^{2k_2\Delta} = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2 \qquad \text{and} \qquad e^{2k_2\delta_1} = \frac{c_1^2}{2k_1}.$$

<u>The third frame</u> v_g.

Here we are going to move with the velocity of the v_g to see what is going on the particle.

$$\lim_{\substack{v_g \text{ fixed} \\ t\to-\infty}} u_1(x,t) = \lim_{\substack{v_g \text{ fixed} \\ t\to-\infty}} \left\{ -2\frac{\partial^2}{\partial x^2}\log\left[1+\mu_2(x,t)\right] \right\}$$

$$= -2\frac{\partial^2}{\partial x^2}\log\left[1+e^{8\bar{k}^2 t}\gamma_2\right] = 0. \tag{5.22}$$

In the other hand as $t \to \infty$

$$\lim_{\substack{v_g \text{ fixed} \\ t\to\infty}} u_2(x,t) = \lim_{\substack{v_g \text{ fixed} \\ t\to\infty}} \left\{ -2\frac{\partial^2}{\partial x^2}\log\left[1+\mu_2(x,t)\right] \right\}$$

$$= \lim_{\substack{v_g \text{ fixed} \\ t\to\infty}} \left\{ -2\frac{\partial^2}{\partial x^2}\log\left[1+e^{8\bar{k}^2 t}\gamma_2\right] \right\}$$

$$= \lim_{\substack{v_g \text{ fixed} \\ t\to\infty}} \left\{ -2\frac{\partial^2}{\partial x^2}\left[\log\left(e^{4\bar{k}^2 t}\right)+\log\left(e^{-4\bar{k}^2 t}+\gamma_2\right)\right] \right\}$$

$$= \lim_{\substack{v_g \text{ fixed} \\ t\to\infty}} \left\{ -2\frac{\partial^2}{\partial x^2}\log\left(\gamma_2\right) \right\}$$

$$= \bar{g}\ .$$

This completes the proof of the above theorem.

Example 5.1 The interaction of two solitons

Let $\lambda_1 = -1,\ \lambda_2 = -4$ and $c_1 = \sqrt{6},\ c_2 = 2\sqrt{3}$ be the scattering data for the two solitons which we obtained in chapter three. Therefore, the soliton matrix A takes the following form

$$A = \begin{pmatrix} \frac{c_1^2}{2k_1}e^{2k_1 v_1} & \frac{c_1 c_2}{k_1+k_2}e^{k_1 v_1 + k_2 v_2} \\ \frac{c_1 c_2}{k_1+k_2}e^{k_1 v_1 + k_2 v_2} & \frac{c_2^2}{2k_2}e^{2k_2 v_2} \end{pmatrix}$$

where

i. $\frac{c_1^2}{2k_1} = \frac{6}{2} = 3$, $\quad \frac{c_1 c_2}{k_1+k_2} = \frac{6\sqrt{2}}{3} = 2\sqrt{2}$, $\quad \frac{c_2^2}{2k_2} = \frac{12}{4} = 3$.

ii. $2k_1v_1 = 2(x - 4t) = 2x - 8t$.

iii. $k_1v_1 + k_2v_2 = (x - 4t) + 2(x - 16t) = 3x - 36t$.

iv. $2k_2v_2 = 4(x - 16t) = 4x - 64t$.

$$A = \begin{pmatrix} 3e^{2x-8t} & 2\sqrt{2}e^{3x-36t} \\ 2\sqrt{2}e^{3x-36t} & 3e^{4x-64t} \end{pmatrix}$$

$$Tr(A) = 3e^{2x-8t} + 3e^{4x-64t}.$$

$$Det(A) = 9e^{6x-72t} - 8e^{6x-72t} = e^{6x-72t}.$$

The eigenvalues for the above matrix are:

$$\begin{aligned}\mu_1(x,t) &= \frac{1}{2}\left(Tr(A) + \sqrt{[Tr(A)]^2 - 4\det(A)}\right) \\ &= \frac{1}{2}\left(3e^{2x-8t} + 3e^{4x-64t} + \sqrt{\left[3e^{2x-8t} + 3e^{4x-64t}\right]^2 - 4e^{6x-72t}}\right) \\ &= \frac{1}{2}\left(3e^{2x-8t} + 3e^{4x-64t} + \sqrt{9e^{4x-16t} + 14e^{6x-72t} + 9e^{8x-128t}}\right).\end{aligned}$$

$$\begin{aligned}\mu_2(x,t) &= \frac{1}{2}\left(Tr(A) - \sqrt{[Tr(A)]^2 - 4\det(A)}\right) \\ &= \frac{1}{2}\left(3e^{2x-8t} + 3e^{4x-64t} - \sqrt{\left[3e^{2x-8t} + 3e^{4x-64t}\right]^2 - 4e^{6x-72t}}\right) \\ &= \frac{1}{2}\left(3e^{2x-8t} + 3e^{4x-64t} - \sqrt{9e^{4x-16t} + 14e^{6x-72t} + 9e^{8x-128t}}\right).\end{aligned}$$

Using the above eigenvalues, the decay eigenfunctions become,

$$u_n(x,t) = -2\frac{\partial^2}{\partial x^2}\log\left[1 + \mu_n(x,t)\right] \qquad \text{where } n = 1,2.$$

To find the ghost matrix A_g

$$A_g = \begin{pmatrix} \frac{c_1^2}{2k_1}e^{2k_1v_g} & \frac{c_1c_2}{k_1+k_2}e^{(k_1+k_2)v_g} \\ \frac{c_1c_2}{k_1+k_2}e^{(k_1+k_2)v_g} & \frac{c_2^2}{2k_2}e^{2k_2v_g} \end{pmatrix}$$

Since $k_g^2 = k_1^2 + k_1 k_2 + k_2^2 = 1 + 2 + 4 = 6$

then $A_g = \begin{pmatrix} 3e^{2v_g} & 2\sqrt{2}e^{3v_g} \\ 2\sqrt{2}e^{3v_g} & 3e^{4v_g} \end{pmatrix}$ where $v_g = x - 4k_g^2 t = x - 144t$.

$$Tr(A) = 3e^{2v_g} + 3e^{4v_g} \quad \text{and} \quad Det(A) = 9e^{6v_g} - 8e^{6v_g} = e^{6v_g}.$$

Whence, the eigenvalues are given by

$$\begin{aligned}
\gamma_1 &= \frac{1}{2}\left(Tr(A) + \sqrt{[Tr(A)]^2 - 4\det(A)}\right) \\
&= \frac{1}{2}\left(3e^{2v_g} + 3e^{4v_g} + \sqrt{\left[3e^{2v_g} + 3e^{4v_g}\right]^2 - 4e^{6v_g}}\right) \\
&= \frac{1}{2}\left(3e^{2v_g} + 3e^{4v_g} + \sqrt{9e^{4v_g} - 18e^{6v_g} + 9e^{8v_g}}\right).
\end{aligned}$$

and

$$\begin{aligned}
\gamma_2 &= \frac{1}{2}\left(Tr(A) - \sqrt{[Tr(A)]^2 - 4\det(A)}\right) \\
&= \frac{1}{2}\left(3e^{2v_g} + 3e^{4v_g} - \sqrt{\left[3e^{2v_g} + 3e^{4v_g}\right]^2 - 4e^{6v_g}}\right) \\
&= \frac{1}{2}\left(3e^{2v_g} + 3e^{4v_g} - \sqrt{9e^{4v_g} - 18e^{6v_g} + 9e^{8v_g}}\right).
\end{aligned}$$

Using the above eigenvalues, the ghost particle is given by

$$\begin{aligned}
g(v_g) &= -2\frac{\partial^2}{\partial v_g^2}\log\left[\gamma_1(v_g)\right] \\
&= -2\frac{\partial^2}{\partial v_g^2}\log\left[\frac{1}{2}\left(3e^{2v_g} + 3e^{4v_g} + \sqrt{9e^{4v_g} - 18e^{6v_g} + 9e^{8v_g}}\right)\right].
\end{aligned}$$

and the anti-ghost particle is given by

$$\begin{aligned}
\overline{g}(v_g) &= -2\frac{\partial^2}{\partial v_g^2}\log\left[\gamma_2(v_g)\right] \\
&= -2\frac{\partial^2}{\partial v_g^2}\log\left[\frac{1}{2}\left(3e^{2v_g} + 3e^{4v_g} - \sqrt{9e^{4v_g} - 18e^{6v_g} + 9e^{8v_g}}\right)\right]
\end{aligned}$$

and $v_g = x - 4k_g^2 t = x - 144t$ is the ghost moving frame.

Conclusions

1. Figure 5-1 shows the interaction of two solitons through a series of time and match the results we get them in chapter two.
2. Figure 5-2 shows the movement of the bigger soliton through a series of time and splitting occurs in the fourth frame.
3. Figure 5-3 shows the movement of the smaller soliton through a series of time and splitting occurs in the fourth frame.
4. $\overline{g}(v_g) + g(v_g) = 0.$
5. Both $\overline{g}(v_g),\ g(v_g) = 0$ do not appear in solution due to their cancellation.

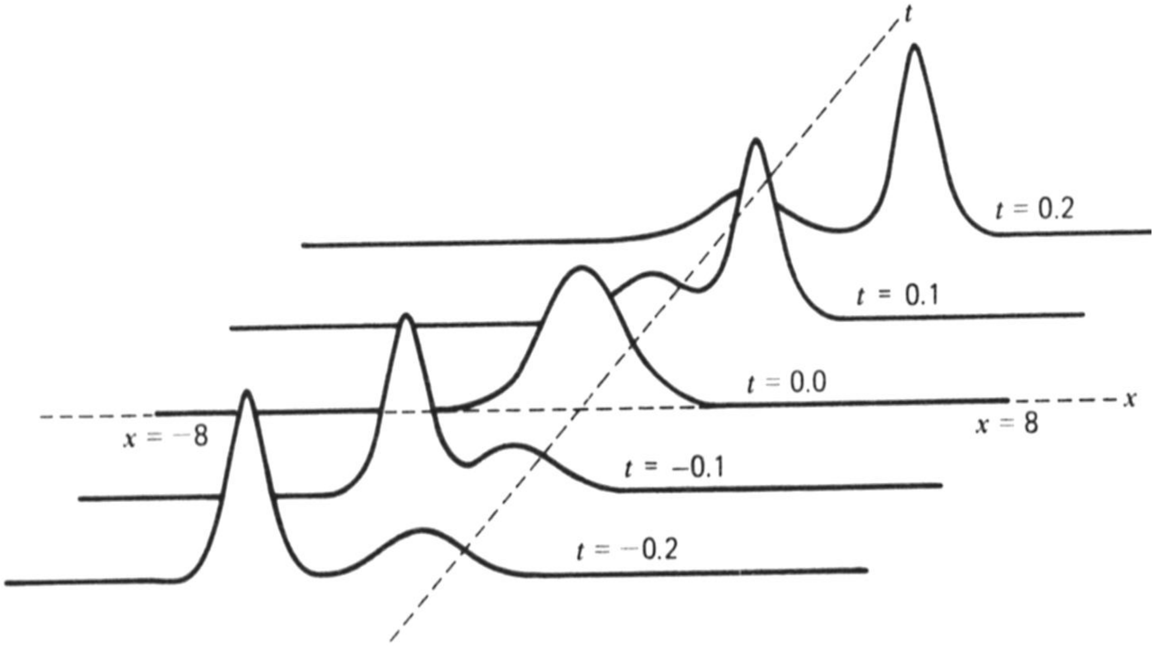


Figure 5-1 The Interaction of Two Soliton Solution for the KdV Equation.

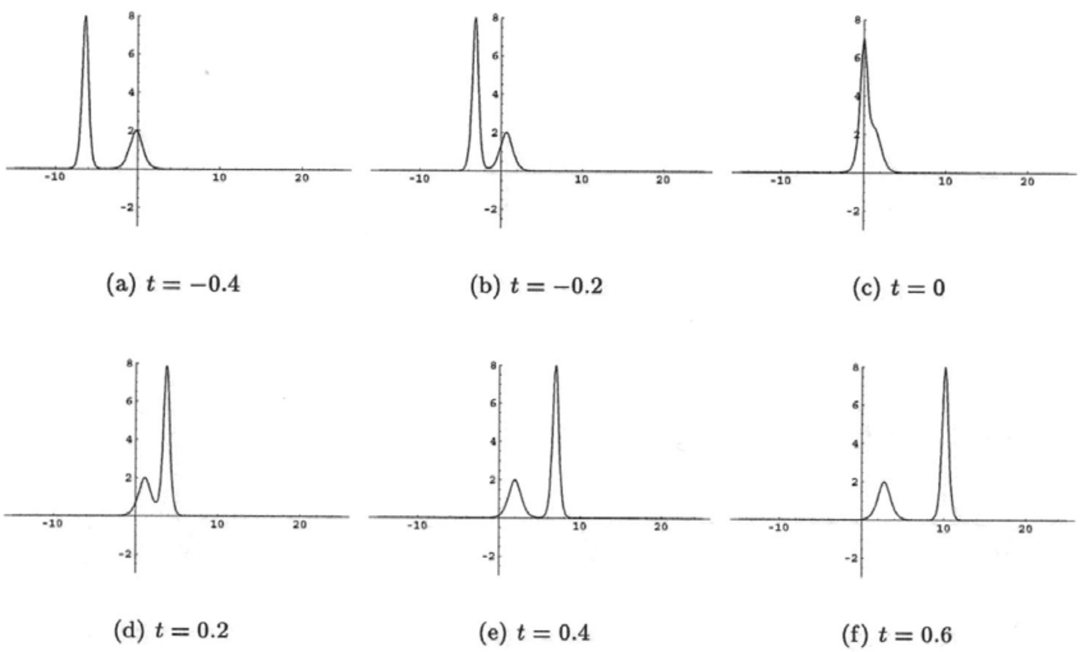


Figure 5-2 Two Soliton Solution Particle Collision for the KdV.

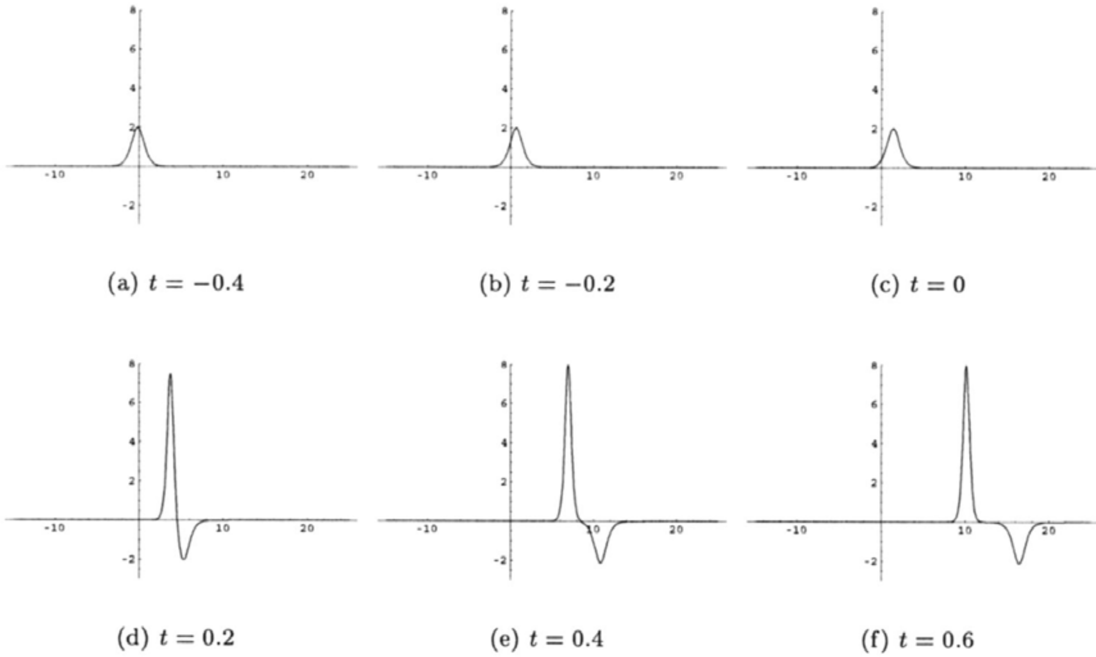


Figure 5-3 The Smaller Soliton for the KdV Equation.

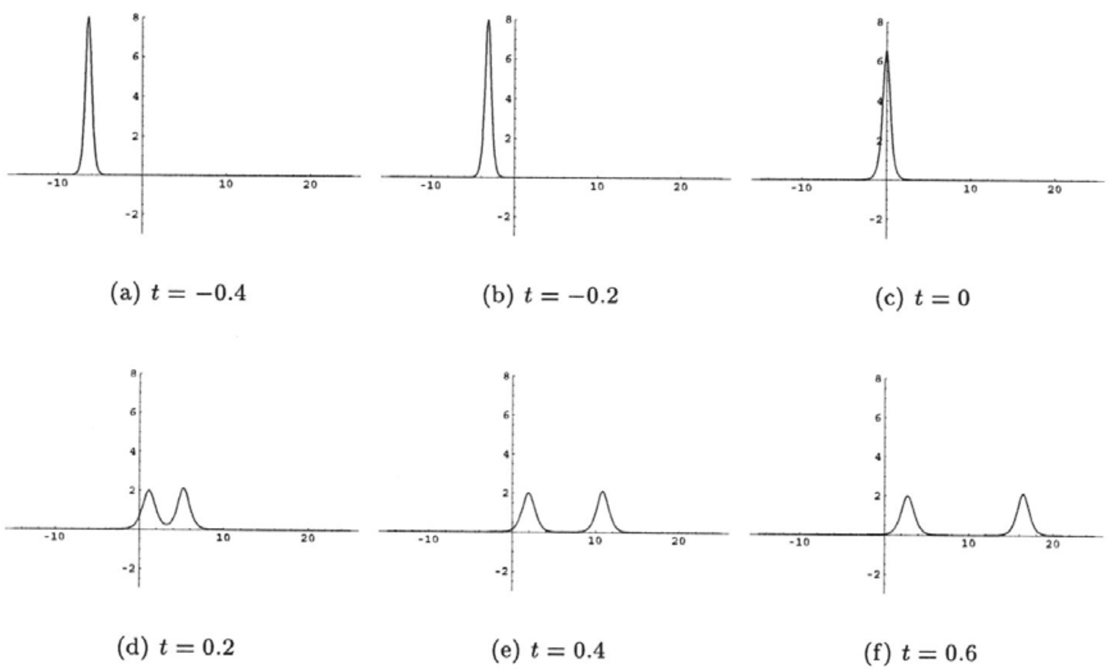


Figure 5-4 The Bigger Soliton for the KdV Equation.

5.5 Decomposition of solution for two solitons

In this section, we try to decompose the decay function for two solitons into soliton and ghost particles. Also, we will show that the sum of the ghost particles result in their fissions. This can be done as follow:

5.5.1 Decomposing the solution into soliton particles and ghost particles

From the equation (5.9), we have

$$u(x,t)=\sum_{i=1}^{2}u_n(x,t)=u_1(x,t)+u_1(x,t)$$

$$=-2\frac{\partial}{\partial x^2}\{Ln(1+\mu_1)\}-2\frac{\partial}{\partial x^2}\{Ln(1+\mu_2)\}$$

$$=-2\left\{\frac{(1+\mu_1)\mu_1''-(\mu_1')^2}{(1+\mu_1)^2}\right\}-2\left\{\frac{(1+\mu_2)\mu_2''-(\mu_2')^2}{(1+\mu_2)^2}\right\}$$

$$=-2\left\{\frac{\mu_1''}{(1+\mu_1)^2}+\frac{\mu_1\mu_1''-(\mu_1')^2}{(1+\mu_1)^2}\right\}-2\left\{\frac{\mu_2''}{(1+\mu_2)^2}+\frac{\mu_2\mu_2''-(\mu_2')^2}{(1+\mu_2)^2}\right\}$$

$$=-2\left\{\frac{\mu_1''}{(1+\mu_1)^2}+\frac{\left(e^{8\bar{k}^2t}\right)^2\left[\gamma_1\gamma_1''-(\gamma_1')^2\right]}{\left(1+e^{8\bar{k}^2t}\gamma_1\right)^2}\right\}-2\left\{\frac{\mu_2''}{(1+\mu_2)^2}+\frac{\left(e^{8\bar{k}^2t}\right)^2\left[\gamma_2\gamma_2''-(\gamma_2')^2\right]}{\left(1+e^{8\bar{k}^2t}\gamma_2\right)^2}\right\}$$

$$=-2\left\{\frac{\mu_1''}{(1+\mu_1)^2}+\frac{\gamma_1\gamma_1''-(\gamma_1')^2}{\left(e^{-8\bar{k}^2t}1+\gamma_1\right)^2}\right\}-2\left\{\frac{\mu_2''}{(1+\mu_2)^2}+\frac{\gamma_2\gamma_2''-(\gamma_2')^2}{\left(e^{-8\bar{k}^2t}+\gamma_2\right)^2}\right\}. \qquad (5.23)$$

Conclusion:

From (5.23), we notice that the decay function consists of two solitons particles and two ghost particles which agree with graphs obtained in section 5.4.

5.5.2 Asymptotic behavior with respect to ghost moving frame V_g

Case (i) Assume that V_g is fixed and $t \to -\infty$ and using (5.23),

$$\lim_{\substack{v_g\,fixed \\ t\to-\infty}} u(x,t)$$

$$= \lim_{\substack{v_g\,fixed \\ t\to-\infty}} \left\{ -2\left\{ \frac{\mu_1''}{(1+\mu_1)^2} + \frac{\gamma_1\gamma_1'' - (\gamma_1')^2}{\left(e^{-8\bar{k}^2 t}1+\gamma_1\right)^2} \right\} - 2\left\{ \frac{\mu_2''}{(1+\mu_2)^2} + \frac{\gamma_2\gamma_2'' - (\gamma_2')^2}{\left(e^{-8\bar{k}^2 t}+\gamma_2\right)^2} \right\} \right\}$$

$$= 0. \tag{5.24}$$

Case (ii) Assume that V_g is fixed and $t \to +\infty$. In the light of this (5.23) gives,

$$\lim_{\substack{v_g\,fixed \\ t\to\infty}} u(x,t)$$

$$= \lim_{\substack{v_g\,fixed \\ t\to\infty}} \left\{ -2\left\{ \frac{\mu_1''}{(1+\mu_1)^2} + \frac{\gamma_1\gamma_1'' - (\gamma_1')^2}{\left(e^{-8\bar{k}^2 t}1+\gamma_1\right)^2} \right\} - 2\left\{ \frac{\mu_2''}{(1+\mu_2)^2} + \frac{\gamma_2\gamma_2'' - (\gamma_2')^2}{\left(e^{-8\bar{k}^2 t}+\gamma_2\right)^2} \right\} \right\}$$

$$= -2\left\{ \frac{\gamma_1\gamma_1'' - (\gamma_1')^2}{(\gamma_1)^2} \right\} - 2\left\{ \frac{\gamma_2\gamma_2'' - (\gamma_2')^2}{(\gamma_2)^2} \right\}$$

$$= -2\left\{ \frac{\partial}{\partial x^2} ln(\gamma_1) \right\} - 2\left\{ \frac{\partial}{\partial x^2} ln(\gamma_2) \right\}$$

$$= g(\gamma_1) + \bar{g}(\gamma_2). \tag{5.25}$$

Conclusions:

1. (5.24) indicates that there is no generation of ghost particles before soliton interaction.
2. (5.25) indicates that the creation of the ghost particles happen as time goes $t \to +\infty$.
3. The sum of the ghost particles results in their fissions.

5.5.3 The sum of the ghost particles and their fissions

From (5.25), the sum of ghost particles is given by:

$$
\begin{aligned}
g(\gamma_1) + \overline{g}(\gamma_2) &= -2\left\{\frac{\partial}{\partial x^2} ln(\gamma_1)\right\} - 2\left\{\frac{\partial}{\partial x^2} ln(\gamma_2)\right\} \\
&= -2\frac{\partial}{\partial x^2}\{ln(\gamma_1) + ln(\gamma_2)\} \\
&= -2\frac{\partial}{\partial x^2} ln(\gamma_1\gamma_2) \\
&= -2\frac{\partial}{\partial x^2} ln(det(A_g)) \\
&= -2\frac{\partial}{\partial x^2} ln(\left[\left(\frac{c_1^2}{2k_1}\frac{c_2^2}{2k_2} - \frac{c_1^2 c_2^2}{2(k_1+k_2)}\right)e^{2(k_1+k_2)v_g}\right]) \\
&= 0. \qquad (5.26)
\end{aligned}
$$

Conclusion:

(5.26) indicates that the sum of the ghost particles results in their fissions and as a result of that they disappear from the solution of the KdV equation (see figure 5-5).

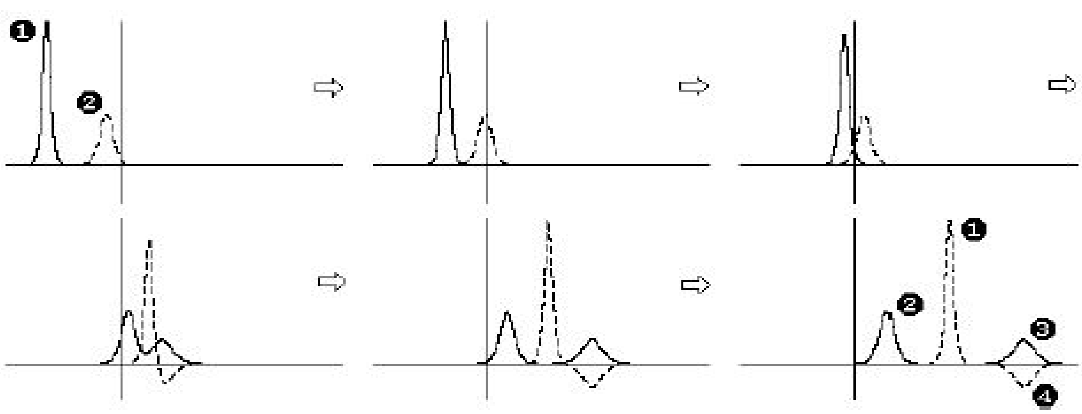


Figure 5-5 Ghost Particles and their Fissions.

APPENDIX (I)

A SOLUTION FOR THE KDV EQUATION OF THE FORM $z(x-ct)$

Consider the KdV equation

$$u_t(x,t)-6u(x,t)u_x(x,t)+u_{xxx}(x,t)=0. \tag{1}$$

In order to find a solitary wave solution for the KdV equation, we may try to look for a solution of the form $u(x,t)=z(x-ct)=z(\xi)$. In the light of this, the standard KdV-equation becomes,

$$-c\frac{dz}{d\xi}+6z\frac{dz}{d\xi}+\frac{d^3z}{d\xi^3}=0. \tag{2}$$

From (2) we notice that it is in the form of total derivative. Therefore, we can integrate it directly to obtain

$$-cz+3z^2+\frac{d^2z}{d\xi^2}=A \tag{3}$$

where A is a constant of integration. In order to obtain a first order differential of z, we multiply the above equation by $\frac{dz}{d\xi}$ to make it in the form of a total derivative

$$-cz\frac{dz}{d\xi}+3z^2\frac{dz}{d\xi}+\frac{1}{2}\left(\frac{dz}{d\xi}\right)^2\frac{dz}{d\xi}=A\frac{dz}{d\xi}. \tag{4}$$

Integrating (4) we get

$$\frac{-c}{2}z^2+z^3+\frac{1}{2}\left(\frac{dz}{d\xi}\right)^2=Az+B \tag{5}$$

where B is a constant of integration. Since, we require z and all its derivatives to vanish as $z\to\pm\infty$, therefore, we must have $A=0\,\&\,B=0.$

Thus the above equation becomes,

$$\frac{-c}{2} z^{2} + z^{3} + \frac{1}{2}\left(\frac{dz}{d\xi}\right)^{2} = 0, \tag{6}$$

which can be written as

$$d\xi = \frac{dz}{z^{2}\sqrt{c-2z}}. \tag{7}$$

Integrating using trigonometric substitution we get,

$$s = \frac{1}{2} c \ \operatorname{sech}^{2}(\theta).$$

In the light of above (7) becomes

$$\begin{aligned}
\int_{\xi_0}^{\xi} dy &= \int_{\theta_0}^{\theta_1} \frac{ds}{s\sqrt{c-2s}} \\
&= \int_{\theta_0}^{\theta_1} \frac{-c \ \sinh(\theta) d\theta}{\frac{1}{2} c \ \operatorname{sech}^{2}(\theta)\sqrt{c} \tanh(\theta)\cosh^{3}(\theta)} \\
&= \frac{-2}{\sqrt{c}} \int_{\theta_0}^{\theta_1} d\theta
\end{aligned}$$

$$\xi - \xi_0 = \frac{-2}{\sqrt{c}}(\theta_1 - \theta_0). \tag{8}$$

By substituting the original variables into equation (8), we obtain

$$z(\xi) = \frac{1}{2} c \ \operatorname{sech}^{2}\left[\frac{1}{2}\sqrt{c}\,(\xi - \xi_0) - \theta_0\right]. \tag{9}$$

Since the constants can be chosen in a variety of ways therefore, by requiring the solution to attain its maximum at the origin, we obtain the simple firm

$$z(\xi) = \frac{1}{2} c \ \operatorname{sech}^{2}\left[\frac{1}{2}\sqrt{c}\,(\xi)\right]. \tag{10}$$

By substituting the original variables equation (10) can be written as

$$u(x,t)=\frac{1}{2}c\ \text{sech}^2\left[\frac{1}{2}\sqrt{c}\left(x-ct\right)\right] \tag{11}$$

which is given in Figure A.

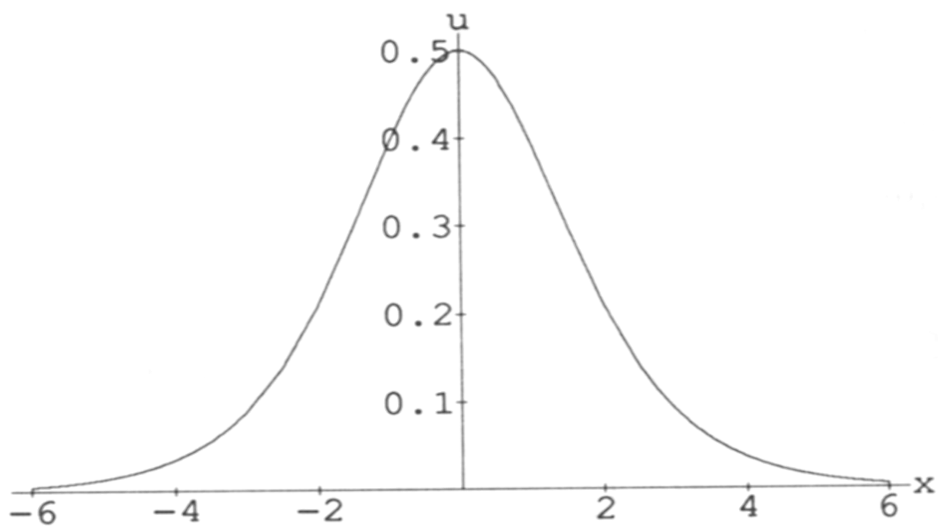


Figure A. The Profile of the Above Solution.

Remarks

1. Since the quantity c appears in the argument of the solution as a square root, therefore, this quantity must be chosen to be positive to meet the conditions (i.e. the function and its first two derivatives vanishes as $\xi \to \pm\infty$ i.e. the solitary wave moves only in the direction of the positive x-axis)
2. The amplitude is proportional to the velocity of the solitary wave. In other words, larger amplitude moves faster than the smaller amplitude solitary wave.

APPENDIX (II)

NON-SOLITARY OSCILLATION IN KDV EQUATION

As we have seen in chapter three through some examples that if the initial condition, $u_0(x)$, yields N discrete eigenvalues $\{\lambda_1, \lambda_2, ..., \lambda_n\}$ and the reflection coefficient is $R(k,t) = 0$ then, standard KdV equation has only N-soliton solution. However, we will see in this appendix that the oscillatory part of the solution comes from the condition $R(k,t) \neq 0$. In fact, the best example that illustrates this idea is the delta function potential which leads to both discrete as well as continuous spectrum.

Example Delta Function Potential

Consider the following initial distribution for the potential

$$u(x,0) = -A\,\delta(x),$$

where the $\delta(x)$ is the Dirac's delta function.

Solution:

We solve the direct scattering problem by substituting the initial condition into the equation

$$-\frac{d^2 w(x)}{dx^2} - A\,\delta(x) w(x) = \lambda w(x), \tag{1}$$

The above equation can be written as,

$$w_{xx} = -(\lambda + A\,\delta(x))w. \tag{2}$$

By integrating both sides we obtain,

$$\int_{-\varepsilon}^{\varepsilon} w_{xx}\, dx = -\int_{-\varepsilon}^{\varepsilon} (\lambda + A\,\delta(x))w \; dx$$
$$\Rightarrow [w_x]_{-\varepsilon}^{\varepsilon} \sim A\,w(0) \text{ as } \varepsilon \to 0. \tag{3}$$

To find the bound states ($\lambda < 0$) we let $\lambda = -k^2$ where k is a positive real number

$$\lambda(0) = -k^2$$
$$\therefore\; w_{xx} - k^2 w = 0 \quad for\ x \to \pm\infty \tag{4}$$

The above equation can be easily seen to yield

$$w(x) = \begin{cases} e^{-kx} & as \quad x > 0. \\ e^{kx} & as \quad x < 0. \end{cases} \tag{5}$$

Using jump condition for w_x at $x = 0$ we get,

$$[w_x]_{-0}^{0} = (-k) - (k) = -A\,w(0) = -A \Rightarrow 2k = A \Rightarrow k = \frac{1}{2}A$$

$$\lambda = -\frac{1}{4}A^2. \tag{6}$$

To find the normalization constant, we use the definition in (chapter two),

$$c(0) \;=\; \frac{1}{\int_{-\infty}^{\infty} w^2(x)} = \frac{1}{\int_{-\infty}^{0} e^{2kx} dx + \int_{0}^{\infty} e^{-2kx} dx} = \frac{1}{2\int_{0}^{\infty} e^{-2kx} dx} = \frac{1}{2}A$$

$$c(0) = \frac{1}{2}A. \tag{7}$$

Note that the discrete spectrum will be empty if $A < 0$ and to find the unbound states, we put $\lambda(0) = k^2$ where k is a positive real number

$$\begin{aligned} w(x) &= \begin{cases} e^{-ikx} + b(k,0)e^{ikx} & as \quad x \to +\infty. \\ a(k,0)e^{-ikx} & as \quad x \to -\infty. \end{cases} \\ &= \begin{cases} e^{-ikx} + b(k,0)e^{ikx} & as \quad x \to +\infty. \\ [1 + b(k,0)]e^{-ikx} & as \quad x \to -\infty. \end{cases} \end{aligned} \tag{8}$$

In order to find the reflection and transmission coefficients, we apply the continuity condition at the origin

$$\left[w_x\right]_{-0}^{0} = (-ik + ikb)e^0 - (-ik)(1+b)e^0 = -Aw(0) = -A(1+b).$$

$$(-ik + ikb)e^0 - (-ik)(1+b) = -A(1+b).$$

$$b(k,0) = -\frac{A}{A+2ik} \qquad \& \qquad a(k,0) = 1 - \frac{A}{A+2ik}. \tag{9}$$

Using the definition in chapter two, we obtain the evolution equation

$$\begin{array}{ll} i)\ \lambda(t) = -\frac{1}{2}A^2. & ii)\ b(k,t) = -\frac{Ae^{8ik^3t}}{A+2ik}. \\ iii)\ a(k,t) = 1 - \frac{A}{A+2ik}. & iv)\ c(t) = -\frac{A}{2}e^{A^3t}. \end{array} \tag{10}$$

Now substituting these equation into the kernel , we obtain

$$B(\xi,t) = \sum_{m=1}^{N} c_m^2(t)\exp(-k_m\xi) + \frac{1}{2\pi}\int_{-\infty}^{\infty} R(k,t)\exp(ik\xi)dk$$

$$B(\xi,t) = C_1(t)\exp(-k_1\xi) - \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{Ae^{8ik^3t+ik\xi}}{A+2ik}dk. \tag{11}$$

Substituting the above kernel (11) into the Gel'fand and Levitan equation,

$$K(x,y,t) + B(x+y,t) + \int_{x}^{\infty} B(y+z,t)K(x,z,t)dz = 0 \tag{12}$$

Unfortunately, we cannot solve the above integral equation in term of well-known analytic function. However, we can approximate the solution numerically (see figure B).

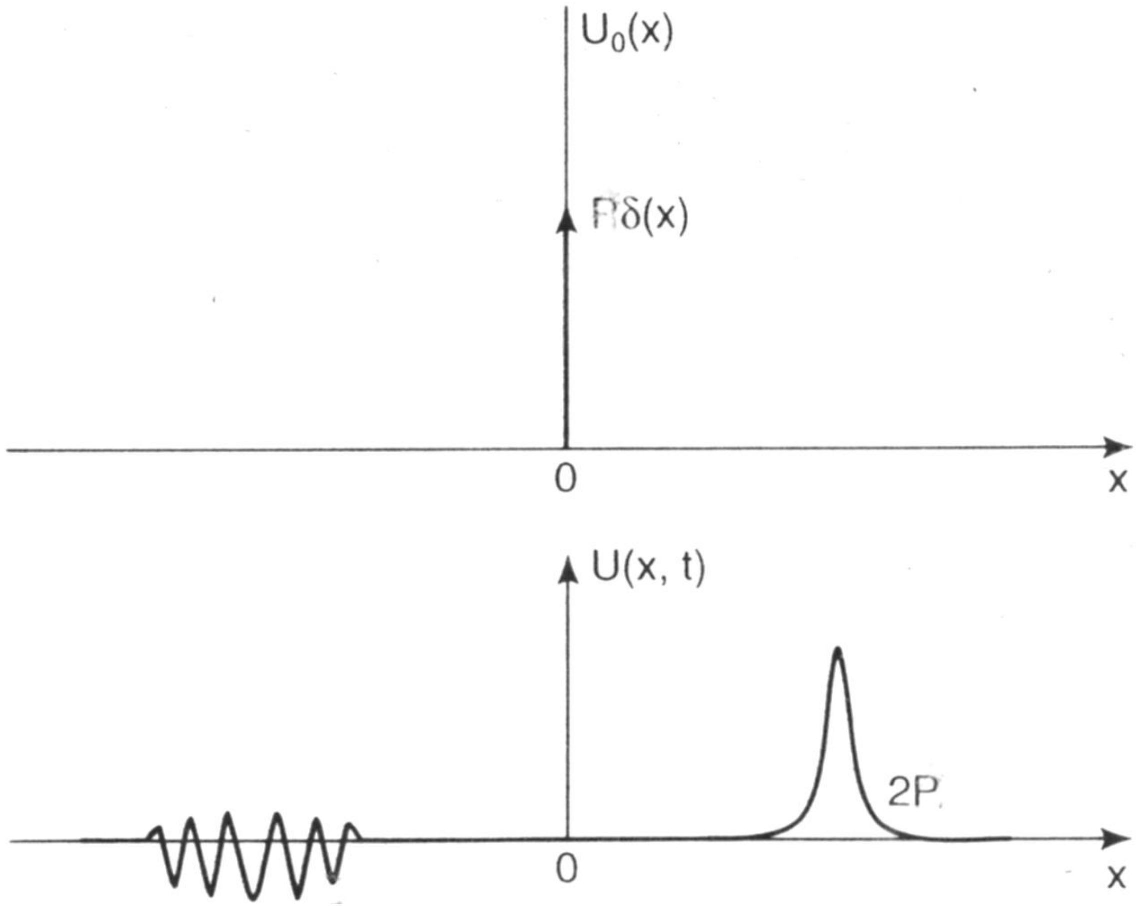


Figure B. The Profile for Scattering by Delta Function.

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