WAVELETS AND SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS

BY

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In the name of Allah, the most gracious, the most merciful

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THESIS ABSTRACT

Full Name of Student: **Mohammed Hussain Al-Lail**

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In this thesis, we present continuous and discrete wavelet transform. The Multiresolution Analysis (MRA) and compactly supported wavelets are introduced. Different approaches of using wavelets to solve differential equations are presented. Finally, some work that we have done during our research is discussed.

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خلاصة الرسالة

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في هذه الرسالة سوف نستعرض المويجات وتحويليها المتصل والمنفصل. ثم سنعرض التحليل المتعدد (Multiresolution Analysis) باستخدام المويجات. بعد ذلك سوف نناقش عدة طرق لحل المعادلات التفاضلية باستخدام المويجات. وأخيرا سنناقش بعض النتائج التي حصلنا عليها خلال البحث.

درجة الماجستير في العلوم الرياضية

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PREFACE

Wavelet analysis has been one of the major research areas in science and engineering in the last 15 years. More and more, mathematicians and scientists are joining this exciting area. Wavelet analysis has had a great impact on areas such as approximation theory, harmonic analysis, differential and integral equations, and scientific computation. It has shown great potential in applications to information technology such as data compression, image processing, and computer graphics.

The work is organized as follows. Chapter 1 gives general introduction to wavelets, basic idea of Fourier transforms, and continuous and discrete wavelet transform. In Chapter 2, the idea of multiresolution analysis and construction of wavelets are presented. This chapter includes decomposition and reconstruction algorithms for wavelets. Finally, Chapter 3 presents different apporoches of using wavelets to solve ordinary differential equations. Some work that we have done during our research is disscussed.

Chapter 1
WAVELET TRANSFORM

1.1 Introduction

Wavelet theory involves representing general functions in terms of simpler building blocks at different scales and positions. The fundamental idea behind wavelets is to analyze according to scale. Wavelets are mathematical tools that cut up data or functions or operators into different frequency components, and then study each component with a resolution matching to its scale.

Everywhere around us there are signals that can be analyzed. For example, there are human speech, engine vibrations, medical images, financial data, music, and many other type of signals. Wavelet analysis is a new and promising set of tools and techniques for analyzing these signals.

In the history of mathematics, wavelet analysis shows many different origins. Much of the work was performed in the 1930s [19]. Before 1930, the main branch of mathematics leading to wavelets began with Joseph Fourier with his theory of frequency analysis. He asserted in 1807 that any 2π-periodic function $f(x)$ could be represented by the sum

$$f(x) = a_0 + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx) \tag{1.1}$$

which is called Fourier series. The coefficients a_k and b_k are calculated by

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx, \tag{1.2}$$

$$a_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(kx)dx, \quad k = 1, 2, \cdots, \tag{1.3}$$

$$b_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(kx)dx \qquad k = 1, 2, \cdots. \tag{1.4}$$

Fourier's statement played an essential role in the evolution of the ideas mathematicians had about the functions. He opened up the door to a new functional universe.

The first mention of wavelets appeared in an appendix to the thesis of A. Haar (1909). Haar asked himself this question in his thesis: Does there exist another orthonormal system $h_1, h_2, ..h_n,$ of functions defined on [0, 1] such that, for any continuous function f defined on [0,1], the series

$$< f, h_o > h_o + < f, h_1 > h_1 + \cdots\cdots + < f, h_n > h_n + \cdots\cdots \tag{1.5}$$

converges to f uniformly on $[0, 1]$? Here the inner product $<\,,\,>$ is defined as

$$< f, g > = \int_0^1 f(x)\bar{g}(x)dx \tag{1.6}$$

where $\bar{g}(x)$ is the complex conjugate of $g(x)$.

In 1909, Haar discovered the simplest solution and at the same time opened a route leading to wavelets.

Haar introduced the functions h and $h_n(x)$ as

$$h(x) = \begin{cases} 1, & 0 \le x < \frac{1}{2}, \\ -1, & \frac{1}{2} \le x < 1, \\ 0, & \text{otherwise}, \end{cases} \tag{1.7}$$

$$h_n(x) = 2^{j/2}h(2^j x - k),$$

where $n = 2^j + k$, $j \geq 0$, $0 \leq k < 2^j$, j, k, n are integers. To complete the set, let $h_0(x) = 1$ on [0,1). Then the sequence

$$h_0, h_1, \cdots\cdots, h_n, \cdots\cdots$$

is an orthonormal basis for $L^2[0,1]$. $h(x)$ defined in (1.7) is known as the Haar wavelet. One property of the Haar wavelet is that it has compact support, which means that it vanishes outside a finite interval. Unfortunately, Haar wavelets are not continuously differentiable which somewhat limits their applications [7], [11].

1.2 Fourier Analysis

Fourier's representation of functions as a superposition of sines and cosines has become very important for both the analytic and numerical solution of differential equations and for the analysis and treatment of communication signals [13].

Let $L^2(0, 2\pi)$ denote the space of all measurable functions defined on the interval $(0, 2\pi)$ with the following condition:

$$\int_0^{2\pi} |f(x)|^2 \, dx < \infty.$$

This collection is often called the space of 2π-periodic square integrable functions. Any function f in $L^2(0, 2\pi)$ can be represented in the form

$$f(x) = \sum_{-\infty}^{\infty} c_n e^{inx}, \tag{1.8}$$

where the constants c_n are called Fourier coefficients of f, and can be calculated by

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} dx. \tag{1.9}$$

The Fourier transform is used to analyze the frequency content of a signal in the time domain. It transforms a function in the time domain into a function in the frequency domain. The signal can then be analyzed for its frequency content. An inverse Fourier transform takes the data from the frequency domain into the time domain.

A function f is considered to be a *square-integrable* function if

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx < \infty.$$

The function space $L^2(R)$ is the space of all square integrable functions defined on R.

$$L^2(R) = \left\{ f : \int_{-\infty}^{\infty} |f(x)|^2 \, dx < \infty \right\}$$

with the L^2- norm defined by

$$\|f\|_2 = \left[\int_{-\infty}^{\infty} |f(x)|^2 \, dx \right]^{\frac{1}{2}} < \infty.$$

Elements of $L^2(R)$ are called square integrable functions. Many functions in physics and engineering are square integrable.

Definition 1.1 Let $f, g \in L^2(R)$ then, the inner product is defined by

$$< f, g >= \int_{-\infty}^{\infty} f(x)\bar{g}(x)dx, \tag{1.10}$$

where $\bar{g}(x)$ is the complex conjugate of $g(x)$. It is clear that

$$< f, f >= \|f\|_2^2 \, .$$

Definition 1.2 The Fourier transform of a function $f \in L^2(R)$ defined by

$$(Ff)(w) = \hat{f}(w) = \int_{-\infty}^{\infty} f(x)e^{-iwx}dx, \tag{1.11}$$

x is called the time variable and w is called frequency variable. The Fourier transformation (F) takes $L^2(R)$ onto itself.

Definition 1.3 The inverse Fourier transform (F^{-1}) of $g(w)$ is defined by

$$(F^{-1}g)(x) = \check{g}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(w)e^{iwx}dw. \tag{1.12}$$

As we said Fourier analysis is a mathematical technique for transforming our view of the signal from time domain to frequency domain.

Theorem 1.1 *(Parseval's Formula) [7], [13]* Suppose $f, g \in L^2(R)$. Then we have

$$< f, g >= \frac{1}{2\pi} < \hat{f}, \hat{g} > . \tag{1.13}$$

For many signals, Fourier analysis is extremely useful because the signal's frequency content is of great importance. So why do we need other techniques, like wavelet analysis?

The Fourier transform of a signal does not contain any local information and it just enables us to investigate problems either in the time domain or in the frequency domain, but not simultaneously in both domains. These are the major weaknesses of the Fourier transform analysis.

In transforming to the frequency domain, time information is lost. When looking at a Fourier transform of a signal, it is impossible to tell when a particular event took place. If the signal properties do not change much over time, that is, if it is what is called a stationary signal, the weaknesses of the Fourier transform isn't very important. However, most interesting signals contain numerous non-stationary characteristics. These characteristics are often the most important part of the signal, and Fourier analysis is not suited to detecting them.

In an effort to correct the weaknesses of Fourier transform, Dennis Gabor (1946) introduced the windowed Fourier transform to measure localized frequency components of sound wave. He adapted the Fourier transform to analyze only a small section of the signal at a time, a technique called windowing the signal. Windowed Fourier Transform maps a signal into a two-dimensional function of time and frequency. The Windowed Fourier transform can be used to give information about signals simultaneously in the time domain and in the frequency domain. Gabor first introduced the windowed Fourier transform by using a Gaussian distribution function as a window function. His major idea was to use a time-localization window function $g(x-t)$ for extracting local information from the Fourier transform of a signal, where parameter t is used to translate the window in order to cover the whole time domain. The idea is to use this window function in order to localize the Fourier transform, then shift the window to another position, and so on. See Figure 1.1.

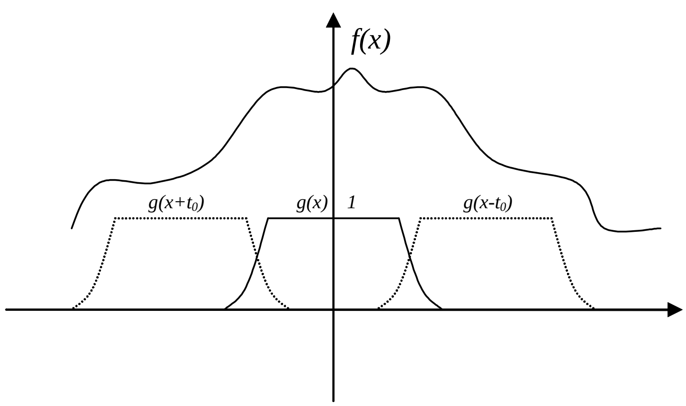

Figure 1.1. The windowed Fourier transform.

In the windowed Fourier transform, the function $f(x)$ is multiplied with the window function $g(x)$ and the Fourier coefficients of the product $f(x)g(x)$ are computed. Then, the procedure is repeated for translated versions of the windows, $g(x-t_0)$, $g(x-2t_0)$, $\cdots$.

Definition 1.4 The Windowed Fourier Transform of a function f with respect to a window function g denoted by $G(f)(w,t)$ is defined by

$$G(f)(w,t) = \tilde{f}_g(w,t) = \int_{-\infty}^{\infty} f(x)g(x-t)e^{-iwx}dx \tag{1.14}$$

where $f, g \in L^2(R)$.

Clearly, the windowed Fourier transform of a given function f depends on both time t and frequency w. For more detail see [13].

Definition 1.5 The inversion formula for the Windowed Fourier transform (G^{-1}) is given by

$$G^{-1}\left(\tilde{f}_g(w,t)\right) = f(x) = \frac{1}{2\pi \|g\|^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{f}_g(w,t)\overline{g(x-t)}e^{iwt}dtdw.$$

With the Windowed Fourier Transform, the input signal $f(x)$ is chopped up into sections, and each section is analyzed for its frequency content separately.

The Windowed Fourier Transform provides some information about both when and at what frequencies a signal event occurs. The drawback of the Windowed Fourier Transform is that once you choose a size for the window, that window is the same for all frequencies. For more details [13], [19].

1.3 Continuous Wavelet Transforms

Unlike Fourier analysis, in which we analyze signals using sines and cosines, now we use wavelet functions. One of the main reasons for the discovery of wavelets and wavelet transforms is that the Fourier transform analysis does not contain the local information of signals. Wavelet analysis is a new method for solving difficult problems in mathematics, physics, and engineering [7], [11], [13].

Definition 1.6 A wavelet is a function $\psi \in L^2(R)$ which satisfies the condition

$$C_\psi = \int_{-\infty}^{\infty} \frac{\left|\hat{\psi}(\xi)\right|^2}{|\xi|} d\xi < \infty, \tag{1.15}$$

where $\hat{\psi}(\xi)$ is the Fourier transform of $\psi(x)$.

The condition (1.15) is called the wavelet admissibility condition and it is required for finding the inverse of the continuous wavelet transform.

Based on the idea of wavelets as a family of functions constructed from translation and dilation of a single function ψ called the mother wavelet, we define family of wavelets by

$$\psi_{a,b}(x) = \frac{1}{\sqrt{|a|}} \psi(\frac{x-b}{a}), \quad a, b \in R,\ a \neq 0, \tag{1.16}$$

where a is called a scaling parameter which measures the degree of compression or scale, and b is translation parameter which determines the time location for the wavelet.

Remark 1.1

1. Scaling a wavelet simply means stretching (or compressing) it. If $|a| < 1$, the wavelet $\psi_{a,b}(x)$ is the compressed version of the mother wavelet $\psi(x)$ and corresponds mainly to higher frequencies. However, at a large scale, the wavelet $\psi_{a,b}(x)$ is stretched version of the mother wavelet $\psi(x)$ and corresponds lower frequencies.

2. As the scale a decreases, the resolution in the time domain increases (the time resolution becomes finer) while that in the frequency domain increases (the frequency resolution becomes coarser).

3. Since $\psi(x) \in L^2(R)$, then $\psi_{a,b}(x) \in L^2(R)$ for all a, b, because

$$\begin{aligned} \left\|\psi_{a,b}(x)\right\|^2 &= \frac{1}{|a|}\int_{-\infty}^{\infty}\left|\psi(\frac{x-b}{a})\right|^2 dx \\ &= \int_{-\infty}^{\infty}|\psi(x)|^2\,dx = \|\psi\|^2. \end{aligned} \tag{1.17}$$

4. The Fourier transform of $\psi_{a,b}(x)$

$$\hat{\psi}_{a,b}(w) = \frac{1}{\sqrt{|a|}}\int_{-\infty}^{\infty}\psi(\frac{x-b}{a})e^{-iwx}dx = \sqrt{|a|}e^{-iwb}\hat{\psi}(aw). \tag{1.18}$$

We sketch a typical mother wavelet in the Figure 1.2 (a). Different values of the parameter b represent the time localization center, and each $\psi_{a,b}(x)$ is localized around the center $x = b$. As a scale parameter a varies, wavelet $\psi_{a,b}(x)$ covers different frequency ranges. Large values of $|a|$ ($|a| \gg 1$) result in very wide windows and correspond to small frequencies Figure 1.2 (b). However, small values of $|a|$ ($|a| \ll 1$) result in very narrow windows and correspond to high frequencies as shown in Figure 1.2 (c).

Definition 1.7 Let ψ be a wavelet. The continuous wavelet transform T_ψ of $f \in L^2(R)$ with respect to the wavelet ψ is defined on $L^2(R)$ by

$$\begin{aligned} (T_\psi f)(a,b) &= < f, \psi_{a,b} >= \int_{-\infty}^{\infty} f(x)\overline{\psi_{a,b}(x)}dx \\ &= \frac{1}{\sqrt{|a|}}\int_{-\infty}^{\infty} f(x)\overline{\psi(\frac{x-b}{a})}dx, \end{aligned} \tag{1.19}$$

where $a \in R\backslash\{0\}, b \in R$.

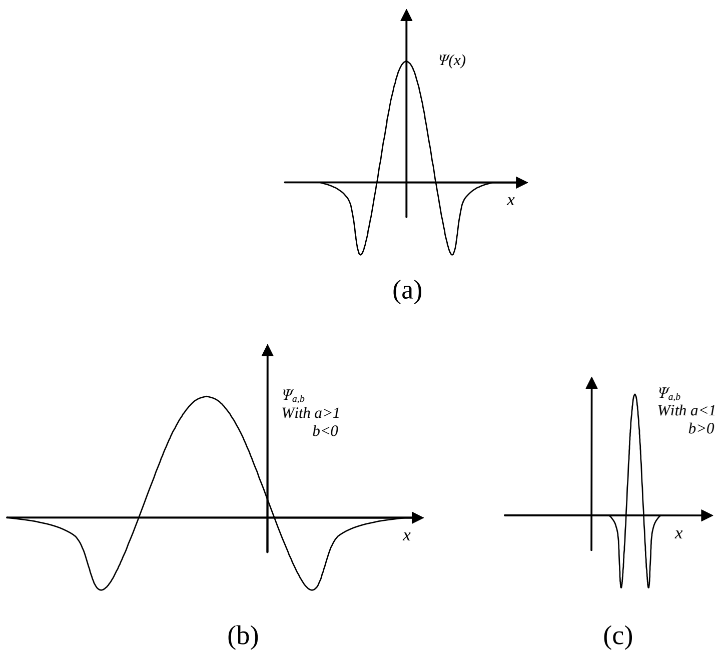

Figure 1.2. Typical mother wavelet.

Example 1.1 The Haar wavelet is one of the most fundamental examples that illustrates major features of the general wavelet theory. It is defined by

$$\psi(x) = \begin{cases} 1, & 0 \le x < \frac{1}{2} \\ -1, & \frac{1}{2} \le x < 1 \\ 0, & \text{otherwise.} \end{cases} \tag{1.20}$$

The Haar wavelet has compact support [0,1] and satisfies

$$\int_{-\infty}^{\infty} \psi(x) dx = 0,$$

and

$$\int_{-\infty}^{\infty} |\psi(x)|^2 \, dx = 1.$$

This wavelet is well localized in the time domain, but it is discontinuous at $x = 0, \frac{1}{2}, 1$. The Fourier transform of the Haar wavelet is calculated as follows:

$$\begin{aligned}\hat{\psi}(\xi) &= \int_0^{\frac{1}{2}} e^{-i\xi x} dx - \int_{\frac{1}{2}}^{1} e^{-i\xi x} dx \\ &= \frac{1}{-i\xi}\left\{\left[e^{-i\xi x}\right]_0^{\frac{1}{2}} - \left[e^{-i\xi x}\right]_{\frac{1}{2}}^{1}\right\} \\ &= \frac{i}{\xi}\left\{2e^{-\frac{i\xi}{2}} - e^{-i\xi} - 1\right\} \\ &= \frac{4\sin^2(\frac{\xi}{4})}{\xi} e^{-\frac{i}{2}(\xi-\pi)}.\end{aligned}$$

The $\psi(x)$ and $\left|\hat{\psi}(\xi)\right|$ are sketched in Figure 1.3 and Figure 1.4 respectively.

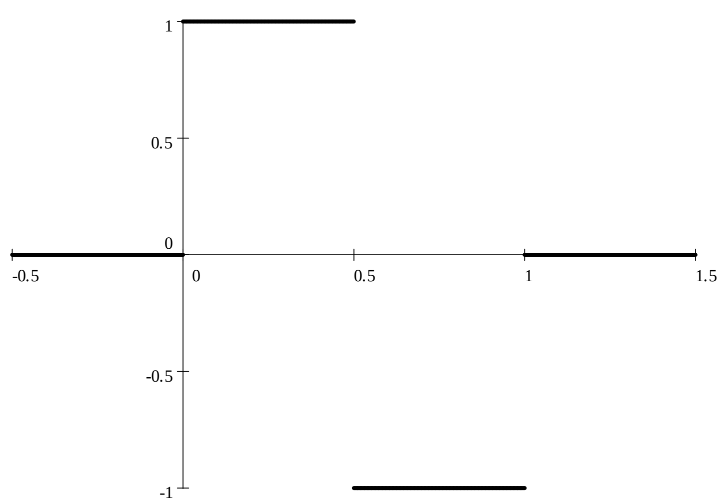

Figure 1.3. The Haar wavelet $\psi(x)$.

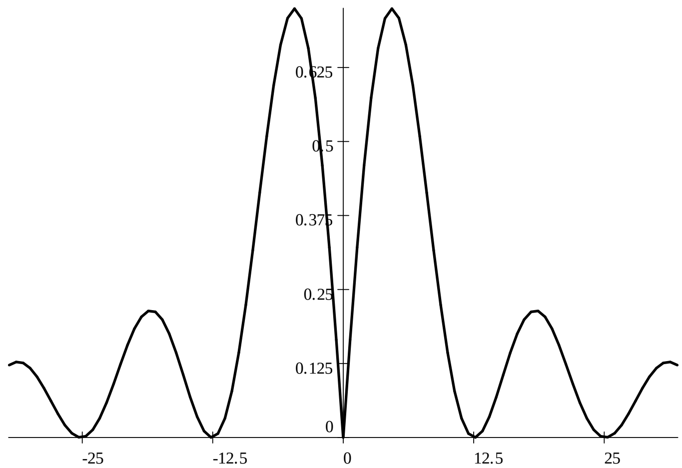

Figure 1.4. The Fourier Transform of Haar wavelet $\left|\widehat{\psi}\right|$.

The following theorem is useful to generate new wavelets.

Theorem 1.2 If ψ is a wavelet and φ a bounded integrable function, then the convolution function $\psi * \varphi$ is a wavelet [13], [26].

Proof. Since,

$$\begin{aligned}
\int_{-\infty}^{\infty} |\psi * \varphi(x)|^2 \, dx &= \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} \psi(x-y)\varphi(y)dy \right|^2 dx \\
&\leq \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |\psi(x-y)| \, |\varphi(y)| \, dy \right)^2 dx \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |\psi(x-y)| \sqrt{|\varphi(y)|}\sqrt{|\varphi(y)|}dy \right)^2 dx \\
&\leq \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |\psi(x-y)|^2 \, |\varphi(y)| \, dy \int_{-\infty}^{\infty} |\varphi(y)| \, dy \right) dx \\
&\leq \int_{-\infty}^{\infty} |\varphi(y)| \, dy \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\psi(x-y)|^2 \, |\varphi(y)| \, dx dy \\
&= \left(\int_{-\infty}^{\infty} |\varphi(y)| \, dy \right)^2 \int_{-\infty}^{\infty} |\psi(x)|^2 \, dx < \infty,
\end{aligned}$$

we have, $\psi * \varphi \in L^2(R)$. Moreover,

$$\begin{aligned} C_{\psi*\varphi} &= \int_{-\infty}^{\infty} \frac{\left|\widehat{\psi*\varphi}(\xi)\right|^2}{|\xi|} d\xi = \int_{-\infty}^{\infty} \frac{\left|\hat{\psi}(\xi)\hat{\varphi}(\xi)\right|^2}{|\xi|} d\xi \\ &= \int_{-\infty}^{\infty} \frac{\left|\hat{\psi}(\xi)\right|^2 |\hat{\varphi}(\xi)|^2}{|\xi|} d\xi \leq \sup |\hat{\varphi}(\xi)|^2 \int_{-\infty}^{\infty} \frac{\left|\hat{\psi}(\xi)\right|^2}{|\xi|} d\xi < \infty. \end{aligned}$$

Therefore, the convolution function $\psi * \varphi$ is a wavelet. ■

Example 1.2 The convolution of the Haar wavelet with the following function

$$\varphi(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

generate a wavelet as shown in Figure 1.5.

Example 1.3 By convolution of the Haar wavelet with $\varphi(x) = e^{-x^2}$, we get an infinitely differentiable (or smooth) wavelet , as shown in Figure 1.6.

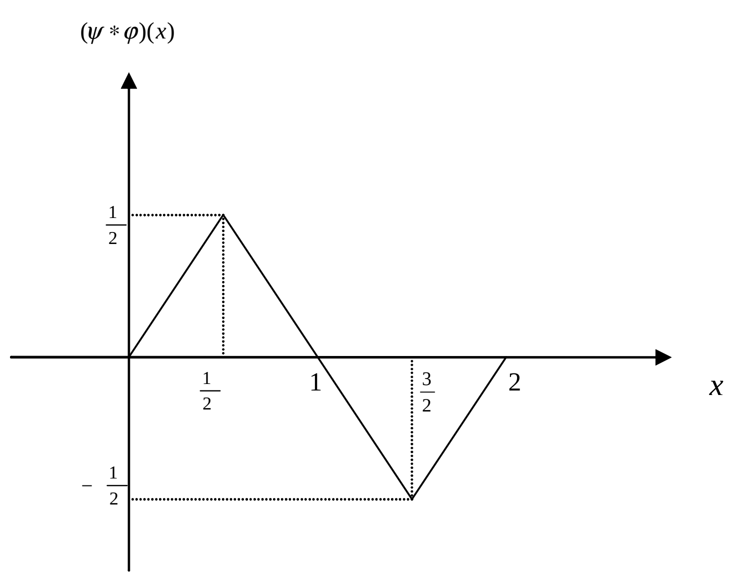

Figure 1.5. The convolution of the Haar wavelet with $\varphi(x)$.

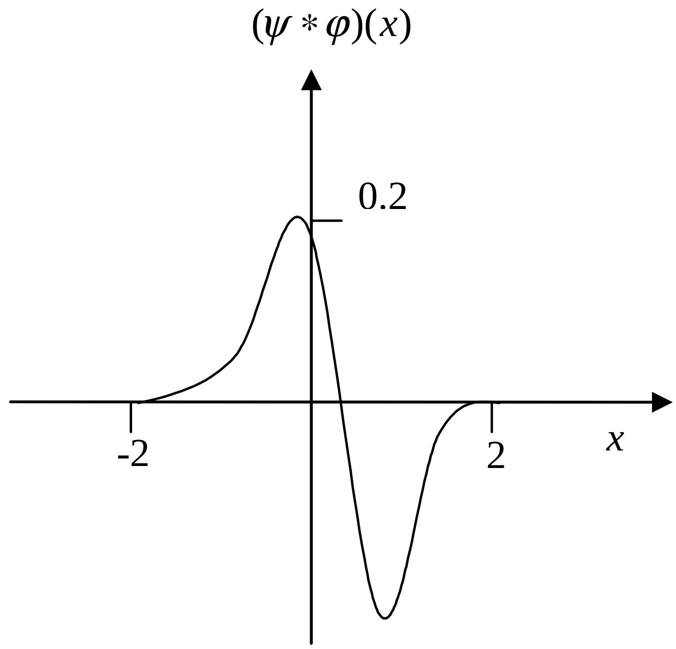

Figure 1.6. The convolution of Haar wavelet with $\varphi(x) = e^{-x^2}$.

Definition 1.8 The k^{th} moment of a wavelet ψ is defined by

$$m_k = \int_{-\infty}^{\infty} x^k \psi(x) dx. \tag{1.21}$$

A wavelet ψ has n vanishing moments if

$$\int_{-\infty}^{\infty} x^k \psi(x) dx = 0 \quad \text{for} \quad k = 0, 1, \ldots, n. \tag{1.22}$$

Or, equivalently,

$$\left[\frac{d^k \hat{\psi}(\xi)}{d\xi^k} \right]_{\xi=0} = 0 \quad \text{for} \quad k = 0, 1, \ldots, n. \tag{1.23}$$

Since the wavelet transform is expressed as the inner product of f with $\psi_{a,b}$, it is linear. The following properties can be proved by using the properties of the inner product.

Let ψ and φ be wavelets and let f and g be functions of $L^2(R)$. Then the following relations hold [13], [26] :

1. *(Isometry)* The wavelet transform $T_\psi f$ is an isometry, that is,

$$\|(T_\psi f)(a,b)\|_{L^2} = \|f\|_{L^2}\,. \tag{1.24}$$

2. *(Translation)*

$$T_\psi(S_c f)(a,b) = T_\psi f(a, b-c) \tag{1.25}$$

where S_c is the translation operator defined by

$$S_c f(x) = f(x-c).$$

2. *(Dilation)*

$$T_\psi(D_c f)(a,b) = \frac{1}{\sqrt{c}} T_\psi f(\frac{a}{c}, \frac{b}{c}), \quad c > 0, \tag{1.26}$$

where c is a positive number and D_c is a dilation operator defined by

$$D_c f(x) = \frac{1}{c} f(\frac{x}{c}).$$

3. *(Antilinearity)*

$$(T_{(\alpha\psi+\beta\varphi)} f)(a,b) = \alpha(T_\psi f)(a,b) + \beta(T_\varphi f)(a,b), \tag{1.27}$$

where α and $\beta \in R$.

4. *(Symmetry)*

$$(T_\psi \varphi)(a,b) = (T_\varphi \psi)(\frac{1}{a}, -\frac{b}{a}), a \neq 0. \tag{1.28}$$

5. *(Parity)*

$$T_{P\psi}(Pf)(a,b) = (T_\psi f)(a,-b), \tag{1.29}$$

where P is the parity operator defined by $Pf(x) = f(-x)$.

6.

$$T_{S_c\psi}(f)(a,b) = (T_\psi f)(a, b+ac), \tag{1.30}$$

7.

$$T_{D_c\psi}(f)(a,b) = \frac{1}{\sqrt{c}} T_\psi f(ac, b) \tag{1.31}$$

Theorem 1.3 *(Parseval's Formula for Wavelet Transforms) [7], [13].* Let ψ be a wavelet. Then, for any functions $f, g \in L^2(R)$, the following formula holds:

$$< f, g >= \frac{1}{C_\psi} \int_{-\infty}^{\infty} \frac{da}{a^2} \int_{-\infty}^{\infty} (T_\psi f)(a,b)\overline{(T_\psi g)(a,b)}db, \tag{1.32}$$

where

$$0 < C_\psi = \int_{-\infty}^{\infty} \frac{\left|\hat{\psi}(w)\right|^2}{|w|} dw < \infty.$$

Proof. By Parseval's Formula for Fourier transform Theorem 1.1, we have

$$\begin{aligned} (T_\psi f)(a,b) &= < f, \psi_{a,b} > \\ &= \frac{1}{2\pi} < \hat{f}, \hat{\psi}_{a,b} > \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(w)\sqrt{|a|}e^{ibw}\overline{\hat{\psi}(aw)}dw, \end{aligned} \tag{1.33}$$

and

$$\begin{aligned} \overline{(T_\psi g)(a,b)} &= \overline{< g, \psi_{a,b} >} \\ &= \int_{-\infty}^{\infty} \bar{g}(x)\frac{1}{\sqrt{|a|}}\psi(\frac{x-b}{a})dx \\ &= \frac{1}{2\pi}\overline{< \hat{g}, \hat{\psi}_{a,b} >} \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{\hat{g}(\eta)}\sqrt{|a|}e^{-ib\eta}\hat{\psi}(a\eta)d\eta. \end{aligned} \tag{1.34}$$

Then, by using (1.33) and (1.34) we have

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(T_\psi f)(a,b)\overline{(T_\psi g)(a,b)}\frac{dbda}{a^2} \\
=&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(w)\frac{e^{ibw}}{\sqrt{|a|}}\overline{\hat{\psi}(aw)}dw\frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{\hat{g}(\eta)}\frac{e^{-ib\eta}}{\sqrt{|a|}}\hat{\psi}(a\eta)d\eta\frac{dbda}{a^2} \\
=&\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dbda}{a^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|a|\,\hat{f}(w)\overline{\hat{g}(\eta)}\overline{\hat{\psi}(aw)}\hat{\psi}(a\eta)e^{ib(w-\eta)}dwd\eta,
\end{aligned}$$

by interchanging the order of integration, we have

$$\begin{aligned}
=&\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{da}{|a|}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\hat{f}(w)\overline{\hat{g}(\eta)}\overline{\hat{\psi}(aw)}\hat{\psi}(a\eta)dwd\eta\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ib(w-\eta)}db\right] \\
=&\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{da}{|a|}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\hat{f}(w)\overline{\hat{g}(\eta)}\overline{\hat{\psi}(aw)}\hat{\psi}(a\eta)dwd\eta\,[\delta_{\eta-w}] \\
=&\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{da}{|a|}\int_{-\infty}^{\infty}\hat{f}(w)\overline{\hat{g}(w)}\overline{\hat{\psi}(aw)}\hat{\psi}(aw)dw, \\
=&\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{da}{|a|}\int_{-\infty}^{\infty}\hat{f}(w)\overline{\hat{g}(w)}\left|\hat{\psi}(aw)\right|^2dw,
\end{aligned}$$

interchange the order of integration again and put $aw = t$, we get

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(T_\psi f)(a,b)\overline{(T_\psi g)(a,b)}\frac{dbda}{a^2} \\
=&\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(w)\overline{\hat{g}(w)}dw\int_{-\infty}^{\infty}\left|\hat{\psi}(t)\right|^2\frac{dt}{|t|}, \\
=&C_\psi\frac{1}{2\pi}<\hat{f},\hat{g}>=C_\psi<f,g>.\blacksquare
\end{aligned}$$

By takin $g = f$, we have

Corollary 1.4 If $f \in L^2(R)$, then

$$C_\psi \|f\|^2 = C_\psi \int_{-\infty}^{\infty} |f(x)|^2 \, dx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |(T_\psi f)(a,b)|^2 \, \frac{dbda}{a^2}. \tag{1.35}$$

This means that except for the factor C_ψ, the wavelet transform is an isometry.

Theorem 1.5 *(Inversion Formula) [7], [13], [26].* If $f \in L^2(R)$, then f can be reconstructed by the formula

$$f(x) = \frac{1}{C_\psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (T_\psi f)(a,b) \psi_{a,b}(x) \frac{dbda}{a^2}, \tag{1.36}$$

where $(T_\psi f)(a,b)$ is a wavelet transform of f and C_ψ is admissibility condition.

Proof. For any $g \in L^2(R)$, we have from Theorem 1.3

$$\begin{aligned} C_\psi \quad < \quad f, g >&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (T_\psi f)(a,b) \overline{(T_\psi g)(a,b)} \frac{dbda}{a^2} \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left((T_\psi f)(a,b) \int_{-\infty}^{\infty} \bar{g}(x) \psi_{a,b}(x) dx \right) \frac{dbda}{a^2} \end{aligned}$$

$$\begin{aligned} C_\psi \quad < \quad f, g >&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (T_\psi f)(a,b) \psi_{a,b}(x) \frac{dbda}{a^2} \right) \bar{g}(x) dx \\ &= \quad < \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (T_\psi f)(a,b) \psi_{a,b}(x) \frac{dbda}{a^2}, g(x) > . \end{aligned}$$

or

$$< C_\psi f - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (T_\psi f)(a,b) \psi_{a,b}(x) \frac{dbda}{a^2}, g >= 0, \text{ for all } g \in L^2(R).$$

Therefore,

$$C_\psi f - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (T_\psi f)(a,b) \psi_{a,b}(x) \frac{dbda}{a^2} = 0$$

or

$$f = \frac{1}{C_\psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (T_\psi f)(a,b) \psi_{a,b}(x) \frac{dbda}{a^2}. \blacksquare$$

1.4 Discrete Wavelet Transform

In many applications [1-5], [15], [18-21], [25], [27], data are represented by a finite number of values, so it is useful to consider a discrete version of the continuous wavelet transform (1.19), by assuming that a and b take only integer values. For a wavelet ψ we can define

$$\psi_{j,k}(x) = a_0^{\frac{j}{2}} \psi(a_0^j x - kb_0), \tag{1.37}$$

where j and $k \in \mathbb{Z}$ and $a_0 > 1$ and $b_0 > 0$ are fixed constants. Then Wavelet transform defined by (1.19) becomes

$$\begin{aligned} (T_\psi f)(j,k) &= < f, \psi_{j,k}(x) > \\ &= a_0^{\frac{j}{2}} \int_{-\infty}^{\infty} f(x) \overline{\psi(a_0^j x - kb_0)} dx. \end{aligned} \tag{1.38}$$

There are two questions:

Q1) Does the sequence $\{< f, \psi_{j,k}(x) >\}_{j,k \in \mathbb{Z}}$ characterize the function f ?

Q2) Is it possible to express any $f(x)$ as the superposition

$$f(x) = \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} < f, \psi_{j,k}(x) > \psi_{j,k}(x)? \tag{1.39}$$

The answer is positive if the wavelets $\psi_{j,k}(x)$ form a complete orthonormal system in $L^2(R)$. For computational efficiency, $a_0 = 2$ and $b_0 = 1$ are commonly used, then

$$\psi_{j,k}(x) = 2^{\frac{j}{2}} \psi(2^j x - k) \tag{1.40}$$

Definition 1.9 *(Riesz Basis) [13], [17].* A sequence of vectors $\{\phi_n\}_{n=1,2,\cdots}$ in a Hilbert space H is called a Riesz basis if for every $f \in H$ there exists a unique sequence $\{c_n\}_{n=1,2,\cdots} \in l^2$, such that, $f = \sum_{n=1}^{\infty} c_n\phi_n$, and there exist two positive constants A and B, where $0 < A \leq B < \infty$, independent of $f \in H$, such that

$$A\sum_{n=1}^{\infty} |c_n|^2 \leq \|f\|^2 \leq B\sum_{n=1}^{\infty} |c_n|^2 .$$

Definition 1.10 A wavelet is a function $\psi \in L^2(R)$ such that the family of functions $\psi_{j,k}$ defined by

$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k), \tag{1.41}$$

where j and k are arbitrary integers, is an orthonormal basis in the Hilbert space $L^2(R)$.

This definition means the following:

1. Orthonormal family of $\{\psi_{j,k}\}$, that is,

$$< \psi_{j,k}, \psi_{m,n} >= \int_{-\infty}^{\infty} \psi_{j,k}(x)\bar{\psi}_{m,n}(x)dx = \delta_{j,m}\delta_{k,n} \tag{1.42}$$

   where j, k, m, n are integer , $\delta_{j,m}$ and $\delta_{k,n}$ are Kronecker delta.

2. If $f \in L^2(R)$ then it can be written as

$$f(x) = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} < f, \psi_{j,k} > \psi_{j,k}(x), \tag{1.43}$$

   where $\mathbb{Z}$ denotes the set of integers.

3. The factor $2^{j/2}$ is included so that the L^2 norm will be the same for all j, k that is, $\psi_{j,k}$ are normalized $\|\psi_{j,k}\| = \|\psi\| = 1$.

4. The wavelet $\psi_{0,1} = \psi$ is called the basic wavelet or mother wavelet.

Definition 1.11 Wavelet coefficients of a function $f \in L^2(R)$, denoted by $c_{j,k}$ are defined as the inner product of f with $\psi_{j,k}$

$$c_{j,k} =< f, \psi_{j,k} >= \int_{-\infty}^{\infty} f(x)\psi_{j,k}(x)dx \tag{1.44}$$

The series

$$\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} < f, \psi_{j,k} > \psi_{j,k}(x),$$

is called the wavelet series of $f \in L^2(R)$. The expression

$$f(x) = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} < f, \psi_{j,k} > \psi_{j,k}(x),$$

is called the wavelet representation of f.

It may be observed that the wavelet coefficient $c_{j,k}$ is the wavelet transform of f with respect to ψ at the point $(2^{-j}, k2^{-j})$:

$$c_{j,k} = (T_\psi f)(2^{-j}, k2^{-j}).$$

Chapter 2
MULTIRESOLUTION ANALYSIS AND THE CONSTRUCTION OF WAVELET

The objective of this chapter is to construct a wavelet system, which is a complete orthonormal set in $L^2(R)$.

2.1 Multiresolution Analysis (MRA)

The idea of multiresolution analysis is to represent a function (or signal) f as a limit of successive approximations, each of which is a finer version of the function f. The basic principle of the multiresolution analysis (MRA) deals with the decomposition of the whole function space into individual subspaces $V_n \subset V_{n+1}$ [7], [13], [17] and [22].

Definition 2.1 *(Multiresolution Analysis)*. A multiresolution analysis (MRA) of $L^2(R)$ is defined as a sequence of closed subspaces V_j of $L^2(R)$, $j \in \mathbb{Z}$, that satisfy the following properties:

1. Monotonicity

$$V_j \subset V_{j+1}, \quad \text{for all } j \in \mathbb{Z}, \tag{2.1}$$

2. Dilation property

$$f(x) \in V_j \Leftrightarrow f(2x) \in V_{j+1} \quad \text{for all } j \in \mathbb{Z}, \tag{2.2}$$

3. Intersection property

$$\bigcap_{j\in\mathbb{Z}} V_j = \{0\}, \tag{2.3}$$

4. Density property

$$\bigcup_{j\in\mathbb{Z}} V_j \text{ is dense in } L^2(R), \tag{2.4}$$

5. Existence of scaling function. There exists a function $\phi \in V_0$, such that $\{\phi(x-n) : n \in \mathbb{Z}\}$ is an orthonormal basis for V_0.

$$V_0 = \left\{ \sum_{k\in\mathbb{Z}} \alpha_k \phi(x-k) : \{\alpha_k\}_{k\in\mathbb{Z}} \in l^2(\mathbb{Z}) \right\}. \tag{2.5}$$

Density property means that for any $f \in L^2(R)$, there exists a sequence $\{f_n\}_{n=1}^{\infty}$ such that each $f_n \in \bigcup_{j\in\mathbb{Z}} V_j$ and $\{f_n\}_{n=1}^{\infty}$ converges to f in $L^2(R)$, that is, $\|f_n - f\| \to 0$ as $n \to \infty$.

The function ϕ is called the *scaling function* or *father wavelet* of the given MRA. Sometimes condition (2.5) is relaxed by assuming that $\{\phi(x-n) : n \in \mathbb{Z}\}$ is a Riesz basis for V_0. In this case, we have a multiresolution analysis with a Riesz basis. Dilation condition (2.2) implies that $f\ (x) \in V_j \Leftrightarrow f(2^m x) \in V_{j+m}$ for all $j, m \in \mathbb{Z}$. In particular $f(x) \in V_0 \Leftrightarrow f(2^j x) \in V_j$. Let

$$\phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k), \tag{2.6}$$

The orthonormality of the set $\{\phi(x-n) : n \in \mathbb{Z}\}$ implies that for each $j \in \mathbb{Z}$, $\{\phi_{j,k}(x), k \in \mathbb{Z}\}$ is an orthonormal set, because changing variables shows that for $j, k, m \in \mathbb{Z}$,

$$< \phi_{j,k}, \phi_{j,m} >=< \phi_{0,k}, \phi_{0,m} >$$

Then $\{\phi_{j,k}(x), k \in \mathbb{Z}\}$ is an orthonormal basis for V_j. It follows that for each $j \in \mathbb{Z}$,

$$V_j = \left\{ \sum_{k \in \mathbb{Z}} \alpha_k \phi_{j,k}(x) : \{\alpha_k\}_{k \in \mathbb{Z}} \in l^2(\mathbb{Z}) \right\}. \tag{2.7}$$

Define the orthogonal projection operator P_j from $L^2(R)$ onto V_j by

$$P_j(f)(x) = \sum_{k \in \mathbb{Z}} < f, \phi_{j,k} > \phi_{j,k}(x), \tag{2.8}$$

then the conditions (2.3) and (2.4) give that

$$\lim_{j \to \infty} P_j(f) = f \tag{2.9}$$

and

$$\lim_{j \to -\infty} P_j(f) = 0$$

The projection $P_j(f)$ can be considered as an approximation of f at the scale 2^{-j}. Therefore, the successive approximations of a given function f are defined as the orthogonal projections $P_j(f)$ onto the space V_j. We can choose $j \in \mathbb{Z}$ such that $P_j(f)$ is a good approximation of f [7], [11], [26].

Theorem 2.1 Suppose $\phi \in L^2(R)$ such that $\hat{\phi}$ is bounded, $\left|\hat{\phi}\right|$ is continuous at 0. Also, suppose that for each $j \in \mathbb{Z}$, $\left\{\phi_{j,k} : k \in \mathbb{Z}\right\}$ is an orthonormal set. Let V_j defined by (2.7). Then, the following two conditions are equivalent:

$$\hat{\phi}(0) \neq 0,$$

$$\bigcup_{j \in \mathbb{Z}} V_j \text{ is dense in } L^2(R).$$

Moreover, when either is the case, $\left|\hat{\phi}(0)\right| = 1$. The proof can be found in [11].

Example 2.1 Let V_j be the space of all function in $L^2(R)$ which are constant on intervals of the form $I_{j,k} = [2^{-j}k, 2^{-j}(k+1)]$, $k \in \mathbb{Z}$.

$$V_j = \{f \in L^2(R) : f = \text{constant on } I_{j,k}, \forall k \in \mathbb{Z}\}.$$

Then, $\{V_j, j \in \mathbb{Z}\}$ is an MRA.

Obviously, $V_m \subset V_{m+1}$, because any function that is constant on intervals of length 2^{-m} is automatically constant on intervals of half that length. The space V_0 contains all functions $f(x)$ in $L^2(R)$ that are constant on $k \leq x < k+1$. The function $f(2x)$ in V_1 is then constant on $\frac{k}{2} \leq x < \frac{k+1}{2}$. A sample function in spaces V_{-1}, V_0 and V_1 are shown in the Figure 2.1.

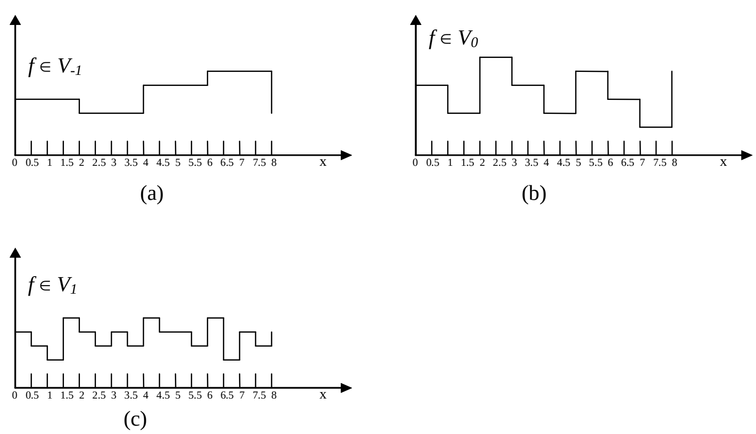

Figure 2.1. A sample function in spaces V_{-1}, V_0 and V_1.

We can take the scaling function to be $\phi = \chi_{[0,1]}$, where $\chi_{[0,1]}$ denotes the characteristic function of $[0, 1]$. This MRA is related to the Haar basis. Figure 2.2 shows the projection of some function f on the Haar spaces V_0 and V_1.

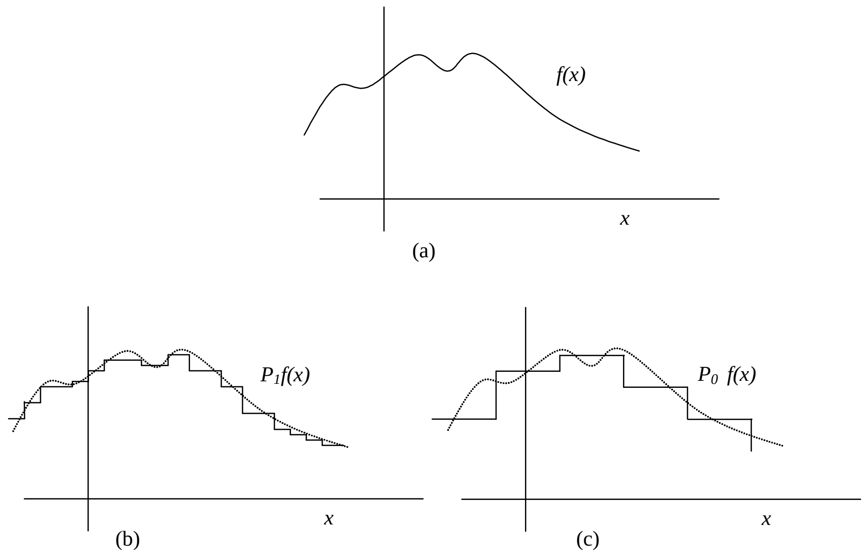

Figure 2.2. The projection of some function f on the Haar spaces V_0 and V_1.

Theorem 2.2 If $\phi \in L^2(R)$, then the system $\{\phi_{0,k} = \phi(x-k) : k \in \mathbb{Z}\}$, is an orthonormal system if and only if

$$\sum_{k\in\mathbb{Z}} \left|\hat{\phi}(w+2\pi k)\right|^2 = 1 \text{ for } a.e.\ w \in R.$$

Proof. The Fourier transform of $\phi_{0,k} = \phi(x-k)$ is $\hat{\phi}_{0,k}(w) = e^{-ikw}\hat{\phi}(w)$. By using Parseval's formula for Fourier transform (Theorem 1.1), we have

$$\begin{aligned}
< \ & \phi_{0,n}, \phi_{0,m} >=< \phi_{0,0}, \phi_{0,m-n} >= \frac{1}{2\pi} < \hat{\phi}_{0,0}, \hat{\phi}_{0,m-n} > \\
= \ & \frac{1}{2\pi}\int_R e^{-i(m-n)w}\left|\hat{\phi}(w)\right|^2 dw \\
= \ & \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\int_{2\pi k}^{2\pi(k+1)} e^{-i(m-n)w}\left|\hat{\phi}(w)\right|^2 dw \\
= \ & \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\int_0^{2\pi}\left|\hat{\phi}(\mu+2\pi k)\right|^2 e^{-i(m-n)\mu} d\mu \\
= \ & \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{k=-\infty}^{\infty}\left|\hat{\phi}(\mu+2\pi k)\right|^2\right) e^{-i(m-n)\mu} d\mu
\end{aligned}$$

Thus, it follows from the completeness of $\{e^{-in\mu}, n \in \mathbb{Z}\}$ in $L^2(0, 2\pi)$ that

$$< \phi_{0,n}, \phi_{0,m} >= \delta_{n,m}$$

if and only if

$$\sum_{k=-\infty}^{\infty} \left|\hat{\phi}(\mu + 2\pi k)\right|^2 = 1 \text{ for } a.e.\ \mu \in R. \blacksquare$$

Theorem 2.3 For any two functions $\phi, \psi \in L^2(R)$, the sets of functions $\{\phi_{0,n} = \phi(x - n), n \in \mathbb{Z}\}$ and $\{\psi_{0,m} = \psi(x - m), m \in \mathbb{Z}\}$ are biorthogonal, that is,

$$< \phi_{0,n}, \psi_{0,m} >= 0 \quad \text{for all } n, m \in \mathbb{Z},$$

if and only if

$$\sum_{k=-\infty}^{\infty} \hat{\phi}(w + 2\pi k)\overline{\hat{\psi}(w + 2\pi k)} = 0 \quad \text{almost everywhere } w \in R.$$

Proof. Applying similar argument to those stated in the proof of Theorem 2.2 to obtain

$$\begin{aligned}
< \quad & \phi_{0,n}, \psi_{0,m} >=< \phi_{0,0}, \psi_{0,m-n} >= \frac{1}{2\pi} < \hat{\phi}_{0,0}, \hat{\psi}_{0,m-n} > \\
= \quad & \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i(m-n)w} \hat{\phi}(w)\overline{\hat{\psi}(w)} dw \\
= \quad & \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \int_{2\pi k}^{2\pi(k+1)} e^{-i(m-n)w} \hat{\phi}(w)\overline{\hat{\psi}(w)} dw \\
= \quad & \frac{1}{2\pi} \int_{0}^{2\pi} e^{-i(m-n)w} \sum_{k=-\infty}^{\infty} \hat{\phi}(w + 2\pi k)\overline{\hat{\psi}(w + 2\pi k)} dw
\end{aligned}$$

Thus,

$$< \phi_{0,n}, \psi_{0,m} >= 0 \quad \text{for all } n, m \in \mathbb{Z},$$

if and only if

$$\sum_{k=-\infty}^{\infty} \hat{\phi}(w + 2\pi k)\overline{\hat{\psi}(w + 2\pi k)} = 0 \quad \text{almost everywhere.} \blacksquare$$

If we only assume that $\left\{\phi_{0,n} = \phi(x-n), n \in \mathbb{Z}\right\}$ is a Riesz basis for V_0, we can find a function $\gamma \in V_0$ such that $\{\gamma(x-n), n \in \mathbb{Z}\}$ is an orthonormal basis for V_0. This is an easy consequence of the following theorem and corollary [17].

Theorem 2.4 Suppose $\phi \in L^2(R)$ is such that the set of translates $\{\phi(x-n),\ n \in \mathbb{Z}\}$ form a Riesz basis of the closed subspace of $L^2(R)$ that they spans; that is,

$$A \sum_{n\in\mathbb{Z}} |c_n|^2 \leq \left\| \sum_{n\in\mathbb{Z}} c_n \phi(x-n) \right\|_2^2 \leq B \sum_{n\in\mathbb{Z}} |c_n|^2\,, \tag{2.10}$$

where the constants A and B satisfy $0 < A \leq B < \infty$ and are independent of $\{c_n\}_{n\in\mathbb{Z}} \in l^2(\mathbb{Z})$. Let

$$\sigma_\phi(w) = \left(\sum_{n\in\mathbb{Z}} \left| \hat{\phi}(w + 2\pi n) \right|^2 \right)^{1/2} \tag{2.11}$$

Then $\sqrt{A} \leq \sigma_\phi(w) \leq \sqrt{B}$ for almost every $w \in R$.

Corollary 2.5 Suppose $\phi \in L^2(R)$ is such that the set of translates $\{\phi(x-n), n \in \mathbb{Z}\}$ form a Riesz basis of the V_0. Then $\{\gamma(x-n), n \in \mathbb{Z}\}$ is an orthonormal basis of V_0, with

$$\hat{\gamma}(w) = \frac{\hat{\phi}(w)}{\sigma_\phi(w)}, \tag{2.12}$$

and $\sigma_\phi(w)$ is given by (2.11).

Proof. From the Theorem 2.4, $\frac{1}{\sigma_\phi(w)}$ is bounded with

$$0 < \frac{1}{\sqrt{B}} \leq \frac{1}{\sigma_\phi(w)} \leq \frac{1}{\sqrt{A}} \quad \text{for } a.e\ w \in R,$$

$\hat{\gamma}(w)$ and, hence, γ belong to $L^2(R)$. Since $\sigma_\phi(w)$ is 2π-periodic we can find two sequences $\{a_n\}_{n\in\mathbb{Z}}$ and $\{b_n\}_{n\in\mathbb{Z}} \in l^2(\mathbb{Z})$ such that

$$\frac{1}{\sigma_\phi(w)} = \sum_{n\in\mathbb{Z}} a_n e^{-inw} \tag{2.13}$$

and

$$\sigma_\phi(w) = \sum_{n\in\mathbb{Z}} b_n e^{-inw} \tag{2.14}$$

for almost everywhere $w \in [-\pi, \pi)$. Thus,

$$\hat{\gamma}(w) = \hat{\phi}(w) \sum_{n\in\mathbb{Z}} a_n e^{-inw} \tag{2.15}$$

and

$$\hat{\phi}(w) = \hat{\gamma}(w) \sum_{n\in\mathbb{Z}} b_n e^{-inw}. \tag{2.16}$$

Taking inverse Fourier transforms for (2.15) and (2.16) gives

$$\gamma(x) = \sum_{n\in\mathbb{Z}} a_n \phi(x-n)$$

and

$$\phi(x) = \sum_{n\in\mathbb{Z}} b_n \gamma(x-n),$$

with convergence in $L^2(R)$. Thus,

$$\gamma(x) \in \overline{span\{\phi(x-n), n\in\mathbb{Z}\}}$$

and

$$\phi(x) \in \overline{span\{\gamma(x-n), n\in\mathbb{Z}\}.}$$

Furthermore, From the definition of $\hat{\gamma}(w)$ (2.12) and the 2π-periodicity of $\sigma_\phi(w)$ we obtain

$$\sum_{n\in\mathbb{Z}} |\hat{\gamma}(w+2\pi n)|^2 = \sum_{n\in\mathbb{Z}} \frac{\left|\hat{\phi}(w+2\pi n)\right|^2}{|\sigma_\phi(w+2\pi n)|^2} = \frac{1}{|\sigma_\phi(w)|^2} \sum_{n\in\mathbb{Z}} \left|\hat{\phi}(w+2\pi n)\right|^2 = 1$$

Then, by Theorem 2.2 the system $\{\gamma(x-n), n\in\mathbb{Z}\}$ is an orthonormal basis of V_0.

2.2 Construction of Wavelets from a Multiresolution Analysis

We now pass to the construction of orthonormal wavelets from an MRA. The real importance of a multiresolution analysis lies in the simple fact that it enables us to construct an orthonormal basis for $L^2(R)$ [7], [11], [13], [17]. In order to prove this statement, we first assume that $\{V_m\}$ is a multiresolution analysis. Since $V_0 \subset V_1$, we define W_0 as the orthogonal complement of V_0 in V_1; that is, $V_1 = V_0 \bigoplus W_0$. Since $V_m \subset V_{m+1}$, we define W_m as the orthogonal complement of V_m in V_{m+1} for every $m \in \mathbb{Z}$ so that we have

$$V_{m+1} = V_m \bigoplus W_m \quad \text{for each } m \in \mathbb{Z}.$$

Since V$_m \to \{0\}$ as $m \to -\infty$, we see that

$$V_{m+1} = V_m \bigoplus W_m = \bigoplus_{l=-\infty}^{m} W_l \quad \text{for all } m \in \mathbb{Z}.$$

Since $\bigcup\limits_{j \in \mathbb{Z}} V_j$ is dense in $L^2(R)$, we may take the limit as $m \to \infty$ to obtain

$$L^2(R) = \bigoplus_{l=-\infty}^{\infty} W_l \tag{2.17}$$

To find an orthonormal wavelet, therefore, all we need to do is to find a function $\psi \in W_0$ such that $\{\psi(x-k) : k \in \mathbb{Z}\}$ is an orthonormal basis for W_0. In fact, if this is the case, then $\left\{\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k) : k \in \mathbb{Z}\right\}$ is an orthonormal basis for W_j for all $j \in \mathbb{Z}$ due to the condition (2.2) in the definition of multiresolution analysis and definition of W_j. Hence

$$\left\{\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k) : k,\ j \in \mathbb{Z}\right\}$$

is an orthonormal basis for $L^2(R)$, which shows that ψ is an orthonormal wavelet on R. Consider $V_1 = V_0 \bigoplus W_0$ and observe that $\phi_{0,0} = \phi \in V_0 \subset V_1$. By (2.5) we can express this function in terms of the basis

$$\left\{\phi_{1,n}(x) = \sqrt{2}\phi(2x-n) : n \in \mathbb{Z}\right\}$$

to obtain

$$\phi(x) = \sqrt{2}\sum_{n\in\mathbb{Z}} \alpha_n \phi(2x-n) \tag{2.18}$$

where $\alpha_n =< \phi, \phi_{1,n} >= \sqrt{2}\int_R \phi\, \bar{\phi}(2x-n)dx$; the convergence in (2.18) is in $L^2(R)$ and $\sum_{n\in\mathbb{Z}} |\alpha_n|^2 < \infty$.

Taking Fourier transforms of (2.18), we obtain

$$\begin{aligned}\hat{\phi}(w) &= \frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}} \alpha_n e^{\frac{-iwn}{2}} \hat{\phi}(\frac{w}{2}) \\ &= \hat{\phi}(\frac{w}{2})\hat{m}(\frac{w}{2}),\end{aligned} \tag{2.19}$$

where

$$\hat{m}(w) = \frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}} \alpha_n e^{-iwn} \tag{2.20}$$

is a 2π -periodic function. The function $\hat{m}(w)$ is called low pass filter associated with the scaling function ϕ.

Lemma 2.6 The low pass filter satisfies the following property

$$|\hat{m}(w)|^2 + |\hat{m}(w+\pi)|^2 = 1 \quad a.e.\ w \in R. \tag{2.21}$$

Proof. By using the relation (2.19) and substitute in the Theorem 2.2 we get

$$1 = \sum_{k=-\infty}^{\infty} \left|\hat{\phi}(w+2\pi k)\right|^2 = \sum_{k=-\infty}^{\infty} \left|\hat{\phi}(\frac{w}{2}+\pi k)\right|^2 \left|\hat{m}(\frac{w}{2}+\pi k)\right|^2.$$

This is true for any w and hence, replacing w by $2w$ gives

$$1 = \sum_{k=-\infty}^{\infty} \left|\hat{\phi}(w + \pi k)\right|^2 |\hat{m}(w + \pi k)|^2 .$$

We now sum the above formula separately over the even integers and over the odd integers and using the 2π periodic property of the function $\hat{m}$ and the Theorem 2.2 to obtain

$$\begin{aligned} 1 &= \sum_{k=-\infty}^{\infty} \left|\hat{\phi}(w + 2\pi k)\right|^2 |\hat{m}(w + 2\pi k)|^2 \\ &\quad + \sum_{k=-\infty}^{\infty} \left|\hat{\phi}(w + (2k+1)\pi)\right|^2 |\hat{m}(w + (2k+1)\pi))|^2 \\ &= \sum_{k=-\infty}^{\infty} \left|\hat{\phi}(w + 2\pi k)\right|^2 |\hat{m}(w)|^2 + \sum_{k=-\infty}^{\infty} \left|\hat{\phi}(w + 2\pi k + \pi)\right|^2 |\hat{m}(w + \pi)|^2 \\ &= |\hat{m}(w)|^2 \sum_{k=-\infty}^{\infty} \left|\hat{\phi}(w + 2\pi k)\right|^2 + |\hat{m}(w + \pi)|^2 \sum_{k=-\infty}^{\infty} \left|\hat{\phi}(w + 2\pi k + \pi)\right|^2 \\ &= |\hat{m}(w)|^2 .1 + |\hat{m}(w + \pi)|^2 .1 \blacksquare \end{aligned}$$

Lemma 2.7 The function $\hat{\phi}$ can be represented by the infinite product

$$\hat{\phi}(w) = \prod_{k=1}^{\infty} \hat{m}(\frac{w}{2^k}). \tag{2.22}$$

Proof. By using (2.19)

$$\begin{aligned} \hat{\phi}(w) &= \hat{\phi}(\frac{w}{2})\hat{m}(\frac{w}{2}) \\ &= \hat{m}(\frac{w}{2})\hat{m}(\frac{w}{4})\hat{\phi}(\frac{w}{4}) \end{aligned}$$

which is, by the $(k-1)^{th}$ iteration, we get

$$\begin{aligned} \hat{\phi}(w) &= \hat{m}(\frac{w}{2})\hat{m}(\frac{w}{4}) \cdots\cdots \hat{m}(\frac{w}{2^k})\hat{\phi}(\frac{w}{2^k}) \\ &= \hat{\phi}(\frac{w}{2^k}). \prod_{n=1}^{k} \hat{m}(\frac{w}{2^k}). \end{aligned} \tag{2.23}$$

Since $\hat{\phi}(0) = 1$ and $\hat{\phi}(w)$ is continuous, we obtain

$$\lim_{k\to\infty} \hat{\phi}(\frac{w}{2^k}) = \hat{\phi}(0) = 1.$$

Taking the limit of (2.23) gives

$$\hat{\phi}(w) = \hat{\phi}(0)\prod_{n=1}^{\infty} \hat{m}(\frac{w}{2^k}) = \prod_{n=1}^{\infty} \hat{m}(\frac{w}{2^k}). \blacksquare$$

Lemma 2.8 If ϕ is a scaling function for an MRA $\{V_j\}_{j\in\mathbb{Z}}$, and $\hat{m}$ is the associated low-pass filter, then

$$V_0 = \{f : \hat{f}(w) = l(w)\hat{\phi}(w)\ \} \tag{2.24}$$

for some 2π-periodic function $l \in L^2[-\pi,\pi)$ and

$$W_0 = \{f : \hat{f}(w) = e^{i\frac{w}{2}} s(w)\overline{\hat{m}(\frac{w}{2}+\pi)}\hat{\phi}(\frac{w}{2})\} \tag{2.25}$$

for some $2\pi-$periodic function $s \in L^2[-\pi,\pi)$.

Proof. If $g \in V_0$, then

$$g(x) = \sum_{k\in\mathbb{Z}} d_k\phi(x-k) \ \text{ with} \sum_{k\in\mathbb{Z}} |d_k|^2 < \infty\ ,$$

taking the Fourier transform we get

$$g(w) = \hat{\phi}(w)\sum_{k\in\mathbb{Z}} d_k e^{-ikw} = l(w)\hat{\phi}(w)$$

where $l(w) = \sum_{k\in\mathbb{Z}} d_k e^{-ikw}$ is a $2\pi-$ periodic function in $L^2[-\pi,\pi)$.

Conversely, if $l \in L^2[-\pi,\pi)$ and is $2\pi-$ periodic function. Then l can be written as

$$l(w) = \sum_{k\in\mathbb{Z}} c_k e^{-ikw} \ \text{ where} \sum_{k\in\mathbb{Z}} |c_k|^2 \ < \infty.$$

Then the function f defined by

$$\hat{f}(w) = l(w)\hat{\phi}(w),$$

or $f(x) = \sum_{k\in\mathbb{Z}} c_k\phi(x-k)$, which belongs to V_0. We have to prove that $\hat{f}$ is a function in $L^2(R)$. Due to the orthonormality of $\{\phi(x-k) : k \in \mathbb{Z}\}$ we have

$$\begin{aligned}
\int_R \left|\hat{f}(w)\right|^2 dw &= \int_R |l(w)|^2 \left|\hat{\phi}(w)\right|^2 dw \\
&= \sum_{k\in\mathbb{Z}} \int_0^{2\pi} |l(w)|^2 \left|\hat{\phi}(w+2\pi k)\right|^2 dw \\
&= \int_0^{2\pi} |l(w)|^2 dw = \|l\|^2_{L^2[-\pi,\pi)}\,.
\end{aligned}$$

This establishes the characterization of V_0.

Since $f \in W_0$, it follows from $V_1 = V_0 \bigoplus W_0$ that $f \in V_1$ and is orthogonal to V_0.Thus, f can be written as

$$f(x) = \sum_{n\in\mathbb{Z}} c_n\phi_{1,n}(x) = \sqrt{2}\sum_{n\in\mathbb{Z}} c_n\phi(2x-n),$$

taking the Fourier transform we get,

$$\begin{aligned}
\hat{f}(w) &= \frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}} c_n e^{\frac{-inw}{2}}\hat{\phi}(\frac{w}{2}) \\
&= \hat{m}_f(\frac{w}{2})\hat{\phi}(\frac{w}{2}),
\end{aligned} \tag{2.26}$$

where $\hat{m}_f(\frac{w}{2}) = \frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}} c_n e^{\frac{-inw}{2}}$ is a 2π- periodic function belonging to $L^2[-\pi,\pi)$. Since f is orthogonal to V_0 then

$$< f(x), \phi(x-n) >= \frac{1}{2\pi} < \hat{f}(w), e^{-inw}\hat{\phi}(w) >= 0 \;\; \text{for all } n \in \mathbb{Z}.$$

Then we have,

$$\int_{-\infty}^{\infty} \hat{f}(w)e^{inw}\overline{\hat{\phi}(w)}dw = 0, \quad \forall n \in \mathbb{Z}$$

and hence,

$$\begin{aligned}\sum_{k=-\infty}^{\infty}\int_{2\pi k}^{2\pi(k+1)}\hat{f}(w)e^{inw}\overline{\hat{\phi}(w)}dw &= \sum_{k=-\infty}^{\infty}\int_{0}^{2\pi}\hat{f}(w+2\pi k)e^{inw}\overline{\hat{\phi}(w+2\pi k)}dw \\ &= \int_{0}^{2\pi}\left(\sum_{k=-\infty}^{\infty}\hat{f}(w+2\pi k)\overline{\hat{\phi}(w+2\pi k)}\right)e^{inw}dw = 0,\end{aligned}$$

thus, it follows from the completeness of $\{e^{inw}, n \in \mathbb{Z}\}$ in $L^2(0, 2\pi)$ that

$$\sum_{k=-\infty}^{\infty}\hat{f}(w+2\pi k)\overline{\hat{\phi}(w+2\pi k)} = 0. \tag{2.27}$$

By substituting (2.26) and (2.19) into (2.27) we obtain

$$\begin{aligned}&\sum_{k=-\infty}^{\infty}\hat{m}_f(\frac{w}{2}+\pi k)\hat{\phi}(\frac{w}{2}+\pi k)\overline{\hat{\phi}(w+2\pi k)} \\ =& \sum_{k=-\infty}^{\infty}\hat{m}_f(\frac{w}{2}+\pi k)\hat{\phi}(\frac{w}{2}+\pi k)\overline{\hat{\phi}(\frac{w}{2}+\pi k)\hat{m}(\frac{w}{2}+\pi k)} \\ =& \sum_{k=-\infty}^{\infty}\hat{m}_f(\frac{w}{2}+\pi k)\overline{\hat{m}(\frac{w}{2}+\pi k)}\left|\hat{\phi}(\frac{w}{2}+\pi k)\right|^2 = 0,\end{aligned}$$

which is, by splitting the sum into even and odd integers k and then using the 2π- periodic property of functions $\hat{m}$ and $\hat{m}_f$ we obtain

$$\begin{aligned}0 &= \sum_{k=-\infty}^{\infty}\hat{m}_f(\frac{w}{2}+2\pi k)\overline{\hat{m}(\frac{w}{2}+2\pi k)}\left|\hat{\phi}(\frac{w}{2}+2\pi k)\right|^2 \\ &\quad+\sum_{k=-\infty}^{\infty}\hat{m}_f(\frac{w}{2}+\pi+2\pi k)\overline{\hat{m}(\frac{w}{2}+\pi+2\pi k)}\left|\hat{\phi}(\frac{w}{2}+\pi+2\pi k)\right|^2 \\ &= \hat{m}_f(\frac{w}{2})\overline{\hat{m}(\frac{w}{2})}\sum_{k=-\infty}^{\infty}\left|\hat{\phi}(\frac{w}{2}+2\pi k)\right|^2 \\ &\quad+\hat{m}_f(\frac{w}{2}+\pi)\overline{\hat{m}(\frac{w}{2}+\pi)}\sum_{k=-\infty}^{\infty}\left|\hat{\phi}(\frac{w}{2}+\pi+2\pi k)\right|^2.\end{aligned}$$

Due to orthonormality of the system $\{\phi(x-k), k \in \mathbb{Z}\}$ we get

$$\hat{m}_f(\frac{w}{2})\overline{\hat{m}(\frac{w}{2})}.1+\hat{m}_f(\frac{w}{2}+\pi)\overline{\hat{m}(\frac{w}{2}+\pi)}.1 = 0.$$

By replacing w by $2w$ we get

$$\hat{m}_f(w)\overline{\hat{m}(w)} + \hat{m}_f(w+\pi)\overline{\hat{m}(w+\pi)} = 0 \tag{2.28}$$

or equivalently,

$$\begin{vmatrix} \hat{m}_f(w) & \overline{\hat{m}(w+\pi)} \\ -\hat{m}_f(w+\pi) & \overline{\hat{m}(w)} \end{vmatrix} = 0.$$

This can be interpreted as the linear independence of two vectors

$$\begin{bmatrix} \hat{m}_f(w) \\ -\hat{m}_f(w+\pi) \end{bmatrix} \text{ and } \begin{bmatrix} \overline{\hat{m}(w+\pi)} \\ \overline{\hat{m}(w)} \end{bmatrix}.$$

Hence, there exists a function $\lambda(w)$ such that

$$\hat{m}_f(w) = \lambda(w)\overline{\hat{m}(w+\pi)} \quad \text{almost everywhrere.} \tag{2.29}$$

Since both $\hat{m}_f(w)$ and $\hat{m}(w)$ are 2π- periodic functions, so is $\lambda(w)$. Further, substituting (2.29) into (2.28) gives

$$\lambda(w)\overline{\hat{m}(w+\pi)\hat{m}(w)} + \lambda(w+\pi)\overline{\hat{m}(w)\hat{m}(w+\pi)} = 0,$$

or

$$\lambda(w) + \lambda(w+\pi) = 0 \ \text{ for } a.e. \ w \in [-\pi, \pi).$$

Thus, there exist a 2π-periodic function $s \in L^2[-\pi,\pi)$ defined by

$$\lambda(w) = e^{iw}s(2w). \tag{2.30}$$

Substituting (2.30) in (2.29) gives

$$\hat{m}_f(w) = e^{iw}s(2w)\overline{\hat{m}(w+\pi)}. \tag{2.31}$$

Finally, substituting (2.31) in (2.26) we get

$$\begin{aligned}\hat{f}(w) &= \hat{m}_f(\frac{w}{2})\hat{\phi}(\frac{w}{2}) \\ &= e^{i\frac{w}{2}}s(w)\overline{\hat{m}(\frac{w}{2}+\pi)}\hat{\phi}(\frac{w}{2}).\end{aligned} \tag{2.32}$$

This completes the proof. ■

Note, similarly, we have

$$W_j = \{f : \hat{f}(2^j w) = e^{i\frac{w}{2}}s(w)\overline{\hat{m}(\frac{w}{2}+\pi)}\hat{\phi}(\frac{w}{2})\ \}, \tag{2.33}$$

for some 2π-periodic function $s \in L^2[-\pi,\pi)$.

If we define ψ by

$$\hat{\psi}(w) = e^{i\frac{w}{2}}\overline{\hat{m}(\frac{w}{2}+\pi)}\hat{\phi}(\frac{w}{2}) \tag{2.34}$$

that is, $s \equiv 1$ in (2.32) we claim that we have found an orthonormal wavelet we are looking for. In fact, all orthonormal wavelets in W_0 can be characterized as follows:

Proposition 2.9 If ϕ is a scaling function for an MRA $\{V_j\}_{j\in\mathbb{Z}}$, and $\hat{m}$ is the associated low-pass filter, then a function $\psi \in W_0$ is an orthonormal wavelet for $L^2(R)$ if and only if

$$\hat{\psi}(w) = e^{i\frac{w}{2}}v(w)\overline{\hat{m}(\frac{w}{2}+\pi)}\hat{\phi}(\frac{w}{2}) \tag{3.35}$$

*a.e.*on R, for some 2π-periodic function v such that

$$|v(w)| = 1 \ \ a.e. \text{ on } [-\pi,\pi).$$

Proof. It is clear that $\psi \in W_0$. Now for any $g \in W_0$, by our characterization of W_0, there is a 2π-periodic $s \in L^2[-\pi,\pi)$ such that

$$\begin{aligned}\hat{g}(w) &= e^{i\frac{w}{2}}s(w)\overline{\hat{m}(\frac{w}{2}+\pi)}\hat{\phi}(\frac{w}{2}) = \frac{s(w)}{v(w)}e^{i\frac{w}{2}}v(w)\overline{\hat{m}(\frac{w}{2}+\pi)}\hat{\phi}(\frac{w}{2}) \\ &= \frac{s(w)}{v(w)}\hat{\psi}(w) = s(w)\bar{v}(w)\hat{\psi}(w).\end{aligned}$$

Since $s\bar{v} \in L^2[-\pi, \pi)$, we can write $s(w)\bar{v}(w) = \sum_{k\in\mathbb{Z}} c_k e^{-iwk}$, where $\sum_{k\in\mathbb{Z}} |c_k|^2 < \infty$. Then

$$\hat{g}(w) = \sum_{k\in\mathbb{Z}} c_k e^{-iwk}\hat{\psi}(w).$$

or

$$g(x) = \sum_{k\in\mathbb{Z}} c_k \psi(x-k),$$

which proves that $\{\psi(x-k), k \in \mathbb{Z}\}$ generates W_0. The orthonormality of $\{\psi(x-k), k \in \mathbb{Z}\}$ can be proved by showing that $\hat{\psi}$ satisfies the equality in Theorem 2.2.

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} \left|\hat{\psi}(w+2\pi k)\right|^2 &= \sum_{k\in\mathbb{Z}} \left| e^{i\frac{w+2\pi k}{2}} v(w+2\pi k)\overline{\hat{m}(\frac{w+2\pi k}{2}+\pi)}\hat{\phi}(\frac{w+2\pi k}{2})\right|^2 \\
&= \sum_{k\in\mathbb{Z}} \left|\hat{m}(\frac{w}{2}+\pi k+\pi)\right|^2 \left|\hat{\phi}(\frac{w}{2}+\pi k)\right|^2 \\
&= \sum_{k\in\mathbb{Z}} \left|\hat{m}(\frac{w}{2}+2\pi k+\pi)\right|^2 \left|\hat{\phi}(\frac{w}{2}+2\pi k)\right|^2 \\
&\quad + \sum_{k\in\mathbb{Z}} \left|\hat{m}(\frac{w}{2}+2\pi k+2\pi)\right|^2 \left|\hat{\phi}(\frac{w}{2}+2\pi k+\pi)\right|^2 \\
&= \left|\hat{m}(\frac{w}{2}+\pi)\right|^2 \sum_{k\in\mathbb{Z}} \left|\hat{\phi}(\frac{w}{2}+2\pi k)\right|^2 \\
&\quad + \left|\hat{m}(\frac{w}{2})\right|^2 \sum_{k\in\mathbb{Z}} \left|\hat{\phi}(\frac{w}{2}+2\pi k+\pi)\right|^2 \\
&= \left|\hat{m}(\frac{w}{2}+\pi)\right|^2 .1 + \left|\hat{m}(\frac{w}{2})\right|^2 .1 = 1,
\end{aligned}$$

where we have summed over the even and odd integers separately, and using the 2π-periodicity of $\hat{m}$, Theorem 2.2 for ϕ and (2.21) for $\hat{m}$. Now we have to show that all orthonormal wavelets ψ in W_0 are described by (2.35). For an $\psi \in W_0$, by Lemma 2.8, there must be a 2π-periodic function $v \in L^2[-\pi, \pi)$ such that

$$\hat{\psi}(w) = e^{i\frac{w}{2}} v(w)\overline{\hat{m}(\frac{w}{2}+\pi)}\hat{\phi}(\frac{w}{2}).$$

If ψ is an orthonormal wavelet, then the orthonormality of $\{\psi(x-k), k \in \mathbb{Z}\}$ gives us

$$\begin{aligned} 1 &= \sum_{k\in\mathbb{Z}} \left|\hat{\psi}(w+2\pi k)\right|^2 = \sum_{k\in\mathbb{Z}} |v(w)|^2 \left|\hat{m}(\frac{w}{2}+\pi k+\pi)\right|^2 \left|\hat{\phi}(\frac{w}{2}+\pi k)\right|^2 \\ &= |v(w)|^2 \left(\sum_{k\in\mathbb{Z}} \left|\hat{m}(\frac{w}{2}+\pi)\right|^2 \left|\hat{\phi}(\frac{w}{2}+2\pi k)\right|^2 \right. \\ &\quad \left. + \sum_{k\in\mathbb{Z}} \left|\hat{m}(\frac{w}{2})\right|^2 \left|\hat{\phi}(\frac{w}{2}+2\pi k+\pi)\right|^2 \right) \\ &= |v(w)|^2 \left(\left|\hat{m}(\frac{w}{2}+\pi)\right|^2 + \left|\hat{m}(\frac{w}{2})\right|^2 \right) = |v(w)|^2, \end{aligned}$$

where we have summed over the even and odd integers separately, and using the 2π-periodicity of $\hat{m}$, Theorem 2.2 for ϕ and (2.21) for $\hat{m}$. Finally, if $\{\psi(x-k), k \in \mathbb{Z}\}$ is an orthonormal basis for W_0, then $\{\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k),\ k \in \mathbb{Z}\}$ is an orthonormal basis for W_j. Hence, (2.17) shows that ψ is an orthonormal wavelet for $L^2(R)$. ■

This proposition completes the construction of a wavelet from an MRA. Let us for simplicity consider the wavelet given by (2.34) (in terms of Proposition 2.9 this means $v(w) \equiv 1$). Since ψ belongs to V_1 it can be written as

$$\psi(x) = \sqrt{2} \sum_{n\in\mathbb{Z}} d_n \phi(2x-n).$$

In fact, there is a way of writing d_n in terms of α_n's that determined $\hat{m}(w)$. From (2.34), (2.20) and (2.19) we obtain

$$\begin{aligned} \hat{\psi}(w) &= e^{i\frac{w}{2}} \overline{\hat{m}(\frac{w}{2}+\pi)} \hat{\phi}(\frac{w}{2}) \\ &= e^{i\frac{w}{2}} \sum_{n\in\mathbb{Z}} \overline{\frac{\alpha_n}{\sqrt{2}} e^{-in(\frac{w}{2}+\pi)}} \hat{\phi}(\frac{w}{2}) \end{aligned}$$

$$\begin{aligned}\hat{\psi}(w) &= \frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}}\bar{\alpha}_n e^{i\frac{w}{2}(n+1)}e^{i\pi n}\hat{\phi}(\frac{w}{2}) \\ &= \frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}}(-1)^n\bar{\alpha}_n e^{i\frac{w}{2}(n+1)}\hat{\phi}(\frac{w}{2}). \end{aligned} \tag{2.36}$$

Taking the inverse Fourier transform this gives us

$$\psi(x) = \sqrt{2}\sum_{n\in\mathbb{Z}}(-1)^n\bar{\alpha}_n\phi(2x+(n+1)),$$

which is, by putting $n=-(k+1)$,

$$\begin{aligned}\psi(x) &= \sqrt{2}\sum_{k\in\mathbb{Z}}(-1)^{k-1}\bar{\alpha}_{-k-1}\phi(2x-k) \\ &= \sqrt{2}\sum_{n\in\mathbb{Z}}d_n\phi(2x-n), \end{aligned} \tag{2.37}$$

where the coefficients d_n are given by

$$d_n = (-1)^{n-1}\bar{\alpha}_{-n-1.} \tag{2.38}$$

Thus, the representation (2.37) of a mother wavelet ψ has the same structure as that of the scaling function ϕ given by (2.18).

The mother wavelet ψ associated with a given MRA is not unique. Let $v(w)=e^{-iNw}$ for some $N\in\mathbb{Z}$. Substituting in the Proposition 2.9 we obtain

$$\begin{aligned}\hat{\psi}(w) &= e^{i\frac{w}{2}}v(w)\overline{\hat{m}(\frac{w}{2}+\pi)}\hat{\phi}(\frac{w}{2}) \\ &= e^{i\frac{w}{2}}e^{-iNw}\overline{\sum_{n\in\mathbb{Z}}\frac{\alpha_n}{\sqrt{2}}e^{-in(\frac{w}{2}+\pi)}}\hat{\phi}(\frac{w}{2}) \\ &= \frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}}\bar{\alpha}_n e^{i\frac{w}{2}(n+1-2N)}e^{i\pi n}\hat{\phi}(\frac{w}{2}) \\ &= \frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}}(-1)^n\bar{\alpha}_n e^{i\frac{w}{2}(n+1-2N)}\hat{\phi}(\frac{w}{2}). \end{aligned}$$

Taking the inverse Fourier transform this gives us

$$\psi(x) = \sqrt{2}\sum_{n\in\mathbb{Z}}(-1)^n\bar{\alpha}_n\phi(2x + (n + 1 - 2N)),$$

which is, by putting $n = -(k + 1 - 2N)$, we get

$$d_n = (-1)^{n-1}\bar{\alpha}_{2N-n-1}, \tag{2.39}$$

which defines another wavelet. Also if we put $v(w) = -e^{-iw}$ we get

$$d_n = (-1)^n\bar{\alpha}_{1-n}. \tag{2.40}$$

Any one of d_n in (2.38), (2.39) or (2.40) can be used to find a mother wavelet. If ϕ has a compact support (the support of ϕ is contained in a finite interval), only the finite numbers of α_n are not zero, then ψ is represented by finite linear combination of $\{\phi_{1,n} = \sqrt{2}\phi(2x - n), \in \mathbb{Z}\}$.

2.3 Compactly Supported Wavelets

In this section we will present that for any non negative integer n there exists an orthonormal wavelet ψ with compact support such that all the derivatives of ψ up to order n exist and are bounded. Daubechies (1988, 1992) first developed the theory and constructed orthonormal wavelets with compact support [8], [10], [11]. Wavelet with compact support can be constructed to have a given number of derivatives and a given number of vanishing moments. Daubechies wavelets are family of orthogonal wavelets indexed by N where N is the number of vanishing wavelet moments.

Daubechies has constructed, for an arbitrary integer N, an orthonormal basis for $L^2(R)$ of the form

$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k),\ j,k \in \mathbb{Z}$$

that satisfy the following properties:

1. The support of ψ is contained in $[-N+1, N]$. To emphasize this point, ψ is often denoted by ψ_N .

2. ψ_N has γN continuous derivatives, where $\gamma = (1 - \frac{1}{2}\log_2 3) = 0.20752$, for large N [17]. Hence, a C^N compactly supported wavelet has a support whose measure is, roughly, $5N$.

3. ψ_N has N vanishing moments

$$\int_{-\infty}^{\infty} x^k \psi(x)dx = 0 \quad for \quad k = 0, 1, \ldots, N.$$

Or, equivalently,

$$\left[\frac{d^k\hat{\psi}(\xi)}{d\xi^k}\right]_{\xi=0} = 0 \quad \text{for} \quad k = 0, 1, \ldots, N. \tag{2.41}$$

In fact, we have the following theorem [10], [11], [26] :

Theorem 2.10 *(Daubechies)* There exists a constant K such that for each $N = 2, 3, \cdots$, there exists an MRA with the scaling function ϕ and associated wavelet ψ such that

1. ϕ and $\psi \in C^N$.

2. ϕ and ψ are compactly supported such that supp ϕ and supp ψ are contained in $[-KN, KN]$.

3. $\int_{-\infty}^{\infty} \psi(x)dx = \int_{-\infty}^{\infty} x\psi(x)dx = \cdots = \int_{-\infty}^{\infty} x^N \psi(x)dx = 0.$

We refer to [11] for a proof of the theorem.

We assume that the scaling function ϕ satisfies

$$\phi(x) = \sqrt{2} \sum_{n\in\mathbb{Z}} \alpha_n \phi(2x - n) \tag{2.42}$$

where $\alpha_n =< \phi, \phi_{1,n} >= \sqrt{2} \int_R \phi\, \bar{\phi}(2x-n)dx$; the convergence is in $L^2(R)$ and $\sum\limits_{n\in\mathbb{Z}} |\alpha_n|^2 < \infty$.

If the scaling function ϕ has compact support, then only a finite number of α_n have nonzero values. The associated low pass filter $\hat{m}(w)$,

$$\hat{m}(w) = \frac{1}{\sqrt{2}} \sum_{n\in\mathbb{Z}} \alpha_n e^{-iwn} \tag{2.43}$$

is a trigonometric polynomial and it satisfies (2.21) with $\hat{m}(0) = 1$ and $\hat{m}(\pi) = 0$. The wavelet ψ is given by the formula (2.34) with $\left|\hat{\phi}(0)\right| = 1$. The Fourier transform $\hat{\psi}$ of order N is N times continuously differentiable and it satisfies the moment condition (2.41) that is

$$\hat{\psi}^{(k)}(0) = 0 \quad for \quad k = 0, 1, \ldots, m. \tag{2.44}$$

It follows that if ϕ and $\psi \in C^m$ then the low pass filter $\hat{m}$ has a zero at $w = \pi$ of order $(m+1)$. This means that $\hat{m}$ must be of the form

$$\hat{m}(w) = \left(\frac{1 + e^{-iw}}{2}\right)^{m+1} \hat{L}(w), \tag{2.45}$$

with $\hat{L}$ being a trigonometric polynomial. In addition to the orthogonality condition (2.21), we assume

$$\hat{m}(w) = \left(\frac{1+e^{-iw}}{2}\right)^N \hat{L}(w), \tag{2.46}$$

where $\hat{L}$ is 2π-periodic and $\hat{L} \in C^{N-1}$. Writing

$$\begin{aligned} M(w) &= |\hat{m}(w)|^2 = \hat{m}(w)\hat{m}(-w) = \left(\frac{1+e^{-iw}}{2}\right)^N \hat{L}(w) \left(\frac{1+e^{iw}}{2}\right)^N \hat{L}(-w) \\ &= \left(\cos^2(\frac{w}{2})\right)^N \left|\hat{L}(w)\right|^2 = \left(\cos^2(\frac{w}{2})\right)^N Q(\cos w), \end{aligned} \tag{2.47}$$

where $\left|\hat{L}(w)\right|^2$ is a polynomial in $\cos w$, that is,

$$\left|\hat{L}(w)\right|^2 = Q(\cos w).$$

Since $1 - \cos w = 2\sin^2(\frac{w}{2})$, we can write

$$M(w) = \left(\cos^2(\frac{w}{2})\right)^N Q(1 - 2\sin^2(\frac{w}{2})) = (1-x)^N P(x), \tag{2.48}$$

where $x = \sin^2(\frac{w}{2})$ and P is a polynomial in x.

We next use the formula

$$\cos^2(\frac{w+\pi}{2}) = \sin^2(\frac{w}{2}) = x$$

and

$$\begin{aligned} \left|\hat{L}(w+\pi)\right|^2 &= Q(-\cos w) = Q(2x-1) \\ &= Q(1 - 2(1-x)) = P(1-x). \end{aligned} \tag{2.49}$$

This equality, together with (2.48) and (2.21) imply that P must satisfy the equality

$$(1-x)^N P(x) + x^N P(1-x) = 1. \tag{2.50}$$

Since $(1-x)^N$ and x^N are relatively prime, then, by Bezout's theorem (for more details [11]), there is a unique polynomial $P_N(x)$ of degree $\leq N-1$ that satisfies (2.50). An explicit solution for $P_N(x)$ is given by

$$P_N(x) = \sum_{k=0}^{N-1} \begin{pmatrix} N-1+k \\ k \end{pmatrix} x^k, \tag{2.51}$$

since

$$P_N(x) = Q(1-2x) = Q(\cos w) = \left|\hat{L}(w)\right|^2, \tag{2.52}$$

we can find $\hat{L}(w)$ from $P_N(x)$ by using the following lemma:

Lemma 2.11 (Riez-Spectral Factorization). If

$$\hat{A}(w) = \sum_{k=0}^{n} a_k \cos^k w, \tag{2.53}$$

where $a_k \in R$ and $a_k \neq 0$, and if $\hat{A}(w) \geq 0$ for real w with $\hat{A}(0) = 1$, then there exists a trigonometric polynomial

$$\hat{L}(w) = \sum_{k=0}^{n} b_k e^{-ikw} \tag{2.54}$$

with real coefficients b_k with $\hat{L}(0) = 1$ such that

$$\hat{A}(w) = \hat{L}(w)\hat{L}(-w) = \left|\hat{L}(w)\right|^2$$

is identically satisfied for w.

For a proof of the lemma see [11]. Note that the factorization of $\hat{A}(w)$ is not unique. For a given N, then $\hat{A}(w)$ is a polynomial of degree $(N-1)$ in $\cos w$ and $\hat{L}(w)$ is a polynomial of degree $(N-1)$ in e^{-iw} . Then the low pass filter $\hat{m}(w)$ in (2.46) is of degree $(2N-1)$ in e^{-iw}. The support of the scaling function ϕ_N is $[0, 2N-1]$. More details [2], [3], [10], [11] and references given therein. Some Daubechies wavelet are drawn in the following figures:

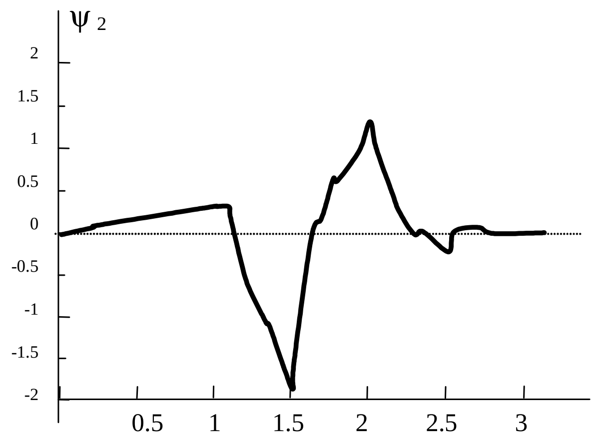

Figure 2.3. Daubechies wavelet ψ_2.

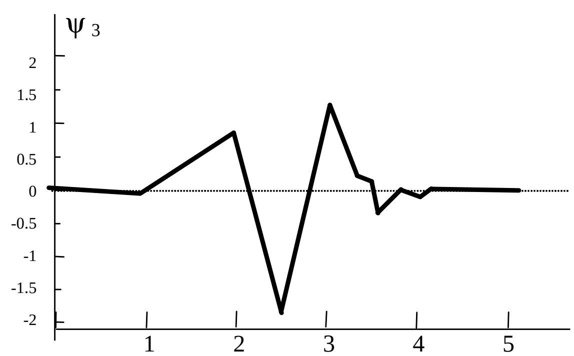

Figure 2.4. Daubechies wavelet ψ_3.

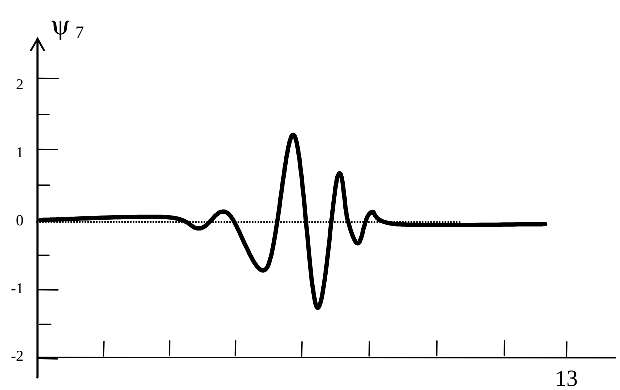

Figure 2.5. Daubechies wavelet ψ_7.

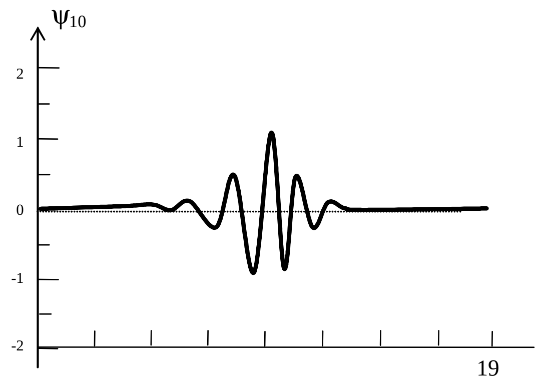

Figure 2.6. Daubechies wavelet ψ_{10}.

2.4 Decomposition and Reconstruction algorithms for wavelets

The multiresolution analysis (MRA) is well adapted to image analysis. The spaces V_j that appeared in the definition of an MRA can be interpreted as spaces where an approximation to the image at the j^{th} level is obtained. In addition, the detail in the approximation occurring in V_j, that is not in V_{j-1}, is stored in the spaces W_{j-1} which satisfy $V_j = V_{j-1} \bigoplus W_{j-1}$. This leads to efficient decomposition and reconstruction algorithms [4], [7], [13], [26]. Chose an MRA with scaling ϕ and wavelet ψ.

Definition 2.2 Define the approximation operators P_j, $j \in \mathbb{Z}$ from $L^2(R)$ onto V_j by

$$P_j f(x) = \sum_{k\in\mathbb{Z}} < f, \phi_{j,k} > \phi_{j,k}(x)$$

and define the detail operator Q_j by

$$\begin{aligned} Q_j f(x) &= P_{j+1} f(x) - P_j f(x) \\ &= \sum_{k\in\mathbb{Z}} < f, \psi_{j,k} > \psi_{j,k}(x), \end{aligned}$$

where

$$\phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k),$$

$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k).$$

Let f be a function defined on R. Since $\lim_{j\to\infty} P_j f(x) = f(x)$ in $L^2(R)$ norm, we can choose $j \in \mathbb{Z}$ such that $P_j f$ is a good approximation of f. Thus, we have

$$f(x) \simeq P_j f(x) = \sum_{k\in\mathbb{Z}} c_{j,k}\phi_{j,k}(x), \tag{2.55}$$

the coefficients

$$c_{j,k} =< f, \phi_{j,k} >, \quad j,k \in \mathbb{Z}.$$

Since we have the orthogonal direct sum decomposition

$$V_j = V_{j-1} \bigoplus W_{j-1},$$

we can also use the bases for V_{j-1} and W_{j-1}; that is , we use

$$\{\phi_{j-1,k}\}_{k\in\mathbb{Z}} \bigcup \{\psi_{j-1,k}\}_{k\in\mathbb{Z}},$$

then

$$\begin{aligned} P_j f(x) &= \underbrace{\sum_{k\in\mathbb{Z}} < f, \phi_{j-1,k} > \phi_{j-1,k}(x)}_{P_{j-1}f} \\ &\qquad + \underbrace{\sum_{k\in\mathbb{Z}} < f, \psi_{j-1,k} > \psi_{j-1,k}(x).}_{Q_{j-1}f} \end{aligned} \tag{2.56}$$

The decomposition formula starts with the coefficients relative to the first basis in (2.55) and uses them to calculate the coefficients relative to the second basis in (2.56). The reconstruction formula does the reverse.

Recall that $\phi \in V_0 \subset V_1$, so that

$$\phi(x) = \sum_{k\in\mathbb{Z}} \alpha_k \phi_{1,k}(x) = \sqrt{2}\sum_{k\in\mathbb{Z}} \alpha_k \phi(2x-k),$$

where

$$\alpha_k = \int_{-\infty}^{\infty} \phi(x)\overline{\phi(2x-k)}dx.$$

Then for V_j and V_{j-1} we have

$$\begin{aligned}\phi_{j,k}(x) &= 2^{j/2}\phi(2^j x-k) = 2^{j/2}\sqrt{2}\sum_{n\in\mathbb{Z}} \alpha_n \phi(2^{j+1}x-2k-n)\\ &= \sum_{n\in\mathbb{Z}} \alpha_n \phi_{j+1,2k+n}(x)\end{aligned}$$

That is

$$\phi_{j-1,k}(x) = \sum_{n\in\mathbb{Z}} \alpha_n \phi_{j,2k+n}(x) \tag{2.57}$$

and similarly, $\psi \in W_0 \subset V_1$ and hence

$$\psi(x) = \sum_{k\in\mathbb{Z}} d_k \phi_{1,k}(x) = \sqrt{2}\sum_{k\in\mathbb{Z}} d_k \phi(2x-k),$$

$$\psi_{j,k}(x) = \sum_{n\in\mathbb{Z}} d_n \phi_{j+1,2k+n}(x)$$

or

$$\psi_{j-1,k}(x) = \sum_{n\in\mathbb{Z}} d_n \phi_{j,2k+n}(x), \tag{2.58}$$

where d_k is chosen as in (2.40)

$$d_k = (-1)^k\overline{\alpha_{1-k}},$$

so that the coefficient d_k do not require further computations.

What we want to do is to decompose the sequence

$$C^j = \left\{c_{j,k} =< f, \phi_{j,k} >, k\in\mathbb{Z}\right\},$$

into sequences C^{j-1} and q^{j-1}.

Now by using (2.57) we obtain

$$\begin{aligned} c_{j-1,k} &= < f, \phi_{j-1,k} >=< f, \sum_{n\in\mathbb{Z}} \alpha_n \phi_{j,2k+n} > \\ &= \sum_{n\in\mathbb{Z}} \overline{\alpha_n} < f, \phi_{j,2k+n} > \\ &= \sum_{n\in\mathbb{Z}} \overline{\alpha_n} c_{j,2k+n} \\ &= \sum_{n\in\mathbb{Z}} \overline{\alpha_{n-2k}} c_{j,n}. \end{aligned} \tag{2.59}$$

This shows that the coefficients $c_{j-1,k}$ of the lowest resolution V_{j-1} can be obtained from the coefficients $c_{j,k}$ of the V_j and the low-pass filter coefficients α_k.

The rest of the terms, which contain the "details" in passing fromV_{j-1} to V_j, are contained in W_{j-1},

$$\begin{aligned} Q_{j-1}f(x) &= P_j f(x) - P_{j-1}f(x) \\ &= \sum_{k\in\mathbb{Z}} q_{j-1,k} \psi_{j-1,k}(x), \end{aligned}$$

where

$$q_{j-1,k} =< f, \psi_{j-1,k} > .$$

Using (2.58) we obtain

$$\begin{aligned} q_{j-1,k} &= < f, \psi_{j-1,k} > \\ &= < f, \sum_{n\in\mathbb{Z}} d_n \phi_{j,2k+n} > \\ &= \sum_{n\in\mathbb{Z}} \overline{d_n} < f, \phi_{j,2k+n} > \\ &= \sum_{n\in\mathbb{Z}} \overline{d_n} c_{j,2k+n} \\ &= \sum_{n\in\mathbb{Z}} (-1)^n \alpha_{1-n+2k} c_{j,n}. \end{aligned} \tag{2.60}$$

Thus we have decomposed C^j into sequences C^{j-1} and q^{j-1}. The process can be continued with C^{j-1} to obtain the decomposition algorithm given in the Figure 2.7.

$$
\begin{array}{ccccccccccc}
 & & q^{j-1} & & q^{j-2} & & \cdots & & & q^{j-m} & \\
 & \nearrow & & \nearrow & & \nearrow & \cdots & & \nearrow & & \nearrow \\
C^j & \rightarrow & C^{j-1} & \longrightarrow & C^{j-2} & \longrightarrow & \cdots & C^{j-m+1} & \longrightarrow & C^{j-m} & \longrightarrow
\end{array}
$$

Figure 2.7. Decomposition algorithm.

Reconstruction C^j from the sequences $q^{j-1}, q^{j-2}, \cdots\cdots, q^{j-m}$ and C^{j-m}. By induction, it is enough to consider the reconstruction of C^j from q^{j-1} and C^{j-1}. Since

$$P_j f(x) = P_{j-1} f(x) + Q_{j-1} f(x)$$

By using (2.57) and (2.58) we obtain

$$
\begin{aligned}
\sum_{k\in\mathbb{Z}} c_{j,k}\phi_{j,k} &= \sum_{k\in\mathbb{Z}} c_{j-1,k}\phi_{j-1,k} + \sum_{k\in\mathbb{Z}} q_{j-1,k}\psi_{j-1,k} \\
&= \sum_{k\in\mathbb{Z}} c_{j-1,k}\left(\sum_{n\in\mathbb{Z}} \alpha_n \phi_{j,2k+n}\right) + \sum_{k\in\mathbb{Z}} q_{j-1,k}\left(\sum_{n\in\mathbb{Z}} d_n \phi_{j,2k+n}\right) \\
&= \sum_{k\in\mathbb{Z}} c_{j-1,k}\left(\sum_{n\in\mathbb{Z}} \alpha_{n-2k} \phi_{j,n}\right) + \sum_{k\in\mathbb{Z}} q_{j-1,k}\left(\sum_{n\in\mathbb{Z}} d_{n-2k} \phi_{j,n}\right) \\
&= \sum_{n\in\mathbb{Z}} \left(\sum_{k\in\mathbb{Z}} c_{j-1,k}\alpha_{n-2k} + \sum_{k\in\mathbb{Z}} q_{j-1,k} d_{n-2k}\right)\phi_{j,n}
\end{aligned}
$$

$$\sum_{k\in\mathbb{Z}} c_{j,k}\phi_{j,k} = \sum_{n\in\mathbb{Z}} \underbrace{\left(\sum_{k\in\mathbb{Z}} c_{j-1,k}\alpha_{n-2k} + \sum_{k\in\mathbb{Z}} q_{j-1,k} d_{n-2k}\right)}_{c_{j,n}}\phi_{j,n},$$

Hence

$$
\begin{aligned}
c_{j,n} &= \sum_{k\in\mathbb{Z}} \left(c_{j-1,k}\alpha_{n-2k} + q_{j-1,k} d_{n-2k}\right) \\
&= \sum_{k\in\mathbb{Z}} \left(c_{j-1,k}\alpha_{n-2k} + (-1)^n q_{j-1,k}\overline{\alpha_{1+2k-n}}\right), \qquad (2.61)
\end{aligned}
$$

Formula (2.61) allows us to add the sequences q^{j-1} and C^{j-1} to obtain C^j. If we start this process with C^{j-m} and q^{j-m} and we also know the "details" $q^{j-m+1}, q^{j-m} \cdots\cdots q^{j-1}$, we have the construction algorithm given in the Figure 2.8.

$$\begin{array}{ccccccccccc}
q^{j-m} & & q^{j-m+1} & & & & \cdots & & q^{j-1} & & \\
& \searrow & & \searrow & & & \cdots & \searrow & & \searrow & \\
C^{j-m} & \longrightarrow & C^{j-m+1} & \longrightarrow & C^{j-m+2} & & \cdots & \longrightarrow & C^{j-1} & \longrightarrow & C^j
\end{array}$$

Figure 2.8. Reconstruction algorithm.

2.5 Biorthogonal Wavelets

The orthogonality property puts a strong limitation on the construction of wavelets. It is known that the Haar wavelet is the only real valued wavelet that is compactly supported, symmetric and orthogonal [10].

Definition 2.3 *(Biorthogonal Wavelets)* Two function $\psi, \tilde{\psi} \in L^2(R)$ are called biorthogonal wavelets if each one of the set $\{\psi_{j,k} : j,k \in \mathbb{Z}\}$ and $\{\tilde{\psi}_{j,k} : j,k \in \mathbb{Z}\}$ is a Riesz basis of $L^2(R)$ and they are biorthogonal,

$$\left\langle \psi_{j,k}, \tilde{\psi}_{l,m} \right\rangle = \delta_{j,l}\delta_{k,m} \quad \text{for all } j,l,k,m \in \mathbb{Z}. \tag{2.62}$$

Recall from section 2.1. A multiresolution analysis (MRA) of $L^2(R)$ is defined as a sequence of closed subspaces V_j of $L^2(R)$, $j \in \mathbb{Z}$, that satisfies the following properties:

1. *Monotonicity*

$$V_j \subset V_{j+1}, \quad \text{for all } j \in \mathbb{Z},$$

2. *Dilation property*

$$f(x) \in V_j \Leftrightarrow f(2x) \in V_{j+1} \quad \text{for all } j \in \mathbb{Z},$$

3. *Intersection property*

$$\bigcap_{j \in \mathbb{Z}} V_j = \{0\},$$

4. *Density property*

$$\bigcup_{j \in \mathbb{Z}} V_j \text{ is dense in } L^2(R),$$

5. *Existence of scaling function*. There exists a function $\phi \in V_0$, such that the set of functions$\left\{\phi_{j,l}(x) = 2^{j/2}\phi(2^j x - l) : l \in \mathbb{Z}\right\}$ is a Riesz basis of V_j.

As a result, there is a sequence $\{h_k : k \in \mathbb{Z}\}$ such that the scaling function satisfies a refinement equation

$$\phi(x) = 2\sum_{n \in \mathbb{Z}} h_n \phi(2x - n). \tag{2.63}$$

Define W_j as a complementary space of V_j in $V_{j+1,}$ such that $V_{j+1} = V_j \bigoplus W_j$, and consequently,

$$L^2(R) = \bigoplus_{l=-\infty}^{\infty} W_l.$$

A function ψ is a wavelet if the set of function $\{\psi(x - l) : l \in \mathbb{Z}\}$ is a Riesz basis of W_0. Then the set of wavelet functions $\left\{\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k) : j, k \in \mathbb{Z}\right\}$ is a Riesz basis of $L^2(R)$. Since the wavelet is an element of V_1 then it satisfies the relation

$$\psi(x) = 2\sum_{n \in \mathbb{Z}} g_n \phi(2x - n). \tag{2.64}$$

There are dual functions $\tilde{\phi}_{j,l}$ and $\tilde{\psi}_{j,l}$ exist so that the projection operators P_j and Q_j onto V_j and W_j, respectively are given by

$$P_j(f)(x) = \sum_{k\in\mathbb{Z}} < f, \tilde{\phi}_{j,k} > \phi_{j,k}(x),$$

and

$$Q_j(f)(x) = \sum_{k\in\mathbb{Z}} < f, \tilde{\psi}_{j,k} > \psi_{j,k}(x),$$

then we have

$$f = \sum_{j,k\in\mathbb{Z}} < f, \tilde{\psi}_{j,k} > \psi_{j,k}.$$

Here the definitions of $\tilde{\phi}_{j,k}$ and $\tilde{\psi}_{j,k}$ are similar to those for $\phi_{j,k}$ and $\psi_{j,k}$. Then, the basis functions and dual functions are biorthogonal [20] ,

$$< \phi_{j,l}, \tilde{\phi}_{j,k} >= \delta_{l,k} \quad \text{and} \quad < \psi_{j,l}, \tilde{\psi}_{m,k} >= \delta_{j,m}\delta_{l,k}. \tag{2.65}$$

Note that if the basis functions are orthogonal, they coincide with the dual function and the projections are orthogonal as in section 2.1.

The dual scaling function and wavelet satisfy

$$\tilde{\phi}(x) = 2\sum_{n\in\mathbb{Z}} \tilde{h}_n \tilde{\phi}(2x-n), \qquad \tilde{\psi}(x) = 2\sum_{n\in\mathbb{Z}} \tilde{g}_n \tilde{\phi}(2x-n), \tag{2.66}$$

and

$$\tilde{\phi}(2x-n) = \sum_{l\in\mathbb{Z}} h_{n-2l}\tilde{\phi}(x-l) + \sum_{l\in\mathbb{Z}} g_{n-2l}\tilde{\psi}(x-l). \tag{2.67}$$

Taking the Fourier transform of the refinement equations (2.63) and (2.64) gives

$$\hat{\phi}(w) = h(\frac{w}{2})\hat{\phi}(\frac{w}{2}) \qquad \text{with} \quad h(w) = \sum_{n\in\mathbb{Z}} h_n e^{-inw} \tag{2.68}$$

and

$$\hat{\psi}(w) = g(\frac{w}{2})\hat{\phi}(\frac{w}{2}) \qquad \text{with} \quad g(w) = \sum_{n\in\mathbb{Z}} g_n e^{-inw}. \tag{2.69}$$

Here h and g are 2π-periodic functions. Similarly, for dual functions. Taking the Fourier transform of (2.66) gives

$$\widehat{\tilde{\phi}}(w) = \tilde{h}(\frac{w}{2})\widehat{\tilde{\phi}}(\frac{w}{2}) \qquad \text{with} \quad \tilde{h}(w) = \sum_{n\in\mathbb{Z}} \tilde{h}_n e^{-inw} \tag{2.70}$$

and

$$\widehat{\tilde{\psi}}(w) = \tilde{g}(\frac{w}{2})\widehat{\tilde{\phi}}(\frac{w}{2}) \qquad \text{with} \quad \tilde{g}(w) = \sum_{n\in\mathbb{Z}} \tilde{g}_n e^{-inw}. \tag{2.71}$$

A necessary condition for biorthogonality is then [8], [20],

$$\forall w \in R : \tilde{m}(w)\overline{m^t(w)} = 1,$$

where

$$m(w) = \begin{bmatrix} h(w) & h(w+\pi) \\ g(w) & g(w+\pi) \end{bmatrix}$$

and

$$\tilde{m}(w) = \begin{bmatrix} \tilde{h}(w) & \tilde{h}(w+\pi) \\ \tilde{g}(w) & \tilde{g}(w+\pi) \end{bmatrix}.$$

The existence of the dual filters is guaranteed by the following lemma:

Lemma 2.12 The space generated by the set of functions $\{\psi_{j,l} : l \in \mathbb{Z}\}$ complements V_j in V_{j+1} if and only if $\delta(w) = \det m(w)$ does not vanish [20].

Chapter 3
WAVELETS AND DIFFERENTIAL EQUATIONS

Many applications of mathematics require the numerical approximation of solutions of differential equations. In this chapter we will present different approaches of using wavelets in the solution of boundary value ordinary differential equations. We consider the class of ordinary differential equation of the form

$$Lu(x) = f(x) \quad \text{for } x \in [0,1], \quad \text{where } L = \sum_{j=0}^{m} a_j(x)D^j,$$

and with appropriate boundary conditions on $u(x)$ for $x = 0, 1$. There are two major solution techniques. First, if the coefficients $a_j(x)$ of the operator are constants, then the Fourier transform is well suited for solving these equations because that the complex exponentials are eigenfunctions of a constant coefficient operator and they form an orthogonal system. As a result, the operator becomes diagonal in the Fourier basis and can be inverted trivially. If the coefficients are not constant finite element or finite difference methods can be used [14]. We focus here on finite element methods.

3.1 Wavelet- Galerkin Methods for Differential Equations

In this section we will describe how to use wavelets to find the numerical solution of ordinary differential equations. The classical Galerkin methods have the disadvantage that the stiffness matrix becomes ill conditioned as the problem size grows. To overcome this disadvantage, we use wavelets as basis functions in a Galerkin method. Then, the results is a linear system that is sparse because of the compact support of the wavelets, and that, after preconditioning, has a condition number independent of problem size because of the multiresolution structure. We will see that using wavelets in conjunction with the Galerkin method gives the two main desired properties for the associated linear system : sparseness and low condition number [5], [12], [14], [18], [20] and [29].

The methods for numerically solving a linear ordinary differential equation come down to solving a linear system of equations, or equivalently, a matrix equation $Ax = y$. For the system to have a unique solution x for every y if and only if A is an invertible matrix. However, in applications there are further issues that are of crucial importance. One of these has to do with the condition number of a matrix A which measures the stability of the linear system $Ax = y$. Let us see an example [14].

Example 3.1 Consider the linear system $Ax = y$, where $x, y \in \mathbb{C}^2$, and

$$A = \begin{bmatrix} 5.95 & -14.85 \\ 1.98 & -4.94 \end{bmatrix}.$$

The determinant of A is 0.01, which is not 0, so A is invertible. For

$$y = \begin{bmatrix} 3.05 \\ 1.02 \end{bmatrix},$$

the solution to $Ax = y$ is

$$x = \begin{bmatrix} 8 \\ 3 \end{bmatrix},$$

however, if

$$y' = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

Then the solution to $Ax' = y'$ is

$$x' = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

Note that y and y' are close but x and x' are far apart. A linear system for which this happens is called ill conditioned. In this case, small errors in the data y can lead to large errors in the solution x. This is undesirable in applications.

Definition 3.1 Let A be an $n \times n$ matrix. Define $\|A\|$ called the operator norm, or just the norm, of A by

$$\|A\| = \sup \frac{\|Az\|}{\|z\|},$$

where the supremum is taken over all nonzero vector in $\mathbb{C}^n$.

Equivalently

$$\|A\| = \sup \left\{ \|Az\| : \|z\| = 1, z \in \mathbb{C}^n \right\}.$$

Definition 3.2 *(Condition number of a matrix)* Let A be an $n \times n$ matrix. Define $C_{\#}(A)$, the condition number of the matrix A, by

$$C_{\#}(A) = \|A\| \left\|A^{-1}\right\|.$$

if A is not invertible, set $C_{\#}(A) = \infty$.

Note that the condition number $C_{\#}(A)$ is scale invariant [6], that is for $c \neq 0$,

$$C_{\#}(cA) = C_{\#}(A).$$

Lemma 3.1 Suppose that A is an $n \times n$ normal invertible matrix. Let

$$|\lambda|_{\max} = \max \left\{ |\lambda| : \lambda \text{ is an eigenvalue of } A \right\}$$

and

$$|\lambda|_{\min} = \min \left\{ |\lambda| : \lambda \text{ is an eigenvalue of } A \right\}.$$

Then

$$C_{\#}(A) = \frac{|\lambda|_{\max}}{|\lambda|_{\min}}.$$

The condition number of A measures how unstable the linear system $Ax = y$ is under perturbation of the data y. In applications, a small condition number (i.e., near 1) is desirable [14]. If the condition number of A is high, we would like to replace the linear system $Ax = y$ by an equivalent system $Mz = v$ whose matrix M has a low condition number.

We consider the class of ordinary differential equations (known as Sturm-Liouville equations) of the form

$$\begin{aligned} Lu(t) &= -a(t)u''(t) - a'(t)u'(t) + b(t)u(t) \\ &= -\frac{d}{dt}\left(a(t)\frac{du}{dt}\right) + b(t)u(t) = f(t), \text{ for } 0 \le t \le 1, \end{aligned} \tag{3.1}$$

with Dirichlet boundary conditions

$$u(0) = u(1) = 0.$$

Here $a, b,$ and f are given real-valued functions and we wish to solve for u. We assume f and b are continuous and a has a continuous derivative on $[0, 1]$ (this always means a one-sided derivative at the endpoints). Note that L may be a variable coefficient differential operator because $a(t)$ and $b(t)$ are not necessarily constant. We assume that the operator is uniformly elliptic which means that there exist finite constants $C_1, C_2,$ and C_3 such that

$$0 < C_1 \le a(t) \le C_2 \text{ and } 0 \le b(t) \le C_3, \tag{3.2}$$

for all $t \in [0, 1]$. By a result in the theory of ordinary differential equations, there is a unique function u satisfying equation (3.1) and the boundary conditions $u(0) = u(1) = 0$.

For the Galerkin method [14], [15], we suppose that $\{v_j\}_j$ is a complete orthonormal system for $L^2[0,1]$, and that every v_j is C^2 on $[0,1]$ and it satisfies

$$v_j(0) = v_j(1) = 0. \tag{3.3}$$

We select some finite set Λ of indices j and consider the subspace

$$S = span\left\{v_j\ ;\ j \in \Lambda\right\}.$$

We look for an approximation to the solution u of equation (3.1) of the form

$$u_s = \sum_{k\in\Lambda} x_k v_k\ \in S, \tag{3.4}$$

where each x_k is a scalar. These coefficients should be determined such that u_s behaves like the true solution u on the subspace S, that is

$$\langle Lu_s, v_j\rangle = \langle f, v_j\rangle \qquad \text{for all } j \in \Lambda. \tag{3.5}$$

By linearity, it follows that

$$\langle Lu_s, g\rangle = \langle f, g\rangle \qquad \text{for all } g \in S.$$

Note that the approximate solution u_s automatically satisfies the boundary conditions $u_s(0) = u_s(1) = 0$ because of equation (3.3).

Substituting (3.4) in equation (3.5), we get

$$\left\langle L\left(\sum_{k\in\Lambda} x_k v_k\right), v_j\right\rangle = \langle f, v_j\rangle \qquad \text{for all } j \in \Lambda,$$

or

$$\sum_{k\in\Lambda} \langle Lv_k, v_j\ \rangle\, x_k = \langle f, v_j\rangle \qquad \text{for all } j \in \Lambda. \tag{3.6}$$

Let x denote the vector $(x_k)_{k\in\Lambda}$, and y be the vector $(y_k)_{k\in\Lambda}$, where $y_k = \langle f, v_j\rangle$. Let A be the matrix with rows and columns indexed by Λ, that is, $A = [a_{j,k}]_{j,k\in\Lambda,}$ where

$$a_{j,k} = \langle Lv_k, v_j\rangle . \tag{3.7}$$

Thus, equation (3.6) is the linear system of equations

$$\sum_{k\in\Lambda} a_{j,k}x_k = y_j \quad \text{for all } j \in \Lambda,$$

or

$$Ax = y. \tag{3.8}$$

In the Galerkin method, for each subset Λ we obtain an approximation $u_s \in S$, by solving the linear system (3.8) for x and using these components to determine u_s by equation (3.4). We expect that as we increase our set Λ in some systematic way, our approximation u_s will converge to the exact solution u.

Our main concern is the nature of the linear system (3.8) that results from choosing a wavelet basis for the Galerkin method. There are two properties that we would like the matrix A in the linear system (3.8) to have. First, we would like A to have a small condition number to obtain stability of the solution under small perturbations in the data. Second, we would like A to be sparse for quick calculations [14], [15].

There is a way of modifying the wavelet system for $L^2(R)$ so as to obtain a complete orthonormal system

$$\{\psi_{j,k}\}_{(j,k)\in\Gamma} \tag{3.9}$$

for $L^2[0,1]$. More details [1], [12], [15] and references given therein. The set Γ is a certain subset of $\mathbb{Z} \times \mathbb{Z}$. For each $(j,k) \in \Lambda$, $\psi_{j,k} \in C^2$ and satisfies the boundary conditions

$$\psi_{j,k}(0) = \psi_{j,k}(1) = 0.$$

The wavelet system $\{\psi_{j,k}\}_{(j,k)\in\Gamma}$ also satisfies the following estimate: There exist constants $C_4, C_5 > 0$ such that for all functions g of the form

$$g = \sum_{j,k} c_{j,k}\psi_{j,k} \tag{3.10}$$

where the sum is finite, we have

$$C_4 \sum_{j,k} 2^{2j} |c_{j,k}|^2 \leq \int_0^1 |g'(t)|^2 \, dt \leq C_5 \sum_{j,k} 2^{2j} |c_{j,k}|^2 . \tag{3.11}$$

An estimate of this form is called a norm equivalence. It states that up to the two constants, the quantities $\sum_{j,k} 2^{2j} |c_{j,k}|^2$ and $\int_0^1 |g'(t)|^2 \, dt$ are equivalent.

For wavelets we write equation (3.4) as

$$u_s = \sum_{(j,k)\in\Lambda} x_{j,k}\psi_{j,k},$$

and equation (3.6) as

$$\sum_{(j,k)\in\Lambda} \langle L\psi_{j,k}, \psi_{l,m}\rangle x_{j,k} = \langle f, \psi_{l,m}\rangle \quad \text{for all } (l,m) \in \Lambda, \tag{3.12}$$

for some finite set of indices Λ. We can write (3.12) as matrix equation of the form $Ax = y$, where the vectors $x = (x_{j,k})_{(j,k)\in\Lambda}$ and $y = (y_{j,k})_{(j,k)\in\Lambda}$ are indexed by the pairs $(j,k) \in \Lambda$, and the matrix

$$A = [a_{l,m;j,k}]_{(l,m),(j,k)\in\Lambda}$$

defined by

$$a_{l,m;j,k} = \langle L\psi_{j,k}, \psi_{l,m}\rangle \tag{3.13}$$

has its rows indexed by the pairs $(l,m) \in \Lambda$ and its columns indexed by the pairs $(j,k) \in \Lambda$.

As suggested, we would like A to be sparse and have a low condition number. A itself does not have a low condition number, however, we can replace the system $Ax = y$ by an equivalent system $Mz = v$, for which the new matrix M has low condition number. To get this, first define the diagonal matrix

$$D = [d_{l,m;j,k}]_{(l,m),(j,k)\in\Lambda}$$

by

$$d_{l,m;j,k} = \left\{ \begin{array}{ll} 2^j & \text{if } (l,m) = (j,k) \\ 0 & \text{if } (l,m) \neq (j,k) \end{array} \right\}. \tag{3.14}$$

Define $M = [m_{l,m;j,k}]_{(l,m),(j,k)\in\Lambda}$ by

$$M = D^{-1}AD^{-1}. \tag{3.15}$$

By writing this out, we get

$$m_{l,m;j,k} = 2^{-j-l}a_{l,m;j,k} = 2^{-j-l}\left\langle L\psi_{j,k}, \psi_{l,m} \right\rangle. \tag{3.16}$$

Then, the system $Ax = y$ is equivalent to

$$D^{-1}AD^{-1}Dx = D^{-1}y,$$

if we put $z = Dx$ and $v = D^{-1}y$, we get

$$Mz = v. \tag{3.17}$$

The norm equivalence (3.11) has the consequence that the system (3.17) is well conditioned as we will see in Theorem 3.3. The following lemma is needed to prove Theorem 3.3. It explains the need for the uniform ellipticity assumption (3.2).

Lemma 3.2 Let L be a uniformly elliptic Sturm-Liouville operator (i.e., an operator as defined in equation (3.1) satisfying relation (3.2)). Suppose g is C^2 on $[0,1]$ and satisfies $g(0) = g(1) = 0$. Then

$$C_1 \int_0^1 |g'(t)|^2 \, dt \leq \langle Lg, g \rangle \leq (C_2 + C_3) \int_0^1 |g'(t)|^2 , \tag{3.18}$$

where C_1, C_2, and C_3 are the constants in relation (3.2).

Proof. Observe that

$$\begin{aligned} \langle -(ag')', g \rangle &= \int_0^1 -(ag')'(t)\bar{g}(t)dt \\ &= \int_0^1 a(t)g'(t)\overline{g'(t)}dt \\ &= \langle ag', g' \rangle , \end{aligned}$$

by integration by parts (the boundary term is 0 because $g(0) = g(1) = 0$). Therefore,

$$\langle Lg, g \rangle = \langle -(ag')' + bg, g \rangle = \langle ag', g' \rangle + \langle bg, g \rangle .$$

Hence, by relation (3.2),

$$\begin{aligned} C_1 \int_0^1 |g'(t)|^2 \, dt &\leq \int_0^1 a(t) \, |g'(t)|^2 \, dt \\ &= \int_0^1 a(t)g'(t)\overline{g'(t)}dt = \langle ag', g' \rangle . \end{aligned} \tag{3.19}$$

Also by relation (3.2),

$$0 \leq \int_0^1 b(t) \, |g(t)|^2 \, dt = \langle bg, g \rangle .$$

Adding the above inequalities gives

$$C_1 \int_0^1 |g'(t)|^2 \, dt \leq \langle Lg, g \rangle ,$$

which is the left half of relation (3.18). For the other half, note that by relation (3.2),

$$\langle ag', g' \rangle = \int_0^1 a(t) \left| g'(t) \right|^2 dt \le C_2 \int_0^1 \left| g'(t) \right|^2 dt. \tag{3.20}$$

Also note that because $g(0) = 0$,

$$g(t) = \int_0^t g'(s) ds,$$

by the fundamental theorem of calculus. Hence by the Cauchy-Schwarz inequality for the function $g'\chi_{[0,\,t]}$ and $\chi_{[0,\,t]}$ where $\chi_{[0,\,t]}$ is

$$\chi_{[0,\,t]}(x) = \begin{cases} 1 & \text{for } x \in [0,t] \\ 0 & \text{for } x \notin [0,t] \end{cases},$$

we get

$$\left| g(t) \right|^2 \le \left(\int_0^t \left| g'(s) \right|^2 ds \right) \left(\int_0^t 1 ds \right) \le \int_0^1 \left| g'(s) \right|^2 ds,$$

for every $t \in [0, 1]$. Therefore

$$\int_0^1 \left| g(t) \right|^2 dt \le \int_0^1 \left| g'(s) \right|^2 ds \int_0^1 dt = \int_0^1 \left| g'(s) \right|^2 ds. \tag{3.21}$$

Hence, by (3.2),

$$\langle bg, g \rangle = \int_0^1 b(t) \left| g(t) \right|^2 dt \le C_3 \int_0^1 \left| g(t) \right|^2 dt \le C_3 \int_0^1 \left| g'(t) \right|^2 dt.$$

Adding this result and relation (3.20) gives the right side of relation (3.18). ■

Theorem 3.3 Let L be a uniformly elliptic Sturm-Liouville operator. Let $\{\psi_{j,k}\}_{(j,k)\in\Gamma}$ be a complete orthonormal system for $L^2[0,1]$ such that each $\psi_{j,k}$ is C^2, satisfies $\psi_{j,k}(0) = \psi_{j,k}(1) = 0$, and such that the norm equivalence (3.11) holds. Let Λ be a finite subset of Γ. Let M be the matrix defined in equation (3.15). Then the condition number of M satisfies

$$C_{\#}(M) \le \frac{(C_2 + C_3)\, C_5}{C_1 C_4}, \tag{3.22}$$

for any finite set Λ, where C_1, C_2, and C_3 are the constants in relation (3.2), C_4 and C_5 are the constants in relation (3.11).

Proof. Let $z = (z_{j,k})_{(j,k)\in\Lambda}$ be any vector with $\|z\| = 1$. For D as in equation (3.14), let $w = D^{-1}z$; that is, $w = (w_{j,k})_{(j,k)\in\Lambda}$, where

$$w_{j,k} = 2^{-j} z_{j,k}.$$

Define

$$g = \sum_{(j,k)\in\Lambda} w_{j,k}\psi_{j,k}.$$

Then by equation (3.16),

$$\begin{aligned} \langle Mz, z\rangle &= \sum_{(l,m)\in\Lambda} (Mz)_{l,m}\overline{z_{l,m}} \\ &= \sum_{(l,m)\in\Lambda}\sum_{(j,k)\in\Lambda} \left\langle L\psi_{j,k}, \psi_{l,m}\right\rangle 2^{-j} z_{j,k} 2^{-l}\overline{z_{l,m}} \\ &= \left\langle L\left(\sum_{(j,k)\in\Lambda} w_{j,k}\psi_{j,k}\right), \sum_{(l,m)\in\Lambda} w_{l,m}\psi_{l,m}\right\rangle = \langle Lg, g\rangle, \end{aligned}$$

since $2^{-j}z_{j,k} = w_{j,k}$ and $2^{-l}z_{l,m} = w_{l,m}$. Applying Lemma 3.2 and relation (3.11) gives

$$\langle Mz, z\rangle = \langle Lg, g\rangle \le (C_2 + C_3)\int_0^1 |g'(t)|^2\,dt \le (C_2+C_3)C_5 \sum_{(j,k)\in\Lambda} 2^{2j}|w_{j,k}|^2,$$

and

$$\langle Mz, z\rangle = \langle Lg, g\rangle \ge C_1\int_0^1 |g'(t)|^2\,dt \ge C_1C_4 \sum_{(j,k)\in\Lambda} 2^{2j}|w_{j,k}|^2.$$

However,

$$\sum_{(j,k)\in\Lambda} 2^{2j}|w_{j,k}|^2 = \sum_{(j,k)\in\Lambda} |z_{j,k}|^2 = \|z\| = 1.$$

So for any z with $\|z\| = 1$,

$$C_1C_4 \le \langle Mz, z\rangle \le (C_2 + C_3)C_5.$$

If λ is an eigenvalue of M, we can normalize the associated eigenvector z so that $\|z\| = 1$, to obtain

$$\langle Mz, z\rangle = \langle \lambda z, z\rangle = \lambda \langle z, z\rangle = \lambda \|z\|^2 = \lambda.$$

Therefore, every eigenvalue λ of M satisfies

$$C_1C_4 \leq \lambda \leq (C_2 + C_3)C_5. \tag{3.23}$$

Note that M is Hermitian and hence normal, so by Lemma 3.1, $C_{\#}(M)$ is the ratio of the largest eigenvalue to the smallest. Then condition (3.22) holds. ■

Thus the matrix in the system $Mz = v$ has a condition number bounded independently of the set Λ. As a result, if we increase Λ to approximate our solution with more accuracy, the condition number remains bounded.

Note that the matrices that obtained by using finite differences are sparse, but they have large condition numbers [14]. Using the Galerkin method with the Fourier system, we can obtain a bounded condition number but the matrix is not sparse. Using the Galerkin method with a wavelet system, we obtain both advantages [2], [5], [12] and [14].

3.2 Biorthogonal Wavelets Diagonalizing The Differential Equations

The derivative operator is not diagonal in a wavelet basis [3], [12], [28]. However, we can make differential operator diagonal by using two pairs of biorthogonal or dual bases of compactly supported wavelets [12]. In this case, we have two related multiresolution spaces $\{V_j\}$ and $\{\tilde{V}_j\}$ such that

$$V_{j+1} \subset V_j, \quad \text{and} \quad \tilde{V}_{j+1} \subset \tilde{V}_j, \quad \text{for all } j \in \mathbb{Z},$$

corresponding to two scaling functions $\phi, \tilde{\phi}$ and two wavelets $\psi, \tilde{\psi}$. They are defined by two trigonometric polynomials m_0 and $\tilde{m}_0$, satisfying

$$m_0(w)\overline{\tilde{m}_0(w)} + m_0(w+\pi)\overline{\tilde{m}_0(w+\pi)} = 1. \tag{3.24}$$

Then we have

$$\widehat{\phi}(w) = \frac{1}{\sqrt{2\pi}} \prod_{j=1}^{\infty} m_0(2^{-j}w), \tag{3.25}$$

$$\widehat{\tilde{\phi}}(w) = \frac{1}{\sqrt{2\pi}} \prod_{j=1}^{\infty} \tilde{m}_0(2^{-j}w), \tag{3.26}$$

also, we have

$$\hat{\psi}(w) = e^{-\frac{iw}{2}}\overline{\tilde{m}_0(\frac{w}{2}+\pi)}\hat{\phi}(\frac{w}{2}), \tag{3.27}$$

and

$$\widehat{\tilde{\psi}}(w) = e^{-\frac{iw}{2}}\overline{m_0(\frac{w}{2}+\pi)}\widehat{\tilde{\phi}}(\frac{w}{2}), \tag{3.28}$$

with

$$< \psi_{j,k}, \tilde{\psi}_{m,n} >= \delta_{j,m}\delta_{k,n}, \tag{3.29}$$

where

$$\psi_{j,k}(x) = 2^{-j/2}\psi(2^{-j}x-k),$$

$$\tilde{\psi}_{j,k}(x) = 2^{-j/2}\tilde{\psi}(2^{-j}x-k).$$

If $\psi \in C^{L-1}(R)$, then $\tilde{\psi}$ must have L vanishing moments see [8] and [11], that is

$$\int_{-\infty}^{\infty} x^l \tilde{\psi}(x)dx = 0 \quad \text{for} \quad l = 0, 1, \ldots, L-1. \tag{3.30}$$

Or, equivalently,

$$\left[\frac{d^l}{dw^l} m_0\right]_{w=\pi} = 0 \quad \text{for} \quad l = 0, 1, \ldots, L-1. \tag{3.31}$$

Which implies that m_0 should be divisible by $(1 + e^{-iw})^L$ [11]. The same for $\tilde{\psi}$, reverse the roles of $\psi, \tilde{\psi}$ and $m_0, \tilde{m}_0$. Then m_0 and $\tilde{m}_0$ can be written as [12]

$$m_0(w) = (\cos\frac{w}{2})^L e^{-ir\frac{w}{2}} P(\cos w), \tag{3.32}$$

$$\tilde{m}_0 = (\cos\frac{w}{2})^{\tilde{L}} e^{-ir\frac{w}{2}} \tilde{P}(\cos w), \tag{3.33}$$

where

$$r = \begin{cases} 1, & \text{if } L \text{ and } \tilde{L} \text{ are odd,} \\ 0, & \text{if } L \text{ and } \tilde{L} \text{ are even,} \end{cases}$$

here L and $\tilde{L}$ must have the same parity see [11] and [12]. The polynomials P and $\tilde{P}$ satisfy the equation

$$(1+x)^K P(x)\tilde{P}(x) + (1-x)^K P(-x)\tilde{P}(-x) = 2^K, \tag{3.34}$$

where $L + \tilde{L} = 2K$. Now, if we split $2K$ into a different sum $2K = L^* + \tilde{L}^*$, gives different m_0^* and $\tilde{m}_0^*$, but P and $\tilde{P}$ can be left unchanged. Substituting (3.32) and (3.33) into (3.25) and (3.26) gives

$$\widehat{\phi}(w) = \frac{1}{\sqrt{2\pi}} e^{-ir\frac{w}{2}} \left[\frac{\sin(\frac{w}{2})}{(\frac{w}{2})}\right]^L \prod_{j=1}^{\infty} P(\cos 2^{-j} w), \tag{3.35}$$

$$\widehat{\tilde{\phi}}(w) = \frac{1}{\sqrt{2\pi}} e^{-ir\frac{w}{2}} \left[\frac{\sin(\frac{w}{2})}{(\frac{w}{2})}\right]^{\tilde{L}} \prod_{j=1}^{\infty} \tilde{P}(\cos 2^{-j} w),$$

where we used $\prod_{j=1}^{\infty}\cos(2^{-j}\alpha) = \frac{\sin(\alpha)}{\alpha}$. Also, substitution (3.32) and (3.33) into (3.27) and (3.28) gives

$$\hat{\psi}(w) = \frac{(i)^r}{\sqrt{2\pi}}\left[\sin\frac{w}{4}\right]^{\tilde{L}}\left[\frac{\sin\frac{w}{4}}{\frac{w}{4}}\right]^{L}\prod_{j=2}^{\infty}P(\cos 2^{-j}w), \tag{3.36}$$

$$\widehat{\tilde{\psi}}(w) = \frac{(i)^r}{\sqrt{2\pi}}\left[\sin\frac{w}{4}\right]^{L}\left[\frac{\sin\frac{w}{4}}{\frac{w}{4}}\right]^{\tilde{L}}\prod_{j=2}^{\infty}\tilde{P}(\cos 2^{-j}w). \tag{3.37}$$

Multiplying (3.36) by iw gives

$$iw\hat{\psi}(w) = \frac{4(i)^{r+1}}{\sqrt{2\pi}}\left[\sin\frac{w}{4}\right]^{\tilde{L}+1}\left[\frac{\sin\frac{w}{4}}{\frac{w}{4}}\right]^{L-1}\prod_{j=2}^{\infty}P(\cos 2^{-j}w). \tag{3.38}$$

The Fourier transform of the derivative ψ' of ψ is simply $iw\hat{\psi}(w)$. Then (3.38) can be written as

$$\widehat{\psi'(w)} = \frac{4(i)^{r+1}}{\sqrt{2\pi}}\left[\sin\frac{w}{4}\right]^{\tilde{L}+1}\left[\frac{\sin\frac{w}{4}}{\frac{w}{4}}\right]^{L-1}\prod_{j=2}^{\infty}P(\cos 2^{-j}w). \tag{3.39}$$

Up to multiplicative constant 4, this is exactly the Fourier transform of the wavelet ψ^* which corresponded to the same P, $\tilde{P}$ and K in (3.34), but with the choice $L^* = L-1$, $\tilde{L}^* = \tilde{L}+1$, we have

$$m_0^*(w) = \left[\cos\frac{w}{2}\right]^{L-1} e^{-\frac{i(1-r)w}{2}}P(\cos w),$$

$$\tilde{m}_0^*(w) = \left[\cos\frac{w}{2}\right]^{\tilde{L}+1} e^{-\frac{i(1-r)w}{2}}\tilde{P}(\cos w).$$

It follows that if we construct two pairs of biorthogonal wavelet bases, one using $\psi, \tilde{\psi}$, and the other using $\psi^*, \tilde{\psi}^*$, then we have

$$\psi' = 4\psi^*,$$

or

$$(\psi_{j,k})' = 2^{-j}4\psi^*_{j,k},$$

and hence

$$\left\langle \frac{d}{dx}\psi_{j,k}, \tilde{\psi}^*_{m,n} \right\rangle = 2^{-j}4\delta_{j,m}\delta_{k,n}.$$

This means that we have diagonalized the derivative operator. Note that this is not a "true" diagonalization because we use two different bases. However, this means that we can find the wavelet coefficients of $f\,'$ *i.e.,*

$$\left\langle \frac{d}{dx}f, \psi_{j,k} \right\rangle = 2^{-j}4\left\langle f, \psi^*_{j,k} \right\rangle.$$

For mor details see [12] and the references therein.

Another approach of diagonalizing the differential operator, using wavelets, is by constructing biorthogonal wavelets with respect to the inner product defined by the operator [19], [20].

We consider the class of ordinary differential equation of the form

$$Lu(x) = f(x) \quad \text{for } x \in [0,1], \text{ where } L = \sum_{j=0}^{m} a_j(x)D^j, \tag{3.40}$$

and with appropriate boundary conditions on $u(x)$ for $x = 0, 1$.

Define the operator inner product associated with an operator L by

$$\langle\langle u, v\rangle\rangle = \langle Lu, v\rangle.$$

An approximate solution of u can be found with a Petrov-Galerkin method, *i.e.* consider two spaces S and S^* and look for a solution $u \in S$ such that

$$\langle\langle u, v\rangle\rangle = \langle f, v\rangle,$$

for all v in S^*. If S and S^* are finite dimensional spaces with the same dimension, this leads to a linear system of equations. The matrix of this system, also referred to as the stiffness matrix, has as elements the operator inner products of the basis functions of S and S^*.

We assume that L is self-adjoint and positive definite and, in particular, we can write

$$L = V^* V,$$

where V^* is the adjoint of V. We call V the square root operator of L. Suppose that $\{\Psi_{j,l}\}$ and $\left\{\Psi^*_{j,l}\right\}$ are bases for S and S^* respectively. The entries of the stiffness matrix are then given by

$$\left\langle\left\langle \Psi_{j,l}, \Psi^*_{m,n} \right\rangle\right\rangle = \left\langle L\Psi_{j,l}, \Psi^*_{m,n} \right\rangle = \left\langle V\Psi_{j,l}, V\Psi^*_{m,n} \right\rangle .$$

Now, the idea is to let

$$\Psi_{j,l} = V^{-1}\psi_{j,l} \quad \text{and} \quad \Psi^*_{j,l} = V^{-1}\tilde{\psi}_{j,l},$$

where ψ and $\tilde{\psi}$ are the wavelets of a classical multiresolution analysis. We will call the Ψ and Ψ^* functions the operator wavelets. Then the operator wavelet are biorthogonal with respect to the operator inner product. We want the operator wavelets to be compactly supported and to be able to construct compactly supported operator scaling functions $\Phi_{j,l}$. The analysis is relatively straight forward for simple constant coefficient operators such as the Laplace and polyharmonic operator [20].

Example 3.2 *(Laplace operator)* Consider the one dimensional Laplace operator

$$L = -D^2$$

Then the square root operator V is

$$V = D.$$

The associated operator inner product is

$$\langle\langle u, v\rangle\rangle = \langle Lu, v\rangle = \langle Vu, Vv\rangle = \langle u', v'\rangle .$$

Since the action of V^{-1} is taking the antiderivative, we define the operator wavelets as

$$\Psi(x) = \int_{-\infty}^{x} \psi(t)dt, \quad \text{and} \quad \Psi^*(x) = \int_{-\infty}^{x} \tilde{\psi}(t)dt. \tag{3.41}$$

Note that the operator wavelets $\Psi(x)$ and $\Psi^*(x)$ are compactly supported because the integral of the original wavelets has to vanish. Also translation and dilation invariance is preserved, so we define

$$\Psi_{j,l}(x) = \Psi(2^j x - l) \quad \text{and} \quad \Psi^*_{j,l}(x) = \Psi^*(2^j x - l).$$

Now,

$$\left\langle\left\langle \Psi^*_{j,l}(x), \Psi_{m,n}(x)\right\rangle\right\rangle = \left\langle V\Psi^*_{j,l}(x), V\Psi_{m,n}(x)\right\rangle = 2^j \delta_{j,m}\delta_{l,n} \quad \text{for } j, l, m, n \in \mathbb{Z}.$$

This means that the stiffness matrix is diagonal with powers of 2 on its diagonal. We now need to find an operator scaling function Φ. The antiderivative of the original scaling function is not compactly supported and hence not suited. To find an operator scaling function Φ convolute the original scaling function with the indicator function $\chi_{[0,1]}$,

$$\Phi = \phi * \chi_{[0,1]}, \tag{3.42}$$

and define

$$\Phi_{j,l}(x) = \Phi(2^j x - l).$$

Similarly for the dual functions

$$\Phi^* = \tilde{\phi} * \chi_{[0,1]}.$$

Now, define

$$V_j = clos\ span\ \{\Phi_{j,k} : k \in \mathbb{Z}\}$$

and

$$W_j = clos\ span\ \{\Psi_{j,k} : k \in \mathbb{Z}\}\,.$$

We want to show that $V_j \subset V_{j+1}$ and W_j complements V_j in V_{j+1}. By taking the Fourier transform of (3.41) and (3.42) we get

$$\hat{\Phi}(w) = \frac{1-e^{-iw}}{iw}\hat{\phi}(w) \quad \text{and} \quad \hat{\Psi}(w) = \frac{1}{iw}\hat{\psi}(w). \tag{3.43}$$

A simple calculation shows that the operator scaling function satisfies the following equation

$$\hat{\Phi}(w) = H(\frac{w}{2})\hat{\Phi}(\frac{w}{2}) \quad \text{with} \quad H(w) = \frac{1+e^{-iw}}{2}h(w). \tag{3.44}$$

Consequently, $V_j \subset V_{j+1}$. Also

$$\hat{\Psi}(w) = G(\frac{w}{2})\hat{\Phi}(\frac{w}{2}) \quad \text{with} \quad G(w) = \frac{1}{2(1-e^{-iw})}g(w), \tag{3.45}$$

where $h(w)$ and $g(w)$ are defined as in (2.68) and (2.69) respectively. This implies that $W_j \subset V_{j+1}$. To prove that W_j is complements V_j in V_{j+1} we have to prove that

$$\Delta(w) = \det \begin{bmatrix} H(w) & H(w+\pi) \\ G(w) & G(w+\pi) \end{bmatrix}$$

does not vanish. In fact,

$$\begin{aligned}
\Delta(w) &= H(w)G(w+\pi) - H(w+\pi)G(w) \\
&= \frac{1+e^{-iw}}{2}h(w)\frac{1}{2(1-e^{-i(w+\pi)})}g(w+\pi) - \\
&\qquad\qquad \frac{1+e^{-i(w+\pi)}}{2}h(w+\pi)\frac{1}{2(1-e^{-iw})}g(w) \\
&= \frac{1}{4}h(w)g(w+\pi) - \frac{1}{4}h(w+\pi)g(w) = \frac{1}{4}\delta(w),
\end{aligned}$$

where $\delta(w) = h(w)g(w+\pi) - h(w+\pi)g(w)$, and this cannot vanish since ϕ and ψ generate a multiresolution analysis. Then W_j is complements V_j in V_{j+1} by Lemma 2.12. The construction of the dual functions Φ^* and Ψ^* from $\tilde{\phi}$ and $\tilde{\psi}$ is completely similar. The coefficients of the trigonometric functions H, H^*, G and G^* now define a fast wavelet transform.

Now, we will describe the algorithm in the case of periodic boundary conditions. This implies that the basis functions on the interval $[0, 1]$ are just the periodization of the basis functions on the real line.

Let $S = V_n$ and consider the basis $\{\Phi_{n,l} : 0 \leq l \leq 2^n\}$. Define vectors b and x such that

$$b_l = \left\langle f, \Phi^*_{n,l} \right\rangle, \quad \text{and} \quad u = \sum_{l=0}^{2^n-1} x_l \Phi_{n,l}. \tag{3.46}$$

The Galerkin method with this basis then yields a system

$$Ax = b \quad \text{with} \quad A_{k,l} = \langle\langle \Phi_{n,l}, \Phi_{n,k} \rangle\rangle . \tag{3.47}$$

The matrix A is not diagonal and the condition number grows as $O(2^{2n})$ [20]. Now, consider the decomposition

$$V_n = V_0 \oplus W_0 \oplus \cdots \oplus W_{n-1},$$

and the corresponding wavelet basis. The space V_0 has dimension one and contains constant functions. We now switch to a one index notation such that the sets

$$\left\{1, \Psi_{j,l} : 0 \leq j < n, 0 \leq l < 2^j\right\} \quad \text{and} \quad \{\Psi_k : 0 \leq k < 2^n\}$$

coincide. Now, define the vectors $\tilde{b}$ and $\tilde{x}$ such that

$$\tilde{b} = \langle f, \Psi^*_l \rangle \quad \text{and} \quad u = \sum_{l=0}^{2^n-1} \tilde{x}_l \Psi_l. \tag{3.48}$$

There exists matrices T and T^* [20] such that

$$\tilde{b} = T^* b \quad \text{and} \quad x = T\tilde{x}.$$

The matrix T^* corresponds to the fast wavelet transform decomposition with filters H^* and G^* and T corresponds to reconstruction with filters H and G. In the wavelet basis the system becomes

$$\tilde{A}\tilde{x} = \tilde{b} \quad \text{with} \quad \tilde{A} = T^* A T$$

and

$$\tilde{A}_{k,l} = \langle\langle \Psi_{n,l}, \Psi_{n,k} \rangle\rangle .$$

Since $\tilde{A}$ is diagonal, it can be trivially inverted and the solution is then given by

$$x = T\tilde{A}^{-1} T^* b.$$

Example 3.3 *(The polyharmonic operator)* The polyharmonic equation is defined as

$$-u^{(2m)} = f, \tag{3.49}$$

then the square root operator is $V = D^m$. The operator scaling function Φ is m times the convolution of the original scaling function ϕ with the indicator function $\chi_{[0,1]}$ and the operator wavelet Ψ is m times the antiderivative of the original wavelet ψ. In order to get a compactly supported wavelet, the original wavelet now needs to have at least m vanishing moments [20]. The construction and algorithm are similar to the case of the Laplace operator as in Example 3.2.

Example 3.4 *(The Helmholz operator)* The one dimensional Helmholz operator is defined by

$$L = -D^2 + k^2. \tag{3.50}$$

Without loss of generality assume that $k = 1$ which can always be obtained from transformation. The square root operator is

$$V = D + 1 = e^{-x} D e^{x} \tag{3.51}$$

and

$$V^{-1} = e^{-x} D^{-1} e^{x}. \tag{3.52}$$

Note that $V^{-1}\psi$ will not necessarily give a compactly supported function because $e^{x}\psi_{j,l}$ in general does not have a vanishing integral. Therefore we let

$$\Psi_{j,l} = V^{-1} e^{-x} \psi_{j,l} = e^{-x} D^{-1} \psi_{j,l}. \tag{3.53}$$

If $\psi_{j,l}$ has a vanishing integral, then $\Psi_{j,l}$ is compactly supported.

In order to diagonalize the stiffness matrix, the original wavelets now need to be orthogonal with respect to a weighted inner product with weight function e^{-2x}

$$\begin{aligned} \big\langle\big\langle \Psi_{j,l}, \Psi^{*}_{m,n} \big\rangle\big\rangle &= \big\langle V\Psi_{j,l}, V\Psi^{*}_{m,n} \big\rangle \\ &= \big\langle e^{-x}\psi_{j,l}, e^{-x}\psi_{m,n} \big\rangle \\ &= \int_{-\infty}^{\infty} e^{-2x} \psi_{j,l}(x) \tilde{\psi}_{m,n}(x) dx. \end{aligned} \tag{3.54}$$

To find the wavelet let

$$\text{supp } \psi_{j,l} = [2^{-j} l, 2^{-j}(l+1)].$$

Then the orthogonality of the wavelets on each level immediately follows from their disjoint support. To get orthogonality between two different levels, we need that V_j is orthogonal to W_m for $m \geq j$ or

$$\int_{-\infty}^{\infty} e^{-2x} \phi_{j,l}(x) \tilde{\psi}_{m,n}(x) dx = 0 \quad \text{for } m \geq j.$$

Now, let the scaling function coincide with e^{2x} on the support of the finer scale wavelets,

$$\phi_{j,l}(x) = e^{2x}\chi_{[j,l]}, \tag{3.55}$$

where $\chi_{[j,l]}$ is the indicator function on the interval $[2^{-j}l, 2^{-j}(l+1)]$, normalized such that the integral of the scaling functions is constant. As in the Haar case we choose the wavelets as

$$\psi_{j,l} = \phi_{j+1,2l} - \phi_{j+1,2l+1}, \tag{3.56}$$

so that they have vanishing integral. The orthogonality between levels now follows from the fact that the scaling functions coincide with e^{2x} on the support of the finer scale wavelets, and from the vanishing integral of the wavelets

$$\int_{-\infty}^{\infty} e^{-2x}\phi_{j,l}(x)\tilde{\psi}_{m,n}(x)dx = \int_{-\infty}^{\infty} \chi_{[j,l]}\tilde{\psi}_{m,n}(x)dx = \int_{-\infty}^{\infty} \tilde{\psi}_{m,n}(x)dx = 0.$$

One can see that the operator wavelets are now piecewise combinations of e^{x} and e^{-x}. The operator scaling functions are chosen as

$$\Phi_{j,l} = e^{-x}D^{-1}(\phi_{j,l} - \phi_{j,l+1}) \tag{3.57}$$

so that

$$\Psi_{j,l} = \Phi_{j+1,2l}. \tag{3.58}$$

With the right normalization, one gets

$$\Phi_{j,l}(x) = \begin{cases} \frac{\sinh(x-l2^{-j})}{\sinh(2^{-j})} & \text{for } x \in [l2^{-j}, (l+1)2^{-j}] \\ \frac{\sinh((l+2)2^{-j}-x)}{\sinh(2^{-j})} & \text{for } x \in [(l+1)2^{-j}, (l+2)2^{-j}] \\ 0 & \text{elsewhere.} \end{cases}$$

The operator scaling functions on one level are translates of each other but the ones on different levels are no longer dilates of each other. They are supported on the same sets.

The operator scaling functions satisfy a relation

$$\Phi_{j,l} = \sum_{k=0}^{2} H_k^j \Phi_{j+1,2l+k}, \tag{3.59}$$

where

$$H_0^j = H_2^j = \frac{\sinh(2^{-j-1})}{\sinh(2^{-j})} \quad \text{and} \quad H_1^j = 1.$$

The Helmholz operator in this bases of hyperbolic wavelets is diagonal . So we can conclude that a wavelet transform can diagonalize constant coefficient operators similar to the Fourier transform. The resulting algorithm is faster ($O(N)$ instead of $O(NlogN)$) [20].

Now, how to use wavelets for variable coefficient operator. Consider the following operator

$$L = -Dp^2(x)D, \tag{3.60}$$

where p is sufficiently smooth and positive. The square root operator is now

$$V = pD \quad \text{and} \quad V^{-1} = D^{-1}\frac{1}{p}. \tag{3.61}$$

The analysis is similar to the case of the Helmholz operator. Applying V^{-1} directly to a wavelet does not yield a compactly supported function. Therefore we take

$$\Psi_{j,l} = V^{-1}p\psi_{j,l} = D^{-1}\psi_{j,l}. \tag{3.62}$$

Then,

$$\begin{aligned} \langle\langle \Psi_{j,l}, \Psi^*_{m,n} \rangle\rangle &= \langle V\Psi_{j,l}, V\Psi^*_{m,n} \rangle \\ &= \langle p\psi_{j,l}, p\psi_{m,n} \rangle \\ &= \int_{-\infty}^{\infty} p^2 \psi_{j,l}(x)\tilde{\psi}_{m,n}(x)dx, \end{aligned} \tag{3.63}$$

which implies that the wavelets need to be biorthogonal with respect to a weighted inner product with p^2 as weight function. We use the same trick as for Helmholz operator. Let the scaling function $\phi_{j,l}$ coincide with $\frac{1}{p^2}$ on the interval $[2^{-j}l, 2^{-j}(l+1)]$

$$\phi_{j,l} = \frac{1}{p^2}\chi_{[j,l]}, \tag{3.64}$$

and normalize them such that they have a constant integral. We then take the wavelets

$$\psi_{j,l} = \phi_{j+1,2l} - \phi_{j+1,2l+1},$$

so they have vanishing integral and the operator wavelet are compactly supported. The operator wavelets $\Psi_{j,l}$ are now piecewise functions that locally look like

$$AP + B$$

where P is the antiderivative of $\frac{1}{p^2}$. The operator wavelets are neither dilates nor translates of one function, since their behavior locally depends on p [20]. The coefficients in the fast wavelet transform are now different every where and they depend in a very simple way on the Haar wavelet transform of $\frac{1}{p^2}$. Then, the entries of diagonal stiffness matrix can be calculated from the wavelets transform of $\frac{1}{p^2}$ we refer for more details to [20] and the references cited therein.

Let us take a numerical example [20]. By solving the equation

$$-De^{x^2}Du(x) = \frac{e^{x^2}(\sin(x)(3x^2-2)+\cos(x)(2x-2x^3))}{x^3},$$

with $u(0) = 1$ and $u(1) = \sin(1)$, the exact solution is given by $u(x) = \frac{\sin x}{x}$. The L_∞ error of the numerically computed solution is function of the number of levels (l) is given in Table 3.1.

l	l_∞ error
1	1.22×10^{-2}
2	3.37×10^{-3}
3	8.66×10^{-4}
4	2.18×10^{-4}
5	5.45×10^{-5}
6	1.36×10^{-5}

Table 3.1.

Note that each time the number of levels is increased the error is divided almost by a factor of 4, which agrees with the $O(h^2)$ convergence. For more details see [20].

3.3 Discussion

In this section we present some work that we have done during our research.

3.3.1 Differential and Integral Equations

In this subsection the relation between some differential equations and the integral equations is given. The differential equations can be transformed into the integral equations by using the continuous wavelet transform. An abstract proof of the following lemma can be found in [16] but here we present our proof.

Lemma 3.4 Let $\psi \in L^2(R)$, with $0 < C_\psi < \infty$, then for any $f \in L^2(R)$ we have

$$f^{(k)}(x) = \frac{1}{C_\psi} \int_{-\infty}^{\infty} \frac{da}{a^2} \int_{-\infty}^{\infty} \left\langle f, \psi_{a,b} \right\rangle a^{-k} \psi_{a,b}^{(k)}(x) db, \tag{3.65}$$

where C_ψ is admissibility condition defined by

$$C_\psi = \int_{-\infty}^{\infty} \frac{\left|\hat{\psi}(\xi)\right|^2}{|\xi|} d\xi < \infty, \tag{3.66}$$

and

$$\psi_{a,b}(x) = \frac{1}{\sqrt{|a|}}\psi(\frac{x-b}{a}), \quad a, b \in R,\ a \neq 0,$$

Proof. By using the Parseval's Formula for Wavelet Transforms Theorem 1.3 For any $g \in L^2(R)$, we have

$$< f^{(k)}, g >= \frac{1}{C_\psi}\int_{-\infty}^{\infty}\frac{da}{a^2}\int_{-\infty}^{\infty}\left\langle f^{(k)}, \psi_{a,b}\right\rangle \overline{\langle g, \psi_{a,b}\rangle} db \tag{3.67}$$

By using Parseval's Formula for Fourier Transform Theorem 1.1 we have

$$< f^{(k)}, g >= \frac{1}{2\pi} < \widehat{f^{(k)}}, \widehat{\psi_{a,b}} >= \frac{(iw)^k}{2\pi} < \widehat{f}, \hat{\psi}_{a,b} > \tag{3.68}$$

Then (3.67) becomes

$$\begin{aligned}
< \ f^{(k)}, g >&= \frac{1}{C_\psi}\int_{-\infty}^{\infty}\frac{da}{a^2}\int_{-\infty}^{\infty}\frac{1}{(2\pi)^2}(iw)^k < \widehat{f}, \hat{\psi}_{a,b} > \overline{\left\langle \hat{g}, \hat{\psi}_{a,b}\right\rangle} db \\
&= \frac{1}{C_\psi}\int_{-\infty}^{\infty}\frac{da}{a^2}\int_{-\infty}^{\infty}\frac{a^{-k}}{(2\pi)^2} < \widehat{f}, \hat{\psi}_{a,b} >\overline{< \hat{g}, (iwa)^k\hat{\psi}_{a,b} >} db \\
&= \frac{1}{C_\psi}\int_{-\infty}^{\infty}\frac{da}{a^2}\int_{-\infty}^{\infty}\frac{a^{-k}}{(2\pi)^2} < \widehat{f}, \hat{\psi}_{a,b} >\overline{< \hat{g}, \widehat{\psi^{(k)}_{a,b}} >} db.
\end{aligned}$$

Again using Parseval's Formula for Fourier Transform Theorem 1.1 we have

$$\begin{aligned}
< \ f^{(k)}, g >&= \frac{1}{C_\psi}\int_{-\infty}^{\infty}\frac{da}{a^2}\int_{-\infty}^{\infty} a^{-k} < f, \psi_{a,b} >\overline{< g, \psi^{(k)}_{a,b} >} db \\
&= \frac{1}{C_\psi}\int_{-\infty}^{\infty}\frac{da}{a^2}\int_{-\infty}^{\infty} a^{-k} < f, \psi_{a,b} > \left(\int_{-\infty}^{\infty}\bar{g}(x)\psi^{(k)}_{a,b}(x)dx\right) db \\
&= \int_{-\infty}^{\infty}\left(\frac{1}{C_\psi}\int_{-\infty}^{\infty}\frac{da}{a^2}\int_{-\infty}^{\infty} a^{-k} < f, \psi_{a,b} > \psi^{(k)}_{a,b}(x) db\right)\bar{g}(x)dx \\
&= < \frac{1}{C_\psi}\int_{-\infty}^{\infty}\frac{da}{a^2}\int_{-\infty}^{\infty} a^{-k} < f, \psi_{a,b} > \psi^{(k)}_{a,b} db, g >,
\end{aligned}$$

where we have interchanged integral in the third step. Then we have

$$< f^{(k)} - \frac{1}{C_\psi} \int_{-\infty}^{\infty} \frac{da}{a^2} \int_{-\infty}^{\infty} a^{-k} < f, \psi_{a,b} > \psi_{a,b}^{(k)} db, g >= 0,$$

for all $g \in L^2(R)$. Then

$$f^{(k)} - \frac{1}{C_\psi} \int_{-\infty}^{\infty} \frac{da}{a^2} \int_{-\infty}^{\infty} a^{-k} < f, \psi_{a,b} > \psi_{a,b}^{(k)} db = 0$$

or

$$f^{(k)} = \frac{1}{C_\psi} \int_{-\infty}^{\infty} \frac{da}{a^2} \int_{-\infty}^{\infty} < f, \psi_{a,b} > a^{-k} \psi_{a,b}^{(k)} db. \blacksquare$$

Now, consider the following class of differential equations

$$\sum_{k=0}^{n} a_k(x) y^{(k)} = b(x), \tag{3.69}$$

$\{a_k(x); k = 0, 1, \cdots, n\} \subset L^\infty(R)$, $\{y^{(k)}; k = 0, 1, \cdots, n\} \subset L^2(R)$, $b(x) \in L^2(R)$.

Let $\{\psi^{(k)}; k = 0, 1, \cdots, n\} \subset L^2(R)$ with supp$(\psi) \subset [-L, L]$. According to Lemma 3.4 we have

$$y^{(k)}(x) = \frac{1}{C_\psi} \int_{-\infty}^{\infty} \frac{da}{a^2} \int_{-\infty}^{\infty} \langle y, \psi_{a,b} \rangle a^{-k} \psi_{a,b}^{(k)}(x) db, \tag{3.70}$$

$\{$for $k = 0, 1, \cdots, n\}$. Then (3.69) becomes

$$\frac{1}{C_\psi} \int_{-\infty}^{\infty} \frac{da}{a^2} \int_{-\infty}^{\infty} \langle f, \psi_{a,b} \rangle \sum_{k=0}^{n} a^{-k} a_k(x) \psi_{a,b}^{(k)}(x) db = b(x). \tag{3.71}$$

Then the differential equation (3.69) is equivalent to integral equation (3.71).

Example 3.5 Consider the following differential equation

$$\sum_{k=0}^{n} a_k(x) y^{(k)} = b(x), \tag{3.72}$$

$\{b(x),\ a_k(x);\ k = 0, 1, \cdots, n\} \subset C[-\pi, \pi]$, $\{y^{(k)}; k = 0, 1, \cdots, n\} \subset L^2(R)$. If $x \notin [-\pi, \pi]$ let $b(x) = a_k(x) = y^{(k)} = 0$ for $k = 0, 1, \cdots, n$. Then $\{b(x),\ a_k(x);\ k =$

$0, 1, \cdots, n\} \subset L^\infty(R)$ and $\{b(x), a_k(x), y^{(k)}; k = 0, 1, \cdots, n\} \subset L^2(R)$. Define ψ by

$$\psi(x) = \begin{cases} \cos x & x \in [-\pi, \pi] \\ 0 & x \notin [-\pi, \pi] \end{cases},$$

ψ is drawn in Figure 3.1. ψ is a wavelet because

$$\hat{\psi}(w) = \frac{1}{\sqrt{2\pi}} \left[\frac{\sin(w+1)\pi}{w+1} + \frac{\sin(w-1)\pi}{w-1} \right],$$

and

$$0 < C_\psi < \infty.$$

Then the continuous wavelet transform of y with respect to the wavelet ψ is

$$\begin{aligned} (T_\psi y)(a,b) &= \int_{-\infty}^{\infty} y(z)\overline{\psi_{a,b}(z)}dz = \frac{1}{\sqrt{|a|}} \int_{-\infty}^{\infty} y(z)\psi(\frac{z-b}{a})dz \\ &= \frac{1}{\sqrt{|a|}} \int_{-|a|\pi+b}^{|a|\pi+b} y(z)\cos(\frac{z-b}{a})dz. \end{aligned} \tag{3.73}$$

Now, by using Lemma 3.4, (3.72) becomes

$$\begin{aligned} b(x) &= \frac{1}{C_\psi} \int\limits_{-\infty}^{\infty} \frac{da}{a^2} \left(\int\limits_{-|a|\pi+x}^{|a|\pi+x} \left[\frac{1}{\sqrt{|a|}} \int_{-|a|\pi+b}^{|a|\pi+b} y(z)\cos(\frac{z-b}{a})dz \right] \right. \\ &\qquad \left. \sum_{k=0}^{n} a_k(x) \frac{1}{\sqrt{|a|}} a^{-k} \cos(\frac{x-b}{a} + k\frac{\pi}{2}) db \right). \end{aligned} \tag{3.74}$$

Then in order to solve the differential equation (3.72) we only need to solve the integral equation in (3.74) [16].

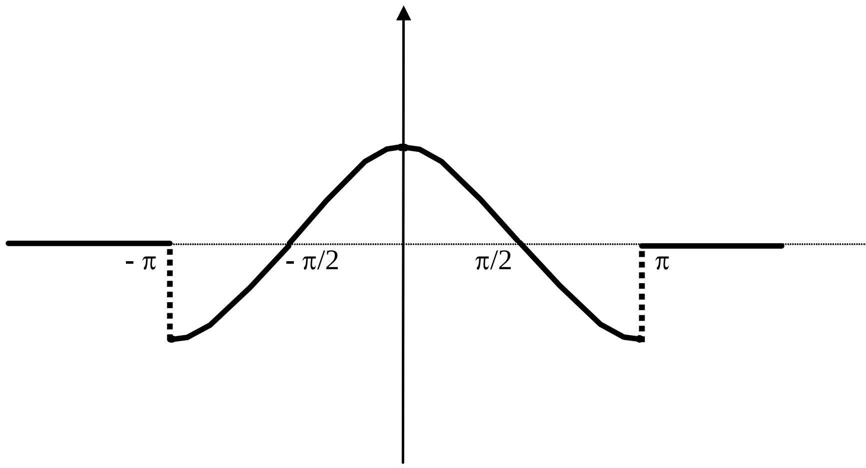

Figure 3.1. Wavelet in Example 3.5.

3.3.2 Using Difference Equations

Suppose ϕ is a scaling function for a multiresolution analysis $\{V_j\}_{j\in\mathbb{Z}}$.

$$V_j = \left\{ \sum_{k\in\mathbb{Z}} \alpha_k \phi_{j,k}(x) : \{\alpha_k\}_{k\in\mathbb{Z}} \in l^2(\mathbb{Z}) \right\},$$

where

$$\phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k).$$

The orthogonal projection operator P_j from $L^2(R)$ onto V_j is defined by

$$P_j(f)(x) = \sum_{k\in\mathbb{Z}} < f, \phi_{j,k} > \phi_{j,k}(x),$$

also we have

$$\lim_{j\to\infty} P_j(f) = f.$$

The projection $P_j(f)$ can be considered as an approximation of f at the scale 2^{-j}. Therefore, the successive approximations of a given function f are defined as the orthogonal projections $P_j(f)$ onto the space V_j. We can choose $j \in \mathbb{Z}$ such that $P_j(f)$ is a good approximation of f. For very large j we can approximate $f(x)$ by $P_j(f)$ that is

$$f(x) \approx P_j(f)(x) = \sum_{k\in\mathbb{Z}} \alpha_{j,k}\phi_{j,k}(x), \tag{3.75}$$

where

$$\alpha_{j,k} =< f, \phi_{j,k} >,$$

and

$$\phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k).$$

From the definition of the derivative we have

$$f'(x) = \lim_{j\to\infty} \frac{f(x+\frac{1}{2^j}) - f(x)}{\frac{1}{2^j}},$$

Again for large j we can approximate $f'(x)$ by

$$f'(x) \approx 2^j \left[f(x+\frac{1}{2^j}) - f(x) \right], \tag{3.76}$$

substituting (3.75) into (3.76) we get

$$\begin{aligned}
f'(x) &\approx 2^j \left[f(x+\frac{1}{2^j}) - f(x) \right] \\
&= 2^j \left[\sum_{k\in\mathbb{Z}} \alpha_{j,k} 2^{j/2}\phi(2^j(x+\frac{1}{2^j}) - k) - \sum_{k\in\mathbb{Z}} \alpha_{j,k} 2^{j/2}\phi(2^j x - k) \right] \\
&= 2^j \left[\sum_{k\in\mathbb{Z}} \alpha_{j,k} 2^{j/2}\phi(2^j x + 1 - k) - \sum_{k\in\mathbb{Z}} \alpha_{j,k} 2^{j/2}\phi(2^j x - k) \right] \\
&= 2^j \left[\sum_{k\in\mathbb{Z}} (\alpha_{j,k+1} - \alpha_{j,k})\, \phi_{j,k}(x) \right]. \qquad (3.77)
\end{aligned}$$

Let V_j be the space of all function in $L^2(R)$ which are constants on intervals of the form $I_{j,k} = [2^{-j}k, 2^{-j}(k+1)]$, $k \in \mathbb{Z}$.

$$V_j = \{f \in L^2(R) : f = \text{constant on } I_{j,k}, \forall k \in \mathbb{Z}\}.$$

Then $\{V_j, j \in \mathbb{Z}\}$ is an MRA see Example 2.1. The scaling function is given by

$$\phi = \chi_{[0,1]}.$$

Now, consider a simple differential equation

$$f'(x) + bf(x) = 0, \quad f(0) = f_0, \tag{3.78}$$

where b is a constant real number. The exact solution of the differential equation (3.78) is

$$f(x) = f_0 e^{-bx}.$$

Now, substituting (3.75) and (3.77) into (3.78) yields

$$2^j \left[\sum_{k\in\mathbb{Z}} (\alpha_{j,k+1} - \alpha_{j,k})\, \phi_{j,k} \right] + b \sum_{k\in\mathbb{Z}} \alpha_{j,k}\phi_{j,k}(x) = 0$$

$$\sum_{k\in\mathbb{Z}} \left(2^j \alpha_{j,k+1} + (b - 2^j)\alpha_{j,k}\right) \phi_{j,k} = 0, \tag{3.79}$$

taking the inner product with $\phi_{j,n}$ in (3.79) we get

$$2^j \alpha_{j,n+1} + (b - 2^j)\alpha_{j,n} = 0$$

or

$$\alpha_{j,n+1} = (1 - \frac{b}{2^j})\alpha_{j,n}. \tag{3.80}$$

Solving the difference equation in (3.80) we get

$$\alpha_{j,n} = (1 - \frac{b}{2^j})^n \alpha_{j,0}, \tag{3.81}$$

where

$$
\begin{aligned}
\alpha_{j,0} &= < f, \phi_{j,0} >= \int_{-\infty}^{\infty} f(x)\phi_{j,0}(x)dx \\
&= \int_{0}^{2^{-j}} f(x)2^{j/2}\phi(2^j x)dx \\
&= 2^{j/2} f(0) \int_{0}^{2^{-j}} \phi(2^j x)dx \\
&= 2^{j/2} f(0) \int_{0}^{2^{-j}} 1dx \\
&= 2^{-j/2} f(0) = 2^{-j/2} f_{_0}. \qquad (3.82)
\end{aligned}
$$

Since $f(x)$ is continuous and the integration is taken over a small interval $[0, 2^{-j}]$, we can approximate $f(x)$ by $f(0)$ for very large j. Similarly for $\alpha_{j,k}$ we have

$$
\begin{aligned}
\alpha_{j,k} &= < f, \phi_{j,k} >= \int_{-\infty}^{\infty} f(x)\phi_{j,k}(x)dx \\
&= \int_{2^{-j}k}^{2^{-j}(k+1)} f(x)2^{j/2}\phi(2^j x - k)dx \\
&= 2^{-j/2} f(2^{-j}k). \qquad (3.83)
\end{aligned}
$$

Then, from (3.81), (3.82) and (3.83) we get

$$
f(2^{-j}k) = f_{_0}(1 - \frac{b}{2^j})^k. \qquad (3.84)
$$

Let $k \to 2^j x$, then (3.84) becomes

$$
f(x) = f_{_0}(1 - \frac{b}{2^j})^{2^j x}, \qquad (3.85)
$$

for very large j. Take the limit in (3.85) as $j \to \infty$ we get

$$
f(x) = \lim_{j\to\infty} f_{_0}(1 - \frac{b}{2^j})^{2^j x} = f_{_0} e^{-bx}, \qquad (3.86)
$$

which coincides with the exact solution of (3.78).

3.3.3 Expansion of Derivative

In this subsection we will prove that for certain functions the derivative can be written as

$$f'(x) = \sum_{n\in\mathbb{Z}} t_n f(x-n),$$

where $t_n \in R$ for all $n \in \mathbb{Z}$.

Let P_k be a space of polynomial which has degree less than or equal to k. Then t_n can be found by solving a system of linear equations. For example for $f \in P_2$. One can prove that

$$f'(x) = \frac{1}{2}f(x+1) - \frac{1}{2}f(x-1).$$

For $f \in P_4$. We have

$$f'(x) = \frac{-1}{12}f(x+2) + \frac{2}{3}f(x+1) - \frac{2}{3}f(x-1) + \frac{1}{12}f(x-2).$$

Lemma 3.5 Let $f \in L^2(R)$, and $\hat{f}$ does not vanish in $[-\pi, \pi]$ almost everywhere, then f' can be written in the following form:

$$f'(x) = \sum_{n\in\mathbb{Z}} t_n f(x-n), \tag{3.87}$$

where

$$t_n = \left\{ \begin{array}{ll} \frac{(-1)^n}{n}, & n \neq 0, n \in \mathbb{Z} \\ 0, & n = 0 \end{array} \right\}.$$

Proof. Taking the Fourier Transform of (3.87) we get

$$(iw)\hat{f}(w) = \sum_{n\in\mathbb{Z}} t_n e^{-iwn}\hat{f}(w)$$

Since $\hat{f}(w) \neq 0$ $a.e.$ $w \in (-\pi, \pi)$, then by cancelling $\hat{f}(w)$ from both sides we get

$$(iw) = \sum_{n\in\mathbb{Z}} t_n e^{-iwn}. \tag{3.88}$$

Taking inner product with $e^{-iwm}, m \neq 0$ in (3.88) we get

$$\int_{-\pi}^{\pi}(iw)e^{iwm}dw = \int_{-\pi}^{\pi}\sum_{n\in\mathbb{Z}} t_n e^{-iwn}e^{iwm}dw$$

Then

$$\frac{2\pi\left(\cos\pi m\right)m - 2\sin\pi m}{m^2} = \sum_{n\in\mathbb{Z}} t_n \int_{-\pi}^{\pi} e^{-iwn}e^{iwm}dw$$

Because $\{e^{-iwn}\}$ are orthogonal in $(-\pi,\pi)$ and m is integer we have

$$\frac{2\pi(-1)^m}{m} = 2\pi t_m.$$

Thus,

$$t_m = \frac{(-1)^m}{m}.$$

If $m = 0$, then $t_0 = 0$. ■

Example 3.6 Let $f(x) = \frac{\sin x}{x}$

$$\sum_{n\in\mathbb{Z}} t_n f(x-n) = \sum_{n\neq 0}\frac{(-1)^n}{n}\frac{\sin(x-n)}{x-n} = \frac{\cos x}{x} - \frac{\sin x}{x^2} = f\,'(x).$$

Similar results can be found with higher derivatives. For second derivative we have

$$f\,''(x) = \sum_{n\in\mathbb{Z}} r_n f(x-n),$$

where

$$r_n = \left\{\begin{array}{lr} \frac{2(-1)^{n+1}}{n^2}, & n\neq 0, n\in\mathbb{Z} \\ -\frac{\pi^2}{3}, & n=0 \end{array}\right\}.$$

For third derivative

$$f\,'''(x) = \sum_{n\in\mathbb{Z}} r_n f(x-n),$$

where

$$r_n = \left\{\begin{array}{lr} (-1)^n(\frac{6}{n3} - \frac{\pi^2}{n}), & n\neq 0, n\in\mathbb{Z} \\ 0, & n=0 \end{array}\right\}.$$

For fourth Derivative we get

$$f^{(4)}(x) = \sum_{n \in \mathbb{Z}} r_n f(x-n),$$

where

$$r_n = \left\{ \begin{array}{ll} 4(-1)^n(\frac{\pi^2}{n^2} - \frac{6}{n^4}), & n \neq 0, n \in \mathbb{Z} \\ \frac{\pi^4}{5}, & n = 0 \end{array} \right\}.$$

And higher derivatives can be obtained in a similar procedure.

There are four main properties of wavelets; namely, they are local in both space and frequency, they satisfy biorthogonality conditions, they provide a multiresolution structure and fast transform algorithms are available. Because of these properties wavelets have proven to be useful in the solution of ordinary differential equations. As proposed by several researchers, wavelets can be used as basis functions in Galerkin method. This has proven to work and the results in a linear system that is sparse because of the compact support of wavelets, and that, after preconditioning, has a condition number independent of problem size because of the multiresolution structure. By using two pairs of biorthogonal compactly supported wavelets, derivative operator can be diagonalized [12]. Like the Fourier transform, wavelets can diagonalize constant coefficient operators. The resulting algorithm is slightly faster [20]. Even non constant coefficient operators can be diagonalized with the right choice of basis which yields a much faster algorithm than classical iterative methods.

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