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This thesis, written by **Sogome Suraizou** under the direction of his thesis supervisor and approved by his thesis committee, has been presented to and accepted by the Dean of Graduate Studies, in partial fulfillment of the requirements for the degree of **MASTER OF SCIENCE IN MATHEMATICAL SCIENCES.**

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Abstract

Name: Sogome Suraizou

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Since Seidenberg's work on the dimension of polynomial rings in 1952, we know that Noetherian domains have the Krull dimension coincide with the valuative dimension (i.e., are Jaffard domains in non-Noetherian jargon). It is still an open problem to know whether this holds for arbitrary finite-dimensional Krull domains. However, non-Noetherian finite-dimensional Krull domains are rare in the literature and all happen to be Jaffard. Some of them arise as symbolic blow-up algebras or symbolic Rees algebras. Our purpose in this MS thesis is to have a close look at various works on this subject, towards a better understanding of non-Noetherian finite-dimensional Krull structures; particularly, those issued from blow-up and Rees algebras.

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1 Chapter 1

Introduction

It is well known that for finite-dimensional Noetherian or Prüfer domains the valuative dimension coincides with the Krull dimension (i.e., Jaffard domains). However, it is still an open problem to compute the valuative dimension of an arbitrary finite-dimensional Krull domain.

Bouvier's conjecture [17] sustains that

"`there is a Krull domain (or UFD)` R `with`

$1 + \dim(R) \lneqq \dim(R[X]) < \infty$." In [11], a diagram puts this conjecture in its spectral context and hence shows how it naturally arose. In particular, it reveals how the classes of Noetherian domains, Prüfer domains, UFDs, Krull domains, and PVMDs interact with the notion of Jaffard domain as well as with the spectrum-related S-domain properties of Kaplansky. See Figure 1. For more details about the above concepts, we refer to [2, 9, 26, 27, 30]. Any unreferenced material is standard, as in [18, 28, 31].

Since Seidenberg's work on the dimension of polynomial rings, we know that Noetherian domains are Jaffard (1952-54), so that one has to dig beyond the classic context of Noetherianness. However, non-Noetherian finite-dimensional Krull domains are rare in the literature [4, 10, 14, 19, 32, 38, 39], and all happen to be (locally) Jaffard [11]. Some of them arise as symbolic blow-up algebras and symbolic Rees algebras [14, 38, 39]. Our purpose in this MS thesis is to

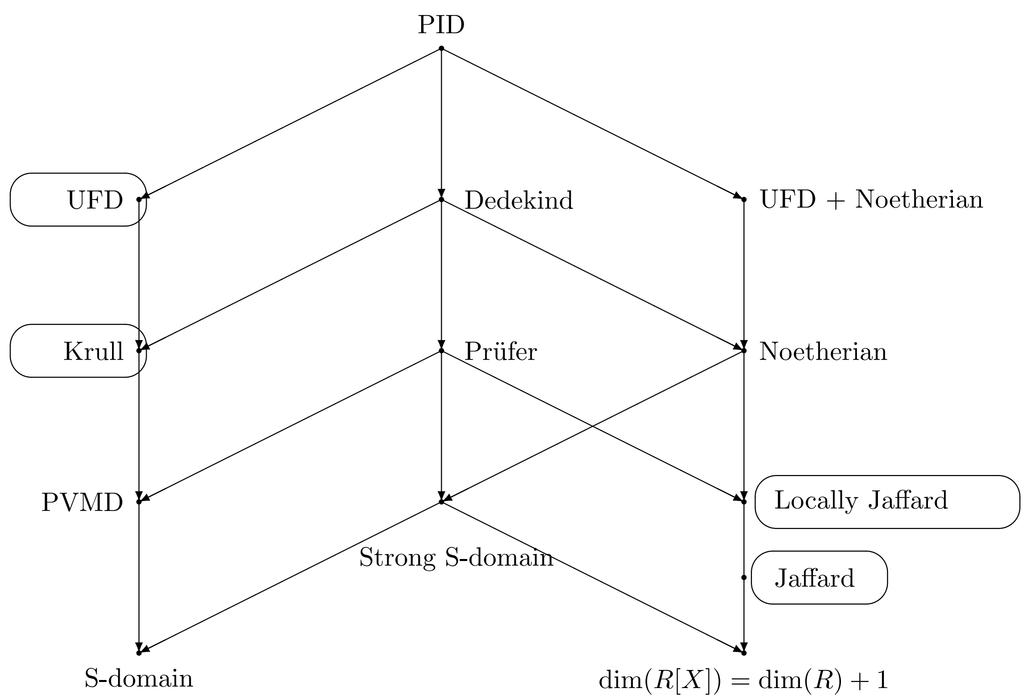

Figure 1: `Diagram of Implications`

have a close look at various works on this subject, towards a better understanding of non-Noetherian finite-dimensional Krull structures; particularly, those issued from blow-up and Rees algebras.

In 1958, Rees constructed in [37] a first counter-example to the Zariski-Hilbert problem also called the generalized 14th problem of Hilbert (initially posed at the Second International Congress of Mathematicians at Paris in 1900). His construction gave rise to (what is now called) Rees algebras. Since then, these special graded algebras has been capturing the interest of many mathematicians, particularly commutative algebraists and geometers.

Let A be an integral domain, t an indeterminate over A, and $P \in \mathrm{Spec}(A)$. For each $n \in \mathbb{Z}$, set $P^{(n)} := P^n A_P \cap A$, the nth symbolic power of P, where $P^n := A$ for each $n \leq 0$. We define the following graded algebras:

- $\bigoplus_{n\in\mathbb{N}} P^n t^n = A[Pt, ..., P^n t^n, ...]$ the blow-up algebra of P,
- $\bigoplus_{n\in\mathbb{N}} P^{(n)} t^n = A[P^{(1)}t, ..., P^{(n)}t^n, ...]$ the symbolic blow-up algebra of P,
- $\bigoplus_{n\in\mathbb{Z}} P^n t^n = A[t^{-1}, Pt, ..., P^n t^n, ...]$ the Rees algebra of P,
- $\bigoplus_{n\in\mathbb{Z}} P^{(n)} t^n = A[t^{-1}, P^{(1)}t, ..., P^{(n)}t^n, ...]$ the symbolic Rees algebra of P.

In 1970, based on Rees' work, Eakin and Heinzer constructed in [14] a (first example of a) 3-dimensional non-Noetherian Krull domain. It arose as a symbolic Rees algebra. It is proved in [9] that this construction yields locally Jaffard domains.

In 1973, Hochster studied in [22] criteria for P^n to equal the symbolic power $P^{(n)}$ (i.e., P^n primary) for each positive integer n within Noetherian contexts. One of his applications asserts that $P^{(n)} = P^n$ when P is a prime ideal of R generated by an R-sequence. One has then to go beyond these contexts to avoid a collapse between blow-up or Rees algebras and symbolic blow-up or Rees algebras.

In 1985, Roberts constructed in [38] a second counter-example to the generalized 14th problem of Hilbert which also answered an open question (due to Cowsik) about the existence of regular local rings in which the symbolic blow-up of a prime ideal is not finitely generated over the ground ring. In 1990, Roberts in [39] provided a new counter-example to the generalized 14th problem of Hilbert with, in addition, a solution to Cowsik's question in a complete regular local ring. His example arose as a symbolic blow-up algebra that is a 3-dimensional non-Noetherian Krull domain. The problem of whether these settings yield (locally) Jaffard domains is treated in details in [9].

An integral domain R satisfies PIT (resp., GPIT) if every minimal prime over a principal (resp., n-generated) ideal has height ≤ 1 (resp., $\leq n$). Here PIT stands for "Principal Ideal Theorem" and GPIT for " Generalized Principal Ideal Theorem." Both Noetherian and Krull domains satisfy PIT. Moreover, while Noetherian rings satisfy GPIT (by Krull's altitude theorem), Krull domains don't. Eakin-Heinzer's symbolic Rees algebra (mentioned above) provided a first counter-example in this regard, which was later developed by Anderson-Dobbs-Eakin-Heinzer in 1990 [3].

This MS thesis traverses five sections along with an introduction and a preliminaries section. Each section is devoted to an original research paper. Section 3 studies Jaffard domains (see definition above). First, we present transfer results to ring extensions associated with the basic ring. Then we examine the possible transfer of the Jaffard property to various pullback contexts. The study is then backed with examples and counter-examples. Section 4 deals with the GPIT property (see definition above). Three paragraphs subsequently investigate this notion in the contexts of integral extensions, homomorphic images , and monoid domains. A fourth paragraph builds an example (mentioned above) of a local three-dimensional Krull domain not satisfying GPIT, hence not Noetherian. Section 5 studies the n-symbolic powers of prime ideals (see definition above). It investigates criteria for symbolic powers of a prime ideal to collapse to the ordinary powers. In this case, Symbolic Rees or blow-up algebras merely reduce to Rees or blow-up algebras. Sections 6 and 7 examines in details the Rees-Eakin-Heinzer's construction and Roberts' construction (mentioned above), respectively. We close the thesis with some open problems and conjectures.

2 Chapter 2

Preliminaries

In this section, we treat different categories of rings, namely Krull rings, graded rings, complete regular local rings, Rees algebras and blow-up algebras, and application of (Krull) dimension theory to theses rings. Then we will give formulas which relate the dimension of a Noetherian ring with any of its extension, and finish up with formulas to compute the dimension of the associated graded rings as well as Rees algebras and blow-up algebras.

2.1 Krull rings

Definition 2.1 ([31]) . Let A be an integral domain and K its field of fractions. We write $K^\star$ for the multiplicative group of K. We say that A is a Krull ring if there is a family $\mathfrak{F} = \{R_\lambda\}_{\lambda\in\Lambda}$ of DVRs of K such that the following two conditions hold, where we write v_λ for the normalized additive valuation corresponding to R_λ:

(1) $A = \bigcap_\lambda R_\lambda$,

(2) for every $x \in K^\star$, there are at most a finite number of $\lambda \in \Lambda$ such that $v_\lambda(x) \neq 0$.

Example 2.2 (1) Any UFD is a Krull ring.

(2) If R is a Krull ring, then $R[X]$ and $R[[X]]$ are Krull rings.

(3) If K is a field and $\{X_i\}_{i=1}^{\infty}$ is a collection of indeterminates over K, then $K[\{X_i\}_{i=1}^{\infty}]$ is a Krull ring.

Proposition 2.3 . *Let R be a Noetherian domain such that its localization at any prime is integrally closed. Then R is a Krull ring.*

Proof. See [31, Theorem 12.4]. □

Proposition 2.4 ([31, Theorem 12.4]) . *Let A be an integral domain, K its field of fractions and L an extension field of K. If $\{A_i\}_{i\in I}$ is a family of Krull rings contained in L and satisfying the two conditions:*

(1) $A = \bigcap A_i$ *and*

(2) *given any* $0 \neq a \in A$, *we have* $aA_i = A_i$ *for all, but finitely many* i,

then A is a Krull ring.

Proof. For each i, $A_i \cap K$ is a Krull ring, so that $A = \bigcap(A_i \cap K) = \bigcap A_{ij}$, where A_{ij} are DVRs of K. Next, Condition (2) shows that any $a \in A$ is a non-unit in a finite number of A_i; therefore there exists a finite number of j such that $v_{ij}(a) \neq 0$. So, if $b = \frac{b_1}{b_2} \in K^\star$, then $v_{ij}(b) = v_{ij}(b_1) - v_{ij}(b_2) \neq 0$ for a finite number of i and j. □

2.2 Graded ring, Hilbert function, and Samuel function

Definition 2.5 ([31]) Let G be an abelian semigroup with identity element 0. A graded (or $G-$graded) ring is a ring R together with a direct sum decomposition of R as an additive group $R = \bigoplus_{i \in G} R_i$ satisfying $R_i R_j \subset R_{i+j}$. Similarly, a graded $R-$module is an $R-$module M together with a direct sum decomposition $M = \bigoplus_{i \in G} M_i$ satisfying $R_i M_j \subset M_{i+j}$. An element $x \in M$ is homogeneous of degree i if it belongs to M_i. A submodule of M is homogeneous (or graded) if it can be generated by homogeneous elements.

Lemma 2.6 * *Let R be a graded ring and M a Noetherian (resp., an Artinian) graded $R-$module. Then M_n is*
a Noetherian (resp., an Artinian) R_0-module for all n.

Proof. We will give a proof only for the Noetherian case. But before, notice that if R is a graded ring, M a graded $R-$module and N an R_0-submodule of M_n for some n, then $RN \cap M_n = N$. Indeed, $N \subset RN$ since $1 \in R$; so that $N \subset M_n \cap RN$.

Let $z \in M_n \cap RN$ and assume that $z \in M_n \setminus N$. Then $z \in RN$ implies that $z = \sum r_i n_i$, $r_i \in R$ and $n_i \in N$. Since M_n is homogeneous, each $r_i n_i \in M_n$ so that $r_i n_i \in M_n \cap RN$. Let $r_j n_j$ be the first term of z not in N. Then $r_j \notin R_0$. But $n_j \in N \subset M_n$ implies that $r_j n_j \notin M_n$, which contradicts the assumption. Now, let $N_1 \subset N_2 \subset \ldots$ be an ascending chain of
R_0-submodules of M_n. Then $RN_1 \subset RN_2 \subset \ldots$ is an ascending chain of $R-$submodules of M and so must stabilize. Contracting back to M_n, we conclude that the chain $N_1 \subset N_2 \subset \ldots$ stabilizes. □

Proposition 2.7 ([1, Proposition 10.7]) *An*

$\mathbb{N}-$*graded ring* $R = \bigoplus R_n$ *is Noetherian if and only if* R_0 *is Noetherian and* R *is finitely generated as a ring over* R_0.

Proof. If R_0 is Noetherian and R is a f.g. R_0-algebra, then R is Noetherian by Hilbert basis theorem. Conversely, $R_+ = \bigoplus_{n>0} R_n$ is an ideal of R; so that it is f.g. by homogeneous elements $y_1, y_2, \ldots, y_k$, with $\deg y_i = s_i$. Then $R_0 \cong \frac{R}{\bigoplus_{n>0} R_n}$ is Noetherian. Let

$s = \max\{s_1, s_2, \ldots, s_k\}$ and put $N = R_1 \oplus R_2 \oplus \ldots \oplus R_s$. By Lemma 2.6, each R_i is Noetherian R_0-module and so N is finitely generated as R_0-module. Since N is homogeneous, let $x_1, x_2, \ldots, x_t$ be the homogeneous generators of N. Clearly,

$(x_1, x_2, \ldots, x_t)R = (y_1, y_2, \ldots, y_k)R = R_+$. Next, we show that $R_0[R_+] = R_0[x_1, x_2, \ldots, x_t]$. Since both are homogeneous it will be sufficient to show that they have the same component in each degree n. $n = 0$ is a trivial case. Assume that the equality holds for all components of degree less than or equal to $n-1$. Let $r \in (R_0[R_+])_n$. If $n \leq s$, then $r \in R_0x_1 + \ldots + R_0x_t \subset R_0[x_1, \ldots, x_t]$. If $n > s$, then since $r \in R_+R = (x_1, \ldots, x_t)R$, there exists homogeneous elements $u_1, \ldots, u_t \in R$ such that $r = \sum u_i x_i$. Therefore, $\deg u_i + \deg x_i = n$ for all i, i.e., $\deg u_i = n - \deg x_i \leq n$, and the inductive hypothesis implies that $u_i \in R_0[x_1, \ldots, x_t]$. Hence $r \in (R_0[x_1, \ldots, x_t])_n$. The reverse inclusion is trivial and since $R = R_0[R_+]$, we obtain our desired result. □

Lemma 2.8 * *Let* R *be a graded ring and* M *a graded* $R-$*module. Then* M

is simple as an $R-$module if and only if M is simple as an R_0-module.

Proof. Suppose that M is simple as an $R-$module. Since M is cyclic, we have $M \cong R/\mathcal{N}$ as a graded $R-$module, for some homogeneous maximal ideal $\mathcal{N} = \text{Ann}(M)$. But then $\mathcal{N} = \ldots \oplus R_{-2} \oplus R_{-1} \oplus m \oplus R_1 \oplus R_2 \oplus \ldots$ for some maximal ideal m of R_0. Thus, $M \cong R/\mathcal{N} \cong R_0/m$ and so M is simple as R_0-module. The converse is straightforward. □

Lemma 2.9 ⋆ *Let R be a graded ring and M a graded $R-$module such that $\ell(M) = n$. Then there exists a chain $M = M_0 \supset M_1 \supset M_2 \supset \ldots \supset M_n = (0)$ of submodules of M such that M_i/M_{i+1} is simple and M_i is graded for all i.*

Proof. If $n = 0, 1$ the result is trivial, so suppose $n > 1$. By induction, it is enough to show that there exists a non-zero proper graded submodule of M. Let $x \in M$ be a non-zero homogeneous element. If $xR \neq M$, we are done, so suppose that $Rx = M$. Then $M \cong R/I$ as graded $R-$module, where $I = (0 :_R x)$. Thus, $\ell(R/I) = n$ and so R/I is Artinian. All the maximal ideals of R/I are homogeneous (since they are minimal). If the only maximal ideal of R/I is (0), then $n = \ell(R/I) = 1$, a contradiction. Therefore, there exists a non-zero homogeneous element $r \in R \setminus I$ such that $r + I$ is not a unit in R/I. Set $y = rx$ and $N = Ry$. Then N is a non-zero proper graded submodule of M. □

Lemma 2.10 ⋆ *Let R be a graded ring and M a graded $R-$module. Then*

$\ell_R(M) = \ell_{R_0}(M) = \sum_n \ell_{R_0}(M_n)$.

Proof. If $\ell_R(M) = \infty$, then $\ell_{R_0}(M) = \infty$ since M is also an R_0-module. Suppose that $\ell_R(M) = n < \infty$. Then by Lemma 2.9, there exists a composition series

$M = M_0 \supset M_1 \supset M_2 \supset \ldots \supset M_n = (0)$, where M_i/M_{i+1} are graded simple $R-$modules. By Lemma 2.8, these modules are simple R_0-modules. Hence $\ell_{R_0}(M) = n$. Now, let $M_+ = \bigoplus_{n\geq 0} M_n$ and $M_- = \bigoplus_{n<0} M_n$, then $0 \longrightarrow M_- \longrightarrow M \longrightarrow M_+ \longrightarrow 0$ is an exact sequence and since the length is an additive function, it follows that $\ell(M) = \ell(M_-) + \ell(M_+)$. For each integer k, $M_{n<-k}$

$\longrightarrow M_{n\leq -k} \longrightarrow M_{-k} \longrightarrow 0$ and $M_{n>k} \longrightarrow M_{n\geq k} \longrightarrow M_k \longrightarrow 0$ are exact sequences, so that $\ell(M_{n\geq k}) = \ell(M_k) + \ell(M_{n>k})$ and $\ell(M_{n\leq -k}) = \ell(M_{-k}) + \ell(M_{n<-k})$. Letting k vary in $\mathbb{N}$, we obtain the second equality. $\square$

Lemma 2.11 * *Let R be a $\mathbb{Z}-$graded ring which is not a field and assume that the only homogeneous ideals of R are (0) and R. Then $R = K[t^{-1}, t]$, where $K = R_0$ is a field and t is a homogeneous element of R that is transcendental over K.*

Proof. If u is a non-zero homogeneous element of R, then $(u) = R$ (i.e., there exists $v \in R$ such that $uv = 1$). It follows that every non-zero homogeneous element of R is a unit. So R_0 is a field. As R is not a field, then $R \neq R_0$.

So there exists some $0 \neq t \in R_n$ $(n \neq 0)$. Since t is a unit, $t^{-1} \in R_{-n}$. Assume that n is the smallest integer such that $R_n \neq 0$. Notice that if R is a reduced graded ring, where R_0 is a field, and if $u \in R_n \setminus \{0\}$ with $n \neq 0$, then u is transcendental over R_0. Indeed, if $\sum a_k u^k = 0$ for $a_k \in R_0$, then $a_k u^k = 0 \ \ \forall \ k$ i.e., $u^k = 0$ since a_k is a unit. But this is not possible since R has no nilpotent element. Thus t is transcendental over R_0. Now, we show that any homogeneous element of R_m has the form ct^i for some i and some $c \in R_0$. The case $0 \leq m < n$ is trivial (if $m < n$, then $0 = 0t^i$). If $x \in R_n$, then $xt^{-1} \in R_0$ i.e., $xt^{-1} = r_0$ and $x = r_0 t$. Assume inductively that the result is true for all indices $\leq m - 1$ with $m > n$. Let $u \in R_m$. Then $t^{-1}u \in R_{m-n}$ and $0 \leq m - n < m$. Thus, $ut^{-1} = ct^i$. Multiplying by t, we get the desired result. We argue similarly for elements of negative degree. □

Suppose that M is a graded $R-$module and N is an $R-$submodule of M. We denote by $N^\star$ the $R-$submodule of M generated by all the homogeneous elements contained in N.

Lemma 2.12 ([44, Lemma 3]) . *Let R be a $\mathbb{Z}-$graded ring and P a non-homogeneous prime ideal of R. Then $P^\star$ is prime and there are no prime ideals properly between P and $P^\star$.*

Proof.* Let x, y be homogeneous elements in R such that $xy \in P^\star$ and $x \notin P^\star$. Since $P^\star \subset P$, then $y \in P$ and therefore $y \in P^\star$ because it is homogeneous.

Now, let x, y be non homogeneous such that $xy \in P^\star$ and $x, y \notin P^\star$. Set $x = \sum x_i$ and $y = \sum y_i$. Let i_0 and i_1 be respectively the first indices such that $x_{i_0} \notin P^\star$ and $y_{i_1} \notin P^\star$. Then $xy \in P^\star$ implies that $x_{i_0} y_{i_1} \in P^\star$, so that x_{i_0} or y_{i_1} must be in $P^\star$; which is absurd. Thus $P^\star$ is prime. By passing to $R/P^\star$, we may assume that R is a domain and $P^\star = (0)$. Let S be the set of all non-zero homogeneous elements of R. Since $P \cap S = \emptyset$, PR_S is a non-zero prime ideal of R_S. Thus, in the local graded ring $R_S = \bigoplus (R_S)_n$, where $(R_S)_n = \{\frac{r}{s} \in R_S \mid r \text{ and } s \text{ are homogeneous and } \deg r - \deg s = n\}$, the non-zero homogeneous elements are units. By Lemma 2.11, $R_S = K[t^{-1}, t]$. Since $\dim(K[t^{-1}, t]) = 1$, there are no primes properly between (0) and PR_S. Contracting back to R, we get the desired result. □

Theorem 2.13 ((Matijevic-Roberts) [31, Exercise 13.6]) . *Let R be a graded ring and P a non-homogeneous prime ideal of R. then if* $\operatorname{ht}(P) < \infty$, $\operatorname{ht}(P) = \operatorname{ht}(P^\star) + 1$.

Proof. Assume that $\operatorname{ht}(P^\star) = n < \infty$ and argue by induction on n. If $n = 0$, then Lemma 2.12 gives the result. Suppose that $n > 0$ and let $Q \subset P \in \operatorname{Spec}(R)$. It suffices to prove that $\operatorname{ht}(Q) \leq n$. We have $Q^\star \subseteq Q$ and $P^\star \subseteq P$. If $Q^\star = P^\star$, then $Q = P^\star$ by Lemma 2.12. Otherwise $\operatorname{ht}(Q^\star) \leq n - 1$. By induction hypothesis $\operatorname{ht}(Q) \leq n$. □

Corollary 2.14 **: Let R be a graded ring and M a finitely generated graded*

$R-$module. Let P be an element of the support of M, where P is not homogeneous. Then $\dim(M_P) = \dim(M_{P^\star}) + 1$.

Proof. As M_P is a finitely generated R_P-module,

$\dim(M_P) = \dim(R_P/\operatorname{Ann}(M_P)) = \dim(R/\operatorname{Ann}(M))_P$. By passing to

$R/\operatorname{Ann}_R(M)$, we may assume that $\operatorname{Ann}(M) = 0$. Thus, $\dim(M_P) = \dim(R_P) = \operatorname{ht}(P) \ \forall\, P \in \operatorname{Supp}(M)$. □

Corollary 2.15 **: Let R be an $\mathbb{N}-$graded ring. then if R is locally finite dimensional,* $\dim(R) = \operatorname{Max}\{\operatorname{ht}(M) \mid M$ *a homogeneous maximal ideal*$\}$.

Proof. Let N be a non-homogeneous maximal ideal of R. Then $\operatorname{ht}(N^\star) = \operatorname{ht}(N) - 1$, by Theorem 2.13. Since $N^\star$ is homogeneous and R is $\mathbb{N}-$graded, $N^\star$ is contained in a homogeneous maximal ideal M (such maximal can be obtained by considering any maximal of R_0 which contains $N^\star \cap R_0$). But $M \neq N^\star$ implies that $\operatorname{ht}(M) \geq \operatorname{ht}(N^\star) + 1 = \operatorname{ht}(N)$. □

Definition 2.16 ([31]) . Let R be a Noetherian

$\mathbb{N}-$graded ring and M a finitely generated

graded $R-$module. Suppose that R_0 is Artinian. Then $\ell_{R_0}(M_n) < \infty$ for all n. We define the *Hilbert function* $H_M : \mathbb{N} \longrightarrow \mathbb{N}$ of M by $H_M(n) = \ell_{R_0}(M_n)$ for all n and the *Poincaré series (or Hilbert series)* of M by $P(M,t) = \sum_{n\in\mathbb{N}} H_M(n)t^n \ \in \mathbb{Z}[[t]]$.

Proposition 2.17 $\star$: *Let $R = K[X_1, \ldots, X_d]$ be a polynomial ring over a field K and $degX_i = 1$ for all i. Then $H_R(n) = C^{d-1}_{n+d-1}$ for all $n \geq 0$.*

Proof. We use induction on $n+d$. If $n = 0$ or $d = 1$, then the result is trivial since in the first case $\dim_K(K) = 1$ and in the second one $R_n = K < X^n >$ and $C^0_n = 1$. Suppose that $n > 0$ and $d > 1$. Let $S = K[X_1, \ldots, X_{d-1}]$ and consider the exact sequence

$0 \longrightarrow R_{n-1} \xrightarrow{X_d} R_n \longrightarrow S_n \longrightarrow 0$, where X_d is the multiplication by X_d(a group homomorphism). Then using [1, Proposition 6.10] and the fact that the length is additive, we get:

$$\begin{aligned} H_R(n) = \dim_K(R_n) &= \dim_K(R_{n-1}) + \dim_K(S_n) \\ &= C^{d-1}_{n+d-2} + C^{d-2}_{n+d-2} \\ &= C^{d-1}_{n+d-1}. \end{aligned}$$

□

Theorem 2.18 . *Let $R = \bigoplus_{n \geq 0} R_n$ be a Noetherian graded ring and let M be a finitely generated graded $R-$module. Suppose that $R = R_0[x_1, \ldots, x_r]$ with x_i of degree d_i, and that R_0 is Artinian and $P(M,t)$ is the Hilbert series. Then $P(M,t) = \frac{f(t)}{\prod_{i=1}^{r}(1-t^{d_i})}$, where $f(t)$ is a polynomial with coefficients in $\mathbb{Z}$.*

Proof. [1, Theorem 11.1]. □

Many information on the value of $\ell(M_n)$ can be obtained from the above theorem. A simple case is when $d_1 = \ldots = d_r = 1$, so that R is generated over R_0 by elements of degree 1. In this case, Theorem 2.18 gives $P(M,t) = f(t)(1-t)^{-d}$ with $d \geq 0$ and $f(1) \neq 0$ if $d > 0$. We then write $d = d(M)$.

Since $(1-t)^{-1} = 1 + t + t^2 + \ldots$, we can repeatedly differentiate both sides to get

$(1-t)^{-d} = \sum_{n=0}^{\infty} C_{n+d-1}^{d-1} t^n$. If $f(t) = a_0 + a_1 t + \ldots + a_s t^s$, then the identification after rewriting $f(t)(1-t)^{-d}$ gives $\ell(M_n) = a_0 C_{d+n-1}^{d-1} + a_1 C_{n+d-2}^{d-1} + \ldots + a_s C_{n+d-s-1}^{d-1}$, where we set $C_m^{d-1} = 0$ for $m < d-1$.

So $\ell(M_n)$ can be arranged as a polynomial in n with rational coefficients, say $\varphi(n)$. Then

$\varphi(X) = \frac{f(1)}{(d-1)!} X^{d-1}$+(terms of lower degree). As $C_m^{d-1} = \frac{m(m-1)\ldots(m-d-2)}{(d-1)!}$ for $m \geq 0$, we get the following Corollary:

Corollary 2.19 ([31]) . *If $d_1 = \ldots = d_r = 1$ in Theorem 2.18 and $d = d(M)$ is defined as above, then there is a polynomial $\varphi_M(X)$ of degree $d-1$ with rational coefficients such that for $n \geq s+1-d$, we have $\ell(M_n) = \varphi_M(n)$, where s is the degree of the polynomial $(1-t)^d P(M,t)$.*

Definition 2.20 ([31]) . The polynomial φ_M in Corollary 2.19 is called the *Hilbert polynomial* of the graded module M.

Definition 2.21 ([1]) . A chain $M = M_0 \supseteq M_1 \supseteq \ldots \supseteq M_n \supseteq \ldots$, where the

M_n are $R-$submodules of M, is called a filtration of M and denoted by (M_n). It is a $q-$filtration, where q is an ideal of R, if $qM_n \subseteq M_{n+1}$ for all n, and a stable $q-$filtration if $qM_n = M_{n+1}$ for all sufficiently large n.

Proposition 2.22 ([1, Proposition 11.4]) . *Let (R, m) be a Noetherian local ring and q an $m-$primary ideal. Suppose that M is a finitely generated $R-$module with a stable $q-$filtration (M_n). Then:*

(1) *M/M_n has finite length for each $n \geq 0$.*

(2) *For all sufficiently large n this length is a polynomial $g(n)$ of degree $\leq s$, where s is the least number of generators of q.*

(3) *The degree and leading coefficient of $g(n)$ depend only on M and q, but not on the filtration.*

Proof. (1) Let $G(R) = \bigoplus q^n/q^{n+1}$ and

$G(M) = \bigoplus M_n/M_{n+1}$. Since m is the unique minimal prime of q, $\dim(R/q) = 0$ and thus R/q is Artinian. By [1, Proposition 10.22], each $G_n(M) = M_n/M_{n+1}$ is a Noetherian $R/q-$module. Notice that a f.g. $R-$module E with annihilator A satisfies a.c.c (resp., d.c.c) if and only if R/A satisfies the same condition. Thus, by [1, Proposition 6.10], $\ell(M_n/M_{n+1}) < \infty$. Hence $\ell(M/M_n) = \sum_{r=1}^{n} \ell(M_{r-1}/M_r) < \infty$.

(2) Let $q = (x_1, \ldots, x_s)$. Then $G(R) = (R/q)[\overline{x}_1, \ldots, \overline{x}_s]$, where $\overline{x}_i \in q/q^2$; thus of degree 1. By Corollary 2.19, $\ell(M_n/M_{n+1}) = \varphi(n)$ for large n, with $\deg \varphi \leq s - 1$. But $\ell(M/M_{n+1}) = \ell(M/M_{n_0}) + \sum_{i \geq n_0} \ell(M_i/M_{i+1})$. Each

$\ell(M_i/M_{i+1}),\ i \geq n_0$ is a polynomial and by (1) $\ell(M/M_{n_0})$ is a constant.

(3) Let $(\tilde{M_n})$ be another stable $q-$filtration of M and let $\tilde{g}(n) = \ell(M/\tilde{M_n})$. By [1, Lemma 10.6], $\exists\ n_0 \in \mathbb{N}$ such that $M_{n+n_0} \subseteq \tilde{M}_n$ and $\tilde{M}_{n+n_0} \subseteq M_n\ \forall\ n \geq 0$. Thus, $\tilde{g}(n) \leq g(n+n_0)$ and $g(n) \leq \tilde{g}(n+n_0)$. So $\frac{g(n)}{g(n+n_0)} \leq \frac{g(n)}{\tilde{g}(n)} \leq \frac{\tilde{g}(n+n_0)}{\tilde{g}(n)}$ and $\lim_{n\longrightarrow\infty} \frac{g(n)}{\tilde{g}(n)} = 1$. $\square$

More generally, let R be a Noetherian semilocal ring, and J the Jacobson radical of R. Let I be an ideal of R such that $J^\nu \subset I \subset J$ for some $\nu > 0$. We call I an *ideal of definition*. The $I-$adic and $J-$adic topologies then coincide. If we set $gr_I(M) = \bigoplus_{n\geq 0} I^nM/I^{n+1}M$, then $gr_I(M)$ is a graded module over $gr_I(R) = \bigoplus_{n\geq 0} I^n/I^{n+1}$. For brevity, write $gr_I(R) = R'$ and $gr_I(M) = M'$. Then the ring $R'_0 = R/I$ is Artinian. If $I = \sum_{i=1}^r x_iR$ and ξ_i is the image of x_i in I/I^2, then $R' = R'_0[\xi_1, \ldots, \xi_r]$. Indeed, if $Q \in \mathrm{Spec}(R/I)$, then $Q = P/I$ for some $P \in \mathrm{Spec}(R)$, with $I \subset P$. Then $J^\nu \subset P$, which implies that $P = m$, for some maximal m. Then R/I is Artinian since $\dim(R/I) = 0$. If also $M = \sum_{i=1}^s Rw_i$, then $M' = \sum R'\overline{w}_i$ (where $\overline{w}_i$ is the image of w_i in M/IM). Indeed, for any $k, l \in \mathbb{N}, (I^k/I^{k+1})(I^lM/I^{l+1}M) \subset I^{k+l}M/I^{k+l+1}M$ so that $\sum R'\overline{w}_i \subseteq M'$. Now, if $z \in M'$, then $z = \sum \overline{z}_n$, where $\overline{z}_n \in I^nM/I^{n+1}M$. Let z_n be a representative of $\overline{z}_n$, then $z = \sum_{r,j,k} \beta_{jk}\alpha_{nj}^{r_1}\alpha_{nj}^{r_2}\ldots\alpha_{nj}^{r_n}w_k$, where $\beta_{jk} \in R, \alpha_{nj}^{r_i} \in I$ and $\sum r_i = n$. The class of z will then be obtained by taking the product of the classes of each $\alpha_{nj}^{r_i} \mod I^2$ and $w_k \mod IM$.

Set $\chi_M^I(n) = \ell(M/I^{n+1}M)$. Then $\chi_M^m(n) = \chi_M(n)$ is called the *Samuel func-*

tion of the $R-$module M.

In what follows (R, m) is a Noetherian local ring. Let q be an $m-$primary ideal of R. Set $\delta(R)$ = least number of generators of q and $d(R)$ =degree of the polynomial $g(n)$ in Proposition 2.22(2), for $M = R$. This degree does not depend on the $m-$ primary ideal q.

Proposition 2.23 ([1, Proposition 11.7]) . *$\delta(R) \geq d(R)$.*

Proof. It follows from Proposition 2.22 (2) with $R = M$. □

Proposition 2.24 ([1, Proposition 11.8]) . *Let M be a finitely generated $R-$module and $x \in R$ a non-zero-divisor in M. Set $M' = M/xM$. Then $d(M') \leq d(M) - 1$.*

Proof. Set $N = xM$. Then $N \cong M$. By [1, Proposition 10.9], $(N_n = N \cap q^n M)$ is a $q-$filtration of N. Next, consider the induced exact sequence $0 \longrightarrow N/N_n \longrightarrow M/q^n M \longrightarrow M'/q^n M' \longrightarrow 0$. Since the length is additive, $\ell(N/N_n) - \chi^q_M(n) + \chi^q_{M'}(n) = 0$. By Proposition 2.22 (3), $\ell(N/N_n)$ and $\chi^q_M(n)$ have the same leading monomial. □

Corollary 2.25 ([1, Corollary 11.9]) . *Let x be a non-zero divisor in R. Then $d(R/(x)) \leq d(R) - 1$.*

Proposition 2.26 ([1, Proposition 11.10]) . *$d(R) \geq \dim(R)$.*

Proof. By induction on $d(R) = \deg \ell(R/m^n)$. Assume that $d(R) = 0$. Then $\ell(R/m^n)$ is constant for all large n. From $\ell(R/m^{n+1}) = \ell(R/m^n) + \ell(m^n/m^{n+1})$, we get $\ell(m^n/m^{n+1}) = 0$. Then $m^n = m^{n+1}$ for large n, so that by Nakayama's Lemma, $m^n = 0$ and R is an Artinian ring. Now, suppose that $d(R) > 0$. Then let $P_0 \subset P_1 \subset \ldots \subset P_r$ be any chain of prime ideals in R. Choose $x \in P_1 \setminus P_0$ and let $\overline{x} \neq 0 \in R/P_0 = R'$. By Corollary 2.25, $d(R'/(\overline{x})) \leq d(R')-1$. Let m' be the maximal ideal of the local ring R' and consider $R \overset{\phi}{\twoheadrightarrow} R'$ and $R/m^n \overset{\psi}{\twoheadrightarrow} R'/(m')^n$. From $\ell(\ker\psi)+\ell(R'/(m')^n) = \ell(R/m^n)$, we get $\ell(R'/(m')^n) \leq \ell(R/m^n)$, i.e., $d(R') \leq d(R)$. Thus, $d(R'/(\overline{x})) \leq d(R') - 1 \leq d(R) - 1$. By induction hypothesis $\dim(R'/(\overline{x})) \leq d(R'/(\overline{x}))$. Therefore $\dim(R'/(\overline{x})) \leq d(R) - 1$. Since there exists a one-to-one correspondence between primes of R' and those of R which contain x, we have $r - 1 \leq d(R) - 1$. Since r is arbitrarily chosen, it follows that $\dim(R) \leq d(R)$.

□

Proposition 2.27 ([1, Proposition 11.13]) . *If* $\dim(R) = d$, *then there exists an*

$m-$*primary ideal in* R *generated by* d *elements* $x_1, \ldots, x_d$ *and therefore* $\dim(R) \geq \delta(R)$.

Proof. We are going to construct inductively the elements x_i with the property that every prime ideal which contains $(x_1, \ldots, x_k)$ has height $\geq k$. Let x_1 be an element not contained in any minimal prime of R and suppose for $k > 0$ that

$x_1, \ldots, x_{k-1}$ are constructed. Let P_j $(1 \leq j \leq s)$ be the minimal prime ideals (if any) of $(x_1, \ldots, x_{k-1})$ which have height exactly $k-1$. Since R is local, then $m \neq P_j$ $(1 \leq j \leq s)$ so that $m \neq \bigcup_{j=1}^{s} P_j$. But $\bigcup P_j$ is an ideal so that we can find $x_k \in m \setminus \bigcup P_j$. Now, let Q be a prime ideal containing $(x_1, \ldots, x_k)$. then Q contains a minimal prime of $(x_1, \ldots, x_{k-1})$ say, q. If $q = P_j$ for some $j \leq s$ then we are done. If $q \neq P_j \quad \forall j \leq s$, then by induction hypothesis $\mathrm{ht}(q) > k-1$ and we are done. Now, if P is a minimal prime of $(x_1, \ldots, x_d)$, then $\mathrm{ht}(P) \geq d$. So $P = m$ and P is $m-$primary since m is maximal. $\square$

Theorem 2.28 . *Let (R, m) be a Noetherien local ring. Then* $\delta(R) = \dim(R) = d(R)$.

Proof. Proposition 2.22, 2.23, 2.26 and 2.27. $\square$

2.3 Inverse limit, completion, and regular local rings

A *directed set* is a partially ordered (say, by $\leq$) set Ω such that for any $\lambda, \mu \in \Omega$, there exists $\nu \in \Omega$ with $\lambda \leq \nu$ and $\mu \leq \nu$. Let Ω be a directed set and assume that for each element $\lambda \in \Omega$, we associate an $R-$module M_λ, and whenever $\lambda \leq \nu$, there is an $R-$linear map $f_{\nu\lambda} : M_\nu \longrightarrow M_\lambda$ such that $f_{\lambda\lambda} = 1$ and $f_{\lambda\nu} \circ f_{\nu\mu} = f_{\lambda\nu}$ for $\mu \leq \nu \leq \lambda$.

If these conditions hold, we say that $\mathfrak{F} = \{M_\lambda; f_{\lambda\mu}\}$ is an inverse system. Next, let φ be a map from a set M_∞ to $\mathfrak{F}$ i.e., a family of maps $\varphi_\lambda : M_\infty \longrightarrow M_\lambda$ satisfying $f_{\lambda\nu} \circ \varphi_\nu = \varphi_\lambda$ for $\nu \leq \lambda$. M_∞ will be called *inverse limit* or *projective* *limit* of $\mathfrak{F}$, and we write

$$M_\infty = \varprojlim M_\lambda,$$

if for any set X and any map $\psi = \{\psi_\lambda\} : X \longrightarrow \mathfrak{F}$, there exists a unique map $h : X \longrightarrow M_\infty$ such that $\psi_\lambda = \varphi_\lambda \circ h$ for all λ.

Now, assume that $\mathfrak{F}$ is an inverse system and each $M_\lambda \in \mathfrak{F}$ is a submodule of M such that $\lambda \leq \mu$ implies that $M_\lambda \supseteq M_\mu$. Then taking $\mathfrak{F}$ as a system of neighborhoods of 0 makes M into a topological group under addition. This topology is separated if and only if $\bigcap_\lambda M_\lambda = 0$ and $M/\bigcap_\lambda M_\lambda$ is called the separated module associated with M. If $\varphi_{\lambda\mu} : M/M_\mu \longrightarrow M/M_\lambda$ is the canonical map, then $\{M/M_\lambda; \varphi_{\lambda\mu}\}$ is an inverse system of $R-$modules. Its inverse limit is called the *completion* of M and it is written $\hat{M}$. If I is an ideal of R and $M_\lambda = I^\lambda M,\ \lambda \in \mathbb{N}$, then $\mathfrak{F}$ is called the $I-$adic topology on M. If in addition, we give R the $I-$adic topology, then $\hat{R}$ and $\hat{M}$ are the $I-$adic completions.

Theorem 2.29 . *Let R be a Noetherian ring and let $m = (a_1, \ldots, a_n)$ be an ideal of R. Let $\hat{R} = \hat{R}_m$ be the completion of R with respect to m.*

(1) *If M is a finitely generated $R-$module, then the natural map*

$$\hat{R} \otimes_R M \longrightarrow \varprojlim M/m^j M \ := \hat{M}$$

is an isomorphism. In particular, if S is a ring that is finite as an $R-$module, *then* $\hat{R} \otimes_R S$ *is the completion of* S *with respect to the powers of the ideal* mS.

(2) $\hat{R}$ *is flat as an* $R-$*module.*

(3) $\hat{R} \cong R[[X_1, \ldots, X_n]]/(X_1 - a_1, \ldots, X_n - a_n)$.

Proof. See [15, Theorem 7.2] and [31, Theorem 8.12]. □

Theorem 2.30 . *Let* R *be a Noetherian local ring of dimension* d*,* m *its maximal ideal and* $K = R/m$*. Then the following statements are equivalent:*

(i) $gr_m(R) \cong K[t_1, \ldots, t_d]$*, where the* t_i *are independent indeterminates;*

(ii) $\dim_K(m/m^2) = d$*;*

(iii) m *can be generated by* d *elements.*

Proof. See [1, Proposition 11.22]. □

A local ring satisfying the conditions of Theorem 2.30 is called a *regular local ring.*

2.4 Dimension of extension rings

Let $\varphi : A \longrightarrow B$ be a homomorphism of rings. For $P \in \mathrm{Spec}(A)$, write $K(P) = A_P/PA_P \cong A_P \otimes_A A/P$. Then the ring $B \otimes_A K(P)$ will be called the fibre ring over P and $\mathrm{Spec}(B \otimes_A K(P))$ the fibre of φ over P. For more details on the following propositions refer to [31, Section 15].

Proposition 2.31 *Let* $\varphi : A \longrightarrow B$ *be a homomorphism of Noetherian rings. Let* P *be a prime ideal of* B *and* $p = P \cap A$*. Then:*

(1) $\mathrm{ht}(P) \leq \mathrm{ht}(p) + \dim(B_P/pB_P)$.

(2) *If φ is flat, or more generally if the going-down theorem holds between A and B, then equality holds in* (1).

Proof. (1) From $\mathrm{ht}(p) = \dim(A_p)$, we can assume that $(A, \mathcal{M})$ and $(B, \mathcal{N})$ are local with $\mathcal{M}B \subset \mathcal{N}$ and have to show that $\dim(B) \leq \dim(A) + \dim(B/\mathcal{M}B)$. Let $x_1, \ldots, x_r$ be a system of parameters of A and $y_1, \ldots, y_s$ in B such that their images in $B/\mathcal{M}B$ form a system of parameters of $B/\mathcal{M}B$. Then $\sum x_i A$ and $\sum y_i B + \mathcal{M}B$ are respectively $\mathcal{M}-$primary and $\mathcal{N}/\mathcal{M}B-$primary. By [1, Proposition 7.14], there exist μ and ν such that $\mathcal{M}^{\nu} \subset \sum x_i A$ ($\mathcal{M}^{\nu}B \subset \sum x_i B$) and $\mathcal{N}^{\mu} \subset \sum y_i B + \mathcal{M}B$. Thus, for suitable μ and ν, $\mathcal{N}^{\mu\nu} \subset \sum x_i B + \sum y_i B$; which implies that $\mathcal{N} \subset \sqrt{\sum x_i B + \sum y_i B} \subset \mathcal{N}$; so that $\sum x_i B + \sum y_i B$ is $\mathcal{N}-$primary and $\dim(B) \leq r + s$.

(2) if B is a flat $A-$algebra, then the going-down theorem holds between A and B. Now, let $\dim(B/\mathcal{M}B) = s$ and let $\mathcal{N} = p_0 \supset p_1 \ldots \supset p_s \supset \mathcal{M}B$ be a chain of prime ideals. By [1, Theorem 1.17], $\mathcal{M} \subset \mathcal{M}^{ec}$ i.e., $\mathcal{M} \subset (\mathcal{M}B)^c$; so that $p_i \cap A = \mathcal{M}$ for $0 \leq i \leq s$ (since $\mathcal{M}$ is maximal). Let $\dim(A) = s$ and $\mathcal{M} = p_0 \supset p_1 \supset \ldots \supset p_r$ be a strictly decreasing chain of prime ideals of A. By the going-down theorem, we can construct a strictly decreasing chain of primes of B $P_s \supset \ldots \supset P_{s+r}$ such that $P_{s+i} \cap A = p_i$. Thus, $\dim(B) \geq r + s$. From (1) we get $\dim(B/\mathcal{M}B) + \dim(A) \leq \dim(B) \leq \dim(A) + \dim(B/\mathcal{M}B)$.
□

Proposition 2.32 . *Let* $\varphi : A \longrightarrow B$ *be a homomorphism of Noetherian rings and suppose that the going-up theorem holds between* A *and* B. *If* p *and* q *are prime ideals such that* $p \supset q$, *then* $\dim(B \otimes_A K(p)) \geq \dim(B \otimes_A K(q))$.

Proof. Set $r = \dim(B \otimes K(q))$ and $\mathrm{ht}(p/q) = s$. Next, let $q = p_0 \subset p_1 \subset \ldots p_s = p$ be a strictly increasing chain of primes of A. From $B \otimes K(q) = B \otimes A_q/qA_q \cong B/qB$, we can find a chain $qB \subset Q_0 \subset Q_1 \subset \ldots \subset Q_r$ of primes in B. As B is considered as a A_q-module, $qB \cap A_q = qA_q$; so that all the primes in the preceding chain lie over q. By GU, there exists a chain $Q_r \subset Q_{r+1} \subset \ldots \subset Q_{r+s}$ of prime ideals of B such that $Q_{r+i} \cap A = p_i$. Then $Q_{r+s} \cap A = p_s = p$ and $\mathrm{ht}(Q_{r+s}/qB) \geq r + s$. Applying Theorem 2.31 to $\varphi^{\#} : A/q \longrightarrow B/qB$, we get $r + s \leq \mathrm{ht}(Q_{r+s}/qB) \leq s + \dim(B_{Q_{r+s}}/pB_{Q_{r+s}})$. Thus, $r \leq \dim(B_{Q_{r+s}}/pB_{Q_{r+s}}) \leq \dim(B \otimes K(p))$. □

Proposition 2.33 . *Let* $\varphi : A \longrightarrow B$ *be a homomorphism of Noetherian rings and suppose that the going-down theorem holds between* A *and* B. *If* p *and* q *are prime ideals of* A *with* $p \supset q$, *then* $\dim(B \otimes_A K(p)) \leq \dim(B \otimes_A K(q))$.

Proof. We may assume that $\mathrm{ht}(p/q) = 1$. It is enough to prove that, given a chain $P_0 \subset P_1 \subset \ldots \subset P_r$ of primes of B lying over p such that $\mathrm{ht}(P_i/P_{i+1}) = 1$, we can construct a chain of prime ideals $Q_0 \subset Q_1 \subset \ldots \subset Q_r$ of B lying over q such that $Q_i \subset P_i$ $(0 \leq i \leq r)$ and $\mathrm{ht}(Q_i/Q_{i+1}) = 1$. By GD, we can find Q_0 since $P_0 \cap A = p$. If $r \geq 1$, then take $x \in p \setminus q$ and let $T_1, \ldots, T_s$ be the minimal prime divisors of $Q_0 + xB$. Since $Q_0 \in \mathrm{Spec}(B)$, we get $\mathrm{ht}(T_i/Q_0) = 1$

and $\mathrm{ht}(P_1/Q_0) \geq 2$. Hence we can choose $y \in P_1 \setminus (\cup T_i)$. Let Q_1 be a minimal prime divisor of $Q_0 + yB$ contained in P_1 (This is possible by [28, Theorem 10]). Then $\mathrm{ht}(Q_1/Q_0) = 1$, $Q_1 \neq T_i \; \forall \; i$ (since $y \notin T_i$) and $\varphi(x) \notin Q_1$. Indeed, if $\varphi(x) \in Q_1$ then $Q_0 + xB \subset Q_1$, i.e., Q_1 will be one of the T_i. Therefore $Q_1 \cap A \neq p$ (since $x \in p$ and $\varphi(x) \notin Q_1$). Since $\mathrm{ht}(p/q) = 1$ and $q \subseteq Q_1 \cap A \in \mathrm{Spec}(A)$ we must have $Q_1 \cap A = q$. To construct Q_2, let $T_1, \ldots, T_l$ be the minimal prime divisors of $Q_1 + xB$. Then $\mathrm{ht}(T_i/Q_1) = 1$ and $\mathrm{ht}(P_2/Q_1) \geq 2$. We choose $z \in P_2 \setminus (\bigcup T_i)$. Then $(z) + Q_1 \subset P_2$; so that P_2 contains a minimal prime Q_2 of $Q_1 + (z)$. Then $\mathrm{ht}\, Q_2/Q_1) = 1$, $Q_2 \neq T_i$ since $z \notin T_i$ and $\varphi(x) \notin Q_2$. But $q \subseteq Q_1 + (z) \cap A \subseteq Q_2 \cap A \subset P_2 \cap A = p$ and $\mathrm{ht}(p/q) = 1$ implies that $Q_2 \cap A = q$.

Likewise, we construct the other Q_i. □

Proposition 2.34 . *Let A be a Noetherian ring and $X_1, \ldots, X_n$ indeterminates over A. Then* $\dim(A[X_1, \ldots, X_n]) = \dim(A) + n$.

Proof. Since $A[X, Y] = A[X][Y]$ and using the Hilbert basis theorem, it is enough to consider the case $n = 1$.

If B is an $A-$module and $A[X]$ is the polynomial ring over A, then $B \otimes_A A[X] \cong B[X]$. Indeed, the map $A[X] \times B \longrightarrow B[X]$, $(aX^k, b) \mapsto abX^k$ is bilinear; so that there exists a unique $A-$linear map $A[X] \otimes B \xrightarrow{\Psi} B[X]$, $aX^k \otimes b \mapsto abX^k$. Next, $B[X] \xrightarrow{\Phi} A[X] \otimes B$, $bX^k \mapsto X^k \otimes b$ is $A-$linear and we have, $\Phi \circ \Psi = Id_{A[X] \otimes B}$ and $\Psi \circ \Phi = Id_{B[X]}$. Thus, $\forall \; P \in \mathrm{Spec}(A)$, $\dim(A[X] \otimes_A$

$K(P)) = \dim(K(P)[X]) = 1$ if we set $B = A[X]$. For a maximal ideal m of A, $A[X]/mA[X] \cong (A/m)[X] = K(m)[X]$. As $A[X]$ is faithfully flat over A, by Proposition 2.31 (2), $\dim(A[X])$

$= \dim(A) + 1$. $\square$

Proposition 2.35 . *Let A be a Noetherian integral domain and B an extension ring of A which is an integral domain. Let $P \in \operatorname{Spec}(B)$ and $p = P \cap A$. Then:*

$\operatorname{ht}(P) + \operatorname{t.d.}_{K(p)} K(P) \leq \operatorname{ht}(p) + \operatorname{t.d.}_A B$, *where* $\operatorname{t.d.}_A B$ *is the transcendental degree of the field of fractions of B over that of A.*

Proof. If $\operatorname{t.d.}_A B = \infty$, then there is nothing to prove.

If $\operatorname{t.d.}_A B + \operatorname{ht}(p) < \infty$, then we may assume B finitely generated over A. Indeed, let m and t be in $\mathbb{N}^\star$ such that $m \leq \operatorname{ht}(P)$ and $t \leq \operatorname{t.d.}_{K(p)} K(P)$. Then there is a chain of prime ideals $P_0 = P \supset P_1 \supset \ldots \supset P_m$ in B. For each $i < m$, choose $a_i \in P_i \setminus P_{i+1}$ and let $c_1, \ldots, c_t \in B$ such that their images modulo P are algebraically independent over (the field of fraction of) A/p. Set $C = A[\{a_i, c_i\}]$. Then $C \subset B$. If the theorem holds for C, then we have $m + t \leq \operatorname{ht}(p) + \operatorname{t.d.}_A C$. Letting m and t vary we obtain the desired result.

By induction, we may assume that B is generated over A by a single element. Replacing A by A_p and B by $B_p = A_p[x]$, we can assume that A is local with maximal ideal p. Set $k = A/p$ and $B = A[X]/Q$ (since every finitely generated algebra is isomorphic to a quotient of the polynomial ring). If $Q =$

(0), then $B = A[X]$; which is faithfully flat over A, hence by Proposition 2.31, $\mathrm{ht}(P) = \mathrm{ht}(p) + \mathrm{ht}(P/pB)$. $B/pB = A/p[X] = k[X]$; thus one-dimensional and therefore $P = pB$ or $\mathrm{ht}(P/pB) = 1$. If $P = pB$ then $\mathrm{ht}(P/pB) = 0$ and $\mathrm{qf}(B/P) = \mathrm{qf}(B/pB) = \mathrm{qf}(K[X]) = K(X)$. Thus, $\mathrm{t.\,d.}_{K(p)} K(P) = 1$. If $P \neq pB$, then $\mathrm{ht}(P/pB) = 1$ and $\mathrm{qf}(B/P) = B_P/pB_P$. Thus $\mathrm{t.\,d.}_{K(p)} K(P) = 0$. In either cases $\mathrm{ht}(P/pB) = 1 - \mathrm{t.\,d.}_{K(p)} K(P)$; so that equality holds in the theorem.

If $Q \neq (0)$, then x is algebraic over A, i.e., B is an algebraic extension and $\mathrm{t.\,d.}_A B = 0$. Since A is a subring of B, then $Q \cap A = (0)$. If K is the field of fractions of A, then $\mathrm{ht}(Q) = \mathrm{ht}(QK[X]) = 1$. Let P' be the reciprocal image of P in $A[X]$. Then $P'/Q = P$ and $K(P) = K(P')$

$$\begin{aligned}\mathrm{ht}(P) &\leq \mathrm{ht}(P') - \mathrm{ht}(Q) = \mathrm{ht}(P') - 1\\ &= \mathrm{ht}(p) + 1 - \mathrm{t.\,d.}_{K(p)} K(P) - 1\\ &= \mathrm{ht}(p) - \mathrm{t.\,d.}_{K(p)} K(P).\end{aligned}$$

□

2.5 The Rees algebra and the blow-up algebra

Let R be a ring, I an ideal of R and t an indeterminate over R. We obtain a graded ring $R_+ \subset R[t]$ by setting $R_+ = R_+(R, I) = \{ \sum c_n t^n \quad | c_n \in I^n\} = \bigoplus_n I^n t^n \subset R[t]$.

We notice first that if $I = (a_1, \ldots, a_r)$, then R_+ can be written $R_+ = R[a_1 t, \ldots, a_r t]$;

so that R_+ is Noetherian if R is Noetherian. Now, let $u = t^{-1}$ and consider $R[t, u]$ as a $\mathbb{Z}-$graded ring in the natural way. The *Rees algebra of R with respect to I* is the graded $R-$subalgebra of $R[t, u]$

$\mathcal{R} = \mathcal{R}(R, I) = R_+[u] = \left\{ \sum c_n t^n \mid \begin{matrix} n \in I^n \text{ for } c_n \geq 0 \\ c_n \in R \text{ for } n \leq 0 \end{matrix} \right\}.$

The *blow-up algebra of I in R* is the $R-$algebra

$B_I R = R \oplus I \oplus I^2 \oplus \ldots \cong R[tI]$ via the correspondence $a_i^k \longmapsto a_i^k t^k$ for all $a \in I^k$ and all $k \geq 1$.

Proposition 2.36 ([15, Exercise 13.8]) . *Let R be a Noetherian ring, I a proper ideal, $\mathcal{R} = \mathcal{R}(A, I)$ and $G = gr_I(R)$. Then:*

(1) $\dim(\mathcal{R}) = 1 + \dim(R)$.

(2) $\dim(B_I(R)) = \max\{\dim(R/P), \quad \dim(R/Q) + 1$, *where P ranges over the minimal primes containing I and Q over the minimal primes not containing $I\}$.*

(3) $\dim(G) = \max\{\dim(R_P) \mid P$ *is a maximal ideal of R containing $I\}$.*

Proof.* First notice that in a $\mathbb{Z}-$graded ring R, if M is a homogeneous maximal ideal of R, then

$M = \ldots \bigoplus R_{-2} \bigoplus R_{-1} \bigoplus m \bigoplus R_1 \bigoplus \ldots$, where m is a maximal ideal of R_0. In the case of $\mathbb{N}-$graded the equivalence holds. Indeed, R/M is graded and its component of degree n is $(R/M)_n = (R_n + M_n)/M_n \cong R_n/R_n \cap M = 0 \ \forall \ n \neq 0$ since in a field only the 0 degree component is $\neq \emptyset$. Thus,

$R_n = R_n \cap M \quad \forall\ n \neq 0$; so that $M = \sum R_n \cap M = \sum_{n \neq 0} R_n \bigoplus R_0 \cap M$. Further, $(R/M)_0 = R/M = (R_0 + M_0)/M_0 \cong R_0/R_0 \cap M$; so that $R_0 \cap M$ is a maximal ideal. In the case of $\mathbb{N}-$graded, $M = m \oplus \sum_{i>0} R_i$ is a homogeneous ideal. If it is not maximal, then it is contain in some homogeneous maximal $\mathcal{M}' = m' \bigoplus R_i$. Thus, $m' \supset m$, which is absurd.

(1) First, we show that the minimal primes of $\mathcal{R}$ are of the form $PR[t, t^{-1}] \cap \mathcal{R}$, where P is a minimal prime of R. Indeed, if we set $S = \{t^k \mid k \in \mathbb{N}\}$, then S is a multiplicative subset of $R[t]$ which does not meet P[t]; so that $P[t, t^{-1}] \cap \mathcal{R}$ is a prime of $\mathcal{R}$. If it were not minimal, then it contains a prime Q whose contraction in R will be equal to P. Then extending back to $\mathcal{R}$ gives a contradiction. Next, if $I' = IR[t, t^{-1}] \cap \mathcal{R}$, then $I' \cap R = I$. Therefore, if $I \neq J$ then $I' \neq J'$. Now, Suppose that $P \in \operatorname{Spec}(\mathcal{R})$ with $P \cap R = p$. Let p'_{0i} be a minimal prime of $\mathcal{R}$ contained in P such that $\operatorname{ht}(P/p'_{0i}) = \operatorname{ht}(P)$. Then $\mathcal{R}/p'_{0i} \supset R/p_{0i}$ and we can apply Proposition 2.35 to get $\operatorname{ht}(P) = \operatorname{ht}(P/p_{0i}) \leq \operatorname{ht}(p/p_{0i}) + 1 - \operatorname{t.d.}_{K(p)} K(P) \leq \operatorname{ht}(p) + 1$. Hence, $\dim(\mathcal{R}) \leq \dim(R) + 1$. Since $\mathcal{R}_S = R[t, t^{-1}]$, we get $\dim(\mathcal{R}) \geq \dim(R[t, t^{-1}]) = \dim(R) + 1$.

(2) The minimal primes of $B_I(R)$ are of the form $PR[t] \cap B_I(R)$, where P is a minimal prime of R. Indeed, the extension of a prime in the polynomial ring is prime and the contraction of any prime is also prime. If $PR[t] \cap B_I(R) \supsetneq Q$, then $Q^c \in \operatorname{Spec}(R)$ and $Q^c \subsetneq P$; which is impossible. Since R is Noetherian, if $I = (a_1, \ldots, a_r)$, then $B_I(R) = R[a_1 t, \ldots, a_r t]$; so that

$\dim(B_I(R)) = \max\{\dim(R[a_1t, \ldots, a_rt]/Q)$, where Q ranges over the minimal primes of $R[a_1t, \ldots, a_rt]\}$. Now, let Q be a minimal prime of $R[a_1t, \ldots, a_rt]$. If Q contains I, then $\overline{a}_i = \overline{0}$. Hence $R[a_1t, \ldots, a_rt]/Q \cong R/P$, where $Q \cap \mathcal{R} = P$. Assume that Q does not contain I. We have $R[a_1t, \ldots, a_rt]/Q \cong (R/P)[\overline{a_1t}, \ldots, \overline{a_rt}]$. But R/P

is Noetherian, then, by [18, Theorem 30.11],

$\dim(B_I(R)/Q) \leq \dim(R/P) + d$, where d is the transcendence degree of $B_I(R)/Q$ over the quotient field of R/P. Since $d = 1$ we get that $\dim(B_I(R)) \leq \dim(R/P) + 1$. Now, let P be a minimal prime ideal of R not containing I. Then there exists an integer $i \leq r$ such that $\overline{a_it} \neq 0$. Then $1 + \dim(R/P) \leq \dim(R/P[\overline{a_it}]) \leq \dim(B_I(R))$.

(3) By Corollary 2.15, $\dim(G) = \max\{\mathrm{ht}(\mathcal{M}) \mid \mathcal{M}$ is homogeneous maximal ideal of $G\}$. Then $\mathcal{M} = m \oplus \sum I^k/I^{k+1}$, where m is a maximal ideal of R/I, i.e., $m = M/I$, $M \in \mathrm{Max}(R)$. Now, let $(0) \subset P_1 \subset \ldots \subset I \subset P_r \subset \ldots \subset P_u$ be a saturated chain of prime ideals in R. Then $(0) \subset P_1 \bigoplus_{n>0} I^n/I^{n+1} \subset \ldots \subset P_r \bigoplus_{n>0} I^n/I^{n+1} \subset \ldots \subset P_u \bigoplus_{n>0} I^n/I^{n+1}$ is saturated chain of prime ideals in G. Therefore

$\mathrm{ht}_R(M) = \dim(R_M) \leq \dim(G)$. The reverse inequality is trivial so that equality holds. □

Corollary 2.37 ([31, Theorem 15.7]) . *Let R be a Noetherian ring, I a proper ideal and $G = gr_I(R)$. Then* $\dim(G) \leq \dim(R)$. *If in addition R is* *local, then* $\dim(G) = \dim(R)$.

Proof. A consequence of Proposition 2.36(3). □

3 Chapter 3

Jaffard Domains

In general, if $X_1, X_2, \ldots, X_r$ are indeterminates over a ring R and if $\dim(R) < \infty$, then

$r + \dim(R) \leq \dim(R[X_1, \ldots, X_r]) \leq r + (r+1)\dim(R)$ (see [41] and [42]) for each positive integer r. In the case where R is a Noetherian domain or a Prüfer domain, Krull and Seidenberg proved respectively that the equality $\dim(R[X_1, \ldots, X_r]) = \dim(R) + r$ holds. The aim of this section is to study the class of rings, besides these two classes, for which this equality holds. We define the valuative dimension of a ring and study its relation with the Krull dimension. For the valuative dimension, it is known that the equality $\dim_v(R[X_1, \ldots, X_r]) = \dim_v(R) + r$ holds for any ring R and any positive integer r (see [26]). We will show that the class of rings satisfying the equality $\dim(R[X_1, \ldots, X_r]) = \dim(R) + r$, for any positive integer r, are those with the property $\dim(R) = \dim_v(R)$. Examples will be given. Throughout this section, the use of dimension without any specification should be taken for Krull dimension and all rings are assumed to be integral domains.

Theorem 3.1 *Let R be a domain which is not a field, K the quotient field of R, L an algebraic extension field of K, and n a positive integer. The following conditions are equivalent:*

(i) *Each* $(L-)$*valuation overring of* R *has dimension at most* n*, and there exists an* $(L-)$

valuation overring of R *having dimension* n*;*

(ii) *Each* $(L-)$*overring of* R *has dimension at most* n*, and there exists an* $(L-)$*overring of* R *having dimension* n*;*

(iii) $\dim(R[X_1, \ldots, X_n]) = 2n$*;*

(iv) $\dim(R[X_1, \ldots, X_r]) = r + n$ *for all* $r \geq n - 1$*.*

Proof. See [18, Theorem 30.9]. □

If the above conditions hold, R is said to have valuative dimension n (in short $\dim_v(R) = n$). If there exists no positive integer n satisfying $(i) - (iv)$, R is said to have infinite valuative dimension (in short $\dim_v(R) = \infty$). For the sake of completeness, each field is assigned valuative dimension 0.

For any ring R, $\dim(R) \leq \dim_v(R)$. If V is a valuation ring, then $\dim(V) = \dim_v(V)$ (since any overring of V is a localization of V). For any ring R, it is known that $\dim_v(R[X_1, \ldots, X_r]) = \dim_v(R) + r$ (see [26, Theorem 2]).

Definition 3.2 ([2, Definition 0.2]) . A domain R is said to be a Jaffard domain if $\dim(R) = \dim_v(R) < \infty$; equivalently, if $\dim(R) < \infty$ and $\dim(R[X_1, \ldots, X_r]) = \dim(R) + r$ for all $r \geq 1$.

Example 3.3 .Finite-dimensional Noetherian domains and Prüfer domains are Jaffard.

3.1 Transfer results

Proposition 3.4 ([2, Proposition 1.1]) . *Let $R \subset S$ be an integral extension of domains. Then S is a Jaffard domain if and only if R is a Jaffard domain.*

Proof. The integrality condition implies that $\dim(R) = \dim(S)$ and $\dim_v(R) = \dim_v(S)$; so that the result follows. □

Proposition 3.5 ([2, Proposition 1.2]) . (1) *If R is a Jaffard domain, then $R[X_1, \ldots, X_r]$ is also a Jaffard domain for each positive integer r.*
(2) *Let R be a domain with $\dim_v(R) = n < \infty$. Then $R[X_1, \ldots, X_r]$ is a Jaffard domain for each positive integer $r \geq n - 1$.*

Proof. (1) R is Jaffard implies that $\dim(R) < \infty$; so that a.c.c on prime ideals holds in R. By [18, Corollary 30.3], a.c.c on prime ideals holds in $R[X]$. Let $\dim(R) = n$. We have $n + r \leq \dim(R[X_1, \ldots, X_r]) \leq \dim_v(R[X_1, \ldots, X_r]) = r + \dim_v(R) = r + n$. Thus, $\dim(R[X_1, \ldots, X_r]) = \dim_v(R[X_1, \ldots, X_r]) = n + r$.
(2) $\dim_v(R[X_1, \ldots, X_r]) = \dim_v(R) + r = r + n$
$= \dim(R[X_1, \ldots, X_r])$, where the last equality is obtained by Theorem 3.1(iv).
□

The Jaffard property is not preserved under localization (see Example 3.33). In order to study those rings for which the Jaffard property is locally preserved, we introduce the following definition.

Definition 3.6 ([2, Definition 1.4]) . A domain R is said to be locally Jaffard if R_P is a Jaffard domain for each $P \in \mathrm{Spec}(R)$.

Proposition 3.7 ([2, Proposition 1.5]) . *Let* R *be a domain with* $\dim_v(R) < \infty$. *Then:*

(1) *R is locally Jaffard if and only if $S^{-1}R$ is a Jaffard domain for each multiplicative closed subset S of R.*

(2) *If R is locally Jaffard, then R is a Jaffard domain.*

Proof.* (1) We only need to prove the "only if" assertion. Let S be a multiplicative closed subset of R. $\dim_v(S^{-1}R) = \sup\{\dim_v((S^{-1}R)_{S^{-1}P}) \mid P \in \mathrm{Spec}(R)$ and $P \cap S = \emptyset\} = \sup\{\dim_v(R_P) \mid P \cap S = \emptyset\} = \sup\{\dim(R_P) \mid P \cap S = \emptyset\} = \dim(S^{-1}R) \leq \dim_v(S^{-1}R)$.

(2)

$$\begin{aligned}
\dim_v(R) &= \sup\{\dim_v(R_P) | P \in \mathrm{Spec}(R)\} \\
&= \sup\{\dim(R_P) \mid P \in \mathrm{Spec}(R)\} \\
&= \dim(R)
\end{aligned}$$

$\square$

Definition 3.8 ([2]) . A domain R is said to be equicodimensional if all its maximal ideals have the same height.

Definition 3.9 ([31]) . A ring R is catenarian if for any prime ideals $P \subset P'$, all saturated chains of prime ideals between P and P' have the same length.

For valuative dimension, we always have $\dim_v(R_P) + \dim_v(R/P) \leq \dim_v(R)$ for any prime P (see [26, Proposition 2]).

Proposition 3.10 ([2, Proposition 1.8]) . *Let R be a finite-dimensional equicodimensional catenarian domain. Then R is locally Jaffard if and only if R is a Jaffard domain.*

Proof. Assume that R is Jaffard and let $P \in \mathrm{Spec}(R)$. Since all chains between any minimal prime ideal and any maximal ideal have the same length, we have $\dim(R) = \dim(R_P) + \dim(R/P) \leq \dim_v(R_P) + \dim_v(R/P) \leq \dim_v(R) = \dim(R)$. Thus, $\dim(R_P) = \dim_v(R_P)$. The converse follows from Proposition 3.7. □

In what follows the rings are not necessarily domains. Let T be a cancellative monoid and set $G = T \times T / \sim$, where $(a, b) \sim (a', b')$ if and only if $a + b' = a' + b$. Let $[a, b]$ denote the class of (a, b). Then G is an abelian group with zero element $[s, s], s \in T$, under the operation $[a, b] + [a', b'] = [a + a', b + b']$. G is called the quotient group of T. If R is a ring, then $R[T]$ is a monoid ring and the group ring $R[G]$ is its quotient. Thus, $\dim(R[G]) \leq \dim(R[T])$. Let S be a submonoid of T. An element $t \in T$ is said to integral over S if there exists $n \in \mathbb{N}$ such that $nt \in S$. If $f \in R[S]$, then $f = \sum f_i X^{s_i}$ and we define the support of f to be the set $\mathrm{Supp}(f) = \{s_i\}$.

Proposition 3.11 ([20, Theorem 12.4]) . *Let T be a cancellative monoid, S a submonoid of T and R a ring. Then, T is integral over S if and only if $R[T]$ is integral over $R[S]$.*

Proof. Let us assume that T is integral over S and suppose $\sum f_i X^{t_i} \in R[T]$. Then for each t_i there exists $n_i \in \mathbb{N}$ such that $n_i t_i \in S$. Therefore $(f_i X^{t_i})^{n_i} \in R[S]$ and $f_i X^{t_i}$ is integral over $R[S]$. Since the set of integral elements is a ring, it follows that $\sum f_i X^{t_i}$ is integral over $R[S]$. Conversely, let show that t is integral over S if X^t is integral over $R[S]$. Let $X^{nt} + \alpha_{n-1} X^{(n-1)t} + \ldots + \alpha_0 = 0$, with $\alpha_i \in R[S]$. Then not all the α_i are zero and therefore, for some $i < n$, $nt \in \mathrm{Supp}(\alpha_i X^{it})$. Consequently, $nt = s + it$ for some $s \in S$ and by the cancellation low on T, $(n-i)t = s \in S$. □

Proposition 3.12 ([19, Proposition 1]) . *Let R be a commutative ring with identity, $\{X_\lambda\}$ be a set of indeterminates over R, and $A = R[\{X_\lambda, X_\lambda^{-1}\}, \lambda \in \Lambda]$. Then $\dim(A) = \dim(R[\{X_\lambda\}])$.*

Proof. If R is infinite-dimensional, then it is clear that both A and $R[\{X_\lambda\}]$ are infinite-dimensional. Indeed, $\forall\ P \in \mathrm{Spec}(R)$, $P[X] \in \mathrm{Spec}(R[X])$ and $P[X] \cap S = \emptyset$, where $S = \{X^k \mid k \in \mathbb{N}\} \subset R[X]$. Thus a chain of prime ideals in R extends to a chain of prime ideals in A. Next, A is a localization of $R[\{X_\lambda\}]$ so that $\dim(A) \leq \dim(R[\{X_\lambda\}])$. Now, if R is finite-dimensional and Λ is infinite, then $R[\{X_\lambda\}]$ and A are infinite-dimensional. For, if $\mathcal{M}$ is a maximal ideal of R and $\{X_{\lambda_i}\}_{i=1}^{\infty}$ is an infinite subset of $\{X_\lambda\}$, then $(\mathcal{M}, X_{\lambda_1} - 1) \subset (\mathcal{M}, X_{\lambda_1} - 1, X_{\lambda_2} - 1) \subset \ldots$ is an infinite saturated chain of prime ideals in $R[\{X_\lambda\}]$ with respect to the multiplicative system generated by $\{X_\lambda\}$. If R is finite-dimensional and $\Lambda = \{1, 2, \ldots, k\}$ is finite, then $\dim(A) \leq \dim(R[\{X_\lambda\}])$ (since A is a localization of $R[\{X_\lambda\}]$). On the other hand, it is shown in [5], that $\dim(R[\{X_\lambda\}]) = \mathrm{ht}(\mathcal{M}[\{X_\lambda\}]) + k$, for some maximal ideal $\mathcal{M}$ of R. Hence, if $P_0 \subset P_1 \subset \ldots \subset P_t = \mathcal{M}[\{X_\lambda\}]$ is a chain of prime ideals of $R[\{X_\lambda\}]$ of length $t = \mathrm{ht}(\mathcal{M}[\{X_\lambda\}])$, then $P_0 \subset P_1 \subset \ldots \subset P_t \subset P_t + (X_1 - 1) \subset \ldots \subset P_t + (X_1 - 1, \ldots, X_k - 1)$ is a chain of primes in $R[\{X_\lambda\}]$, and each of these prime ideals extends to a proper ideal of A since none of them meets S. Consequently, $\dim(A) \geq \dim(R[\{X_\lambda\}])$ and the equality holds. To show that $P_t + (X_{\lambda_1} - 1, \ldots, X_{\lambda_i} - 1)$ is a prime ideal, first we observe that none of the variable X_{λ_i} belongs to P_t and $R[\{X_\lambda\}]/P_t$ is an integral domain. The ring-homomorphism $R[\{X_\lambda\}]/P_t \xrightarrow{\varphi} R/\mathcal{M}; \overline{X}_{\lambda_j} \mapsto 1$ is surjective with $\ker(\varphi) = (\overline{(X_{\lambda_1} - 1)}, \ldots, \overline{(X_{\lambda_i} - 1)})$. The result follows from $(R[\{X_\lambda\}]/P_t)/\ker(\varphi) \cong R[\{X_\lambda\}]/P_t + \ker(\varphi)$ □

Corollary 3.13 ([19, Corollary 1]) . *Let R be a commutative ring with identity and G a torsion-free abelian group with rank α. Then* $\dim(R[G]) = \dim(R[\{X_\lambda\}_{\lambda\in\Lambda}])$, *where* $|\Lambda| = \alpha$.

Proof. Let $\{Y_i\}_{i\in\Lambda}$ be a minimal free subset of G and let $H = \sum_{i\in\Lambda} \mathbb{Z}Y_i$

be the subgroup of G generated by $\{Y_i\}$. Then $G \cong H \oplus G_{tor}$ as $\mathbb{Z}-$module and G is integral over H. Thus by Proposition 3.11, $R[G]$ is integral over $R[H]$ and $\dim(R[G]) = \dim(R[H]) = \dim(R[\{X_\lambda, X_\lambda^{-1}\}_{\lambda\in\Lambda}])$. Indeed, if $H = \mathbb{Z} < Y_i >_{i\in\Lambda}$, then $R[H] \cong R[\{X_\lambda, X_\lambda^{-1}\}_{\lambda\in\Lambda}]$ since $z \in R[H]$ implies that $z = \sum f_i X^{\sum \lambda_i Y_i} = \sum f_i (X^{Y_i})^{\lambda_i}$. Set then $X_\lambda = X^{Y_\lambda}$. If $\alpha = \infty$, then $\dim(R[H]) = \dim(R[X_\lambda]) = \infty$ since $(X_1) \subset (X_1, X_2) \subset (X_1, X_2, X_3) \subset \ldots$ is an infinite chain of prime ideals in $R[H]$. □

Definition 3.14 ([36]) .Let Γ be a grading monoid, i.e., a commutative and associative monoid, with neutral element, which satisfies the cancellative low. The law of composition is written as addition and the neutral element is denoted by 0. The grading monoid Γ is said to be torsionless if from $n\gamma = n\gamma'$, where n is an integer and γ, γ' belong to Γ, follows $\gamma = \gamma'$.

Proposition 3.15 ([20, Theorem 8.1]) . *Let R be a non-trivial ring and S be a semigroup. Then, the semigroup ring $R[S]$ is an integral domain if and only if R is an integral domain and S is torsion-free and cancellative.*

Proposition 3.16 ([20, Theorem 21.4]) . *Let R be a unitary ring, S a cancellative monoid, and G the quotient group of S. Then* $\dim(R[G]) = \dim(R[S])$.

Proposition 3.17 ([2, Proposition 1.14]) . *Let R be a domain and let S be a torsionless grading monoid with quotient group G. If rank$(G) = r$, then*

$\dim(R[S]) = \dim(R[G]) = \dim(R[X_1, \ldots, X_r]) =$

$\dim(R[X_1, X_1^{-1}, \ldots, X_r, X_r^{-1}])$.

Proposition 3.18 ([2, Lemma 1.15]) . *If R is a domain, then* $\dim_v(R[X_1, \ldots, X_r]) = \dim_v(R[X_1, \ldots, X_r, X_1^{-1}, \ldots, X_r^{-1}])$ *for each positive integer r.*

Proof. $R[X_1, \ldots, X_r, X_1^{-1}, \ldots, X_r^{-1}]$ is an overring of $R[X_1, \ldots, X_r]$, so $\dim_v(R[X_1, \ldots, X_r, X_1^{-1}, \ldots, X_r^{-1}]) \leq \dim_v(R[X_1, \ldots, X_r])$. For the reverse inequality, we have

$$\begin{aligned}
\dim_v(R[X_1, \ldots, X_r, X_1^{-1}, \ldots, X_r^{-1}]) &\geq \sup\{\dim(A[X_1, \ldots, X_r, X_1^{-1}, \ldots, X_r^{-1}]) \mid A \textit{ is an overring of } R\} \\
&= \sup\{\dim(A[X_1, \ldots, X_r]) \mid A \textit{ is an overring of } R\} \\
&\geq \sup\{\dim(A) + r \mid A \textit{ overring of } R\} \\
&= \sup\{\dim(A) \mid A \textit{ overring of } R\} + r \\
&= \dim_v(R) + r \\
&= \dim_v(R[X_1, \ldots, X_r]),
\end{aligned}$$

as desired. □

Proposition 3.19 ([2, Proposition 1.17]) .

Let R be a domain with $\dim_v(R) < \infty$ and S a torsionless grading monoid *with quotient group* G *such that rank*$(G) = r < \infty$. *Then* $\dim_v(R[S]) = \dim_v(R[G]) = \dim_v(R[X_1, \ldots, X_r]) = \dim_v(R) + r$.

Proof. Let F be a finitely generated free abelian subgroup of G with rank r. Then $G = F + G_{tor}$ and G/F is a torsion group. $\forall g \in G$, $\exists\ n \in \mathbb{N}$, $ng \in F$; i.e., G is integral over F as monoid and therefore by Proposition 3.11, $R[G]$ is integral over $R[F]$. $R[F] = R[Y_1, Y_1^{-1}, \ldots, Y_r, Y_r^{-1}]$ for some family of indeterminates $\{Y_1, Y_2, \ldots, Y_r\}$. Indeed, if $F = \mathbb{Z} < a_1, \ldots, a_r >$, then $\forall\ y \in R[F]$, $y = \sum X^{\sum \lambda_i a_i} = \sum (X^{a_1})^{\lambda_1} \ldots (X^{a_r})^{\lambda_r}$. Set $Y_i = X^{a_i}$. Let $D = R[Y_1, Y_2, \ldots, Y_r]$. Then $\dim(R[S]) = \dim(R[G]) = \dim(R[F]) = \dim(D)$. Moreover, by integrality

$\dim_v(R[G]) = \dim_v(R[F])$. By Proposition 3.18,

$\dim_v(R[F]) = \dim_v(D) = r + \dim_v(R)$. Now, $R[G]$ is an overring of $R[S]$, so $\dim_v(R[G]) \leq \dim_v(R[S])$. Let V be a valuation overring of $R[S]$. Then for each $i \leq r$, $Y_i \in V$ or $Y_i^{-1} \in V$; hence, by replacing Y_i by Y_i^{-1} if necessary, we may assume that $D \subset V$ and then V is a valuation overring of D so that $\dim(V) \leq \dim_v(D)$. Hence $\dim_v(R[S]) \leq \dim_v(D) = \dim_v(R[G])$, concluding the proof. □

Theorem 3.20 ([2, Corollary 1.18]) . *Let* R *be a domain and let* S *be a torsionless grading monoid with quotient group* G *such that rank*$(G) = r < \infty$. *Then the following statements are equivalent:*

(i) $R[S]$ *is a Jaffard domain;*

(ii) $R[G]$ *is a Jaffard domain;*

(iii) $R[X_1, \dots, X_r]$ *is a Jaffard domain;*

(iv) $R[X_1, X_1^{-1}, \dots, X_r, X_r^{-1}]$ *is a Jaffard domain.*

Proof. It is a consequence of Propositions 3.17 and 3.19. □

Proposition 3.21 ([2, Corollary 1.19]) . *Let R be a domain and let S be a torsionless grading monoid with quotient group G.*

(1) *If R is a Jaffard domain, then $R[S]$ is a Jaffard domain if and only if rank$(G) < \infty$.*

(2) *If* $\dim_v(R) < \infty$*, then $R[S]$ is a Jaffard domain if* $\dim_v(R) - 1 \leq$*rank*$(G) < \infty$.

Proof. (1) Let $\text{rank}(G) < \infty$. By Proposition 3.5, $R[X_1, \dots, X_r]$ is a Jaffard domain for any positive integer r and by Theorem 3.20, $R[S]$ is a Jaffard domain. Conversely, if $R[S]$ is a Jaffard domain, then by Theorem 3.20, $R[G]$ is a Jaffard domain, i.e., $\dim_v(R[G]) < \infty$ and so $\text{rank}(G) < \infty$.

(2) If $\text{rank}(G)$ is finite and $\dim_v(R) - 1 \leq \text{rank}(G) < \infty$, then by Proposition 3.5(2), $R[X_1, \dots, X_r]$ is a Jaffard domain and $R[S]$ is therefore a Jaffard domain by Theorem 3.20. □

3.2 Pullbacks

In this paragraph, we determine necessary and sufficient conditions for certain "pullback-type" constructions to Jaffard domains.

Lemma 3.22 ([2, Lemma 2.1]) . *Let consider the pullback determined by the following diagram of commutative rings:*

$$\begin{array}{ccc} R & \longrightarrow & D \\ \downarrow & & \downarrow \\ T & \overset{\varphi}{\twoheadrightarrow} & K \end{array}$$

where T is a domain, φ is a homomorphism from T onto a field K with $\ker(\varphi) = \mathcal{M}$, *$D$ is a proper subring of K, and $R = \varphi^{-1}(D)$. Then:*

(1) *$\mathcal{M} = (R : T)$ and $R/\mathcal{M} \cong D$.*

(2) *If T is local, then $\mathcal{M}$ is a divided prime ideal of R (i.e., $\mathcal{M}R_{\mathcal{M}} = \mathcal{M}$) and so each prime ideal of R is comparable to $\mathcal{M}$. If, in addition, K is the quotient field of D, then $R_{\mathcal{M}} = T$.*

(3) *If T is local, then* $\dim(R) = \dim(T) + \dim(D)$.

(4) *For each $P \in \operatorname{Spec}(R)$ with $\mathcal{M} \nsubseteq P$, there is a unique $Q \in \operatorname{Spec}(T)$ such that $Q \cap R = P$ and Q satisfies $T_Q = R_P$.*

(5) *If $P \in \operatorname{Spec}(R)$ and $P \supset \mathcal{M}$, then there is a unique $Q \in \operatorname{Spec}(D)$ such that $P = \varphi^{-1}(Q)$. Moreover, the diagram*

$$\begin{array}{ccc} R_P & \longrightarrow & D_Q \\ \downarrow & & \downarrow \\ T_{\mathcal{M}} & \twoheadrightarrow & K \end{array}$$

is a pullback diagram.

(6) *T is integral over R if and only if D is a field and K is algebraic over D.*

Lemma 3.23 ([2, Lemma 2.2]) . *If R is a domain and $\mathcal{M}$ a divided prime ideal of R, then*

$\mathrm{ht}(Q[X_1,\ldots,X_r]) = \mathrm{ht}(Q[X_1,\ldots,X_r]/\mathcal{M}[X_1,\ldots,X_r]) + \mathrm{ht}(\mathcal{M}[X_1,\ldots,X_r])$,

for each positive integer r and each prime ideal Q of R such that $Q \supset \mathcal{M}$.

Proposition 3.24 ([2, Proposition 2.3]) . *Let $(T,\mathcal{M},K)$ be a local domain and $\varphi : T \twoheadrightarrow K$ the canonical surjection. Let $R = \varphi^{-1}(D)$, where D is a proper subring of K with quotient field K. Then:*

(1) $\dim(R[X_1,\ldots,X_r]) = \dim(D[X_1,\ldots,X_r]) +$

$\dim(T[X_1,\ldots,X_r]) - \dim(K[X_1,\ldots,X_r])$, $\forall r \in \mathbb{N}$.

(2) $\dim_v(R) = \dim_v(D) + \dim_v(T)$.

(3) *R is a Jaffard domain if and only if D and T are Jaffard domains.*

Proof. (1) Set $X = \{X_1,\ldots,X_r\}$. Since $\dim(R) = \dim(D) + \dim(T)$; then $\dim(R) < \infty$ if and only if $\dim(D) < \infty$ and $\dim(T) < \infty$. Assume that each domain is finite-dimensional. By [5, Corollary 2.9], there exists a maximal ideal Q of $R[X]$ such that $\dim(R[X]) = \mathrm{ht}(Q) = \mathrm{ht}(q[X]) + r$, where $q = Q \cap R \in \mathrm{Max}(R)$. By Lemma 3.22, $\mathcal{M}$ is comparable to every ideal of R. If $q \subset \mathcal{M}$ then $q[X] \subsetneqq \mathcal{M}[X]$ and $\mathrm{ht}(q[X]) + r < \mathrm{ht}(\mathcal{M}[X]) + r \leq \dim(R[X])$; which is impossible. Thus, $\mathcal{M} \subseteq q$. Since q and Q are maximal, then $\overline{q} = q/\mathcal{M}$ and $(Q/\mathcal{M}[X])$ are maximal in D and $D[X]$, respectively. Since $\mathcal{M}$ is a divided prime ideal of R, by Lemma 3.23 $\mathrm{ht}(q[X]) = \mathrm{ht}(q[X]/\mathcal{M}[X]) + \mathrm{ht}(\mathcal{M}[X])$.

Hence,

$\dim(R[X]) = \text{ht}(Q) = \text{ht}(q[X]/\mathcal{M}[X]) + \text{ht}(\mathcal{M}[X]) + r$. Since $K = \text{qf}(D)$, we get from Lemma 3.22 that $T = R_{\mathcal{M}}$. Thus, $\text{ht}_{R[X]}(\mathcal{M}[X]) = \text{ht}_{R_{\mathcal{M}}[X]}(\mathcal{M}R_{\mathcal{M}}[X]) = \text{ht}_{T[X]}(\mathcal{M}T[X])$. By [5, Corollary 2.10], $\text{ht}(\mathcal{M}[X]) + r = \dim(T[X])$. $\dim(D[X]) = \text{ht}(Q/\mathcal{M}[X]) = \text{ht}(q[X]/\mathcal{M}[X]) + r$. Since K is Noetherian, $\dim(K[X_1, \ldots, X_r]) = \dim(K) + r = r$. Consequently $\dim(R[X]) = \dim(D[X]) + \dim(T[X]) - r$.

(2) Assume that $\dim_v(R) < \infty$. Then $\dim(T) + \dim(D) = \dim(R) < \infty$. Since $R \subset T$ with $\text{qf}(R) = \text{qf}(T)$, $\dim_v(T) < \dim_v(R) < \infty$. If B is an overring of D, then $\varphi^{-1}(B) = A$ is an overring of R and Lemma 3.22 yields $\dim(A) = \dim(B) + \dim(T)$ since $B \subsetneq K$ and T is local. So $\dim(B) \leq \dim(A)$, hence $\dim_v(D) \leq \dim_v(R) < \infty$. Thus, $\dim_v(D)$ and $\dim_v(T)$ are both finite. Let $r \in \mathbb{N}$ such that $r + 1 \geq \max\{\dim_v(R), \dim_v(D), \dim_v(T)\}$. By Theorem 3.1(iv), $\dim(R[X]) = \dim_v(R) + r$, $\dim(D[X]) = \dim_v(D) + r$ and $\dim(T[X]) = \dim_v(T) + r$. Substituting in (1) we get $\dim_v(R) + r = \dim_v(D) + r + \dim_v(T) + r - r = \dim_v(D) + \dim_v(T) + r$. Now, assume that both $\dim_v(D)$ and $\dim_v(T)$ are finite and let $r+1 \geq \max\{\dim_v(D), \dim_v(T)\}$. By (1) and Theorem 3.1(iv), $\dim(R[X]) = \dim(D[X]) + \dim(T[X]) - r = \dim_v(D) + r + \dim_v(T) + r - r = \dim_v(D) + \dim_v(T) + r$. Identifying with Theorem 3.1(iv) , we get that $\dim_v(R) = \dim_v(D) + \dim_v(T)$ is finite, completing the proof.

(3) If D and T are Jaffard, then $\dim_v(R) = \dim_v(D) + \dim_v(T) = \dim(D) + \dim(T) \leq \dim(R)$. Conversely, assume that R is Jaffard. We have $\dim(R) = \dim_v(R) = \dim_v(D) + \dim_v(T) = \dim(T) + \dim(D)$. So $(\dim(T) - \dim_v(T)) + (\dim(D) - \dim_v(D)) = 0$ and each parentheses must be equal to zero, as desired. □

Proposition 3.25 ([2, Proposition 2.5]) .

Let $(T, \mathcal{M}, K)$ be a local domain which is not a field and let $\varphi : T \twoheadrightarrow K$ be the canonical surjection. Let $R = \varphi^{-1}(F)$, where F is a subfield of K. Then:

(1) $\dim_v(R) = \dim_v(T) + \mathrm{t.d.}(K/F)$.

(2) *R is a Jaffard domain if and only if T is a Jaffard domain and K is algebraic over F.*

Proof. If $\mathrm{t.d.}(K/F) = \infty$, the induction below proves that $\dim_v(R) = \infty$. If $\dim_v(T) = \infty$ and $\mathrm{t.d.}(K/F) < \infty$, then $\dim_v(R) = \infty$. Indeed, if $\dim_v(R) < \infty$, then each $K-$overring of R has dimension at most $\dim_v(R)$, i.e, $\dim_v(T) < \infty$; which is absurd. Thus, we may assume that $\dim_v(T)$ and $\mathrm{t.d.}(K/F)$ are both finite. We may also assume that K is a purely transcendental extension of F. Indeed, let $\{Y_1, \ldots, Y_d\}$ be a transcendental basis of K over F and put $L = F(Y_1, \ldots, Y_d)$. Since K is algebraic over L. By Lemma 3.22, T is integral over $A = \varphi^{-1}(L)$, by $\dim_v(A) = \dim_v(T)$. Also $\mathrm{t.d.}(K/F) = \mathrm{t.d.}(L/F)$. Therefore, we can replace T by A and K by L.

(1) Let argue by induction on $d = \mathrm{t.d.}(K/F)$. If $d = 0$, then because of our assumption of purely transcendence, we have $K = F$ and $R = \varphi^{-1}(F) = T$. If $d = 1$, then write $K = F(Y)$ for some $Y = \varphi(y)$, $y \in T \setminus \mathcal{M}$ ($y \notin \mathcal{M}$ since Y is invertible). We claim that $R[y] = \varphi^{-1}(F[Y])$. Indeed, since $\varphi(y) = Y \in F[Y]$ and $\varphi(R) = F$ we get that $R[y] \subset \varphi^{-1}(F[Y])$. Let $x \in \varphi^{-1}(F[Y])$, then $\varphi(x) \in F[Y]$, i.e., $\varphi(x) = \sum \alpha_k Y^k = \sum \alpha_k \varphi(y)^k = \sum \alpha_k \varphi(y^k) = \varphi(z)$, where $z = \sum \beta_k y^k \in R[y]$ and $\varphi(\beta_k) = \alpha_k$. So $x - z \in \ker(\varphi) = \mathcal{M} \subset R$. Thus, $x \in z + R \subset R[y]$ and hence $\varphi^{-1}(F[Y]) \subset R[y]$. Similarly, $\varphi^{-1}(F[Y^{-1}]) = R[y^{-1}]$. By Proposition 3.24(2), these equalities lead to $\dim_v(R[y]) = \dim_v(F[Y]) + \dim_v(T)$ and $\dim_v(R[y^{-1}]) = \dim_v(F[Y^{-1}]) + \dim_v(T)$. Since F is a field, $\dim_v(F[Y]) = \dim_v(F[Y^{-1}]) = 1$, $\dim_v(R[y]) = \dim_v(R[y^{-1}]) = 1 + \dim_v(T)$. Let B be any valuation overring of R. Since $y \in T \subseteq \mathrm{qf}(R)$, this implies that $y \in B$ or $y^{-1} \in B$, i.e., $R[y] \subset B$ or $R[y^{-1}] \subset B$. Thus B is an overring of $R[y]$ or of $R[y^{-1}]$ and $\dim_v(R) = \max\{\dim_v(R[y]), \dim_v(R[y^{-1}])\} = \dim_v(T) + 1$. Now, suppose that the equality asserted in (1) holds whenever the transcendental degree is less than d. Let $K = F(Y_1, \ldots, Y_d)$; $\mathbb{K} = F(Y_1, \ldots, Y_{d-1})$ and $B = \varphi^{-1}(\mathbb{K})$. By induction hypothesis $\dim_v(R) = \dim_v(B) + \mathrm{t.d.}(\mathbb{K}/F) = \dim_v(B) + (d-1)$. Since $K = \mathbb{K}(Y_d)$, applying the case $d = 1$ we get $\dim_v(B) = \dim_v(T) + 1$ and the equality in (1) follows.

(2) By Lemma 3.22, we have $\dim(R) = \dim(F) + \dim(T) = \dim(T)$. From (1) $\dim_v(R) = \dim_v(T) + \mathrm{t.d.}(K/F) = \dim(R) = \dim(T)$ if and only if

t. d.$(K/F) = 0$ and $\dim_v(T) = \dim(T)$. □

Theorem 3.26 ([2, Theorem 2.6]) . *Let $(T, \mathcal{M}, k)$ be a local domain which is not a field and $T \overset{\varphi}{\twoheadrightarrow} k$ the canonical surjection. Let $R = \varphi^{-1}(D)$, where D is any subring of k. Let F be the quotient field of D. Then:*

(1) $\dim_v(R) = \dim_v(D) + \dim_v(T) + \text{t. d.}(k/F)$.

(2) *R is a Jaffard domain if and only if D and T are each Jaffard domains and k is algebraic over F.*

Theorem 3.27 ([2, Theorem 2.11]) . *Let T be a domain with maximal ideal $\mathcal{M}$, $K = T/\mathcal{M}$ and $\varphi : T \twoheadrightarrow K$ the canonical surjection. Let D be a proper subring of K with quotient field F. Put $R = \varphi^{-1}(D)$ and $d =$ t. d.(K/F). Then:*

(1) $\dim(R) = \max\{\dim(T), \dim(D) + \dim(T_{\mathcal{M}})\}$.

(2) $\dim_v(R) = \max\{\dim_v(T), \dim_v(D) + \dim_v(T_{\mathcal{M}}) + d\}$.

Proof. (1) $\dim(R) = \max\{\sup\{\dim(R_P) \mid \mathcal{M} \not\subseteq P\}, \sup\{\dim(R_P) \mid \mathcal{M} \subset P\}\}$. If $\mathcal{M} \subset P$, then by Lemma 3.22(5), there exists $Q \in \text{Spec}(D)$ with $P = \varphi^{-1}(Q)$ and since $T_{\mathcal{M}}$ is local $\dim(R_P) = \dim(T_{\mathcal{M}}) + \dim(D_Q)$. As P varies, Q also varies in the same order and taking the sup yields $\dim(R) = \dim(D) + \dim(T_{\mathcal{M}})$. If $\mathcal{M} \not\subseteq P$, then by Lemma 3.22(4), there exists $Q \in \text{Spec}(T)$ with $R_P = T_Q$ and therefore $\dim(R) = \dim(T)$. Finally the equality in (1) follows.

(2) $\dim_v(R) = \max\{\sup\{\dim_v(R_P)$

$\mid \mathcal{M} \not\subseteq P\}, \sup\{\dim_v(R_P) \mid \mathcal{M} \subset P\}\}$. If $\mathcal{M} \not\subseteq P$, then $\dim_v(R_P) = \dim_v(T_Q)$ and taking the sup gives $\dim_v(R) = \dim_v(T)$. If $\mathcal{M} \subset P$, then applying Theorem 3.26(1) to the diagram in Lemma 3.22(5) yields $\dim_v(R_P) = \dim_v(T_{\mathcal{M}}) + \dim_v(D_Q) + \mathrm{t.\,d.}(k/F)$. Making P vary give $\dim_v(R) = \dim_v(D) + \dim_v(T_{\mathcal{M}}) + d$, completing the proof. □

Corollary 3.28 ([2, Corollary 2.12]) . *With the same hypothesis as in Theorem 3.27:*

(1) *R is a locally Jaffard domain if and only if D and T are locally Jaffard domains and K is algebraic over F.*

(2) *If T is a locally Jaffard domain with* $\dim_v(T) < \infty$, *D is a Jaffard domain, and K is algebraic over F, then R is a Jaffard domain.*

Proof. (1) Let assume that R is locally Jaffard and consider $P \in \mathrm{Spec}(T)$. Then

$$\begin{array}{ccc} R_{P\cap R} & \twoheadrightarrow & D_S \\ \downarrow & & \downarrow \\ T_P & \twoheadrightarrow & K \end{array}$$

, where $S = (K \setminus \varphi(P)) \cap D$ is a pullback diagram, so by Lemma 3.22(3), $\dim(R_{P\cap R}) = \dim(T_P) + \dim(D_S)$. Since R is locally Jaffard, this is equal to $\dim_v(R_{P\cap R})$ and then by Theorem 3.26(1), $\dim_v(R_{P\cap R}) = \dim_v(D_S) + \dim_v(T_P) + \mathrm{t.\,d.}(K/F)$. Therefore, we must have

$\text{t. d.}(K/F) = 0$, $\dim_v(T_P) = \dim(T_P)$ and $\dim_v(D_S) = \dim(D_S)$. Now, if $Q \in \text{Spec}(D)$, then $\varphi^{-1}(Q) \in \text{Spec}(T)$ and the above argument applies. Conversely, let $P \in \text{Spec}(R)$. If $P \supset \mathcal{M}$, then applying Theorem 3.26(1) to the pull-back diagram in Lemma 3.22(5), we get $\dim_v(R_P) = \dim_v(D_Q) + \dim_v(T_{\mathcal{M}}) + \text{t. d.}(K/F)$. Since T, D are locally Jaffard and $\text{t. d.}(K/F) = 0$, Lemma 3.22(3) implies that R is locally Jaffard. If $P \not\supseteq \mathcal{M}$, then by Lemma 3.22(4), there exists $Q \in \text{Spec}(T)$ such that $T_Q = R_P$ and since T is locally Jaffard it follows that R is locally Jaffard.

(2) By Proposition 3.7(2) and Theorem 3.27(2),

$\dim_v(R) = \max\{\dim(T), \dim(D) + \dim(T_{\mathcal{M}})\} = \dim(R)$. □

Proposition 3.29 ([2, Proposition 2.14]) . *Let V be a non trivial valuation domain of the form $V = K + \mathcal{M}$, where K is a field and $\mathcal{M}$ is the maximal ideal of V. Let $R = D + \mathcal{M}$, where D is a proper subring of K. Let F be the quotient field of D and let $d = \text{t. d.}(K/F)$. Then:*

(1) $\dim_v(R) = \dim_v(D) + \dim(V) + d$.

(2) *R is a Jaffard domain if and only if D is a Jaffard domain, V is finite-dimensional and K is algebraic over F.*

Proof.* (1) Let consider the diagram of pullbacks

$$\begin{array}{ccc} & R & \longrightarrow & D \\ & \downarrow & & \downarrow \\ A = F + \mathcal{M} & & \longrightarrow & F \\ & \downarrow & & \downarrow \\ & V & \longrightarrow & K \end{array}$$

By Proposition 3.24, $\dim_v(R) = \dim_v(D) + \dim_v(A)$ and by Proposition 3.25, $\dim_v(A) = \dim_v(V) + \text{t.d.}(K/F) = \dim(V) + \dim(K/F)$, proving (i) .

(2) Follows from (1) and Lemma 3.22(3). □

Proposition 3.30 ([2, Proposition 2.15]) . *Let K be a field, D a subring of K with quotient field F, $R = D + XK[X]$, and $d = \text{t.d.}(K/F)$. Then:*

(1) $\dim(R) = \dim(D) + 1$.

(2) $\dim_v(R) = \dim_v(D) + d + 1$.

(3) *R is a Jaffard domain if and only if D is a Jaffard domain and K is algebraic over F.*

Proof.* Consider the diagram

$$\begin{array}{ccc} R & \longrightarrow & D \\ \downarrow & & \downarrow \\ K[X] & \longrightarrow & K \end{array}$$

(1) and (2) are consequences of Theorem 3.27 and Proposition 3.10, if we set $\mathcal{M} = XK[X]$, $T = K[X]$ and observe that $\dim(K[X]) = \dim(K[X]_{(X)}) = 1$.
(3) $\dim(R) = \dim_v(R)$ if and only if $\dim(D) + 1 = \dim_v(D) + d + 1$, i.e., $\dim(D) = \dim_v(D) + d$. Since $\dim(D) \leq \dim_v(D)$, the equality $\dim(D) = \dim_v(D) + d$ will hold if and only if $d = 0$ and $\dim(D) = \dim_v(D)$. □

3.3 Applications and Examples

Example 3.31 ([2, Example 3.1 (a)]) . For each positive integer n, there exists a finite-dimensional non-Jaffard domain R such that $\dim_v(R) - \dim R = n$.

Proof. Let $X_1, X_2, \ldots, X_{n+1}$ be $n+1$ indeterminates over a field K. Let $L = K(X_1, \ldots, X_n)$, $V = L[X_{n+1}]_{(X_{n+1})} = L + X_{n+1}V$. Put $\mathcal{M} = X_{n+1}V$ and $R = K + \mathcal{M}$. We claim that R is the required ring. Indeed, V is a discrete valuation domain. $\mathrm{Spec}(V) = \mathrm{Spec}(R)$. Indeed, R is a subring of V so that every prime ideal of V contract to a prime ideal in R. On the other hand, let $P \in \mathrm{Spec}(R)$. Let show that P is an ideal of V. Let $x_0 \in \mathcal{M} \setminus P$, $z \in V$ and $y \in P$. $zx_0 \in \mathcal{M} \subset R$ implies that $(zx_0)y = x_0(zy) \in P$. But $x_0 \notin P$ and P is prime. Therefore $zy \in P$, as desired. Now, let $x, y \in V$ such that $xy \in P$. If x or y is not in $\mathcal{M}$, then x^{-1} or y^{-1} is in V. So $x^{-1}(xy) = y$ or $y^{-1}(xy) = x$ is in P. If $x \in \mathcal{M}$ and $y \in \mathcal{M}$, then $x, y \in R$ and since P is a prime ideal in R we have $x \in P$ or $y \in P$, as desired. Thus $\dim(V) = \dim(R) = 1$. By Proposition 3.25, $\dim_v(R) = \dim_v(V) + \mathrm{t.d.}(L/K) = \dim(V) + n = 1 + n$.

Thus, $\dim_v(R) - \dim(R) = n$, as desired. $\square$

Example 3.32 ([2, Example 3.1 (b)]) . There exists a domain R such that $\dim(R) = 1$ and $\dim_v(R) = \infty$.

Proof. Let $Y, X_1, X_2, \ldots$ be indeterminates over a field K. Put $L = K(X_1, X_2, \ldots)$; so that $\text{t.d.}(L/K) = \infty$. Set $V = L[Y]_{(Y)} = L + YV$ and $R = K + YV$. By Proposition 3.25, $\dim_v(R) = \dim_v(V) + \text{t.d.}(L/K) = \infty$. But $\dim(R) = \dim(V) + \dim(K) = 1$. $\square$

Example 3.33 ([2, Example 3.2]) . There exists a two-dimensional Jaffard domain R such that R is not locally Jaffard.

Proof. Let k be a field and X_1, X_2, Y indeterminates over k. Set $V_1 := k(X_1, X_2)[Y]_{(Y)} = k(X_1, X_2) + M_1$ and $A := k(X_1) + M_1$, where $M_1 = YV_1$. Let (V, M) be a one-dimensional valuation domain of the form $V = k(Y) + M$ such that $k(Y[X_1, X_2]) \subset V \subset k(X_1, X_2, Y)$. Consider the two-dimensional valuation ring $V_2 := k[Y]_{(Y)} + M = k + M_2$ with maximal ideal $M_2 = Yk[Y]_{(Y)} + M$ (M is a prime ideal of V_2 which is strictly contained in M_2). V_1 and V_2 are incomparable. First, notice that V_1 is local and Bezout and therefore is a valuation with $\dim(V_1) = 1$. If V_1 and V_2 were comparable, then we must have $V_1 = (V_2)_M$. $Y \notin M$ and $Y \in V_2$ implies that $\frac{1}{Y} \in (V_2)_M$, but $\frac{1}{Y} \notin V_1$. By [34, Theorem 11.11] and [28, Theorem 64], $B := V_1 \cap V_2$ is a two-dimensional Prüfer domain domain with two maximal ideals, say N_1 and N_2, $B_{N_1} = V_1$

and $B_{N_2} = V_2$. Finally, put $R := A \cap V_2$. $R \subset B$ is an integral ring extension. Indeed, if $z \in B$, then $\sum_{k=0}^{n} \lambda_k z^k = 0$, where $\lambda_k \in B$. If λ_k is a unit in V_2, then $\lambda_k \in k \subset k(X_1) \subset A$. If λ_k is not a unit in V_2, then $\lambda_k \in M_1 \cap M_2 \subset A$. Thus R has exactly two maximal ideals $\mathcal{M}_1 = N_1 \cap R$ and $\mathcal{M}_2 = N_2 \cap R$ with $R_{\mathcal{M}_1} = A_{\mathcal{M}_1} \cap B_{\mathcal{M}_1} = A \cap B_{N_1} = A$ and $R_{\mathcal{M}_2} = V_2$. By Lemma 3.22, $\dim(A) = 1$ and therefore $\dim(R) = \max\{\dim(R_{\mathcal{M}_1}), \dim(R_{\mathcal{M}_2})\} = 2$ and $\dim_v(R) = \max\{\dim_v(R_{\mathcal{M}_1}), \dim_v(R_{\mathcal{M}_2})\} = 2$. Thus R is Jaffard but not locally Jaffard, since $\dim(R_{\mathcal{M}_1}) = \dim(A) = 1 \neq \dim_v(R_{\mathcal{M}_1}) = \dim_v(A) = 2$.

□

Proposition 3.34 ([2, Proposition 3.3]) . *Let* X_1, $\ldots, X_r$ *be finitely many indeterminates over a domain* R *which is not a field. Assume that* $\dim_v(R) = n < \infty$. *Then:*

(1) *If* $R[X_1, \ldots, X_r]$ *is a Jaffard domain, then* $r \geq \frac{n - \dim(R)}{\dim(R)}$.

(2) *Assume that* $\dim(R) = 1$. *Then* $R[X_1, \ldots, X_r]$ *is a Jaffard domain if and only if* $r \geq n - 1$.

Proof. (1) For every $r \geq 0$ we have $\dim(R[X_1, \ldots, X_r]) \leq r + (r+1)\dim(R)$. Since $\dim_v(R[X_1, \ldots, X_r]) =$ $\dim(R[X_1, \ldots, X_r]) = \dim_v(R) + r \leq r + (r+1)\dim(R)$, then $\frac{n - \dim(R)}{\dim(R)} \leq r$, as desired.

(2) If $\dim(R) = 1$, then $r \geq n-1$ by (1). Conversely if $r \geq n-1$, $R[X_1, \ldots, X_r]$ is a Jaffard domain by Proposition 3.5(2). □

Example 3.35 ([2, Example 3.4]) . For each positive integer r, there exists a finite-dimensional non-Jaffard domain R such that r is the least positive integer m for which the polynomial ring $R[X_1, \ldots, X_m]$ is a Jaffard domain.

Proof. We consider the ring R in Example 3.31 with $\dim(R) = 1$. Thus, $\dim_v(R) = r + 1$. $R[X_1, \ldots, X_m]$ is a Jaffard domain if and only if $m \geq \dim_v(R) - 1 = r$ by Proposition 3.34. □

Example 3.36 ([2, Example 3.5]) . There exists a non-Jaffard domain T with $n = \dim_v(T) < \infty$ and a positive integer $r < n - 1$ such that the polynomial ring $T[X_1, \ldots, X_r]$ is a Jaffard domain.

Proof. Consider the ring R of Example 3.31 with
$\dim(R) = 1$ and $\dim_v(R) = 3$. Consider indeterminates $Y, Y_1, \ldots, Y_r$ over R and put $T = R[Y]$. By [2, Theorem 1.10], R is not a strong $S-$domain. In this case, $\dim(T) = \dim(R[Y]) = 3$ since $2 \lneq \dim(R[Y]) \leq 3$ (see [28, Theorem 38]). By Theorem 3.1, $\forall\, r \geq \dim_v(R) - 1 = 2,\ \dim(R[Y_1, \ldots, Y_r]) = r + \dim_v(R) = r + 3$. For $r \geq 1,\ \dim(T[Y_1, \ldots, Y_r]) = \dim(R[Y, Y_1, \ldots, Y_r]) = (r + 1) + 3 = r + 4$. Further, $\dim_v(T) = \dim_v(R[Y]) = 1 + \dim_v(R) = 4 > 3 = \dim(T)$. Thus, T is not Jaffard. Since $\dim(T[Y_1]) = \dim(R[Y, Y_1]) = 5$, and $\dim_v(T[Y_1]) = \dim_v(R[Y, Y_1]) = 2 + \dim_v(R) = 5$; then $T[Y_1]$ is Jaffard. □

4 Chapter 4

On the Generalized Principal Ideal Theorem and Krull Domains

An integral domain R satisfies PIT (resp., GPIT) if every minimal prime over a principal (resp., n-generated) ideal has height ≤ 1 (resp., $\leq n$). Here PIT stands for "Principal Ideal Theorem" and GPIT for " Generalized Principal Ideal Theorem." Both Noetherian and Krull domains satisfy PIT. Moreover, while Noetherian rings satisfy GPIT (by Krull's altitude theorem), Krull domains don't. Eakin-Heinzer's symbolic Rees algebra (mentioned above) provided a first counter-example in this regard, which was later developed by Anderson-Dobbs-Eakin-Heinzer in 1990 [3]. A ring R of dimension 2 satisfies GPIT if and only if it satisfies PIT. Indeed, it is obvious that GPIT implies PIT. Conversely, if P is minimal over $(a_1, \ldots, a_n)$, then two cases are possible:

(1) If $n \geq 2$, then $\mathrm{ht}(P) \leq \dim(R) = 2 \leq n$.

(2) If $n = 1$, then we are done by PIT.

4.1 GPIT and integrality

In this paragraph, we study the stability of GPIT under integral extension.

Proposition 4.1 ([3, Proposition 2.1]) . *Let $R \subset T$ be an extension of domains. Then:*

(1) *Suppose $R \subset T$ satisfies LO, INC and GD. If T satisfies GPIT, then R*

satisfies GPIT.

(2) *Suppose $R \subset T$ satisfies GU and INC. If T satisfies GPIT, then R satisfies GPIT.*

(3) *If T is integral over R and T satisfies GPIT, then R satisfies GPIT.*

Proof.* (1) Let $I = (a_1, \ldots, a_n)R$ be an $n-$generated ideal of R and P a minimal prime ideal of I. By LO, there exists $Q \in \text{Spec}(T)$ such that $Q \cap R = P$ and Q is minimal over the $n-$generated ideal IT. Since T satisfies GPIT, $\text{ht}(Q) = m \leq n$. Let $Q \supset Q_{m-1} \supset \ldots \supset Q_0 = (0)$ be a saturated chain of prime ideals in T. By GD, there exists a chain $P \supset P_{m-1} \supset \ldots \supset P_0 = (0)$ in R such that $P_i = Q_i \cap R$. Moreover, this chain is saturated since the extension $R \subset T$ satisfies INC. Thus $\text{ht}(P) \leq n$, as desired.

(2) It follows from (1) since GU implies LO.

(3) It is a consequence of (2) since an integral extension satisfies INC and GU.

□

Proposition 4.2 ([3, Theorem 2.2]) . *For a domain R, the following conditions are equivalent:*

(i) *If $u_1, \ldots, u_n$ are finitely many elements of a domain which contains R and is integral over R, then $R[u_1, \ldots, u_n]'$ satisfies GPIT.*

(ii) *If R is a subring of a domain T which is integral over R, then T satisfies GPIT.*

Proof. (ii) $\Longrightarrow$(i) by setting $T = R[u_1, \ldots, u_n]'$. Suppose that the converse fails. Then there exists $P \in \mathrm{Spec}(T)$ such that P is minimal over some $n-$generated ideal J of T and $\mathrm{ht}(P) > n$. Let $J = (u_1, \ldots, u_n)T$. We replace R with $R[u_1, \ldots, u_n]$ and set $I = \sum_{i=1}^{n} Ru_i$. Then $J = IT$ and T is integral over R. In order to reach a contradiction, we have to show that R' does not satisfy GPIT. Let $P = P_0 \supset P_1 \supset \ldots \supset P_{n+1}$ be a saturated chain of prime ideals in T. Since $R \subset T$ is an integral extension, if φ denotes the projection $\mathrm{qf}(R) \longrightarrow T$, then the extension ring, $S = R'T = \varphi(R')$, inside $\mathrm{qf}(T)$ is integral over $\varphi(R) \subset T$ (see [34, Theorem 10.13]). Thus $T \subset S$ verifies LO, GU and INC and so there exists a chain $N = N_0 \supset \ldots \supset N_{n+1}$ of prime ideals in S lying over the given chain $\{P_i\}$ with N minimal over $\sum Su_i = IS = JS$. Put $Q_i = N_i \cap R' \in \mathrm{Spec}(R')$. Since S is integral over R', it follows via INC that $\{Q_i\}$ consists of $n+2$ distinct primes of R', whence $\mathrm{ht}(Q_0) > n$. By [34, Theorem 10.13], $R' \subset S$ satisfies GD. As N is minimal over $\sum Su_i$, it now follows via GD that Q_0 is minimal over $\sum R'u_i$. In particular, R' does not satisfy GPIT, as desired. □

Lemma 4.3 *Let R be a domain. Then R satisfies GPIT if and only if R_M satisfies GPIT for each maximal ideal M of R.*

Proof.* Let M be a maximal ideal of R and
$J = (\frac{a_1}{s_1}, \ldots, \frac{a_n}{s_n})R_M$ an $n-$generated ideal of R_M. Consider the ideal $I = (a_1, \ldots, a_n)R$, then $IR_M = J$ and by assumption, $\mathrm{ht}(I) \leq n$. Next, none of the

a_i is inside $R \setminus M$, so that if P is a minimal prime of I, then PR_M is a minimal prime of J and therefore $\text{ht}_{R_M}(J) = \text{ht}_{R_M}(PR_M) = \text{ht}_R(P) \leq n$. Conversely, let Q be a minimal prime ideal over $(b_1, \ldots, b_r)R$. Then $Q \subseteq M$ for some maximal ideal M of R and $QR_M \in \text{Spec}(R_M)$ is minimal over $(\frac{b_1}{1}, \ldots, \frac{b_r}{1})R_M$. Therefore

$\text{ht}_R(Q) = \text{ht}_{R_M}(QR_M) \leq r$. □

Proposition 4.4 ([3, Corollary 2.3]) . *If $R \subset T$ is an integral extension of domains and R is Noetherian, then T satisfies GPIT.*

Proof. By Proposition 4.2, it suffices to show that $R[u_1, \ldots, u_n]'$ satisfies GPIT if $u_1, \ldots, u_n$ are integral elements of a domain containing R.

As $R[u_1, \ldots, u_n]$ is Noetherian, we may replace R by $R[u_1, \ldots, u_n]$, and then only show that the integral closure $R' = T$of R satisfies GPIT. Without loss of generality, we may assume that R is local. Hence, by [21, Chapter 0, Corollary 23.2.5], R has a finitely generated (integral) overring S such that the canonical map $\text{Spec}(T) \longrightarrow \text{Spec}(S)$ is injective. Since S is Noetherian and $T = S'$, we may replace R with S. Consider a prime ideal P of T such that P is minimal over a finitely generated ideal $(v_1, v_2, \ldots, v_k)$ of T and put $A = R[v_1, v_2, \ldots, v_k]$. Since $R \subset A \subset T$ and $\text{Spec}(T) \longrightarrow \text{Spec}(R)$ is injective, then $\text{Spec}(T) \longrightarrow \text{Spec}(A)$ is injective. For, if two different primes of T contract to the same prime in A, they must contract to the same prime in R, which is impossible. Since T is integral over A, the extension $A \subset T$ satisfies LO and GU. The GD holds between A and T. Indeed, let $p \subset q$ in $\mathrm{Spec}(A)$ and $Q \in \mathrm{Spec}(T)$ such that $Q \cap A = q$. Then by LO, there exists $P \in \mathrm{Spec}(T)$ with $P \cap A = p$ and by GU, there exists $Q' \in \mathrm{Spec}(T)$ such that $P \subset Q'$ and $Q' \cap A = q$. Now, since $\mathrm{Spec}(T) \hookrightarrow \mathrm{Spec}(A)$ is injective, $Q' \cap A = Q \cap A = q$ implies that $Q = Q'$. Hence $Q = P \cap A$ is minimal over $\sum Av_i$. As A is Noetherian, it satisfies GPIT, so that $\mathrm{ht}(Q) \leq k$. By [28, Theorem 45], $\mathrm{ht}(P) \leq k$. Hence T satisfies GPIT. □

Thus, the integral closure of a Noetherian domain satisfies GPIT. The following example points out the importance of the Noetherian hypothesis.

Example 4.5 ([3, Example 2.4]) . There exists a local two-dimensional domain R satisfying GPIT and an element u such that $R' = R[u]$ and R' does not satisfy (G)PIT.

Proof. Let x and y be algebraically independent indeterminates over a field K. Put $D = K[x,y]_{(x-1,y-1)}$ and $\mathcal{M}_1 = (x-1, y-1)D$. Then $D/\mathcal{M}_1 \cong K$ and clearly $D = K + \mathcal{M}_1$. Next, consider the rank 2 valuation v of $K(x,y)$ over K, with value group $\mathbb{Z} \oplus \mathbb{Z}$ lexicographically ordered, defined by $v(x) = (1,0)$, $v(y) = (0,1)$. The corresponding valuation ring is $V = K[y]_{(y)} + xK[x,y]_{(x)}$, which can be written as $V = K + \mathcal{M}_2$, where $\mathcal{M}_2 = yV$ is the maximal ideal of V. Indeed, $K[y]_{(y)}$ is a valuation ring since it is local and Bezout. Therefore V is also a valuation since $V \subset K(y) + xK[x,y]_{(x)}$ and $\mathrm{qf}(K[y]_{(y)}) = K(y)$. Let W be the valuation ring associated to v. Then $V \subseteq W$, i.e., $W = V_Q$, where

$Q \in \operatorname{Spec}(V)$. Note that $\dim(V) = 2$. Let $(0) \subset P \subset m$ be a saturated chain of prime ideals in V. Then $Q \neq (0)$ since W is not a field. If $Q = P$, then $\dim(W) = 1$ and in that case $\operatorname{rank}(W) = 1$; which is not the case. So $Q = m$ and it follows that $W = V$, as desired. Put $J = \mathcal{M}_1 \cap \mathcal{M}_2$ and $R = K + J$.
R is the required ring. First, note that $R' = D \cap V$. Indeed, since D and V are integrally closed and contain R, $R' \subseteq D \cap V$. For the reverse inclusion, it suffices to show that each $t \in D \cap V$ is integral over R. We observe that J is a common ideal for $D \cap V$ and R, so that $D \cap V$ is an overring of R. This assure us that $D \cap V \subseteq \operatorname{qf}(R)$. Let $t = a_1 + m_1 = a_2 + m_2$, with $a_i \in K$ and $m_i \in \mathcal{M}_i$, $i = 1, 2$. Then $(t - a_1)(t - a_2) = m_1 m_2 \in \mathcal{M}_1 \cap \mathcal{M}_2 = J \subset R$, and so t is a root of the monic polynomial $Z^2 - (a_1 + a_2)Z + a_1 a_2 - m_1 m_2 \in R[Z]$. Hence $D \cap V \subseteq R'$. Next, we show that R is local with maximal ideal J. Let $r \in J$. Then since D and V are local rings, $(1+r)$ is invertible in each of them; so that we can write $(1+r)^{-1} = 1 - r(1+r)^{-1} \in 1 + J$ so that $(1+r)$ is a unit of R, and $r \in \mathrm{J}(R)$, the Jacobson radical of R. Since J is maximal in R, then $J = \mathrm{J}(R)$, as desired. Consider the primes $P_1 = \mathcal{M}_1 \cap R'$ and $P_2 = \mathcal{M}_2 \cap R'$. We have $P_1 \cap R = \mathcal{M}_1 \cap D \cap V \cap R = \mathcal{M}_1 \cap (K + \mathcal{M}_1) \cap (K + \mathcal{M}_2) \cap R = J$ since $\mathcal{M}_i \cap K = \emptyset$. Similarly, $P_2 \cap R = J$. As each of these meets R in J, it follows via integrality that P_1 and P_2 are maximal in R' since they contract to the maximal ideal J of R. Moreover, $P_1 \neq P_2$ since $x \in P_2 \setminus P_1$. Next, we prove that $R'_{P_1} = D$ and $R'_{P_2} = V$. For any multiplicatively closed subset S of R', we have $R'_S = D_S \cap V_S$.

Let S_i denote $R' \setminus P_i$, then $D_{S_1} = (K[x,y]_{(x-1,y-1)})_{S_1} = D$ and $V_{S_2} = V$ and the result will follow if we show that $V_{S_1} = D_{S_2} = K(x,y)$. Consider any nonzero $g \in K[x,y]$. If $v(g) = (i,j)$, then $v(\frac{g}{x^i y^j}) = v(g) - iv(x) - jv(y) = 0$; so that $\frac{g}{x^i y^j}$ is a unit and $gV = x^i y^j V$. Further, $V[(xy)^{-1}] = K(x,y)$; for, if $\frac{h}{g} \in K(x,y)$, then $g = \alpha x^i y^j$, where α is a unit in V. So $\frac{h}{g} = \frac{\alpha^{-1}h}{x^i y^j} \in V[(xy)^{-1}]$. As $xy \in R' \setminus P_1 = S_1$, we have $V_{S_1} = K(x,y)$ as desired (because if we set $S = \{(xy)^n, n \in \mathbb{N}\}$, then $S \subset S_1$ and $K(x,y) = V_S \subseteq V_{S_1} \subseteq K(x,y)$). Moreover, for g as above, $h = g(x^i y^j)^{-1} \in (V \setminus \mathcal{M}_2) \cap D = S_2$, since $\frac{g}{x^i y^j}$ is a unit in V and $x^i y^j \notin (x-1, y-1)$. Therefore h is invertible in D_{S_2} and $g^{-1} = (x^i y^j)^{-1} h^{-1} \in D_{S_2}$. Thus, $D_{S_2} = K(x,y)$, as desired. Since V is a two-dimensional valuation domain, V does not satisfy PIT (indeed, let $(0) \subset Q \subset \mathcal{M}_2$ be the Spectrum of V and choose an element $a \in \mathcal{M}_2 \setminus Q$ such that $\mathcal{M}_2$ is minimal over the ideal (a)). As $R'_{P_2} = V$, it follows from [7, Proposition 3.1(a)] that R' does not satisfy PIT. Since $x \in R' \setminus R$ we have $R \neq R'$. Moreover, $x(x-1) = m \in \mathcal{M}_1 \cap \mathcal{M}_2 = J \subset R$ so that x is a solution of $Z^2 - Z + m = 0$. We claim that $R[x] = R'$ (in other words, $u = x$ satisfies the assertion): From the equality $R'_{P_1} = D$, we get $K \cong D/\mathcal{M}_1 = R'_{P_1}/P_1 R'_{P_1} \cong R'/P_1$ (since P_1 is maximal in R'). Whence $R' = K + P_1$. Similarly, $R'_{P_2} = V$ leads to $K \cong V/\mathcal{M}_2 \cong R'/P_2$ and $R' = K + P_2$. Since $xP_1 \subset x\mathcal{M}_1 \subset \mathcal{M}_1$ and $xP_1 \subset \mathcal{M}_2 V = \mathcal{M}_2$ ($x \in \mathcal{M}_2$ and $P_1 \subset R' \subset V$), we have $xP_1 \subset J$, so that $xR' = xK + xP_1 \subset xR + J \subset R[x]$.

As $(x-1) \in R' \cap \mathcal{M}_1 = P_1$ and $P_1P_2 \subset P_1 \cap P_2 = \mathcal{M}_1 \cap \mathcal{M}_2 \cap R' = J \cap R' = J$, we also have $(x-1)R' = (x-1)K + (x-1)P_2 \subset R[x] + P_1P_2 = R[x]$. Hence, $R' \subset xR' + (x-1)R' \subset R[x]$. So $R[x] = R'$ as claimed. Since R' is integral over R, any maximal ideal $\mathcal{M}$ of R' meets R in $J = \mathcal{M}_1 \cap \mathcal{M}_2 = P_1 \cap P_2$ (See [31, Lemma 2]). It follows that $\mathcal{M}$ contains (and hence equals to) one of the ideals P_1 or P_2. Thus, P_1 and P_2 are the only maximal ideals of R', and so $\dim(R') = \sup\{\dim(R'_{P_1}), \dim(R'_{P_2})\} = \sup\{\dim(D), \dim(V)\} = 2$. By integrality, $\dim(R) = \dim(R') = 2$. Next, we show that R satisfies GPIT. Since R is a two-dimensional, it is sufficient to show that R verifies PIT. This amount to show that J is not minimal over any nonzero principal ideal Rs with $s \in J \setminus \{0\}$. In the Noetherian ring D, the prime ideal $P_1D = P_1R'_{P_1}$ has height 2 and contains s, but cannot be minimal over Ds by PIT. Hence $Ds \subset QD \subsetneqq P_1D$ for some prime Q of R'. By INC, we have $Q \cap R \subsetneqq P_1 \cap R = J$. As $s \in QD \cap R = Q \cap R$, J is not minimal over Rs. □

4.2 GPIT and homomorphic images

Proposition 4.6 ([3, Proposition 3.1]) . *If R is a ring such that R/P satisfies GPIT for each minimal prime P of R, then R satisfies GPIT.*

Proof. Suppose the assertion fails. Then there exist $Q \in \operatorname{Spec}(R)$ such that Q is minimal over some $n-$generated ideal I of R and $\operatorname{ht}(Q) = k > n$. Pick a chain $Q = P_0 \supset P_1 \supset P_2 \ldots \supset P_k = P$ of distinct primes in R. Then P is a minimal prime. Since the ring R/P satisfies GPIT and its prime $\overline{Q} = Q/P$

is minimal over the $n-$generated ideal $(I + P)/P$, it follows that $\text{ht}(\overline{Q}) \leq n$. However, the chain $\{P_i/P\}$ of distinct primes reveals that $\text{ht}(\overline{Q}) > n$, a contradiction. □

If R is a ring and n is a non-negative integer, we shall say that R satisfies $n-$PIT in case $\text{ht}(P) \leq n$ for each $P \in \text{Spec}(R)$ which is minimal over an $n-$generated ideal of R. Evidently, R satisfies GPIT if and only if R satisfies $n-$PIT for al $n \geq 0$.

Lemma 4.7 ([3, Lemma 3.4]) . *Let R be a ring satisfying $k-$PIT for some $k > 0$. Let I be an ideal of R generated by an $R-$sequence $y_1, y_2, \ldots, y_n$ for some $n < k$. Then $\overline{R} = R/I$ satisfies $(k-n)-$PIT.*

Proof. We proceed by induction on n. If $n = 1$, then y_1 been a nonzerodivisor, it lies in no minimal prime of R (cf. [28, Theorem 84]). Assume that $R/(y_1)R$ does not satisfy $(k-1)-$PIT, i.e., there exists a prime ideal P in R such that $\text{ht}(P/(y_1)) > k-1$ and $P/(y_1)$ is minimal over $(\overline{x_1}, \ldots, \overline{x_{k-1}})$, where $\overline{x_i}$ denotes the class of x_i in $R/(y_1)$. Let $P \supset P_1 \supset \ldots \supset P_k \supset \ldots \supset P_l$ be a chain of distinct primes in R containing y_1 and P_l be a minimal prime ideal. Then $\text{ht}_R(P) \geq k+1$, which is impossible since P is minimal over $(x_1, \ldots, x_{k-1}, y_1)$ and R satisfies $k-$PIT. Now, assume the result is true for all integers $r < n$. Let $I = (y_1, \ldots, y_n)R$ be an ideal of R generated by an $R-$sequence. Set $A = R/(y_1, \ldots, y_{n-1})$. Since $y_1, \ldots, y_n$ is an $R-$sequence, $\overline{y_n}$ is a nonzerodivisor in A. Since for any ideal L and J of A, $(A/L)/(J/L) \cong A/L+J$, we obtain from the case $n=1$ that R/I satisfies $(k-n)$- PIT. □

Proposition 4.8 ([3, Theorem 3.3]) . *If a ring R satisfies GPIT and I is an ideal of R generated by an*
$R-$sequence, then R/I satisfies GPIT.

Proof. Let $I=(y_1,\ldots,y_m)$ be an ideal of R generated by an $R-$sequence and $n\in\mathbb{N}$. As R satisfies GPIT, it satisfies $(m+n)-$PIT and so by Lemma 4.7, R/I satisfies $n-$PIT. Since n was arbitrarily chosen, the desired result follows.
□

4.3 GPIT and monoid domains

Proposition 4.9 ([3, Theorem 4.2]) . *Let G be a nonzero torsion-free abelian group with finite rank n and let R be a domain. Then the following conditions are equivalent:*

(i) $R[X_1,\ldots,X_n]$ *satisfies GPIT;*

(ii) $R[X_1,X_1^{-1},\ldots,X_n,X_n^{-1}]$ *satisfies GPIT;*

(iii) $R[G]$ *satisfies GPIT.*

Proof. Let S be a multiplicatively closed subset of R and $J=(\frac{a_1}{s_1},\ldots,\frac{a_n}{s_n})R_S$. Consider the ideal $I=(a_1,\ldots,a_n)$ of R. Then $\mathrm{ht}_{R_S}(IR_S)=\mathrm{ht}_{R_S}(J)=\mathrm{ht}_{R_S}(PR_S)=\mathrm{ht}_R(P)$, where $P\in\mathrm{Spec}(R),\ P\cap S=\emptyset$ and P minimal over I. Thus GPIT is preserved under localization and so (i) implies (ii) .

(ii) $\Longrightarrow$(iii) Suppose that $B = R[G]$ does not satisfy GPIT. Then there exist $f_1 \ldots, f_r \in B$ and a prime ideal P of B with $\text{ht}(P) > r$ such that P is minimal over $(f_1, \ldots, f_r)B$. Let $G = F + G_{tor}$, where F is a free abelian subgroup of G with $\text{rank}(F) = n$ and $f_1, \ldots, f_r \in A = R[F]$. Set $Q = P \cap A$. Then since B is integral over A, we have $\text{ht}(Q) > r$. By (ii) , A satisfies GPIT and so there exists a prime ideal $Q_1 \in \text{Spec}(A)$ with $(f_1, \ldots, f_r)A \subset Q_1 \subsetneq Q$. B is a flat $A-$module so that the extension $A \subset B$ satisfies GD. Thus there exists a prime ideal P_1 of B with $(f_1, \ldots, f_r)B \subset P_1 \subsetneq P$, contradicting the minimality of P.

(iii) $\Longrightarrow$(ii) Let $G = F + G_{tor}$, where F is a free abelian subgroup of G with $\text{rank}(F) = n$. Then G/F is a torsion group, and $A = R[X_1, X_1^{-1}, \ldots, X_r, X_r^{-1}] \cong R[F] \subseteq R[G]$. Moreover, the extension $R[F] \subseteq R[G]$ is integral. By Proposition 4.1(3), A satisfies GPIT, as desired.

(ii) $\Longrightarrow$(i) Set $A = R[X_1, \ldots, X_r]$ and let $I = (f_1, \ldots, f_r)A$ be an ideal of A with minimal prime ideal P. We may assume that $X_i \notin P$ for $1 \leq i \leq r$. Otherwise, apply the translation $X_i \mapsto X_i + a_i$ for $0 \neq a_i \in R$. Let $B = R[X_1, X_1^{-1}, \ldots, X_r, X_r^{-1}]$. Then since B is a localization of A, PB is minimal over IB and therefore $\text{ht}(P) = \text{ht}(PB) \leq r$. □

Proposition 4.10 ([3, Corollary 4.3]) . *Let R be a domain. Then $R[G]$ satisfies GPIT for each torsionfree abelian group G if and only if $R[X_1, \ldots, X_n]$ satisfies GPIT for each positive integer n.*

Proof. The "only if" part is a consequence of Proposition 4.9. Conversely, suppose that $R[G]$ does not satisfy GPIT, for some torsion free abelian group G. Then there exist $f_1, f_2, \ldots, f_r$ in $R[G]$ and a prime ideal P of $R[G]$ such that P is minimal over $(f_1, f_2, \ldots, f_r)$ with $\mathrm{ht}(P) > r$. Let $Q' \,\mathrm{Spec}(R[G])$ such that $P \subsetneq Q'$ and let $g \in Q' \backslash P$. Write $f_k = \sum \lambda_{j_k} X^{\sum_i l_{ij_k} a_{ij_k}}$ and $g = \sum \beta_j X^{\sum_i h_{ij} b_{ij}}$. Consider $F = \mathbb{Z} < a_{ij_k}, b_{ij} >$. Then F is finitely generated subgroup of G with finite rank, say $\mathrm{rank}(F) = n$ such that $f_1, f_2, \ldots, f_r \in A = R[F]$. $Q = P \cap A$ is minimal prime over $(f_1, \ldots, f_r)A$ with $\mathrm{ht}(Q) > r$ since the extension $R[F] \subseteq R[G]$ satisfies INC. By Proposition 4.9, $R[X_1, \ldots, X_n]$ does not satisfy GPIT, which contradicts our assumption. □

Proposition 4.10 may be used to construct examples of non-Noetherian domains which satisfy GPIT. In particular, the group ring $k[G]$ satisfies GPIT for any field k and torsionfree abelian group G. In [20, Section 14] and [19, Theorem 2], this result has been used to construct finite-dimensional non-Noetherian UFD's. Note that a consequence of [20, Theorems 16.2, 14.7, 14.10 and 14.15] is that, $k[G]$ is a Krull domain $\Longleftrightarrow$ $k[G]$ is a UFD $\Longleftrightarrow$ each rank-one subgroup of G is cyclic. Thus, any non-Noetherian Krull domains constructed as group ring over a field satisfy GPIT.

4.4 A Krull domain not satisfying GPIT

By Proposition 4.4, any Krull ring obtained as the integral closure of a Noetherian normal domain satisfies GPIT. In this paragraph we will give the construction of a local three-dimensional Krull domain whose maximal ideal $\mathcal{M}$ is the radical of a 2-generated ideal. Thus, this domain does not satisfy GPIT.

Example 4.11 ([3, Example 5.1]) . We take $(R, \mathcal{M})$ to be a two-dimensional integrally closed Noetherian local domain having a height 1 prime ideal P such that P is not the radical of a principal ideal. By [31, Theorem 11.2], R/P is a DVR. Let $P^{(n)} = P^n R_P \cap R$ denote the nth symbolic power of P. Set $B = R[t^{-1}, Pt, \ldots, P^{(n)}t^n, \ldots]$, the symbolic Rees ring with respect to P. Then B is a three-dimensional non-Noetherian Krull ring. Let us show that $\mathcal{N} = (\mathcal{M}, t^{-1}, Pt, \ldots, P^{(n)}t^n, \ldots)$ is a height 3 maximal ideal of B. Since B is a $\mathbb{Z}-$graded domain with the homogeneous terms of degree n consisting of $P^{(n)}t^n$ if $n > 0$ and Rt^n if $n \leq 0$, it follows that $\mathcal{N}$ is a maximal ideal of B. Consider the multiplicatively closed subset $S = R \setminus P$. By [31, Theorem 11.2], R_P is a DVR with maximal ideal $PR_P = (\pi)$, with $\pi \in P$; so that $P^n R_P = (\pi^n)$ for all $n \geq 1$ and hence $B_S = R_P[t^{-1}, \pi t]$. Let

$$\begin{aligned}\varphi : & R_P[u,v] \longrightarrow R_P[t^{-1}, \pi t] \\ & u \longmapsto t^{-1} \\ & v \longmapsto \pi t \\ & r \longmapsto r, \ \forall r \in R_P.\end{aligned}$$

Then φ is a surjective R_P-homomorphism . Indeed, for any $\sum r_i(t^{-1})^{\alpha_i}(\pi t)^{\beta_i}$, there exists $\sum r_i u^{\alpha_i} v^{\beta_i} \in R_P[u,v]$ which is its preimage by φ. Next, $\varphi(uv - \pi) = 0$, so that $(uv-\pi) \subseteq \ker(\varphi)$. On the other hand, let $\sum r_i(t^{-1})^{\alpha_i}(\pi t)^{\beta_i} = 0$. This can be rewritten as $\sum r_i(\pi)^{\beta_i} t^{\gamma_i} = 0$ $(\star)$, where $\gamma_i \in \mathbb{Z}$. Let $\gamma = \sup\{\gamma_i\}$. Then by multiplying $(\star)$ by $t^{-\gamma}$ and setting $g(uv) = f(u,v) = \sum r_i(uv)^{\beta_i}$, we have $g(\pi) = 0$, i.e., $f = g \in (uv - \pi)$. Now, observe that $\sum r_i(t^{-1})^{\alpha_i}(\pi t)^{\beta_i} = h(u,v)$, where $h(u,v) = f(u,v)(u^{\gamma})$. Thus $\ker(\varphi) = (uv - \pi)$ and $B_S \cong R_P[u,v]/(uv - \pi)$. Since R_P is a Noetherian ring, it satisfies GPIT and therefore $\text{ht}(uv - \pi) \leq 1$. Moreover, this height is exactly one since R_P ia a domain and $uv - \pi \neq 0$. Now, $R_P[u,v]/(uv - \pi) \cong R_P[t^{-1}, \pi t]$ implies that $\dim(R_P[u,v]/(uv - \pi)) = \dim(R_P[t^{-1}, \pi t]) = 2$ since $R_P[t^{-1}] \subseteq R_P[t^{-1}, \pi t] \subseteq R_P[t, t^{-1}]$ and R_P is a Jaffard domain with $\dim(R_P) = 1$. Now, since $P \subsetneq \mathcal{M}$, $(t^{-1}, \pi t)B \subset \mathcal{N}$ is a saturated chain of primes in B. Thus, $\text{ht}(\mathcal{N}) \geq 3$. Next, $R[t^{-1}] \subseteq B \subseteq R[t, t^{-1}]$ so that $\dim(B) \leq \dim_v(B) \leq \dim_v(R[t^{-1}]) = \dim_v(R[t, t^{-1}]) = 3$ and therefore $\text{ht}(\mathcal{N}) = 3$. $B_{\mathcal{N}}$ is the required ring. In order to conclude the proof, we need to show that $\mathcal{N}$ is the radical of a 2-generated ideal. Set $I = (t^{-1})B$, a homogeneous ideal. We have $A = B/(t^{-1})B = R/P \oplus P/P^{(2)} \oplus \ldots \oplus P^{(n)}/P^{(n+1)} \oplus \ldots$ is graded ($A_n = B_n/(I)_n = B_n/B_n \cap I = P^{(n)}t^n/P^{(n+1)}t^n$). Let $K = R_P/PR_P$ be the quotient field of R/P. Then $A_S = B_S/IB_S = (R_P[u,v]/(uv-\pi))/((u, uv-\pi)/(uv-\pi)) \cong R_P[u,v]/(u, uv-\pi) = R_P[u,v]/(u,\pi) \cong R_P[v]/(\pi)[v] = R_P[v]/PR_P[v] = K[v]$ since $(u,v) = (u, uv-\pi)$. Thus, we have $(R/P)[v] \subset A \subset K[v]$. But then $A_n = K < v^n > \cap A$; so that $A = \oplus F_n v^n$, where $F_n = \{x \in K |\ xv^n \in A\} = (A : v^n)$ is a fractional ideal of R/P. Let $\overline{x} = x + P$ generate the maximal ideal of R/P, for some $x \in \mathcal{M}$. $\overline{a}F_n = (\overline{x}^{s_i})$ for some integer s_i. If $\overline{a}$ is a unit in R/P, then $F_n = (\overline{x}^{s_i})$; if not $F_n = (\overline{x}^{s_i - s_j})$. Let $F_n = (\overline{x}^{-d_n})$. Then the F_n are all comparable (precisely they form an increasing sequence with respect to inclusion) and the sequence $\{d_n\}$ is non-decreasing. We will show that the maximal ideal of A, namely $\mathcal{N}/(t^{-1})$ is the radical of $\overline{x}A$. It will therefore follow that $\mathcal{N} = \mathrm{rad}_B(t^{-1}, x)$, as desired. It suffices to show that $F_n v^n \subset \mathrm{rad}(\overline{x})$ for each $n \geq 1$, for as A is an $\mathbb{N}-$graded ring, $(\overline{x}) \bigoplus_{n>0} F_n v^n \subseteq \mathrm{rad}(\overline{x})$ is a maximal homogeneous ideal and therefore $\mathrm{rad}(\overline{x}A) = (\overline{x})R/P \bigoplus F_n v^n$. Let $F_{nm}v^{nm} = (\overline{x}^{-d_{nm}})v^{nm}$ and $F_n v^n = (\overline{x}^{-d_n})v^n$, then $\overline{x}^{-1}(F_n v^n)^m = (\overline{x}^{-md_n - 1})v^{nm}$. We claim that there exists $m \geq 1$ such that $\overline{x}^{-1}(F_n v^n)^m \subset F_{nm}v^{nm}$. Notice that our claim is equivalent to $\exists\ m \geq 1$ such that $-md_n - 1 \geq -d_{nm}$. Indeed, if this is not the case, then for each $m \geq 1$,

$$md_n + 1 > d_{nm} \iff md_n \geq d_{nm} \iff F_{nm} \subseteq (F_n)^m \subseteq (F_{nm}).$$

Thus $(F_n)^m = F_{nm}$. For $n = 1$, $F_m = (F_1)^m$, $\forall\ m \geq 1$, so that $A = (R/P)[F_1 v]$. For $n = 2$, $F_{2m} = (F_2)^m$, $\forall\, m \geq 1$. For $m < k < 2m$, $2m - k > m$ and we have $F_{2m-k} \subseteq F_{2m} \subseteq (F_2)^m$ and $v^{2m-k} = (v^m)^q v^r$ with $0 \leq r < m$. For $2m < k < 2(m+1)$, we have $k = (2m) + r$ with $r < m$ so that $F_k \subseteq F_{2(m+1)} = (F_2)^{m+1}$ and $v^k = (v^2)^m v^r$. In all cases, $F_k v^k \subset A = (R/P)[F_1 v, F_2 v^2]$. The same calculation applies to any integer n to give $A = (R/P)[\{F_i v^i : 1 \leq i \leq n\}]$. But R/p is Noetherian so that A is Noetherian, a contradiction.

5 Chapter 5

The nth Symbolic Power of a Prime Ideal

Throughout this paragraph, R is a Noetherian domain and P is a proper prime ideal. We will give some conditions under which the equality $P^{(n)} = P^n$, where $P^{(n)} = P^n R_P \cap R$, holds for all integer n in R. As an application, we will show that this is always the case if the prime ideal P is generated by an $R-$sequence.

5.1 Criteria for equality of ordinary and symbolic power

If $Q \in \operatorname{Spec}(R)$ such that $P^n \subset Q$, then $P \subset Q$; so that P is the minimal prime ideal of P^n. This shows that the $P-$primary component of P^n is the same in all minimal primary decomposition of P^n. If $P^n = q_1 \cap q_2 \cap \ldots q_r$ is a primary decomposition of P^n, where q_i is p_i-primary, then since P is the minimal prime ideal of P^n, localizing at $S = R \setminus P$ and contracting back to R implies that $P^{(n)}$ is the $P-$primary component in the primary decomposition of P^n. As a first criteria we get:

$P^{(n)} = P^n$ if and only if P^n is P- primary. The following example shows that P^n is not always $P-$primary.

Example 5.1 ([35, Example 3]) . Let K be a field and let

$P = \{f(x,y,z) \in K[x,y,z] \mid f(t^3,t^4,t^5) = 0\}$. Let

φ: $K[x,y,z] \longrightarrow K[t]$ defined by $f(x,y,z) = f(t^3,t^4,t^5)$. Then $K[x,y,z]/P \cong$

$\varphi(K[x,y,z]) \subset K[t]$ and P is prime (since $K[t]$ is an integral domain). Let $f_1 = y^2 - xz$; $f_2 = yz - x^3$; $f_3 = z^2 - x^2y$.

Claim $P = (f_1, f_2, f_3)$. $\forall\ f \in K[x,y,z]$, $f = g \bmod (f_1, f_2, f_3)$, where $g = x^2A(z) + xyB(z) + xC(z) + yD(z) + E(z) \in K[y,z][x]$. Indeed, consider f as an element of $T[x]$, $T = K[y,z]$. Then $f_2 \in T[x]$ and the Euclidean division gives $f = (x^3 - yz)g + r$, with $r = 0$ or $\deg_x r \leq 2$. So if $r \neq 0$, we can write $r = A_2(y,z)x^2 + A_1(y,z)x + A_0(y,z)$ with $A_i(y,z) \in K[y,z]$. Similarly, $f_1 \in B[y]$, $B = K[x,z]$ and the Euclidean division gives $A_i(y,z) = (y^2 - xz)g_i + r_i(x,z)$ with $\deg_y(r_i) \leq 1$ in $K[x,z][y]$ and $\deg_x(r_i) \leq 1$. Thus $r = (x^2g_2 + xg_1 + g_0)f_1 + r_2x^2 + r_1x + r_0$. We have $r_i = (a_i(z)x + b_i(z))y + c_i(z)$ and $r = (a_2(z)x + b_2(z))yx^2 + c_2(z)x^2 + (a_1(z)x + b_1(z))yx + c_1(z)x + (a_0(z)x + b_0(z))y + c_0(z)$. Using either of $x^2ya_i(z) = a_i(z)z^2 - f_3a_i(z)$, $x^3a_i(z) = yza_i(z) - f_2a_i(z)$, $x^2yb_i(z) = b_i(z)z^2 - f_3b_i(z)$ or $x^3b_i(z) = yzb_i(z) - f_2b_i(z)$, we get the desired result. Now, if $f \in P$ then $f(t^3, t^4, t^5) = 0$, i.e., $t^6A(t^5) + t^7B(t^5) + t^3C(t^5) + t^4D(t^5) + E(t^5) = 0$ and this new polynomial in t is zero only if all its coefficients are zero. Thus $A = \ldots = E = 0$. The reverse inclusion is trivial; so that $f \in (f_1, f_2, f_3)$. Now, $f_2^2 - f_1f_2 = x(x^5 - 3x^2yz + xy^3 + z^3) \in P^2$. But $x^n \notin P\ \forall\, n$. Thus, if P^2 is $P-$primary, then $(x^5 - 3x^2yz + xy^3 + z^3)$ would be an element of P. But no element in P^2 has degree less than 4, i.e., $P^{(2)} \neq P^2$.

Let $S = R[x_1, \ldots, x_k]$, where x_i are algebraically independents over R and let $P = (p_1, \ldots, p_k)$ be a prime ideal of R. Set $J = \bigcup J_n(v)$ where v denotes the $k-$tuple $p_1, \ldots, p_k$ and $J_n(v)$ the increasing sequence of ideals of S defined recursively as follows : $J_0(v) = (0)$ and $J_{n+1}(v) = (\sum_{i=1}^k s_i x_i \in S / \sum_{i=1}^k s_i p_i \in J_n(v))$.

Proposition 5.2 ([22, Theorem 1]) . *The following conditions on a prime ideal $P = \sum_{i=1}^k p_i R$ in a Noetherian domain R are equivalent:*

(i) $P^n = P^{(n)}$ $\forall n \in \mathbb{N}$ *and the associated graded ring of R_P is a domain;*

(ii) *$PS+J$ is prime;*

(iii) *For some integer $n \geq 0$, $PS + J_n$ is a prime of height k. In this case $PS + J_n = PS + J$;*

(iv) *There is a rank k prime Q of S such that $Q \subset PS + J$. In this case, $Q = PS + J$;*

(v) *z is a prime element in the subring $R[z, p_1/z, \ldots, p_k/z]$ of $R[z, 1/z]$.*

Proof. If $\sum s_i x_i \in J_1$, then $\sum s_i p_i = 0 \in J_1$; so that $\sum s_i x_i \in J_2$. If $\sum s_i x_i \in J_2$, then $\sum s_i p_i \in J_1 \subset J_2$ and $J_2 \subset J_3$. Recursively, we show that $J_n \subset J_{n+1}$. We notice that every element of $R[z, 1/z]$, is uniquely expressible as a polynomial including possibly both positive and negative powers of z with coefficients in R. If $f = \sum r_j z^j$, then $f \in R[z, P/z]$ if and only if for each $j < 0, r_j \in P^{-j}$; so $f \in zR[z, P/z]$ if and only if for $j \leq 0$, $r_j \in P^{-j+1}$.

(i) $\Longrightarrow$(v) Given $x \in R_P$, we denote by $v(x)$ the largest integer n such that $x \in P^n R_P$ and for $x \in \bigcap_n P^n R_P = \{0\}, v(0) = +\infty$. If x is a unit in R_P, then $v(x) = 0$. We define $G(x)$ to be the residue class of x in $P^{v(x)} R_P / P^{v(x)+1} R_P$

and $G(0) = 0$. $xy \in P^{v(x)+v(y)+1}R_P$ implies that $v(xy) > v(x) + v(y)$, which is equivalent to $G(x)G(y) = 0$. Indeed,
$(P^{v(x)}R_P/P^{v(x)+1}R_P)(P^{v(y)}R_P/P^{v(y)+1}R_P) \subset$
$P^{v(x)+v(y)}R_P/P^{v(x)+v(y)+1}R_P$. Now, let $f, g \notin zR[z, P/z]$, i.e., $\exists\ m, n$, with $m, n \leq 0$ such that $r_f \notin P^{-m+1} = P^{(-m+1)}, r_g \notin P^{-n+1}$, for some coefficients r_f and r_g of f and g. We may assume that we have subtract off the monomials rz^j of f, g which are in $zR[z, P/z]$ and therefore assume that $r_f z^n, r_g z^m$ are the lowest degree terms. Therefore, $r_f r_g z^{m+n}$ will be the lowest degree term in fg. To show that $fg \notin zR[z, P/z]$, we will show that $r_f r_g \notin P^{-m-n+1}R_P$. But $r_f \notin P^{-m+1}R_P$ implies that $r_f \notin \bigcap_n P^n R_P = \{0\}$, since R_P is local. Then $G(r_f) \neq 0$. Similarly, $G(r_g) \neq 0$ and $G(r_f)G(r_g) \neq 0$; so that $r_f r_g \notin P^{-m-n+1}R_P$.

(v) $\Longrightarrow$(i) $\forall\ n \in \mathbb{N}, P^n \subset P^{(n)}$. Let $q \in P^{(n)} \backslash P^n$. Since $q \notin P^n$, then we can find an integer $t < n$ and an appropriate $a \in R \backslash P$ such that $q \in P^t \backslash P^{t+1}$ and $aq \in P^n$. Then $\frac{q}{z^t} \in R[z, P/z] \backslash zR[z, P/z]$ since $q \in P^t \backslash P^{t+1}$. Also $a \notin zR[z, P/z]$. But $\frac{aq}{z^t} \in zR[z, P/z]$ and this is impossible by the assumption that $zR[z, P/z]$ is prime. So $P^n = P^{(n)} \forall\ n$. If the associated graded ring of R_P is not a domain, then $R_P / \cap P^n R_P$ is not a domain by [44, Theorem 1]. Therefore there is $a \in P^m R_P \setminus P^{m+1}R_P$ and $b \in P^n R_P \setminus P^{n+1}R_P$, such that $ab \in P^{m+n+1}R_P$. By suitable multiplication we can choose our elements in R. Thus, $ab \in P^{m+n+1}$. Then $\frac{ab}{z^{m+n}} \in zR[z, P/z]$, but $\frac{a}{z^n}$ and $\frac{b}{z^m} \notin zR[z, P/z]$

which is assumed to be prime. Thus, a contradiction.

Let Φ be the $R[z]-$ homomorphism $T = R[z][x_1, \ldots, x_k] \overset{\varphi}{\twoheadrightarrow} R[z][\frac{P}{z}]$; $x_i \mapsto p_i/z$ and denote by $I = \ker \Phi$. Then Φ induces a homomorphism $S \cong S[z]/zS[z] \overset{\overline{\varphi}}{\twoheadrightarrow} R[z, \frac{P}{z}]/zR[z, \frac{P}{z}]; \overline{x_i} \mapsto \overline{\varphi(x_i)}$. Then $J^\star = \ker(\overline{\varphi}) = (I + zT) \bigcap S$. Indeed, let $y \in S$ be an element of $\ker(\overline{\varphi})$. Then $\overline{\varphi(y)} = 0$, i.e., $y \in \ker(\varphi) = I$ or $\varphi(y) \in zR[z, \frac{P}{z}]$. Then $\varphi(y) = z\varphi(y_1) = \varphi(zy_1)$ with $y_1 \in T$. So $y - zy_1 \in I$. In both cases $y \in I + zT$; and since $y \in S$, we get the desired equality.

(v) $\Longleftrightarrow J^\star$ is prime. Indeed, $S/J^\star \cong R[z, \frac{P}{z}]/zR[z, \frac{P}{z}]$.

Claim $J^\star = PS + J$. $\forall\ n \in \mathbb{N}$ let $I_n = (\sum_{i=1}^k (p_i - x_i z)T \colon (z^n T)) = (\sum_{i=1}^k (p_i - x_i z)T \colon (z^{n-1}T)) \colon zT) =$
$(I_{n-1} \colon zT)$ and set $I_\infty = \bigcup_{n \geq 0} I_n$, the stable value of I_n. Thus, $I_0 = (p_1 - x_1 z, \ldots, p_k - x_k z)T$. Let ψ be the $R[z, \frac{1}{z}]-$ homomorphism $T[\frac{1}{z}] \longrightarrow R[z, \frac{1}{z}]$; $x_i \mapsto p_i/z$. Then $\ker(\psi) = I_0 T[\frac{1}{z}]$ and $I = I_\infty$. Indeed, it is trivial that $I \subseteq I_\infty$. If $x \in I_n$, then $xzT \subset I_{n-1}$, i.e., $xz^2 \subset I_{n-2}$, $xz^n T \subset I_0$ or $xT \subset \frac{I_0}{z^n}$. $\forall\ 0 \neq t \in T$, $\psi(xt) = \psi(x)\psi(t) = 0$ and since R is a domain, we must have $\psi(x) = 0$. Now, $\forall\ n \in \mathbb{N}, I_n = I_0 + J_n T$ and $(I_n + zT) \bigcap S = PS + J_n$. It follows that $I = I_0 + JT$ and $J^\star = PS + J$. Notice that if this result is admitted, then $(I_n + zT) \bigcap S = I_0 \bigcap S + (J_n T) \bigcap S + zT \bigcap S = PS + J_n$ and $I = I_\infty = \bigcup I_n = I_0 + \bigcup (J_n)T = I_0 + JT$. Therefore $J^\star = (I + zT) \bigcap S = (I_0 + JT + zT) \bigcap S = PS + J$. Let us prove the equality $I_n = I_0 + J_n T\ \forall\ n \in \mathbb{N}$. We proceed by induction on n. The

case where $n = 0$ is an obvious. Assume that the result is true for n and let us show that $I_{n+1} = I_0 + J_{n+1}T$. Let $y \in I_0 + J_{n+1}T$, $y = i_0 + j_{n+1}$ with $j_{n+1} = \sum_{j=1}^{r} \lambda_j \sum_{i=1}^{k} s_{ij}x_i, \sum_i^k s_{ij}p_i \in J_n$ and $\lambda_j \in T$. Then $\sum_{i\geq 0} s_{ij}(p_i - x_i z) \in I_0$ implies that $z(\sum_i x_i s_{ij}) \in \sum_{i\geq 0} s_{ij}p_i + I_0 \subset I_0 + J_nT = I_n$. So $z(\sum_{i\geq 0} s_{ij}x_i) \in I_n$ implies that $\sum_{i\geq 0} x_i s_{ij} \in I_{n+1}$. Thus, $y \in I_{n+1}$ since $I_0 \subset I_{n+1}$. On the other hand, suppose that $f \in I_{n+1}$, then $zf \in I_n = I_0 + J_nT$ and we can write it as $zf = \sum_i (g_i z + s_i)(p_i - x_i z) + j$, where $j = \sum_i \overline{j}_i z^i \in J_n[z]$; each $\overline{j}_i \in J_n$, $s_i \in S$ and $g_i \in S[z]$. $zf = z\sum_i g_i(p_i - x_i z) + \sum_i s_i p_i + \overline{j}_0 - \sum_i s_i x_i z + \sum_i \overline{j}_i z^i$. The constant term on the right must be therefore equal to 0, i.e., $\sum_i s_i p_i = -\overline{j}_0 \in J_n$; so that $\sum_i s_i x_i \in J_{n+1}$. Then $f = \sum_i g_i(p_i - x_i z) - \sum_i s_i x_i + \sum_{i\geq 1} \overline{j}_i z^{i-1} \in I_0 + J_{n+1}T$. This proves (ii) $\Longleftrightarrow$(v) .

Now, we show that any minimal prime of $J^\star = PS + J$ has rank k. This shows that $\mathrm{ht}(J^\star) = k$ since it is prime. (ii) implies (iii) since $J_0^\star \subset J_1^\star \subset \ldots \subset J_n^\star = J_{n+1}^\star = J_{n+2}^\star = \ldots$. Necessary $J_n^\star = PS + J_n = J^\star$. (iv) is obtained by putting $Q = J_n^\star$. (iv) implies (v) since $Q = PS + J = J^\star$ and $S/J^\star \cong \frac{R[z,P/z]}{zR[z,P/z]}$. Let P_1 be a minimal prime of $J^\star$ and put $\mathrm{ht}(P_1) = r$. Then $\mathrm{ht}(P_1S[z] + zS[z]) \geq r + 1$ because $T/P_1S[z] + zS[z] \cong S/P_1$; so that $P_1S[z] + zS[z]$ is prime. $T/P_1S[z] = \frac{S[z]}{P_1S[z]} \cong (S/P_1)[z]$ which is integral . Thus, $P_1S[z]$ is prime. By [28, Theorem 37], $\mathrm{ht}(P_1S[z] + zS[z]) = r + 1$. It is minimal over $J^\star S[z] + zS[z] = ((I + zT)\bigcap S)T + zS[z] = I + zT$. But, $\mathrm{ht}(I) = k$. Indeed, $I\bigcap(S = \{1, z^{-1}, z^{-2}, \ldots\}) = \emptyset$. Thus, $\mathrm{ht}_T(I) = \mathrm{ht}_{T_S}(I_S) =$

$\text{ht}(IS[z, \frac{1}{z}]) = \text{ht}(I_0T[1/z]) = k.$ □

Example 5.3 ([22, Application and Example]) . If $P = (a_1, a_2, \dots, a_k)$ such that $a_1, \dots, a_k$ is an $R-$sequence and if P is prime, then $P^{(n)} = P^n$ for all n. Indeed, in this case $PS + J = PS$ is prime (since $J_n(v) = 0$ for all n and the extension of a prime in the polynomial ring is also prime) and the result follows from (ii) $\Longrightarrow$(i) .

Proposition 5.4 ⋆ *Let R be an integral domain.*

If $P \in \text{Spec}(R)$ is an invertible ideal, then $P^{(n)} = P^n$ for all integer n.

Proof. First, notice that if M, N are $R-$module and $M \xrightarrow{f} N$ is an $R-$ isomorphism, then $S^{-1}M \xrightarrow{S^{-1}f} S^{-1}N$ is also an isomorphism for any multiplicatively closed subset S of R. So it is sufficient to show the equality in R_M, where M is a maximal ideal of R. But P invertible implies that PR_M is invertible. Moreover PR_M is principal in R_M. Therefore without loss of generality, we may assume that R is local and P is principal, say $P = (x)R$. In this case x is a prime element in R. Now, let $z \in P^{(n)}$. $z = \lambda x^m$, with $m \geq n$ and $\lambda = \frac{a}{s} \in R_P$, i.e., $zs = ax^m$ and therefore $z = z_1x$ (since x does not divide s). This reduces to $z_1s = ax^{m-1}$ and inductively we see that $z \in (x^n) = P$, as desired. □

6 Chapter 6

Finite-Dimensional non-Noetherian Krull Domains I: Rees-Eakin-Heinzer's construction

If K is a field and $\{X_i\}_{i=1}^{\infty}$ a collection of indeterminates over K, then $K[\{X_i\}_{i=1}^{\infty}]$ is a Krull ring. Thus, Krull rings need be neither finite-dimensional nor

Noetherian. Our purpose in this section is to give a method to construct finite-dimensional, non-Noetherian Krull rings. An analysis of such method reveals conditions under which the rings may be Noetherian. We make use of these conditions to give an example of a non-Noetherian, two-dimensional Krull ring.

6.1 A non-Noetherian three-dimensional symbolic Rees algebra

Let R be a Noetherian Krull domain, P a height 1 prime ideal of R and v the valuation, with value group $\mathbb{Z}$, associated with P. Let t be an indeterminate over R. Let $B = \{\sum_{i=-p}^{q} c_i t^i \mid c_i \in R \text{ and if } i > 0, \text{ then } v(c_i) \geq i\}$. If $z = c_i t^i \in B,\ i > 0$, then $v(c_i) = l \geq i > 0$. Let $PR_P = (x)$ with $v(x) = 1$. Then $c_i \in (PR_P)^l \cap R = P^{(l)}$, so that B is a graded ring with $P^{(i)}t^i$ as the set of homogeneous elements of degree i for $i > 0$ and Rt^i the set of homogeneous elements of $\deg i$ if $i \leq 0$. Then $B = R[Pt, P^{(2)}t^2, \ldots, t^{-1}]$ is a Krull ring. Consider the ring $A' = R[Pt, t^{-1}]$, where $Pt = \{pt \mid p \in P\}$. By Hilbert basis theorem, A' is Noetherian, so that its integral closure A is a Krull ring. Let $p_1, p_2, \ldots, p_r$ be the finite set of height one prime ideals in A for which $v_{P_j}(t^{-1}) > 0$. Now, let $v^\star$ be the valuation on the quotient field of A defined by setting $v^\star(\sum c_i t^i) = \inf\{v(c_i) - i\}$ for any element $\sum c_i t^i$ of B since $A' \subset B$. Let $R^\star$ denote the valuation ring of $v^\star$. Then $R^\star$ is a Krull ring. To show this, we will show that the maximal ideal of $R^\star$ has height one and thus $R^\star$ will be a DVR. Let $m^\star$ be the maximal ideal of R. Then $T = R_P[Pt, t^{-1}] \subseteq A_{m^\star}$. Set $P' = m^\star A_{m^\star} \cap T$. Then $T_{P'} \subseteq A_{m^\star}$. Since T is a Noetherian ring, if we show that P' is a minimal prime ideal, then it will follow that $A_{m^\star}$ is an overring of a one-dimensional Noetherian ring; thus a one-dimensional Noetherian ring itself. Since T is Noetherian, it verifies PIT; so that it is sufficient to show that P' is principal. $v^\star(t^{-1}) = 1 > 0$ implies that $t^{-1}T \subseteq P'$. Next, if $x = \sum a_i t^i \in P'$, then $v^\star(a_i t^i) > 0$ for each i. Hence $a_i t^i \in P'$. But $v^\star(a_i t^i) = v(a_i) - i$ implies $a_i t^i = a_t^{v(a_i)} t^{i - v(a_i)} \in t^{-1}T$. Thus $P' = t^{-1}T$, as desired. $R[t, t^{-1}] = R[t]_S$, where $S = \{t^k, k \in \mathbb{N}\}$, is a Krull ring, since R is a Krull ring. By [40], $B = R^\star \cap R[t, t^{-1}]$ is a Krull ring. Let v' denotes the canonical extension of the valuation v to $R[t^{-1}]$ (i.e., $v'(\sum_{i=0}^{r} c_i t^i) = \inf\{v(c_i)\}$). Let Q denote the center of the essential valuation v' on B, i.e., $Q = B \cap Q'$, where Q' is the maximal ideal of the valuation ring of v'.

Lemma 6.1 ([14, Lemma 2.3]) . *Let m be a positive integer such that $m! \geq m^3$. If s is a positive integer such that $s \geq 2(m!)$ and if $s = s_1 + 2s_2 + \ldots + ms_m$*

where each s_i is a nonnegative integer, then $m! = t_1 + 2t_2 + \ldots + mt_m$, where each t_i is an integer such that $0 \leq t_i \leq s_i$.

Proposition 6.2 ([14, Theorem 2.2]) . *The following statements are equivalent:*

(i) *There exists a positive integer m such that for each integer $s > m$:*
$$P^{(s)} = \sum_{\substack{n_1+n_2+\ldots+n_t=s \\ 1\leq n_i\leq m}} P^{(n_1)}P^{(n_2)}\ldots P^{(n_t)}$$
(ii) *There exists a positive integer k such that for $n \geq 1$*
$P^{(nk+l)} = ((P^{(k)})^{n-1})(P^{(k+l)})$ *with $0 \leq l \leq k$.*

(iii) *B is a finite ring extension of R.*

(iv) *B is Noetherian.*

(v) *The minimal prime ideal Q of B has a finitely generated primary ideal.*

Proof. (i) $\Longrightarrow$ (ii) Note that $s_1 + 2s_2 + \ldots + ms_m = \underbrace{1 + 1 + \ldots + 1}_{s_1 times} + \underbrace{2 + 2 + \ldots + 2}_{s_2 times} + \ldots + \underbrace{m + m + \ldots + m}_{s_m times}$, so that $P^{(s)} = \sum T_\alpha$, where each T_α has the form

$(P^{(1)})^{s_1}(P^{(2)})^{s_2}\ldots(P^{(m)})^{s_m}$ with $s = s_1 + 2s_2 + \ldots + ms_m$. Let $k = m!$, where m is an integer such that $m! > m^3$. Then by Lemma 6.1, for any $s \geq 2(m!)$, $s = \sum_{i=1}^m is_i$ we have $m! = \sum_{i=1}^m it_i$, $1 \leq t_i \leq s_i$. Thus, $s_i = t_i + l_i$, for some integer l_i, and

$T_\alpha = (P^{(1)})^{t_1}(P^{(2)})^{t_2}\ldots(P^{(m)})^{t_m}(P^{(1)})^{s_1-t_1}\ldots(P^{(m)})^{s_m-t_m}$. Since $k = m! > m$, by (i) , $P^{(k)} = \sum_{n_1+\ldots+n_u=k} P^{(n_1)}\ldots P^{(n_u)}$; which contains $(P^{(1)})^{t_1}\dots(P^{(m)})^{t_m}$. Similarly, since $s-k>m$, we have $P^{(s-k)} = \sum_{n_1+\dots+n_v=s-k} P^{(n_1)}\dots P^{(n_v)}$; which contains $(P^{(1)})^{s_1-t_1}\dots(P^{(m)})^{s_m-t_m}$. Thus, $T_\alpha \subseteq P^{(k)}P^{(s-k)}$ and $P^{(s)} = P^{(k)}P^{(s-k)}$ since the reverse inclusion is trivial. If $(s-k)\geq 2k$, we apply this conclusion to $P^{(s-k)}$ to get $P^{(s-k)} = P^{(k)}P^{(s-2k)}$ and this procedure continues till we reach an exponent less than $2k$, in which case, we will have $P^{(s)} = (P^{(k)})^u P^{(k+v)}$, where $v = s-(u+1)k$, (ii) then follows.

(ii) $\Longrightarrow$ (iii) For any $m>k$, $m = nk+l$ with $0\leq l<k$, so that $P^{(m)} = \underbrace{(P^{(k)})^{n-1}}_{\subset P^{(k)}} P^{(k+l)}$. Next, as R is Noetherian, each $P^{(i)}$ is finitely generated and therefore

$B = R[Pt, P^{(2)}t^2, \dots, P^{(2k)}t^{2k}, t^{-1}]$ is a finite extension of R.

(iii) $\Longrightarrow$ (iv) Follows from Hilbert basis Theorem.

(iv) $\Longrightarrow$ (v) Trivial since any ideal of B is finitely generated.

(v) $\Longrightarrow$ (i) Let first show that in the Krull ring B, if $\mathrm{ht}(Q)=1$, then the set $\{Q^{(n)}\}_{n=1}^{\infty}$ of symbolic powers of Q is precisely the set of $Q-$primary ideals. It is clear that for any integer n, $Q^{(n)}$ is $Q-$primary. On the other hand, let P' be a $Q-$primary ideal of B. Then $P'\subset Q$, so that $P'B_Q \subset QB_Q$. Since B_Q is a DVR, there exists an integer k such that $P'B_Q = (\xi)^k = Q^kB_Q$, where $QB_Q = (\xi)$. Contracting back to B, we get $P' \subseteq P'B_Q\cap B = Q^{(k)}$. Let $z\in P'B_Q\cap B$, $z = \frac{x}{s}$, where $x\in P'$ and $s\notin Q$. Then $zs\in P'$ which is $Q-$primary. Since $P'\subset Q$, then $s^n\notin P'$ for all n, so that $z\in P'$. It follows that $P' = P'B_Q \cap B = Q^{(k)}$, as desired. Note that if $A \twoheadrightarrow B$ is a surjective homomorphism of rings and I an ideal of A, then I has a primary decomposition if and only if I^e has a primary decomposition. Now, since $B \longrightarrow B_Q$ is surjective and B_Q is Noetherian it follows that Q^n has a primary decomposition. Next, since Q is a minimal prime ideal it is homogeneous, so that Q^n is also homogeneous. Thus all the primary components of Q^n can be chosen to be homogeneous and therefore $Q^{(n)}$ is homogeneous. Let z be an element of degree m in $Q^{(n)}$. Since z is an element of degree m in B, then $z = ct^m$, where $c \in P^{(m)}$. By (v) , if $Q^{(n)}$ is finitely generated, then it has a finite basis formed by homogeneous elements, say $c_1t^{n_1}, c_2t^{n_2}, \ldots, c_rt^{n_r}$. Let $m = \max\{n_1, n_2, \ldots, n_r, n\}$. For $s > m$, $P^{(s)}t^s$ is the set of homogeneous elements of degree s in $Q^{(n)}$, so that $P^{(s)}t^s \subset Q^{(n)} = \sum_{i=1}^r c_it^{n_i}B$ and thus $P^{(s)}t^s \subseteq \sum_{i=1}^r c_it^{n_i}P^{(s-n_i)}t^{s-n_i}$, $c_i \in P^{(n_i)}$. Hence, $P^{(s)} \subseteq \displaystyle\sum_{i+j=s} P^{(i)}P^{(j)}$ and $1 \leq i \leq m$ for each i. If $j \geq m$, then we apply the same argument to $P^{(j)}$ to get $P^{(j)} \subseteq \displaystyle\sum_{u+v=j} P^{(u)}P^{(v)}$ with $1 \leq u \leq m$. We continue to iterate this argument till we reach $P^{(s)} \subseteq \displaystyle\sum_{n_1+n_2+\ldots+n_t=s} P^{(n_1)} \ldots P^{(n_r)}$. The reverse inclusion is trivial if we observe that for any i, j, $P^{(i)}P^{(j)} = [(PR_P)^i]^c[(PR_P)^j]^c \subset [(PR_P)^i(PR_P^j)]^c = [(PR_P)^{i+j}]^c$. $\square$

6.2 A non-Noetherian two-dimensional Krull ring

Let k be a field of characteristic p. Let K be an extension of k, $x_1, x_2, \ldots, x_n \in K$ and B a subset of K. We say that $x_1, x_2, \ldots, x_n$ are $p-$*independent* over k if $[K^p(k, x_1, \ldots, x_n) : K^p(k)] = p^n$, and B is $p-$independent over k if any finite subset of B is $p-$independent. If, in addition, $K = K^p(k, B)$, then B is called a $p-$basis of the extension K/k.

Let K be field of characteristic 2 with a countably infinite two-basis $\{p_j\}_{j=0}^{\infty}$ over its subfield k where $K^2 \subset k$. Partition $\{p_j\}$ into two disjoint collections $\{b_j\}_{j=0}^{\infty}$ and $\{c_j\}_{j=0}^{\infty}$. Now, inductively define the fields K_i by $K_0 = k$ and $K_{i+1} = K_i(b_i, c_i)$. Then $K_i \subset K_{i+1}$ and $[K_{i+1} : K_i] = 4$. Moreover, if $x \in K_i$, then $x^2 \in k$.

Example 6.3 ([14, Example 3.1]) .

Let x and y be transcendental over K and denote by $R^\star$ the power series ring $K[[x, y]]$. Denote the subring $K_0[[x, y]]$ by R and the element $b_i x + c_i y$ by d_i. Then $T = R[\{d_i\}]$ is a non-Noetherian, local two-dimensional Krull ring.

First, we observe that $R^\star$ is integral over T. Indeed, if $x \in R^\star$ then the coefficients of x^2 are in $K^2 \subset k$; so that x is zero of $F(Z) = Z^2 - x^2 \in T[Z]$. Then $\dim(T) = \dim(R^\star) = 2$. By integrality, T is local since any maximal ideal of T must the contraction of some maximal of $R^\star$. Next, $R^\star$ is a UFD, so that it is Krull. Since $\mathrm{qf}(T) \subset \mathrm{qf}(R^\star)$, if we show that T is integrally closed, then we will have $T = R^\star \cap \mathrm{qf}(T)$, and therefore a Krull ring.

Lemma 6.4 ([14, Lemma 3.2]) . *The ring $K_i[[x, y]][d_i]$ is integrally closed.*

Proof. Let $A_i = K_i[[x, y]]$ and F_i the quotient field of A_i. Each A_i is integrally closed since it is a power series over a field. If $z \in A_{i+1}$, then $z = \sum \alpha_{i_1 i_2} x^{i_1} y^{i_2}$, where $\alpha_{i_1 i_2} \in K_{i+1}$. $1, b_i, c_i$ and $b_i c_i$ form a free module basis for A_{i+1} over A_i. Suppose that a is an element of $F_i(d_i)$ which is integral over $A_i[d_i]$. Since $A_i[d_i] \subset A_{i+1}$ and A_{i+1} is integrally closed, it must be true that $a \in A_{i+1}$. Further, $d_i \in A_i \subset F_i$, so d_i is algebraic over F_i. Therefore $F[d_i] = F(d_i)$ and $[F(d_i) : F_i] = 2$ since $A_i \subset k[[x, y]] = R \subset R^\star$. Thus $1, d_i$ is a basis of $F(d_i)$ over F_i and a has two representations:

$a = \frac{g_1 + g_2(d_i)}{g_3}(a = \lambda_1 + \lambda_2 d_i,\ \lambda_i \in F_i, i.e., \lambda_i = \frac{u_i}{v_i}$ with $u_i, v_i \in K_i[[x, y]])$; and $a = g_4 + b_i g_5 + c_i b_i g_6 + c_i g_7$, where each $g_i \in A_i$. Since $1, b_i, c_i b_i$ and c_i are linearly independent over A_i, we can equate coefficients and get:

$g_1 = g_3 g_4, x g_2 = g_3 g_5$ and $y g_2 = g_3 g_7$. Thus, $g_1/g_3 \in A_i$ and $g_3 g_5 g_7 = x g_2 g_7 = y g_2 g_5$, i.e., $x g_7 = y g_5$ in a UFD. This implies that $g_5 \in (x)$; so that $g_2/g_3 = g_5/x \in A_i$. Hence $a \in A_i[d_i]$. □

Let $T_0 = R$ and define inductively $T_{i+1} = T_i[d_i]$. Then each T_i is integrally closed. Notice that $T = \cup_{i=1}^{\infty} T_i$. Indeed, if $i = 0$, the result is trivial. Assume that T_n is integrally closed and let us show that T_{n+1} is integrally closed.

For each i, $T_i \subset K_i[[x,y]]$ so that $T_{n+1} \subset K_n[[x,y]][d_n]$ which is integrally closed by Lemma 6.4. Suppose that ξ is an element of the quotient field of T_{n+1} which is integral over T_{n+1}. $\xi \in \mathrm{qf}(T_{n+1}) = \mathrm{qf}(T_n[d_n]) = \mathrm{qf}(T_n)(d_n)$ so that $\xi = \lambda_1 + \lambda_2 d_n$ with $\lambda_i \in \mathrm{qf}(T_n)$. Thus $\lambda_i = \frac{u_i}{v_i}$, $u_i, v_i \in T_n \subset K_n[[x,y]]$. Therefore $\xi = \frac{f_{n+1}+f'_{n+1}d_n}{g_{n+1}}$, where $f_{n+1}, f'_{n+1}, g_{n+1} \in K_n[[x,y]]$ and f_{n+1}/g_{n+1} and f'_{n+1}/g_{n+1} are in $\mathrm{qf}(T_n)$. $\xi \in K_n[[x,y]][d_n]$ since $\mathrm{qf}(T_{n+1}) \subset \mathrm{qf}(K_n[[x,y]][d_n])$, i.e., $\frac{f_{n+1}}{g_{n+1}}, \frac{f'_{n+1}}{g_{n+1}} \in K_n[[x,y]]$ since $1, d_n$ form a basis over $K_n[[x,y]]$. Also $K_n[[x,y]]$ is integral over T_n. But T_n is integrally closed by assumption so that f_{n+1}/g_{n+1}, f'_{n+1}/g_{n+1} are in T_n and ξ is in T_{n+1}. Therefore $T = \bigcup T_i$ is integrally closed. To complete the proof we will prove that T is not Noetherian.

Lemma 6.5 ([14, Lemma 3.3]) . *Suppose that R and S are domains with R integrally closed and S integral over R. Let L denote the quotient field of R and suppose $\{y_i\}_{i=0}^{\infty}$ is a collection of elements of S such that $[L(\{y_i\}) : L(\{y_i\}_{i\neq j})] = [L(y_j) : L] > 0$. If $D = R[\{y_i\}]$ and B denotes the ideal of D generated by the collection $\{y_i\}$, then B has a finite basis if and only if $B = D$.*

R is integrally closed and $R^\star$ is integral over R. Since $1, d_i$ is a basis of $K[[x,y]][d_n]$ over $K[[x,y]]$, it follows that $[F_0(d_i) : F_0] = 2$. Moreover since $\{d_i\}$ is a sub-family of a two-basis, we have $[F_0(\{d_i\}) : F_0(\{d_i\}_{i\neq j})] = 2^1$. Now, if T is Noetherian, by Lemma 6.5, it would be equal to the ideal generated by the collection $\{d_i\}$. But T is local with maximal ideal containing $\sum d_i T$ since each d_i is a non-unit elements, so that the equality cannot hold.

7 Chapter 7

Finite-Dimensional non-Noetherian Krull Domains II: Roberts' construction

Hilbert's fourteenth problem, in a modern terminology, can be stated as follows: let F be a field of characteristic zero, let R denote the polynomial ring in n variables, and let K be a subfield of the field of fractions of R. Then the question is whether or not the ring $K \cap R$ is a finitely generated algebra over F. The problem was generalized by Zariski, who asked whether the conclusion was true when the condition that R be a polynomial ring was weakened to the assumption that R be an integrally closed domain which is finitely generated over F. The first counterexample on this question was given by Rees, who solved the problem using a symbolic blow-up of a prime ideal. The aim of this section is to build an example of a prime ideal in a complete local ring whose symbolic blow-up is not finitely generated.

7.1 A non finitely generated blow-up algebra

Let F be a field of characteristic zero, X, Y, Z, S, T, U, V indeterminates over F. Let $R_0 = F[X, Y, Z]$ and $m_0 = (X, Y, Z)$ a maximal ideal of R_0. Set $R = R_0[S, T, U, V]$. Then R is a graded ring with R_n, the set of homogeneous elements of degree n, formed by the monomials of degree n. Let $\hat{R}_0 = F[[X, Y, Z]]$ with maximal ideal $\hat{m}_0 = (X, Y, Z)$ and $\hat{I}$ the ideal of $\hat{R}_0$

generated by $X^{t+1}, Y^{t+1}, Z^{t+1}, X^tY^tZ^t$, where t is an integer greater than or equal to 2. Let $\hat{R} = \hat{R}_0[[S,T,U,V]]$. If $\hat{R}_i$ denote the part of degree i in S,T,U,V of $\hat{R}$, then we have $\hat{R} = \prod_{i\geq 0} \hat{R}_i$ (since any power series is an infinite sequence) and each $\hat{R}_i$ is an $\hat{R}_0-$module. Moreover $\hat{R}_1$ is free with basis S,T,U,V. For each non-negative integer n, we let $\hat{I}^n$ denote the nth power of $\hat{I}$, and let $\widehat{R(\hat{I})} = \prod_{i\geq 0} \hat{I}^n$ denote the completed Rees ring of $\hat{I}$. Then $\hat{R}_1/\hat{\mathcal{M}} \cong \hat{I}$ for some $\hat{R}_0-$submodule $\hat{\mathcal{M}}$ of $\hat{R}_1$ and this extends to an algebra homomorphism $\hat{\psi}$ from $\hat{R}$ onto $\widehat{R(\hat{I})}$. Let $\hat{P} = \ker(\hat{\psi})$. Since $\widehat{(R\hat{I})}$ is a domain and $\widehat{R(\hat{I})} \cong \hat{R}/\hat{P}$, we get that $\hat{P}$ is prime. Let $\mathcal{M}$ be the kernel of the R_0-module homomorphism $R_1 \overset{\psi}{\twoheadrightarrow} I$. We agree that, in the sequel all notations without the $\hat{}$ will refer to the non-complete case. Let $\overline{S^n(\mathcal{M})} = \{\alpha \in R_n |\ m_0^k\alpha \subseteq S^n(\mathcal{M})$ for some integer $k\}$.

Lemma 7.1 *Let F be a field and X,Y,Z,S,T,U,V be indeterminates over F. Let $R_0 = F[X,Y,Z]$,*
$I = (X^{t+1}, Y^{t+1}, Z^{t+1}, X^tY^tZ^t)R_0$ and $R_1 = R_0S + R_0T + R_0U + R_0V$. Consider the homomorphism of free R_0-modules $\varphi : R_1 \longrightarrow I$ such that $\varphi(S) = X^{t+1}$, $\varphi(T) = Y^{t+1}$, $\varphi(U) = Z^{t+1}$ and $\varphi(V) = X^tY^tZ^t$. Then $\mathcal{M} := \ker(\varphi) = R_0(X^{t+1}T - Y^{t+1}S) + R_0(X^{t+1}U - Z^{t+1}S) + R_0(Y^{t+1}U - Z^{t+1}T) + R_0(XV - Y^tZ^tS) + R_0(YV - X^tZ^tT) + R_0(ZV - X^tY^tU)$.

Proof. It suffices to show that $\mathcal{M} \subseteq R_0(X^{t+1}T - Y^{t+1}S) + R_0(X^{t+1}U - Z^{t+1}S) + R_0(Y^{t+1}U - Z^{t+1}T) + R_0(XV - Y^tZ^tS) + R_0(YV - X^tZ^tT) + R_0(ZV -$

X^tY^tU). Let $H = fS+gT+hU+lV \in \mathcal{M}$ with $f,g,h,l \in R_0$. Then $fX^{t+1}+gY^{t+1}+hZ^{t+1}+lX^tY^tZ^t = 0$ (*). Then $f(X,Y,0)X^{t+1}+g(X,Y,0)Y^{t+1} = 0$. So $f(X,Y,0) = Y^{t+1}f'(X,Y)$ and $g(X,Y,0) = -X^{t+1}f'(X,Y)$. We have $f = f(X,Y,0)+Zf_1(X,Y,Z)$ and $g = g(X,Y,0)+Zg_1(X,Y,Z)$, hence $H = f(X,Y,0)S+Zf_1S+g(X,Y,0)T+Zg_1T+hU+lV = Y^{t+1}f'(X,Y)S+Zf_1S-X^{t+1}f'(X,Y)T+Zg_1T+hU+lV = f'(X,Y)(Y^{t+1}S-X^{t+1}T)+Zf_1S+Zg_1T+hU+lV$. Applying again φ we get $Zf_1X^{t+1}+Zg_1Y^{t+1}+hZ^{t+1}+lX^tY^tZ^t = 0$. Then $f_1X^{t+1}+g_1Y^{t+1}+hZ^t+lX^tY^tZ^{t-1} = 0$. Applying a similar method as to (*), we obtain $Zf_1S+Zg_1T+hU+lV = Zf_1'(X,Y)(Y^{t+1}S-X^{t+1}T)+Z^2f_2S+Z^2g_2T+hU+lV$, and thus $H = f''(X,Y,Z)(Y^{t+1}S-X^{t+1}T)+Z^2f_2S+Z^2g_2T+hU+lV$. We can iterate this process until we obtain $H = F_1(X,Y,Z)(Y^{t+1}S-X^{t+1}T)+Z^tf_2S+Z^tg_2T+hU+lV$. Applying φ and canceling Z^t, we get $f_2X^{t+1}+g_2Y^{t+1}+hZ+lX^tY^t = 0$. Then $f_2(X,0,Z)X^{t+1}+h(X,0,Z)Z = 0$. Then $f_2(X,0,Z) = Zf_2'(X,Z)$ and $h(X,0,Z) = -f_2'(X,Z)X^{t+1}$. Hence $H_1 = Z^tf_2S+Z^tg_2T+hU+lV = Z^t(f_2(X,0,Z)+Yf_3)S+Z^tg_2T+(h(X,0,Z)+Yh_1)U+lV = f_2'(X,Z)(Z^{t+1}S-X^{t+1}U)+YZ^tf_3S+Z^tg_2T+Yh_1U+lV$. Iterating this process as above, we obtain $H_1 = F_2(X,Y,Z)(Z^{t+1}S-X^{t+1}U)+Y^tZ^tf_4S+Z^tg_2T+Y^th_2U+lV$. Applying φ to H_1 we get $Y^tZ^tf_4X^{t+1}+Z^tg_2Y^{t+1}+Y^th_2Z^{t+1}+lX^tY^tZ^t = 0$, and thus $f_4X^{t+1}+g_2Y+h_2Z+lX^t = 0$. Then $g_2(0,Y,Z)Y+h_2(0,Y,Z)Z = 0$, so that $g_2(0,Y,Z) = Zg_2'(Y,Z)$ and $h_2(0,Y,Z) = -Yg_2'(Y,Z)$. Hence $H_2 = Y^tZ^tf_4S+Z^tg_2T+Y^th_2U+$

$lV = Y^tZ^tf_4S + Z^t(g_2(0,Y,Z) + Xg_3)T + Y^t(h_2(0,Y,Z) + Xh_3)U + lV = Y^tZ^tf_4S + g_2'(Y,Z)(Z^{t+1}T - Y^{t+1}U) + XZ^tg_3T + XY^th_3U + lV$. We iterate this process until we get $H_2 = Y^tZ^tf_4S + G_2(X,Y,Z)(Z^{t+1}T - Y^{t+1}U) + X^tZ^tg_4T + X^tY^th_4U + lV$. Now, let $H_3 = Y^tZ^tf_4S + X^tZ^tg_4T + X^tY^th_4U + lV$. Applying φ and canceling $X^tY^tZ^t$ we get $f_4X + g_4Y + h_4Z + l = 0$, so that $l = -f_4X - g_4Y - h_4Z$. It follows that $H_3 = f_4(Y^tZ^tS - XV) + g_4(X^tZ^tT - YV) + h_4(X^tY^tU - ZV)$. Consequently, $H \in R_0(X^{t+1}T - Y^{t+1}S) + R_0(X^{t+1}U - Z^{t+1}S) + R_0(Y^{t+1}U - Z^{t+1}T) + R_0(XV - Y^tZ^tS) + R_0(YV - X^tZ^tT) + R_0(ZV - X^tY^tU)$, as desired.

□

Let $S(\mathcal{M})$ denote the R_0-subalgebra generated by $\mathcal{M}$. Then $S(\mathcal{M}) = F[X,Y,Z](\mathcal{M}) = F[X,Y,Z,X^{t+1}T - Y^{t+1}S, X^{t+1}U - Z^{t+1}S, Y^{t+1}U - Z^{t+1}T, XV - Y^tZ^tS, YV - X^tZ^tT, ZV - X^tY^tU]$ and regarded as a graded ring we can rewrite it as $S(\mathcal{M}) = \bigoplus_{i\geq 0} S^i(\mathcal{M})$. Let K be the field of fraction of $S(\mathcal{M})$.

Proposition 7.2 ([39, Theorem 1]) . *With the notation above, the ring $K \cap R$ is not finitely generated as an algebra over F.*

Lemma 7.3 ([39, Lemma 1]) . *With the notation above, we have* $K \cap R = \bigoplus_{n\geq 0} \overline{S^n(\mathcal{M})}$.

Proof. If $\alpha \in \overline{S^n(\mathcal{M})}$, then , in particular, $X^m\alpha \in \overline{S^n(\mathcal{M})}$ for some m. Since $X \in S(\mathcal{M})$ and $X^m = X^nX^{m-n}$, this implies that $\alpha \in K$. $\overline{S^n(\mathcal{M})} \subseteq R$ by definition. On the other hand, suppose that some element α of R is in K. let

$S = \{X^k,\ k \in \mathbb{N}\}$ be a multiplicative closed subset of R. Note that $\mathcal{M}_X = \mathcal{M}_S$ is a (free) direct summand of the free module R_{1_X}. In fact, localizing with X the exact sequence $0 \longrightarrow \mathcal{M} \longrightarrow R_1 \longrightarrow I \longrightarrow 0$, we get the exact sequence $0 \longrightarrow \mathcal{M}_X \longrightarrow (R_1)_X \longrightarrow R_0 \longrightarrow 0$. Next, R_0 is projective (since free) and therefore we obtain $(R_1)_X = \mathcal{M}_X \oplus R_0$. Likewise, we get a similar result when localizing by inverting the elements Y and Z. Moreover, $S(\mathcal{M})_X = (R_0)_X[\mathcal{M}]$, $X^{t+1}(Y^{t+1}U - Z^{t+1}T) = Z^{t+1}(Y^{t+1}S - X^{t+1}T) + Y^{t+1}(X^{t+1}U - Z^{t+1}S)$, $X(YV - X^tZ^tT) = Y(XV - Y^tZ^tS) - Z^t(X^{t+1}T - Y^{t+1}S)$ and $X(ZV - X^tY^tU) = Z(XV - Y^tZ^tS) - Y^t(X^{t+1}U - Z^{t+1}S)$. We have $\frac{\alpha}{1} \in S(\mathcal{M})_X$, i.e., some power of X times α belongs to $S(\mathcal{M})$. Since the same applies when we localize by inverting the element Y or Z, we conclude that for some $k \in \mathbb{N}$ sufficiently large $m_0^k\alpha \subseteq S(\mathcal{M})$ and $\alpha \in \overline{S(\mathcal{M})}$, as desired. $\square$

Thus, Proposition 7.2 will be obtained if we show that $\bigoplus \overline{S^n(\mathcal{M})}$ is not finitely generated. This will be done if we show that for each n there is an element of $\overline{S^n(\mathcal{M})}$ which is not in the subalgebra generated by all the $\overline{S^i(\mathcal{M})}$ for $i < n$. Consider the R_0-module homomorphism $R_n \xrightarrow{\phi_n} R_{n-1}$ defined by evaluating to reduce degree. For example for $n = 1$, ϕ_1 is the map from R_1 to R_0 which maps the generators of R_1 onto the generators of I defined above. Then for $n > 1$ we define ϕ_n by letting $\phi_n(m_1 \dots m_n) = \sum_{j=1}^{n} m_1 \dots \phi_1(m_j) \dots m_n$. This map is well-defined since R_n is the nth symmetric power of R_1.

In terms of monomials, ϕ_n is defined by letting $S^aT^bU^cV^d$ go to the element

$a(X^{t+1})S^{a-1}T^bU^cV^d + b(Y^{t+1})S^a$

$T^{b-1}U^cV^d + c(Z^{t+1})S^aT^bU^{c-1}V^d + d(X^tY^tZ^t)S^aT^bU^cV^{d-1}$. This new expression of ϕ_n defines a matrix which represents ϕ_n. To simplify this matrix, we give a third representation of ϕ_n in terms of divided powers. Since the ground field has characteristic zero, an alternative basis for R_n is obtained by replacing the monomial $S^aT^bU^cV^d$ by its multiple $(\frac{1}{a!b!c!d!})S^aT^bU^cV^d$, which we denote by $S^{(a)}T^{(b)}U^{(c)}V^{(d)}$. Thus, $\phi_n(S^{(a)}T^{(b)}U^{(c)}V^{(d)}) =$

$(X^{t+1})S^{(a-1)}T^{(b)}U^{(c)}V^{(d)} + (Y^{t+1})S^{(a)}T^{(b-1)}U^{(c)}V^{(d)} +$

$(Z^{t+1})S^{(a)}T^{(b)}U^{(c-1)}V^{(d)} + (X^tY^tZ^t)S^{(a)}T^{(b)}U^{(c)}V^{(d-1)}$.

The advantage is that all the integer coefficient in the matrix of ϕ_n are replaced by ones.

Let $T_1 := Y^{t+1}S - X^{t+1}T$, $T_2 := X^{t+1}U - Z^{t+1}S$ and $T_3 := XV - Y^tZ^tS$. Then $\mathcal{M}_X = (R_0)_XT_1 + (R_0)_XT_2 + (R_0)_XT_3$. Further $Y^{t+1}U - Z^{t+1}T, YV - X^tZ^tT, ZV - X^tY^tU \in (R_0)_X[T_1, T_2, T_3]$. It follows that $S(\mathcal{M})_X = (R_0)_X[T_1, T_2, T_3]$ (*). It is then easy to see that $R_X = S(\mathcal{M})_X[S]$. Set $T_4 := \frac{S}{X^{t+1}}$. Hence $R_X = (R_0)_X[T_1, T_2, T_3, T_4]$ (**) with T_1, T_2, T_3 form a basis of $\mathcal{M}_X$. Consequently, $K \cap R_X = K \cap S(\mathcal{M})_X[S] = S(\mathcal{M})_X$ since S is transcendental over K (easy to see by (*) and (**) and $\operatorname{t.d.}(R : R_0) = 4$). Also, note that since $S(\mathcal{M})_X = R_{0X}[T_1, T_2, T_3]$, we have $S^n(\mathcal{M})_X = R_{0X}[T_1, T_2, T_3] \cap R_{nX}$.

Lemma 7.4 $\ker(\phi_n) = \overline{S^n(\mathcal{M})}$.

Proof. Let $\phi_{nX} : R_{nX} \longrightarrow R_{(n-1)X}$ be the module homomorphism obtained after localization. From the above discussion, we have $R_X = R_{0X}[T_1, T_2, T_3, T_4]$ with T_1, T_2, T_3 form a basis of $\mathcal{M}_X$ and $\phi_{nX}(T_4) = 1$. Then ϕ_{nX} is simply partial differentiation with respect to T_4 (in fact, $\phi_{nX}(T_1^a T_1^b T_3^c T_4^d) = dT_1^a T_1^b T_3^c T_4^{d-1}$ since $T_1, T_2, T_3 \in \ker(\phi_1)$). It follows that $\ker(\phi_{nX}) = R_{0X}[T_1, T_2, T_3] \cap R_{nX}$. On the other hand,

$$\begin{aligned}
\overline{S^n(\mathcal{M})}_X &= \{\alpha \in R_{nX} : m_0^k \alpha \subseteq S^n(M)_X \ for\ some\ k \in \mathbb{N}\} \\
&= \{\alpha \in R_{nX} : m_0^k \alpha \subseteq R_{0X}[T_1, T_2, T_3]\} \\
&= \{\alpha \in R_{nX} : \alpha \in R_{0X}[T_1, T_2, T_3]\} \\
&= R_{0X}[T_1, T_2, T_3] \cap R_{nX} \\
&= S^n(\mathcal{M})_X.
\end{aligned}$$

Hence $\ker(\phi_{nX}) = \overline{S^n(\mathcal{M})}_X = S^n(\mathcal{M})_X$. Similarly, we get $\ker(\phi_{nY}) = \overline{S^n(M)}_Y = S^n(\mathcal{M})_Y$ and $\ker(\phi_{nZ}) = \overline{S^n(\mathcal{M})}_Z = S^n(\mathcal{M})_Z$. Now, let $\alpha \in \overline{S^n(\mathcal{M})}$. Then $\alpha \in (\overline{S^n(\mathcal{M})})_X = \ker(\phi_{nX})$. Thus there exists an integer r such that $X^r \alpha \in \ker(\phi_n)$, so that $X^r \phi_n(\alpha) = 0$ since $\alpha \in R_n$. Hence $\phi_n(\alpha) = 0$ and thus $\alpha \in \ker(\phi_n)$. Conversely, let $\alpha \in \ker(\phi_n)$. Then $\alpha \in \ker(\phi_{nX}) \cap \ker(\phi_{nY}) \cap \ker(\phi_{nZ})$. Then $\alpha \in S^n(\mathcal{M})_X \cap S^n(\mathcal{M})_Y \cap S^n(\mathcal{M})_Z$. It follows that there exists an integer r such that $X^r \alpha, Y^r \alpha, Z^r \alpha \in S^n(\mathcal{M})$. Thus $m_0^{3r} \alpha \subseteq S^n(\mathcal{M})$. Then $\alpha \in \overline{S^n(\mathcal{M})}$, as desired.

□

Lemma 7.5 ([39, Lemma 2]) . *For all $n > 0$, we have $\overline{S^n(\mathcal{M})} \subseteq m_0 R_n$, where m_0 is the maximal ideal of R_0.*

Proof. By contraposition, let $\xi \in R_n \backslash m_0 R_n$ with $\phi_n(\xi) = 0$. We will show that this is not possible. Let $\alpha \in m_0 R_{m_0} R_n \cap R_n$. Then there exists $f \in R_0 - m_0$ such that $f\alpha \in m_0 R_n$. Thus $f(0,0,0)\alpha(0,0,0,S,T,U,V) = 0$. Since $f \notin m_0$, we get $f(0,0,0) \neq 0$ and hence
$\alpha(0,0,0,S,T,U,V) = 0$. Then $\alpha \in m_0 R_n$. It follows that $m_0 R_{m_0} R_n \cap R_n = m_0 R_n$. Consequently, it suffices to show that $\overline{S^n(\mathcal{M})}_{m_0 R_{m_0}} \subseteq m_0 R_{m_0} R_n$, in other words we may suppose that (R_0, m_0) is local. Then we may assume that one of the monomials of ξ has coefficient 1. Let $S^a T^b U^c V^d$ be a monomial of ξ whose coefficient is 1. Suppose for example that $a > 0$, since $a + b + c + d \geq 1$. In $\phi_n(\xi)$, the monomial $S^{a-1} T^b U^c V^d$ will comme from different monomials of degree n and so will have coefficient $[laX^{t+1}$+(something in the ideal $(Y^{t+1}, Z^{t+1}, X^t Y^t Z^t))]$, $l \in \mathbb{N}$. Since the variables S, T, U, V are algebraically independent over R_0, $\phi_n(\xi) = 0$ will imply that X^{t+1} is in the ideal $(Y^{t+1}, Z^{t+1}, X^t Y^t Z^t)$, which is not true. Similarly, if we assume the other exponents to be positive, then we reach a contradiction. □

Lemma 7.6 ([39, Lemma 3]) . *For all n there is an element of $\overline{S^n(\mathcal{M})}$ whose V^n coefficient is X.*

Proposition 7.7 ([39, Theorem 2]) *The symbolic blow-up $\bigoplus_{i\geq 0} \hat{P}^{(i)}$ is not finitely generated algebra over $\hat{R}$.*

Proof. Since $\hat{P}$ is homogeneous prime ideal, an element of $\hat{R} = \prod \hat{R}_n$ is in $\hat{P}$ if and only if each $\hat{R}_n-$component is in $\hat{P}$. Next, we have $\hat{P} \cap \hat{R}_0 = 0$ and $\hat{P} \cap \hat{R}_1 = \hat{\mathcal{M}}$. Indeed, since we have an R_0-homomorphism, only the element zero in $\hat{R}_0$ is mapped to zero. The second equality is a consequence of the algebra extension. Now, if $\alpha \in P^n$, then $\alpha = \sum_i \alpha_1^i \ldots \alpha_n^i$, where $\alpha_j^i \in P$. Thus, since $P \cap R_0 = (0)$, $\deg(\alpha) \geq n$ and P^n has non-zero components only in degree greater than or equal to n. Moreover, $P^n \cap \hat{R}_n = S^n(\hat{\mathcal{M}})$. Indeed, $\alpha \in P^n \cap R_n$ implies that $\alpha = \sum_i \alpha_1^i \ldots \alpha_n^i$, $\alpha_j^i \in \hat{R}_1$ and so $\alpha \in S^n(\hat{\mathcal{M}})$. On the other hand, if $\alpha \in S^n(\hat{\mathcal{M}})$ which is the R_0-algebra generated by $\hat{\mathcal{M}}$, then $\alpha = \sum_i \lambda \alpha_{i_1} \ldots \alpha_{i_n}$, $\alpha_{i_j} \in \hat{\mathcal{M}} \subset \hat{R}_1$ and $\lambda \in R_0$. Thus $\alpha \in (\hat{R}_1)^n \subset \hat{R}_n$. The result then follows since $\hat{\mathcal{M}} \subset \hat{P}$. After localizing at X, Y or Z, P is generated by a regular sequence, so that its symbolic powers are equal to its ordinary powers; hence an element α of $\hat{R}_n$ is in $P^{(n)}$ if and only if after localizing at X, Y or Z the image of α in the localization is in P^n, i.e., some power of $\hat{m}_0$ times α is in P^n. Then $P^{(n)} \cap \hat{R}_n = \overline{S^n(\mathcal{M})}$. Now, if an element α of $P^{(n)} \cap \hat{R}_n$ were in the subalgebra of $\bigoplus_{i\geq 0} P^{(i)}$ generated by the $P^{(i)}$ for $i < n$, then $\alpha = \displaystyle\sum_{1\leq i_j < n} a x_{i_1} x_{i_2} \ldots 5 x_{i_r}$, where $a \in R_0$ and $x_{i_j} \in P^{(i_j)}$, i.e., its belongs to a finitely generated $R-$algebra, which is a UFD (thus integrally closed and f.g. $R-$module). If $\alpha \in R < \lambda_1, \ldots, \lambda_r >$, with degree $\lambda_i \in P^{(n_i)}$, $n_i < n$, then

$\alpha \in \sum \lambda_i(\bigoplus P^{(i)}) = \sum \lambda_i P^{(n-n_i)}$. then we could write it as a sum of elements in the ideals $P^{(i)}P^{(j)}$, where $0 < i, j < n$ and $i + j = n$. Since $P^{(i)} \cap \hat{R}_k = 0$ for $k < i$, and similarly for j, the only elements that could give a non-zero contribution are those in $(P^{(i)} \cap \hat{R}_i)(P^{(j)} \cap \hat{R}_j) = \overline{S^i(\hat{\mathcal{M}})}\ \overline{(S^j(\hat{\mathcal{M}}))}$. taking the contraposition of this implication, we see that an element of $\overline{S^n(\hat{\mathcal{M}})}$ which is not in the subalgebra of $\bigoplus_{i \geq 0} \overline{S^i(\hat{\mathcal{M}})}$ generated by components of degree less than n also gives an element of $P^{(n)}$ which is not in the subalgebra of $\bigoplus_{i \geq 0} P^{(i)}$ generated by components of lower degree, so $\bigoplus_{i \geq 0} P^{(i)}$ is not finitely generated algebra over $\hat{R}$. $\square$

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