On the Margin Stability of Interval Matrices

by

Muhammad Yousuf

A Thesis Presented to the

FACULTY OF THE COLLEGE OF GRADUATE STUDIES

KING FAHD UNIVERSITY OF PETROLEUM & MINERALS

DHAHRAN, SAUDI ARABIA

In Partial Fulfillment of the
Requirements for the Degree of

MASTER OF SCIENCE

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This thesis written by MUHAMMAD YOUSUF under the direction of his Thesis Advisor and approved by his Thesis Committee, has been presented to and approved by the Dean of College of Graduate Studies, in partial fulfillment of the requirements for the degree of MASTER OF SCIENCE IN MATHEMATICS.

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Dedication

This thesis is dedicated to my parents Muhammad Ishaq and Saleema Bibi.

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All praise to the almighty Allah, the lord of the world, with whose gracious help it was possible to accomplish this work. May peace and Allah's blessing be upon Muhammad the last of his messengers.

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Abstract

In the first part of this thesis, some necessary and sufficient conditions for h-stability of a class of interval matrices are developed. It is proved that the problem of determining the h-stability of an interval matrix is related to that of existence and positiveness of the inverse of a point matrix constructed from the given interval matrix. In the second part some results on positive definiteness and stability of interval matrices are extended to h-positive definiteness and h-stability. In the third part some verifiable necessary and sufficient conditions for regularity of interval matrices are established. Based on these conditions an algorithm for determining the regularity of interval matrices is developed which lead to several algorithms for checking the stability and positive definiteness of interval matrices.

ملخص البحث

في الجزء الأول من هذه الرسالة أوجدنا بعض الشروط الضرورية والكافية للأتزان المقطعي لفصل من المصفوفات والتي تتكون من عناصر كل واحد منهم عبارة عنه فترة متصلة من الأعداد الحقيقية. تدعى مصفوفة من هذا النوع بمصفوفة فتريه. لقد برهنا أنه خاصية الاتزان المقطعي لمصفوفة فترية لها علاقة بوجود و ايجابية المعكوس لمصفوفة معينة. الجزء الثاني خاص بتعميم بعض النتائج الخاصة بالاتزان و الايجابيه المطلقة للمصفوفات الفطرية. و في الجزء الثالث تعرضنا لموضوع أنتظام المصفوفات الفترية و علاقته بخاصيتي الاتزان و الايجابية المطلقة. و اخيرا تم تصميم بعض الطرفة العدديه لأختبار ما أ ذا كانت مصفوفة فتريه معنية تحقق أي منه الخواص سابقه الذكر.

Chapter 1

Introduction

1.1 Interval Matrices

An interval matrix is a matrix whose elements are intervals, e.g., a real matrix whose entries are known within certain closed intervals or a matrix whose entries are continuous functions of some parameter $x \in [a, b]$. Thus an interval matrix, denoted by $A^I = [B, C]$, is a set of real matrices defined as

$$A^I = \{A = [a_{ij}] : b_{ij} \leq a_{ij} \leq c_{ij},\ i,\ j = 1, 2, , ..., n\}, \tag{1.1}$$

where $B = [b_{ij}]$ and $C = [c_{ij}]$ are $n \times n$ real matrices [13].

A square matrix $D = [d_{ij}]$ is said to be a corner matrix of an interval matrix A^I, if $d_{ij} = b_{ij}$ or c_{ij} for all i, j. An interval matrix A^I is said to be regular if each $A \in A^I$ is nonsingular. A^I is said to be symmetric if both B and C are symmetric. Notice that a symmetric interval matrix may contain nonsymmetric matrices. Thus a symmetric interval matrix may have complex eigenvalues.

A square matrix A is said to be stable with stability margin h or h-stable if for all eigenvalues λ of A, $Re[\lambda] < -h$, $h \geq 0$. An interval matrix A^I is said to be stable with stability margin h or h-stable if each $A \in A^I$ is h-stable. A^I is said to be stable if it is h-stable with $h = 0$.

A non-singular $M-$matrix M is a real matrix of the form $M = pI - N$, $p > 0$, $N \geq 0$ with $p > \rho[N]$, where $N \geq 0$ means that all elements of N are nonnegative (such a matrix N is called nonnegative matrix), and $\rho[N]$ denotes the spectral radius of N, i.e., absolute value of the eigenvalue of N of largest magnitude. It follows that the diagonal elements of a non-singular $M-$matrix M are positive, while the off-diagonal elements are nonpositive. There are more than fifty equivalent characterizations of

$M-$matrices [8].

An $n \times n$ matrix A is said to be positive definite if $x^T A x > 0$ for all $x \in R^n$ such that $x \neq 0$. An interval matrix A^I is said to be positive definite if each $A \in A^I$ is positive definite.

A system of linear interval equations

$$A^I X = b^I \tag{1.2}$$

where A^I is an interval matrix and b^I an interval vector, is a system of linear equations in which the coefficients of variables and right-hand-side constants are not determined exactly but are known only to lie within certain closed intervals (obtained for example as a result of roundoff, truncation or data errors). Such a system of linear interval equations represents a family of linear systems of equations which can be obtained from it by fixing coefficients and right-hand values in the prescribed intervals. The matrix of coefficients of a system of linear interval equations is an interval matrix A^I and such a system has a unique solution under the assumption that A^I is regular [22]. Thus to check the solvability of such a system one has to determine the regularity of the coefficients interval matrix A^I.

Many practical problems require extensive numerical computations involving quantities determined experimentally by approximate measurements with some estimate of accuracy of the measured values. A typical calculation will begin with some numbers known only to a certain number of significant decimal digits. Results computed from such an inexact initial data will also be of limited precision. It is of great practical importance to be able to access the accuracy of such results. Uncertainties enter the model in the form of linearizations, approximation measurement errors or unmodelled dynamics. Some of the uncertainties may be removed by higher order approximations. But it is inevitable to endup with some of the uncertainties. It is therefore advantageous to be able to guarantee the stability and performance of the system in the face of unknown uncertainties. This is generally achieved by specifying bounds within which the parameters are allowed to vary and establishing conditions under which the required stability or performance property is maintained for all values of the system parameters in the specified range.

In the control system literature it has been recognized that a good control system must be robust, i.e., it must remain stable for all values of the system parameters within certain range. Interval matrices have recently been used to model parametric variations in the linear time-invariant systems

$$\dot{X}(t) = A\,X(t) \tag{1.3}$$

described in state-space form under data perturbation [7], [9] and [11].

There are different approaches to check the stability of different types of interval matrices. These approaches usually consist of checking that some test matrices, constructed from the original interval matrix, satisfy certain conditions. Some of these approaches are discussed in the following sections.

1.2 Stability: Gershgorin's Theorem and its extension

Gershgorin's Theorem is useful in determining regions containing eigenvalues of matrices. Based on Gershgorin's theorem, Heinen [1], gave some sufficient condition for the stability of an interval matrix $A^I = [B, C]$ which requires the upper limits of the diagonal intervals to be all negative, i.e., $c_{ii} < 0$ for all i. Argoun [2] suggested an extension of Gershgorin's theorem based on which Juang and Shao [9] gave useful sufficient conditions for the stability of interval matrices which removed the restriction that the $c_{ii} < 0$ for all i. These conditions were improved by Chen [3]. Using the fact that similar matrices have the same eigenvalues, some tightest sufficient conditions based on Gershgorin's theorem and its extension were given by Chen in [3] (by tighter conditions, we mean that conditions which lead to reduced (Gershgorin's) regions containing eigenvalues). In [3] it is also established that the stability of an interval matrix A^I is related to the characterization of a certain non-singular M-matrix constructed from A^I. Some properties of M-matrices were also used for testing stability of interval matrices [5], [6], [7]. The numerical implementation of the conditions given in [3] require finding the spectral radius of certain matrices constructed from the given interval matrix. If these matrices have precisely one dominant eigenvalue, one can find that eigenvalue using the power method [17], [18], [19]. In [8] several conditions equivalent to the conditions in [3] are given which replace the problem of finding the spectral radius by that of finding the inverses of some test matrices constructed from the given interval matrix.

1.3 Stability: Positive definiteness and Regularity

We can write an interval matrix $A^I = [B, C]$ as

$$A^I = [A^m - \Delta, A^m + \Delta], \tag{1.4}$$

where $A^m = \frac{1}{2}(B + C)$ and $\Delta = \frac{1}{2}(C - B)$. Notice that $A^I = [A^m - \Delta . A^m + \Delta]$ is symmetric if both A^m and Δ are symmetric.

In [23], Z. C. Shi and W. B. Gao, proved some necessary and sufficient conditions for the positive definiteness of an interval matrix using its corner matrices. The number of test matrices in these results was $2^{\frac{n(n-1)}{2}}$. Using the fact that a symmetric matrix A is stable if and only if $-A$ is positive definite, D. Hertz [16], proved that every symmetric matrix in a symmetric interval matrix A^I is stable if and only if every test matrix in a finite subset of corner matrices of A^I is stable. The cardinality of this subset was 2^{n-1}. These results were improved by Rohan [4]. Rohan proved that stability (positive definiteness) of this subset implies stability (positive definiteness) of the whole A^I. Rohan also proved that A^I is positive definite if and only if A^I_s is positive definite, where A^I_s is a symmetric interval matrix constructed from A^I. In [4] it is also proved that a symmetric interval matrix is positive definite if and only if it is regular and contains at least one positive definite matrix. Since A^I and the corresponding A^I_s are equivalently positive definite, one can verify positive definiteness of a general interval matrix using corresponding symmetric interval matrix. However, this result is not as simply implemented, because verifying the regularity of an interval matrix is generally a difficult problem. For example see [22], where a number of necessary and sufficient conditions for regularity are given, all of which require computations of at least 2^{n-1} operations such as evaluating determinants, solving systems of linear equations, inverting matrices and so on. Now we mention some results due to Rohan [4] which relates stability, positive definiteness and regularity of interval matrices. In contrast to the previous results, where an interval matrix A^I and the corresponding symmetric interval matrix A_s^I are equivalently positive definite, here only stability of A_s^I implies stability of A^I. Thus if A_s^I is not stable one cannot comment on the stability of A^I. Therefore for stability, the investigation is restricted only to symmetric interval matrices. Using positive definiteness, Rohan [4] proved that a symmetric interval matrix A^I is stable if and only if all matrices in a finite set of cardinality $2^{n-1}$of matrices in A^I are stable. Using a previous result for positive definiteness and regularity, Rohan [4] also proved that a symmetric interval matrix A^I is stable if and only if it is regular and contains at least one stable matrix.

For the necessary and sufficient stability conditions of an interval matrix A^I, formulated in terms of checking stability of a finite subset of matrices in A^I, given by Soh [23], Hertz [16] and Rohan [4], the set of test matrices increases exponentially with the matrix size. Therefore these conditions are hard to apply for higher order interval matrices. In [14], Rohan proposed a branch-and-bound type algorithm for checking the stability of a symmetric interval matrix, based on necessary and/or sufficient stability conditions. This algorithm cannot be expected to circumvent exponentially in the verification process in general, but due to built-in branch-and-bound strategy it drastically reduces the amount of computations in many cases. In contrast to the stability checks based on sufficient conditions only, this algorithm always yields a result: in a finite number of steps it either verifies the stability of A^I or finds an unstable matrix in A^I.

In [11], Wang, Michel and Liu developed necessary and sufficient conditions for the stability and using these conditions they developed an algorithm for checking the stability of an interval matrix. This algorithm requires the solution of Lyapunov equation, which is a matrix equation of order $\frac{n(n-1)}{2} \times \frac{n(n-1)}{2}$, where n is the size of interval matrix A^I [21].

In the present study, our plan is:

1. To examine the necessity of some of the sufficient conditions for $h-$stability of interval matrices given by Chen[3] and their numerical implementations.

2. To extend the results given in [4] for positive definiteness and stability to $h-$positive definiteness and $h-$stability for symmetric interval matrices and develop some numerical algorithms based on these results.

3. To establish some verifiable necessary and sufficient conditions for the regularity and stability of interval matrices and develop numerical algorithms based on these conditions.

Chapter 2

Stability, Positive Definiteness and Regularity of Interval Matrices: Review of some existing results

2.1 Introduction

In this chapter, we review some results that are generally used for stability analysis of interval matrices. In section 2, several sufficient conditions for the h-stability of interval matrices using Gershgorin's theorem and its extension are presented. Some important characterizations of nonsingular M-matrices are also presented in this section. Furthermore, as proved in [3], it turns out that the problem of determining the stability of an interval matrix is related to that of characterizations of a certain nonsingular M-matrix. An important consequence of such conditions is that they may potentially result in many more sufficient conditions, because a nonsingular M-matrix is abundant in its characterizations [8]. In section 3, some characterizations of positive definiteness and stability of interval matrices, due to Rohan [4], are presented. In section 4, we present some necessary and/or sufficient stability conditions and algorithms based on them [4].

2.2 Marginal Stability: Gershgorin's Theorem and its Extension

In this section we present the sufficient conditions for $h-$stability of interval matrices based on Gershgorin's Theorem and its extension, due to Heinen [1] and Chen [3].

2.2.1 Gershgorin's Theorem and its extension

In this subsection we state the well known Gershgorin's theorem which has been used in forming some sufficient conditions for the $h-$stability of interval matrices [1], [3].

Theorem 1 *(Gershgorin's Theorem)[19] For an $n \times n$ matrix A, every eigenvalue λ of A must be contained in at least one of the discs characterized by rows of A as*

$$|\lambda - a_{ii}| \leq \sum_{j=1, j \neq i}^{n} |a_{ij}|, \quad i = 1, 2, ..., n. \tag{2.1}$$

Remark 1 *It follows from (2.1) that real part of each eigenvalue λ of A must satisfy one of the following conditions*

$$Re[\lambda] \leq a_{ii} + \sum_{j=1, j \neq i}^{n} |a_{ij}|, \ i = 1, 2, ..., n. \tag{2.2}$$

Remark 2 *Since eigenvalues are invariant under similarity transformations, inequality (2.2) can be tightened by using matrix scaling which is as follows. Let*

$$\Re = \{R = diag(r_1, r_2, ..., r_n) : r_i > 0, \ i = 1, 2, ..., n\}. \tag{2.3}$$

Then A and $R^{-1}AR$ have the same eigenvalues for all $R \in \Re$, where

$$(R^{-1}AR)_{ij} = \begin{cases} a_{ii} & \text{if } i = j, \\ \frac{r_j}{r_i} a_{ij} & \text{if } i \neq j. \end{cases}$$

Thus for all $R \in \Re$, the real part of each eigenvalue λ of A must satisfy

$$Re[\lambda] \leq a_{ii} + \sum_{j=1, j \neq i}^{n} \frac{r_j}{r_i} |a_{ij}|, \ i = 1, 2, ..., n. \tag{2.4}$$

It follows from (2.4) that

$$Re[\lambda] \leq a_{ii} + \min_{R \in \Re} \sum_{j=1, j \neq i}^{n} \frac{r_j}{r_i} |a_{ij}|, \ i = 1, 2, ..., n. \tag{2.5}$$

Remark 3 *Extension of Gershgorin's Theorem [2]*

Any matrix $A \in A^I$ can be written in the perturbed form as $A = \bar{A} + \delta A$ where $|\delta A| < \Delta A$. $\bar{A}$ and ΔA are given by

$$\bar{A} = \left[\frac{1}{2}(b_{ij} + c_{ij})\right] \quad , \qquad \Delta A = \left[\frac{1}{2}(c_{ij} - b_{ij})\right].$$

Let T be a similarity transformation such that $T^{-1}\,\bar{A}\,T = A_J$, where A_J is the Jordan form of $\bar{A}$. Let $\bar{\lambda}_1, \bar{\lambda}_2, ..., \bar{\lambda}_n$ denote the eigenvalues of $\bar{A}$, and $\Lambda \equiv diag[\bar{\lambda}_i]$ and let

$$F = A_J - \Lambda + |T^{-1}|\,\Delta A\,|T\,| = [f_{ij}].$$

It follows from Theorem 1 and the triangular inequality that every eigenvalue λ of A must be contained in at least one of the discs

$$\left|\lambda - \bar{\lambda}_i\right| \leq \sum_{j=1, j\neq i}^{n} f_{ij} \qquad i = 1, 2, ..., n.$$

2.2.2 Sufficient Conditions, The Extreme Matrix

Based on Gershgorin's Theorem, Heinen [1] and Chen [3] gave the following sufficient conditions for the h−stability of interval matrices.

Theorem 2 *Consider the interval matrix $A^I = [B, C]$ and let $h \geq 0$. Then*

1. *A^I is h−stable if*

$$c_{ii} + \sum_{j=1, j\neq i}^{n} max\left\{|b_{ij}|, |c_{ij}|\right\} < -h\,, \quad i = 1, 2, ..., n. \tag{2.6}$$

2. *A^I is $h-$stable if there exists an $R \in \Re$ such that*

$$c_{ii} + \sum_{j=1, j \neq i}^{n} \frac{r_j}{r_i} \max\{|b_{ij}|, |c_{ij}|\} < -h, \ i = 1, 2, \ldots, n. \tag{2.7}$$

Now corresponding to A^I, define a matrix $W_h = [w_{ij}]$ by

$$w_{ij} = \begin{cases} 0 & \text{if } i = j, \\ \frac{\max\{|b_{ij}|, |c_{ij}|\}}{|c_{ii} + h|} & \text{if } i \neq j. \end{cases} \tag{2.8}$$

Then as given in [3], the following Corollary is a compact version of Theorem 2.

Corollary 1 *Suppose that for interval matrix A^I, $c_{ii} + h < 0$, $i = 1, 2, \ldots, n$, for some $h \geq 0$. Then*

1. *A^I is h-stable if $\|W_h\|_\infty < 1$.*

2. *A^I is $h-$stable if there exists an $R \in \Re$ such that $\|R^{-1}W_hR\|_\infty < 1$.*

It is now clear that the problem of finding the tightest condition for stability is translated into that of finding the minimum of $\|R^{-1}W_hR\|_\infty$ over the set $\Re$. Chen [3] solved this problem by using the following lemma from [8].

Lemma 1 *Suppose that A is a nonnegative matrix. Then*

$$\rho[A] = \min_{R \in \Re} \left\|R^{-1}AR\right\|. \tag{2.9}$$

Since $\forall R \in \Re$, A and $R^{-1}AR$ have the same eigenvalues, Chen [3] found the following tightest condition for stability of interval matrices by applying Corollary 1 and Lemma 1 to the matrix W_h.

Theorem 3 *Suppose that $c_{ii} + h < 0$, $i = 1, 2, \ldots, n$ and for some $h \geq 0$. Then the interval matrix A^I is $h-$stable if $\rho[W_h] < 1$.*

An interesting consequence of this theorem, given by Chen [3], is that the condition $\rho[W_h] < 1$ coincides with the characterizations of a certain nonsingular M-matrix. In what follows we present these results.

Theorem 4 *The following statements are equivalent for a real matrix* $M = [m_{ij}]$ *with* $m_{ii} > 0$ *for* $i = 1, 2, ..., n$, *and* $m_{ij} \leq 0$ *for* $i \neq j$, $i, j = 1, 2, ..., n$.

1. *M is a nonsingular $M-$matrix.*

2. *There exists an $R \in \Re$ such that*

$$m_{ii}r_i > \sum_{j=1, j\neq i}^{n} r_j \left|m_{ij}\right|, \qquad i = 1, 2, ..., n. \tag{2.10}$$

3. *There exists an $R \in \Re$ such that $MR + RM^T$ is positive definite.*

4. *The real part of each eigenvalue of M is positive.*

Now define $U_h = [u_{ij}]$ by

$$u_{ij} = \begin{cases} |c_{ii} + h| & \text{if } i = j, \\ -max\{|b_{ij}|, |c_{ij}|\} & \text{if } i \neq j. \end{cases} \tag{2.11}$$

Using this definition and Theorem 4, Chen [3] obtained the following result which relates the problem of finding the tightest condition for h-stability to the characterization of U_h as being a nonsingular $M-$matrix.

Theorem 5 *Let W_h and U_h be as defined above. Then $\rho[W_h] < 1$ if and only if U_h is a nonsingular M matrix.*

Remark 4 *It follows that A^I is $h-$stable if $c_{ii} + h < 0$, $h \geq 0$, for all i and U_h is a nonsingular $M-$matrix.*

2.2.3 Sufficient Conditions, The Perturbed Matrix

The sufficient conditions given by the above results are rather restrictive because they require $c_{ii} + h < 0$, for all i. Using the extension of Gershgorin's Theorem, given in Remark 3, Chen [3] obtained the sufficient conditions given in the following theorem that relax this restriction.

Theorem 6 *Suppose that* $Re[\bar{\lambda_i}] < -h, \ \ i = 1, 2, ..., n, \ h \geq 0$. *Then*

1. A^I *is* $h-$*stable if*

$$Re[\bar{\lambda_i}] + \sum_{j=1, j\neq i}^{n} f_{ij} < -h, \qquad i = 1, 2, ..., n. \tag{2.12}$$

2. A^I *is* $h-$*stable if there exists an* $R \in \Re$ *such that*

$$Re[\bar{\lambda_i}] + \sum_{j=1, j\neq i}^{n} \frac{r_j}{r_i} f_{ij} < -h, \qquad i = 1, 2, ..., n. \tag{2.13}$$

Define $\Gamma_h = [\gamma_{ij}]$ by

$$\gamma_{ij} = \frac{f_{ij}}{|Re[\bar{\lambda_i}] + h\,|}. \tag{2.14}$$

Then the following compact version of Theorem 6 was also given by Chen in [3].

Corollary 2 *Suppose that* $Re[\bar{\lambda_i}] + h < 0, \ i = 1, 2, ..., n, \ h \geq 0$. *Then*

1. A^I *is* $h-$*stable if* $\|\Gamma_h\|_\infty < 1$.

2. A^I *is* $h-$*stable if there exists an* $R \in \Re$ *such that* $\|R^{-1}\Gamma_h R\|_\infty < 1$.

Again using Lemma 1 and Corollary 2, Chen [3] gave the tightest condition for the $h-$stability of A^I as stated in the following theorem.

Theorem 7 *Suppose that* $Re[\bar{\lambda_i}] + h < 0$, $i = 1, 2, ..., n$, $h \geq 0$. *Then* A^I *is* $h-$*stable if* $\rho[\Gamma_h] < 1$.

Chen [3] also proved that this result also coincides with the characterization of a certain nonsingular $M-$matrix.

Corollary 3 *Let* Γ_h *be defined as above. Then* $\rho[\Gamma_h] < 1$ *if and only if* $I - \Gamma_h$ *is a nonsingular* $M-$*matrix.*

2.3 Positive Definiteness and Stability

A summary of the results obtained in [4] for positive definiteness and stability of interval matrices is as follows:

1. An interval matrix has one of these properties if and only if it is true for a finite subset of test matrices constructed from the given interval matrix.

2. A symmetric interval matrix is positive definite (stable) if and only if it is regular and contains at least one symmetric positive definite (stable) matrix.

First we present some notations and a proposition used by Rohan to prove the results stated above. Corresponding to any interval matrix $A^I = [A^m - \Delta, A^m + \Delta]$ one can construct a symmetric interval matrix

$$A_s^I = [A^{m'} - \Delta', A^{m'} + \Delta'] \tag{2.15}$$

where $A^{m'}$ and Δ' are given by

$$A^{m'} = \frac{1}{2}(A^m + A^{mT}), \text{ and } \Delta = \frac{1}{2}(\Delta + \Delta^T). \tag{2.16}$$

Clearly, if $A \in A^I$, then $\frac{1}{2}(A+A^T) \in A^I_s$ and A^I is symmetric if and only if $A^I = A^I_s$.

Define an indexing set

$$Y = \{z \in R^n : |z_j| = 1 \; for \; j = 1, 2, ..., n\}, \tag{2.17}$$

i.e., Y is the set of all vectors in R^n with components as ± 1 and hence its cardinality is 2^n. For each $z \in Y$, let T_z be an $n \times n$ diagonal matrix with diagonal vector z and A_z represents an $n \times n$ matrix defined by

$$A_z = A^m - T_z \Delta T_z. \tag{2.18}$$

It is clear that for each (i, j),

$$(A_z)_{ij} = (A^m)_{ij} - z_i \Delta_{ij} z_j = \begin{cases} (A^m - \Delta)_{ij} & \text{if } z_i z_j = 1 \\ (A^m + \Delta)_{ij} & \text{if } z_i z_j = -1 \end{cases}. \tag{2.19}$$

Hence for each $z \in Y$, $A_z \in A^I$. Because $A_{-z} = A_z$, the number of mutually different matrices A_z is at most 2^{n-1}. If A^I is symmetric then each A_z is symmetric.

Define a function $f : R^{n\times n} \to R^1$ by

$$f(A) = \min_{||x||=1} x^T A x \quad \text{for } A \in R^{n\times n}. \tag{2.20}$$

Since for any square matrix A, $g(x) = x^T A x$ is continuous on the unit sphere in R^n, then f is well defined. The following Proposition sums up some basic properties of f that were used by Rohan in [4] to prove the main results.

Proposition 1 *The function f has the following properties:*

(i) $f(A) = f(\frac{1}{2}(A+A^T))$ *for each* $A \in R^{n\times n}$;

(ii) $f(A) = \lambda_{min}(A)$ *for each symmetric* $A \in R^{n\times n}$;

(iii) $|f(A+D)-f(A)| \leq \rho(\frac{1}{2}(D+D^T))$ *for each* $A, D \in R^{n\times n}$;

(vi) f *is continuous in* $R^{n\times n}$;

(v) for each interval matrix A^I *we have*

$$min\{f(A) : A \in A^I\} = min\{f(A_z) : z \in Y\};$$

(vi) for each interval matrix A^I *we have*

$$min\{f(A) : A \in A^I\} = min\{f(A) : A \in A^I_s\};$$

(vii) each interval matrix $A^I = [A^m - \Delta, A^m + \Delta]$ *satisfies*

$$min\{f(A) : A \in A^I\} \geq f(A^m) - \rho(\Delta');$$

(viii) if A *is symmetric and* $f(A) = 0$, *then* A *is singular.*

2.3.1 Positive Definiteness of Interval Matrices

From (2.20) it is clear that A is positive definite if $f(A) > 0$ for all $x \in R^n$, such that $||x|| = 1$. As a consequence of Proposition 1, Rohan [4] obtained the following result which relates the positive definiteness of an interval matrix A^I and corresponding symmetric interval matrix A^I_s.

Theorem 8 *Let* A^I *be a square interval matrix. Then the followings are equivalent:*

(a) A^I *is positive definite,*

(b) A^I_s *is positive definite,*

(c) A_z *is positive definite for each* $z \in Y$.

Remark 5 *The assertion (c) shows that positive definiteness of an interval matrix can be verified by testing 2^{n-1} matrices, taken from A^I, for positive definiteness. The equivalence "(a) $\Leftrightarrow$ (b)" reveals another property that verification of positive definiteness of A^I can always be performed by inspecting the associated symmetric interval matrix A^I_s. Hence for positive definiteness we can restrict our attention to symmetric interval matrices only.*

The next result describes the relation between positive definiteness and regularity of a symmetric interval matrix [4].

Theorem 9 *A symmetric interval matrix A^I is positive definite if and only if it is regular and contains at least one positive definite matrix.*

The necessary and sufficient condition in Theorem 9 requires only one matrix to be checked for positive definiteness. However, the result is not as pleasant as it might seem. Verifying regularity of an interval matrix is generally a difficult problem as can be seen in [22].

2.3.2 Stability of Interval Matrices

In this subsection, we present the results given in [4] for stability of symmetric interval matrices. These results turn out to be closely connected to the contents of the previous section due to the well known result that a symmetric matrix A is stable if and only if $-A$ is positive definite. In contrast to the results for positive definiteness, where the matrices $A_z = A^m - T_z \Delta T_z$ were employed, Rohan [4] characterized the stability in terms of the matrices

$$\bar{A}_z = A^m + T_z \Delta T_z, \quad z \in Y.$$

Obviously, $\bar{A}_z \in A^I$ and all $\bar{A}_z$ are symmetric if A^I is symmetric.

Theorem 10 *Let $A^I = [A^m - \Delta, A^m + \Delta]$ be a symmetric interval matrix. Then the following assertions are equivalent:*

(a) A^I is stable,

(b) $A^I_o = [-A^m - \Delta, -A^m + \Delta]$ is positive definite,

(c) $\bar{A}_z$ is stable for each $z \in Y$.

In Theorem 8 it has been shown that problem of checking positive definiteness of a general interval matrix can be equivalently formulated in terms of the problem of checking the associated symmetric interval matrix. But this nice property is not true for stability, where only one implication is true [4].

Theorem 11 *If A^I_s is stable then A^I is stable.*

Now we present the corresponding versions of Theorem 9 for the case of stability as given in [4].

Theorem 12 *A symmetric interval matrix A^I is stable if and only if it is regular and contains at least one stable symmetric matrix.*

2.4 Algorithm for Checking Stability

In this section a branch-and bound type algorithm for checking stability of symmetric interval matrices is presented, which is based on necessary and/or sufficient stability conditions [14]. As proved in [14] this algorithm cannot be expected to circumvent exponentially in the verification process in general, but due to the built-in branch-and-bound strategy it essentially reduces the amount of computations in many cases. This algorithm always either verifies the stability of a symmetric interval matrix A^I or finds an unstable matrix in A^I.

Let $A^I = [B, C]$ be an $n \times n$ symmetric interval matrix. Define the set

$$Y_o = \{z \in R^n : z_i \in \{-1, 0, 1\} \text{ for each } j\}. \tag{2.21}$$

Using this set define two real matrices

$$(A_z)_{ij} = \begin{cases} \frac{1}{2}(b_{ij} + c_{ij}) & \text{if } z_i z_j = 0 \text{ and } i \neq j \\ c_{ij} & \text{if } z_i z_j = 1 \text{ or } i = j \\ b_{ij} & \text{if } z_i z_j = -1 \end{cases}, \tag{2.22}$$

and

$$(\Delta_z)_{ij} = \begin{cases} \frac{1}{2}(c_{ij} - b_{ij}) & \text{if } z_i z_j = 0 \text{ and } i \neq j \\ 0 & \text{if } z_i z_j \neq 0 \text{ or } i = j \end{cases}, \tag{2.23}$$

where $i, j = 1, 2, \ldots, n$. Since A^I is symmetric, therefore for each $z \in Y_o$, both the matrices A_z and Δ_z are symmetric, $A_z \in A^I$ and $\Delta_z \geq 0$.

Now consider the indexing set

$$Y = \{z \in R^n : z_i \in \{-1, 1\} \text{ for each } i\}.$$

Clearly $Y \subset Y_o$. Using Y and A_z Rohan [14] proved the following necessary and sufficient condition for stability of symmetric interval matrices.

Theorem 13 *A symmetric interval matrix A^I is stable if and only if A_z is stable for each $z \in Y$, $z_1 = 1$.*

The following sufficient condition for stability of interval matrices is also given by Rohan in [14].

Theorem 14 *A symmetric interval matrix* $A^I = [A^m - \Delta, A^m + \Delta]$ *is stable if the matrix* $A^m + ||\Delta||_1 I$ *is stable.*

Checking the stability of a symmetric matrix A can be performed by computing $\lambda_{max}(A)$ and checking its negativity. Now define a sign vector $sgn\, x$ by

$$(sgn\, x)_i = \begin{cases} 1 & \text{if } x_i \geq 0 \\ -1 & \text{if } x_i < 0 \end{cases} . \tag{2.24}$$

Clearly $sgn\, x \in Y$ for each x. Rohan [14] used this definition to construct the algorithm described below which generates a sequence of matrices A_z with ascending values of $\lambda_{max}(A_z)$ and verifies the instability of A^I.

Problem 1: Check the instability of symmetric interval matrix $A^I = [A^m - \Delta, A^m + \Delta]$.

Algorithm 1:

1. Compute $\lambda_{max}(A_z)$ and corresponding eigenvector x.

2. **repeat**

   $z = sgn\, x$

   compute $\lambda_{max}(A_z)$ and corresponding eigenvector x

   until $(\lambda_{max}(A_z) \geq 0$ or $\Delta_{ij} z_i x_i z_j x_j \geq 0$ for each $i, j)$

3. **if** $\lambda_{max}(A_z) \geq 0$ **then** { A^I is unstable} **else** {instability is not verified}, **Stop.**

The algorithm 2, which is the main algorithm, is described as follows [14]:

Starting from A^I it proceeds towards the matrices A_z, $z \in Y$, $z_1 = 1$ used in the necessary and sufficient conditions of Theorem 13. This is done by starting from $z^* = (1, 0, ..., 0)^T$ and replacing zeros in the $z's$ by -1 or 1. If for some $z \in Y_o$, the symmetric interval matrix $[A_z^m - \Delta_z, A_z^m + \Delta_z] \subseteq A^I$ is found to be unstable by using Algorithm 1, then this algorithm terminates with the result that A^I is unstable else it checks the stability of $A_z^m + ||\Delta_z||_1 I$. If $A_z^m + ||\Delta_z||_1 I$ is stable, then the interval matrix $[A_z^m - \Delta_z, A_z^m + \Delta_z]$ is stable and is removed from further considerations. If $A_z^m + ||\Delta_z||_1 I$ is not stable, then replace some zero entries in the current vector z by -1 or 1. In order to do this, first find indices where perturbation is maximum in $[A_z^m - \Delta_z, A_z^m + \Delta_z]$. That is find (i, j), $i < j$, satisfying

$$(\Delta_z)_{ij} = \max_{k < m} (\Delta_z)_{km}. \tag{2.25}$$

Since A_z^m is stable, $A_z^m + || \Delta_z||_1 I$ is not stable, and Δ_z is symmetric, therefore $(\Delta_z)_{ij} > 0$ and $z_i z_j = 0$. Set $h = i$ if $z_i = 0$ and $h = j$ otherwise, so that $z_h = 0$. Construct two new vectors $z^1$and $z^2 \in Y_o$ by

$$z^1 = \begin{cases} -1 & \text{if } i = h \\ z_k & \text{otherwise} \end{cases} \quad \text{and} \quad z^2 = \begin{cases} 1 & \text{if } i = h \\ z_k & \text{otherwise} \end{cases}, \tag{2.26}$$

and put them into the set L of items to be tested.

Problem 2: Check the stability of symmetric interval matrix $A^I = [A^m - \Delta, A^m + \Delta]$.

Algorithm2:

1. Put the vector $(1, 0, ..., 0)^T$ in the set L and make unstable = false.

2. **repeat**

   remove the top most entry from L

   compute A^m_z and Δ_z

   verify the instability of $[A^m_z - \Delta_z, A^m_z + \Delta_z]$ by using Algorithm 1.

   if $[A^m_z - \Delta_z, A^m_z + \Delta_z]$ is unstable **then** unstable = true

   else

   if $A^m_z + ||\Delta_z||_1 I$ is unstable **then**

   find i, j satisfying (2.25)

   if $z_i = 0$ **then** $h = i$ **else** $h = j$

   construct $z^1$and z^2 by (2.26)

   insert z^1 and z^2 into the set L

   until ($L = \Phi$ or unstable)

3. **if** unstable = true **then** { A^I is unstable} else {A^I is stable}. **Stop.**

Chapter 3

Stability, Positive Definiteness and Regularity of Interval Matrices: Some Further Extensions

3.1 Introduction

This chapter is organized as follows. In section 2, based on some results from [3], we prove the necessity of a sufficient condition for h-stability of an interval matrix and establish a result which relates the problem of determining the stability of an interval matrix A^I to that of the inversion of a point matrix constructed from A^I. In section 3, we extend the results given by Rohan [4] for positive definiteness and stability to h-positive definiteness and h-stability of interval matrices. In section 4, first we state some well-known results for linear operators and establish some new necessary and sufficient conditions for the regularity of interval matrices. Using these regularity conditions, we establish necessary and sufficient conditions for stability of symmetric interval matrices which are also sufficient conditions for any interval matrix. Based on these conditions, we develop algorithms for checking the regularity and stability of interval matrices. We demonstrate the effectiveness of our algorithms by some numerical examples.

3.2 Gershgorin's Theorem and Stability

Let $A^I = [B, C]$ be an interval matrix and define the matrix $E = [e_{ij}]$ by

$$e_{ij} = \begin{cases} c_{ii} & \text{if } i = j \\ \max\{|b_{ij}|, |c_{ij}|\} & \text{if } i \neq j \end{cases} \tag{3.1}$$

Remark 6 *An interval matrix $A^I = [B, C]$ is $h-$stable for some $h \geq 0$, if and only if the interval matrix $\bar{A}^I = [B + hI, C + hI]$ is stable.*

Theorem 15 *Let $A^I = [B, C]$ be an interval matrix with $c_{ii} < 0$ for all $i = 1, 2, \ldots, n$ and $b_{ij} + c_{ij} \geq 0$ for all $i \neq j$. Then A^I is stable if and only if E is stable.*

Proof. The condition $b_{ij} + c_{ij} \geq 0$ implies $E \in A^I$. Therefore stability of A^I implies stability of E.

To prove the converse suppose that E is stable. Then real part of each eigenvalue of E is negative, therefore real part of each eigenvalue of U is positive. Then by Theorem 4, U is a nonsingular M-matrix and by Theorem 5, $\rho[W] < 1$. Hence by Theorem 3, A^I is stable. ■

Using Remark6 and Theorem 15, we have the following corollary for $h-$stability.

Corollary 4 *Let $A^I = [B, C]$ be an interval matrix with $c_{ii} + h < 0$ for all $i = 1, 2, \ldots, n$ and $b_{ij} + c_{ij} \geq 0$ for all $i \neq j$. Then A^I is $h-$stable if and only if E is $h-$stable.*

Proof. Suppose that A^I is $h-$stable, then $\bar{A}^I$ is stable and $E + h \in \bar{A}^I$. Thus $E + h$ is stable and hence E is $h-$stable.

Conversely if E is $h-$stable and $E + h$ is stable and by Theorem15 $\bar{A}^I$ is stable, which implies A^I is $h-$stable.

From the interval matrix A^I defined by (1.1) for which $c_{ii} + h < 0$, $h \geq 0$, define a matrix $V_h = [v_{ij}]$ by

$$v_{ij} = \begin{cases} 1 & \text{if } i = j, \\ \frac{max\{|b_{ij}|, |c_{ij}|\}}{c_{ii} + h} & \text{if } i \neq j. \end{cases} \qquad (3.2)$$

From this definition it is clear that $v_{ij} \leq 0$ for $i \neq j$.

The next result relates the problem of checking the $h-$stability of an interval matrix A^I to that of checking the inverse of a point matrix constructed from A^I. Its proof is based upon the following lemma due to Verga [8].

Lemma 2 *Let $A \geq 0$ be an $n \times n$ matrix. Then the following are equivalent:*

1. *$\rho[A] < 1$,*

2. *$I - A$ is nonsingular and $(I - A)^{-1} \geq 0$.*

Theorem 16 *Let A^I be an interval matrix with $c_{ii} + h < 0$ for all $i = 1, 2, \ldots, n$, $h \geq 0$. Then A^I is h-stable if V_h is nonsingular and $(V_h)^{-1} \geq 0$.*

Proof. Since $c_{ii} + h < 0$, therefore $I - V_h = W_h$ and $W_h \geq 0$, where W_h is defined by (2.8). Then by Lemma 2, $V_h = I - W_h$ being nonsingular and $(V_h)^{-1} = (I - W_h)^{-1} \geq 0$ implies that $\rho[W_h] < 1$. Hence by Theorem 3, A^I is $h-$stable. ■

3.3 h-Positive Definiteness and h-Stability

In this section, we prove some results which extend the results on the positive definiteness and stability of symmetric interval matrices, given in [4], to h-positive definiteness and h-stability, $h \geq 0$.

Definition 1 *A square matrix A is said to be positive definite with positive definiteness margin h, for some $h \geq 0$, or h-positive definite if $x^T Ax > h$, for each $x \in R^n$ such that $||x|| = 1$.*

Remark 7 *Thus in view of function defined by (2.20), A is h-positive definite if $f(A) = \min_{\|x\|=1} x^T Ax > h$. An interval matrix A^I is said to be h-positive definite if each $A \in A^I$ is h-positive definite.*

3.3.1 h-Positive Definiteness

As a consequence of Proposition 1, we have the following characterizations of h-positive definiteness of interval matrices.

Proposition 2 *Let A^I be a square interval matrix and $h \geq 0$. Then the following assertions are equivalent:*

(a) A^I is h-positive definite,

(b) A^I_s is h-positive definite,

(c) A_z is h-positive definite for each $z \in Y$.

Proof. By definition A^I is h-positive definite if and only if

$$min\{f(A) : A \in A^I\} > h$$

holds. Then equivalence of (a) and (b), and of (a) and (c) follows from assertion (vi) and (v) of Proposition1 respectively. ■

Remark 8 *Part (c) of Proposition 2 shows that h-positive definiteness of an interval matrix can be verified by testing 2^{n-1} matrices from A^I for h-positive definiteness. The equivalence of (a) and (b) reveals that h-positive definiteness of any interval matrix A^I can be checked by inspecting the associated symmetric interval matrix A^I_s*

defined by (2.15). Hence for h−positive definiteness, we may restrict our attention to symmetric interval matrices.

Theorem 17 *An interval matrix $A^I = [A^m - \Delta, A^m + \Delta]$ is h-positive definite, for some $h \geq 0$, if and only if the interval matrix $\tilde{A}^I = [A^m - hI - \Delta, A^m - hI + \Delta]$ is positive definite.*

Proof. For $A \in A^I$, write $\tilde{A} = A - hI \in \tilde{A}^I$. Then A^I is h-positive definite if and only if

$$f(A) = \min_{\|x\|=1} x^T A x > h \qquad \forall A \in A^I$$

if and only if

$$f(\tilde{A}) = f(A - hI) = \min_{\|x\|=1} x^T A x - h > 0$$

if and only if $\tilde{A}$ is h−positive definite if and only if $\tilde{A}^I$ is h−positive definite. ■

Using Theorem 9 and 17, we have the following corollary.

Corollary 5 *A symmetric interval matrix $A^I = [A^m - \Delta, A^m + \Delta]$ h-positive definite if and only if the symmetric interval matrix $\tilde{A}^I = [A^I - hI - \Delta, A - hI + \Delta]$ is regular and contains at least one positive definite matrix.*

The necessary and sufficient conditions of the above corollary require only one matrix to check for positive definiteness. However verifying regularity of an interval matrix is generally a difficult problem as it can be seen from [22], where a number of necessary and sufficient regularity conditions are given, all of which require testing at least 2^{n-1} quantities of some sort (as evaluating determinant, solving systems of linear equations, inverting matrices etc.).

3.3.2 h-Stability

We remark that a symmetric matrix A is h-stable if and only if $-A$ is h-positive definite. Also it follows from Corollary 5 that $A^I = [A^m - \Delta, A^m + \Delta]$ is $h-$stable if and only if $\bar{A}_o^I = [-A^m - hI - \Delta, -A^m - hI + \Delta]$ is positive definite if and only if $\bar{A}_z$ is h-stable for each $z \in Y$, where A_z, $z \in Y$ are as defined in Section 2 of Chapter 2.

In Proposition 2, we showed that h-positive definiteness of a general interval matrix A^I is equivalent to the $h-$positive definiteness of A_s^I. Unfortunately this Proposition does not hold for h-stability. For $h-$stability we have the following theorem.

Theorem 18 *An interval matrix A^I is h-stable if the corresponding symmetric interval matrix A_s^I is h-stable.*

Proof. Let λ be an eigenvalue of $A \in A^I$. Then $\frac{1}{2}(A+A^T) \in A_s^I$ and by the Bandixon theorem ([18], pp.395), we have

$$Re[\lambda] \le \lambda_{max}\left(\frac{1}{2}(A + A^T)\right) < -h.$$

Hence A^I is h-stable. ■

Example 1 : *The interval matrix $A^I = [A^m - \Delta, A^m + \Delta]$ where*

$$A^m = \begin{bmatrix} -2.5 & 4.5 & 5 & 0.25 \\ -3.5 & -3.5 & -3.5 & 1.5 \\ -4.5 & 2.5 & -4.5 & -0.5 \\ -0.45 & 0.5 & 1.5 & -3.25 \end{bmatrix} \quad \text{and} \quad \Delta = \begin{bmatrix} 0.5 & 0.5 & 1 & 1.25 \\ 0.5 & 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0.5 & 0.5 \\ 0.55 & 0.5 & 0.5 & 0.75 \end{bmatrix}$$

is stable. The corresponding symmetric interval matrix $A_s^I = [A^{m'} - \Delta', A^{m'} + \Delta']$ is *not stable because it contains*

$$\begin{bmatrix} -2 & 1 & 1 & 0.8 \\ 1 & -3 & 0 & 1.5 \\ 1 & 0 & -4 & 1 \\ 0.8 & 1.5 & 1 & -2.5 \end{bmatrix}$$

which has a positive eigenvalue 0.0631.

Theorem 19 *A symmetric interval matrix $A^I = [A^m - \Delta, A^m + \Delta]$ is h-stable if and only if the symmetric interval matrix $\tilde{A}^I = [A^m + hI - \Delta, A^m + hI + \Delta]$ is stable.*

Proof. For $z \in Y$, write $\tilde{A}_z = \bar{A}_z + hI$, where $\bar{A}_z \in A^I$. Then A^I is h-stable $\bar{A}_z$ is $h-$stable if and only if $\lambda + h < 0$ for all eigenvalues λ of $\bar{A}_z$ if and only if $\tilde{A}_z$ is stable if and only if $\tilde{A}^I$ is stable. ■

Corollary 6 *A symmetric interval matrix $A^I = [A^m - \Delta, A^m + \Delta]$ is h-stable if and only if $\tilde{A}^I = [A^m + hI - \Delta, A^m + hI + \Delta]$ is regular and contains at least one h-stable symmetric matrix.*

Remark 9 *The Algorithm 2 given in Chapter 2 is to verify the stability of a symmetric interval matrix. Using the same algorithm, we can verify the $h-$stability of a symmetric interval matrix A^I, if we store eigenvalues computed in each iteration and take maximum of these eigenvalues. This maximum will be the margin of stability if it is negative otherwise A^I will be unstable.*

3.4 Regularity and Stability

In this section, we first state and prove some well known results for bounded linear operators [15]. These results are used in the proofs of the main results of the section. These results provide necessary and sufficient conditions for the regularity and stability of interval matrices.

For a bounded linear operator T on a normed linear space X, $\rho(T)$ denotes the set of all numbers λ in the complex plane for which $R(\lambda; T) \equiv (\lambda I - T)^{-1}$ exist as a bounded operator. $\rho(T)$ is called the resolvent of T. The complement of $\rho(T)$, denoted by $\sigma(T)$, is called the spectrum of T. In the case when $X = R^n$ the spectrum of T consists of all eigenvalues of T.

Lemma 3 *[15] Let $\ell(X)$ be the set of bounded linear operators on a normed linear space X. Then the set G of elements in $\ell(X)$, which have inverses in $\ell(X)$, is an open set with respect to the uniform topology on $\ell(X)$. Furthermore for $A \in G$ the sphere*

$$\{B : ||A - B|| < ||A^{-1}||^{-1}\} \tag{3.3}$$

is contained in G and the inverse of an element B of this sphere is given by

$$B^{-1} = A^{-1} \sum_{n=0}^{\infty} [(A - B)A^{-1}]^n. \tag{3.4}$$

Proof. *Let $||I - B|| < 1$, so that the series $S = \sum_{n=0}^{\infty} (I - B)^n$ converges. Since*

$$SB = BS = [I - (I - B)]S = \sum_{n=0}^{\infty} (I - B)^n - \sum_{n=1}^{\infty} (I - B)^n = I \tag{3.5}$$

It follows that $B^{-1} = S$ and $\{B : ||I - B|| < 1\} \subset G$. Now let $A \in G$, and let

$||A-B|| < ||A^{-1}||^{-1}$ *Then*

$$||I-BA^{-1}|| = ||(A-B)A^{-1}|| < 1.$$

Hence BA^{-1} *has an inverse in* $\ell(X)$, *given by the series*

$$\left(BA^{-1}\right)^{-1} = \sum_{n=0}^{\infty} [(A-B)A^{-1}]^n \tag{3.6}$$

and this in turn implies

$$B^{-1} = A^{-1} \sum_{n=0}^{\infty} [(A-B)A^{-1}]^n. \tag{3.7}$$

■

Corollary 7 *[15]Let* T, T_1 *be in* $\ell(X)$, $\lambda \in \rho(T)$ *and* $||T - T_1|| < ||R(\lambda;T)||^{-1}$. *Then* $\lambda \in \rho(T_1)$ *and*

$$R(\lambda;T_1) = R(\lambda;T) \sum_{n=0}^{\infty} [(T_1 - T)\ R(\lambda;T)]^n.$$

Proof. $\lambda \in \rho(T) \Rightarrow R(\lambda;T) = (\lambda I - T)^{-1}$ *exist as a bounded operator. Let* $A = \lambda I - T$ *and* $B = \lambda I - T_1$. *Then* $A^{-1} = (\lambda I - T)^{-1}$ *exist and* $R(\lambda;T) = A^{-1}$. *Therefore*

$$||T - T_1|| = ||(\lambda I - T_1) - (\lambda I - T)|| = ||B - A|| < ||A^{-1}||^{-1} \tag{3.8}$$

$\Rightarrow B$ *has inverse in* $\ell(X)$. *That is* $B^{-1} = (\lambda I - T_1)^{-1} = R(\lambda;T_1)$ *exist as a bounded operator. Therefore* $\lambda \in \rho(T_1)$.

Now

$$B^{-1} = A^{-1} \sum_{n=0}^{\infty} [(A-B)A^{-1}]^n \tag{3.9}$$

$\Rightarrow$

$$R(\lambda;T_1) = R(\lambda;T) \sum_{n=0}^{\infty} [(T_1 - T)\ R(\lambda;T)]^n. \tag{3.10}$$

■

Remark 10 *It follows from above corollary that if $\lambda \notin \sigma(A)$ and $\|A - A_1\| < \|(\lambda I - A)^{-1}\|^{-1}$ then $\lambda \notin \sigma(A_1)$. Hence if A is nonsingular and $\|A - A_1\| < \|A^{-1}\|^{-1}$ then A_1 is nonsingular.*

3.4.1 Necessary and Sufficient Conditions for Regularity

We now establish some necessary and sufficient conditions for the regularity of an interval matrix $A^I = [B, C]$. First we prove a sufficient condition that A^I is regular if it satisfies the following conditions:

$(H.1)$ $A^m = \frac{1}{2}(B + C)$ is nonsingular,

$(H.2)$ $\|C - B\|_\infty < \frac{2}{\|(A^m)^{-1}\|_\infty}$.

Lemma 4 *If the interval matrix $[B, C]$ satisfies assumptions $(H.1)$ and $(H.2)$, then it is regular.*

Proof. To show that $[B, C]$ is regular, we have to show that each $A \in [B, C]$ is nonsingular. Let $A \in [B, C]$ be any point matrix and $\Delta A = A - A^m$. Then $\Delta A = [\Delta a_{ij}]$ satisfies

$$|\Delta a_{ij}| = \left| a_{ij} - \frac{1}{2}(b_{ij} + c_{ij}) \right| \leq \frac{1}{2}(c_{ij} - b_{ij}), \quad 1 \leq i, j \leq n,$$

which implies that

$$\|A - A^m\|_\infty = \|\Delta A\|_\infty \leq \frac{1}{2}\|C - B\|_\infty < \frac{1}{\|(A^m)^{-1}\|_\infty}. \tag{3.11}$$

It follows from (3.11) and Remark 10 that A is nonsingular. Since $A \in [B, C]$ is arbitrary, $[B, C]$ is regular. ■

In developing our necessary conditions for regularity of interval matrix A^I we will need the following lemma.

Lemma 5 *Suppose that* $[B, C]$ *is regular. Then there exist a constant* $r > 0$ *such that for any subinterval matrix* $[B_o, C_o] \subseteq [B, C]$, $[B_o, C_o]$ *satisfies assumptions* $(H.1)$ *and* $(H.2)$ *as long as* $\|C_o - B_o\|_\infty < r$.

Proof. By supposition $[B, C]$ is regular, therefore each $A \in [B, C]$ is nonsingular. Thus for each $A \in [B, C]$ there exist a nonsingular matrix $D = D(A)$ such that

$$AD = I. \tag{3.12}$$

Since $[B, C]$ is a compact set in $R^{n \times n}$ and since every continuous function on a compact set assumes its minimum value, there exist a constant $r > 0$ such that

$$r \leq \frac{1}{\|D(A)\|_\infty} < \frac{2}{\|D(A)\|_\infty} \quad \text{for all } A \in [B, C].$$

Any $[B_o, C_o] \subseteq [B, C]$, with $\|C_o - B_o\|_\infty < r$ satisfies

$$\|C_o - B_o\|_\infty < \frac{2}{\|D\|_\infty},$$

where $D = D(A_o^m)$, $A_o^m = \frac{1}{2}(B_o + C_o)$.

The regularity of $[B, C]$ together with the above condition imply that $[B_o, C_o]$ satisfies assumptions $(H.1)$ and $(H.2)$. ■

Using Lemmas 3 and 4 we now prove the main results of this section. Theorem 20 provides a necessary and sufficient condition for the regularity of an interval matrix, Theorem 21 gives some necessary and sufficient conditions for the stability of a symmetric interval matrix and Theorem 22 gives a sufficient condition for the stability of any interval matrix.

Theorem 20 *An interval matrix $[B, C]$ is regular if and only if it contains at least one nonsingular matrix and there are finitely many subinterval matrices $[B_i, C_i] \subseteq [B, C]$, $1 \le i \le k$, such that*

$$[B, C] = \bigcup_{i=1}^{k} [B_i, C_i], \tag{3.13}$$

and for each $1 \le i \le k$, $[B_i, C_i]$ satisfies assumptions $(H.1)$ and $(H.2)$.

Proof. (Sufficiency) Assume that $[B_i, C_i]$ satisfies assumptions $(H.1)$ and $(H.2)$ for each $1 \le i \le k$. Then by Lemma 4, $[B_i, C_i]$ is regular for each $1 \le i \le k$ and equation (3.13) implies that $[B, C]$ is regular.

(Necessity) Given that $[B, C]$ is regular. Then by Lemma 5, there exist a constant $r > 0$ such that any subinterval matrix $[B_o, C_o] \subseteq [B, C]$, satisfies assumptions $(H.1)$ and $(H.2)$ as long as $\|C_o - B_o\|_\infty < r$.

Since $[B, C]$ is a hyperrectangle in R^{n^2}, we can subdivide it into a finite number of hyperrectangles $[B_i, C_i]$, $1 \le i \le k$ such that $\|C_i - B_i\|_\infty < r$ for each $1 \le i \le k$. (Notice that each $A \in R^{n \times n}$ satisfies $\|A\|_\infty \le n \max_{1 \le i \le k} |a_{ij}|$). Therefore by Lemma 5, all the subinterval matrices $[B_i, C_i]$, $1 \le i \le k$ satisfy assumptions $(H.1)$ and $(H.2)$.

■

3.4.2 Necessary and Sufficient Conditions for Stability

The proof of the following result uses Theorem 12.

Theorem 21 *A symmetric interval matrix $[B, C]$ is stable if and only if it contains at least one stable matrix and there are finitely many subinterval matrices $[B_i, C_i]$, $1 \le$*

$i \leq k$ such that

$$[B, C] = \bigcup_{i=1}^{k} [B_i, C_i]$$

and for each $1 \leq i \leq k$, $[B_i, C_i]$ satisfies assumptions $(H.1)$ and $(H.2)$.

Proof. (Sufficiency) Since every stable matrix is nonsingular. It follows that if $[B, C]$ contains a stable matrix and $[B_i, C_i]$ satisfies assumptions $(H.1)$ and $(H.2)$ for each $1 \leq i \leq k$, then $[B, C]$ is regular. Hence by Theorem 8 of [3] it follows that $[B, C]$ is stable.

(Necessity) If $[B, C]$ is stable then it is regular and proof follows form Theorem 20. ∎

Remark 11 : *Given any interval matrix $A^I = [B, C]$, we can construct a symmetric interval matrix $A_s^I = [B', C']$, where*

$$B' = \frac{1}{2}(B + B^T) \quad \text{and} \quad C' = \frac{1}{2}(C + C^T) \tag{3.14}$$

Using a result given in [3] which states that stability of symmetric interval matrix A_s^I implies the stability of A^I, the following result follows Theorem 21.

Theorem 22 *If $A_s^I = [B', C']$ contains at least one stable matrix and there are finitely many subinterval matrices $[B_i', C_i']$, $1 \leq i \leq k$, such that*

$$[B', C'] = \bigcup_{i=1}^{k} [B_i', C_i']$$

and each of $[B_i', C_i']$ satisfy assumptions $(H.1)$ and $(H.2)$ then A^I is regular.

3.4.3 Necessary and Sufficient Conditions for Positive Definiteness

The proof of the following result uses Theorem 9.

Theorem 23 *A symmetric interval matrix $[B, C]$ is positive definite if and only if it contains at least one positive definite matrix and there are finitely many subinterval matrices $[B_i, C_i]$, $1 \leq i \leq k$ such that*

$$[B, C] = \bigcup_{i=1}^{k} [B_i, C_i]$$

and for each $1 \leq i \leq k$, $[B_i, C_i]$ satisfies assumptions $(H.1)$ and $(H.2)$.

Proof. (Sufficiency) Since every positive definite matrix is nonsingular. It follows that if $[B, C]$ contains a positive definite matrix and $[B_i; C_i]$ satisfies assumptions $(H.1)$ and $(H.2)$ for each $1 \leq i \leq k$, then $[B, C]$ is regular. Hence by Theorem 9 it follows that $[B, C]$ is positive definite.

(Necessity) If $[B, C]$ is positive definite then it is regular and proof follows form Theorem 20. ■

Remark 12 *Since any interval matrix A^I and corresponding symmetric interval matrix A^I_s are equivalently positive definite, therefore using the above result one can verify the positive definiteness of a general interval matrix.*

3.4.4 Algorithms

In this section we develop three algorithms which are based on Theorem 20, 21 and 22 for testing regularity of interval matrices and stability of symmetric interval matrices.

We demonstrate the effectiveness of our algorithms by some numerical examples.

The first algorithm is designed to check the regularity of a general interval matrix $A^I = [B, C]$. This algorithm determines the regularity of A^I by verifying the assumptions (H.1) and (H.2). If these assumptions are satisfied, then the algorithm terminates with the result that A^I is regular. If (H.1) is not satisfied, then the algorithm terminates with the result that A^I is not regular. If (H.1) is satisfied but (H.2) is not satisfied, then we subdivide the interval matrix A^I into two equal subinterval matrices and repeat the above process for each subinterval matrix. The algorithm continues in this manner until each subinterval matrix of $[B, C]$ is determined to be regular or at least one of the subinterval matrices is determined to be not regular.

Using the same algorithm with some modifications, we write second algorithm to check the regularity of given symmetric interval matrix A^I and stability of one point matrix in A^I to verify the stability A^I.

Our third algorithm construct a symmetric interval matrix A^I_s from a given interval matrix A^I and then using the second algorithm checks the stability of A^I_s.

The first algorithm answers

Problem 1 Given an interval matrix $A^I = [B, C]$ with $B = [b_{ij}]$ and $C = [c_{ij}]$, determine its regularity under the assumption that B and C are nonsingular.

Algorithm 1

1. Initialization: $B_1 = [b^1_{ij}] = B$, $C_1 = [c^1_{ij}] = C$ and $K_1 = \{1\}$.

2. Let $K = K_1$.

3. For every $k \in K$, compute $A^m_k = \frac{1}{2}(B + C) = [a^k_{ij}]_{n\times n}$ and check its non-

singularity. If for any k, A_k^m is singular, terminate the algorithm with the message that $[B,C]$ is not regular.

4. If for every k, A_k^m is nonsingular, find the inverse V_k of A_k^m.

5. For every $k \in K$, compute $P_k = C_k - B_k = [c_{ij}^k]_{n\times n}$, $\alpha_k = \|P_k\|_\infty$ and $\beta_k = \frac{2}{\|V_k\|_\infty}$.

6. If for every $k \in K$, $\alpha_k < \beta_k$, the interval matrix is regular. Stop. Otherwise, determine $K_o = \{k \in K : \alpha_k \geq \beta_k\}$.

7. For every $k \in K_o$, find the maximal element p_{rs}^k of the matrix P_k and partition $[B_k, C_k]$ into two interval matrices $[D_k, E_k]$ and $[F_k, G_k]$ where $D_k = B_k$, $G_k = C_k$, $E_k = [e_{ij}^k]$, and $F_k = [f_{ij}^k]$ with

$$e_{ij}^k = \begin{cases} a_{ij}^k & if\ i = r \text{ and } j = s \\ c_{ij}^k & \text{otherwise} \end{cases}$$

and

$$f_{ij} = \begin{cases} a_{ij}^k & if\ i = r \text{ and } j = s \\ b_{ij}^k & \text{otherwise} \end{cases}.$$

8. Relabel the set $\{[D_k, E_k], [F_k, G_k], k \in K_o\}$ using $\{[B_k, C_k], k \in K_1\}$ where $K_1 = \{1, 2, \ldots, N\}$, N is the number of subinterval matrices in step 7.

9. Go to step 2.

The second algorithm answers deal with the following problem

Problem 2 Given a symmetric interval matrix $A^I = [B, C]$ with $B = [b_{ij}]$ and $C = [c_{ij}]$, determine stability of A^I under the assumption that B and C are stable.

Algorithm 2

1. Check the stability of any point matrix in $A^I = [B, C]$, e.g., $A^m = \frac{1}{2}(B + C)$. If any eigenvalue of A^m is positive terminate the algorithm with the result that $[B, C]$ is not stable otherwise go to step 2.

2. Check the regularity of symmetric interval matrix $[B, C]$, using algorithm 1 with step 7 replaced by step 7′:

   For every $k \in K_o$, find the maximal element p_{rs}^k of the matrix P_k and partition $[B_k, C_k]$ into two interval matrices $[D_k, E_k]$ and $[F_k, G_k]$ where $D_k = B_k$, $G_k = C_k$, $E_k = [e_{ij}^k]$, and $F_k = [f_{ij}^k]$ with

$$e_{ij}^k = \begin{cases} a_{ij}^k & \text{if } i = r \text{ and } j = s \text{ or if } i = s \text{ and } j = r \\ c_{ij}^k & \text{otherwise} \end{cases}$$

   and

$$f_{ij}^k = \begin{cases} a_{ij}^k & \text{if } i = r \text{ and } j = s \text{ or if } i = s \text{ and } j = r \\ b_{ij}^k & \text{otherwise} \end{cases}.$$

Remark 13 *Algorithm 2 can also be used to verify the positive definiteness of a symmetric interval matrix A^I by checking one symmetric point matrix in A^I for positive definiteness instead of checking for stability.*

The third algorithm answers:

Problem 3. Given an interval matrix $A^I = [B, C]$ with $B = [b_{ij}]$ and $C = [c_{ij}]$. Construct corresponding symmetric interval matrix $A_s^I = [B', C']$ and determine stability of A^I.

Algorithm 3:

1. Compute $B' = \frac{1}{2}(B + B^T)$ and $C' = \frac{1}{2}(C + C^T)$.

2. Check the stability of $A^I_s = [B', C']$ using Algorithm 2.

3. If A^I_s turns out to be stable then A^I is also stable else no conclusion.

Chapter 4

Examples

Example 1: For the interval matrix $A^I = \begin{bmatrix} [-5,-3] & [1,2] \\ [4,5] & [-6,-4] \end{bmatrix}$ verify the stability by using Theorem 15.

Sol: For $h = 0.298$, we get $E_h = \begin{bmatrix} -2.702 & 2 \\ 5 & -3.702 \end{bmatrix} \in A^I$ and $W_h = \begin{bmatrix} 0 & 0.7402 \\ 1.3506 & 0 \end{bmatrix}$.
with $\rho[W_h] = 0.9998598 < 1$, which shows that A^I is stable.

Example 2: The interval matrix $A^I = [B, C]$, with B and C given by

$$B = \begin{bmatrix} -159 & -59 & 4 & -19 & 67 & 98 & -135 \\ -59 & -247 & -105 & -34 & -33 & -83 & -71 \\ 4 & -105 & -288 & 102 & 43 & -179 & 106 \\ -19 & -34 & 102 & -123 & -52 & 25 & -129 \\ 67 & -33 & 43 & -52 & -253 & -39 & 12 \\ 98 & -83 & -179 & 25 & -39 & -287 & 88 \\ -135 & -71 & 106 & -129 & 12 & 88 & -271 \end{bmatrix} \text{ and}$$

$$C = \begin{bmatrix} -159 & -56 & 8.5 & -14.5 & 80.5 & 111.5 & -129 \\ -56 & -244 & -103.5 & -32.5 & -24 & -77 & -65 \\ 8.5 & -103.5 & -277.5 & 106.5 & 52 & -176 & 107.5 \\ -14.5 & -32.5 & 106.5 & -115.5 & -52 & 31 & -115.5 \\ 80.5 & -24 & 52 & -52 & -250 & -28.5 & 16.5 \\ 111.5 & -77 & -176 & 31 & -28.5 & -275 & 101.5 \\ -129 & -65 & 107.5 & -115.5 & 16.5 & 101.5 & -262 \end{bmatrix}$$

is found to be stable, using Algorithm 2 of chapter 2, with stability margin $h = -5.6152$.

Example 3: The interval matrix $A^I = [B, C]$, where B and C are given by

$$B = \begin{bmatrix} -159 & -59 & 4 & -19 & 67 & 98 & -135 \\ -59 & -247 & -105 & -34 & -33 & -83 & -71 \\ 4 & -105 & -288 & 102 & 43 & -179 & 106 \\ -19 & -34 & 102 & -123 & -52 & 25 & -129 \\ 67 & -33 & 43 & -52 & -253 & -39 & 12 \\ 98 & -83 & -179 & 25 & -39 & -287 & 88 \\ -135 & -71 & 106 & -129 & 12 & 88 & -271 \end{bmatrix} \quad \text{and}$$

$$C = \begin{bmatrix} -159 & -55 & 10 & -13 & 85 & 116 & -127 \\ -55 & -243 & -103 & -32 & -21 & -75 & -63 \\ 10 & -103 & -274 & 108 & 55 & -175 & 108 \\ -13 & -32 & 108 & -113 & -52 & 33 & -111 \\ 85 & -21 & 55 & -52 & -249 & -25 & 18 \\ 116 & -75 & -175 & 33 & -25 & -274 & 106 \\ -127 & -63 & 108 & 111 & 18 & 106 & -259 \end{bmatrix}$$

is found to be unstable, using Algorithm 2 of Chapter 2, with a point matrix

$$U = \begin{bmatrix} -159 & -59 & 10 & -13 & 85 & 116 & -135 \\ -59 & -243 & -105 & -34 & -33 & -83 & -63 \\ 10 & -105 & -274 & 108 & 55 & -175 & 106 \\ -13 & -43 & 108 & -113 & -52 & 33 & -129 \\ 85 & -33 & 55 & -52 & -249 & -25 & 12 \\ 116 & -83 & -175 & 33 & -25 & -274 & 88 \\ -135 & -63 & 106 & -129 & 12 & 88 & -259 \end{bmatrix}$$

having positive eigenvalue 1.6108.

Example 4: The interval matrix $A^I = \begin{bmatrix} [-3,-2] & [4,5] & [4,6] & [-1,1.5] \\ [-4,-3] & [-4,-3] & [-4,-3] & [1,2] \\ [-5,-4] & [2,3] & [-5,-4] & [-1,0] \\ [-1,0.1] & [0,1] & [1,2] & [-4,-2.5] \end{bmatrix}$ is determined to be regular by using Algorithm 1of Chapter 3, which determines the regularity of A^I by checking only 11 matrices A_k^m.

Example 5: The interval matrix $A^I = \begin{bmatrix} [1,3] & [2,4] & [0,2] & [1,3] \\ [-3,-1] & [4,5] & [-2,0] & [4,6] \\ [2,4] & [7,8] & [1.5,2.5] & [1,2] \\ [5,7] & [8,10] & [2,4] & [6,8] \end{bmatrix}$ is determined to be not regular by using Algorithm 1 of Chapter 3, which determines the nonregularity of A^I by checking 4098 matrices A_k^m and gives a singular matrix

$$A_o = \begin{bmatrix} 1.5 & 3.5 & 0.5 & 2.5 \\ -1.5 & 4.5 & -1.5 & 4.5 \\ 2.5 & 7.5 & 2 & 1.5 \\ 6.5 & 9.5 & 2.5 & 7 \end{bmatrix}.$$

Example 6: The symmetric interval matrix

$$A^I = \begin{bmatrix} [-3,-2] & [0,1] & [-0.5,1] & [-1,0.28] \\ [0,1] & [-4,-3] & [-1,0] & [0.5,1.5] \\ [-0.5,1] & [-1,0] & [-5,-4] & [0,1] \\ [-1,0.28] & [0.5,1.5] & [0,1] & [-4,-2.5] \end{bmatrix}$$

is regular and its mid point matrix

$$A^m = \begin{bmatrix} -2.5 & 0.5 & 0.25 & -0.36 \\ 0.5 & -3.5 & -0.5 & 1 \\ 0.25 & -0.5 & -4.5 & 0.5 \\ -0.36 & 1 & 0.5 & -3.25 \end{bmatrix}$$

is stable with maximum eigenvalue −2.2773 .It follows that the symmetric interval matrix A^I is stable.

Example 7: The symmetric interval matrix $A^I = [B, C]$, with B and C given by

$$B = \begin{bmatrix} -2.149 & -3.72 & -11.48 \\ -2.048 & -4.59 & -8.61 \\ -1.947 & -5.46 & -5.740 \end{bmatrix}, \text{ and}$$

$$C = \begin{bmatrix} 3.851 & 2.280 & -5.48 \\ 3.952 & 1.41 & -2.61 \\ 4.053 & 0.540 & 0.260 \end{bmatrix}$$

is not stable. It is found to be not regular using Algorithm 2 of Chapter 3, and contains a singular point matrix $\begin{bmatrix} 0.851 & -0.72 & -8.48 \\ 0.952 & -1.59 & -5.61 \\ 1.053 & -2.46 & -2.74 \end{bmatrix}$.

Example 8: The symmetric interval matrix $A^I_s = [B', C']$, where B' and C' given by

$$B' = \begin{bmatrix} -3 & 0 & -0.5 & -1 \\ 0 & -4 & -1 & 0.5 \\ -0.5 & -1 & -5 & 0 \\ -1 & 0.5 & 0 & -4 \end{bmatrix}, \text{ and } C' = \begin{bmatrix} -2 & 1 & 1 & 0.6 \\ 1 & -3 & 0 & 1.5 \\ 1 & 0 & -4 & 1 \\ 0.6 & 1.5 & 1 & -2.5 \end{bmatrix}, \text{ is constructed}$$

from the interval matrix A^I given in Example 1 and is found to be stable using Algorithm 3 of Chapter 3. Thus A^I given in Example 1 is also stable.

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