

# Nonholonomic Motion Planning for Wheeled Mobile Systems using Geometric Phases

by

Emad Ibrahim Al-Regib

A Thesis Presented to the

FACULTY OF THE COLLEGE OF GRADUATE STUDIES

KING FAHD UNIVERSITY OF PETROLEUM & MINERALS

DHAHRAN, SAUDI ARABIA

In Partial Fulfillment of the  
Requirements for the Degree of

**MASTER OF SCIENCE**

In

**MECHANICAL ENGINEERING**

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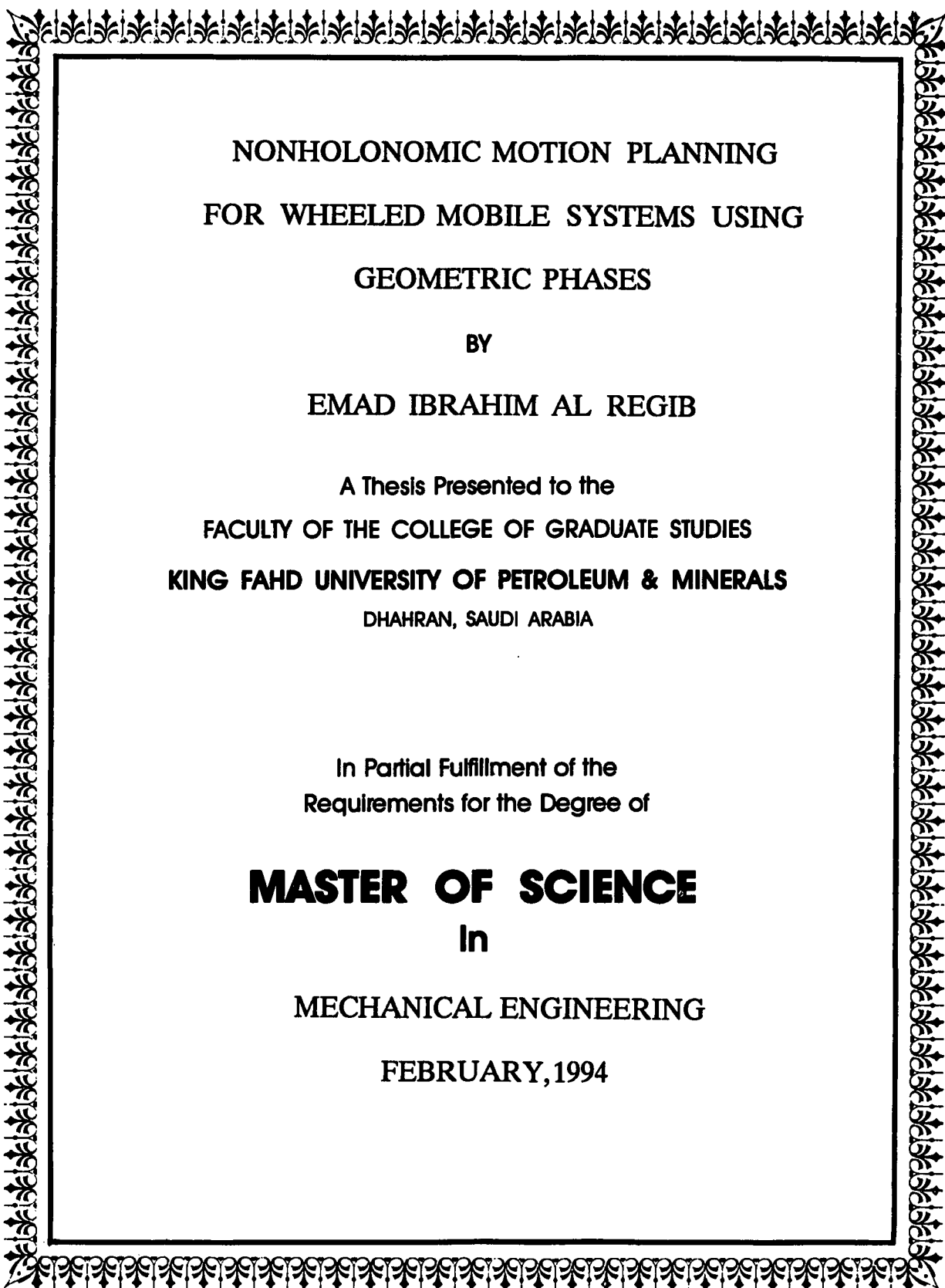
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using geometric phases**

**Al Regib, Emad Ibrahim, M.S.**

**King Fahd University of Petroleum and Minerals (Saudi Arabia), 1994**

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**EMAD IBRAHIM AL-REGIB**

**MECHANICAL ENGINEERING**

**FEBRUARY 1994**

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**To my mother**

**and**

**in memory of my father**

## **Acknowledgment**

First and for most my deep appreciation goes to my thesis advisor Dr. Mahmut Reyhanoglu for his constant help, guidance and the great amount of precious time he devoted to my academic development.

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## Abstract

**Name:** Emad Ibrahim AL-Regib  
**Title:** Nonholonomic Motion Planning  
for Wheeled Mobile Systems  
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*In this thesis, the motion planning for wheeled mobile systems with nonholonomic constraints is studied. Such systems, in general, admit local representations in which the constraint equations are cyclic in certain variables. A nonlinear control system model describing the controlled motion of a wheeled mobile system with driving and steering inputs is first presented. State space and input space transformations are introduced to obtain a nonlinear control system in a normal form which is referred to as "Caplygin form ." The structure of the Caplygin form equations allows identification of a "base space", on which a set of decoupled controllable dynamics is defined. A general motion planning approach is then described. The motion planning strategy first transfers a given initial configuration to the origin of the base space and then causes the system to track a closed path in the base space that produces a desired "geometric phase". i.e. a desired net change in the system configuration. It is shown that this motion planning approach constitutes a powerful analytic method for solving the motion planning problem associated with a large class of wheeled mobile systems including a car pulling  $n$  trailers. Results are illustrated through simulations of several examples of wheeled mobile systems.*

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## خلاصة الرسالة

إسم الطالب : عماد إبراهيم الرقب  
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 التخصص : هندسة ميكانيكية.  
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في هذا البحث تتم دراسة التخطيط الحركي للإنظمة المتحركة ذات الدواليب و المقيدة الحركة بمعادلات لاهولونية (غير قابلة للتكامل). المعادلات المقيدة لحركة هذه الأنظمة تتميز بخاصية التماثل بالنسبة لمتغيرات معينة. أولاً ، يتم عرض نموذج تحكم غير خطي يصف كيفية التحكم بحركة هذه الأنظمة عن طريق القيادة و التوجيه. ثم يتم تقديم التحويلات المطلوبة للحصول على نظام تحكم غير خطي في شكل عادي يُدعى (كابليجن) . إن تركيب معادلات هذا الشكل يسمح بتعيين فراغ أساسي تُعرف عليه مجموعة ديناميكا غير متقارنة. إستراتيجية تخطيط الحركة يتم عرضها بعد ذلك. هذه الاستراتيجية تُنقل أولاً حالة أولية معينة إلى نقطة أصل الفراغ الأساسي ثم تجعل الحالة تتعقب مساراً مفلقاً في الفراغ الأساسي الذي ينتج التغيير المطلوب في شكل النظام. إن استراتيجية تخطيط الحركة المقترحة تمثل أسلوب تحليلي لحل مشكلة التخطيط الحركي لعدد كبير من الأنظمة المتحركة ذات الدواليب و منها مركبة تجر عدة مقطورات. نتائج المحاكاة مرفقة للتوضيح.

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# Chapter 1

## INTRODUCTION

In many cases, the very construction of a mechanical system precludes arbitrary motion of its individual parts. In such cases, it is customary to say that constraints are imposed on the system. If the constraints are holonomic, i.e. integrable in the sense that they can be reduced to geometric constraints, then it is possible to characterize the system configuration by a smaller number of generalized coordinates. However, in case of nonholonomic constraints (those that are nonintegrable in the sense that they cannot be reduced to geometric constraints) it is impossible to reduce the number of generalized coordinates. Unlike a holonomic constraint which reduces both the dimension of the configuration space and number of degrees of freedom, a nonholonomic constraint reduces

only the number of degrees of freedom (not the dimension of the configuration space). For instance, a car can move forward and backward, and can make turns. However, it cannot move sideways. Nevertheless, it can take any position and orientation in the configuration space.

Interest in mobile systems is growing rapidly because of the very broad range of their potential applications. The challenge is that these systems move intelligently so that they can perform various actions without human intervention. The growing use of wheeled mobility applications such as transportation vehicles, surveillance and ferrying mobile robots motivates the need for developing motion planning strategies for systems with nonholonomic constraints.

The subject of discussion in this thesis is the nonholonomic motion planning (NMP) problem for wheeled mobile systems. The NMP problem involves finding a path which links the initial and final system configurations (positions) while satisfying the nonholonomic constraints to which the system is subjected to.

In the past few years, there has been a great deal of interest in the

generation of NMP algorithms for wheeled mobile systems. Early work concentrated on search methods [2], [9] for the generation of such algorithms. Quite recently, nonlinear control theory was introduced into this field. This led to the study of navigating mobile systems using analytical techniques rather than algorithmic or heuristic motion planning. Laffarriere and Susmann [10] introduced Lie algebraic methods. Murray and Sastry [12] explored the use of sinusoids for steering the so-called chained form systems. Sinusoids were also used in a method proposed by Sussmann and Liu [21]. Their method was completely general in that it applied to any controllable nonlinear system without drift.

A car pulling  $n$  trailers constitutes a typical example of nonholonomic wheeled mobile systems. The nonholonomic constraints in this example arise from constraining each pair of wheels to roll without slipping. Note that in this example each trailer adds one nonholonomic constraint, thereby giving rise to a wheeled mobile system with  $n + 2$  nonholonomic constraints. This well-known  $n$ -trailer problem has indeed generated some fascinating new ideas in the robotics field. It is not a "toy problem" since efforts are underway to automate baggage handling by carts or trucks pulling multiple trailers in airports. Divelbiss and Wen [8] explored a computational approach to the  $n$ -trailer problem, which uses

gradient decent in a discretized input space. Recently, Tilbury, Murray and Sastry [22] have used the machinery of exterior differential forms for converting the constraints for an  $n$ -trailer system into Goursat normal form which is the dual to the chained form. They have proposed three different methods for steering the chained form system.

Although nonholonomic wheeled mobile systems are completely controllable, they cannot be steered to a desired configuration (or position) using time-invariant continuous feedback control. Consequently, the problem of controlling a wheeled mobile system has been addressed from two points of view:

1. Open-loop control strategies: construction of open loop controls to steer the mobile system from any initial configuration (or position) to any prescribed configuration. Exploration of the problem from this perspective can be found in [8], [10], [11], [12], [22] and references therein.
2. Feedback control strategies : construction of discontinuous time-invariant feedback or smooth time-varying feedback laws that steer

the system to a desired configuration. Canudas and Sordalen [6] used discontinuous time-invariant feedback control to steer a mobile robot. Samson and Ait-Abderrahim [20] proposed a continuous time-varying feedback control law to steer a mobile robot. Recently, Pomet [14] gave a systematic way to design smooth time-varying feedback control laws which can be applied to any wheeled mobile system.

This thesis is devoted to study the NMP problem for wheeled mobile systems which can be transformed to a special form which is referred to as Caplygin form. A motion planning strategy developed by Reyhanoglu, McClamroch and Bloch [19] for Caplygin systems is utilized. This strategy is based on using geometric phases which have proved useful in a variety of nonlinear control problems associated with nonholonomic systems [3], [19].

The organization of the thesis is as follows: Chapter II summarizes some notations of differential calculus and differential geometry. In Chapter III models of wheeled mobile systems are presented with a summary of theoretical results on control and stabilization of such systems. Chapter IV is devoted to the synthesis of both open loop controllers and discon-

tinuous state feedback controllers. Practical examples illustrating the results of this thesis are included in Chapter V. Chapter VI contains conclusions and remarks on future research directions.

## Chapter 2

# MATHEMATICAL BACKGROUND

In this chapter, we present the general mathematical background on which our development in later chapters is based. General notation and definitions from differential calculus and differential geometry are introduced. Readers who are familiar with differential geometry can use this chapter as a reference for the notation and can proceed directly to the following chapters. For full details, see [1],[7],[13].

## 2.1 Differentiability, Manifolds,

### Submanifolds

Let  $A$  be an open subset of  $R^n$  and  $h : A \rightarrow R$  a function. The value of  $h$  at  $x = (x_1, \dots, x_n)$  is denoted  $h(x) = h(x_1, \dots, x_n)$ . Let  $k > 0$  be a positive integer. The function  $h$  is called  $C^k$  ( $k$  times continuously differentiable) if it possesses continuous partial derivatives of all orders  $\leq k$  on  $A$ . If  $h$  is  $C^k$  for all  $k$  then  $f$  is said to be  $C^\infty$ . A mapping  $H : A \rightarrow R^m$  is a collection  $(H_1, \dots, H_m)$  of functions  $H_i : A \rightarrow R$ . The mapping  $H$  is  $C^k$  if all  $H_i$ 's are  $C^k$ . A mapping  $P : A \rightarrow R^{m_1 \times m_2}$  is an  $m_1 \times m_2$  matrix of functions  $P_{ij} : A \rightarrow R$ . The mapping  $P$  is  $C^k$  if all  $P_{ij}$ 's are  $C^k$ . We use the notation  $P'$  to denote the transpose of  $P$ .

A set  $S$  with a topology is called a *topological space*. A *neighborhood* of a point  $p$  of a topological space is any open set which contains  $p$ . If  $S_0$  is a subset of a topological space  $S$ , there is a unique open set, denoted  $int(S_0)$  and called the *interior* of  $S_0$ , which is contained in  $S_0$  and contains any other open set contained in  $S_0$ . We say that  $S_0$  has *empty interior* with respect to  $S$  if  $S_0$  contains no open set of  $S$  other than the empty set  $\emptyset$ . Let  $S_1$  and  $S_2$  be topological spaces and  $F$  a mapping

$F : S_1 \rightarrow S_2$ . The mapping  $F$  is *continuous* if the inverse image of every open set of  $S_2$  is an open set of  $S_1$ . The mapping  $F$  is *open* if the image of an open set of  $S_1$  is an open set of  $S_2$ . The mapping  $F$  is a *homeomorphism* if it is a bijection and both continuous and open.

A *locally Euclidean space*  $E$  of dimension  $n$  is a topological space such that for each  $p \in E$  there exists a homeomorphism  $\varphi$  mapping some open neighborhood of  $p$  onto an open set in  $R^n$ . A *manifold*  $M$  of dimension  $n$  is a topological space which is a locally Euclidean space of dimension  $n$ , is Hausdorff (any two different points have disjoint neighborhoods) and has a countable basis. A *coordinate chart* on a manifold  $M$  is a pair  $(U, \phi)$ , where  $U$  is an open set of  $M$  and  $\phi$  a homeomorphism of  $U$  onto an open set of  $R^n$ . Sometimes  $\phi$  is represented as  $(\phi_1, \dots, \phi_n)$ , where  $\phi_i : U \rightarrow R$  is called the  $i$ -th *coordinate function*. If  $p \in U$ , the  $n$ -tuple of real numbers  $(\phi_1(p), \dots, \phi_n(p))$  is called the set of *local coordinates* of  $p$  in the coordinate chart  $(U, \phi)$ .

Let  $(U, \phi)$  and  $(V, \psi)$  be two coordinate charts on a manifold  $M$ , with  $U \cap V \neq \emptyset$ . Let  $(\psi_1(p), \dots, \psi_n(p))$  be the set of coordinate functions

associated with the mapping  $\psi$ . The homeomorphism

$$\psi \circ \phi^{-1} : \phi(U \cap V) \rightarrow \psi(U \cap V)$$

taking, for each  $p \in U \cap V$ , the set of local coordinates  $(\phi_1(p), \dots, \phi_n(p))$  into the set of local coordinates  $(\psi_1(p), \dots, \psi_n(p))$ , is called a *coordinate transformation* on  $U \cap V$ . Clearly,  $\phi \circ \psi^{-1}$  gives the inverse mapping, which expresses  $(\phi_1(p), \dots, \phi_n(p))$  in terms of  $(\psi_1(p), \dots, \psi_n(p))$ . We say that two coordinate charts  $(U, \phi)$  and  $(V, \psi)$  are  *$C^\infty$ -compatible* if, whenever  $U \cap V \neq \emptyset$ , the coordinate transformation  $\psi \circ \phi^{-1}$  is a diffeomorphism.

A  *$C^\infty$  atlas* on a manifold  $M$  is a collection  $\{(U_i, \phi_i)\}_{i \in I}$ , where  $I$  is an index set, of pairwise  $C^\infty$ -compatible coordinate charts with the property that  $\cup_{i \in I} U_i = M$ . An atlas is *complete* if it is not properly contained in any other atlas. A *smooth ( $C^\infty$ ) manifold* is a manifold equipped with a complete  $C^\infty$  atlas.

Now assume that  $M_1$  and  $M_2$  are smooth manifolds of dimension  $n$ . Then a bijective mapping  $F : M_1 \rightarrow M_2$  is a *diffeomorphism* if  $F$  is bijective and both  $F$  and  $F^{-1}$  are smooth mappings. Two manifolds  $M_1$  and  $M_2$  are *diffeomorphic* if there exists a diffeomorphism  $F : M_1 \rightarrow M_2$ .

## 2.2 Tangent Spaces, Vector Fields, Distributions

Let  $M$  be a smooth manifold of dimension  $n$ . Let  $T_x M$  denote the *tangent space* to  $M$  at a point  $x \in M$ . Then the *tangent bundle* of  $M$  is  $TM = \bigcup_{x \in M} T_x M$ , the union of tangent spaces. A *vector field*  $\tau$  on  $M$  is a smooth map which assigns to each point on  $x \in M$  a tangent vector  $\tau(x) \in T_x M$ . In local coordinates, we represent  $\tau$  as a column vector whose elements depends on  $x$ :

$$\tau(x) = (\tau_1(x), \dots, \tau_n(x))' .$$

Alternatively, considering  $\tau$  as a differential operator, we write

$$\tau(x) = \tau_1(x) \frac{\partial}{\partial x_1} + \dots + \tau_n(x) \frac{\partial}{\partial x_n} .$$

The symbol  $\frac{\partial}{\partial x_i}$  is to be thought of as a basis element for the tangent space with respect to a given set of local coordinates. A *distribution* assigns a subspace of the tangent space to each point in  $M$  in a smooth way. A distribution can be defined by a set of smooth vector fields  $\tau^1, \dots, \tau^r$ . In this case we define the distribution as

$$\Delta = \text{span}\{\tau^1, \dots, \tau^r\}$$

where we take the span over the set of smooth real-valued functions on  $M$ . At any point  $x \in M$  the distribution is a linear subspace of the

tangent space, i.e.

$$\Delta(x) = \text{span}\{\tau^1, \dots, \tau^r\}(x) \subset T_x M .$$

Given two smooth vector fields  $X = \sum_{i=1}^n X_i \frac{\partial}{\partial x_i}$  and  $Y = \sum_{i=1}^n Y_i \frac{\partial}{\partial x_i}$  on  $M$ , we define a new vector field, denoted as  $[X, Y]$  and called the *Lie bracket* of  $X$  and  $Y$ , as

$$[X, Y] = \sum_{j=1}^n \left( \sum_{i=1}^n \frac{\partial Y_j}{\partial x_i} X_i - \frac{\partial X_j}{\partial x_i} Y_i \right) \frac{\partial}{\partial x_j} .$$

Using a Taylor expansion up to order two, one can prove the following geometric characterization of the Lie bracket:

$$[X, Y](p) = \lim_{s \rightarrow 0^+, t \rightarrow 0^+} \frac{1}{st} \left( e^{tY} e^{sX} e^{-tY} e^{-sX} p - p \right) ,$$

where the map  $t \mapsto e^{tX} p$  denotes the integral curve of  $X$  that goes through  $p$  at time  $t = 0$ .

Let  $V(M)$  denote the linear space of smooth vector fields defined on  $M$ . Then  $V(M)$  with the Lie bracket operation is a *Lie algebra*. In particular, it can be shown that the map  $(X, Y) \mapsto [X, Y]$  is bilinear, skew-commutative and satisfies the *Jacobi identity*

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 , \quad \forall X, Y, Z \in V .$$

A linear subspace  $L \subset V(M)$  is called a *Lie subalgebra* if

$$[X, Y] \in L, \forall X, Y \in L.$$

Let  $\{X^i \mid 1 \leq i \leq q\}$  be a finite set of vector fields and  $L_1, L_2$  two subalgebras of  $V(M)$  which contain the vector fields  $X^1, \dots, X^q$ . Clearly, the intersection  $L_1 \cap L_2$  is again a subalgebra of  $V(M)$  and contains  $X^1, \dots, X^q$ . Thus we conclude that there exists a unique subalgebra  $L$  of  $V(M)$  which contains  $X^1, \dots, X^q$  and has the property of being contained in all the subalgebras of  $V(M)$  which contain the vector fields  $X^1, \dots, X^q$ . This subalgebra is referred to as the *smallest subalgebra* of  $V(M)$  which contains the vector fields  $X^1, \dots, X^q$ . By an inductive argument using the Jacobi identity, it can be shown that every element of  $L$  is a linear combination of repeated Lie brackets of the form  $[X^{i_k}, [\dots [X^{i_2}, X^{i_1}] \dots]]$  where  $X^{i_j}, 1 \leq i_j \leq q$ , is in the set  $\{X^i \mid 1 \leq i \leq q\}$  and  $1 < j < \infty$ . With the subalgebra  $L$  we may associate a distribution  $\mathcal{L}$  in a natural way:

$$\mathcal{L}(x) = \text{span}\{X(x) \mid X \in L\}, x \in M.$$

## 2.3 Cotangent Spaces, Covector Fields, Codistributions

Associated with the tangent space  $T_x M$  is its dual space  $T_x^* M$ , the *cotangent space* to  $M$  at  $x \in M$ . The *cotangent bundle* of  $M$  is  $T^* M = \bigcup_{x \in M} T_x^* M$ , the union of cotangent spaces. Just as we defined vector fields on  $T_x M$ , on  $T_x^* M$  we can define a *covector field* (one-form)  $\omega$ . In local coordinates, we represent  $\omega$  as a row vector whose elements depends on  $x$ :

$$\omega(x) = (\omega_1(x), \dots, \omega_n(x)) .$$

Alternatively, we write

$$\omega(x) = \omega_1(x)dx_1 + \dots + \omega_n(x)dx_n .$$

The symbols  $dx_i$  represent the basis dual to the basis  $\frac{\partial}{\partial x_i}$  on  $T_x M$  and are defined as

$$dx_i \cdot \frac{\partial}{\partial x_j} = \delta_{ij}$$

where  $\delta_{ij}$  is the Kronecker delta. A one-form acts on a vector field to give a real valued function on  $M$ ,

$$\omega \cdot \tau = \left( \sum_{i=1}^n \omega_i dx_i \right) \cdot \left( \sum_{j=1}^n \tau_j \frac{\partial}{\partial x_j} \right) = \sum_{i=1}^n \tau_i \omega_i .$$

A *codistribution* assigns a subspace of the cotangent space to each point in  $M$  in a smooth way. A codistribution can be defined by a set of smooth

covector fields  $\omega^1, \dots, \omega^s$ . In this case we define the codistribution as

$$\Omega = \text{span}\{\omega^1, \dots, \omega^s\}$$

where we take the span over the set of smooth real-valued functions on  $M$ . At any  $x \in M$  the codistribution is a linear subspace of the cotangent space, i.e.

$$\Omega(x) = \text{span}\{\omega^1, \dots, \omega^s\}(x) \subset T_x^*M .$$

It is possible to construct a codistribution starting from a given distribution, and conversely. The natural way to do this is the following: given a distribution  $\Delta$ , for each  $x \in M$  consider the *annihilator* of  $\Delta(x)$ , that is the set of all covectors which annihilate all vectors in  $\Delta(x)$

$$\Delta^\perp(x) = \{\omega \in T_x^*M \mid \omega \cdot v = 0, \forall v \in \Delta(x)\} .$$

Since  $\Delta^\perp(x)$  is a subspace of  $T_x^*M$ , this construction identifies exactly a codistribution  $\Delta^\perp$ , called the *annihilator* of  $\Delta$ . Conversely, given a codistribution  $\Omega$ , we can construct a distribution, denoted  $\Omega^\perp$  and called the annihilator of  $\Omega$ , so that for each  $x \in M$

$$\Omega^\perp(x) = \{v \in T_x M \mid \omega \cdot v = 0, \forall \omega \in \Omega(x)\} .$$

## Chapter 3

# MODELS OF WHEELED-MOBILE SYSTEMS

### 3.1 General Models of Wheeled Mobile Systems

We consider the class of wheeled mobile systems with linear velocity constraints of the form

$$w^i(p)\dot{p} = 0 \quad i = 1, \dots, m . \quad (3.1)$$

Here  $p$  is an  $n$ -vector of position (configuration) variables,  $\dot{p}$  is an  $n$ -vector of velocity variables, and  $w^i(p)$  is a smooth ( $C^\infty$ )  $n$ -row vector. These are constraints on the velocities of the system. In some cases, the constraints may be explicitly integrable, giving constraints of the form

$$h_i(p) = c_i$$

for some constant  $c_i$ . If this is possible, motion of the system is restricted to a level surface of  $h_i$ . Such a constraint is said to be holonomic. By choosing coordinates for the surface, configuration space methods can be applied.

We assume that the row vectors  $w^i(p)$ ,  $i = 1, \dots, m$  are linearly independent, i.e. these row vectors constitute  $n$  linearly independent smooth covector fields defined on the configuration space; these covector fields span a codistribution  $\Omega$  and the annihilator of the codistribution  $\Omega$ , denoted  $\Omega^\perp$ , is spanned by  $(n - m)$  linearly independent smooth vector fields  $g_j$ ,  $j = 1, \dots, n - m$ , satisfying

$$w^i \cdot g_j = 0 \quad i = 1, \dots, m \quad j = 1, \dots, n - m . \quad (3.2)$$

**Definition 1** [3] *Consider the following nondecreasing sequence of locally defined distributions*

$$N_1 = \Omega^\perp ,$$

$$N_k = N_{k-1} + \text{span}\{[X, Y] \mid X \in N_1, Y \in N_{k-1}\}, k \geq 2 .$$

*There exists an integer  $k^*$  such that*

$$N_k = N_{k^*} .$$

*for all  $k < k^*$ . If  $\dim N_k = n$  and  $k^* > 1$  then the constraints (3.1) are called (completely) nonholonomic and the smallest (finite) number  $k^*$  is called the degree of nonholonomy.*

In this thesis it is assumed that constraint equations (3.1) are completely nonholonomic with nonholonomy degree  $k^*$ . Note that for this to hold  $(n - m)$  must be strictly greater than one. Note also that since the constraints are nonholonomic, there is in fact no explicit restriction on the values of the configuration variables.

It will be convenient to convert problems with nonholonomic constraints into steering problems for nonlinear control systems. Roughly speaking, we would like to convert the constraint specification from describing the directions in which the system cannot move to those in which it can. To

be more specific, let  $G(p)$  denote the  $n \times (n - m)$  matrix function whose columns are  $g_1, \dots, g_{n-m}$  and  $J(p)$  denote the  $m \times n$  matrix function whose rows are  $w^1, \dots, w^m$ . Then the equation (3.2) can be written as

$$J(p)G(p) = 0 ,$$

that is,  $G(p)$  is the annihilator matrix of  $J(p)$ . It is clear that admissible velocities are those which are in the null space of  $J(p)$ , i.e.

$$J(p)\dot{p} = 0 ,$$

or, equivalently, those which are in the range space of  $G(p)$ . Consequently, we have

$$\dot{p} = G(p)u = \sum_{i=1}^m g_i(p)u_i , \quad (3.3)$$

for some  $u \in R^{n-m}$ , which is regarded as input vector. Note that equation (3.3) describes the kinematics of the system.

## 3.2 Controllability and Stabilizability

### Results

Let  $p^0$  and  $p^d$  denote an arbitrary initial configuration and a desired configuration respectively. The system (3.3) is said to be controllable if there exists a  $T > 0$  and an open-loop control  $t \mapsto (p^0, p^d, t)$ . defined on

$[0, T]$ , such that equation (3.3) satisfies  $p(0) = p^0$  and  $p(T) = p^d$ .

Controllability of the system can be characterized in terms of the Lie algebra generated by the vector fields  $g_i$ . Let  $\mathcal{C}$  denote the smallest subalgebra of vector fields (on the configuration space) that contains  $g_1, \dots, g_{n-m}$  and let  $C$  denote the corresponding distribution:

$$C(p) = \text{span}\{X(p) \mid X \in \mathcal{C}\}$$

for any  $p$  in the configuration space. It is a well-known fact that if  $\dim C(p) = n$  for all points, then the system (3.3) is completely controllable.

The following result is obtained for wheeled mobile systems.

**Theorem 1** *Under the assumptions stated above, the nonholonomic wheeled mobile system defined by equation (3.3) is controllable.*

**Proof 1** *Since we have assumed that the system is (completely) nonholonomic with nonholonomy degree  $k^*$ , dimension of the space spanned by the vector fields*

$$\{g_{i_1}, [g_{i_2}, g_{i_1}], \dots, [g_{i_{k^*}}, [\dots [g_{i_2}, g_{i_1}] \dots]], i_j = 1, \dots, n-m, j = 1, \dots, k^*\}$$

is  $n$  at any point  $p$ . This implies that  $\dim C(p) = n$  for all  $p$ . Consequently, the system (3.3) is controllable.

We next consider the stabilizability properties of the system (3.3). Let  $p^d$  denote a desired configuration. The system is said to be asymptotically stabilizable to  $p^d$  if there exists a feedback control  $p \mapsto u(p)$  that steers any point in the neighborhood of  $p^d$  to  $p^d$ .

The following negative result is obtained for wheeled mobile systems.

**Theorem 2** *Let  $p^d$  denote a desired configuration. The nonholonomic wheeled mobile system defined by equation (3.3), is not asymptotically stabilizable to  $p^d$  using time-invariant continuous feedback.*

**Proof 2** *A necessary condition for the existence of time-invariant continuous feedback law for system (3.3) is that the image of the mapping*

$$(p, u) \mapsto G(p)u$$

*contains some neighborhood of zero [24], [5]. Let  $e_0$  be a nonzero vector which is linearly independent from  $g_1(0), \dots, g_{n-m}(0)$ . By continuity, there is an  $\epsilon > 0$  such that for all  $(p, u)$  with  $\|p\| < \epsilon$ , the vector  $G(p)u$  is different for  $\lambda e_0$  in  $R$ . Therefore, the map*

$$(p, u) \mapsto G(p)u$$

*does not map the neighborhood  $[-\epsilon, \epsilon]^{2n-m}$  of the origin in  $R^{2n-m}$  into a neighborhood of the origin in  $R^n$ . Thus, the necessary condition is not satisfied. Hence, the system (3.3) cannot be asymptotically stabilized to  $p^d$  using a time-invariant continuous feedback law.*

This negative result implies that for feedback stabilization of a desired configuration one should restrict the study to the classes of feedback controllers that are not continuous time-invariant. Clearly, traditional methods are of no use. However, as in [3], [4], [6], [16], [19], discontinuous time-invariant feedback controllers or, as in [14], smooth time-varying feedback controllers may be constructed.

## Chapter 4

# NONHOLONOMIC MOTION PLANNING PROBLEM

### 4.1 Nonholonomic Motion Planning

#### Framework

Let  $p^0$  and  $p^d$  denote a pair of initial configuration and desired (goal) configuration. The nonholonomic motion planning (NMP) problem is the problem of determining a motion  $(p(t), \dot{p}(t))$  on an interval  $[0, t_f]$ , for a given  $t_f$ , such that  $p(0) = p^0$  and  $p(t_f) = p^d$ .

Since the commonly-used mobile systems described by equation (3.3) are nonholonomic, these wheeled mobile systems are controllable, as indicated in the previous chapter. Clearly, complete controllability of these systems implies the existence of solutions to the above control problem.

In general, the constraint equations for wheeled mobile systems are cyclic in certain variables. This property guarantees the existence of local diffeomorphisms (coordinate changes) and feedback transformations under which the equation (3.3) takes the form

$$\dot{p} = \begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} = \begin{bmatrix} I \\ B(p_1) \end{bmatrix} u, \quad (4.1)$$

which is referred to as “ Caplygin form ”. Here  $p = (p_1, p_2)$ ,  $p_1 \in R^{n-m}$  and  $p_2 \in R^m$  denotes a partition of the position vector.  $I$  is the  $m \times m$  identity matrix and  $B(p_1)$  is an  $m \times (n - m)$  matrix function. As a consequence,  $p_1$  constitutes a reduced configuration space for the system described by equation(4.1). This reduced configuration space is referred to as the “ base space ” in [19]. Throughout the thesis we refer to  $p_1$  as the base (state) vector and  $p_2$  as the fiber (state) vector.

Note that the motion in the base space is governed by  $(n - m)$  decoupled

integrators and the governing equation for fiber vector is given by

$$\dot{p}_2 = B(p_1)\dot{p}_1 \quad (4.2)$$

The interpretation of equation (4.2) is that the nonholonomic constraints constrain the rate of change of the fiber variables as the indicated function of the base variables and their rate of change; the integrated form of equation (4.2) gives an expression for the change in the fiber variables in terms of an integral involving only the base variables.

Consider equation (4.2). Assume that the base variables are controlled in such a way that  $p_1(t)$ ,  $0 \leq t_1 \leq t \leq t_2$ , describes a closed path  $\gamma$  in the base space. Integrating both sides of equation (4.2) from  $t = t_1$  to  $t = t_2$  and using the fact that  $dp_1 = \dot{p}_1 dt$ , we obtain

$$p_1(t_2) - p_1(t_1) = \oint_{\gamma} B(p_1) dp_1 \quad (4.3)$$

By the nonintegrability property, the above integral is in fact path dependent. In differential geometry the quantity

$$\alpha(\gamma) = \oint_{\gamma} B(p_1) dp_1 \quad (4.4)$$

is referred to as the geometric phase (or holonomy) of the closed path  $\gamma$ . This quantity depends only on the geometry of the closed path and is independent of the speed at which the path is traversed. The geometric

phase is reflected in the fact that traversing a closed path in the base space yields a non-closed path in the full configuration space. A variety of nonlinear control problems have been solved using ideas based on the use of geometric phase concepts [3], [4], [19].

We now describe a constructive approach for obtaining a solution for the NMP problem. For simplicity, a control strategy which transfers any initial configuration to the zero configuration will be considered.

Let  $p^0 = (p_1^0, p_2^0)$  denote an initial configuration. We now describe two steps involved in construction of a control strategy which transfers the initial state to the origin.

- **Step 1** : *Bring the system to the origin of the  $p_1$  base space, i.e. find a control which transfers the initial state  $(p_1^0, p_2^0)$  to  $(0, p_2^1)$  in a finite time, for some  $p_2^1$ .*
- **Step 2** : *Traverse a closed path (or a series of closed paths) in the  $p_1$  base space to produce a desired geometric phase (holonomy) in the  $(p_1, p_2)$  configuration space, i.e. find a control which transfers  $(0, p_2^1)$  to  $(0, 0)$ .*

Note that transferring the system to the origin in the base space and

tracking the prespecified closed path  $\gamma$  can be accomplished trivially since the base space equations constitute  $(n - m)$  decoupled integrators. However, selection of a closed path  $\gamma$  which produces the desired geometric phase requires special consideration.

## 4.2 Geometric Phase Computations

The desired geometric phase condition is obtained by integrating equation (4.2) from  $t = t_1$  to  $t = t_f$  and letting  $p_2(t_1) = p_2^1$ ,  $p_2(t_f) = 0$  as

$$-p_2^1 = \oint_{\gamma} B(p_1) dp_1 \quad , \quad (4.5)$$

where  $\gamma$  denotes a closed path (or concatenation of a series of closed paths) beginning and ending at the origin in the base space. For several specific examples, explicit characterization of a closed path  $\gamma$  satisfying the desired geometric phase equation (4.5) can be found in [19]. It has been demonstrated that the NMP problem for a Caplygin system can be reduced to the problem of finding a closed path in the base space which satisfies the geometric phase condition [19].

We first describe the class of closed paths considered in this thesis. Let  $a = (a_1, \dots, a_{n-m})$  and  $b = (b_1, \dots, b_{n-m})$  denote two displacement vec-

tors in the  $p_1$  base space and let  $\gamma(a, b)$  denote the closed path (in the base space) formed by the line segments from  $p_1 = 0$  to  $p_1 = a$ , from  $p_1 = a$  to  $p_1 = a + b$ , from  $p_1 = a + b$  to  $p_1 = b$ , and from  $p_1 = b$  to  $p_1 = 0$ . In what follows, we consider choosing the closed path  $\gamma$ , which satisfies the geometric phase condition, from the parameterized family

$$\{\gamma(a, b) \mid a, b \in R^{n-m}\} .$$

**Remark 1** *In general, more than one closed path may be required to produce the desired geometric phase; the results can be extended to such cases as follows:*

Let  $\gamma^k(a^k, b^k) \circ \dots \circ \gamma^1(a^1, b^1)$  denote a concatenation of  $k$  closed paths parameterized by  $(a^i, b^i) \in R^{n-m} \times R^{n-m}$ ,  $i = 1, \dots, k$ . Then it is clear that the geometric phase is given by

$$\sum_{i=1}^k \oint_{\gamma^i(a^i, b^i)} B(p_1) dp_1 .$$

In the subsequent development of this thesis, we demonstrate that such characterization can be accomplished for a large class of wheeled mobile systems including a car pulling  $n$  trailers. However, some problems may require a general computational approach. In the following, we derive relationships between the geometric phase and the assumptions associated

with the distribution defined by the constraints. We show that these relationships can be utilized to approximately characterize the required closed path.

Note that in the subsequent development  $I_r$  denotes the set of indices  $\{1, \dots, r\}$ .

For a Caplygin system with nonholonomy degree  $k^*$ , the tangent space to the configuration space at a configuration  $p$  is spanned by the following set of vector fields:

$$\begin{aligned}
 g_j(p) &= \begin{bmatrix} e_j \\ B_j(p_1) \end{bmatrix}, \quad j \in I_{n-m}; \\
 [g_j, g_i] &= \begin{bmatrix} 0 \\ H_{ji}(p_1) \end{bmatrix}, \quad i, j \in I_{n-m}; \\
 &\vdots \\
 [g_{i_{k^*-2}}, \dots, [g_{i_1}, [g_j, g_i]] \dots] &= \begin{bmatrix} 0 \\ \frac{\partial^{k^*-2} H_{ji}(p_1)}{\partial p_{1i_{k^*-2}} \dots \partial p_{1i_1}} \end{bmatrix}, \quad i, j, i_1, \dots, i_{k^*-2} \in I_{n-m};
 \end{aligned}$$

where  $e_j$  is the  $j$ th standard basis vector of  $R^{n-m}$  and  $B_j$  is the  $j$ th column,

$$H_{ji}(p_1) = \begin{bmatrix} \frac{\partial B_{1i}(p_1)}{\partial p_{1j}} - \frac{\partial B_{1j}(p_1)}{\partial p_{1i}} \\ \cdot \\ \cdot \\ \cdot \\ \frac{\partial B_{mi}(p_1)}{\partial p_{1j}} - \frac{\partial B_{mj}(p_1)}{\partial p_{1i}} \end{bmatrix}, \quad i, j \in I_{n-m};$$

and

$$\frac{\partial^k H_{ji}(p_1)}{\partial p_{1i_k} \cdots \partial p_{1i_1}} = \begin{bmatrix} \frac{\partial^k H_{ji}^1(p_1)}{\partial p_{1i_k} \cdots \partial p_{1i_1}} \\ \cdot \\ \cdot \\ \cdot \\ \frac{\partial^k H_{ji}^m(p_1)}{\partial p_{1i_k} \cdots \partial p_{1i_1}} \end{bmatrix}, \quad i, j, i_1, \dots, i_k \in I, \quad k \in I_{k-2}.$$

**Proposition 1** Consider the normal form equations for a Caplygin system. There exist  $m$  points  $\bar{p}_1^k$ ,  $k \in I_m$ , in the  $p_1$ -base space such that

$$\text{rank}\{H_{i_k j_k}(\bar{p}_1^k), k \in I_m, i_k, j_k \in I_{n-m}\} = m.$$

where  $H(\bar{p}_1)$  denote  $m \times m$  constant matrix with columns:

$$H_{i_1 j_1}(\bar{p}_1^1), \dots, H_{i_m j_m}(\bar{p}_1^m).$$

We next show that the above result can be utilized to approximately characterize a closed path, traversal of which gives a specified geometric phase.

Let  $\bar{p}_1^k$ ,  $k \in I_m$ , be  $m$  points satisfying Proposition (1). Denote by  $\gamma(\bar{p}_1^k, c_k)$  a square closed path based at  $\bar{p}_1^k$  (i.e. starting and ending at  $\bar{p}_1^k$ ) and lying on the  $p_{1i_k} - p_{1j_k}$  plane with size  $c_k$ . Let  $\Gamma(\bar{p}_1^k, c_k, z_k)$  denote a curve in the base space with the following three segments:

1. The first segment links  $p_1 = 0$  to  $p_1 = \bar{p}_1^k$ .
2. The second segment traverses  $\gamma(\bar{p}_1^k, c_k)$   $z_k$  times, counter-clockwise if  $z_k > 0$  and clockwise if  $z_k < 0$ .
3. The last segment is the negative of the first segment, and this links  $\bar{p}_1^k$  back to  $q_1 = 0$ .

Let  $\alpha(\bar{p}_1^k, c_k, z_k)$  denote the geometric phase of the path  $\Gamma(\bar{p}_1^k, c_k, z_k)$ . It is clear that

$$\alpha(\bar{p}_1^k, c_k, z_k) = z_k \int_{S_k} H_{i_k j_k}(p_1) dp_{1i_k} dp_{1j_k}$$

where  $S_k$  denotes the area enclosed by  $\gamma(\bar{p}_1^k, c_k)$ . Note that for sufficiently small  $c_k$ ,  $\{c_k^2 H_{i_k j_k}(\bar{p}_1^k) + O(c_k^2), k \in I_m\}$  forms a basis of  $R^m$ .

By Taylor theorem,  $H_{i_k j_k}(p_1)$  can be expressed as:

$$H_{ij}(p_1) = H_{ij}(\bar{p}_1) + \frac{\partial H_{ij}(\xi)}{\partial p_{1i}}(p_{1i} - \bar{p}_{1i}) + \frac{\partial H_{ij}(\xi)}{\partial p_{1j}}(p_{1j} - \bar{p}_{1j}) .$$

where  $\xi$  is some point in the neighborhood of  $\bar{p}_1$ . Using the fact that

$$\int_{\bar{p}_{1j}}^{\bar{p}_{1j}+c} \int_{\bar{p}_{1i}}^{\bar{p}_{1i}+c} H_{ij}(p_1) dp_{1i} dp_{1j} = H_{ij}(\bar{p}_1) c^2 + \frac{1}{2} \left( \frac{\partial H_{ij}(\hat{p}_1)}{\partial p_{1i}} + \frac{\partial H_{ij}(\hat{p}_1)}{\partial p_{1j}} \right) c^3 ,$$

where  $\hat{p}_1$  is some point in the base space, we obtain

$$\alpha(\bar{p}_1^k, c_k, z_k) = \sum_{k=1}^m (H_{i_k j_k}(\bar{p}_1^k) c_k^2 z_k + \frac{1}{2} D_{i_k j_k}(\hat{p}_1) c_k^3 z_k) ,$$

where  $D_{i_k j_k}(\hat{p}_1) = \frac{\partial H_{i_k j_k}(\hat{p}_1)}{\partial p_{1i_k}} + \frac{\partial H_{i_k j_k}(\hat{p}_1)}{\partial p_{1j_k}}$  for all  $i_k, j_k \in I_m$

In matrix notation the above expression can be written as:

$$\alpha = H(\bar{p}_1) C^2 z + \frac{1}{2} D(\hat{p}_1) C^3 z$$

where  $D(p_1)$  denote  $m \times m$  matrix function with columns

$$D_{i_1 j_1}(p_1), \dots, D_{i_m j_m}(p_1)$$

and

$$C = \text{Diag}(c_1, \dots, c_m)$$

$$z = (z_1, \dots, z_m)' \in Z^m$$

where  $Z$  is the set of integers.

Let  $S = C^2z$ . We now describe a method to obtain a control that steers  $(p_1^0, p_2^0)$  to  $(0, 0)$  with any arbitrary prescribed error  $\epsilon$ . We will use the norm  $\|x\|_\infty = \max_i |x_i|$  for any vector  $x$ .

- **Step 1** : steer the system from  $(p_1^0, q_2^0)$  to  $(0, p_2^1)$  for some  $p_2^1$ . Set  $\alpha^i = -p_2^i$ ,  $i=1$ , and go to step 2.
- **Step 2** : compute  $S^i = H^{-1}(\bar{p}_1)\alpha^i$  and select  $z_k^i, c_k^i$  as in Remark (2) below. Then traverse  $\Gamma^i(\bar{p}_1^1, c_1^i, z_1^i), \dots, \Gamma^i(\bar{p}_1^m, c_m^i, z_m^i)$ . Let  $(0, p_2^{i+1})$  be the resulting point and let  $\alpha^{i+1} = -p_2^{i+1}$ .
- **Step 3** : if  $\|\alpha^{i+1}\|_\infty \leq \epsilon$  stop. Else  $i : i + 1$  and go to step 2.

**Remark 2** Note that at the  $i^{\text{th}}$  step traversing  $\Gamma^i(\bar{p}_1^k, c_k^i, z_k^i)$ ,  $k = 1, \dots, m$ , yields:

$$p_2^{i+1} = p_2^i + H(\bar{p}_1)(C^i)^2 z^i + \frac{1}{2}D(\hat{p}_1)(C^i)^3 z^i$$

and if we choose

$$(C^i)^2 z^i = -H^{-1}(\bar{p}_1)p_2^i,$$

then

$$\|p_2^{i+1}\|_\infty \leq \frac{1}{2}d_m c_{\max}^i \ell_m \|p_2^i\|_\infty,$$

where

$$d_m = \max_{p_1} \max_i \sum_j |D_{ij}(p_1)|,$$

$$\ell_m = \max_i \sum_j [H_{ij}^{-1}(\bar{p}_1)] ,$$

$$c_{\max}^i = \max_j | c_j^i | .$$

*Selecting*

$$c_{cr} = c_{\max}^i = \frac{1}{d_m \ell_m} .$$

*yields*

$$\| p_2^{i+1} \|_{\infty} \leq \frac{1}{2} \| p_2^i \|_{\infty} ,$$

*which implies*

$$\| p_2^i \|_{\infty} \leq \frac{1}{2^i} \| p_2^1 \|_{\infty} .$$

*Note that with the above choice, we reach  $\| p_2 \|_{\infty} \leq \epsilon$  in  $k \geq \log_2 \frac{\| p_2^1 \|_{\infty}}{\epsilon}$  steps.*

*We can compute a set of parameters  $z_k^i$  and  $c_k^i$  as follows:*

$$z_k^i = Z\left(\frac{S_k^i}{c_{cr}^2}\right)$$

*where  $Z(a)$  denotes the integer which is closest to  $a$  and which satisfies*

$$| Z(a) | \geq | a | .$$

*Then, it is clear that  $c_k^i = \sqrt{\frac{S_k^i}{z_k^i}}$ , will be a solution to our problem. Note that in the above  $\text{sign } z_k^i = \text{sign } S_k^i$ .*

### 4.3 Open-Loop Approach

Open-loop approach seek to find a bounded sequence of control inputs steering the mobile robot from any initial position to any other arbitrary configuration i.e. solving Motion Planning Problem. This approach is based on determining a motion  $p(t)$  on an interval  $[0, t_f]$ , such that  $p(t)$  satisfies the equation (3.3), with  $p(0) = p^0$ ,  $p(t_f) = p^d$ , for some open loop control function

$$t \mapsto u(t) = (u_1(t), \dots, u_{n-m}(t))$$

We now describe an open loop controller construction to transfer  $p^0$  to the origin in time exactly  $t_f$ , where  $t_f > 0$  is arbitrary. Let  $p'_1$  and  $p''_1$  denote two points in the  $p_1$  base space and let

$$u_{(t', t'')}(p'_1, p''_1, t) = \frac{\pi(p''_1 - p'_1)}{2(t'' - t')} \sin\left(\frac{\pi(t - t')}{(t'' - t')}\right) \quad (4.6)$$

denote an open loop control function defined on  $[t', t'']$ ,  $0 \leq t' \leq t''$ .

For step 1, the following open loop control function

$$u_{(0, t_1)}(p_1^0, 0, t) = -\frac{p_1^0 \pi}{2t_1} \sin\left(\frac{\pi t}{t_1}\right), \quad (4.7)$$

transfers the system from  $(p_1^0, p_2^0)$  to  $(0, p_2^1)$  for some  $p_2^1$ .

For step 2, Suppose, for notational simplicity, that the desired geometric phase condition (4.5) is satisfied by a single closed path  $\gamma(a, b)$ . Then the open loop controller defined as the concatenation of

$$\begin{aligned}
 u_{[t_1, t_2]}(0, a, t) &= \frac{a\pi}{2(t_2 - t_1)} \sin\left(\frac{\pi(t - t_1)}{(t_2 - t_1)}\right), \\
 u_{[t_2, t_3]}(a, a + b, t) &= \frac{b\pi}{2(t_3 - t_2)} \sin\left(\frac{\pi(t - t_2)}{(t_3 - t_2)}\right), \\
 u_{[t_3, t_4]}(a + b, b, t) &= -\frac{a\pi}{2(t_4 - t_3)} \sin\left(\frac{\pi(t - t_3)}{(t_4 - t_3)}\right), \\
 u_{[t_4, t_f]}(b, 0, t) &= -\frac{b\pi}{2(t_f - t_4)} \sin\left(\frac{\pi(t - t_4)}{(t_f - t_4)}\right).
 \end{aligned} \tag{4.8}$$

transfers the system from  $(0, p_2^1)$  to  $(0, 0)$ . Here we have used a partition  $0 \leq t_1 \leq t_2 \leq t_3 \leq t_4 \leq t_f$  of the time interval  $[0, t_f]$ .

Combining the open loop controls (4.7)-(4.8), we can describe an open loop control as

$$u(p^0, t) = \begin{cases} u_{[t_1, t_2]} ; t \in [0, t_1) , \\ u_{[t_2, t_3]} ; t \in [t_1, t_2) , \\ u_{[t_3, t_4]} ; t \in [t_2, t_3) , \\ u_{[t_4, T]} ; t \in [t_4, t_f) . \end{cases} \tag{4.9}$$

It is clear that the constructed control (4.9) transfers any initial position to the origin at time  $t_f$ . The resulting motion is obtained by solving an initial value problem using this control. This completes the solution of the NMP problem. It is important to emphasize that the above construction is based on a priori selection of a closed path consisting of four straight line segments in the base space. Selection of such paths simplifies computation of the controls; however other path selections could be made. There are infinitely many choices for control functions which accomplish the above strategy, and the total time required is arbitrary.

## 4.4 Closed-Loop Approach

Closed-loop approach consists of designing feedback controllers that asymptotically steers the nonholonomic wheeled-mobile system to the origin. Although nonholonomic wheeled mobile systems are locally controllable, there is no continuous time-invariant state feedback law which can locally stabilize this class of systems. However, a discontinuous feedback approach developed in [3] for general Caplygin dynamic systems can be utilized here for nonholonomic wheeled-mobile systems.

We now describe the ideas that are employed to construct such a feedback

which does achieve the desired objective. This construction procedure is essentially an implementation of the local asymptotic stabilization of the previously described open-loop control in a (necessarily) discontinuous state feedback form.

Now let  $\pi_s$  denote the projection map  $\pi_s : (p_1, p_2) \mapsto (p_1)$ . In order to construct a feedback control algorithm to accomplish the previously introduced two steps, we first define a feedback function  $U^{p_1^*}(\pi_s p)$  which satisfies: for any  $\pi_s p(t_0)$  there is  $t_1 \geq t_0$  such that the unique solution of

$$\dot{p}_1 = U^{p_1^*}(\pi_s p) ,$$

satisfies  $\pi_s p(t_1) = p_1^*$ . Note that the feedback function is parameterized by the vector  $p_1^*$ . Moreover, for each  $p_1^*$ , there exists such a feedback function. One such feedback function  $U^{p_1^*}(\pi_s p)$  is given as

$$U^{p_1^*}(\pi_s p) = -|p_1 - p_1^*|^\lambda \text{sign}(p_1 - p_1^*) ,$$

where  $0 \leq \lambda < 1$  is an arbitrary constant.

We specify the control algorithm, with values denoted by  $u^*$ , according to the following construction, where  $p$  denotes the “current state” :

Control algorithm for  $u^*$  :

**Step 0 :** Choose  $(a^*, b^*)$  to achieve the desired holonomy.

**Step 1 :** Set  $u^* = U^{a^*}(\pi_s p)$ , until  $\pi_s p = a^*$ ; then go to Step 2;

**Step 2 :** Set  $u^* = U^{a^*+b^*}(\pi_s p)$ , until  $\pi_s p = a^* + b^*$ ; then go to Step 3;

**Step 3 :** Set  $u^* = U^{b^*}(\pi_s p)$ , until  $\pi_s p = b^*$ ; then go to Step 4;

**Step 4 :** Set  $u^* = U^0(\pi_s p)$ , until  $\pi_s p = 0$ ; then go to Step 0;

We here assumed that the desired geometric phase can be obtained by a single closed path. Clearly the above algorithm can be modified to account for cases for which more than one closed path is required as we show in Remark (1).

Note that the control algorithm is constructed by appropriate switchings between members of the parameterized family of feedback functions. On each cycle of the algorithm the particular functions selected depend on the closed path parameters  $a^*, b^*$ , computed in Step 0, to correct for errors in  $p_2$ .

The control algorithm can be initialized in different ways. The most natural is to begin with Step 4 since  $u^*$  in that step does not depend on

the closed path parameters; however, many other initializations of the control algorithm are possible.

Justification that the constructed control algorithm asymptotically stabilizes the origin follows as a consequence of the construction procedure: that switching between feedback functions guarantees that the proper closed path (or a sequence of closed paths) is traversed in the base space so that the origin is necessarily reached in a finite time.

It is important to emphasize that the above construction is based on the a priori selection of simply parameterized closed paths in the base space. The above selection simplifies the tracking problem in the base space, but other path selections could be made and they would, of course, lead to a different feedback strategy from that proposed above.

# Chapter 5

## APPLICATIONS

### 5.1 Motion Planning for a Synchro-Drive Car

Consider a synchro-drive car with two driving wheels mounted on the same axis, controlled independently, plus one free wheel as shown in Figure 5.1. The position of the wheel axis center is denoted by  $(x, y)$ , the car orientation is denoted by  $\theta$ .

Note that motion of the vehicle is subject to the scalar nonholonomic constraint:

$$\dot{x} \sin \theta - \dot{y} \cos \theta = 0 . \tag{5.1}$$

This constraint simply states that the vehicle can only move in the direction normal to the axis of the driving wheels [23].

The kinematic equations can be written as

$$\begin{aligned}\dot{x} &= v \cos \theta , \\ \dot{y} &= v \sin \theta , \\ \dot{\theta} &= \omega ,\end{aligned}\tag{5.2}$$

where the control variables  $v$  and  $\omega$  denote the tangent and angular velocities, respectively, and are related to the wheel velocities in the following manner :

$$\begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} 0.5 & 0.5 \\ 0.5/d & -0.5/d \end{bmatrix} \begin{bmatrix} v_R \\ v_L \end{bmatrix},\tag{5.3}$$

where  $v_L$  and  $v_R$  denote the tangent velocities of left and right driving wheels, respectively, and  $d$  equals half the distance between the wheels.

**Remark 3** *Note that the conditions  $v_R = r\dot{\phi}_R$  and  $v_L = r\dot{\phi}_L$ , where  $r$  is the radius of each of the wheels and  $\dot{\phi}_R, \dot{\phi}_L$  are rolling angular velocities of the right and left wheels, respectively, guarantee that the two wheels roll without slipping. That is, the following two nonholonomic constraints are also satisfied:*

$$\dot{x} \cos \theta + \dot{y} \sin \theta + d\dot{\theta} = r\dot{\phi}_R ,$$

$$\dot{x} \cos \theta + \dot{y} \sin \theta - d\dot{\theta} = r\dot{\phi}_L .$$

Thus, if the vehicle is controlled in a way that

$$\begin{aligned} \dot{\phi}_R &= \frac{v + d\omega}{r} , \\ \dot{\phi}_L &= \frac{v - d\omega}{r} , \end{aligned}$$

then the vehicle will move in the direction normal to the axis of the driving wheels while the wheels roll without slipping.

The coordinate change

$$\begin{aligned} \eta_1 &= x , \\ \eta_2 &= \tan \theta , \\ \eta_3 &= y , \end{aligned} \tag{5.4}$$

transforms the kinematic equations into to the Caplygin form:

$$\begin{aligned} \dot{\eta}_1 &= u_1 , \\ \dot{\eta}_2 &= u_2 , \\ \dot{\eta}_3 &= \eta_2 u_1 . \end{aligned} \tag{5.5}$$

where

$$\begin{aligned} u_1 &= v \cos \theta , \\ u_2 &= \omega \sec^2 \theta . \end{aligned} \tag{5.6}$$

Consequently, we obtain a system with base space  $(\eta_1, \eta_2)$ .

The following conclusions are based on the analysis of the above equations.

**Proposition 2** *Let  $\eta^e = (\eta_1^e, \eta_2^e, \eta_3^e)$  denote a desired configuration.*

*The following properties are true for the synchro-drive car kinematics:*

1. *The system is completely controllable since the space spanned by the vector fields*

$$g_1, g_2, [g_1, g_2]$$

*has dimension 3 at any configuration,*

2. *There is no time-invariant continuous feedback which asymptotically stabilizes  $\eta^e$ ,*
3. *There is a time-invariant discontinuous feedback which asymptotically stabilizes  $\eta^e$ .*

Consider the problem of transferring the system from  $(\eta_1^0, \eta_2^0, \eta_3^0)$  to the origin  $(0, 0, 0)$ . Clearly, the system can be transformed from  $(\eta_1^0, \eta_2^0, \eta_3^0)$  to  $(0, 0, \eta_3^1)$  in arbitrary time  $t_1$  for some  $(\eta_3^1)$ . Let  $\gamma^1(a, b)$  denote a rectangular closed path with corner points

$$\{(0, 0), (a, 0), (a, b), (0, b)\} ,$$

By evaluating the integral in (4.5) in closed form we obtain

$$\eta_3^1 = -ab . \tag{5.7}$$

We just choose  $a$  and  $b$  arbitrary but different than zero to characterize the closed path  $\gamma$  which solves the desired geometric phase condition.

Then, the open-loop controller or the discontinuous feedback algorithm developed previously achieves the goal.

We present a representative simulation example, transferring the initial position  $(x^0, y^0, \theta^0) = (0, 1, 0)$  to the origin using the open-loop approach. The time responses for  $x, y, \theta$  are shown in Figure 5.2. The time responses for the input driving velocity  $v$  and the input steering velocity  $\omega$  are shown in Figure 5.3. In Figure 5.4 the configuration of the Synchron-drive car is shown for a sequence of time instants. The car path is shown in Figure 5.5. Figure 5.6 illustrates the motion in the base space.

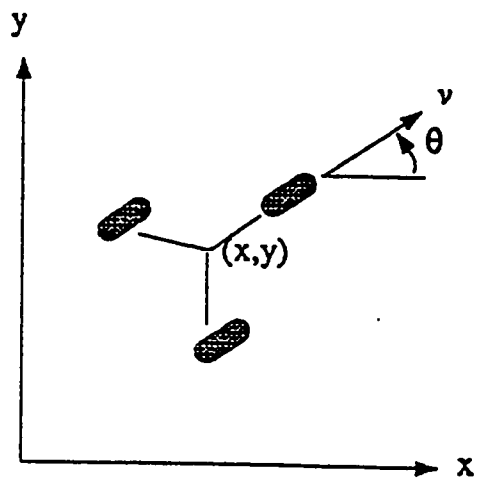


Figure 5.1: The synchro-drive car.

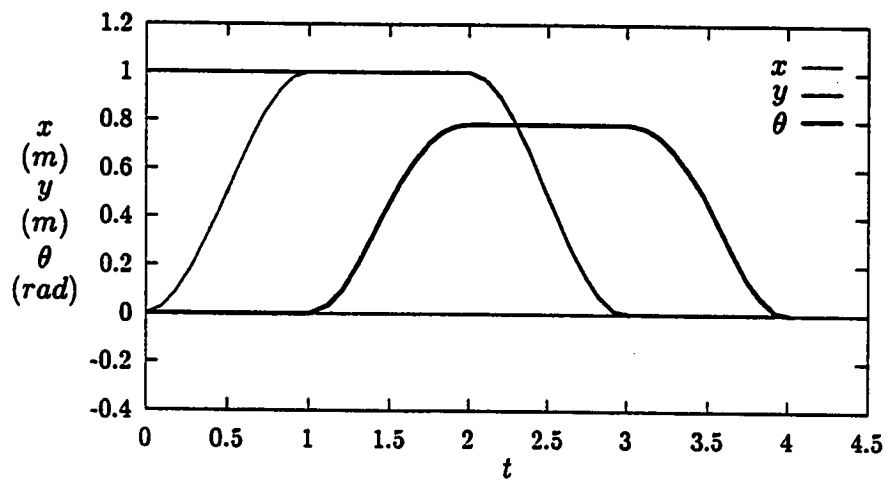


Figure 5.2: Time responses for configuration variables of the synchro-drive car.

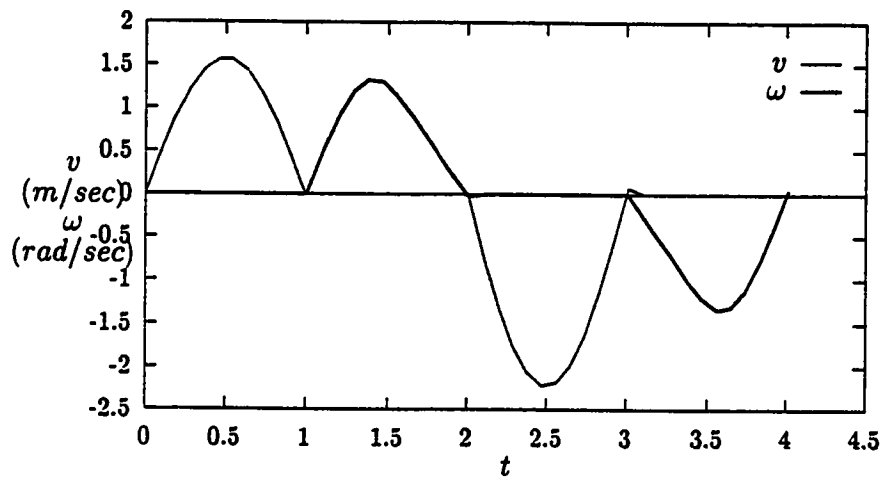


Figure 5.3: Control inputs of the synchro-drive car.

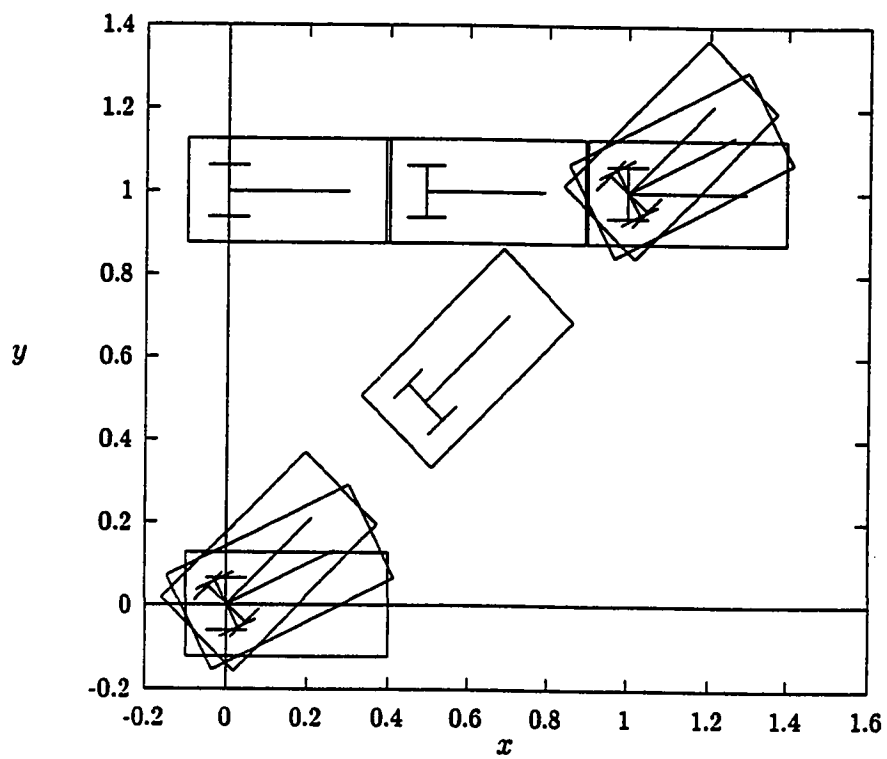


Figure 5.4: Configurations of the synchro-drive car.

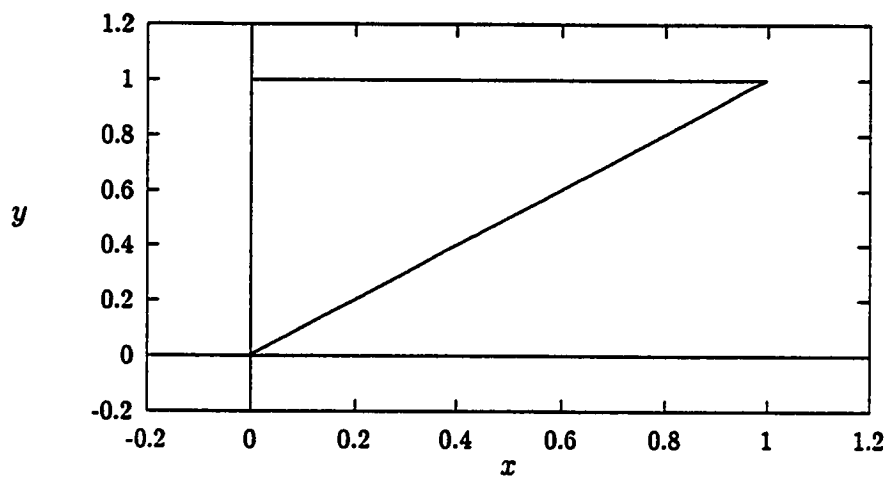


Figure 5.5: Path of the synchro-drive car.

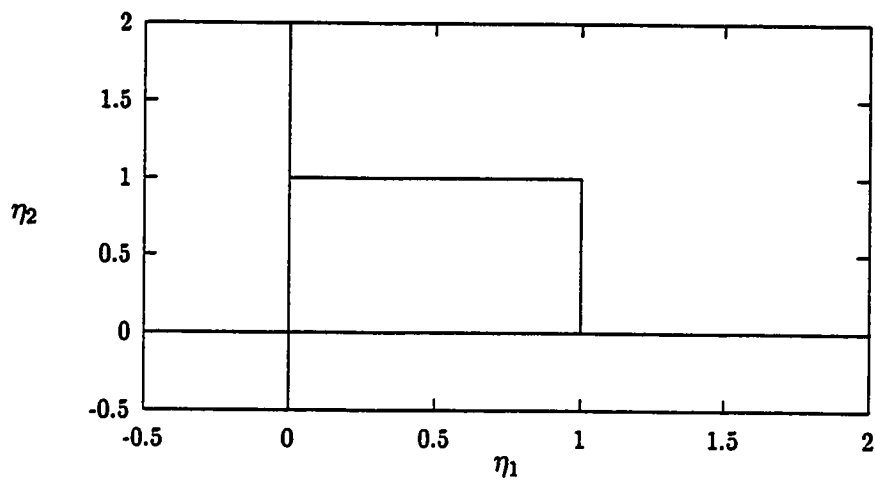


Figure 5.6: Motion in base space for the synchro-drive car.

## 5.2 Motion Planning for a Front-Wheel-Drive Car

Consider a front-wheel-drive car with two front-driving-and-steering-wheels and two rear wheels constrained to roll without slipping as shown in Figure 5.7.

Note the motion of the car is subjected to the following two nonholonomic constraints:

$$\begin{aligned} \dot{x} \sin \theta - \dot{y} \cos \theta &= 0 , \\ \dot{x} \tan \phi - d(\dot{\theta} \cos \theta) &= 0 . \end{aligned} \tag{5.8}$$

The kinematic equations can be written as

$$\begin{aligned} \dot{x} &= v_0 \cos \theta , \\ \dot{y} &= v_0 \sin \theta , \\ \dot{\theta} &= (v_0/d) \tan \phi , \\ \dot{\phi} &= \omega , \end{aligned} \tag{5.9}$$

where  $v_0$  is the rear axle velocity related to the driving velocity  $v$  by  $v = v_0 \sec \phi$ ,  $\omega$  denote the steering velocity,  $\theta$  denotes the orientation of the car with respect to a line parallel to the  $x$  axis and  $\phi$  denotes the

orientation of the front wheel with respect the car.

As shown by Tilbury, Murray and Sastry [22], the coordinate change

$$\begin{aligned}
 z_1 &= x , \\
 z_2 &= (1/d) \tan \phi \sec^3 \theta , \\
 z_3 &= \tan \theta , \\
 z_4 &= y ,
 \end{aligned}
 \tag{5.10}$$

transforms the kinematic equations into chain form:

$$\begin{aligned}
 \dot{z}_1 &= u_1 , \\
 \dot{z}_2 &= u_2 , \\
 \dot{z}_3 &= z_2 u_1 , \\
 \dot{z}_4 &= z_3 u_1 .
 \end{aligned}
 \tag{5.11}$$

Using a new coordinate change given by:

$$\begin{aligned}
 \eta_1 &= z_1 , \\
 \eta_2 &= z_2 , \\
 \eta_3 &= z_3 , \\
 \eta_4 &= z_4 - z_1 z_3 ,
 \end{aligned}
 \tag{5.12}$$

the kinematic equations can be locally described in the following Caply-

gin form:

$$\begin{aligned}
 \dot{\eta}_1 &= u_1 , \\
 \dot{\eta}_2 &= u_2 , \\
 \dot{\eta}_3 &= \eta_2 u_1 , \\
 \dot{\eta}_4 &= -\eta_1 \eta_2 u_1 ,
 \end{aligned} \tag{5.13}$$

where

$$\begin{aligned}
 u_1 &= v \cos \theta \cos \phi , \\
 u_2 &= (3v/d^2) \sin \theta \sec^4 \theta \sin \phi \tan \phi + (\omega/d) \sec^3 \theta \sec^2 \phi .
 \end{aligned} \tag{5.14}$$

Consequently, we obtain a system with base space  $(\eta_1, \eta_2)$ .

The following conclusions are based on the analysis of the above equations.

**Proposition 3** *Let  $\eta^e = (\eta_1^e, \eta_2^e, \eta_3^e, \eta_4^e)$  denote a desired configuration. The following properties are true for the front-wheel-drive car kinematics:*

1. *The system is completely controllable since the space spanned by the vector fields*

$$g_1 \cdot g_2, [g_1, g_2], [[g_1, g_2], g_1]$$

*has dimension 4 at any configuration,*

2. *There is no time-invariant continuous feedback which asymptotically stabilizes  $\eta^e$ ,*

3. *There is a time-invariant discontinuous feedback which asymptotically stabilizes  $\eta^e$ .*

Consider the problem of transferring the system from  $(\eta_1^0, \eta_2^0, \eta_3^0, \eta_4^0)$  to the origin  $(0, 0, 0, 0)$ . Clearly, the system can be transferred from  $(\eta_1^0, \eta_2^0, \eta_3^0, \eta_4^0)$  to  $(0, 0, \eta_3^1, \eta_4^1)$  in arbitrary time  $t_1$  for some  $(\eta_3^1, \eta_4^1)$ . Let  $\gamma^1$  and  $\gamma^2$  denote two rectangular closed paths with corner points

$$\{(0, 0), (a_i, 0), (a_i, b_i), (0, b_i)\}, \quad i = 1, 2,$$

and let  $\gamma$  denote their concatenation. By evaluating the integral in (4.5) in closed form we obtain

$$\begin{bmatrix} \eta_3^1 \\ \eta_4^1 \end{bmatrix} = \begin{bmatrix} -a_1 & -a_2 \\ +\frac{1}{2!}a_1^2 & +\frac{1}{2!}a_2^2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}. \quad (5.15)$$

We just choose  $(a_1, a_2)$  with  $a_1 \neq a_2$  and then solve the resulting linear equations for  $(b_1, b_2)$ , thereby characterizing a closed path  $\gamma$  which solves the desired geometric phase condition. Then, the open loop controller or the discontinuous feedback algorithm defined previously achieves the goal.

We present a representative simulation example, transferring the initial position  $(x^0, y^0, \theta^0, \phi^0) = (0, 1, 0, 0)$  to the origin using the open-loop approach. The time responses for  $x, y, \theta, \phi$  are shown in Figure 5.8. The

time responses for the input driving velocity  $v$  and the input steering velocity  $\omega$  are shown in Figure 5.9. In Figure 5.10 the configuration of the car is shown for a sequence of time instants. The car path is shown in Figure 5.11. Figure 5.12 illustrates the motion in the base space.

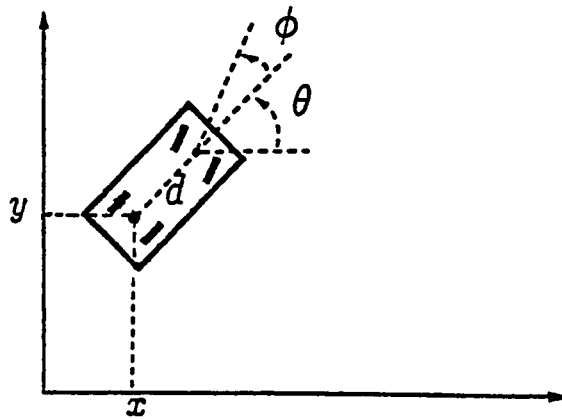


Figure 5.7: The front-wheel-drive car.

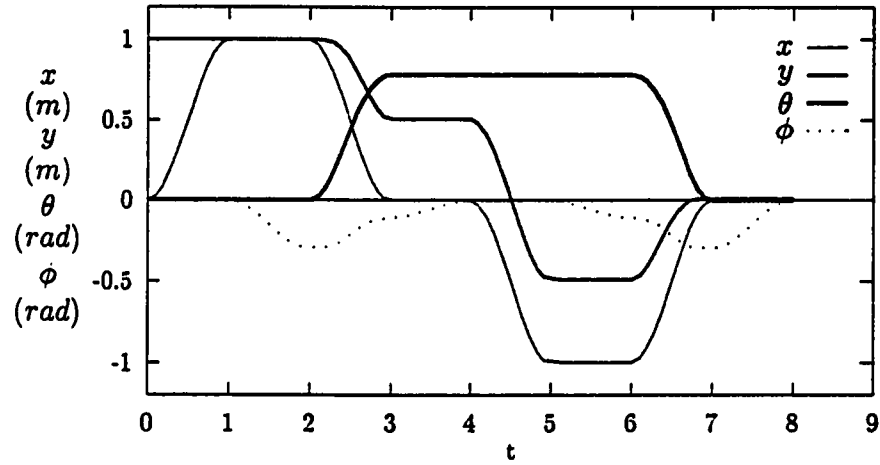


Figure 5.8: Time responses for configuration variables of the front-wheel-drive car.

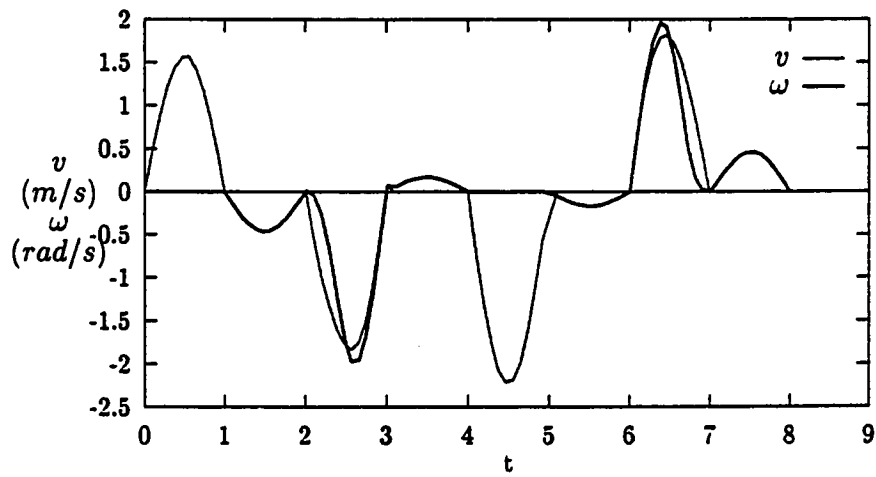


Figure 5.9: Control inputs of the front-wheel-drive car.

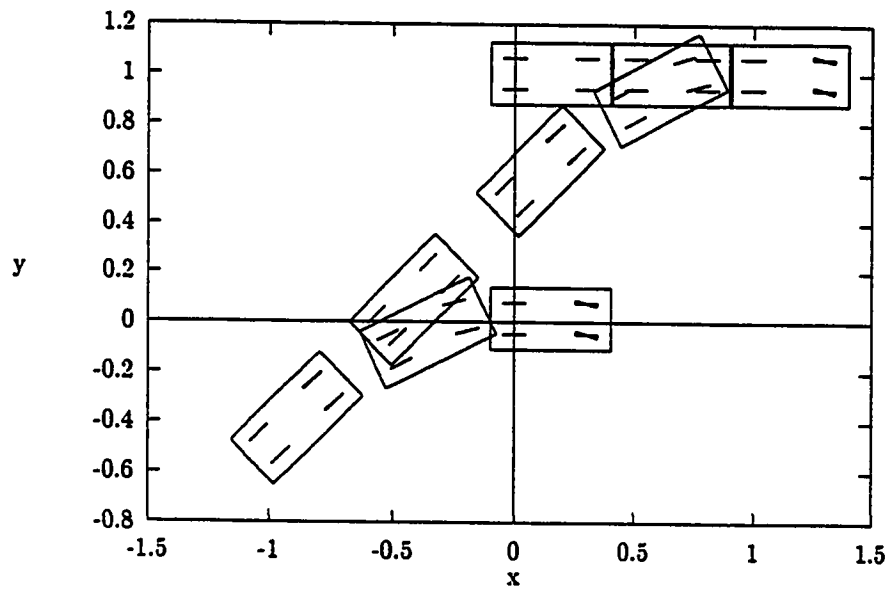


Figure 5.10: Configurations of the front-wheel-drive car.

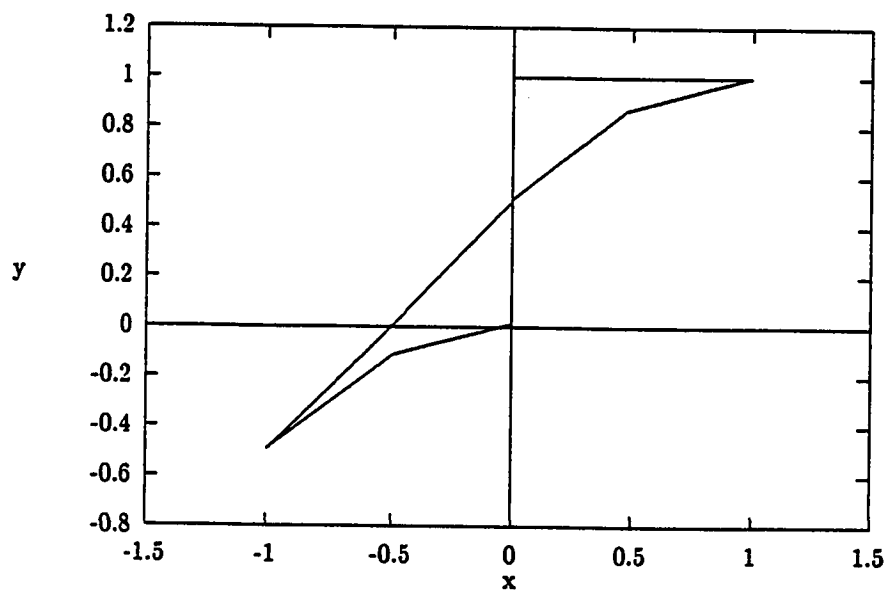


Figure 5.11: Path of the front-wheel-drive car.

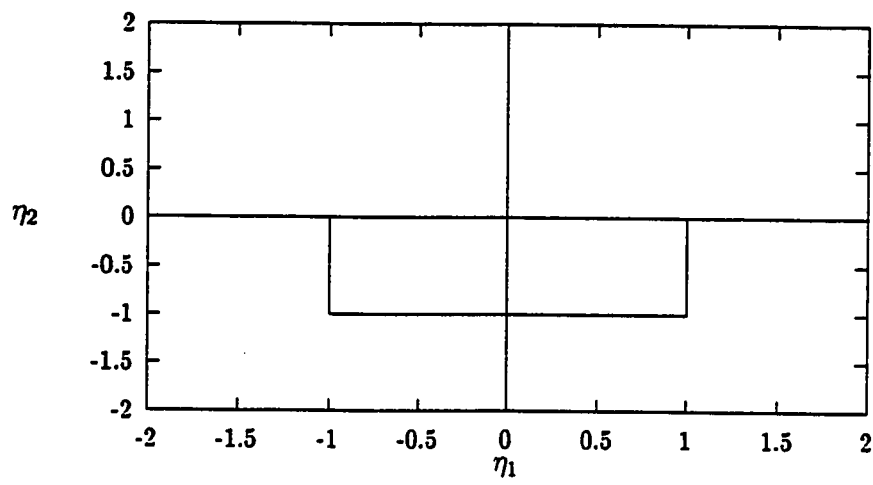


Figure 5.12: Motion in base space for the front-wheel-drive car.

### 5.3 Motion Planning for a Car Pulling $n$ Trailers

Consider a car with  $n$  trailers as shown in Figure 5.13. The hitch of trailer  $i$  is attached to the center of the rear axle of trailer  $i - 1$ . The wheels are aligned with the body of the trailer. The positive parameters  $d_0, d_1, \dots, d_n$  denote the lengths of the wheelbases. The two control inputs are the driving velocity  $v$  and the steering velocity  $\omega$  of the front wheels of the car. The constraints are based on allowing the wheels to roll and spin without slipping. For steering the front wheels of the car, the derivation is simplified by assuming them as a single wheel at the midpoint of the axle. As displayed in Figure 5.13, we denote by  $P_i$  the midpoint of the wheel axle of trailer  $i$  and by  $(x_i, y_i)$  its cartesian coordinates. Consequently, for  $i = 1, \dots, n$ , we have:

$$\begin{aligned} x_i &= x_0 - \sum_{j=1}^i d_j \cos \theta_j , \\ y_i &= y_0 - \sum_{j=1}^i d_j \sin \theta_j . \end{aligned} \tag{5.16}$$

The nonholonomic constraints arise from the fact that the wheels of the car and trailers are constrained to roll without slipping which implies that the velocity of each body in the direction perpendicular to its wheel

must be zero. The constraint for the  $n^{\text{th}}$  trailer is given by

$$\dot{x}_n \sin \theta_n - \dot{y}_n \cos \theta_n = 0 . \quad (5.17)$$

To obtain the constraint for the other trailers, let  $v_i$  denote the velocity of the  $i^{\text{th}}$  trailer. Figure 5.14 shows that the direction of  $v_{i+1}$  is along the direction of the hitch joining the  $(i+1)^{\text{th}}$  trailer to  $i^{\text{th}}$  trailer so that

$$v_{i+1} = \cos(\theta_{i+1} - \theta_i) v_i . \quad (5.18)$$

Note that the velocity of the  $i^{\text{th}}$  trailer is given by

$$v_i = \dot{x}_i \cos \theta_i + \dot{y}_i \sin \theta_i . \quad (5.19)$$

It can be shown that if the  $i^{\text{th}}$  trailer rolls without slipping then we must have

$$d_i \dot{\theta}_i \cos(\theta_i - \theta_{i-1}) + v_i \sin(\theta_i - \theta_{i-1}) = 0 , \quad (5.20)$$

for  $i = 1, \dots, n$ . The  $n$  constraint equations for the  $n$  trailers are given by the above expressions (5.20) and the two constraint equations for the car can be written as

$$\begin{aligned} \dot{x}_0 \sin \theta_0 - \dot{y}_0 \cos \theta_0 &= 0 , \\ \dot{x}_0 \tan \phi - d_0 \dot{\theta}_0 \cos \theta_0 &= 0 . \end{aligned} \quad (5.21)$$

The relations (5.16) constitute  $2n$  holonomic constraints and hence the dimension of the configuration space is  $n + 4$ . Since the positions  $(x_i, y_i)$  for  $i \geq 1$  can be expressed in terms of  $x_0, y_0, \theta_0, \dots, \theta_i$ . By symmetry,  $(x_i, y_i)$  for  $i < n$  can also be expressed in terms of  $x_n, y_n, \theta_n, \theta_{n-1}, \dots, \theta_i$ . Following [22], we consider  $(x_n, y_n, \theta_n, \theta_{n-1}, \dots, \theta_0, \phi) \in \mathbb{R}^2 \times (S^1)^{n+2}$  as the state. Consequently, the system kinematic equations are given as:

$$\begin{aligned}
\dot{x}_n &= v_n \cos \theta_n , \\
\dot{y}_n &= v_n \sin \theta_n \\
\dot{\theta}_n &= (v_n/d_n) \tan(\theta_{n-1} - \theta_n) , \\
\dot{\theta}_i &= (v_n/d_i) \prod_{j=i+1}^n \sec(\theta_{j-1} - \theta_j) \tan(\theta_{i-1} - \theta_i) , \quad i = n - 1, \dots, 1 , \\
\dot{\theta}_0 &= (v_n/d_0) \prod_{j=1}^n \sec(\theta_{j-1} - \theta_j) \tan \phi , \\
\dot{\phi} &= \omega ,
\end{aligned} \tag{5.22}$$

where  $v_n$  is the  $n^{\text{th}}$  trailer velocity related to the driving velocity  $v$  of the car by  $v = v_n \prod_{i=1}^n \sec(\theta_{i-1} - \theta_i) \sec \phi$ ,  $\omega$  denote the steering velocity,  $\theta_i$  denotes the orientation of the  $i^{\text{th}}$  trailer with respect to a line parallel to the  $x$  axis,  $\theta_0$  denotes the orientation of the car with respect to a line parallel to the  $x$  axis and  $\phi$  denotes the orientation of the front wheel with respect to the car.

As shown in [22], the coordinate change

$$\begin{aligned}
 z_1 &= x_n \\
 z_{n+4} &= y_n \\
 z_{n+3} &= \tan \theta_n \\
 z_{n+2} &= (1/d_n) \tan(\theta_{n-1} - \theta_n) \sec^3 \theta_n , \\
 z_i &= \dot{z}_{i+1} / (v_n \cos \theta_n) , \quad i = n + 1, \dots, 2 ,
 \end{aligned} \tag{5.23}$$

transforms the above kinematic equations into the chain form:

$$\begin{aligned}
 \dot{z}_1 &= u_1 , \\
 \dot{z}_2 &= u_2 , \\
 \dot{z}_i &= z_{i-1} u_1 , \quad i = 3, \dots, n + 4 ,
 \end{aligned} \tag{5.24}$$

where

$$\begin{aligned}
 u_1 &= v_n \cos \theta_n , \\
 u_2 &= \dot{z}_2 .
 \end{aligned} \tag{5.25}$$

Using a new coordinate change given by

$$\begin{aligned}
 \eta_1 &= z_1 , \\
 \eta_2 &= z_2 , \\
 \eta_i &= z_i + \sum_{j=3}^{i-1} \frac{(-1)^{i-j}}{(i-j)!} z_1^{i-j} z_j , \quad i = 3, \dots, n + 4 ,
 \end{aligned} \tag{5.26}$$

the kinematic equations can be locally described in the following Caply-

gin form:

$$\begin{aligned}\dot{\eta}_1 &= u_1, \\ \dot{\eta}_2 &= u_2, \\ \dot{\eta}_i &= \frac{(-1)^{i-3}}{(i-3)!} \eta_1^{i-3} \eta_2 u_1, \quad i = 3, \dots, n+4.\end{aligned}\tag{5.27}$$

Consequently, we obtain a system with base space  $(\eta_1, \eta_2)$ .

The following conclusions are based on the analysis of the above equations.

**Proposition 4** *Let  $\eta^e = (\eta_1^e, \eta_2^e, \dots, \eta_{n+4}^e)$  denote a desired configuration. The following properties are true for the front-wheel-drive car kinematics:*

1. *The system is completely controllable since the space spanned by the vector fields*

$$g_1, g_2, [g_1, g_2], \dots, \underbrace{[g_1, [g_1, \dots, [g_1, g_2] \dots]]}_{\text{length}=n+2}$$

*has dimension  $n + 4$  at any configuration,*

2. *There is no time-invariant continuous feedback which asymptotically stabilizes  $\eta^e$ ,*
3. *There is a time-invariant discontinuous feedback which asymptotically stabilizes  $\eta^e$ .*

Consider the problem of transferring the system from  $(\eta_1^0, \eta_2^0, \dots, \eta_{n+4}^0)$  to  $(0, 0, \dots, 0)$ . Clearly, the system can be transformed from  $(\eta_1^0, \eta_2^0, \dots, \eta_{n+4}^0)$  to  $(0, 0, \eta_3^1, \dots, \eta_{n+4}^1)$  in arbitrary time  $t_1$  for some  $(\eta_3^1, \dots, \eta_{n+4}^1)$ . Let  $\gamma^1, \dots, \gamma^{n+2}$  denote  $n+2$  rectangular closed paths with corner points

$$\{(0, 0), (a_i, 0), (a_i, b_i), (0, b_i)\}, \quad i = 1, \dots, n+2,$$

and let  $\gamma$  denote their concatenation. By evaluating the integral in (4.5)

in closed form we obtain

$$\begin{bmatrix} \eta_3^1 \\ \eta_4^1 \\ \vdots \\ \eta_{n+4}^1 \end{bmatrix} = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n+2} \\ +\frac{1}{2!}a_1^2 & +\frac{1}{2!}a_2^2 & \cdots & +\frac{1}{2!}a_{n+2}^2 \\ \vdots & \vdots & \cdots & \vdots \\ \frac{(-1)^{n+2}}{(n+2)!}a_1^{n+2} & \frac{(-1)^{n+2}}{(n+2)!}a_2^{n+2} & \cdots & \frac{(-1)^{n+2}}{(n+2)!}a_{n+2}^{n+2} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n+2} \end{bmatrix}. \quad (5.28)$$

Let  $A(a)$  denote the  $(n+2) \times (n+2)$  matrix in the above equation. It can be shown that

$$\det A(a) = \left( \prod_{k=1}^{n+2} \frac{(-1)^k}{k!} a_k \right) \left( \prod_{i>j} (a_i - a_j) \right),$$

which is nonzero provided that  $a_1, \dots, a_{n+2}$  are nonzero and distinct real numbers. Thus we just choose  $(a_1, \dots, a_{n+2})$  with  $a_i \neq a_j \neq 0$ , for all  $i, j$  and then solve the resulting linear equations for  $(b_1, \dots, b_{n+2})$ , thereby characterizing a closed path  $\gamma$  which solves the desired geometric phase condition.

We present a representative simulation example for a car pulling one trailer. For this example, the configuration space is  $(x_1, y_1, \theta_1, \theta_0, \phi)$ . We consider transferring the initial position  $(x_1^0, y_1^0, \theta_1^0, \theta_0^0, \phi^0) = (0, 2, 0, 0, 0)$  to the origin. The time responses of the configuration variables  $x_1, y_1, \theta_1, \theta_0, \phi$  are shown in Figure 5.15. The time responses for the input driving velocity  $v$  and the input steering velocity  $\omega$  are shown in Figure 5.16. In Figure 5.17 the configuration of the car is shown for a sequence of time instants. The trailer path is shown in Figure 5.18. Figure 5.19 illustrates the motion in the base space.

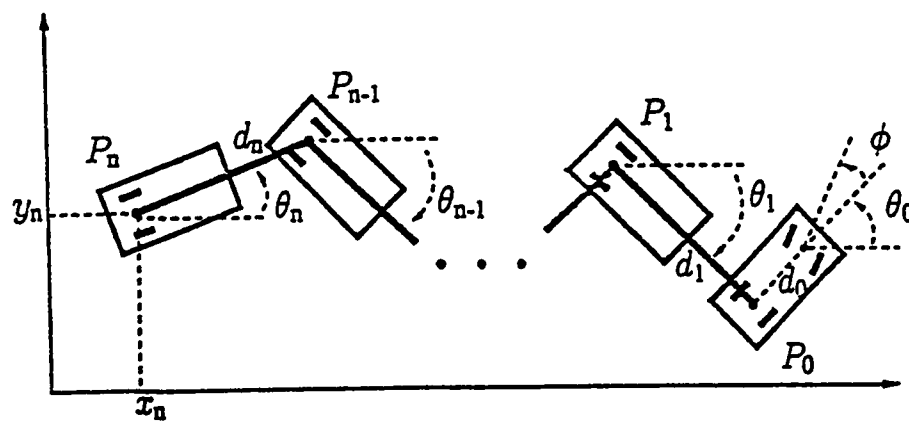


Figure 5.13: The car with  $n$  trailers.

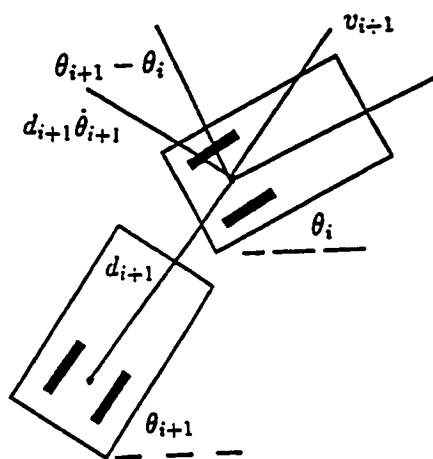


Figure 5.14: The definitions of the angles and the velocities of the  $i^{\text{th}}$  trailer.

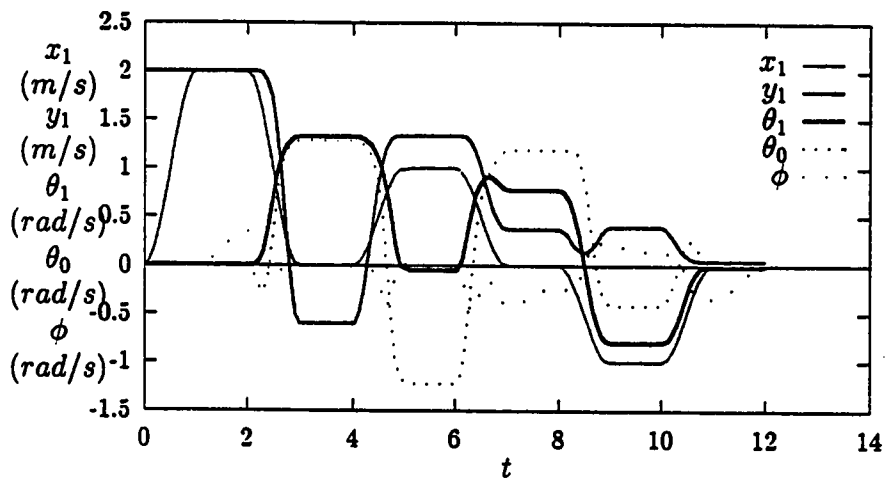


Figure 5.15: Time responses for configuration variables of the car with a trailer.

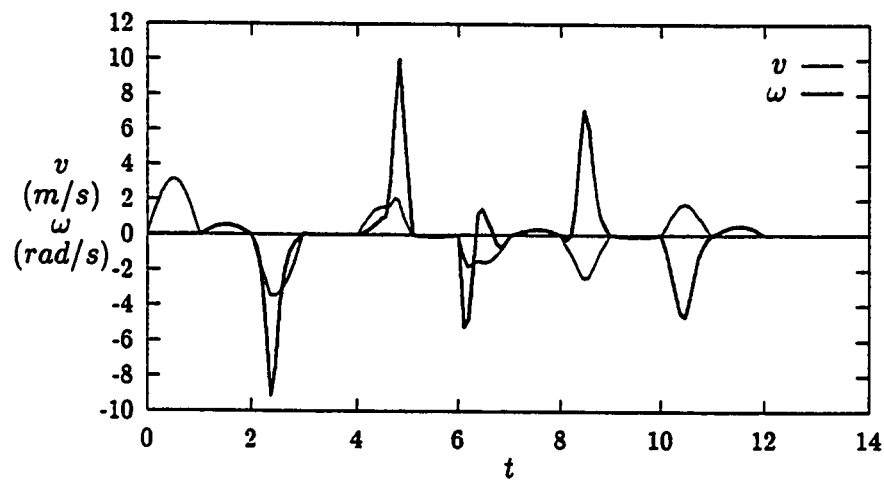


Figure 5.16: Control inputs of the car with a trailer.

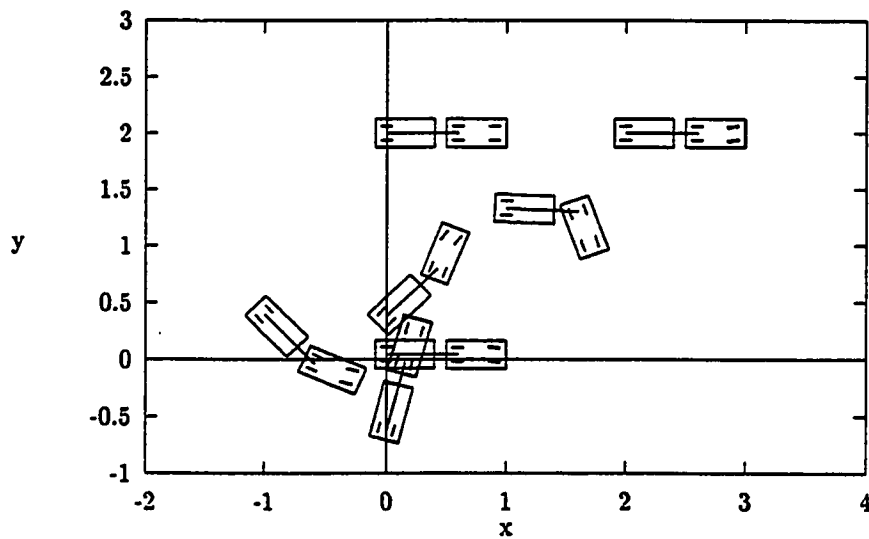


Figure 5.17: Configurations of the car with a trailer.

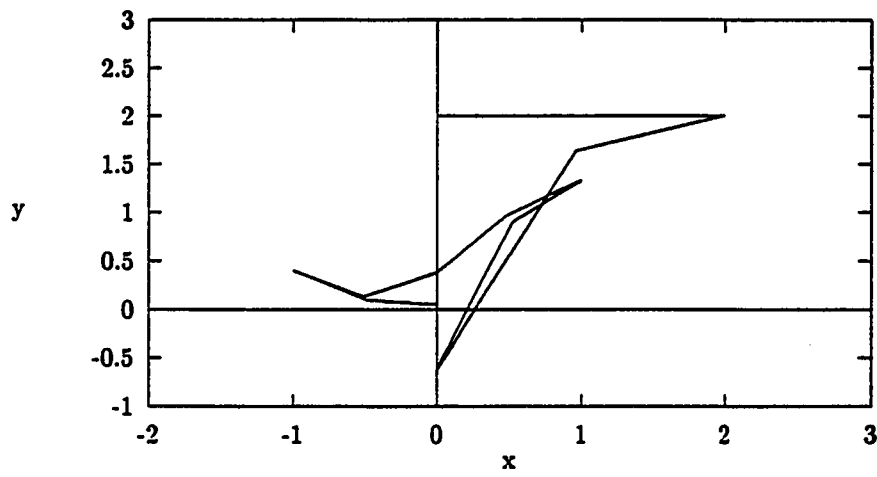


Figure 5.18: Path of the trailer.

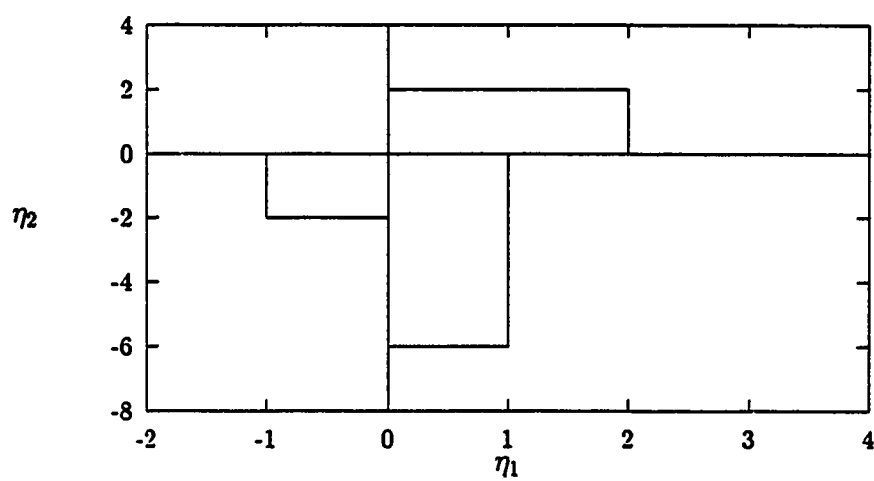


Figure 5.19: Motion in base space for the car with a trailer .

## Chapter 6

# CONCLUSIONS AND FUTURE RESEARCH

### 6.1 Conclusions

We have studied control issues for wheeled mobile systems and we have derived a number of fundamental results. Although nonholonomic wheeled mobile systems are completely controllable, they cannot be asymptotically stabilized to a desired configuration using time-invariant continuous feedback control. A nonholonomic motion planning approach using geometric phases has been presented. It has been demonstrated that the approach constitutes an analytical method for the motion planning problem

associated with a large class of wheeled mobile systems. In particular, a closed-form solution to the well-known multi-trailer problem has been presented. Results have been illustrated through simulations of three examples: a synchro-drive car, a front-wheel drive car and a car pulling a trailer.

We also present a computational approach for the geometric phase of wheeled-mobile systems in Chaplygin form. This study should provide a useful framework for those interested in the design, modeling, and control of mobile robots. Our general approach can easily be applied to any wheeled-mobile system.

## 6.2 Future Research

In this thesis we have considered only the kinematics of wheeled-mobile systems. Future research considers control representations necessarily include nontrivial drift vector fields i.e., considers the dynamic relations to describe such systems.

We have considered discontinuous state feedback controllers for asymptotic stabilization. Future research includes using smooth time-varying feedback controllers as an alternative by extending Pomet's [14] results to wheeled mobile systems.

In this thesis we have not considered optimality issues. It would be natural, for instance, to formulate an optimal motion planning problem for nonholonomic wheeled mobile systems considered in this thesis. Such a formulation would include specification of a performance measure for the motion and it could also include specification of limits on velocities, accelerations, and control inputs.

In this thesis we have not considered obstacle avoidance problem in generating a trajectory for mobile systems. Such problems would, no doubt,

represent a substantial challenge; the result of this thesis can be viewed as a necessary background for future work in this direction.

# Nomenclature

$\Delta$	distribution
$\Omega$	codistribution
$\tau$	vector field
$\omega$	covector field
$[\tau_1, \tau_2]$	Lie bracket of $\tau_1$ and $\tau_2$
$\delta_{ij}$	Kronecker delta
$p$	configuration vector
$p_1$	base vector
$p_2$	fiber vector
$G(p)$	$n \times (n - m)$ matrix function
$g_i(p)$	columns of matrix function $G(p)$
$J(p)$	$m \times n$ matrix function
$k^*$	nonholonomy degree
$u_i$	control inputs
$t$	time
$B(p_1)$	$m \times (n - m)$ matrix function
$U_i(\pi, p)$	feedback functions
$\gamma$	closed path

$\alpha$	geometric phase
$c$	size of the closed path
$c_{cr}$	critical size of the close path
$\eta_i$	configuration variables for chain form
$z_i$	configuration variables for Caplygin form

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