ACOUSTIC WAVEFIELD IN AN INHOMOGENEOUS OCEAN

BY

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To my family

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I am most grateful to Almighty Allah, the Beneficient, the Merciful, for enabling me to complete this work. May peace and Allah's blessing be upon Muhammad, the last of His messengers, and his family.

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التخصـــص : رياضيات

تاريخ الشهادة : صفر / ١٤٢١ هـ

في هذه الرسالة نشتق معادلات الحقل الصوتي في نموذج ذات طبقتين قــاع الطبقة السفلة له ذات شكل متغير غير حاد . وفي هذا البحث نقوم أيضا بدراســة نفس النموذج ولكن مع تغير العدد الموجي له مع العمق وحالـــة عامــة لسـلوك القـــاع. ونقوم كذلك في هذه الرسالة باشتقاق عوامل الارتبـــاط لنفــس النمــوذج باستعمال طريقة الارتباط المعدلة وأسس التبديل . كذلك قمنـــا باشــتقاق عوامــل الارتداد الحجمي والسطحي لنموذج عشوائي ذات طبقـــة واحـــدة ورقــم موجـــي عشوائي السلوك .

THESIS ABSTRACT

Name of Student : **AHMED MOHAMMED AL-MARZOUG**

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In this thesis, we derive the acoustic pressure formulations for the two-layer model with the sea bottom of smooth irregular shape. We also discuss the two-layer model with the non-horizontal seabed and the depth-dependent wavenumber for the more general boundary conditions at the seabed. We also derive the mode-coupling coefficients for the same model using the modified mode-coupling technique and also the reciprocity principle. We also consider the stochastic model arising from the irregular surface and bottom of the ocean and the volume and surface scattering coefficients are obtained in this case for a random wavenumber.

DOCTOR OF SCIENCE DEGREE

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CHAPTER I

INTRODUCTION

1.1 Literature Review

The study of acoustic wave propagation in ocean waveguides has attracted considerable attention due to its applications in geophysics, underground acoustics and marine detection problems. There have been several studies using the homogeneous model of the ocean in which the density and hence the acoustic speed is assumed to be uniform. In such a case, the acoustic pressure, which is the exact solution of the acoustic wave equation, can be obtained using separation of variables resulting in the Helmholtz equations (Ahluwalia and Keller, [1]). While the speed of the sound propagation in the homogeneous ocean is nearly uniform, the small variations in speed may occur due to change in temperature, salinity, depth or other inhomogeneities. The change of density with depth of the ocean water leads to a depth- dependent velocity model.

M.D. Duston, G. Verma and D.H. Wood [7], and Weder Ricardo [20] used perturbation theory to study the problem of variable speed with depth. They used normal modes, involving the eigenvalues and eigenfunctions of a depth-dependent ordinary differential equation and obtained corrections to the solution of the perturbed problem arising from depth-dependent properties.

In many cases, it has been observed that the variation in the physical characteristics is not continuous as depth changes, but change occurs in a discontinuous way. These properties remain piecewise constant within layers and only change across the interface (Etter, [8]). Boyles' [4] has studied a model consisting of two or more homogeneous layers to account for such situations. Boyles model will be discussed in detail in Section 1.3. Holford [12], Y.L.L. et al. [13] obtained a closed form solution of the Helmholtz equation for the case of a point source in a layered medium with refractive index variation using perturbation theory.

In addition to nonhomogeneous models of the ocean, we sometimes need to study the acoustic field due to a source. The presence of the source makes the wave equation inhomogeneous and the separation of variables to obtain a range equation and a depth equation fails. In addition to failure of separation of variables in the presence of source, we observe that it can only be applied to the wave equation if the boundary surface of the problem coincides with the coordinate surface.

Zaman et al. [25] have studied the propagation of Love-type waves in a similar problem arising from propagation of horizontally polarized shear waves (Love-waves) in a wave-guide type layer overlying a half space. In order to avoid the separation of variables, they have used the Fourier transform in one-space variable. The transformed problem is then solved using Green's function method. Thus, if we are to consider the presence of a source or an ocean model with a rough sea surface with depth as well as range-dependent speed profile, we may follow Zaman et al. [25] by using Fourier or Laplace transforms, and Green's function together with perturbation theory.

The variation of the density and wavenumber may not always be determinatistic. In some cases of interest, these variations may occur randomly, thus giving rise to a random or stochastic wave equation. Ogilvy J.A. [18] has discussed the theory of wave scattering from rough surfaces by perturbation theory and Kirchoff method.

Recently, there has been interest in couple-mode theory to handle media with irregular interfaces or a range-dependent rigid boundary (Godin, [11) and (Fawcett [9]). The theory originated by Pierce [19] and and extended by Milder [15] depends on expressing acoustic propagation pressure as a series of modal amplitude functions times vertical eigenfunction. The conventional application of the mode theory which depends on a direct substitution of the series representation for the pressure field in the wave equation leads to erroneous results as discussed by Godin [11]. A careful implementation of mode-coupling theory for a rough surface model is by using the reciprocity principle or the divergence theorem (Fawcett, [9], Brekhovskikh and Godin [5]) and (Godin [11]). Some interesting features of the acoustic waves can be studied due to the bottom interactions or irregular sea bed.

In this work, we intend to study and develop ways to treat situations of practical interest involving:

(a) variable wave number (i.e., variable velocity, or density or both) with the source present;

(b) rough surfaces;

(c) non-planar ocean layers;

(d) mode-coupling theory; and

(e) volume and surface scattering.

In cases (a) to (c), we are going to employ Fourier transform and Green's function. In (d) and (e), we will use the mode-coupling theory using reciprocity principle or the divergence theorem to derive coupling coefficients as well as volume and surface scattering coefficients.

In Chapter One, we give literature review, and discuss conditions under which separation of variables method can be applied and present Boyles model consisting of two homogeneous ocean layers. In Chapter two, we derive the acoustic pressure formulations for the two-layer model, but with sea bottom of smooth irregular shape. Numerical results describing the effect of smooth undulation on the acoustic pressure are given. We also discuss the two-layer model with the non-horizontal seabed and the depth-dependent for the more general boundary conditions at the seabed. For this, we allow the seabed to satisfy reflecting or impedance boundary conditions. In Chapter three, we derive the mode-coupling coefficients for the same model as given in Chapter two, using the modified mode-coupling technique. Also, the general mode-coupling theory using the reciprocity principle is used to derive mode-coupling coefficients for a layered model with irregular interfaces and rigid boundary conditions for the sea bottom. Numerical results for an irregular layer model are given. In Chapter four, we consider the stochastic model arising from the irregular surface and bottom of the ocean. The volume and surface scattering coefficients are obtained in this case for a random wavenumber.

1.2 Separation of Variables and Basic Equations

By this method, the original partial differential equation in several independent variables is separated into a set of ordinary differential equations, each having one independent variable. The following theorem states the conditions under which separation of variables can occur.

Theorem (1.2.1) (Boyles, [4]). *Consider the most general form of a linear homogeneous partial differential equation of the second order in two independent variables*

$$a_{11}\frac{\partial^2 a}{\partial \xi^2} + 2a_{12}\frac{\partial^2 u}{\partial \xi \partial \eta} + a_{22}\frac{\partial^2 u}{\partial \eta^2} + a_{10}\frac{\partial u}{\partial \xi} + a_{01}\frac{\partial u}{\partial \eta} + a_{00}u = 0, \tag{1.2.1}$$

where the a_{ik} are functions of ξ and η. This partial differential equation can be reduced to two ordinary differential equations of the second order, both containing an arbitrary parameter λ by Bernoulli's separation method if there exists a transformation

$$x = x(\xi, \eta) \tag{1.2.2}$$

$$y = y(\xi, \eta), \tag{1.2.3}$$

whose Jacobian does not vanish; so, the resulting equation

$$A_{11}\frac{\partial^2 u}{\partial x^2} + A_{22}\frac{\partial^2 u}{\partial y^2} + A_{10}\frac{\partial u}{\partial x} + A_{01}\frac{\partial u}{\partial y} + A_{00}u = 0 \tag{1.2.4}$$

does not contain a term of the type $\frac{\partial^2 u}{\partial x \partial y}$ and, if there exists a function $B(x, y)$ so that $A_{11}/B = C_{11}, A_{10}/B = C_{10}$ are functions of x only, and if $A_{22}/B = C_{22}$, $A_{01}/B = C_{01}$ are functions of y only: and if A_{00}/B can be split up into a function of x and a function of y, as $A_{00}/B = C_{00}^{(1)}(x) + C_{00}^{(2)}(y)$.

The wave equation that involves density variation and existence of source of strength Q is (Boyles [4])

$$\nabla^2 p - \left(\frac{1}{\rho}\right) \nabla \rho \cdot \nabla p - \frac{1}{c^2(x,y,z)} \frac{\partial^2 p}{\partial t^2} = -\frac{\partial Q}{\partial t}, \tag{1.2.5}$$

where $c(x, y, z)$ is the propagation sound velocity.

If we assume that the density of the ocean is independent of spatial coordinates and the source strength Q is time-independent, equation (1.2.5) becomes

$$\nabla^2 p - \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} = 0. \tag{1.2.6}$$

Assume a solution of the form

$$p(r,t) = p(r)T(t). \tag{1.2.7}$$

Here, $p(r)$ is a function only of the spatial coordinates x, y, and z where the position vector

$$r = xe_1 + ye_2 + ze_3, \tag{1.2.8}$$

and $T(t)$ is a function only of the time.

We substitute the expression for $p(r,t)$ in equation (1.2.6), to obtain

$$T \nabla^2 p = \frac{1}{c^2} p \frac{\partial^2 T}{\partial t^2} \tag{1.2.9}$$

or

$$\left(c^2(z)/p\right) \nabla^2 p = \left(\frac{1}{T}\right) \frac{\partial^2 T}{\partial t^2} = -\omega^2 \tag{1.2.10}$$

where ω is a separation constant. Using equations (1.2.9) and (1.2.10), we get

$$\nabla^2 p + k_0^2 n^2 p = 0 \tag{1.2.11}$$

where $k_0 = \dfrac{\omega}{c_0}$ and $n(z) = \dfrac{c_0}{c(z)}$.

Here, ω is the angular frequency, k_0 is a reference wavenumber, c_0 is a reference speed and $n(z)$ is a refraction index.

In polar coordinates, assuming cylindrical symmetry, (1.2.11) becomes

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} + \frac{\partial^2 p}{\partial z^2} + k_0^2 n^2(z) p = 0. \tag{1.2.12}$$

Equation (1.2.12) is called the Helmholtz equation. Again, we assume a solution of the form $p = R(r)\psi(z)$ and we substitute it in (1.2.12) to get

$$\frac{d^2 R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + k_0^2 \xi^2(z) R = 0 \tag{1.2.13}$$

and

$$\frac{d^2 \psi}{dz^2} + k_0^2 (n^2(z) - \xi^2)\psi = 0 \tag{1.2.14}$$

where ξ is a separation constant.

If we include the source term in (1.2.12), we get

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} + \frac{\partial^2 p}{\partial z^2} + k_0^2 n^2(z) p = -\left(\frac{1}{2\pi r}\right)\delta(r - r_s)\delta(z - z_s). \tag{1.2.15}$$

The separation of variables can not be used in this case and a desired solution of the equation (1.2.15) can be represented in the form (Boyles [4])

$$p(r, r_s, z, z_s) = -\left(\frac{1}{2\pi i}\right)\int_{c_\lambda} G_1(r, r_s, \lambda) G_2(z, z_s, \lambda)\, d\lambda, \tag{1.2.16}$$

where $G_1(r, r_s, \lambda)$ is the radial Green's function and satisfies the following inhomogeneous differential equation:

$$\frac{d^2G_1}{dr^2} + \frac{1}{r}\frac{dG_1}{dr} + k^2\xi^2 G_1 = -\frac{1}{2\pi r}\delta(r - r_s), \tag{1.2.17}$$

and $G_2(z, z_s, \lambda)$ is the z-direction Green's function which satisfies the differential equation

$$\frac{d^2G_2}{dz^2} + \left(k^2 + k^2\xi^2\right)G_2 = -\delta(z - z_s), \tag{1.2.18}$$

and c_λ is a contour in the complex λ plane taken in the positive sense. Thus, to find a solution to the problem involving a source term, we need to solve for two Green's functions $G_1(r, r_s, \lambda)$ and $G_2(z, z_s, \lambda)$ and then integrate along the contours c_λ. Evaluations of such integrals are often non-trivial and need asymptotic methods such as stationary phase or steepest descent.

In what follows, we may consider one or more of the following boundary conditions.

1. free surface at $z = 0$

$$p(x, 0) = 0;$$

2. rigid, boundary condition at the seabed $z = h$

$$\frac{\partial p}{\partial n}(x, h) = 0;$$

3. reflecting type boundary at the seabed $z = h$

$$\frac{\partial p(x, h)}{\partial n} = \alpha p;$$

where α is the reflection coefficient $-1 \le \alpha \le 1$.

1.3 Two Homogeneous Layer Model

In this section, we consider the homogeneous two-layer model discussed by Boyles [4] where the surface is assumed to satisfy pressure-release boundary condition and the bottom satisfies the rigid type condition.

In this model, Boyles [4] assumed ρ_1, c_1 and ρ_2, c_2 to be the constant density and sound velocity in layers 1 and 2, respectively. The point source is assumed to lie at $r = r_s = 0$ and $z = z_s$ as depicted in the diagram below (Figure 1.3.1).

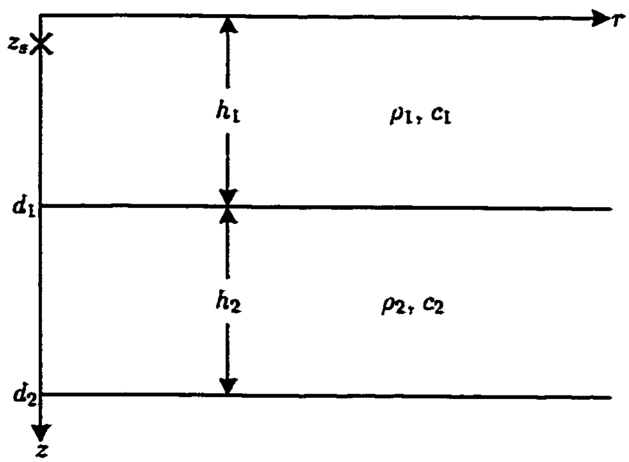

Figure 1.3.1. Coordinate geometry for Boyles' two homogeneous layered model.

The Helmholtz equation in layer 1 is

$$\frac{\partial^2 p^{(1)}}{\partial r^2} + \frac{1}{r}\frac{\partial^{(1)}}{\partial r} + \frac{\partial^2 p^{(1)}}{\partial z^2} + k_1^2 p^{(1)} = -\frac{1}{2\pi r}\delta(r)\delta(z - z_s), \tag{1.3.1}$$

where

$$k_1 = \frac{\omega}{c_1}. \tag{1.3.2}$$

The range-dependent Green's function $G_r^{(1)}(r,\lambda)$ and the depth-dependent Green's function $G_z^{(1)}(z,z_s,\lambda)$ satisfy the following differential equations:

$$\frac{d^2G_r^{(1)}}{dr^2}+\frac{1}{r}\frac{dG_r^{(1)}}{dr}+\lambda G_r^{(1)}=-\frac{1}{2\pi r}\delta(r) \tag{1.3.3}$$

$$\frac{d^2G_z^{(1)}}{dz^2}+(k_1^2-\lambda)G_z^{(1)}=-\delta(z-z_s) \tag{1.3.4}$$

respectively.

The Helmholtz equation in layer 2 is

$$\frac{\partial^2p^{(2)}}{\partial r^2}+\frac{1}{r}\frac{\partial^{(2)}}{\partial r}+\frac{\partial^2p^{(2)}}{\partial z^2}+k_2^2p^{(2)}=0, \tag{1.3.5}$$

where

$$k_2=\frac{\omega}{c_2}. \tag{1.3.6}$$

The range-dependent Green's function $G_r^{(2)}(r,\lambda)$ and the depth-dependent Green's function $G_z^{(2)}(z,z_s,\lambda)$ satisfy the following differential equations

$$\frac{d^2G_z^{(2)}}{dr^2}+\frac{1}{r}\frac{Dg_r^{(2)}}{dr}+\lambda G_r^{(2)}=-\frac{1}{2\pi r}\delta(r) \tag{1.3.7}$$

$$\frac{d^2G_z^{(2)}}{dz^2}+(k_2^2-\lambda)G_z^{(2)}=0, \tag{1.3.8}$$

respectively.

The boundary conditions are

1. the pressure-release free surface at $z=0$ gives

$$p^{(1)}(x,z)=0, \tag{1.3.9}$$

2. continuity of acoustic pressure at the interface $z=d_1$ gives

$$p^{(1)}(x,d_1)=p^{(2)}(x,d_1), \tag{1.3.10}$$

3. continuity of the gradient of acoustic pressure at the interface gives

$$\frac{1}{\rho_1}\frac{\partial^{(1)}(x,d_1)}{\partial z} = \frac{1}{\rho_2}\frac{\partial^{(2)}(x,d_1)}{\partial z}, \tag{1.3.11}$$

4. rigid bottom at $z = d_2$ gives

$$\frac{\partial(x,d_2)}{\partial z} = 0. \tag{1.3.12}$$

$G_r^{(1)}$ and $G_r^{(2)}$ are assumed to satisfy the Sommerfield radiation and the discontinuity conditions

$$1)\quad \lim_{r\to\infty}\sqrt{r}\left[\frac{\partial G_r^{(i)}}{\partial r} - i\sqrt{\lambda}G_r^{(i)}\right] = 0 \tag{1.3.13}$$

$$2)\quad \lim_{\epsilon\to 0}\left[\epsilon\frac{dG_r^{(i)}}{dr}(\epsilon)\right] = -\frac{1}{2\pi}, \tag{1.3.14}$$

where $i = 1,2$.

The solution of equations (1.3.3) and (1.3.7) subject to the boundary conditions is

$$G_r^{(i)}(r,\lambda) = \frac{i}{4}H_0^{(1)}(\sqrt{\lambda}r) \quad i = 1,2. \tag{1.3.15}$$

$H_0^{(1)}$ denotes the Hankel function of zero order of the first kind.

$G_z^{(i)}$ are assumed to satisfy the boundary conditions stated in equations (1.3.9) to (1.3.12).

The general solutions of equations (1.3.4) and (1.3.8) are

$$G_z^{(1)} = A\sin\gamma_1 z + B\cos\gamma_1 z \quad 0 \le z \le z_s \tag{1.3.16}$$

$$= C\sin\gamma_1 z + D\cos\gamma_1 z \quad z_s \le z \le d_1 \tag{1.3.17}$$

$$G_z^{(2)} = E\sin\gamma_2 z + F\cos\gamma_2 z \quad d_1 \le z \le d_2 \tag{1.3.18}$$

where

$$\gamma_1^2 = k_1^2 - \lambda \tag{1.3.19}$$

$$\gamma_2^2 = k_2^2 - \lambda. \tag{1.3.20}$$

Applying the boundary conditions stated in equations (1.3.9) to (1.3.12) and solving for the constants A, B, C, D, E and F using Cramer's rule gives

$$A = \frac{1}{\gamma_1 M}\Big\{\rho_1\gamma_2 \sin\gamma_2 h_2 \sin\gamma_1(h_1 - z_s)) -$$

$$\gamma_1\rho_2\cos\gamma_2 h_2 \cos\gamma_1(d - z_s)\Big\} \tag{1.3.21}$$

$$B = 0 \tag{1.3.22}$$

$$C = -\frac{\sin\gamma_1 z_s}{\gamma_1 M}\Big\{\rho_1\gamma_2\cos\gamma_1 h_1 \sin\gamma_2 h_2 +$$

$$\gamma_1\rho_2 \sin\gamma_1 h_1 \cos\gamma_2 h_2\Big\} \tag{1.3.23}$$

$$D = \frac{\sin\gamma_1 z_s}{\gamma_1} \tag{1.3.24}$$

$$E = -\frac{1}{M}(\rho_2 \sin\gamma_2 d_2 \sin\gamma_1 z_s) \tag{1.3.25}$$

$$F = -\frac{1}{M}(\rho_2 \cos\gamma_2 d_2 \sin\gamma_1 z_s) \tag{1.3.26}$$

where $h_1 = d_1$, $h_2 = d_2 - d_1$ and M is the determinant of the coefficients given by

$$M = \rho_1\gamma_2 \sin\gamma_1 h_1 \sin\gamma_2 h_2 - \rho_2\gamma_1 \cos\gamma_1 h_1 \cos\tan\gamma_2 h_2. \tag{1.3.27}$$

Using the expressions for A, B, C, D, E and F, the depth-dependent Green's function given by equations (1.3.16) to (1.3.18) becomes

$$G_z^{(1)} = \Big\{\rho_1\gamma_2 \sin\gamma_2 h_2 \sin\gamma_1(h_1 - z_s) - \gamma_1\rho_2 \cos\gamma_2 h_2 \cos\gamma_1(h_1 - z_s)\Big\}\frac{\sin\gamma_1 z}{\gamma_1 M} \tag{1.3.28}$$

for $0 \le z \le z_s$, and

$$G_z^{(1)} = \Big\{\rho_1\gamma_2 \sin\gamma_2 h_2 \sin\gamma_1(h_1 - z) - \rho_2\gamma_1 \cos\gamma_2 h_2 \cos\gamma_1(h_1 - z_s)\Big\}\frac{\sin\gamma_1 z_s}{\gamma_1 M} \tag{1.3.29}$$

for $z_s \le z \le d_1$, and

$$G_z^{(2)} = -\frac{1}{M}\rho_2 \sin\gamma_1 z_s \cos\gamma_2(d_2 - z) \quad \text{for} \quad d_1 \le z \le d_2. \tag{1.3.30}$$

The general solution is given by

$$p^{(1)}(r, r_s, z, z_s) = -\frac{1}{2\pi i}\int_{c\lambda} G_r^{(1)}(r, r_s, \lambda) G_z^{(1)}(z, z_s, \lambda) d\lambda. \tag{1.3.31}$$

Using equations (1.3.11) and (1.3.24) and using the residue theorem

$$p^{(1)} = \frac{-i\rho_2\rho_1}{4\rho_1}\sum_{n=1}^{\infty}\frac{\cos\gamma_{2n}h_2 \sin\gamma_{1n}z_s}{\sin\gamma_{1n}h_1[\partial M/\partial\lambda]_{\lambda=\lambda_n}} \sin\gamma_{1n}z H_0^{(1)}(\sqrt{\lambda_n}r), \tag{1.3.32}$$

where

$$\left[\frac{\partial M}{\partial \lambda}\right]_{\lambda=\lambda_n}\left(\frac{-\sin\gamma_{1n}h_1}{\rho_1\rho_2\cos\gamma_{2n}h_2}\right) \equiv \frac{1}{N_n^2} \tag{1.3.33}$$

and

$$\frac{1}{N_n^2} = \frac{h_1}{2\rho_1} - \frac{\sin\gamma_{1n}h_1\cos\gamma_{1n}h_1}{2\gamma_{1n}\rho_1} + \frac{1}{2\rho_2\gamma_{2n}}\sin^2\gamma_{1n}h_1\frac{\sin\gamma_{2n}h_2}{\cos\gamma_{2n}h_2} + \frac{h_2}{2\rho_2}\frac{\sin^2\gamma_{1n}h_1}{\cos^2\gamma_{2n}h_2}. \tag{1.3.34}$$

γ_{1n} satisfies the equation

$$\rho_1\gamma_2 \tan \gamma_2 h_2 = \rho_2\gamma_1 \cot \gamma_1 h_1. \tag{1.3.35}$$

The acoustic pressure solution for the two homogeneous layer model is then

$$p^{(1)} = \frac{i}{4\rho_1}\sum_{i=1}^{\infty} N_n^2 \sin\gamma_{1n}z_s \sin\gamma_{1n}z H_0^{(1)}(\sqrt{\lambda}_n r) \quad 0 \leq z \leq d_1, \tag{1.3.36}$$

and

$$p^{(2)} = \frac{i}{4\rho_1}\sum_{n=1}^{\infty} \frac{N_n^2 \sin\gamma_{1n}h_1}{\cos\gamma_2 h_2} \sin\gamma_{1n}z_s \cos\gamma_{2n}(d_2 - z) H_0^{(1)}(\sqrt{\lambda}_n r) \quad d_1 \leq z \leq d_2. \tag{1.3.37}$$

In Chapter II, we shall study perturbations in the Boyles' model discussed here. We shall consider the depth-dependent properties of the ocean with a seabed that has small undulations. This would mean that the boundary condition $\frac{\partial p}{\partial z} = 0$ is replaced by $\frac{\partial p}{\partial \underline{n}} = 0$, $\underline{n}$ being the normal to the surface of the seabed. Moreover, a more general boundary condition, namely, the reflecting type condition, will be considered in the seabed.

CHAPTER II

PERTURBED LAYERED OCEAN MODEL

In this chapter, we consider perturbations in the two-layered ocean model described in Chapter I. In Section (2.1), the seabed is assumed to be a small smooth undulation which introduces a perturbed boundary condition at the bottom of the sea. In Section (2.2), the boundary conditions at the seabed are taken to be of reflecting type to allow a more realistic situation in which part of the energy is reflected back by the seabed. In Section (2.3), numerical analysis demonstrates the effect of parabolic, quadratic, and linear perturbations in the sea bottom on the acoustic pressure. In Section (2.4), in addition to perturbation in the seabed, the properties of the sea are assumed to be depth-dependent. This introduces the perturbation of the wavenumber.

2.1 A Model Consisting of Two Layers

With an irregular seabed, we consider the problem of an ocean consisting of two layers: top layer is assumed to be flat with density ρ_1 and velocity c_1 and the second layer is assumed to have a non-horizontal with density ρ_2 and velocity c_2. A point source is assumed to be situated at $x = x_s = 0$ and $z = z_s$. We will assume that the source lies in layer 1, but the problem with the source in the second layer can be handled similarly. Figure (2.1.1) shows the geometry of the problem.

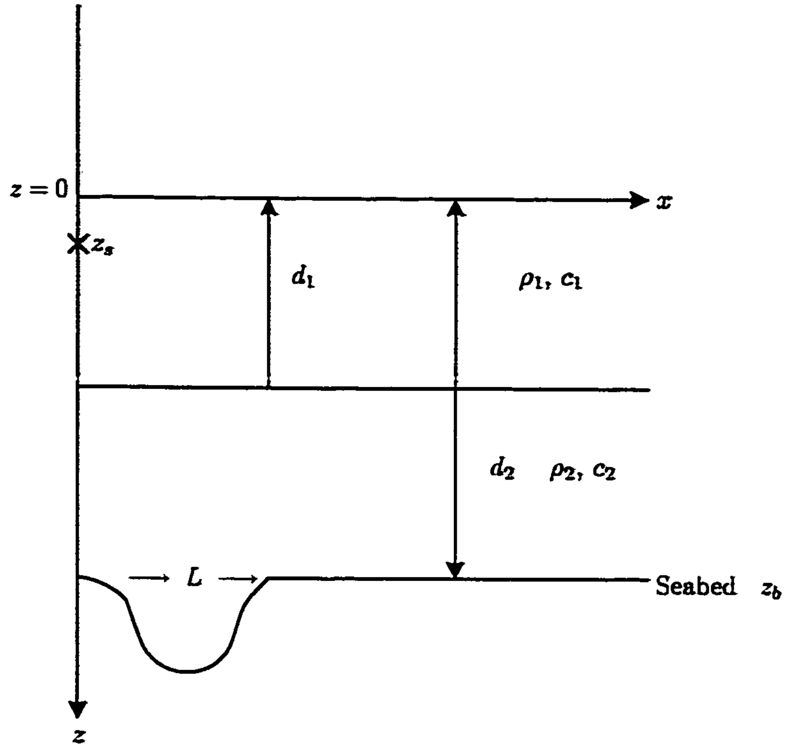

Figure 2.1.1. Coordinate geometry for perturbed two-layered model.

The Boundary Value Problem

The top layer has depth $z = d_1$ and the seabed has depth

$$z_b = \begin{cases} d_2 + \epsilon h(x) & 0 \leq x \leq L \\ d_2 & \text{otherwise} \end{cases} \quad \text{where } 0 < \epsilon \leq 1 \ . \tag{2.1.1}$$

The acoustic pressure $p^{(i)}$, density $\rho^{(i)}$, velocity c_i and the wavenumber k_i refer to the quantities in the i-th layer, $i = 1, 2$. The Helmholtz equation in layer 1 is

$$\frac{\partial^2 p^{(1)}}{\partial x^2} + \frac{\partial^2 p^{(1)}}{\partial z^2} + k_1^2(z) p^{(1)} = -2\pi\delta(x)\delta(z - z_s), \tag{2.1.2}$$

where $k_1 = \frac{\omega}{c_1}$ is wavenumber, ω is the angular frequency and z_s is the source depth.

The Helmholtz equation in layer 2 is

$$\frac{\partial^2 p^{(2)}}{\partial x^2} + \frac{\partial^2 p^{(2)}}{\partial z^2} + k_2^2(z) p^{(2)} = 0, \tag{2.1.3}$$

where $k_2 = \frac{\omega}{c_2}$ is the wavenumber.

In case the seabed is assumed to be rigid, we have the following boundary conditions:

z1) Free surface at $z = 0$

$$p^{(1)}(x, 0) = 0. \tag{2.1.4}$$

z2) Continuity of acoustic pressure at the interface $z = d_1$ gives

$$p^{(1)}(x, d_1) = p^{(2)}(x, d_1). \tag{2.1.5}$$

z3) Continuity of the gradient of acoustic pressure at the interface gives

$$\frac{1}{\rho_1}\frac{\partial p^{(1)}(x, d_1)}{\partial z} = \frac{1}{\rho_2}\frac{\partial p^{(2)}(x, d_1)}{\partial z}. \tag{2.1.6}$$

z4) Rigid bottom at seabed $z = z_b$ gives

$$\frac{\partial p^{(2)}}{\partial n}(x, z_b) = 0. \tag{2.1.7}$$

or

$$\frac{\partial p^{(2)}(x, z_b)}{\partial z} - \epsilon\frac{dh}{dx}\frac{\partial p^{(2)}(x, z_b)}{\partial x} = 0.$$

Solution of the Problem

Let us introduce the one-dimensional Fourier transform pair

$$\hat{f}(\xi, z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x, z)e^{i\xi x}dx,$$

$$f(x, z) = \int_{-\infty}^{\infty} \hat{f}(\xi, z)e^{-i\xi x}d\xi. \tag{2.1.8}$$

Taking the Fourier transform of (2.1.2) and (2.1.3) over x, the equations of motion become

$$\frac{d^2\hat{p}^{(1)}(\xi,z)}{dz^2}+\alpha_1^2\hat{p}^{(1)}(\xi,z)=-\delta(z-z_s), \tag{2.1.9}$$

and

$$\frac{d^2\hat{p}^{(2)}(\xi,z)}{dz^2}+\alpha_1^2\hat{p}^{(2)}(\xi,z)=0, \tag{2.1.10}$$

where $\alpha_i^2=k_i^2-\xi^2,\quad i=1,2$ and the boundary conditions z1) − z4), in the Fourier transform domain become

1F) Free surface at $z=0$

$$\hat{p}^{(1)}(\xi,0)=0. \tag{2.1.11}$$

2F) Continuity of acoustic pressure at the interface $z=d_1$

$$\hat{p}^{(1)}(\xi,d_1)=\hat{p}^{(2)}(\xi,d_1). \tag{2.1.12}$$

3F) Continuity of the derivative of acoustic pressure at the interface

$$\frac{1}{\rho_1}\frac{d\hat{p}^{(1)}(\xi,d_1)}{dz}=\frac{1}{\rho_2}\frac{d\hat{p}^{(2)}(\xi,d_1)}{dz}. \tag{2.1.13}$$

4F) Rigid bottom at seabed

$$\frac{d\hat{p}^{(2)}(\xi,z)}{dz}-\epsilon\left[i\xi H(\xi)*i\xi\hat{p}^{(2)}(\xi,z)\right]=0. \tag{2.1.14}$$

where $H(\xi)$ is the Fourier transform of $h(x)$ and $*$ denotes the convolution operation defined as

$$g(t)=f_1(\xi)*f_2(\xi)=\int_{-\infty}^{\infty}f_1(\xi)f_2(t-\xi)d\xi=\int_{-\infty}^{\infty}f_1(t-\xi)f_2(\xi)d\xi. \tag{2.1.15}$$

We shall use the Green's function to solve our equations of motion in the Fourier transform (2.1.9) and (2.1.10). For more discussion about Green's function, see Appendix A. Equations (2.1.9) and (2.1.10) are satisfied by Green's functions

$$\frac{d^2G^{(1)}(\xi, z_s, z)}{dz^2} + \alpha_1^2 G^{(1)}(\xi, z_s, z) = -\delta(z - z_s), \tag{2.1.16}$$

and

$$\frac{d^2G^{(2)}(\xi, z_s, z)}{dz^2} + \alpha_2^2 G^{(2)}(\xi, z_s, z) = 0, \tag{2.1.17}$$

together with the following boundary conditions

1G) Free surface

$$G^{(1)}(\xi, z_s, 0) = 0. \tag{2.1.18}$$

2G) Continuity of the Green's function at the source location gives

$$G^{(1)}(\xi, z_s, z_s^+) = G^{(1)}(\xi, z_s, z_s^-) \tag{2.1.19}$$

where z_s^- and z_s^+ are above and below the source depth, respectively.

3G) Discontinuity jump in the derivative of Green's function at the source depth gives

$$\frac{dG^{(1)}(\xi, z_s, z_s^+)}{dz} - \frac{dG^{(1)}(\xi, z_s, z_s^-)}{dz} = -1. \tag{2.1.20}$$

4G) Continuity of Green's function at the interface $z = d_1$

$$G^{(1)}(\xi, z_s, d_1) = G^{(2)}(\xi, z_s, d_1). \tag{2.1.21}$$

5G) Continuity of the derivative of Green's function at the interface $z = d_1$

$$\frac{1}{\rho_1}\frac{dG^{(1)}}{dz}(\xi, z_s, d_1) = \frac{1}{\rho_2}\frac{dG^{(2)}}{dz}(\xi, z_s, d_1). \tag{2.1.22}$$

6G) Rigid condition

$$\frac{dG^{(2)}(\xi, z_s, z_b)}{dn} = 0. \tag{2.1.23}$$

Green's function solution for the two layers is

$$G^{(1)}(\xi, z_s, z) = \begin{cases} A\sin\alpha_1 z + B\cos\alpha_1 z & 0 \le z \le z_s \\ C\sin\alpha_1 z + D\cos\alpha_1 z & z_s \le z \le d_1 \end{cases} \tag{2.1.24}$$

for layer 1, and

$$G^2(\xi, z_s, z) = E\sin\alpha_2 z + F\cos\alpha_2 z \quad d_1 \le z \le z_b. \tag{2.1.25}$$

From 1G), we get

$$G^{(1)}(\xi, 0, z_s) = 0 \quad \text{which implies} \quad B = 0. \tag{2.1.26}$$

From 2G), we get

$$A\sin\alpha_1 z_s = C\sin\alpha_1 z_s + D\cos\alpha_1 z_s. \tag{2.1.27}$$

From 3G), we get

$$\alpha_1 C\cos\alpha_1 z_s - \alpha_1 D\sin\alpha_1 z_s - \alpha_1 A\cos\alpha_1 z_s = -1. \tag{2.1.28}$$

From 4G), we obtain

$$C\sin\alpha_1 d_1 + D\cos\alpha_1 d_1 - E\sin\alpha_2 d_1 - F\cos\alpha_2 d_1 = 0. \tag{2.1.29}$$

From 5G), we obtain

$$\frac{1}{\rho_1}\left[\alpha_1 C\cos\alpha_1 d_1 - \alpha_1 D\sin\alpha_1 d_1\right] = \frac{1}{\rho_2}\left[\alpha_2 E\cos\alpha_2 d_2 - \alpha_2 F\sin\alpha_2 d_2\right]. \qquad (2.1.30)$$

From 6G), we get

$$E\left[\alpha_2\cos\alpha_2 z_b - \epsilon R_1(\xi, z_b)\right] - F\left[\alpha_2\sin\alpha_2 z_b + \epsilon R_2(\xi, z_b)\right] = 0, \qquad (2.1.31)$$

where

$$R_1(\xi, z_b) = -\Big(i\xi H(\xi)\Big) * \Big(i\xi\sin\alpha_2 z_b\Big) \qquad (2.1.32)$$

$$R_2(\xi, z_b) = -\Big(i\xi H(\xi)\Big) * \Big(i\xi\cos\alpha_2 z_b\Big). \qquad (2.1.33)$$

If we arrange the equations mentioned above to solve for A, C, D, E, and F, we obtain

1) $A\sin\alpha_1 z_s - C\sin\alpha_1 z_s - D\cos\alpha_1 z_s = 0.$

2) $-\alpha_1 A\cos\alpha_1 z_s\alpha_1 + C\cos\alpha_1 z_s - \alpha_1 D\sin\alpha_1 z_s = 1.$

3) $\frac{1}{\rho_1}\left[\alpha_1 C\cos\alpha_1 d_1 - \alpha_1 D\sin\alpha_1 d_1\right] - \frac{1}{\rho_2}\left[\alpha_2 E\cos\alpha_2 d_2 - \alpha_2 F\sin\alpha_2 d_2\right] = 0.$

4) $C\sin\alpha_1 d_1 + D\cos\alpha_1 d_1 - E\sin\alpha_2 d_1 - F\cos\alpha_2 d_1 = 0.$

5) $EM_1 - FM_2 = 0$ where

$$M_1 = \alpha_2\cos\alpha_2 z_b - \epsilon R_1(\xi, z_b) \qquad (2.1.34)$$

and

$$M_2 = \alpha_2\sin\alpha_2 z_b + \epsilon R_2(\xi, z_b). \qquad (2.1.35)$$

By Cramer's rule, in order for these equations to have a nontrivial solution, the determinant of the coefficients of the unknown quantities A, C, D, E, and

F must vanish. Denoting the determinant by Det, we have

$$\text{Det} = \left(\frac{\alpha_1^2}{\rho_1}\right) M_2 \cos\alpha_1 d_1 \sin\alpha_2 d_1 + \left(\frac{\alpha_1^2}{\rho_1}\right) M_1 \cos\alpha_1 d_1 \cos\alpha_2 d_1$$
$$-\left(\frac{\alpha_1\alpha_2}{\rho_2}\right) M_2 \cos\alpha_2 d_1 \sin\alpha_1 d_1 + \left(\frac{\alpha_1\alpha_2}{\rho_2}\right) M_1 \sin\alpha_2 d_1 \sin\alpha_1 d_1. \quad (2.1.36)$$

This is the characteristic equation and the zeros $\xi^2 = \xi_n^2$, are the eigenvalues of the problem. These eigenvalues are real according to theorem cited below.

Solving for A, we get

$$\begin{aligned} A \;=\; & \left(\frac{1}{\text{Det}}\right) \left[\left(\frac{\alpha_1}{\rho_1}\right) M_2 \sin\alpha_2 d_1 \cos\alpha_1 (z_s - d_1)\right. \\ & + \left(\frac{\alpha_1}{\rho_1}\right) M_1 \cos\alpha_2 d_1 \cos\alpha_1 (z_s - d_1) \\ & + \left(\frac{\alpha_2}{\rho_2}\right) M_2 \cos\alpha_2 d_1 \sin\alpha_1 (z_s - d_1) \\ & + \left(\frac{\alpha_2}{\rho_2}\right) M_1 \sin\alpha_2 d_1 \sin\alpha_1 (d_1 - z_s). \end{aligned} \quad (2.1.37)$$

Solving for C, we get

$$C = \left(\frac{1}{\text{Det}}\right)\left[-\left(\frac{\alpha_1}{\rho_1}\right) \sin\alpha_1 z_s \sin\alpha_1 d_1 (M_2 \sin\alpha_2 d_1 + M_1 \cos\alpha_2 d_1)\right]. \quad (2.1.38)$$

Solving for D, we get

$$\begin{aligned} D \;=\; & \left(\frac{1}{\text{Det}}\right)\left[\left(\frac{\alpha_1}{\rho_1}\right) \sin\alpha_1 z_s \cos\alpha_1 d_1 (M_2 \sin\alpha_2 d_1 + M_1 \cos\alpha_2 d_1)\right. \\ & \left. - \left(\frac{\alpha^2}{\rho_2}\right) M_2 \cos\alpha_2 d_1 \sin\alpha_1 d_1 + \left(\frac{\alpha_2}{\rho_2}\right) M_1 \sin\alpha_2 d_1 \sin\alpha_1 d_1\right]. \end{aligned} \quad (2.1.39)$$

Solving for E, we get

$$E = \left(-\frac{1}{\text{Det}}\right)\left(\frac{\alpha_1}{\rho_1}\right) M_2 \sin\alpha z_s. \quad (2.1.40)$$

Solving for F, we get

$$F = \left(\frac{1}{\text{Det}}\right)\left(\frac{\alpha_1}{\rho_1}\right) M_1 \sin\alpha_1 z_s. \tag{2.1.41}$$

Substituting the values for A, C, D, E and F in equations (2.1.24) and (2.1.25) and taking the inverse Fourier transform, the acoustic pressure for layer 1 is

$$\begin{aligned} p^{(1)}(x,z) &= \int_{-\infty}^{\infty}\left(\frac{\sin\alpha_1 z}{\text{Det}}\right)\left[\left(\frac{\alpha_1}{\rho_1}\right) M_2 \sin\alpha_2 d_1 \cos\alpha_1(z_s - d_1)\right. \\ &+ \left(\frac{\alpha_1}{\rho_1}\right) M_1 \cos\alpha_2 d_1 \cos\alpha_1(z_s - d_1) + \left(\frac{\alpha_2}{\rho_2}\right) M_2 \cos\alpha_2 d_1 \sin\alpha_1(z_s - d_1) \\ &+ \left.\left(\frac{\alpha_2}{\rho_2}\right) M_1 \sin\alpha_2 d_1 \sin\alpha_1(d_1 - z_s)\right] e^{-i\xi x} d\xi \\ &\qquad \text{for } \; 0 \le z \le z_s. \end{aligned} \tag{2.1.42}$$

Using Cauchy's and residue theorems, we have for $0 \le z < z_s$

$$\begin{aligned} p^{(1)}(x,z) &= 2\pi i \sum_{n=1}^{\infty} \left.\frac{\sin\alpha_{1,n} z}{\frac{\partial \text{Det}}{\partial \xi^2}}\right|_{\xi^2=\xi_n^2} \\ &\times \left[\left(\frac{\alpha_{1,r}}{\rho_1}\right) M_{2,n} \sin\alpha_{2,n} d_1 \cos\alpha_{1,n}(z_s - d_1)\right. \\ &+ \left(\frac{\alpha_{1,n}}{\rho_1}\right) M_{1,n} \cos\alpha_{2,n} d_1 \cos\alpha_{1,n}(z_s - d_1) \\ &+ \left(\frac{\alpha_{2,n}}{\rho_2}\right) M_{2,n} \cos\alpha_{2,n} d_1 \sin\alpha_{1,n}(z_s - d_1) \\ &+ \left.\left(\frac{\alpha_{2,n}}{\rho_2}\right) M_{1,n} \sin\alpha_{2,n} d_1 \sin\alpha_{1,n}(d_1 - z_s)\right] e^{-i\xi_n x}. \end{aligned} \tag{2.1.43}$$

For $z_s \le z \le d_1$, we have

$$
\begin{aligned}
p^{(2)}(x,z) &= 2\pi i \sum_{n=1}^{\infty} \frac{1}{\frac{\partial \mathrm{Det}}{\partial \xi^2}}\Bigg|_{\xi^2=\xi_n^2} \\
&\times \left[-\left(\frac{\alpha_1}{\rho_1}\right) \sin\alpha_{1,n} z \sin\alpha_{1,n} z_s \sin\alpha_{1,n} d_1 \right. \\
&\times \left(M_{2,n}\sin\alpha_{2,n}d_1 + M_{1,n}\cos\alpha_{2,n}d_1\right) \\
&+ \left\{ \left(\frac{\alpha_{1,n}}{\rho_1}\right) \sin\alpha_{1,n} z_s \cos\alpha_{1,n} d_1 \left(M_{2,n}\sin\alpha_{2,n}d_1 + M_{1,n}\cos\alpha_{2,n}d_1\right) \right. \\
&- \left(\frac{\alpha_{2,n}}{\rho_2}\right) M_{2,n}\cos\alpha_{2,n}d_1 \sin\alpha_{1,n}d_1 \\
&+ \left. \left. \left(\frac{\alpha_{2,n}}{\rho_2}\right) \left(M_{1,n}\sin\alpha_{2,n}d_1 \sin\alpha_{1,n}d_1\right) \right\} \cos\alpha_{1,n} z \right] e^{-i\xi_n x}. \qquad (2.1.44)
\end{aligned}
$$

For $d_1 \le z \le z_b$, we have

$$
\begin{aligned}
p^{(3)}(x,z) &= 2\pi i \sum_{n=1}^{\infty} \frac{1}{\frac{\partial \mathrm{Det}}{\partial \xi^2}}\Bigg|_{\xi^2=\xi_n^2} \left[-\left(\frac{\alpha_{1,n}}{\rho_1}\right) M_{2,n}\sin\alpha_{2,n} z \sin\alpha_{1,n} z_s \right. \\
&+ \left. M_{1,n}\cos\alpha_{2,n} z \sin\alpha_{1,n} z_s \right] e^{-i\xi_n x}, \qquad (2.1.45)
\end{aligned}
$$

where $\xi^2 = \xi_n^2$ are the zeros of the determinant. Before we determine $\frac{\partial \mathrm{Det}}{\partial \xi^2}$, we give a theorem that studies the nature of zeros for a Sturm-Liouville system.

Theorem (2.1) (Boyles, [4]). *Consider the Sturm-Liouville system*

$$
\begin{aligned}
&\frac{d}{dz}\left[r\frac{dG}{dz}\right] - (q - \lambda\omega)G = 0 \\
&\gamma' G(a) - \frac{\gamma dG(a)}{dz} = 0 \\
&\beta' \frac{dG}{dz}(b) + \beta G(b) = 0,
\end{aligned}
$$

where $r, q,$ and ω are assumed to be real continuous functions with r differentiable when $a \le z \le b$; are independent of λ and are such that $r > 0$, $\omega > 0$. The *coefficients $\gamma, \gamma', \beta, \beta'$ are also independent of λ. Then there exists an infinite set of real numbers $\lambda_1, \lambda_2, \ldots,$ that have no limit except $\lambda = +\infty$. If the corresponding eigenfunctions are $G_1, G_2, \ldots,$ then G_m has exactly $m-1$ zeros in the interval $a < z < b$.*

Now we determine $\dfrac{\partial \text{Det}}{\partial \xi^2}$

$$\begin{aligned}
\frac{\partial \text{Det}}{\partial \xi^2} &= (M_2 \sin \alpha_2 d_1 + M_1 \cos \alpha_2 d_1) \left\{ \left(\frac{-1}{\rho_1}\right) \cos \alpha_1 d_1 \right. \\
&+ \left. \left(\frac{\alpha_1}{2\rho_1}\right) d_1 \sin \alpha_1 d_1 \right\} + \left(\frac{\alpha_1^2}{\rho_1}\right) \cos \alpha_1 d_1 \left\{ \left(\frac{\partial M_2}{\partial \xi^2}\right) \sin \alpha_2 d_1 \right. \\
&- M_2 \left(\frac{d_1}{2\alpha_2}\right) (\cos \alpha_2 d_1) + \left(\frac{\partial M_1}{\partial \xi^2}\right) \cos \alpha_2 d_1 \; n + M_1 \left(\frac{d_1}{\alpha_2}\right) \left. (\sin \alpha_2 d_1) \right\} \\
&+ (M_1 \sin \alpha_2 d_1 - M_2 \cos \alpha_2 d_1) \left\{ - \left(\frac{\alpha_2}{2\alpha_1 \rho_2}\right) \sin \alpha_1 d_1 - \left(\frac{\alpha_1}{2\alpha_2 \rho_2}\right) \sin \alpha_1 d_1 \right. \\
&- \left. \left(\frac{\alpha_2 d_1}{2\rho_2}\right) \cos \alpha_1 d_1 \right\} + \left(\frac{\alpha_1 \alpha_2}{\rho_2}\right) \sin \alpha_1 d_1 \left\{ \left(\frac{\partial M_1}{\partial \xi^2}\right) \sin \alpha_2 d_1 \right. \\
&- M_1 \left(\frac{d_1}{2\alpha_2}\right) (\cos \alpha_2 d_1) - \left(\frac{\partial M_2}{\partial \xi^2}\right) \cos \alpha_2 d_1 - M_2 \left(\frac{d_1}{2\alpha_2}\right) \left. (\sin \alpha_2 d_1) \right\}, \quad (2.1.46)
\end{aligned}$$

where $\dfrac{\partial M_1}{\partial \xi^2}$ and $\dfrac{\partial M_2}{\partial \xi^2}$ can be found from equations (2.1.32) to (2.1.35) as follows:

$$\begin{aligned}
\frac{\partial M_1}{\partial \xi^2} &= -\left(\frac{1}{2\alpha_2}\right) \cos \alpha_2 z_b + z_b \sin \alpha_2 \frac{z_b}{2} \\
&+ \epsilon \left[(\xi H(\xi) * \left(\cos \frac{\alpha_2 z_b}{2\xi} + \left(\frac{z_b \xi^2}{2\alpha_2}\right) \sin \alpha_2 z_b \right) \right] \qquad (2.1.47)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial M_2}{\partial \xi^2} &= -\left(\frac{\xi}{\alpha_2}\right) \sin \alpha_2 z_b - z_b \xi \cos \alpha_2 z_b \\
&- \epsilon \left[(\xi H(\xi)) * \left(\cos \frac{\alpha_2 z_b}{2\xi} + \left(\frac{z_b \xi}{2\alpha_2}\right) \sin \alpha_2 z_b \right) \right]. \qquad (2.1.48)
\end{aligned}$$

In Section (2.2), we study the acoustic pressure for the model in (2.1), but the rigid bottom condition is replaced by the reflecting type condition as this gives a more realistic ocean model.

2.2 Reflecting Type Boundary Condition

In this section, we assume that the sea bottom is of reflecting type as this is a more realistic case. The two-layer model discussed in Section (2.1) is going to be studied with the rigid boundary condition replaced by the reflecting or impedance boundary condition.

To avoid undue repetitions, we do not repeat the boundary conditions (2.1.4) to (2.1.7), their corresponding Fourier transformed boundary conditions (2.1.11) to (2.1.14) and their Green's function boundary conditions (2.1.18) to (2.1.23) which are the same in this case. However, the rigid boundary condition at the seabed

$$\frac{\partial p^{(2)}(x, z_b)}{\partial n} = 0, \tag{2.2.1}$$

is replaced by the reflecting condition

$$\frac{\partial p^{(2)}(x, z_b)}{\partial n} = \alpha p^{(2)}(x, z_b), \tag{2.2.2}$$

where α is the reflection coefficient $-1 \leq \alpha \leq 1$.

Equation (2.2.2) can be rewritten as

$$\frac{\partial p^{(2)}(x, z_b)}{\partial z} - \epsilon \frac{dh}{dz} \frac{\partial p^{(2)}(x, z_b)}{\partial x} = \alpha p^{(2)}(x, z_b). \tag{2.2.3}$$

Taking the Fourier transform of equation (2.2.3) gives

$$\frac{d\hat{p}^{(2)}(\xi, z_b)}{dz} + \epsilon \left[(\xi \hat{h}(\xi)) * (\xi \hat{p}^{(2)}(\xi, z_b)) \right] = \alpha \hat{p}^{(2)}(\xi, z_b). \tag{2.2.4}$$

Equation (2.2.4) can be written as

$$E(\alpha_2 \cos \alpha_2 z_b - \alpha \sin \alpha_2 z_b) - F(\alpha_2 \sin \alpha_2 z_b + \alpha \cos \alpha_2 z_b)$$

$$= -\epsilon \left[(\xi H(\xi)) * (\xi \{E \sin \alpha_2 z_b + F \cos \alpha_2 z_b\}) \right]. \qquad (2.2.5)$$

Equation (2.2.5) can be rewritten as

$$EM_3 - FM_4 = 0, \qquad (2.2.6)$$

where

$$M_3 = \alpha_2 \cos \alpha_2 z_b - \alpha \sin \alpha_2 z_b - \epsilon R_1(\xi, z_b) \qquad (2.2.7)$$

$$M_4 = \alpha_2 \sin \alpha_2 z_b + \alpha \cos \alpha_2 z_b + \epsilon R_2(\xi, z_b), \qquad (2.2.8)$$

and $R_1(\xi, z_b)$ and $R_2(\xi, z_b)$ are as defined in equation (2.1.32) and (2.1.33), respectively. The constants A, C, D, E and F and the determinant Det are the same as determined in Section (2.1), but with the terms M_1 and M_2 replaced by M_3 and M_4, respectively as derived in equations (2.2.7) and (2.2.8).

The acoustic pressure $p^{(1)}, p^{(2)}$ and $p^{(3)}$ will be the same expressions as determined in equation (2.1.1) to (2.1.2) with the terms $\frac{\partial M_1}{\partial \xi^2}$ and $\frac{\partial M_2}{\partial \xi^2}$ replaced by $\frac{\partial M_3}{\partial \xi^2}$ and $\frac{\partial M_4}{\partial \xi^2}$, respectively, where $\frac{\partial M_3}{\partial \xi^2}$ and $\frac{\partial M_4}{\partial \xi^2}$ can be derived from equations (2.2.7), (2.1.8), (2.1.32) and (2.1.33) as follows:

$$\frac{\partial M_3}{\partial \xi^2} = -\left(\frac{1}{2\alpha_2}\right) \cos \alpha_2 z_b + \frac{z_b}{2} \sin \alpha_2 z_b + \frac{\alpha}{2\alpha_2} z_b \sin \alpha_2 z_b$$

$$+ \epsilon \left[(\xi H(\xi)) * \left(\cos \frac{\alpha_2 z_b}{2\xi} + \left(\frac{z_b \xi^2}{2\alpha_2}\right) \sin \alpha_2 z_b \right) \right], \qquad (2.2.9)$$

and

$$\frac{\partial M_4}{\partial \xi^2} = -\left(\frac{\xi}{\alpha_2}\right)\sin\alpha_2 z_b - z_b\xi\cos\alpha_2 z_b + \frac{\alpha z_b}{2\alpha_2}\sin\alpha_2 z_b$$
$$-\epsilon\left[(\xi H(\xi)) * \left(\cos\frac{\alpha_2 z_b}{2\xi} + \left(\frac{z_b\xi}{2\alpha_2}\right)\sin\alpha_2 z_b\right)\right]. \tag{2.2.10}$$

If we take the reflection coefficient to be a random and x dependent function, then $\alpha = \alpha(x,\gamma)$, where γ is a random parameter and the boundary condition (2.2.4) is replaced by

$$\frac{d\hat{p}^{(2)}}{dz} + \epsilon\left[(\xi H(\xi)) * (\xi\hat{p}^{(2)}(\xi, z_b))\right] = \hat{\alpha}(\xi,\gamma) * \hat{p}^{(2)}(\xi, z_b), \tag{2.2.11}$$

where $\hat{\alpha}$ is the Fourier transform of α.

Equation (2.2.6) can be written as

$$EM_5 - FM_6 = 0, \tag{2.2.12}$$

where

$$M_5 = \alpha_2\cos\alpha_2 z_b - \hat{\alpha} * \sin\alpha_2 z_b - \epsilon R_1(\xi, z_b) \tag{2.2.13}$$

$$M_6 = \alpha_2\sin\alpha_2 z_b + \hat{\alpha} * \cos\alpha_2 z_b + \epsilon R_2(\xi, z_b), \tag{2.2.14}$$

and $\frac{\partial M_5}{\partial \xi^2}$ and $\frac{\partial M_6}{\partial \xi^2}$ appearing in $\frac{\partial \text{Det}}{\partial \xi^2}$ can be derived similarly as

$$\frac{\partial M_5}{\partial \xi^2} = -\left(\frac{1}{2\alpha_2}\right)\cos\alpha_2 z_b + \frac{z_b}{2}\sin\alpha_2 z_b + \hat{\alpha} * \left(\frac{z_b}{2\alpha_2}\sin\alpha_2 z_b\right)$$
$$+\epsilon\left[(\xi H(\xi)) * \left(\cos\frac{\alpha_2 z_b}{2\xi} + \left(\frac{z_b\xi^2}{2\alpha_2}\right)\sin\alpha_2 z_b\right)\right], \tag{2.2.15}$$

and

$$\frac{\partial M_6}{\partial \xi^2} = -\left(\frac{\xi}{\alpha_2}\right)\sin\alpha_2 z_b - z_b\xi\cos\alpha_2 z_b + \hat{\alpha} * \left(\frac{z_b}{2\alpha_2}\right)\sin\alpha_2 z_b$$
$$-\epsilon\left[(\xi H(\xi)) * \left(\cos\frac{\alpha_2 z_b}{2\xi} + \left(\frac{z_b\xi}{2\alpha_2}\right)\sin\alpha_2 z_b\right)\right]. \tag{2.2.16}$$

If $\alpha = 0$ (rigid case), then all the equations derived in this section become the same equations as those derived in Section (2.1). In Section (2.3), we apply the acoustic pressure derived in Sections (2.1) and (2.2) for some particular smooth undulation functions. We take the perturbations as sine, linear, and quadratic functions. Also, we implement the reflecting type condition for the same model discussed in this section.

2.3 Numerical Results

In this section, we implement the acoustic pressure equations (2.1.43, 2.1.46, 2.1.47, 2.1.48) derived in Section 2.1 for a two-layered model with smooth undulations in the sea bottom. The seabed perturbation shape is taken to be sine, quadratic and linear functions as these are structurally realistic shapes of ocean layers. The perturbation functions are assumed to be smooth and differentiable possessing Fourier transform. We show the effect of these perturbations on the acoustic pressure and compare the results with the case of horizontal seabed. Also, we plot the case when the sea bottom is of reflecting type condition.

1. Parabolic Perturbation Sea Bottom

If we take the sea bottom as $z_b = d + \epsilon h(x)$, ϵ is a small parameter $(-1 \le \epsilon \le 1)$ and if

$$h(x) = \begin{cases} a \sin\left(\frac{x\pi}{L}\right) & 0 \le x \le L \\ 0 & \text{otherwise} \end{cases},$$

then the Fourier transform of $h(x)$ is

$$H(\xi) = \frac{a}{2\pi^2} L \left[\frac{\xi(e^{i\xi L}+1)}{1-\left(\frac{\xi L}{\pi}\right)^2} \right], \quad \frac{\xi L}{\pi} \neq 1. \tag{2.3.1}$$

Figure 2.3.1 shows the pressure plots for the two-layered model with and without sine undulations in the seabed with range x. We take $L = 20$ meters, $a = 25$, and $\epsilon = 1$ in Figure 2.3.1. There are obvious differences between the sound pressure for the sine perturbed and unperturbed sea bottom. The accuracy of the calculated pressure relies principally on the precision of the calculated eigenvalues.

Figure 2.3.2 shows the effect of ϵ on the pressure amplitude when the range $x = 10$ meters. It is obvious that the pressure amplitude decreases with increasing ϵ as expected.

2. Dipping Sea Bottom

In this case, we take $h(x) = ax$, $0 \leq x \leq L$, $L = 200$ meters and a is the slope for the sea bottom. The Fourier transform of $h(x)$ defined by Equation (2.1.18) is

$$H(\xi) = \frac{a}{2\pi} \left[L\frac{e^{i\xi L}}{i\xi} + \frac{e^{i\xi L}}{\xi^2} - \frac{1}{\xi^2} \right] \quad \xi \neq 0. \tag{2.3.2}$$

Figure 2.3.3 shows the pressure plots when $a = 1$ (bottom sloping at an angle of 45 degrees) and $a = 0$ (horizontal bottom).

3. Quadratic Perturbation Sea Bottom

If we have $h(x)$ defined as

$$h(x) = \begin{cases} x(L-x) & 0 \leq x \leq L \\ 0 & \text{otherwise} \end{cases},$$

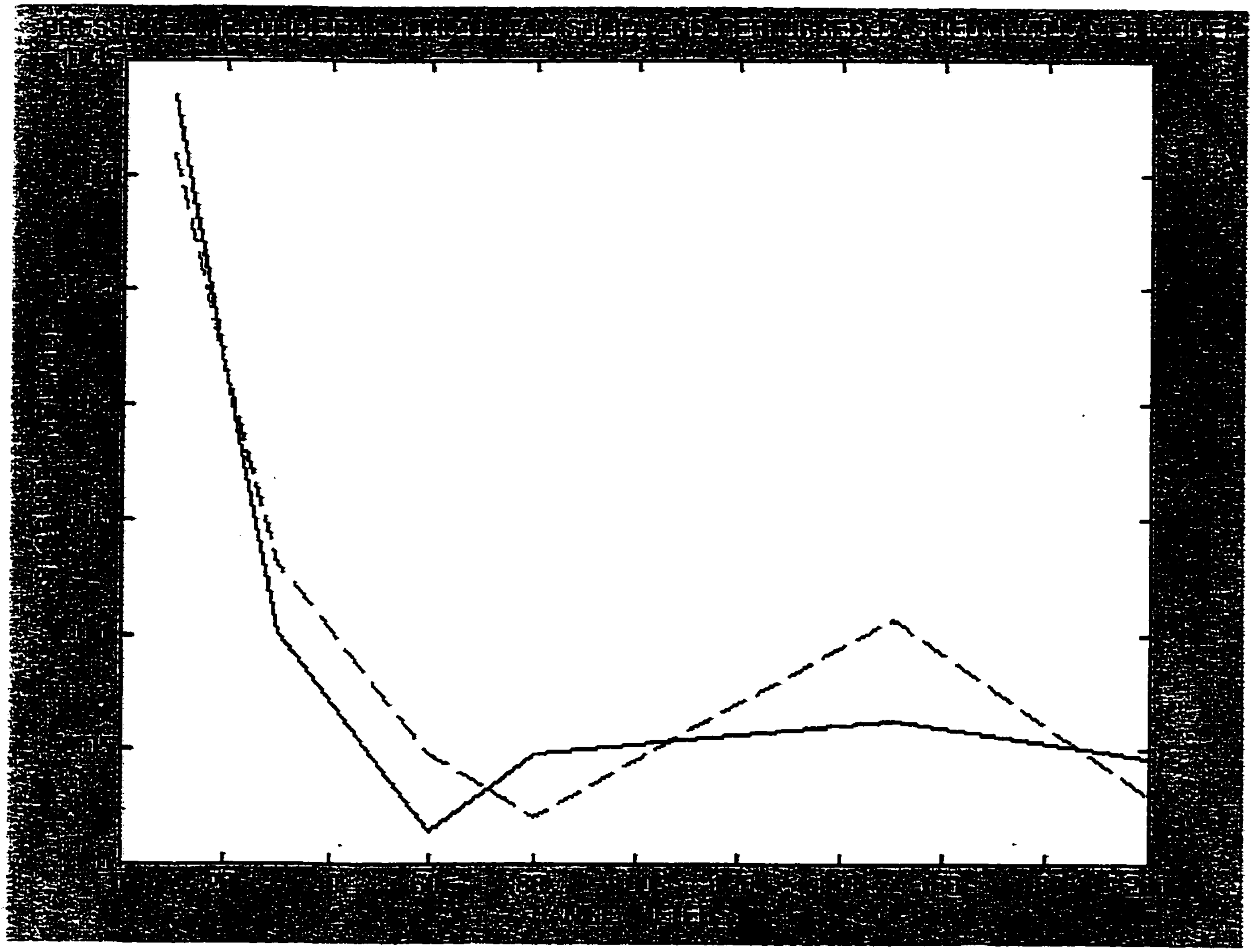

Figure 2.3.1 : Pressure amplitude for horizontal (solid) and sine perturbed (dashed) two layer model.

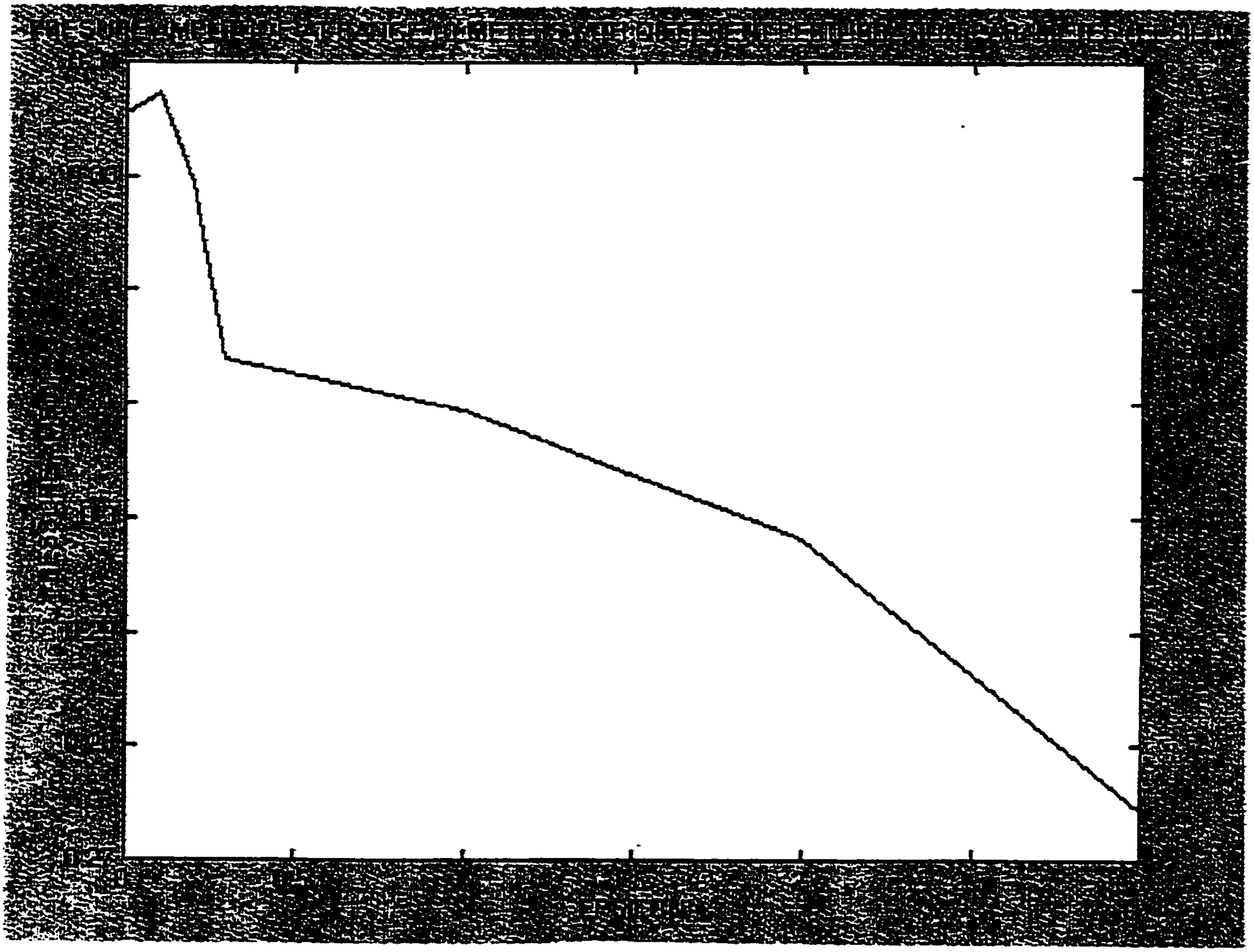

Figure 2.3.2 : Pressure amplitude for different perturbation Parameter values at range 10 meters.

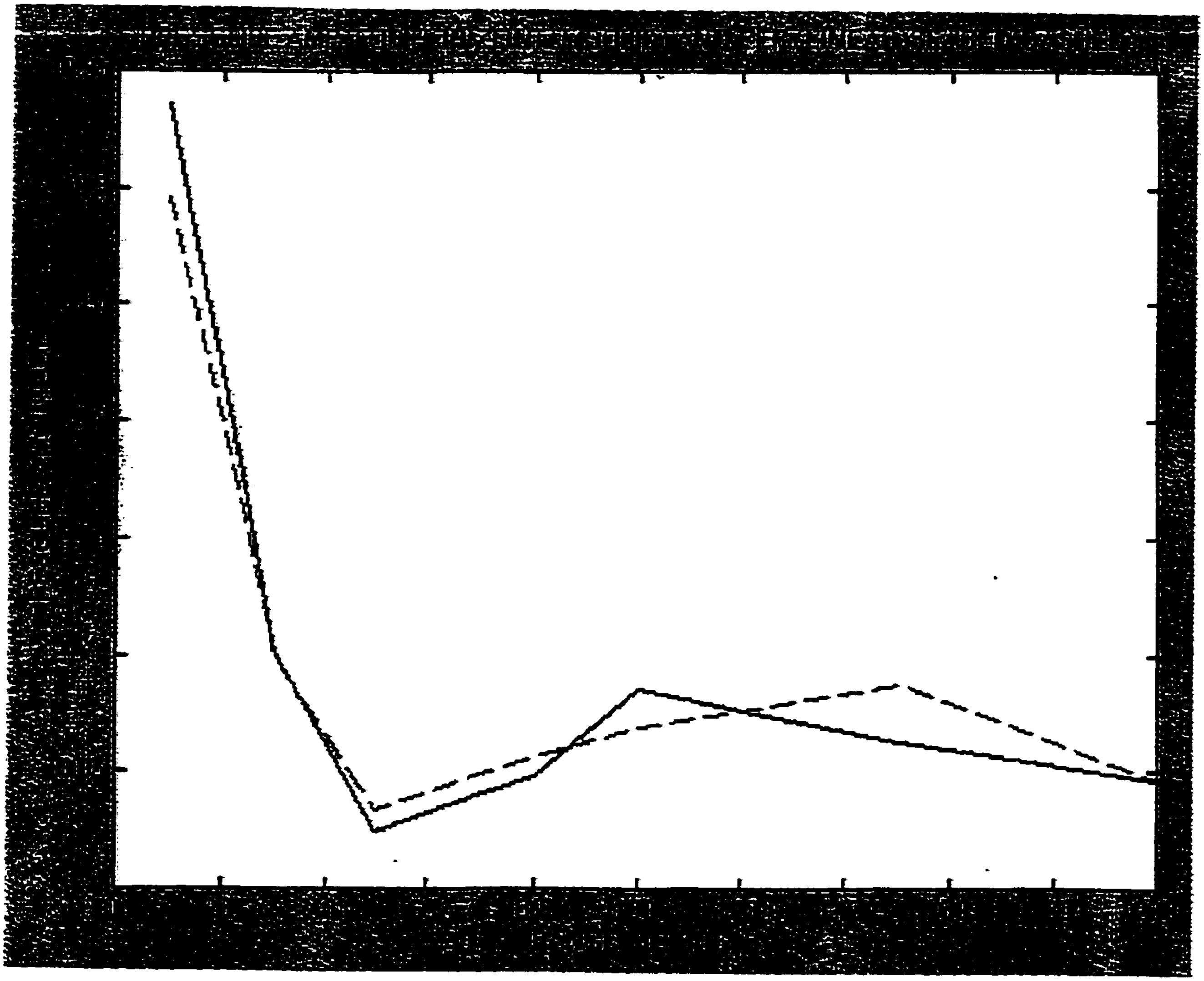

Figure 2.3.3: Pressure amplitude for horizontal (solid) and linear perturbed (dashed) two layer model.

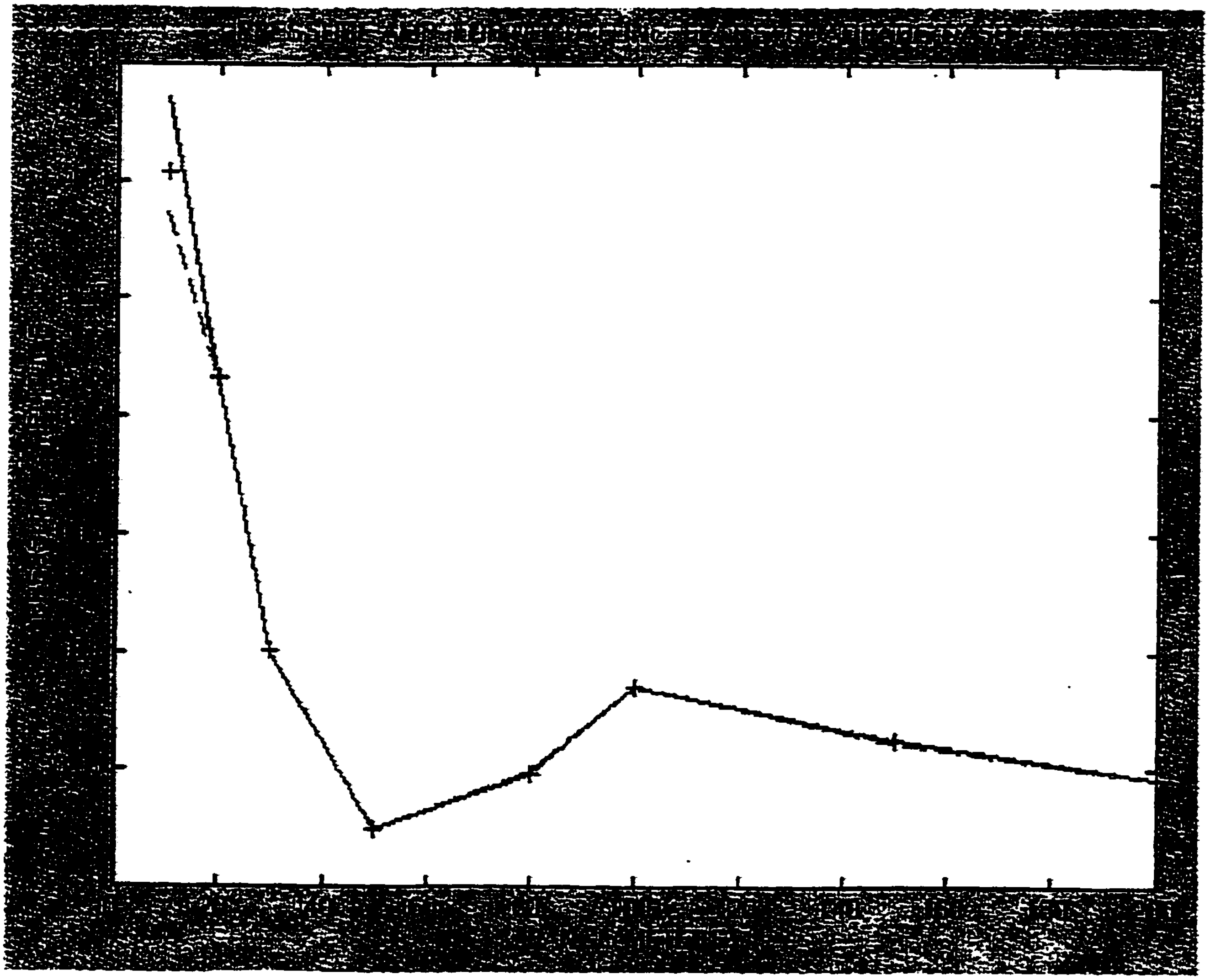

Figure 2.3.4 : Pressure amplitude for horizontal (solid) and quadratic perturbed (dashed) two layer model.

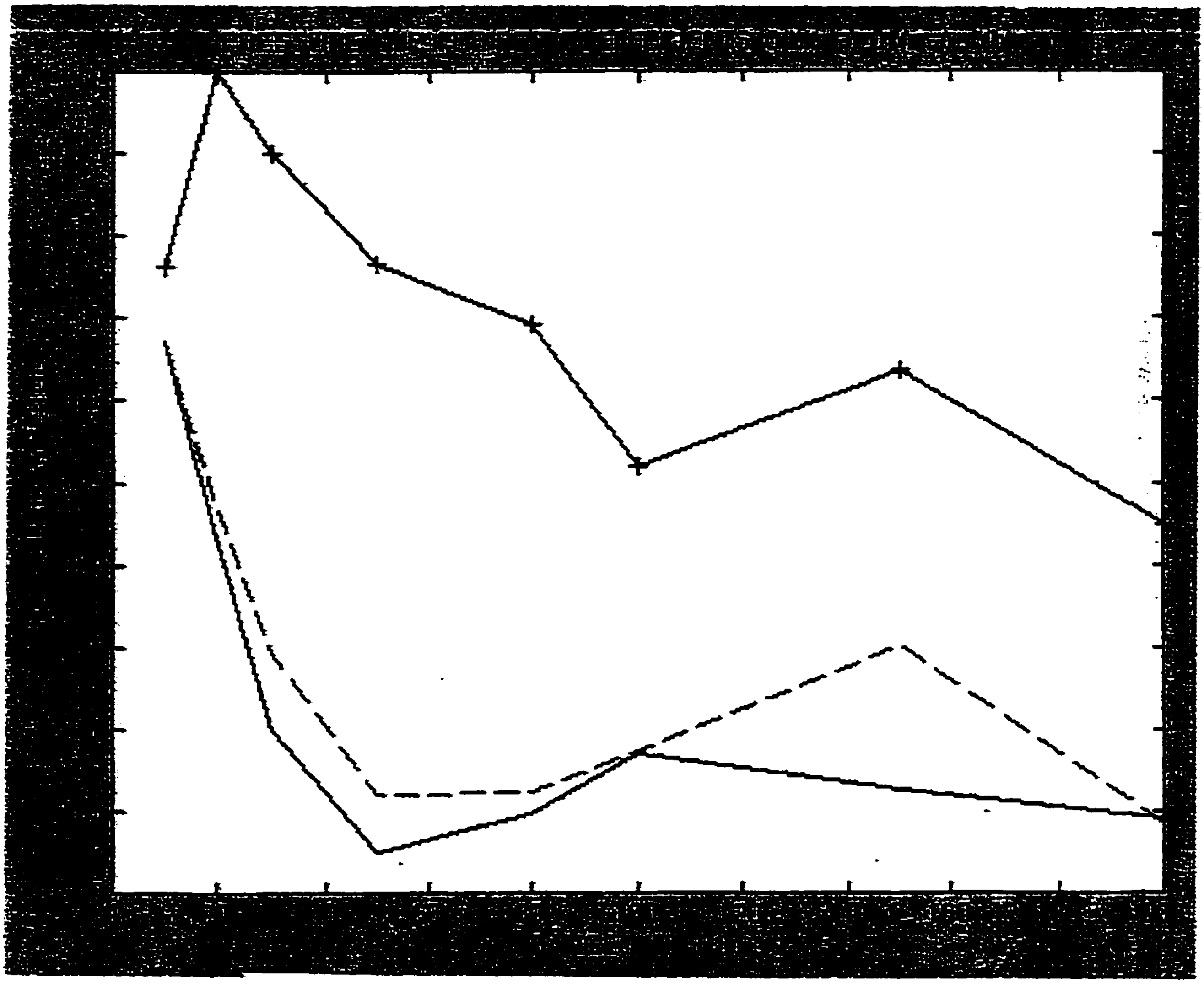

Figure 2.3.5 : Pressure amplitude for reflection coefficients 0(lower), .01(upper), and .1(uppest) for horizontal two layer model.

then $h(x)$ has Fourier transform

$$H(\xi) = \frac{1}{2\pi}\left[-L\frac{e^{i\xi L}}{\xi^2} + \frac{2e^{i\xi L}}{i\xi^3} - \frac{L}{\xi^2} - \frac{2}{i\xi^3}\right] \quad \xi \neq 0. \tag{2.3.3}$$

Figure 2.3.4 shows the plot for the acoustic pressure for the same model with the seabed having guadratic shape defined above. We take $\epsilon = 0$ (horizontal) $\epsilon = .01$ and $\epsilon = .1$. The three plots coincide with each other beyond $L = 20$ meters where the effect of undulation does not exist.

4. Reflecting Type Condition for the Seabed

If we take the sea bottom to be of reflecting type condition, then $\dfrac{\partial p}{\partial z} = \alpha p$ for a horizontal sea bottom, where α is not zero $(-1 \leq \alpha \leq 1)$. Figure 2.3.5 shows three plots for the acoustic pressure $\alpha = 0$ (rigid condition), $\alpha = .01$ and $\alpha = .1$. As α gets larger (away from zero), the pressure values get larger and diverge from the pressure for the rigid case.

2.4 Inhomogeneous Two-Layer Model with the Wave numbers Varying with Depth

In this section, we are going to use the same model as in Section (2.1), but with wavenumbers perturbed vertically. The Helmholtz equation in layer (1) is

$$\frac{\partial^2 p^{(1)}}{\partial x^2} + \frac{\partial^2 p^{(1)}}{\partial z^2} + k^2(z)p^{(1)} = -2\pi\delta(x)\delta(z - z_s), \tag{2.4.1}$$

where the wavenumber $k^2(z)$ for the first layer is

$$k^2(z) = k_1^2 + \epsilon_1\frac{\omega^2}{c_1^2}\,z, \; -1 < \epsilon_1 \leq 1 \text{ and } k_1 = \frac{\omega}{c_1}. \tag{2.4.2}$$

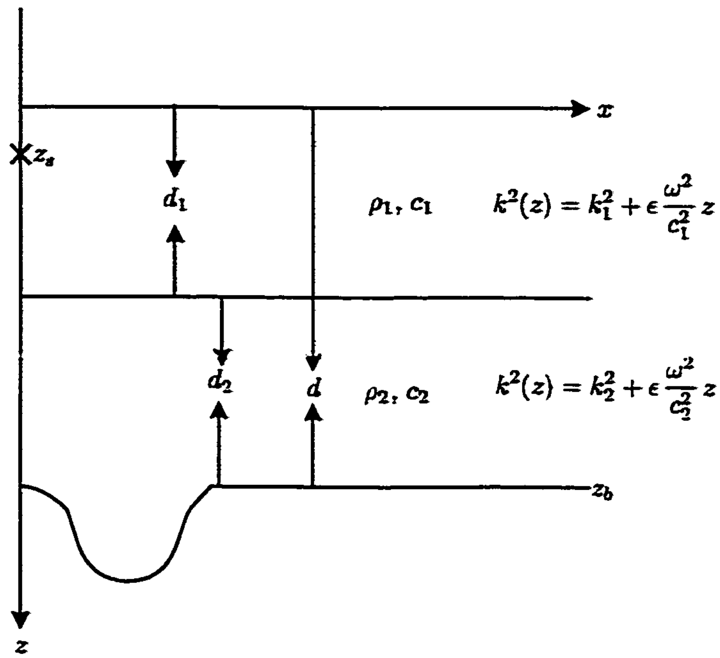

Figure 2.4.1. Coordinate geometry for perturbed two-layered model with the wavenumber varying with depth.

Equation (2.4.1) can be written as

$$\frac{\partial^2 p^{(1)}}{\partial x^2} + \frac{\partial^2 p^{(1)}}{\partial z^2} + k_1^2 p^{(1)} = -2\pi\delta(x)\delta(z - z_s) - \epsilon\frac{\omega^2}{c_1^2} z p^{(1)}. \tag{2.4.3}$$

The Helmholtz equation in layer 2 is

$$\frac{\partial^2 p^{(2)}}{\partial x^2} + \frac{\partial^2 p^{(2)}}{\partial z^2} + k_2^2 p^{(2)} = -\epsilon_2 \frac{\omega^2}{c_2} z p^{(2)}, \tag{2.4.4}$$

where $k_2 = \frac{\omega}{c_2}$ and $-1 < \epsilon_2 \leq 1$.

If we use Fourier transforms of equations (2.4.3) and (2.4.4) over x coordinate, we get

$$\frac{d^2 \hat{p}^{(1)}}{dz^2} + \alpha_1^2 \hat{p}^{(1)} = -\sigma_1(z) \tag{2.4.5}$$

and

$$\frac{d^2 \hat{p}^{(2)}}{dz^2} + \alpha_2^2 \hat{p}^{(2)} = -\sigma_2(z) \tag{2.4.6}$$

where $\alpha_i^2 = k_i^2 - \xi, \quad i = 1, 2$ and

$$\sigma_1(z) = \delta(z - z_s) + \epsilon_1 \frac{\omega^2}{c_1^2} z \hat{p}^{(1)} \tag{2.4.7}$$

$$\sigma_2(z) = -\epsilon_2 \frac{\omega^2}{c_2^2} z \hat{p}^{(2)}. \tag{2.4.8}$$

Green's function $G^{(1)}(\xi, z_1, z)$, and $G^{(2)}(\xi, z_2, z)$ for the first and second layers, respectively, satisfy the following equations:

$$\frac{d^2 G^{(1)}}{dz^2} + \alpha_1^2 G^{(1)} = -\delta(z - z_1) \tag{2.4.9}$$

and

$$\frac{d^2 G^{(2)}}{dz^2} + \alpha_2^2 G^{(2)} = -\delta(z - z_2), \tag{2.4.10}$$

where z_1 is a parameter in the interval $[0, d_1]$ and z_2 is a parameter in $[d_1, d_2]$. The boundary conditions are the same as those stated in Section (2.1).

Since the boundary conditions in our problem are of an inhomogeneous nature, we follow Stakgold [21] and multiply equation (2.4.5) by $G^{(1)}$, equation (2.4.3) by $\hat{p}^{(1)}$ and subtract and integrate over $0 \le z \le d_1$, to obtain

$$\int_0^{d_1} \left(G^{(1)} \frac{d^2 \hat{p}^{(1)}}{dz^2} - \hat{p}^{(1)} \frac{d^2 G^{(1)}}{dz^2} \right) dz$$
$$= \int_0^{d_1} -\sigma_1(z) G^{(1)}(\xi, z_1, z) + \hat{p}^{(1)}(\xi, z) \delta(z - z_1) \Big] \, dz. \tag{2.4.11}$$

Integrating by parts, equation (2.4.11) can be written as

$$G^{(1)}(\xi, z_1, d_1) \frac{d\hat{p}^{(1)}}{dz}(\xi, d_1) - \hat{p}^{(1)}(\xi, d_1) \frac{dG^{(1)}}{dz}(\xi, z_1, d_1)$$
$$= -\int_0^{d_1} \sigma_1(z) G^{(1)}(\xi, z_1, z) dz + \hat{p}^{(1)}(\xi, z_1). \tag{2.4.12}$$

Similarly, for the second layer, we get

$$G^{(2)}(\xi, z_2, z_b)\frac{d\hat{p}^{(2)}}{dz}(\xi, z_b) - \hat{p}^{(2)}(\xi, z_b)\frac{dG^{(2)}}{dz}(\xi, z_2, z_b)$$

$$-\left[G^{(2)}(\xi, z_2, d_1)\frac{d\hat{p}^{(2)}(\xi, d_1)}{dz} - \hat{p}^{(2)}(\xi, d_1)\frac{dG^{(2)}}{dz}(\xi, z_2, d_1)\right.$$

$$= -\int_{d_1}^{z_b} \sigma_2(z)G^{(2)}(\xi, z_2, z)dz + \hat{p}^{(2)}(\xi, z_2). \tag{2.4.13}$$

Applying boundary conditions, equation (2.4.13) can be rewritten as

$$G^{(2)}(\xi, z_2, z_b)\frac{d\hat{p}^{(2)}}{dz}(\xi, z_b) - G^{(2)}(\xi, z_2, d_1)\frac{d\hat{p}^{(2)}}{dz}(\xi, d_1)$$

$$+\hat{p}^{(2)}(\xi, d_1)\frac{dG^{(2)}}{dz}(\xi, z_2, d_1)$$

$$= -\int_{d_1}^{z_b} \sigma_2(z)G^{(2)}(\xi, z_2, z) + \hat{p}^{(2)}(\xi, z_2). \tag{2.4.14}$$

Interchanging z and z_1 in equation (2.4.12), z and z_2 in equation (2.4.14), and using the symmetry of Green's function, we get

$$\hat{p}^{(1)}(\xi, z) = -\hat{p}^{(1)}(\xi, d_1)\frac{dG^{(1)}}{dz}(\xi, z, d_1)$$

$$+G^{(1)}(\xi, z, d_1)\frac{d\hat{p}^{(1)}}{dz}(\xi, d_1) + \int_0^{d_1} \sigma_1(z_1)G^{(1)}(\xi, z, z_1)dz_1, \tag{2.4.15}$$

and

$$\hat{p}^{(2)}(\xi, z) = -G^{(2)}(\xi, z, d_1)\frac{d\hat{p}^{(2)}}{dz}(\xi, d_1)$$

$$+G^{(2)}(z, z_b)\frac{d\hat{p}^{(2)}}{dz}(\xi, z_b) + \hat{p}^{(2)}(\xi, d_1)\frac{dG^{(2)}}{dz}(z, d_1)$$

$$+\int_{d_1}^{z_b} \sigma_2(z_2)G^{(2)}(z, z_2)dz_2. \tag{2.4.16}$$

At $z = d_1$ $\hat{p}^{(1)}(\xi, d_1) = \hat{p}^{(2)}(\xi, d_1)$ and using boundary conditions

$$-\frac{d\hat{p}^{(1)}(d_1)}{dz}\left[G^{(1)}(\xi, d_1, d_1) + \frac{\rho_2}{\rho_1}G^{(2)}(\xi, d_1, d_1)\right]$$

$$= -\int_0^{d_1} \sigma_1(z)G^{(1)}(\xi, d_1, z_1)dz_1 - \int_{d_1}^{z_b} \sigma_2(z_2)G^{(2)}(\xi, d_1, z_2)dz_2$$

$$-\hat{p}^{(1)}(\xi, d_1)\left[\frac{\rho_2}{\rho_1}dG^{(1)}(\xi, d_1, d_1) + \frac{dG^{(1)}}{dz}(\xi, d_1, d_1)\right]$$

$$-G^{(2)}(\xi, d_1, z_b)\left[\epsilon_3(i\xi H(\xi)) * (i\xi\hat{p}^{(2)}(\xi, z_b))\right]$$

or

$$\frac{d\hat{p}^{(1)}}{dz}(\xi, d_1) = \frac{1}{Q}\left[\int_0^{d_1} \sigma_1(z_1)G^{(1)}(\xi, d_1, z_1)dz_1\right.$$

$$-\int_{d_1}^{z_b} \sigma_2(z)G^{(2)}(\xi, d_1, z_2)dz_2$$

$$-\hat{p}^{(1)}(\xi, d_1)\frac{dG^{(1)}}{dz}(\xi, d_1, d_1)\left[\frac{\rho_2}{\rho_1} + 1\right]$$

$$\left.-G^{(2)}(\xi, d_1, z_b)\left[\epsilon_3(\xi H(\xi)) * (\xi p^{(2)}(\xi, z_b)\right]\right., \tag{2.4.17}$$

where

$$Q = -\left[G^{(1)}(\xi, d_1, d_1) + \frac{\rho_2}{\rho_1}G^{(2)}(\xi, d_1, d_1)\right]. \tag{2.4.18}$$

Using equation (2.4.17) in equation (2.4.15), substituting back the value of $\sigma_1(z)$, and using the delta function property, we get

$$\hat{p}^{(1)}(\xi, z) = -\hat{p}^{(1)}(\xi, d_1) dz dG^{(1)}(\xi, z, d_1)$$

$$+ G^{(1)}(z, d_1) \left[\frac{1}{Q} \left\{ \int_0^{d_1} \left[\delta(z_1 - z_s) + \epsilon_1 \frac{\omega^2}{c_1^2} z \hat{p}^{(1)}(\xi, z_1) \right. \right. \right.$$

$$\times \left(G^{(1)}(\xi, d_1, z_1) \right) dz_1 - \int_{d_1}^{z_b} \left[\epsilon_2 \frac{\omega^2}{c_2^2} z_2 \hat{p}^{(2)}(\xi, z_2) \right]$$

$$\times G^{(2)}(\xi, d_1, z_2) dz_2 - \hat{p}^{(1)}(\xi, d_1) \frac{dG^{(1)}(\xi, d_1, d_1)}{dz}$$

$$\times \left[\frac{\rho_2}{\rho_1} + 1 \right] - G^{(2)}(\xi, d_1, z_b) \left[(\epsilon_3 \xi H(\xi)) * \xi \hat{p}^{(2)}(z_b, \xi) \right]$$

$$+ \int_0^{d_1} \left[\delta(z_1 - z_s) + \epsilon_1 \frac{\omega^2}{c_1^2} z_1 \hat{p}^{(1)}(\xi, z_1) \right] G^{(1)}(\xi, z, z_1) dz_1 \qquad (2.4.19)$$

or

$$\hat{p}^{(1)}(\xi, z) = -p^{(1)}(\xi, d_1) \frac{dG^{(1)}}{dz}(\xi, z, d_1)$$

$$+ G^{(1)}(\xi, z, d_1) \left[\frac{1}{Q} \left\{ G^{(1)}(d_1, z_s) \right. \right.$$

$$+ \int_0^{d_1} \left\{ \epsilon_1 \frac{\omega^2}{c_1^2} z_1 \hat{p}^{(1)}(\xi, z_1) \right\} \cdot G^{(1)}(\xi, d_1, z_1) dz_1$$

$$- \int_{d_1}^{z_b} \left[\epsilon_2 \frac{\omega^2}{c_2^2} z_2 \hat{p}^{(2)}(\xi, z_2) \right] \cdot G^{(2)}(\xi, d_1, z_2) dz_2$$

$$- \hat{p}^{(1)}(\xi, d_1) \frac{dG^{(1)}}{dz}(\xi, d_1, d_1) \cdot \left[\frac{\rho_2}{\rho_1} + 1 \right]$$

$$- G^{(2)}(\xi, d_1, z_b) \left\{ \epsilon_3 (\xi H(\xi)) * \xi \hat{p}^{(2)}(\xi, z_b) \right\}$$

$$+ G^{(1)}(\xi, z, z_s) + \int_0^{d_1} \left\{ \epsilon_1 \frac{\omega^2}{c_1^2} z_1 \hat{p}^{(1)}(\xi, z_1) \right\} G^{(1)}(\xi, z, z_1) dz_1. \qquad (2.4.20)$$

As a first approximation, we assume that $\epsilon_1 = \epsilon_2 = \epsilon_3 = 0$, and get

$$\hat{p}^{(1)}(\xi, z) = -\hat{p}^{(1)}(\xi, d_1)\frac{dG^{(1)}}{dz}(z, d_1)$$

$$+G^{(1)}(z, d_1)\left[\frac{1}{Q}\left\{G^{(1)}(\xi, d_1, z_s)\right.\right.$$

$$\left.\left.-\hat{p}^{(1)}(\xi, d_1)\frac{dG^{(1)}}{dz}(\xi, d_1, d_1)\cdot\left(\frac{\rho_2}{\rho_1}+1\right)\right\}\right] + G^{(1)}(\xi, z, z_s), \qquad (2.4.21)$$

where

$$\hat{p}^{(1)}(\xi, d_1) = \frac{1}{s}\left[\frac{1}{Q}\left\{G^{(1)}(\xi, d_1, d_1)G^{(1)}(\xi, d_1, z_s)\right\}\right.$$

$$\left.+G^{(1)}(\xi, d_1, z_s)\right], \qquad (2.4.22)$$

and

$$S = \left[1 + \frac{dG^{(1)}}{dz}(\xi, d_1, d_1) + \frac{1}{Q}\left\{G^{(1)}(\xi, d_1, d_1)\frac{dG^{(1)}}{dz}(\xi, d_1, d_1)\left(\frac{\rho_2}{\rho_1}\right)\right\}\right]. \qquad (2.4.23)$$

Green's functions $G^{(1)}(\xi, z, z_s)$ and $G^{(2)}(\xi, z, z_s)$ are the same Green's functions as those determined in Section (2.1). We take the inverse Fourier transform of Equation (2.4.20) and use the residue theorem to determine $p^{(1)}(x, z)$.

CHAPTER III

MODE-COUPLING THEORY

3.1 Introduction

In this chapter, we will use the mode-coupling theory as initiated by Pierce [19]. We shall, however, use the modification that could handle a perturbed sea bottom satisfying a rigid boundary condition. The conventional mode-coupling technique is only applicable to the flat layered model or perturbation in the layered model free surface only. The modified coupling theory assumes the series expansion of the pressure wavefield in terms of the mode amplitudes times the vertical depth eigenfunctions, but only within the layers. At the boundaries, careful analysis should be carried out to handle these regions. Corrections which are in terms of interface effects are added to the pressure solution for the unperturbed model to give the pressure solution for the perturbed layered model.

3.2 Two-Layer Model With Irregular Surface and Bottom

We assume a point source located in an inhomogeneous oceanic waveguide bounded above by a random time-varying sea surface and bounded below by a non-flat rigid bottom. The source is located at $x = y = 0$, and $z = z_s$. The source signal is assumed to be time-harmonic with angular frequency ω. The orthogonal cartesian coordinate system relative to the waveguide is shown above in Figure (3.2.1).

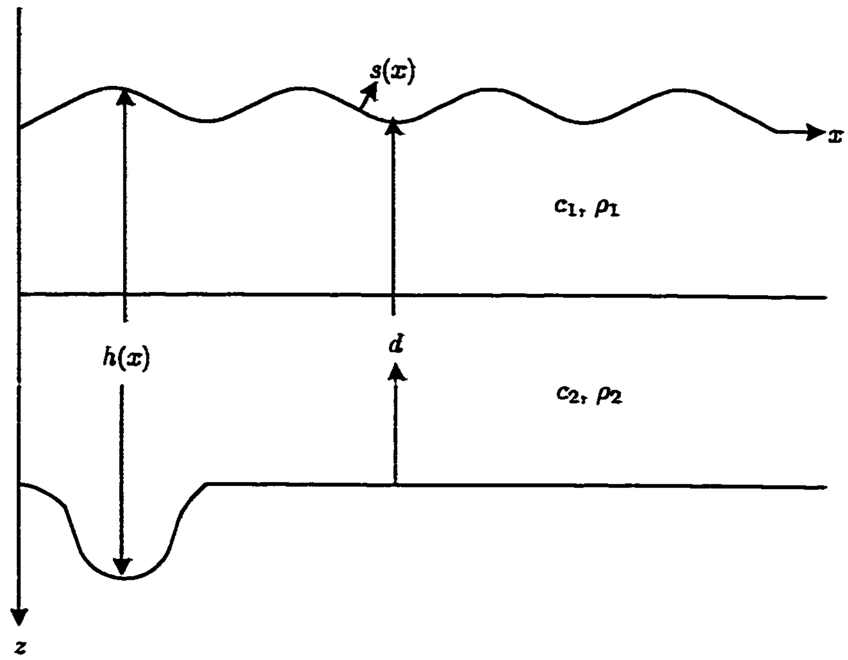

Figure 3.2.1. Coordinate geometry for the two-layered model with random perturbation in the surface and smooth perturbation in the bottom.

We assume that the sea surface can be written as

$$z = s(x, t). \tag{3.2.1}$$

The boundary conditions for the problem are

1. At the free surface $z = s(x)$

$$p^{(1)}(x, y, s(x)) = 0. \tag{3.2.2}$$

2. At the interface $z = d_1$, the same continuity conditions as in Equations (2.2.5) and (2.2.6) are satisfied which are:

$$p^{(1)}(x, y, d_1) = p^{(2)}(x, y, d_1) \tag{3.2.3}$$

$$\frac{1}{\rho_1}\frac{\partial p^{(1)}}{\partial z}(x, y, d_1) = \frac{1}{\rho_2}\frac{\partial p^{(2)}}{\partial z}(x, y, d_1), \tag{3.2.4}$$

3. At the rigid bottom, $z = h(x)$, we have

$$\frac{\partial p^{(2)}}{\partial n}\Big(x, y, h(x)\Big) = 0, \qquad (3.2.5)$$

where n is the outward normal to the surface of the sea bottom.

The speed of sound, c, in waveguide is a function of x and z. The refraction index is denoted by $n(x,z)$ where $n(x,z) = \frac{1}{C(x,z)}$. The wave equation which governs the acoustic field away from the source is

$$\left\{\nabla^2 - \frac{1}{\rho}\nabla\rho.\nabla - n^2\frac{\partial^2}{\partial t^2}\right\}P(r,t) = 0, \qquad (3.2.6)$$

where P is the acoustic pressure.

We assume the narrow-band approximation to the wave equation

$$\frac{\partial^2 P(r,t)}{\partial t^2} \approx -\omega^2 P(r,t) \qquad (3.2.7)$$

which leads to the time-harmonic assumption on source signal

$$P(x,y,z,t) = p(x,y,z,t)e^{i\omega t}. \qquad (3.2.8)$$

Equation (3.2.6) then gives

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} - \frac{1}{\rho}\frac{d\rho}{dz}\frac{\partial p}{\partial z} + k^2(x,z)p = 0, \qquad (3.2.9)$$

where $k(x,z) = \omega n(x,z)$ is the wave number.

3.2.1 Modified Mode-Coupling Approach

To solve the problem stated above using mode coupling initiated by Pierce [19] and used by others later, we have to modify the conventional method so that it accounts for the effect of boundary irregularity. For the above problem, we split the domain into two regions and define the acoustic pressure for the two domains as

$$p(x,y,z) = \begin{cases} \sum_{n=1}^{\infty} F_n(x,y) f_n(x,z) & s \le z < h^- \\ g(x,y,z) & h^- \le z \le h^+ \end{cases} \tag{3.2.10}$$

where the time factor $e^{i\omega t}$ in p, F_n, f_n and g is not included for the sake of simplicity. Here F_n are the mode amplitudes, f_n are the vertical depth modes, and the term $g(x,y,z)$ in equation (3.2.10) is the correction term where $h^- \le z \le h^+$ is the strip around the irregular sea bottom. Due to the special nature of the problem, it needed to account for surface irregularity effects. We substitute equation (3.2.10) in equation (3.2.9) to get

$$\sum_{n=1}^{\infty} \left(\frac{\partial^2 F_n}{\partial x^2} + \frac{\partial^2 F_n}{\partial y^2} + k^2 F_n \right) f_n + 2 \sum_{n=1}^{\infty} \frac{\partial F_n}{\partial x} \frac{\partial f_n}{\partial x}$$
$$+ \sum_{n=1}^{\infty} F_n \left(\frac{\partial^2 f_n}{\partial x^2} + \frac{\partial^2 f_n}{\partial z^2} - \frac{1}{\rho} \frac{\partial \rho}{\partial z} \frac{\partial f_n}{\partial z} \right) = 0 \quad s \le z \le h^- \tag{3.2.11}$$

and

$$\sum_{n=1}^{\infty} F_n \frac{\partial^2 f_n}{\partial x^2} + 2 \frac{\partial F_n}{\partial x} \frac{\partial f_n}{\partial x} + \frac{\partial^2 F_n}{\partial x^2} f_n$$
$$+ \sum_{n=1}^{\infty} \frac{\partial^2 F_n}{\partial y^2} f_n + k^2 \sum_{n=1}^{\infty} F_n f_n$$
$$+ \frac{\partial^2 g}{\partial z^2} - \frac{1}{\rho} \frac{\partial \rho}{\partial z} \frac{\partial g}{\partial z} = 0 \quad h^{-1} \le z \le h^+. \tag{3.2.12}$$

We assume that the local depth modes f_n satisfy

$$\frac{\partial}{\partial z} \left[\frac{1}{\rho(z)} \frac{\partial f_n}{\partial z} \right] + \left[\frac{k^2(x,y) - \xi_n^2(x)}{\rho} \right] f_n = 0, \tag{3.2.13}$$

where ξ_n is the separation constant.

The eigenfunction solution, f_n, of equation (3.2.9) subject to the boundary conditions given by equations (3.2.2) – (3.2.6) form a complete orthonormal system of eigenfunctions relative to th weight functions $\left(\frac{1}{\rho}\right)$ at each point x. The orthonormality condition is

$$\int_s^h \frac{1}{\rho} f_n(x,z) f_m(x,z) dz = \delta_{mn}. \tag{3.2.14}$$

Multiplying equations (3.2.12) and (3.2.13) with $\frac{1}{\rho} f_m$, integrating over z and using the orthonormalization in equation (3.2.15) gives

$$\frac{\partial^2 F_n}{\partial x^2} + \frac{\partial^2 F_n}{\partial y^2} + \xi_n^2 F_n = -\sum_{n=1}^{\infty} F_n \int_s^{h^-} \frac{1}{\rho}\frac{\partial^2 f_n}{\partial x^2} f_m dz$$
$$-2\sum_{n=1}^{\infty} \frac{\partial F_n}{\partial x} \int_s^{h^-} \frac{1}{\rho}\frac{\partial f_n}{\partial x} f_m dz \quad s \le z < h^- \tag{3.2.15}$$

and

$$\frac{\partial^2 F_n}{\partial x^2} + \frac{\partial^2 F_n}{\partial y^2} + k^2 F_n = -\sum_{n=1}^{\infty} F_n \int_{h^-}^{h^+} \frac{1}{\rho}\frac{\partial^2 f_n}{\partial x^2} f_m dz$$
$$-2\sum_{n=1}^{\infty} \frac{\partial F_n}{\partial x} \int_{h^-}^{h^+} \frac{1}{\rho}\frac{\partial f_n}{\partial x} f_m dz$$
$$-f_m \left(\frac{1}{\rho}\frac{\partial p}{\partial z}\right)\Big|_{h^-}^{h^+} + \int_{h^-}^{h^+} \frac{\partial f_m}{\partial z}\left(\frac{1}{\rho}\frac{\partial p}{\partial z}\right) dz. \tag{3.2.16}$$

The third term on the right of equation (3.2.16) introduces interface terms and the third integral on the right of the of equation (3.2.16) does no longer contain interface terms and can be rewritten as

$$\int_{h^-}^{h^+} \frac{\partial f_m}{\partial z}\left(\frac{1}{\rho}\frac{\partial g}{\partial z}\right) dz = g\left(\frac{1}{\rho}\frac{\partial f_m}{\partial z}\right)\Big|_{h_-}^{h^+}$$
$$-\int_{h^-}^{h^+} g\frac{\partial}{\partial z}\left(\frac{1}{\rho}\frac{\partial f_m}{\partial z}\right) dz$$
$$= \int_{h^-}^{h^+} \left(\frac{k^2 - \xi_n^2}{\rho}\right) f_n \sum_{n=1}^{\infty} F_n f_n, \tag{3.2.17}$$

where $p = g = \sum_{n=1}^{\infty} F_n f_n$ has been used.

Equations (3.2.15) – (3.2.17) can be combined to give

$$\frac{\partial^2 F_n}{\partial x^2} + \frac{\partial^2 F_n}{\partial y^2} + \xi_n^2 F_n = -\sum_{n=1}^{\infty} \alpha_{mn} F_n - 2\sum_{n=1}^{\infty} \beta_{mn} \frac{\partial F_n}{\partial x}, \tag{3.2.18}$$

where

$$\begin{aligned} \alpha_{mn} &= A_{mn} - \frac{1}{\rho}\frac{\partial h}{\partial x} f_m \left.\frac{\partial f_n}{\partial x}\right|_{z=h^-}^{z=h^+} \\ A_{mn} &= \int_s^h \frac{1}{\rho} f_m \frac{\partial^2 f_n}{\partial x^2} dz \end{aligned} \tag{3.2.19}$$

and

$$\begin{aligned} \beta_{mn} &= B_{mn} - \frac{1}{\rho}\frac{\partial h}{\partial x} f_m f_n \Big|_{z=h^-}^{z=h^+} . \\ B_{mn} &= \int_s^h \frac{1}{\rho} f_m \frac{\partial f_n}{\partial x} dz. \end{aligned} \tag{3.2.20}$$

Equations (3.3.19) and (3.2.20) give the mode-coupling coefficients α_{mn} and β_{mn}, respectively. α_{mn} and β_{mn} account for sea bottom irregularity effect. When the seabed is horizontal, these coefficients will be the same as given by Boyles [4] in the horizontal layered oceanic model case.

If we differentiate equation (3.2.13) with respect to x

$$\frac{\partial}{\partial x}\left[\frac{\partial}{\partial z}\left(\frac{1}{\rho}\frac{\partial f_n}{\partial z}\right)\right] + \frac{\partial}{\partial x}\left[\frac{k^2(x,z) - \xi_n^2(x)}{\rho(z)}\right] = 0,$$

and then multiply the resulting equation by f_m and integrate over z from $z = s(x)$

to $z = h(x)$ and then use the orthogonality condition in equation (3.2.14), to obtain

$$\int_{s(x)}^{h(x)} \frac{\partial}{\partial z}\left(\frac{1}{\rho}\frac{\partial \dot{f}_n}{\partial z}\right) f_m dz + \int_{s(x)}^{h(x)} \frac{k^2}{\rho} f_m \dot{f}_n dz$$

$$2\int_{s(x)}^{h(x)} \frac{k\dot{k}}{\rho} f_n f_m dz - 2\xi_n \dot{\xi}_n \delta_{nm}$$

$$-k_n^2 \int_{s}^{h(x)} \frac{1}{\rho} f_m \dot{f}_n dz = 0. \tag{3.2.21}$$

If we integrate the first term in equation (3.2.21) twice by parts:

$$\int_{s}^{h(x)} \frac{\partial}{\partial z}\left(\frac{1}{\rho}\frac{\partial \dot{f}_n}{\partial z}\right) f_m dz = \left[\frac{1}{\rho} f_m \frac{\partial \dot{f}_n}{\partial z}\right]\Bigg|_{s(x)}^{h(x)}$$

$$-\left[\frac{1}{\rho}\dot{f}_n \frac{\partial f_m}{\partial z}\right]\Bigg|_{s(x)}^{h(x)} + \int_{s(x)}^{h(x)} \dot{f}_n \frac{\partial}{\partial z}\left(\frac{1}{\rho}\frac{\partial f_m}{\partial z}\right) dz. \tag{3.2.22}$$

Using equation (3.2.13) to rewrite the last term in equation (3.2.22) and then replacing the first term in equation (3.2.21) by the resulting expression, we have

$$\left(\xi_m^2 - \xi_n^2\right) B_{mn} = 2\xi_n \dot{\xi}_n \delta_{nm} + \left[\frac{1}{\rho}\dot{f}_n \frac{\partial f_m}{\partial z}\right]\Bigg|_{s(x)}^{h(x)}$$

$$-\left[\frac{1}{\rho} f_m \frac{\partial \dot{f}_n}{\partial z}\right]_{s(x)}^{h(x)} - 2\int_{s(x)}^{h(x)} \frac{k\dot{k}}{\rho} f_n f_m dz. \tag{3.2.23}$$

Let us find B_{mn}.

Case $m \neq n$. The first term in equation (3.2.23) vanishes

$$\left(\xi_m^2 - \xi_n^2\right) B_{mn} = \left[\frac{1}{\rho}\dot{f}_n \frac{\partial f_m}{\partial z}\right]_{h(x)} - \left[\frac{1}{\rho}\dot{f}_n \frac{\partial f_m}{\partial z}\right]_{s(x)}$$

$$+\left[\frac{1}{\rho} f_m \frac{\partial \dot{f}_n}{\partial z}\right]_{h(x)} + \left[\frac{1}{\rho} f_m \frac{\partial \dot{f}_n}{\partial z}\right]_{s(x)} - 2\int_{s}^{h(x)} \frac{k\dot{k}}{\rho} f_n f_m dz. \tag{3.2.24}$$

Invoking the boundary condition at the ocean's surface, the fourth term on the right of equation (3.2.24) vanishes. Using the boundary condition at sea bottom, the first term on the right of equation (3.2.24) is

$$\left[\frac{1}{\rho}\dot{f}_n\frac{\partial f_m}{\partial z}\right]_{z=h(x)} = 0. \tag{3.2.25}$$

The completeness relation for the eigenfunction f_n is given by

$$\sum_{n=1}^{\infty}\frac{1}{\rho}f_n(x,z)f_n(x,z') = \delta(z-z'). \tag{3.2.26}$$

Multiplying equation (3.2.26) by $\dot{f}_m$ and integrating over z from $z = s$ to $z = h(x)$ results in

$$\begin{aligned}\sum_{n=1}^{\infty} f_n(x,z)\int_{s(x)}^{h(x)}\frac{1}{\rho}\dot{f}_m(x,z')f_n(x,z')dz' \\ = \int_{s}^{h(x)}\dot{f}_m(x,z')\delta(z-z')dz' \\ = \dot{f}_m(x,z),\end{aligned} \tag{3.2.27}$$

or

$$\dot{f}_m = \sum_{n=1}^{\infty} B_{nm}f_n(x,z). \tag{3.2.28}$$

Hence

$$\frac{\partial\dot{f}_m}{\partial z} = \sum_{n=1}^{\infty} B_{nm}\frac{\partial f_n}{\partial z}. \tag{3.2.29}$$

To calculate the second term in equation (3.2.24), we have $f_n\Big(x,s(x)\Big) = 0$ and using the chain rule, we obtain

$$\begin{aligned}\frac{df_n\Big(x,s(x)\Big)}{dx} &= \frac{\partial f_n}{\partial x} + \frac{\partial f_n}{\partial z}\frac{ds(x)}{dx} \\ &= 0.\end{aligned}$$

So

$$\dot{f}_n = \frac{\partial f_n\Big(x,s(x)\Big)}{\partial x} = -\frac{\partial f_n}{\partial z}\frac{ds(x)}{dx}. \tag{3.2.30}$$

Combining all of the above results, the expression for the coupling coefficient $B_{mn}(m \neq n)$ given by equation (3.2.24) gives

$$B_{mn} = \frac{1}{(\xi_n^2 - \xi_m^2)} \left[\frac{1}{\rho(s)} \left\{ \frac{\partial f_m}{\partial z} \frac{\partial f_n}{\partial z} \frac{dz}{dx} \right\}_{z=s(x)} + 2\omega^2 \int_0^h \frac{1}{\rho} n \frac{\partial n}{\partial x} f_m f_n dz. \right. \tag{3.2.31}$$

To get an expression for B_{nm} when $n = m$, we differentiate equation (3.2.32) with respect to x

$$\frac{d}{dx} \left[\int_{s(x)}^{h(x)} \frac{1}{\rho(z)} f_n f_m dz \right] = \int_s^{h(x)} \frac{1}{\rho} \dot{f}_n f_m dz + \int_s^{h(x)} \frac{1}{\rho} f_n \dot{f}_m dz + \left[\frac{1}{\rho(h)} f_n f_m \frac{dh}{dx} \right] - \left[\frac{1}{\rho(s)} f_n f_m \frac{ds}{dx} \right]_{z=s}^{z=h(x)} = 0 \tag{3.2.32}$$

or

$$B_{mn} + B_{nm} = - \left[\frac{1}{\rho(h)} f_n f_m \frac{dh}{dx} \right]_{z=h(x)}. \tag{3.2.33}$$

For $n = m$

$$B_{nn} = -\frac{1}{2} \left[\frac{1}{\rho(h)} f_n^2 \frac{dh}{dx} \right]_{z=h(x)}. \tag{3.2.34}$$

To obtain an expression of the A_{mn} appearing in the second coupling coefficient expression for α_{mn}, we differentiate equation (3.2.19) with respect to x,

$$\dot{B}_{mn} = \int_s^h \frac{1}{\rho} f_m \ddot{f}_m + \int_s^h \frac{1}{\rho} \dot{f}_m \dot{f}_n dz + \left[\frac{1}{\rho} f_m \dot{f}_n \frac{dh}{dx} \right]_{z=h} - \left[\frac{1}{\rho} f_m \dot{f}_n \frac{ds}{dz} \right]_{z=s}. \tag{3.2.35}$$

The fourth term on the right side is zero and equation (3.2.35) gives

$$A_{mn} = \dot{B}_{mn} - \int_s^h \frac{1}{\rho}\dot{f}_m \dot{f}_m dz - \left[\frac{1}{\rho} f_m \dot{f}_n \frac{dh}{dx}\right]_{z=h}. \tag{3.2.36}$$

Using equation (3.2.28), the expression for A_{mn} can be rewritten as

$$A_{mn} = \frac{dB_{mn}}{dx} - \int_s^h \frac{1}{\rho}\left(\sum_{i=1}^{\infty} B_{im} f_i\right)\left(\sum_{j=1}^{\infty} B_{jn} f_j\right) - \left[\frac{1}{\rho} f_m \dot{f}_n \frac{dh}{dx}\right]_{z=h}$$

$$= \frac{dB_{mn}}{dx} - \sum_{i=1}^{\infty} B_{im} B_{in} - \left[\frac{1}{\rho} f_m \dot{f}_n \frac{dh}{dx}\right]_{z=h}. \tag{3.2.37}$$

From equation (3.2.31), we can write the coefficient B_{mn} $(m \neq n)$ as

$$B_{mn} = S_{mn} + N_{mn} \tag{3.2.38}$$

where

$$S_{mn} = \frac{1}{\rho(s)\,(\xi_m^2 - \xi_n^2)} \left\{ \frac{\partial f_m}{\partial z} \frac{\partial f_n}{\partial z} \frac{ds}{dx} \right\}_{z=s} \tag{3.2.39}$$

and

$$N_{mn} = \frac{-2\omega^2}{(\xi_m^2 - \xi_n^2)} \int_s^h \frac{1}{\rho} n \frac{\partial n}{\partial x} f_n f_m dz \tag{3.2.40}$$

and for $m = n$

$$B_{n_n} = -\frac{1}{2}\left[\frac{1}{\rho(h)} f_n^2(h) \frac{dh}{dx}\right] \tag{3.2.41}$$

S_{mn} arises because of coupling due to the rough sea surface, N_{mn} arises because of coupling due to change of refraction index with x. If $m = n$, $B_{nn} \neq 0$ because of change of the seabed with x.

One way to solve the mode-coupling problem numerically is by reformulating the second order coupled differential equations to a system of coupled, first order differential equations. In order to do this, we need to take the Fourier transform of equation (3.2.18) over y to give

$$\frac{d^2\hat{F}_m}{dx^2} + \left(\xi_m^2 - \gamma_2\right)\hat{F}_m = -\sum_{n=1}^{\infty}\alpha_{mn}\hat{F}_n - 2\sum_{n=1}^{\infty}\beta_{mn}\frac{d\hat{F}_n}{dx} \tag{3.2.42}$$

where

$$\hat{F}_m(x,\gamma) = \int_{-\infty}^{\infty} F_m(x,y)e^{-i\gamma y} \tag{3.2.43}$$

and

$$F_m(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{F}_m(x,\gamma)e^{i\gamma y}d\gamma. \tag{3.2.44}$$

Boyles [4] introduced the following transformation

$$\nu(x,y,z,t) = \frac{-i}{\rho\omega}\frac{\partial p}{\partial x}. \tag{3.2.45}$$

Taking the Fourier transform of equations (3.2.10) and (3.2.45) over y gives

$$\hat{p}(x,\gamma,z) = \begin{cases} \sum_{n=1}^{\infty}\hat{F}_n(x,\gamma)f_n(x,z) & s \le z < h^- \\ \hat{g} & h^- \le z \le h^+ \end{cases} \tag{3.2.46}$$

$$\hat{\nu}(x,\gamma,z) = \frac{-i}{\rho\omega}\frac{\partial\hat{p}}{\partial x}. \tag{3.2.47}$$

Since the eigenfunctions f_n form a complete set, therefore

$$\hat{\nu} = \sum_{n=1}^{\infty}\frac{1}{\rho}\hat{\mu}_n(x,\gamma)f_n(x,z). \tag{3.2.48}$$

Multiplying equation (3.2.48) by f_m and integrating and using equation (3.2.14), we get

$$\hat{\mu}_n(x,\gamma) = \int_s^h \hat{\nu}f_n dz. \tag{3.2.49}$$

Substituting equation (3.2.47) in equation (3.2.49), we get

$$\hat{\mu}(x,\gamma) = \int_s^h \frac{-i}{\rho\omega}\left(\sum_{m=1}^{\infty}\frac{\partial \hat{F}_n}{\partial x} f_m + \hat{f}_n \frac{\partial f_m}{\partial x}\right) f_n dz$$

$$= \frac{-i}{\omega}\frac{\partial \hat{F}_n}{\partial x} - \frac{i}{\omega}\sum_{m=1}^{\infty}\hat{F}_n \frac{\partial f_m}{\partial x} f_n dz$$

$$= \frac{1}{i\omega}\frac{\partial \hat{F}_n}{\partial x} + \frac{1}{i\omega}\sum_{m=1}^{\infty} B_{mn}\hat{F}_n. \tag{3.2.50}$$

Using the two variables $\hat{F}_m$ and $\hat{\mu}_m$, equation (3.2.50) can be written as an equivalent system of first-order differential equations

$$\frac{d\hat{F}_m}{dx} - i\omega\hat{\mu}_m + \sum_{n=1}^{\infty}\beta_{mn}\hat{F}_n = 0 \tag{3.2.51}$$

$$i\omega\frac{d\hat{\mu}_m}{dx} + \left(\xi_m^2 - \gamma^2\right)\hat{F}_m + i\omega\sum_{n=1}^{\infty}\beta_{mn}\hat{\mu}_n = 0. \tag{3.2.52}$$

We notice that the mode-coupling coefficient α_{mn} no longer appears in the system of equations (3.2.51) and (3.2.52).

In the next section, we study the generalized mode-coupling method. This method relies on the reciprocity principle instead of vertical eigenfunctions expansion. This method accounts for waveguide irregularity effects as well as variable density.

3.3 Mode-Coupling Theory Using Reciprocity Principle

In the presence of non-horizontal interface or a range-dependent rigid boundary, the term-by-term differentiation with respect to z which is used by the conventional methods is not valid, since otherwise the vertical component of the oscillatory velocity would be continuous at the interface.

In this section, we shall study a more general solution of the mode-coupling problem using the reciprocity principle discussed by Brekhovskikh and Godin [5], and Godin [11]. The reciprocity approach gives a more accurate solution of the two-point boundary value problem. This technique can account for boundary irregularities as well as medium parameters changing with range and depth. The mode-coupling coefficients derived by this technique become the same coupling coefficients derived by the conventional mode-coupling technique when the waveguide layers are horizontal.

Following Pierce [19] and Milder [15], acoustic pressure in an irregular fluid waveguide is represented as an expansion

$$p(x,y,z) = \sum_{m=1}^{\infty} F_m(x,y) f_m(z;x,y), \tag{3.3.1}$$

where f_m are the local vertical eigenfunctions.

The mode amplitudes F_m satisfy the simultaneous coupling equations

$$\nabla_\perp^2 F_n + \xi_n^2 F_n = \sum_{m=1}^{\infty} (\alpha_{nm} F_m + \beta_{nm} \nabla_\perp F_m) \tag{3.3.2}$$

where $\nabla_\perp = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, 0\right)$, $\xi_n(x,y)$ is the propagation constant of the corresponding local mode. α_{nm} and β_{nm} are the coupling coefficients. The vertical eigenfunctions $f_m(z;x,y)$ form a complete set of functions at each cross-section of the waveguide. They satisfy the equation

a)

$$\frac{\partial}{\partial z}\left(\frac{1}{\rho(x,y,z)} \frac{\partial f_m(z,x,y)}{\partial z}\right) + \frac{k^2(x,y,z) - \xi_m^2}{\rho(x,y,z)} f_m(z,x,y) = 0 \tag{3.3.3}$$

at $0 < z < h_1, h_1 < z < h_2 \ldots h_N < z < H$.

b) The continuity conditions

$$f_m\left(z = h_\ell^+(x,y), x, y\right) = f_m\left(z = h_\ell^-(x,y), x, y\right) \tag{3.3.4}$$

c)

$$\frac{1}{\rho(x,y,z)} \frac{\partial f_m(z;x,y)}{\partial z}\bigg|_{z=h_\ell^+(x,y)} = \frac{1}{\rho(x,y,z)} \frac{\partial f_m(z;x,y)}{\partial z}\bigg|_{z=h_\ell^-(x,y)}. \tag{3.3.5}$$

d) The boundary conditions at the boundaries $z = 0$ and $z = H$.

e) The orthogonality relationship

$$(f_n, f_m) = \int \frac{1}{\rho} f_n f_m dz = \delta_{nm}. \tag{3.3.6}$$

Assume a 2-D case, where the waveguide parameters and the acoustic field are independent of y. Following Godin [11], we write the acoustic pressure and x-component of particle velocity in terms of F_m^+ and F_m^-

$$p(x,z) = \sum_{m=1}^{\infty} \left[F_m^+(x) + F_m^-(x)\right] f_m(z;x) \tag{3.3.7}$$

$$v_x(x,z) = \frac{1}{\omega\rho} \sum_{m=1}^{\infty} \left[F_m^+(x) - F_m^-(x)\right] \xi_m(x) f_m(z;x) \tag{3.3.8}$$

where F_m^+ are the amplitudes of the m-th normal mode travelling to the right and F_m^- that of travelling to the left. The acoustic field in a source-free region of the waveguide is uniquely determined by prescribing p and v_x profiles at a vertical cross-section. The mode amplitude F_m^+ and F_m^- satisfy these equations

$$\frac{dF_n^+}{dx} - i\xi_n F_n^+ = \sum_m \left(b_{nm}^+ F_m^+ + b_{nm}^- F_m^-\right) \tag{3.3.9}$$

$$\frac{dF_n^-}{dx} - i\xi_n F_n^- = \sum_m \left(b_{nm}^- F_m^+ + b_{nm}^+ F_m^-\right) \tag{3.3.10}$$

where the matrices b^+_{mn} and b^-_{mn} (coupling coefficients) are given by integrals over z of horizontal derivative of c, p and h_ℓ.

Determination of b^+_{mn} and b^-_{mn}

Since $v_x = \frac{1}{i\omega}\frac{\partial p}{\partial x}$, therefore

$$\frac{1}{\omega\rho}\sum_{m=1}^{\infty}\left(F^+_m(x) - F^-_m(x)\right)\xi_m(x) f_m(z;x) = \frac{1}{i\omega\rho}\times$$

$$\times\left[\sum_{m=1}^{\infty}\frac{\partial}{\partial x}\left(F^+_m(x) + F^-_m(x)\right) f_m(x) + \left(F_m(x) + F^-_m(x)\right)\frac{\partial f_m}{\partial x}\right]\times$$

$$\times\sum_{m=1}^{\infty}\left[f_m(x,z)\frac{\partial}{\partial x}\left(F^-_m(x) + F^-_m(x)\right) - i\xi_m(x)\left(F^+_m(x) - F^-_m(x)\right) f_m(x,z)\right.$$

$$\left. + \left(F^+_m(x) + F^-_m\right)\frac{\partial f_m(x,z)}{\partial x}\right\} = 0. \tag{3.3.11}$$

We consider only the case when the waveguide boundaries are parallel [$H(x)$ =constant], and also impedance of the lower boundary does not depend on x. Then the function $\frac{\partial f_m}{\partial x}$ can be represented as

$$\frac{\partial f_m}{\partial x} = \sum_{n=1}^{\infty} q_{ns}(x) f_n(z;x), \tag{3.3.12}$$

where we have used the fact that f_m is a complete system of functions. The relation in equation (3.3.12) is invalid in a general case of a waveguide. For if the pressure $p\left(z = h(x)\right) = 0$, then

$$\frac{dp}{dx} = \frac{\partial p}{\partial x} + \frac{\partial p}{\partial z}\frac{dz}{dx} \tag{3.3.13}$$

and

$$\frac{\partial p}{\partial x} = -\frac{\partial}{\partial z}\frac{dH(x)}{dx}. \tag{3.3.14}$$

This implies

$$\frac{\partial f_m}{\partial x} = -\frac{\partial f_m}{\partial z}\frac{dH(x)}{dx} \neq 0, \tag{3.3.15}$$

but $\rho = 0$ implies $f_n = 0$ and this is a contradiction. Multiplying the above relation with f_n and using the orthogonality relation (3.3.6), we get

$$B_{nm} = q_{nm} = \int \frac{1}{\rho} f_n \nabla_\perp f_n dz. \tag{3.3.16}$$

Multiplying equation (3.3.11) with f_n and using orthogonality relation and equations (3.3.9) and (3.3.10), we get

$$\frac{\partial}{\partial x}\left(F_m^+ + F_m^-\right) - i\xi_m\left(F_m^+ - F_m^-\right) + \sum_m \left(F_m^+ + F_m^-\right) B_{nm} = 0.$$

This can be rewritten as

$$\sum_n \left(b_{nm}^+ + b_{nm}^-\right)\left(F_m^+ + F_m^-\right) + \sum_m \left(F_m^+ + F_m^-\right) B_{nm} = 0$$

or

$$\sum_m \left(b_{nm}^+ + b_{nm}^- + B_{nm}\right)\left(F_m^+ + F_m^-\right) = 0.$$

From the above, we obtain

$$b_{2n}^+ + b_{nm}^- + B_{nm} = 0. \tag{3.3.17}$$

Godin [11] used the reciprocity principle to derive further equations for the coupling coefficients. The reciprocity principle indicates that surface integral of the vector

$$J = \frac{1}{\rho}(p_1 \nabla p_2 - p_2 \nabla p_1) \tag{3.3.18}$$

is zero for any two acoustic fields p_1 and p_2.

If we set

$$p_1 = \sum \left(F_m^+ + F_m^-\right) f_m \tag{3.3.19}$$

and

$$p_2 = \sum \left(E_n^+ + E_n^-\right) f_n \tag{3.3.20}$$

then

$$\frac{\partial p_1}{\partial x} = i \sum_m \left(F_m^+ - F_m^-\right) \xi_m f_m \tag{3.3.21}$$

and

$$\frac{\partial p_2}{\partial x} = i \sum_n \left(E_n^+ - E_n^-\right) \xi f_n. \tag{3.3.22}$$

Substituting (3.3.21) and (3.3.22) in (3.3.18), then

$$J = \left(\frac{1}{\rho} \sum_m \left(F_m^+ + F_m^-\right) f_m\right) \sum_n i \left(E_n^+ - E_n^-\right) \xi_n f_n$$
$$-\frac{i}{\rho} \sum_m \left(F_m^+ - F_m^-\right) \xi_m f_m \sum_n \left(E_n^+ + E_n^-\right) f_n$$

or

$$J = \left(\sum_m \left(F_m^+ + F_m^-\right) \sum_n i \left(E_n^+ - E_n^-\right) \xi_n\right) \delta_{nm}$$
$$-i \left(\sum_m \left(F_m^+ - F_m^-\right) \xi_m \sum_n \left(E_n^+ + E_n^-\right)\right) \delta_{nm}.$$

If we take the derivative of the vector with respect to x and set it to zero, then we obtain

$$\sum_{m,n} \left[\left(F_n^+ + E_m^+ + F_m^- E_n^-\right) \left(\xi_m b_{nm}^- - \xi_n b_{nm}^-\right)\right.$$
$$\left. + \left(F_n^- E_m^+ - F_n^+ E_m^-\right) \left(\xi_m b_{mn}^+ + \xi_n b_{nm}^+ + \delta_{nm} \frac{\partial \xi_m}{\partial x}\right)\right] = 0. \tag{3.3.23}$$

After some simplifications, we can find that the mode-coupling coefficients satisfy the following equations

$$\xi_m b^-_{mn} - \xi_n b^-_{nm} = 0 \tag{3.3.24}$$

$$\xi_m b^+_{mn} + \xi_n b^+_{nm} + \delta_{mm}\frac{\partial \xi_m}{\partial x} = 0. \tag{3.3.25}$$

From equations (3.3.17), (3.3.24) and (3.3.25)

$$b^\pm_{nm} = \pm\delta_{nm}\frac{1}{2\xi_n}\frac{\partial \xi_n}{\partial x} \pm \frac{\xi_m}{2\xi_n}B_{mn} - \frac{1}{2}B_{nm}. \tag{3.3.26}$$

Equations (3.3.9), (3.3.10) and (3.3.26) represent a more accurate reformulation of the original 2-D boundary value problem.

If we differentiate equations (3.3.9) and (3.3.10) with respect to x and sum the resulting equations, we get

$$\frac{d^2F_n^+}{dx^2} + \frac{d^2F_n^-}{dx^2} + i\xi_n\left(\frac{dF_n^-}{dx} - \frac{dF_n^+}{dx}\right) + \frac{id\xi_n}{dx}\left(F_n^- - F_n^+\right)$$
$$= \sum\left(\frac{d}{dx}\left(b^+_{nm} + b^-_{nm}\right)\right)\left(F_m^+ + F_m^-\right) + \left(b^+_{nm} + b^-_{nm}\right)\left(\frac{dF_m^+}{dx} + \frac{dF_m^-}{dx}\right).$$

Using equations (3.2.9) and (3.2.10), and after collecting F_m^+ and F_m^- together we get

$$\frac{d^2F_n}{dx^2} + \xi^2F_n = -i\xi_n\sum\left(b^+_{nm} - b^-_{nm}\right)\left(F_m^- - F_m^+\right)$$
$$-\frac{id\xi_n}{dx}\left(F_n^- - F+_n\right) + \sum\frac{d}{dx}\left(b^+_{nm} + b^-_{nm}\right)\left(F_m^+ + F_m^+\right)$$
$$+\left(b^+_{nm} + b^-_{nm}\right)\left(\frac{dF_m^+}{dx} + \frac{dF_m^-}{dx}\right) \tag{3.3.27}$$

or

$$\frac{d^2F_n}{dx^2} + \xi^2 F_n = -i\xi_n \sum_m \left(b^+_{nm} - b^-_{nm}\right)\left(\frac{1}{\xi_m}\right)\left[-\left(\frac{dF^+_m}{dx} + \frac{dF^-_m}{dx}\right)\right.$$

$$\left. + \sum_s \left(b^+_{ms} + b^-_{ms}\right)\left(F^+_s + F^-_s\right)\right] + \frac{\partial_{nm}}{\xi_n}\frac{d\xi_n}{dx}\left[\left(\frac{dF^+_n}{dx} + \frac{dF^-_n}{dx}\right)\right.$$

$$\left. - \sum_m \left(b^+_{nm} + b^-_{nm}\right)\left(F^+_m + F^-_m\right)\right]$$

$$+ \sum \frac{d}{dx}\left(b^+_{nm} + b^-_{nm}\right)\left(F^+_m + F^-_m\right)$$

$$+ \left(b^+_{nm} + b^-_{nm}\right)\left(\frac{dF^+_m}{dx} + \frac{dF^-_m}{dx}\right). \qquad (3.3.28)$$

We can finally rewrite equation (3.3.28) as

$$\frac{d^2F_n}{dx^2} + \xi^2 F_n = \sum \alpha_{nm} F_m + \beta_{nm} \cdot \nabla_\perp F_m \qquad (3.3.29)$$

where

$$\alpha_{nm} = \sum_s B_{sn} B_{sm} - \frac{\partial B_{nm}}{\partial x} \qquad (3.3.30)$$

and

$$\beta_{nm} = B_{mn} - B_{nm}. \qquad (3.3.31)$$

Mode-Coupling Coefficients

In order that we express the mode-coupling coefficients β_{nm} and α_{mn} in terms of boundary and interface effects, we rewrite equation (3.3.31) as

$$\begin{aligned}
\beta_{nm} &= -2B_{nm} + B_{mn} + B_{nm} \\
&= -2B_{nm} + \int_0^H \frac{dz}{\rho}\frac{\partial}{\partial x}(f_n f_m) \\
&= -2B_{nm} + \frac{\partial}{\partial x}\int_0^H \frac{dz}{\rho} f_n f_m - \int_0^H dz f_n f_m \frac{\partial}{\partial x}\left(\frac{1}{\rho}\right) \\
&= -2B_{nm} + \left(\frac{1}{\rho}\frac{\partial \rho}{\partial x} f_n, f_m\right) \sum_{j=1}^{N} \frac{\partial h_j}{\partial x}\frac{f_n f_m}{\partial z}\Bigg|_{z=h_j^-}^{z=h_j^+}. \qquad (3.3.32)
\end{aligned}$$

So now the mode-coupling coefficient β_{nm} explicitly includes the boundary, interfaces and density changes. Similarly, the mode-coupling coefficients α_{nm} can be written in terms of environmental effects. From equations (3.3.30)

$$\alpha_{nm} = \sum_s \left(\int \frac{1}{\rho} f_s \frac{\partial f_n}{\partial x} dz\right)\left(\int \frac{1}{\rho} f_s \frac{\partial f_m}{\partial x} d'z\right) - \frac{\partial}{\partial x}\int \frac{1}{\rho} f_n \frac{\partial f_m}{\partial x} dz$$

$$\alpha_{nm} = \int\int \left(\frac{1}{\rho(z,x)}\frac{\partial f_n(z,x)}{\partial x}\right)\left(\frac{1}{\rho(z',x)}\frac{\partial f_m}{\partial x}\right)\left(\sum_s f_s(z;x) f_s(z'x)\right) \times$$
$$\times\, dz dz' - \frac{\partial}{\partial x}\int \frac{1}{\rho} f_n \frac{\partial f_m}{\partial x}.$$

α_{nm} can be written as

$$\begin{aligned}
\alpha_{nm} = &\int \frac{1}{\rho(z,x)}\frac{\partial f_n}{\partial x}\frac{\partial f_m}{\partial x} dz - \left[\int\left(\frac{1}{\rho}\frac{\partial f_n}{\partial x}\frac{\partial f_m}{\partial x} dz + \int f_n \frac{\partial}{\partial x}\left(\frac{1}{\rho}\frac{\partial f_m}{\partial x}\right)\right]\right. \\
&+ \left(\frac{\partial h_1^-}{\partial x}\left(\frac{f_n}{\rho}\frac{\partial f_m}{\partial x}\right)\Bigg|_{z=h_1^-} - \frac{\partial h_1^+}{\partial x}\left(\frac{f_n}{\rho}\frac{\partial f_m}{\partial x}\right)\Bigg|_{z=h_1^+} + \frac{\partial h_2^-}{\partial x}\left(\frac{f_n}{\rho}\frac{\partial f_m}{\partial x}\Bigg|_{z=h_2^-}\right)\right. \\
&+ \cdots + \frac{\partial h_N^-}{\partial x}\left(\frac{f_n}{\rho}\frac{\partial f_m}{\partial x}\right)\Bigg|_{z=h_2^-} - \frac{\partial h_N^+}{\partial x}\left(\frac{f_n}{\rho}\frac{\partial f_m}{\partial x}\right)\Bigg|_{z=h_N^+}).
\end{aligned}$$

So we can write α_{nm} in a more compact form as

$$\alpha_{nm} = -\left(\rho f_n, \frac{\partial}{\partial x}\left(\frac{1}{\rho}\frac{\partial f_n}{\partial x}\right)\right) + \sum_{j=1}^{N} \frac{\partial h_j}{\partial x}\left(\frac{f_n}{\rho}\frac{f_n}{\rho}\frac{\partial f_m}{\partial x}\right)\Bigg|_{z=h_j^-}^{z=h_j^+}. \qquad (3.3.33)$$

In equation (3.3.34), we have used the completeness relationship

$$\sum_{n=1}^{\infty} f_n(z,x) f_n(z_1,x) = \rho(x,z)\delta(z-z_1). \qquad (3.3.34)$$

To express the mode-coupling coefficients B_{nm} in terms of environmental gradients, we need to use the generalized orthogonality relationship which relates fields of eigenmodes in two different guides or cross-sections of the irregular waveguides

$$\int_0^{H(x)} \left[\frac{\partial f_n(z,x,y)}{\partial z} \frac{\partial f_m(z;x_1,y)}{\partial z} \left(\frac{1}{\rho(x_1,y,z)} - \frac{1}{\rho(x,y,z)} \right) + f_n(z,x,y) f_m(z;x,y) \times \right.$$
$$\left. \times \left(\frac{x_m^2(x_1) - k^2(x_1,y,z)}{\rho(x_1,y,z)} - \frac{\xi_n^2(x) - k^2(x,y,z)}{\rho(x,y,z)} \right) dz. \qquad (3.3.35)$$

Differentiating equation (3.3.35) with respect to x_1 and putting $x_1 = x$, we find

$$0 = \int_0^{h_1} + \cdots + \int_{h_N}^{H} dz \frac{\partial f_n}{\partial z}(z;x,y) \frac{\partial f_m(z,x,y}{\partial z} \frac{\partial}{\partial x}\left(\frac{1}{\rho}\right)$$
$$+ f_n(z;x_1,y) \frac{\partial}{\partial x} f_m(z;x,y) \left(\frac{\xi_m^2 - \xi_n^2}{\rho} \right)$$
$$+ f_n f_m \left[\left(\frac{\partial \xi_m^2}{\partial x} - \frac{\partial k^2}{\partial x} \right) \frac{1}{\rho} + (\xi_m^2 - k^2) \left(-\frac{1}{\rho^2} \frac{\partial \rho}{\partial x} \right) \right]$$
$$+ \sum_{j=1}^{N} \frac{\partial h_j}{\partial x} \left[\frac{1}{\rho} \frac{\partial f_m}{\partial z} \frac{\partial f_m}{\partial z} + \frac{k^2 - \xi_m^2}{\partial} f_n f_m \right] \Bigg|_{z=h^-}^{z=h^+}.$$

Or we can rewrite the above equation as

$$(\xi_m^2 - \xi_n^2) B_{mn} + \frac{\partial \xi^2}{\partial x} \delta_{nm} + \sum_{j=1}^{N} \frac{\partial h_j}{\partial x} \left[\frac{1}{\rho} \frac{\partial f_n}{\partial z} \frac{\partial f_m}{\partial z} + \frac{k^2 - \xi_m^2}{\rho} f_n f_m \right] \Bigg|_{z=h_j^-}^{z=h_j^+}$$
$$= \left(\frac{1}{\rho} \frac{\partial \rho}{\partial x} \frac{\partial f_n}{\partial z}, \frac{\partial f_m}{\partial z} \right) + \left(\left[\frac{\partial k^2}{\partial x} + \frac{\xi_m^2 - k^2}{\rho} \frac{\partial \rho}{\partial x} \right] f_n, f_m \right). \qquad (3.3.36)$$

If $n = m$, then equation (3.3.36) gives the derivative of the mode propagation constant $\frac{d\xi_n}{\partial x}$

$$\frac{d\xi_n}{\partial x} = -\frac{1}{2\xi_n} \sum_{j=1}^{N} \frac{\partial h_j}{\partial x} \left[\left(\frac{1}{\rho} \frac{\partial f_n}{\partial z} \right)^2 + \frac{k^2 - \xi_n^2}{\rho} f_n^2 \right] \Bigg|_{z=h_j^-}^{h_j^+}$$
$$+ \left(\frac{1}{\rho} \frac{\partial \rho}{\partial x} \frac{\partial f_n}{\partial z}, \frac{\partial f_n}{\partial z} \right) + \left(\left[\frac{\partial k^2}{\partial x} + \frac{\xi_n^2 - k^2}{\rho} \frac{\partial \rho}{\partial x} \right] f_n, f_n \right). \qquad (3.3.37)$$

If $n \neq m$, then from equation (3.3.36), we can write B_{nm} as

$$B_{nm} = (\xi_n^2 - \xi_m^2)^{-1} \left[\sum_{j=1}^{N} \frac{\partial h_j}{\partial x} \left(\frac{1}{\rho} \frac{\partial f_n}{\partial z} \frac{\partial f_m}{\partial z} + \frac{k^2 - \xi_m^2}{\rho} f_n f_m \right) \Bigg|_{z=h_j^-}^{z=h_j^+} \right.$$
$$\left. - \left(\frac{1}{\rho} \frac{\partial \rho}{\partial x} \frac{\partial f_n}{\partial z}, \frac{\partial f_m}{\partial z} \right) - \left(\left(\frac{\partial k^2}{\partial x} + \frac{\xi_m^2 - k^2}{\rho} \frac{\partial \rho}{\partial x} \right) f_n, f_m \right) \right] \quad (3.3.38)$$

which expresses B_{nm} in terms of waveguide effects.

From equation (3.2.20)

$$B_{nm} + B_{mn} = \int \frac{\partial z}{\rho} \frac{\partial}{\partial x} (f_n f_m)$$

or

$$B_{nm} + B_{mn} = \frac{\partial}{\partial x} \int \frac{dz}{\rho} f_n f_m - \left[\int f_n f_m \frac{\partial}{\partial x} \left(\frac{1}{\rho} \right) dz - \sum_{j=1}^{N} \frac{\partial h_j}{\partial x} \frac{f_n f_m}{\partial} \Bigg|_{z=h_j^-}^{z-h_j^+} \right]. \quad (3.3.39)$$

For $n = m$,

$$B_{nn} = \frac{1}{2} \frac{\partial}{\partial x} (\delta_{nn}) + \left(\frac{f_n}{2\rho} \frac{\partial \rho}{\partial x}, f_n \right) + \sum_{j=1}^{N} \frac{\partial h_j}{\partial x} \frac{f_n^2}{2\rho} \Bigg|_{z=h_j^-}^{z=h_j^+}. \quad (3.3.40)$$

Or we can write B_{nn} as

$$B_{nn} = \left(\frac{f_n}{2\rho} \frac{\partial \rho}{\partial x}, f_n \right) + \sum_{j=1}^{N} \frac{\partial h_j}{\partial x} \frac{f_n^2}{2\rho} \Bigg|_{z=h_j^-}^{z=h_j^+} \quad (3.3.41)$$

If the waveguide has a range-dependent boundary, then we assume that the boundary is at $z = h_N$ and take the limit $\rho \to 0$ or $\rho \to \infty$ at $L_N < z < H$. In this case integrals over $h_n < z < h$ equal zero and summands are zeros at $z = h_N^+$ in equations (3.3.38) and (3.3.41).

3.4 Numerical Discussion

As mentioned in Section 3.2, we rewrite the second-order system of order 2 given in Equation (3.2.18) as a first order system of differential equations of dimension $2N$. To do this, we set

$$q_m = \frac{dF_m}{dx} \tag{3.4.1}$$

and equation (3.2.18) becomes

$$\frac{dq_m}{dx} + k_m^2 F_m = -\sum_{n=1}^{N} (\alpha_m F_n + \beta_{mn} q_n) \tag{3.4.2}$$

where for the one-layer case (case under study) with free surface and rigid bottom conditions

$$\alpha_{mn} = \int_0^L \frac{1}{\rho} f_m(x,z) \frac{\partial^2 f_n}{\partial x^2}(x,z)dz + \frac{1}{\rho} f_m(x,h^-) \frac{\partial f_n}{\partial x}(x,h^-) \frac{\partial h}{\partial x} \tag{3.4.3}$$

and

$$\beta_{mn} = 2\int_0^L \frac{1}{\rho} f_m(x,z) \frac{\partial f_n}{\partial x}(x,z)dz + \frac{1}{\rho} f_n(x,h^-), f_m(x,h^-) \frac{\partial h}{\partial x}. \tag{3.4.4}$$

At the right ($x = x_{\text{left}}$) of the scattering region, we have

$$\begin{aligned} \frac{dF_n}{dx}(x_{\text{left}}) + ik_n(x_{\text{left}})F_n(x_{\text{left}}) &= q_n(x_{\text{left}}) + ik_n(x_{\text{left}})F_n(x_{\text{left}}) \\ &= 2ik_n\gamma_n e^{ik_n x_{\text{left}}} \end{aligned} \tag{3.4.5}$$

where γ_n are the incident coefficients.

At the left ($x = x_{\text{right}}$) of the scattering region, we have

$$q_n(x_{\text{right}}) - ik_n(x_{\text{right}})F_n(x_{\text{right}}) = 0. \tag{3.4.6}$$

Equations (3.4.1), (3.4.2), (3.4.5) and (3.4.6) determine a two-point boundary value problem for (F_m, q_m), $m = 1, \ldots, N$.

For one-layer problem with seabed having smooth undulation

$$h(x) = \begin{cases} 200 - \frac{a}{2}\left[1 + \cos\frac{2\pi x}{100}\right] & -50m < x < 50\,m \\ 200 & \text{elsewhere} \end{cases} ,$$

the eigenfunctions for the constant velocity, rigid waveguide are

$$f_m(x, z) = \sqrt{\frac{2}{h(x)}} \sin\left(\frac{(2n+1)\pi z}{2h(x)}\right).$$

One source of huge computation in source codes is loop iterations, which are common control statements used in scientific programs. Wolfe [24] and Wolf and Lam [23] have developed techniques to parallelize loop iterations using a network of workstations. In our shooting source code that solves the coupled mode system, each iteration in several loops takes substantial time to finish, hence parallelization of the program is demanding.

The Linear Shooting Method

Physical problems that are position-dependent are often demonstrated in terms of differential equations with boundary conditions imposed at more than one point. The two-point boundary-value problem discussed in this work involves a second-order differential equation of the form

$$y'' = f(x, y, y') \quad a \leq x \leq b \tag{3.4.7}$$

together with the boundary conditions $y(a) = \alpha$ and $y(b) = \beta$.

The following theorem gives general conditions that guarantee the existence and uniqueness of the solution to a second-order boundary value problem (Burden and Faires [6]).

Theorem 3.4.1. *Suppose the function f in the boundary-value problem*

$$\begin{gathered} y'' = f(x, y, y') \quad a \le x \le b \\ y(a) = \alpha \\ y(b) = \beta \end{gathered}$$

is continuous on the set

$$D = \{(x, y, y') | a \le x \le b, \quad -\infty < y < \infty, \ -\infty < y' < \infty\},$$

and that $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial y'}$ are also continuous on D. If

(i) $\frac{\partial f}{\partial y}(x, y, y') > 0$ for all $(x, y, y') \in D$, and

(ii) a constant M exists, with $\left|\frac{\partial f}{\partial y'}(x, y, y')\right| \le M$ for all $(x, y, y') \in D$, then the boundary-value problem has a unique solution.

The differential equation

$$y'' = f(x, y, y')$$

is called linear if

$$f(x, y, y') = p(x)y' + q(x)y + r(x).$$

The next corollary ensures a unique solution for the linear boundary-value problem.

Corollary 3.4.2. *If the linear boundary-value problem*

$$y'' = p(x)y' + q(x)y + r(x) \quad a \le x \le b \tag{3.4.8}$$

with

$$y(a) = \alpha$$
$$y(b) = \beta,$$

satisfies

(i) $p(x), q(x)$ *and* $r(x)$ *are continuous on* $[a, b]$,

(ii) $q(x) > 0$ *on* $[a, b]$,

then the problem has a unique solution.

If we consider the initial-value problems

$$y'' = p(x)y' + q(x)y + r(x), \quad a \leq x \leq b \tag{3.4.9}$$

with

$$y(a) = \alpha$$
$$y(b) = 0$$

and

$$y'' = p(x)y' + q(x)y, \quad a \leq x \leq b \tag{3.4.10}$$

with

$$y(a) = 0$$
$$y'(a) = 1,$$

then

$$y(x) = y_1(x) + \frac{\beta - y_1(b)}{y_2(b)} y_2(x) \tag{3.4.11}$$

is the unique solution to our boundary-value problem with $y_2(b) \neq 0$. $y_1(x)$ *and* $y_2(x)$ *are the solutions to equations (3.4.9) and (3.4.10), respectively.*

The shooting method for linear equations is based on the replacement of the boundary-value problem (3.2.18) by the two initial-value problems (3.4.9) and (3.4.10). There are several methods, one of which is the fourth-order Runge-Kutta technique, to find the approximations to $y_1(x)$ and $y_2(x)$. Figure (3.4.1) illustrates the solution method of the boundary-value problem.

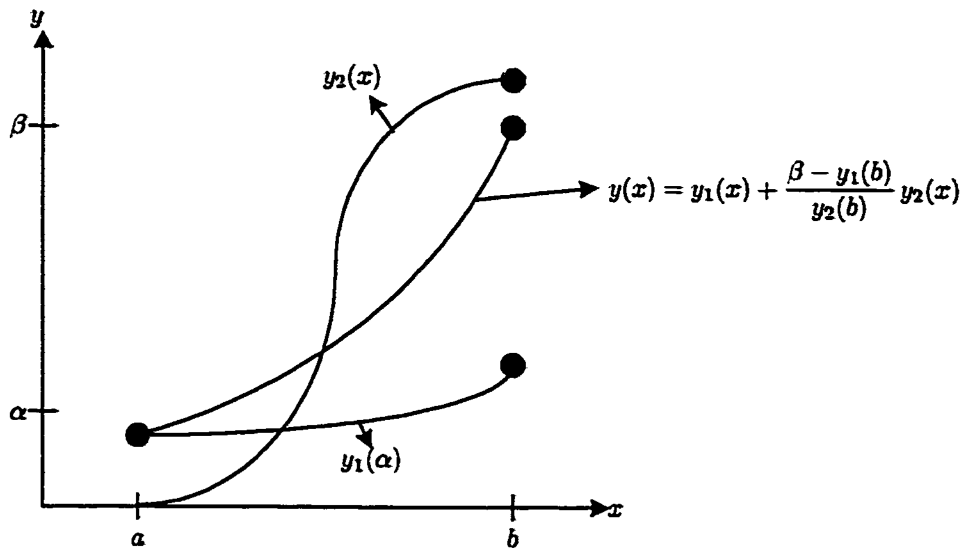

Figure 3.4.1. Solution method of the boundary value problem

In Figure 3.4.2, the pressure amplitude using 2 modes is shown for the perturbed and horizontal one-layer model. The solid curve represents the perturbed bottom model, and the dotted curve represents the flat sea bottom. The two curves almost coincide. Figures 3.4.3 and 3.4.4 show the same case as in Figure 3.4.2, but using 4 and 8 modes. There are visible differences between the two curves.

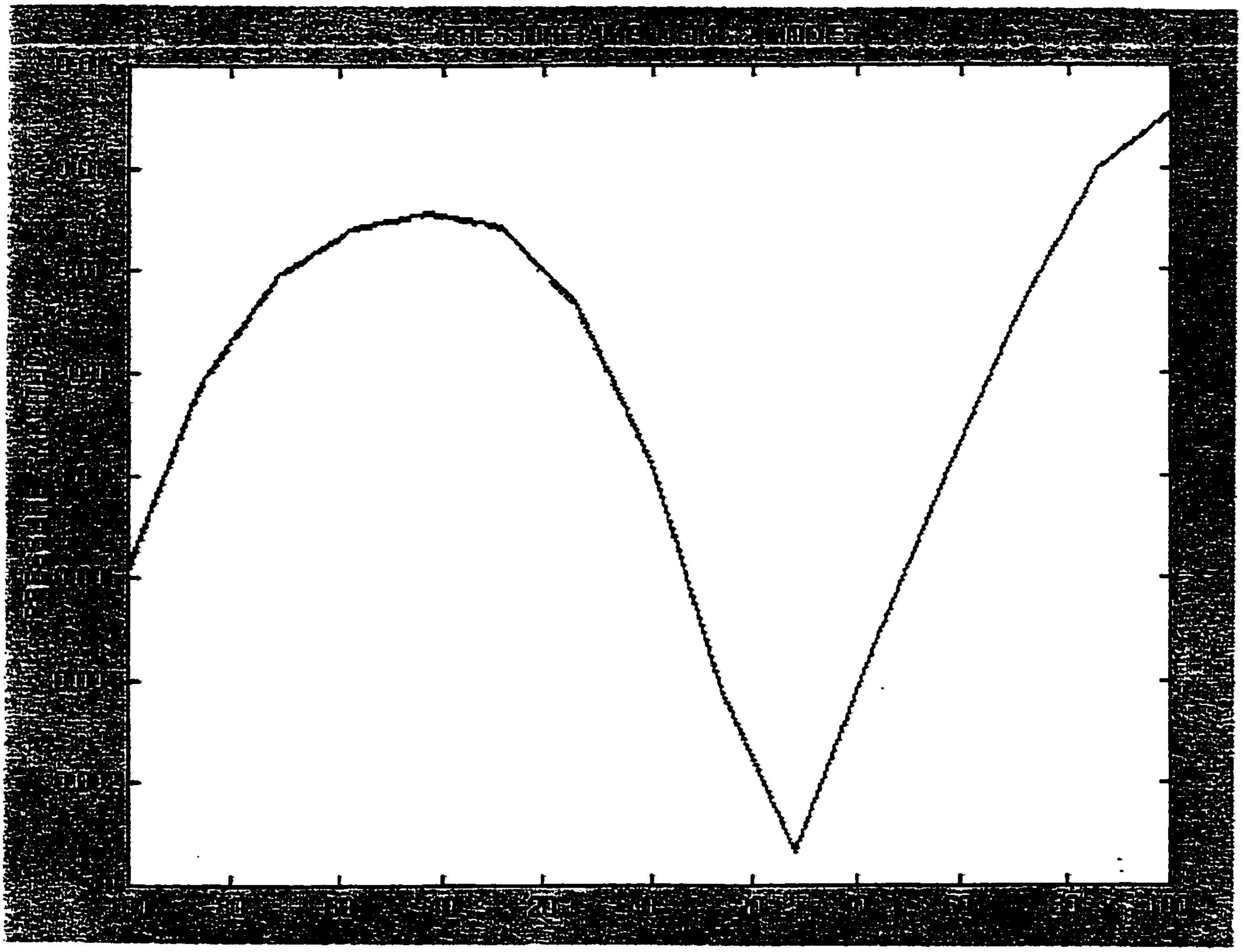

Figure 3.4.2 : Pressure amplitude for horizontal (solid) and perturbed (dashed) one layer model using 2 modes.

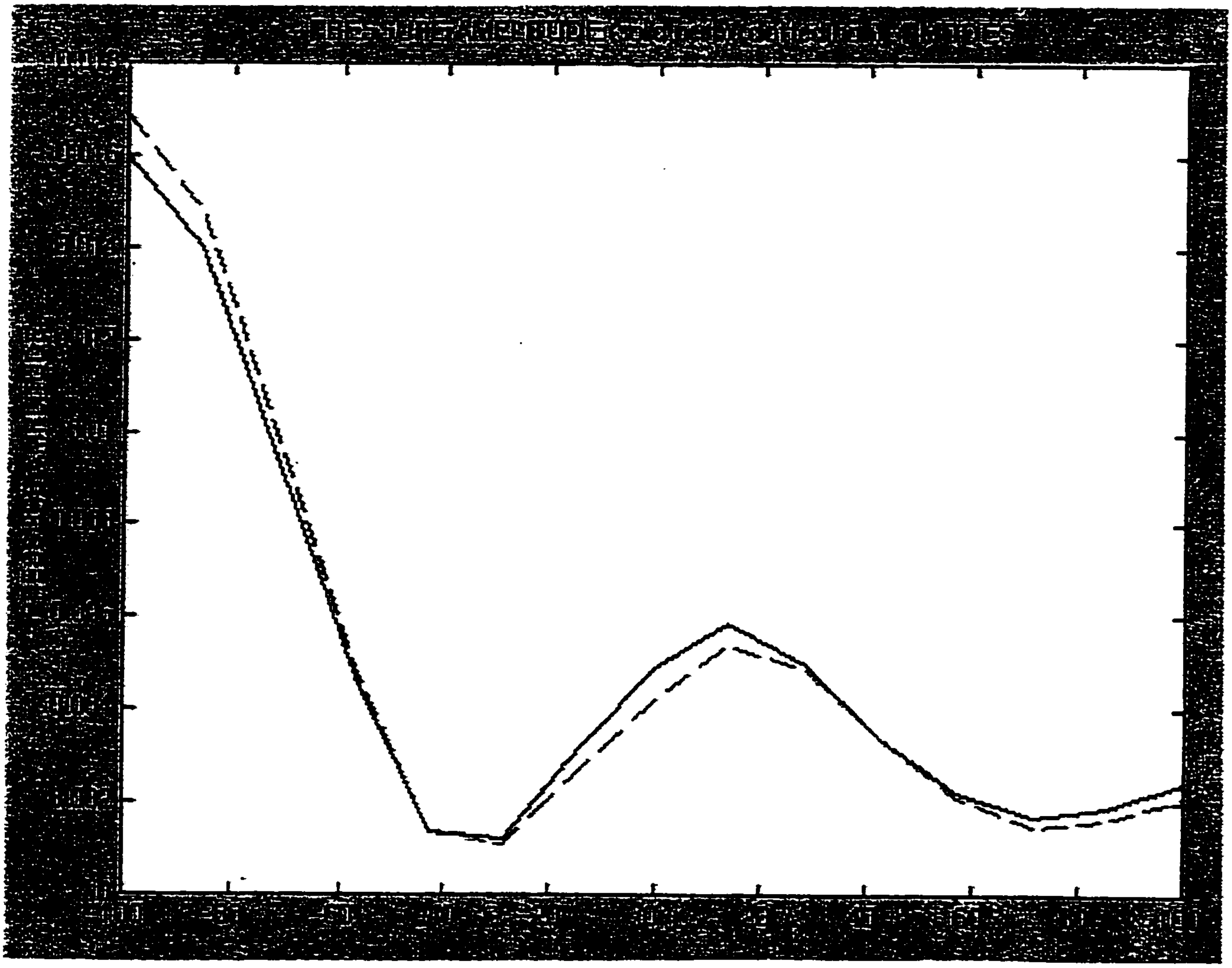

Figure 3.4.3 : Pressure amplitude for horizontal (solid) and perturbed (dashed) one layer model using 4 modes.

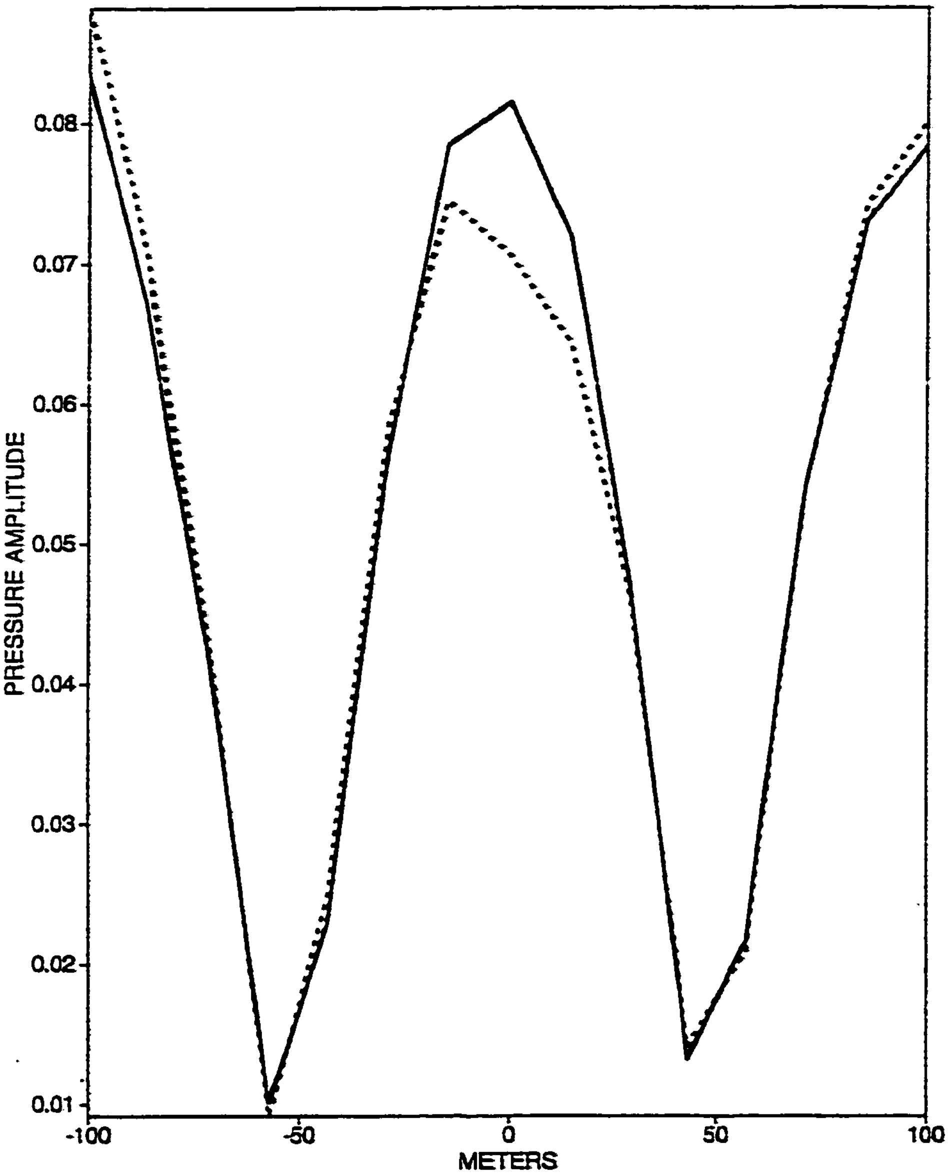

Figure 3.4.4 : Pressure amplitude for horizontal (solid) and perturbed (dashed) one layer model using 8 modes.

CHAPTER IV

ESTIMATION OF SCATTERING BY MODAL COHERENCE

4.1 Introduction

The coherence function associated with propagating acoustic signals in shallow waveguide has been studied, among authors, by Beran and Frankenthal [2, 3], Frankenthal and Beran [10], and McDaniel and McCammon [14] through the modal method. The approach is favorable as it accounts for boundary conditions and the scattering at the boundary. The modal approach basically depends on a modal decomposition of the propagating pressure to derive the modal coherence and hence obtain the overall coherence of the signal. One can think of the modal coherences of the signal as a matrix with the diagonal elements representing the self-modal coherence functions, and the off-diagonal entries representing the cross-modal coherence functions corresponding to two different modes or eigenvalues. In the absence of scattering, only the self-modal coherence functions survive and propagate independently. The overall coherence of the acoustic signal can be computed as a weighted sum of the self- and cross-modal coherence functions. The contribution of the cross-modal coherence functions to the overall coherence is negligible at sufficiently large distances from the source. In Section (4.2), we follow Beran and Frankenthal [2] to derive the scattering coefficients when only volume scattering is involved. Volume scattering effects can be introduced when the sound speed field has random fluctuations about the depth-dependent mean profile. In Section (4.3), we use the reciprocity principle to derive the scattering coefficients when both the volume and surface scattering, in addition to variable density and rigid sea bottom conditions, are involved. In Section (4.4), the range evolution of the transverse horizontal spectra of the modal field coefficients is formulated.

4.2 Problem Formulation

The propagation of acoustic pressure in oceanic medium is governed by the Helmholtz equation as in equation (3.2.9). For a sound speed field that has random fluctuations about a depth-dependent mean profile, we have

$$k^2(\underline{x}) = k^2(z)\left[1 + \mu(\underline{x})\right]$$

where $k(z)$ is the mean wavenumber and $\mu(x)$ is the relative variation about the mean. The modal decomposition solution of the propagation equation is

$$p(\underline{x}) = \sum_{j=1}^{\infty} F_i(r) f_j(z). \tag{4.2.2}$$

The vertical modal eigenfunctions $f_i(z)$ satisfy the following:

$$\left(\frac{d^2}{dz^2} + k^2(z) - \alpha_i^2\right) f_i(z) = 0 \tag{4.2.3}$$

and the boundary conditions are

i) $$f_i\Big(s(x)\Big) = 0. \tag{4.2.4}$$

ii) $$\frac{df_i(d)}{dz} = 0. \tag{4.2.5}$$

Substituting the expansion in equation (4.2.2) in equation (3.2.9) and using the depth-dependent differential equation in (4.2.3), the modal pressures satisfy

$$\left(\frac{\partial^2}{\partial r^2} + \alpha_i^2\right) F_i(r) = -\sum_{j=1}^{\infty} \mu(r)_{ij} F_j(r) \tag{4.2.6}$$

where

$$\mu(r)_{ij} = \int_s^d k^2(z)\mu(x) f_i(z) f_j(z) ds. \tag{4.2.7}$$

The ensemble-averaged coherence function is defined as

$$\Gamma^t(x, y_1, z_1, y_2, z_2) = \langle p(x, y_1, z_1) p^*(x, y_2, z_2)\rangle \tag{4.2.8}$$

where the bracket symbol denotes the ensemble averaging, and $*$ denotes the complex conjugation. The total coherence can be expressed as a weighted sum of the self modal coherences and cross-modal coherences.

We assume that the modal amplitude can be written as

$$F_i = \tilde{F}_i e^{i\alpha_i}. \tag{4.2.9}$$

Substituting equation (4.2.2) in equation (4.2.8) and using equation (4.2.9) gives

$$\Gamma^t(x, y_1, z_1, y_2, z_2) = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} \Gamma_{ij}(x, y_1, y_2) f_i(z) f_j(z) e^{i(\alpha_i - \alpha_j)x} \tag{4.2.10}$$

where

$$\Gamma_{ij}(x, y_1 y_2) = \langle \tilde{F}_i(x_1, y_1) \tilde{F}_j^*(x, y_2)\rangle. \tag{4.2.11}$$

The phase terms appearing in the cross-modal coherences oscillate rapidly at large distances, and this justifies ignoring these terms.

Using the parabolic approximation, equation (4.2.6) can be rewritten as

$$i2\alpha_i \frac{\partial \tilde{F}_i}{\partial x} + \frac{\partial^2 \tilde{F}_i}{\partial y^2} = \sum \tilde{F}_i \mu(r)_{ij} e^{i(\alpha_i - \alpha_j)x}. \tag{4.2.12}$$

If we assume that the scattering is negligible, the coherence can be given by the diffractive equation

$$\frac{\partial}{\partial x} F_{ij}(x, y_1, y_2) = \frac{i}{2} \left(\frac{1}{\alpha_i} \frac{\partial^2}{\partial y_1^2} - \frac{1}{\alpha_j} \frac{\partial^2}{\partial y_2^2} \right) \Gamma_{ij}(x, y_1, y_2). \tag{4.2.13}$$

One can follow the work by Beran and Frankenthal [2] in determining the volume scattering coherence function which is the solution of the equation (4.2.10) iteratively using Green's function of the equation (4.2.12)

$$G(r, r') = -\frac{\sqrt{2}}{4\sqrt{r}} \frac{H(x - x')}{\sqrt{\alpha_i (x - x')}} e^{i\left[\frac{\alpha_i}{2} \frac{(y - y')^2}{x - x'} + \frac{\pi}{4}\right]}. \tag{4.2.14}$$

$\tilde{F}_i$ at the plane $x + \Delta x$ is defined as

$$\tilde{F}_i(x + \Delta x, y) = \tilde{F}_i^o(x + \Delta x, y) + \Delta \tilde{F}_i \tag{4.2.15}$$

where $\Delta \tilde{F}_i$ is the correction term due to the scattering and $\tilde{F}_i^o$ is the scattering free pressure. The total coherence function

$$\Gamma_{ij}(x + \Delta x, y_1, y_2) = \Gamma_{ij}^o + \Delta \Gamma_{ij} \tag{4.2.16}$$

where $\Delta \Gamma_{ij}$ is the scattering term.

Equation (4.2.13) can be rewritten as

$$\Gamma_{ij}(x + \Delta x, y_1, y_2) = \Gamma_{ij}(x, y_1, y_2) + \Delta x \frac{\partial}{\partial x} \Gamma_{ij}(x, y_1, y_2) + \Delta \Gamma_{ij}. \tag{4.2.17}$$

If we confine the propagation to small angles around the x direction and between x and $x + \Delta x$, where Δx is small compared to Fresnel length, but large compared to the wavelength, then the scattering Sc can be defined as

$$\mathrm{Sc} = \lim_{\Delta x \to 0} \frac{\Delta\Gamma_{ij}}{\Delta x} \tag{4.2.18}$$

where $\Delta\Gamma_{ij}$ can be written as

$$\Delta\Gamma_{ij} = \langle \tilde{F}_i^{o} \Delta\tilde{F}_j \rangle + \langle \tilde{F}_j^{o*} \Delta\tilde{F}_i \rangle + \langle \Delta\tilde{F}_i \Delta\tilde{F}_j^{*} \rangle. \tag{4.2.19}$$

$\Delta\tilde{F}(r)$ can be determined in terms of Green's function as

$$\begin{aligned}\Delta\tilde{F}(r) &= \sum_k \int_A dr' G(r,r')\mu_{ij}(r')\tilde{F}_k^{o}(r)e^{i(\alpha_k-\alpha_i)x} \\ &+ \sum_m \sum_k \int_A \int_A dr' dr'' G(r,r')G(r,r'')\mu_{ik}(r')u_{km}(r'') \times \\ &\times \tilde{F}^{o}(r'')e^{i[(\alpha_k-\alpha_i)x'+(\alpha_m-\alpha_k)x'']}.\end{aligned} \tag{4.2.20}$$

where

$$\int_A dr' = \int_z^{z+\Delta z} dz' \int_{-\infty}^{\infty} dx'. \tag{4.2.21}$$

The coherence correction term $\Delta\Gamma_{ij}$ can be obtained as

$$\begin{aligned}\Delta\Gamma_{ij} = \sum_k \sum_m \int_A \int_A dr' dr'' \Big[& G_i(r_1,r')G_j^{*}(r_2,r'')\sigma_{ikjm}(r'-r'') \\ & \Gamma_{km}^{o}(r',r'')e^{i[(\alpha_k-\alpha_i)x'+(\alpha_m-\alpha_j)x'']} \\ & + G_i(r_1,r')G_k(r',r'')\sigma_{ikkm}(r-r') \times \\ & \times \Gamma_{mj}^{o}(r',r_2)e^{i[(\alpha_k-\alpha_i)x'+(\alpha_m-\alpha_k)x'']} + G_j^{*}(r_2,r')G_k^{*}(r',r'')\sigma_{jkkm}(r'-r'') \times \\ & \times \Gamma_{im}^{o}(r_1,r'')e^{-i[(\alpha_k-\alpha_j)x'+(\alpha_m-\alpha_k)x'']} \Big]\end{aligned} \tag{4.2.22}$$

where $\sigma(r, r')$ is defined as

$$\sigma_{ijkm}(r', r'') = \langle \mu_{ij}(r')\mu_{km}(r'') \rangle$$

$$= \int_s^h \int_s^h dz' dz'' k^2(z') k^2(z'') \sigma(y', y'') \times$$

$$\times f_i(z') f_j(z') f_k(z'') f_m(z''). \qquad (4.2.23)$$

Beran and Frankenthal [2] used the saddle point method to rewrite the integrals over y' and y'' in equation (4.2.22) as

$$\Delta\Gamma_{ij} = \frac{1}{4} \sum_k \sum_k \int_z^{z+\Delta z} \int_z^{z+\Delta z} dx' dx'' \left(\frac{1}{\alpha_i \alpha_j} \sigma_{ikjm}(x' - x'', y_1 - y_2) \times \right.$$

$$\times \Gamma^o_{km}(x', x'', y_1, y_2) ei \Big(\alpha_k - \alpha_i) x' - (\alpha_m - \alpha_j) x'' \Big]$$

$$-H(x' - x'') \frac{1}{\alpha_i \alpha_k} \sigma_{ikkm}(x', x'', 0) \Gamma^o_{mj}(x + \Delta x, x', y_1, y_2) \times$$

$$\times e^{i[(\alpha_k - \alpha_i)x' + (\alpha_m - \alpha_k)x'']}$$

$$-H(x' - x'') \frac{1}{\alpha_j \alpha_k} \sigma_{jkkm}(x' - x'', 0) \Gamma^o_{im}(x, \Delta x, x'', y_1, y_2) \times$$

$$\times e^{-i[(\alpha_k - \alpha_j)x' + (\alpha_m - \alpha_k)x'']} \qquad (4.2.24)$$

where $H(x - x'')$ is the Heaviside function.

For forms involving $\dfrac{\alpha(y' - y'')^2}{(x' - x'')}, k\ell_H \gg 1$. For other forms $\Delta x \ll k\ell_H^2$, where ℓ_H is the horizontal correlation length. If we set $x' = x'' = x$, then

$$\Gamma^o_{ij}(x', x'', y_1, y_2) = \Gamma_{ij}(x, y_1, y_2). \qquad (4.2.25)$$

If we assume the following transformation variables

$$u_x = \frac{1}{2}(x' + x'') - x \quad \text{and} \quad s_x = x' - x'', \qquad (4.2.26)$$

then

$$\int_{x}^{x+\Delta x} dx' \int_{x}^{x+\Delta x} dx'' \to \int_{0}^{\Delta x} du_x \int_{-\infty}^{\infty} ds_x. \tag{4.2.27}$$

Using the variable transformations, the scattering term Sc in equation (4.2.26) becomes

$$\begin{aligned} \mathrm{Sc} = & \frac{\delta_{ij}}{4\alpha_i^2} \sum \Gamma_{kk} \int_{-\infty}^{\infty} ds_x e^{i(\alpha_k-\alpha_i)s_x} \sigma_{ikik}(s_x, y_1 - y_2) \\ & + \frac{(1-\delta_{ij})}{4\alpha_i\alpha_j} \int_{-\infty}^{\infty} ds_x \sigma_{iijj}(s_x, y_1 - y_2) \\ & - \Gamma_{ij} \sum_{k=1}^{\infty} \frac{1}{4\alpha_i\alpha_k} \int_0^{\infty} ds_x e^{i(\alpha_k-\alpha_i)s_x} \sigma_{ikki}(s_x, 0). \\ & - \Gamma_{ij} \sum_{k=1}^{\infty} \frac{1}{4\alpha_j\alpha_k} \int_0^{\infty} ds_x e^{-i(\alpha_k-\alpha_j)s_x} \sigma_{jkkj}(s_x, 0). \end{aligned} \tag{4.2.28}$$

From equation (4.2.27), the self modal equation becomes

$$\left[\frac{\partial}{\partial x} - \frac{i}{2\alpha_i}\left(\frac{\partial^2}{\partial y_1^2} - \frac{\partial^2}{\partial y_2^2}\right) + S_{ii}\right]\Gamma_{ii} + \sum_{k \neq i}^{\infty} S_{ik}\Gamma_{ii} = 0, \tag{4.2.29}$$

where S_{ii} are the scattering self modal coefficients given by

$$\begin{aligned} S_{ii}(y_1 - y_2) = & \frac{1}{2\alpha_i^2} \int_0^{\infty} ds_x \left[\sigma_{iiii}(s_x, 0) - \sigma_{iiii}(s_x, y_1 - y_2)\right) \\ & + \frac{1}{2\alpha_i} \sum_{k \neq i}^{\infty} \frac{1}{\alpha_k} \int_0^{\infty} ds_x \cos(\alpha_k - \alpha_i)s_x \sigma_{ikkki}(s_x, 0) \end{aligned} \tag{4.2.30}$$

and S_{ik} the cross-modal coefficients appearing in the self modal equations (4.2.28) are given by

$$S_{ik}(y_1 - y_2) = -\frac{1}{2\alpha_i^2} \int_0^{\infty} ds_x \cos(\alpha_k - \alpha_i)s_x \sigma_{ikik}(s_x, y_1 - y_2) \tag{4.2.31}$$

The equation for the cross-modal coherence is

$$\left[\frac{\partial}{\partial x} - \frac{i}{2}\left(\frac{1}{\alpha_i}\frac{\partial^2}{\partial y_1^2} - \frac{1}{\alpha_j}\frac{\partial^2}{\partial y_2^2}\right) + C_{ij}\right]\Gamma_{ij} = 0, \tag{4.2.32}$$

where the cross-modal scattering coefficients C_{ij} is given by

$$C_{ij}(y_1 - y_2) = \frac{1}{4}\left(\sum_{k=1}^{\infty}\frac{1}{\alpha_i\alpha_k}\int_0^{\infty} ds_x \sigma_{ikki}(s_x, 0) e^{i(\alpha_k - \alpha_i)s_x}\right.$$
$$+\sum_k \frac{1}{\alpha_j\alpha_k}\int_0^{\infty} ds_x \sigma_{jkkj}(s_x, 0) e^{-i(\alpha_k - al_j)s_x}$$
$$\left. -\frac{2}{\alpha_i\alpha_j}\int_0^{\infty} ds_x \sigma_{iijj}(s_x, y_1 - y_2)\right] \tag{4.2.33}$$

Terms in $C_{ij}(y_1 - y_2)$ that involve integrals of exponential terms where $\alpha_k - \alpha_j \neq 0$ and $\alpha_k - \alpha_i \neq 0$, are negligible at large values of $\alpha_k - \alpha_j$ and $\alpha_k - \alpha_i$. That is

$$(\alpha_k - \alpha_i)\,\ell_H \gg 1 \quad k \neq i \tag{4.2.34}$$

The cross-modal scattering coefficients are negligible at large distances due to large oscillations which have zero integrals Beran and Frankenthal [2] give more rigorous proof to rationalize ignoring the scattering cross-modal coefficients. In Section 4.3, we shall include the environmental effect in the same model and derive the mode-scattering coefficients for both of the volume and surface-scattering effects.

4.3 Volume and Surface Scattering

In this work, we study the combined effects of volume and surface scattering of a rough channel surface. We consider a model with variable density as well as variable surface and bottom boundary. The sound propagation is studied in terms of the ensemble-averaged two-point coherence function. The conventional technique which relies on the modal decomposition of the pressure which is initiated by Pierce [19] and used by many other authors in the mode-coupling problems fails to handle this case. To derive the coherence functions for this case, reciprocity principle is employed

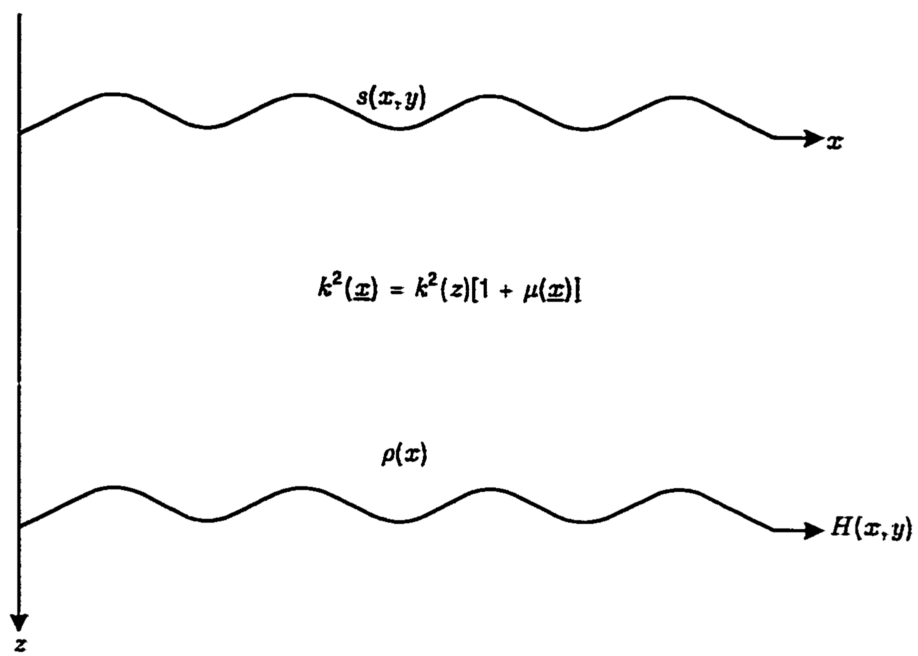

Figure 4.3.1. One-layer model with random wavenumber, variable density with range, and smooth undulation in the surface and bottom.

as discussed in Chapter three.

In the previous section, where the volume scattering is only assumed,

$$\hat{p}(\underline{x}) = \sum_{i=1}^{\infty} \tilde{F}_i(x,y) f_i(z) \tag{4.3.1}$$

and

$$\hat{p}(\underline{x}) = \sum_{i=1}^{\infty} \tilde{F}_i(x,y) f_i(x,y,z) \tag{4.3.2}$$

when a rough surface is included.

The orthogonal eigenfunctions $f_i(x,y,z)$ satisfy the depth-dependent equation. The model geometry is shown in Figure 4.3.1. The boundary conditions are

1) $\hat{p}(s) = 0$

2) $H(x,y) \left. \dfrac{\partial \hat{p}(z)}{\partial n} \right|_{z=H(x,y)} = 0$

with

$$\begin{array}{ll} \text{wavenumber}: & k^2(\underline{x}) = k^2(z)[1+\mu(\underline{x})] \\ \text{density}: & \rho(\underline{x}) \end{array}$$

We use the normalization

$$\int_{S(x,y)}^{(H(x,y)} Y_i^2(x,y,z)dz = H - S. \tag{4.3.3}$$

We will not use the decomposition technique as used by many authors as this will lead to erroneous results; instead the reciprocity principle is adopted. In order to avoid undue repetition, all the details given in Chapter three will not be repeated. The mode amplitude $\tilde{F}_j(x,y)$ satisfy the simultaneous coupling equations

$$\nabla_T^2 \tilde{F}_j(x,y) + \alpha_j^2(x,y)\tilde{F}_j(x,y) = -\sum_{i=1}^{\infty} \nabla_T \tilde{F}_i(x,y) \cdot \nu_{ij}(x,y)$$

$$-\sum_{n=1}^{\infty} \tilde{\Gamma}_i(x,y)\left[\omega_{ij}(x,y) + \mu_{ij}(x,y)\right]. \tag{4.3.4}$$

where

$$\nu_{ij}(x,y) = 2B_{ij} - \left(\frac{1}{\rho(x,y)}(\nabla_T \rho)\, f_i, f_j\right) - \nabla_T h_1 \left.\frac{f_i f_j}{\rho}\right|_{z=h_1^-}^{z=h_1^+} \tag{4.3.5}$$

where $s < h_1 < H$.

$$\begin{array}{ll} h_1^+ & \text{is greater than } H-S \\ h_1^- & \text{is less than } H-S \end{array}$$

and

$$B_{ij} = \frac{1}{H(x,y)-S(x,y)} \int_{S(x,y)}^{H(x,y)} (\nabla_T f_i(x,y,z) f_j(x,y,z)dz \tag{4.3.6}$$

$$\omega_{ij} = \frac{1}{H(x,y)-S(x,y)}\left(\rho f_i, \nabla_T\left(\frac{1}{\rho}\nabla_T h_1\right)\right) - \nabla_T h_1 \left.\left(\frac{1}{\rho} f_i \nabla_T f_j\right)\right|_{z=h_1^-}^{z=h_1^+}$$

$$= \omega_{xij} + \omega_{yij} + \omega_\rho + \omega_h \tag{4.3.7}$$

where $\omega_{xij}, \omega_{yij}$ and ω_p are the Laplacian and density gradient from the first term in equation (4.3.7) and ω_h is the thickness gradient term from the second term in equation (4.3.7).

$$\mu_{ij} = \frac{1}{H(x,y) - S(x,y)} \int_S^H k^2(z)\mu(\underline{x}) f_i(x,y,z) f_j(x,y,z) dz. \qquad (4.3.8)$$

The scattering functions ν_{ij} and ω_{ij} in equations (4.3.5) and (4.3.77) represent the effects of scattering from the rough surface and the scattering term μ_{ij} in equation (4.3.8) represents the effect of volume scattering. Using the approach given by Boyles [4] and used by Beran and Frankenthal [3] and Godin [11] in back-scattering treatment

$$\frac{\partial F^+}{\partial x} - i\xi_{mA} F^+ = \sum_{n=1}^{\infty} \nu_{xnm} F^+ \qquad (4.3.9)$$

$$\frac{\partial F^-}{\partial x} + i\xi_{mA} F^- = \sum_{n=1}^{\infty} \nu_{xnm} F^- \qquad (4.3.10)$$

where

$$\bar{F} = F_m^+ + F_m^- \qquad (4.3.11)$$

$$\bar{\nu}_m = \xi_{mA}(F_m^+ - F_m^-) \qquad (4.3.12)$$

and ν_{xnm} is the x component of ν_{nm} and ξ_{mA} is the eigenvalue.

Substituting equations (10) and (11) in (4)

$$\frac{i\partial \nu_m}{\partial x} + \xi_m^2 \bar{F}_m + i \sum_{n=1}^{\infty} \bar{\nu}_i \frac{\nu_{xmn}}{2} - \frac{1}{H - S)} \frac{\partial (H-S)}{\partial x} \sum_{n=1}^{\infty} \bar{F}_n \nu_{xnm}$$

$$+ \left(\frac{\partial^2 F_m}{\partial y^2} + \sum_{n=1}^{\infty} \bar{F}_n \left(\omega_{ynm} + \omega_\rho + \omega_h + \mu_{nm}\right) + \sum_{n=1}^{\infty} \frac{\partial \bar{F}_n}{\partial y} \nu_{ynm} \right) = 0. \qquad (4.3.13)$$

Substituting equations (4.3.9) – (4.3.10) in equation (4.3.13) and multiplying equations (4.3.9) and (4.3.10) with ξ_{mA} and adding, we get

$$\begin{aligned}
&2i\xi_{mA}\frac{\partial F_m^+}{\partial x}+\xi_{mA}^2\left(F_m^+-F_m^-\right)+\xi_m^2\left(F_m^++F_m^-\right)\\
&\quad+i\sum_{n=1}^{\infty}\left(\xi_{mA}+\xi_{nA}\right)F_n^+\frac{\nu_{xnm}}{2}+\frac{\partial^2}{\partial y^2}\left(F_m^++F_m^-\right)\\
&\quad+\sum_{n=1}^{\infty}\frac{\partial}{\partial y}\left(F_n^++F_n^-\right)\nu_{ynm}+\sum_{n=1}^{\infty}\left(F_n^++F_n^-\right)\left(\omega_{ynm}+\omega_{pnm}+\omega_{hnm}+\mu_{nm}\right)\\
&\quad-\left(\frac{1}{H-S}\frac{\partial(H-S)}{\partial x}\right)\sum_{n=1}^{\infty}\left(F_n^++F_n^-\right)\nu_{xnm}\\
&\quad+i\sum_{n=1}^{\infty}\left(\xi_{mA}-\xi_{nA}\right)F_n^-\frac{\nu_{xnm}}{2}=0. \qquad (4.3.14)
\end{aligned}$$

Similarly F_n^- can be obtained if the difference between the equations is taken.

If we assume that the back-scattered energy is negligible, then we can set $F_n^-=0$, and we have

$$\begin{aligned}
&2i\xi_{mA}\frac{\partial F_m^+}{\partial x}+\xi_{mA}^2\left(F_m^+\right)+\xi_m^2\left(F_m^+\right)+i\sum_{n=1}^{\infty}\left(\xi_{mA}+\xi_{nA}\right)F_n^+\left(\frac{\nu_{xnm}}{2}\right)\\
&\quad+\frac{\partial^2}{\partial y^2}\left(F_m^+\right)+\sum_{n=1}^{\infty}\frac{\partial}{\partial y}\left(F_n^+\right)\nu_{ynm}+\sum_{n=1}^{\infty}F_n^+\left(\omega_{ynm}+\omega_{pnm}+\omega_{hnm}+\mu_{nm}\right)\\
&\quad-\frac{1}{(H-S)}\frac{\partial(H-S)}{\partial x}\sum_{n=1}^{\infty}F_n^+\nu_{xnm}=0. \qquad (4.3.15)
\end{aligned}$$

If we set $F_m^+=\rho_m\exp\left[i\xi_{mA}x\right]$, then equation (4.3.15) becomes

$$2i\xi_{mA}\frac{\partial p_m}{\partial x}+\left[\xi_m^2-\xi_{mA}^2\right]p_m+i\sum_{n=1}^{\infty}(\xi_{mA}+\xi_{nA})\,p_n\left[\exp\left(i\left(\xi_{nA}-\xi_{mA}\right)x\right]\left(\frac{\nu_{xnm}}{2}\right)\right.$$

$$\left.+\frac{\partial^2}{\partial y^2}p_m+\sum_{n=1}^{\infty}\frac{\partial}{\partial y}(p_n)\exp i\left(\xi_{nA}-\xi_{mA}\right)x\right]\nu_{ynm}$$

$$+\sum_{n=1}^{\infty}p_n\left[\exp i\left(\xi_{nA}-\xi_{mA}\right)x\right]\left(\omega_{ynm}+\omega_{\rho nm}+\omega_{hnm}+\mu_{nm}\right)$$

$$-\frac{1}{H-S}\frac{\partial(H-S)}{\partial x}\cdot\sum_{n=1}^{\infty}p_n\left[\exp i\left(\xi_{nA}-\xi_{mA}\right)x\right]\nu_{xnm}=0. \tag{4.3.16}$$

If ν_{xnm} and ν_{ynm} are of the same order, the term $\frac{\partial p_n}{\partial x}$ may be neglected if $\frac{1}{\xi_{nA}}\frac{\frac{\partial p_n}{\partial x}}{p_n}\ll 1$ or $L_{py}\leq\xi_{nA}$ where L_{py} is a characteristic length associated with the variation of p_y in y direction. This implies that the horizontal angular spread of the radiation is small.

If $\frac{\partial p_n}{\partial y}=0$, then equation (4.3.16) becomes

$$2i\xi_{mA}\frac{\partial p_m}{\partial x}+\frac{\partial^2 p_m}{\partial y^2}=-\sum_n\gamma_{mn}(x,y)p_n(x,y)\exp\left[i\left(\xi_{nA}-\xi_{mA}\right)x\right], \tag{4.3.17}$$

where

$$\gamma_{mn}=\left(\xi_{mA}^2-\xi_m^2\right)\delta_{mn}+i\left[\xi_{mA}+\xi_{nA}\right]\frac{\nu_{xnm}}{2}$$
$$-\frac{1}{H-S}\frac{\partial(H-S)}{\partial x}\nu_{xnm}+\omega_{ynm}+\omega_{\rho nm}+\omega_{hnm}+\mu_{nm}, \tag{4.3.18}$$

or

$$\gamma_{mn}(x,y)=\mu_{mn}^s+\mu_{mn}, \tag{4.3.19}$$

where μ_{mn}^s represents the surface-scattering effect, and μ_{mn} represents the volume-scattering effect as mentioned before.

The equations for the coherence functions $\langle F_{mn}\rangle$ can be derived as done in the previous section

$$\left[\frac{\partial}{\partial x}-\left(\frac{i}{2\xi_{mA}}\right)\left(\frac{\partial^2}{\partial y_1^2}-\frac{\partial^2}{\partial y_2^2}\right)+S_1^sC_{mm}\right]\langle F_{mm}\rangle+\sum_{n\neq m}S_2^sC_{mn}\langle F_{nn}\rangle=0, \quad (4.3.20)$$

where the scattering coefficients

$$S_1^sC_{mm}(y_1-y_2)=\frac{1}{2\xi_{mA}^2}\int_0^\infty ds_z\left[\sigma^s_{mmmm}(0,s_z)-\sigma^s_{mmm}(y_1-y_2,s_z)\right]$$
$$+\frac{1}{4\xi_{mA}}\sum_{n\neq m}\frac{1}{\xi_{nA}}\left[\int_0^\infty ds_z\exp\left[i\left(\xi_{nA}-\xi_{mA}\right)s_z\right]\times\right.$$
$$\left.\times\,\sigma^{\mathrm{Sc}}_{mnnm}(0,s_z)+\text{ complex conjugate term}\right], \quad (4.3.21)$$

where

$$Sc^s_{2mn}\left(y_1-y_2\right)=-\frac{1}{4\xi_{mA}^2}\int_0^\infty ds_z\exp\left[i\left(\beta_{nA}-\xi_{mA}\right)s_z\right]\times$$
$$\times\,\sigma^{\mathrm{Sc}}_{mnmn}\left(y_1-y_2,s_z\right)+\text{ complex conjugate term.} \quad (4.3.22)$$

The equation for the cross modes

$$\left[\frac{\partial}{\partial x}-\frac{i}{2}\left(\frac{1}{\xi_{mA}}\frac{\partial^2}{\partial y_1^2}-\frac{1}{\xi_{nA}}\frac{\partial^2}{\partial x_2^2}\right)+C^s_{mn}\right]\langle F_{n,n}\rangle=0$$
$$C^s_{mn}(y_1-y_2)=\frac{1}{4}\left[\frac{1}{\xi_{mA}^2}\int_0^\infty ds_x\sigma_{mmmm}(0,s_x)\right.$$
$$\left.+\frac{1}{\xi_{nA}^2}\int_0^\infty ds_x\sigma^s_{nnnn}(0,s_x)-\frac{2}{\xi_{mA}\xi_{nA}}\int_0^\infty ds_x\sigma^s_{mmnn}(x_1-x_2,s_x)\right]$$
$$+\frac{1}{4}\left[\sum_{q\neq m}\frac{1}{\xi_{mA}\xi_{qA}}\int_0^\infty ds_x\sigma^{sc}_{mqqm}(0,s_x)\exp\Big(i(\xi_{qA}-\xi_{mA})^0s_x\Big)\right.$$
$$\left.+\sum_{q\neq n}\frac{1}{\xi_{nA}\xi_{qA}}\int_0^\infty ds_x\sigma^{sc^*}_{nqqn}(0,s_x)\exp\Big(-i(\xi_{qA}-\xi_{nA})s_x\Big)\right]$$

where

$$\sigma^{sc}_{ijkm}(y',z';y'',z'')=\langle\gamma_{ij}(y',z')\gamma_{km}(y'',z'')\rangle$$

and

$$\sigma^s_{ijkm} = (y', z'; y'', z'') = \langle \gamma_{ij}(y', z') \gamma^*_{km}(y'', z'') \rangle,$$

as derived by Beran and Frankenthal [3], but with different $\gamma_{mn}(y, z)$ due to boundary and density variations.

4.4 Propagation in Random Waveguides

Problem Formulation

We consider the propagation of a monochromatic signal of frequency ω in a one-layer waveguide bounded below by a rigid bottom (not necessarily flat) at $z = H(x)$ and above by a smooth pressure-release top $z = s$. The average sound sped c depends only on z, but its random fluctuations also depend on the horizontal vector $\underline{r}(x, y)$ where x is the range and y is the transverse variable.

The complex amplitudes $\tilde{p}$ satisfy the equation

$$\left[\nabla^2 + k^2(z)\left[1 + \tilde{\mu}(x, y, z)\right]\right] \tilde{p}(x, y, z) = 0, \tag{4.4.1}$$

where $k(z) = \dfrac{\omega}{c(z)}$ is the wavenumber, which measures the depth-dependent average sound speed and $\nabla^2 = \left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}\right)$ and $\tilde{\mu}(x, y, z)$ are the refractivity fluctuations which produce volume scattering. We assume that $\tilde{\mu}(r, z)$, where $r = (x, y)$ is a real random variable that has a variance denoted by $\langle \tilde{\mu}^2 \rangle$ and a correlation function $\tilde{\sigma}$ defined by

$$\tilde{\sigma}(s_x, s_y, s_z) = \langle \tilde{\mu}(x_1, y_1, z_1) \tilde{\mu}(x_2, y_2, z_2) \rangle \tag{4.4.2}$$

where $s_x = x_1 - x_2$, $s_y = y_1 - y_2$ and $s_z = z_1 - z_2$ and the symbol $\langle\ \rangle$ denotes an ensemble average. The transform from $\bar{\mu}(x, y, z)$ to $\mu(x, k, z)$ is defined by

$$\mu(x, k, z) = \int_{-\infty}^{\infty} ds_k e^{-iks_y} \bar{\mu}(x, y, z) \tag{4.4.3}$$

where the symbol $*$ denotes the conjugate.

From equation (4.4.3)

$$\mu(x, k, z) = \mu^*(x, -k, z), \tag{4.4.4}$$

the correlation function

$$\langle \mu(x_1, k_1, z_1)\mu(x_2, k_2, z_2)\rangle = 2\pi\delta(k_1 + k_2)\sigma(k_1, z_1, z_2, s_k) \tag{4.4.5}$$

where

$$\sigma(k, z_1, z_2, s_x) = \int_{-\infty}^{\infty} ds_y e^{-iks_y} \bar{\sigma}(s_y, z_1, z_2, s_x). \tag{4.4.6}$$

Let

$$p(r, z) = \sum F_m(r) f_m(x, z) \tag{4.4.7}$$

where f_m satisfies

$$\frac{\partial}{\partial z}\left(\frac{\partial f_m(z)}{\rho(z)}\right) + \frac{k^2(x, y, z) - \xi_m^2}{\rho(z)} f_m(z) = 0 \quad \text{at } 0 < z > H \tag{4.4.8}$$

and the boundary conditions are:

1. free surface at $z = 0$

$$f_m(0) = 0 \tag{4.4.9}$$

2. rigid bottom at $z = H$

$$\frac{\partial f_m(H)}{\partial z} = 0. \tag{4.4.10}$$

The modal decomposition of the acoustic pressure is

$$p(x, z) = \sum F_m(r) f_m(z) \tag{4.4.11}$$

of equation (4.4.1).

Substituting equation (4.4.7) in (4.4.1) and following the same procedure as outlined in the previous work, we get

$$\left[\frac{\partial^2}{\partial r^2} + \xi_m^2\right] F_m(r) + \sum_k \tilde{\mu}_{mk}(r) F_m(r) = 0 \tag{4.4.12}$$

where

$$\tilde{\mu}_{mk}(r) = \int_0^H k^2(z) f_m(z) f_k(z) \tilde{\mu}(r, z) dz. \tag{4.4.13}$$

The transform $\tilde{\mu}_{mk}(k, z)$ satisfies

$$\mu_{mk}(k, x) = \mu_{mk}^*(-k, x) \tag{4.4.14}$$

and

$$\langle \mu_{mk}(k_1, x_1) \mu_{j\ell}(k_2, x_2) \rangle = \delta(k_1 + k_2) \sigma_{mkj\ell}(k_1, s_x) \tag{4.4.15}$$

where

$$\sigma_{mkj\ell}(k, s_x) = \int_{-\infty}^{\infty} ds_y e^{-iks_y} \tilde{\sigma}_{mkj\ell}(s_y, s_x) \tag{4.4.16}$$

and

$$\tilde{\sigma}_{mkj\ell}(s_y, s_x) = \int_0^H dz \int_0^H dz' k^2(z) k^2(z')$$

$$f_m(z) f_k(z) f_j(z') f_\ell(z') \tilde{\sigma}(s_y, s_x, z, z'). \tag{4.4.17}$$

Defining the particle-velocity vector

$$\underline{\tilde{V}} = -i \frac{\partial}{\partial r} p = \sum_k \tilde{V}_k(r) f_m(z) \tag{4.4.18}$$

one can transform the second-order modal pressure equation (4.4.12) into a two coupled first-order equations. Differentiating equation (4.4.18) with respect to r

$$\frac{\partial}{\partial r} \cdot \underline{\tilde{V}} = -i \frac{\partial^2}{\partial r^2} F_m = i\xi_m^2 F_m + i \sum_k \tilde{\mu}_{mk} F_k \tag{4.4.19}$$

where the vector (gradient) equation for the modal amplitudes

$$\frac{\partial}{\partial r} F_m = i \underline{\tilde{V}}_m. \tag{4.4.20}$$

If we take the Fourier transform over y of equations (4.4.19) and (4.4.20), it gives

$$\tilde{V}_{ym}(x, k) = k \hat{F}_m(x, k) \tag{4.4.21}$$

$$\left(\frac{\partial}{\partial x} \tilde{V}_{xm} + ik \hat{V}_{ym} \right) = i\xi_m^2 \hat{F}_m + i \sum_k \hat{\mu}_{mk} * \hat{F}_m$$

$$\frac{\partial}{\partial x} \hat{v}_{xm}(k, x) = i\eta^2(k) \hat{F}_m(k, x) + i \sum_k \int_{-\infty}^{\infty} \frac{dk'}{2\pi} \mu_{km}(k - k', x) F_k(k', x) \times$$

$$\text{where } \eta_m^n = \xi_m^2 - k^2 \tag{4.4.22}$$

and

$$\frac{\partial}{\partial x} \hat{F}_m(k, z) = i \hat{V}_{xm}(k, x) \tag{4.4.23}$$

Equations (4.4.22) and (4.4.23) can be separated into equations which govern the evolution in x of the transverse spectra $F_m^{\pm}$ of the right and left propagating components of F_m which are defined as

$$\hat{F}_m(k,x) = \hat{F}_m^{+}(k,x) + \hat{F}_m^{-}(k,x) \tag{4.4.24}$$

$$\hat{V}_{xm}(k,x) = \eta_m(k)\left[\hat{F}_m^{+}(k,x) - \hat{F}_m^{-}(k,x)\right]$$

$$\frac{\partial}{\partial x}\hat{F}_x^{\pm} = \frac{\partial}{\partial x}\hat{F}_m \pm \frac{1}{\eta_{m(k)}}\frac{\partial}{\partial x}\hat{V}_{xm}. \tag{4.4.25}$$

In the absence of scattering

$$\frac{\partial}{\partial x}\hat{F}_m^{+} \pm i\eta_m(x)\hat{F}_m^{\pm} = 0 \tag{4.4.26}$$

and in this case

$$\hat{F}_m^{\pm} = F_m^{\pm}\exp\left[\pm i\eta_m(k)x\right]. \tag{4.4.27}$$

For the presence of scattering

$$\frac{\partial}{\partial z}F_m^{\alpha}(k,x) = \alpha\sum_k\sum_{\gamma=\pm}\int_{-\infty}^{\infty}\frac{dk'}{2\pi}R_{mk}^{\alpha\gamma}(k,k',x)p_k^{\gamma}(k',x) \tag{4.4.28}$$

where

$$R_{mk}^{\alpha\gamma}(k,k',x) = \frac{i}{2\eta_m(k)}e^{-i[\alpha\eta_m(k-\gamma\eta_k(k')]x}\mu_{mk}(k-k',x). \tag{4.4.29}$$

Integrating equation (4.4.25) gives

$$F_m^{\alpha}(k,x) = F_m^{\alpha 0}(k,x) + \sum_k\sum_{\gamma=\pm}\int_{x_s}^{\infty}dx' H[\alpha(x-x')] \times$$

$$\times\int_{-\infty}^{\infty}\frac{dk'}{2\pi}R_{mk}^{\alpha\gamma}(k,k',x)F_k^{\gamma}(k',x') \tag{4.4.30}$$

where $F_m^{\alpha 0}(k,x)$ is the field in the absence of scattering and the Heaviside function $H[\alpha(x-x')]$ limits the integration to the x-range for the right or left propagation component.

Equation (4.4.30) gives the formula that governs the evolution in the range x-direction of the transverse wavenumber spectra of the modal coefficients when volume scattering effect is only included.

The iterative perturbation solution of equation (4.4.30) can be written as

$$F_m^{\alpha}(k,x) = F_m^{\alpha 0}(k,x) + \Delta_1 F_m^{\alpha}(k,x) + \Delta_2 F_m^{\alpha}(k,x) + \cdots \tag{4.4.31}$$

where $F_m^{\alpha 0}(k,x)$ is the homogeneous solution of equation (4.4.30) and $\Delta_1 F_m^{\alpha}(k,x)$ and $\Delta_2 F_m^{\alpha}(k,x)$ are the first two iterative corrections given by

$$\Delta_1 F_m^{\alpha}(k,x) = \sum_k \int_{x_0}^{x} dx' \int_{-\infty}^{\infty} \frac{dk'}{2\pi} H\left[\alpha(x-x')\right] R_{mk}^{\alpha\gamma}(k,k',x') F_k^{\alpha}(k',x_0) \tag{4.4.32}$$

where

$$F_k^{\alpha 0}(k,x) = F_k^{\alpha}(k,x_0) \tag{4.4.33}$$

and

$$\begin{aligned}\Delta_2 F_m^{\alpha}(k,x) = &\sum_k \sum_p \int_{x_0}^{x} dx' \int_{x_0}^{x} dx'' \frac{dk'}{2\pi} \times \\ &\times \int \frac{dk''}{2\pi} H\left[\alpha(x-x')\right] H\left[\alpha(x'-x'')\right] R_{mk}^{\alpha\gamma}(k,k',x') \times \\ &\times R_{k\ell}^{\alpha\gamma}(k',k'',x'') F_\ell^{\gamma}(k'',x_0).\end{aligned} \tag{4.4.34}$$

CHAPTER V

CONCLUSIONS AND RECOMMENDATIONS

In this research, we have derived the exact solution for the acoustic pressure in a two-layered model with the seabed perturbed smoothly. The undulations in the seabed shape have been taken to be sine, quadratic and linear functions. The perturbation function should be smooth; i.e., it is differentiable and has Fourier transforms. Also, we take the reflecting type condition for the sea bottom instead of rigid condition, as this is a more realistic model. Also, we derive equations for the pressure for a perturbed two-layered model when the wavenumbers are varying with depth. We use the Fourier transform and the Green functions to determine the sound pressure equations. These equations become the same equations as derived by Boyles [4] when the seabed is assumed to be horizontal and the wavenumbers are assumed piecewise constant functions. We show the effect of perturbations on the acoustic pressure through numerical application of the derived equations. The numerical results show noticeable differences between the calculated pressure for the perturbed and unperturbed seabed models. The accuracy of the technique depends on the accuracy of determination of the zeros of the characteristic function. Consequently, careful analysis should be carried out to estimate these eigenvalues by Newton's method or any other powerful technique.

In Chapter two, we solve the same problem using the modified mode-coupling theory. We have to modify the conventional mode theory by separating the domain of application to two regions; one involves the unperturbed region and the other the perturbed region. In our case the perturbed region involves a strip around the seabed. The perturbed region has to be tackled with more care, as direct substitution of modal decomposition leads to erroneous results. Our application of the modified mode-coupling theory results in seabed effect which would otherwise be neglected by the conventional mode-coupling theory. Also, we followed the work by Godin [11], and Brekhovskikh and Godin [5] to study the more general case which involves any finite number of layers with smooth undulations and density changing with depth and range. The reciprocity principle or the divergence theorem is the basis to derive the mode-coupling coefficients for non-horizontal layered model. This technique gives a more accurate reformulation of the two point boundary value problem than any other methods. However, the reciprocity principle assumes, among other assumptions, that the total field can be split into forward and backward traveling waves. If the sound speed is a slowly varying function or independent of the range, then the total radial field can be split as desired. On the other hand, when there is strong range-dependence in the sound speed, then the decomposition of the radial field will not be meaningful. We use the shooting method to calculate the pressure amplitude for one-layer model with the bottom perturbed smoothly. In order to solve the system of ordinary differential coupling equations representing the mode amplitudes, we initially rewrite the second-order system as a first order system of differential equations of dimension $2N$. We use two, four and eight modes to calculate pressure amplitudes and compare the results of perturbed and unperturbed sea bottom layer.

There are clear differences between the pressure amplitudes for the two cases. The shooting method as any other numerical method has to be used with more care because of convergence issue. Since the mode-coupling two-point solution of the boundary value problem requires large computer storage and computational memory, parallelization technique has been used to speed up the calculation of the modal amplitude when large number of modes (more than four modes) are involved. Using parallelization of 4 CPUs has reduced the clock time from fifteen hours to about 4 hours when we use eight modes. Consequently, if we include many coupling modes in the modal pressure amplitude calculation, the need to devise a faster numerical solving technique or use of parallelization becomes a key issue.

In Chapter four, we calculate the self and cross-modal coherence functions and their scattering coefficients when both the volume and surface scattering are included. The cross modal scattering coefficients are negligible at large propagation distances. We use the results of reciprocity principle for mode coupled perturbed layered model to derive the scattering coefficients when both volume and surface scattering effects are involved and the sea bottom has smooth undulations and the density is varying with depth and range.

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