FRACTIONAL CALCULUS AND SOME OF ITS APPLICATIONS

BY

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Many concepts in mathematics can be generalized .In this thesis, we discuss the generalization of the concept of integrals to include integrals of fractional orders. Historical survey and the contributions of many famous mathematicians have been introduced. Three approaches to the definition of the fractional integral are proved. As a consequence of this definition, fractional derivative is handled. Leibniz's rule, Taylor's series expansion and the order of contact between two functions are also generalized. As an application, we modify an algorithm, which enables us to write two functions in terms of each other. Well-known examples like (sin t, t) and (e^t, t) are discussed. New laboratory experimental examples such as the relation between distance and speed, and current and voltage are found.

CHAPTER 1

INTRODUCTION

1. Introduction

The concept of the differentiation operator $D = d/dx$ is familiar to all who have studied the elemetary calculus. For suitable functions f, the nth derivative of f, namely $D^n f(x) = d^n f(x)/dx^n$ is well-defined provided that n is a positive integer. In 1695 L'Hôpital inquired of Leibniz what meaninig could be ascribed to $D^n f$ if n were a fraction. Since that time the fractional calculus has drawn the attention of many famous mathematicians, such as Euler, Laplace, Fourier, Abel, Liouville, Riemann, and Laurent. But it was not until 1884 that the theory of generalized operators achieved a level in its devolepment suitable as a point of departure for the modern mathematicians. By then the theory had been extended to include operators D^ν, where ν could be rational or irrational, positive or negative, real or complex. Thus the name fractional calculus became somewhat of a misnomer. A better description might be differentiation and integration to an arbitrary order. However, we shall adhere to tradition and refer to this theory as the fractional calculus.

In Chapter II, we briefly trace the historical development of the fractional calculus from Euler to present, and in Chapter III we describe heuristic and mathematical arguments that lead to the present definition of fractional integrals. The first argument is the iterated integral; we begin with a consideration of the n-fold integral then we reduce it to a single integral with a kernel, and finally by finding this kernel we reach our definition. The second argument is the differential equations; we begin with the n-th order derivative operator, we find the solution of $D^n y(x) = f(x)$ by finding the Green's function, then we write this solution as $y(x) = D^{-n} f(x)$. The third argument is the complex variables; we consider the $n-$th derivative of the Cauchy integral formula, we generalize it to fractional order but with different contour.

We define the factional derivative of order ν by the use of the fractional integrals as

$$_cD_x^\nu f(x) = {}_cD_x^n[{}_cD_x^{-u}f(x)]$$

where n is the smallest integer greater than $\Re(\nu)$ and $u = n - \nu$.

In this chapter, we also use Abel's equation to define the left and right sided versions of the fractional derivative.We present a new proof of the generalized Leibniz rule for analytic functions f and g :

$$_aD_x^\alpha(fg) = \sum_{k=0}^{\infty}\begin{pmatrix}\alpha\\ k\end{pmatrix}({}_aD_x^{\alpha-k}f)g^{(k)}$$

At the end of this chapter, we present laws of exponents for: fractional integrals, intergral of derivatives, derivatives of integrals and the law of exponent for fractional derivative.

In Chapter IV, we generalize Taylor series expansion with integral remainder by proving that for f with a summable derivative,

$$f(x) = \sum_{j=-n}^{n-1}\frac{(x-a)^{\alpha+j}}{(\alpha+j)!}({}_aD_x^{\alpha+j}f)(a) + R_n(x), \quad \Re(\alpha) > 0$$

where

$$R_n(x) = I_x^{\alpha+n} D_a^{\alpha+n} f(x)$$

In Chapter V, we concentrate our work in the applications of the fractional calculus : we define the order of contact between two functions, then we apply this definition to a model for the turbulent boundary layer cross flow velocity component, to show that the non integer models "fill the gaps" between the integer models.

At the end of this chapter, we give an algorithm, the functional relations, which enables us to write two functions in terms of each other, we present four examples: two well-known examples; we write $\sin t, t$ and e^t, t in terms of each other and two new: we write the relation between speed and distance and then we write the relation between voltage and current in a simple electrical circuit.

CHAPTER 2

HISTORICAL SURVEY

2. Historical Survey

The question of extension is frequently of great interest to most researchers. Many mathematical concepts and definitions were extended. A well-known example is the extension of real numbers to complex numbers, and another is the extension of factorials of integers to factorials of complex numbers. In generalized integration and differentiation the question of the extension is: Can the derivative of integer order $d^n y/dx^n$ be extended to rational or complex order?

Leibniz invented the notation we used for the derivative and in 1695 L'Hôpital asked him "What if $n = 1/2$?" Leibniz suggested an answer in 1697 in terms of ratios of differentials

$$\frac{d^{\frac{1}{2}}x}{x} = \sqrt{\frac{dx}{x}}$$

Euler in 1730 wrote "When n is a positive integer and p is a function of x, the ratio of $d^n p$ to dx^n can always be expressed algebraically. So that if $n = 2$ and $p = x^3$, then d^2x^3 to dx^2 is $6x$ to 1. But what kind of ratio can then be made if n be a fraction?"

In 1812 Laplace defined a fractional derivative by means of an integral, and in 1819 the first mention of a derivative of arbitrary order appears in a text. Lacroix developed an exercise generalizing from a case of integer order. Starting with $y = x^m$, for a positive integer m, Lacroix easily developed the nth derivative (See[8])

$$\frac{d^n y}{dx^n} = \frac{m!}{(m-n)!}x^{m-n} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)}x^{m-n} \quad for\ m \geq n$$

Then he gave the example for $y = x$ and $n = \frac{1}{2}$, and obtained

$$\frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}} = \frac{2\sqrt{x}}{\sqrt{\pi}}$$

In 1822 Fourier was the next to mention derivatives of arbitrary order. His definition of fractional operations was obtained from his integral representation of $f(x)$ (See [3])

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\alpha)\,d\alpha \int_{-\infty}^{\infty} \cos p(x-\alpha)\,dp$$

but

$$\frac{d^n}{dx^n}\cos p(x-\alpha) = p^n \cos\left[p(x-\alpha) + \frac{1}{2}n\pi\right]$$

for integer n. Formally replacing n with arbitrary u in (2.5), he obtains the generalization

$$\frac{d^u}{dx^u}f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\alpha)\,d\alpha \int_{-\infty}^{\infty} p^u \cos\left[p(x-\alpha) + \frac{1}{2}u\pi\right]\,dp$$

In 1823, Abel's integral equation was given as

$$K = \int_0^x (x-t)^{-\frac{1}{2}} f(t)\,dt \qquad (2.1)$$

Abel wrote the right hand side of his equation (2.1) as

$$\sqrt{\pi}\left[\frac{d^{-\frac{1}{2}}}{dx^{-\frac{1}{2}}}\right] f(x)$$

then he operated on both sides of the equation with $d^{1/2}/dx^{1/2}$ to obtain

$$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}K = \sqrt{\pi} f(x)$$

Mathematicians have described Abel's solution as "elegant". Abel's solution had attracted the attention of Liouville, who made the first major study of fractional calculus. In 1832 he was successful in applying his definitions to problems in potential theory. The starting point for his theoretical development was the known result for derivatives of integer order (See[9])

$$D^m e^{ax} = a^m e^{ax}$$

which he extended in a natural way to derivatives of arbitrary order

$$D^\nu e^{ax} = a^\nu e^{ax}$$

He assumed that the arbitrary derivative of a function $f(x)$ that may be expanded in a series of the form

$$f(x) = \sum_{n=0}^{\infty} C_n e^{a_n x}, \quad \Re(a_n) > 0$$

is

$$D^\nu f(x) = \sum_{n=0}^{\infty} C_n a_n^\nu e^{a_n x} \tag{2.2}$$

The formula (2.2) is known as Liouville's first formula for fractional derivatives. It generalizes to an arbitrary ν, but it has the disadvantage of being applicable only to functions that may be expanded in a series form as given. To obtain his second definition, Liouville started with definite integrals related to the gamma function.

$$I = \int_0^\infty u^{a-1} e^{-xu} du, \quad a > 0, \ x > 0$$

The change of variable $xu = t$ yields

$$I = x^{-a} \int_0^\infty t^{a-1} e^{-t} dt = x^{-a} \Gamma(a)$$

or

$$x^{-a} = \frac{1}{\Gamma(a)} I$$

Apply D^ν to both sides to get

$$D^\nu x^{-a} = \frac{(-1)^\nu}{\Gamma(a)} \int_0^\infty u^{a+\nu-1} e^{-xu} du = \frac{(-1)^\nu \Gamma(a+\nu)}{\Gamma(a)} x^{-a-\nu}, \ a > 0 \qquad (2.3)$$

but, as we see, Liouville's definition is useful only for functions of the type x^{-a} with positive a.

The earliest work that ultimately led to what is now called the Riemann-Liouville defintion appears to be the paper by N.Ya.Sonin in 1869. His starting point was Cauchy's integral formula. In 1868-1872, Letnikov extended Sonin's paper. The nth derivative of Cauchy's integral formula is given by

$$D^n f(z) = \frac{n!}{2\pi i} \int_C \frac{f(\xi)}{(\xi - z)^{n+1}} d\xi \tag{2.4}$$

In 1884 Laurent connected the theory of fractional calculus with the theory of the operators D or d/dx etc. His starting point was also Cauchy's integral formula. His contour was an open circuit on a Riemann surface and he produced the definition

$$_cD_x^{-\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_c^x (x-t)^{\nu-1} f(t)\, dt, \quad \Re(\nu) > 0 \tag{2.5}$$

Usually we have $c = 0$. A sufficient condition that the integral in (2.5) converge is (See[14])

$$f\left(\frac{1}{x}\right) = O(x^{1-\epsilon}), \quad \epsilon > 0$$

Integrable functions with this property are called "functions of Riemann class". For example, constants are of Riemann class, as is x^a with $a > -1$.

If c in the last definition (2.5) is negative infinity, a sufficient condition that the integral converge is that

$$f(-x) = O(x^{-\nu-\epsilon}), \quad \epsilon > 0, \quad x \to \infty$$

Integrable functions with this propety are called "functions of Liouville class". For example, x^{-a} with $a > \nu > 0$ is of Liouville class, but a constant is not. However, if a is between zero and -1, then, depending on the value of ν, the two classes may overlap.

If we let $f(t) = e^{at}$ with the real part of a positive, then

$$_{-\infty}D_x^{-\nu} e^{ax} = a^{-\nu} e^{ax}$$

If we assume that the law of exponents

$$D[D^{-\nu}] = D^{1-\nu}$$

holds, then if $0 < \nu < 1$, we have $\mu = 1 - \nu > 0$ and

$$_{-\infty}D_x^{\mu} e^{ax} = a^{\mu} e^{ax}, \quad \Re(a) > 0.$$

For $f(x) = x^a$ and $\nu > 0$, we have

$$_0D_x^{-\nu} x^a = \frac{\Gamma(a+1)}{\Gamma(a+\nu+1)} x^{a+\nu}, \quad a > -1 \tag{2.6}$$

If $0 < \nu < 1$, then

$$_0D_x^{\nu} x^a = \frac{\Gamma(a+1)}{\Gamma(a-\nu+1)} x^{a-\nu}, \quad a > -1 \tag{2.7}$$

For $f(x) = 1$ and $\nu = 1/2$, we get

$$_0D_x^{\frac{1}{2}}(1) = \frac{1}{\sqrt{\pi x}}$$

The twentieth century:

In the period 1900-1970 a modest amount of published work appeared on the subject of the fractional calculus. The year 1974 saw the first international conference on fractional calculus held at the University of New Haven, Connecticut which was sponnsored by the National Science Foundation.

In the period 1975 to the present, about 400 papers have been published relating to the fractional calculus. In 1980, Nishimoto published a four volume work devoted primarily to applications of the fractional calculus to ordinary and partial differential equations.

The fractional calculus finds use in many fields of science and engineering, including fluid flow, rheology, diffusive transport akin to diffusion, electrical networks, electromagnetic theory, and probability. (see[8])

CHAPTER 3

DEFINITIONS FOR FRACTIONAL ANALYSIS

3. Definitions for fractional analysis

3.1. Fractional integrals

The main objective in this section is to present arguments which should convince the reader that the definition of the fractional integrals that we will use is a feasible entity. We will present here with a complete proof three approaches that lead to the definition of fractional integral which fulfill with our knowledge about the ordinary one. The subject of notation can not be minimized. The concise notation of fractional calculus adds to its elegance. Various authors have used different notation. The one which we prefer was invented by Harold T. Davis. All information can be conveyed by the symbols

$$ {}_cI_x^{\nu} f(x) = {}_cD_x^{-\nu} f(x), \quad \nu \geq 0 \tag{3.1}$$

denoting integration of arbitrary order along the x axis. The subscripts c and x denote the limits of integration of a definite integral which defines fractional integration. The adjoining of these subscripts becomes a vital part of the operator symbol to avoid ambiguities in applications.

We now consider the mathematical problem of defining fractional integration and differentiation. It is clear that the mathematicians mentioned so far were not merely formalizing but were trying to solve a problem which they understood well but did not explicitly formulate. Briefly, what they wanted is this:

For every function $f(z)$ of a sufficiently wide class, and every number ν, irrational, fractional, or complex, a function

$$g(z) = {}_cD_z^\nu f(z) \tag{3.2}$$

should be assigned subject to the following conditions.

•1 The operation ${}_cD_x^\nu$ must produce the same result as ordinary differentiation when ν is a positive integer, with a similar result for ordinary integration when ν is a negative integer.

•2 The operation of order zero ($\nu = 0$) leaves the function unchanged.

•3 The fractional operators must be linear.

$${}_cD_x^{-\nu}[af(x) + bg(x)] = a\,{}_cD_x^{-\nu} f(x) + b\,{}_cD_x^{-\nu} g(x)$$

•4 The law of exponents for integration of arbitrary order holds.

$${}_cD_x^{-\mu}\;{}_cD_x^{-\nu} f(x) = {}_cD_x^{-\mu-\nu} f(x)$$

A definition for fractional integration which fulfills these requirements is named in honor of Riemann and Liouville and is given by

Definition 1. *For $\nu > 0$ and continuous f, we define the fractional integral by:*

$${}_cD_x^{-\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_c^x (x-t)^{\nu-1} f(t)\, dt \tag{3.3}$$

Remark 1. *For $\nu = 0$, we can show that the operator ${}_cD_x^0$ leaves the function* *unchanged, by taking the limit of* ${}_cD_x^{-\nu}f(x)$ *as* $\nu \to 0$*, but we cannot assume* $\nu = 0$ *directly because, even though we can consider* $\Gamma(0) = \infty$*, the integral in (3.3) will diverge like:*

$$ {}_cD_x^0 f(x) = \frac{1}{\Gamma(0)} \int_0^x (x-t)^{-1} f(t)\,dt $$

So we can expan

$$ f(t) = f(x) + (t-x)f'(x) + \dots + \frac{(t-x)^n f^{(n)}(x)}{n!} + \dots $$

then

$$ \begin{aligned} \lim_{\nu\to 0} D_x^{-\nu} f(x) &= \lim_{\nu\to 0} \frac{1}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} f(x)\,dt \qquad \textit{(the other terms will vanish)} \\ &= \lim_{\nu\to 0} \frac{x^\nu f(x)}{\Gamma(\nu+1)} \\ &= f(x) \end{aligned} $$

Remark 2. *We can assume*

$$ \Gamma(n) = \infty \qquad \textit{for } n = 0, -1, -2, -3, \dots. $$

Now we show some of the approaches which lead to this definition.

3.1.1. The iterated integrals

Let us start with the n-fold integral

$$ {}_cD_x^{-n} f(x) = \int_c^x dx_1 \int_c^{x_1} dx_2 \cdots \int_c^{x_{n-1}} f(t)\, dt \tag{3.4} $$

The function f is assumed to be continuous in the interval $[c, b]$ where $b > x$. We can reduce this to a single integral of the form

$$ \int_c^x k_n(x,t)\, f(t)\, dt $$

where the kernel $k_n = k_n(x,t)$ is a function of x, t, and n. So we may write

$$ {}_cD_x^{-\nu} f(x) = \int_c^x k_\nu(x,t) f(t)\, dt, \quad \Re(\nu) > 0 \tag{3.5} $$

In fact, we just use the usual change of variables formula over a triangular region as shown in Fig.1

$$ \int_c^x dx_1 \int_c^{x_1} G(x_1,t)\, dt = \int_c^x dt \int_t^x G(x_1,t)\, dx_1 \tag{3.6} $$

In particular, if $G = f(t)$, then (3.6) becomes

$$ \int_c^x dx_1 \int_c^{x_1} f(t)\, dt = \int_c^x f(t)\, dt \int_t^x dx_1 = \int_c^x (x-t)\, f(t)\, dt \tag{3.7} $$

We can continue in this way to find that

$$k_n(x,t) = \frac{(x-t)^{n-1}}{(n-1)!}$$

and hence

$$_cD_x^{-n} f(x) = \frac{1}{\Gamma(n)} \int_c^x (x-t)^{n-1} f(t)\, dt \tag{3.8}$$

The right hand side of (3.8) is meaningful for any number n whose real part is greater than zero so,

$$_cD_x^{-\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_c^x (x-t)^{\nu-1} f(t)\, dt, \quad \Re(\nu) > 0$$

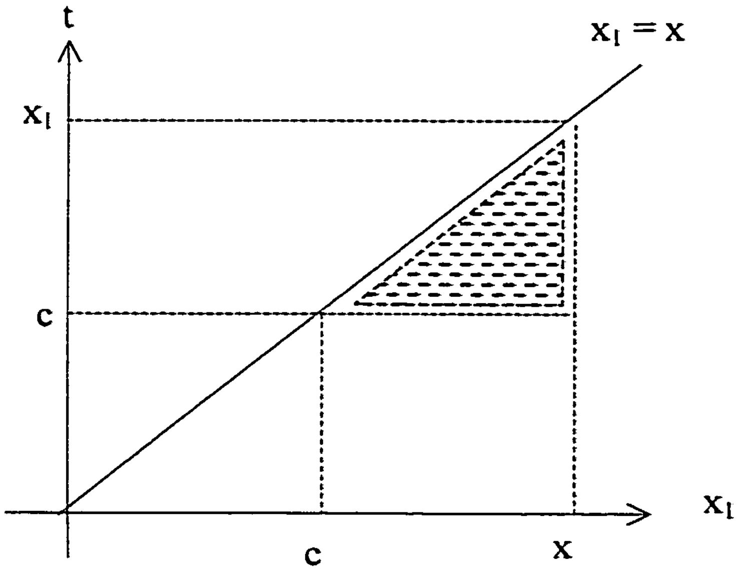


Figure (1)
Triangular region

3.1.2. Differential equations

We will show now how the theory of linear differential equations may be used to arrive at our fundamental definition(3.3).

Suppose

$$L = D^n + p_1(x)D^{n-1} + \ldots + p_n(x)$$

is a linear differential operator whose coefficients $p_i(x)$ are continuous on some interval $\mathcal{I}$. Then if $f(x)$ is continuous on $\mathcal{I}$ and $c \in \mathcal{I}$, we may write

$$Ly(x) = f(x), \quad D^k y(c) = 0, \;\; 0 \leq k \leq n-1$$

which is a linear differential system. The unique solution of this system for all $x \in \mathcal{I}$ is given by

$$y(x) = \int_c^x H(x,\xi)\, f(\xi)\, d\xi \tag{3.9}$$

where H is the one sided Green's function. To derive H, consider

$$\{\phi_1(x), \ldots, \phi_n(x)\}$$

which is the fundamental set of solutions of the homogeneous equation

$$Ly(x) = 0$$

The Green's function may be written explicitly as

$$H(x,\xi) = \frac{(-1)^{n-1}}{W(\xi)} \begin{vmatrix} \phi_1(x) & \phi_2(x) & \dots & \phi_n(x) \\ \phi_1(\xi) & \phi_2(\xi) & \dots & \phi_n(\xi) \\ D\phi_1(\xi) & D\phi_2(\xi) & \dots & D\phi_n(\xi) \\ \dots & \dots & \dots & \dots \\ D^{n-2}\phi_1(\xi) & D^{n-2}\phi_2(\xi) & \dots & D^{n-2}\phi_n(\xi) \end{vmatrix}$$

where the Wronskian is the $n \times n$ determinant

$$W(\xi) = \left|D^i\phi_j(\xi)\right|, \quad 0 \le i \le n-1, \; 1 \le j \le n$$

See [12].

In particular, suppose that $L = D^n$, then the set of fundamental solutions of the homogeneous equation is

$$\{1, x, x^2, \dots, x^{n-1}\}$$

and the one sided Green's function is

$$H(x,\xi) = \frac{(-1)^{n-1}}{W(\xi)} \begin{vmatrix} 1 & x & x^2 & \dots & x^{n-1} \\ 1 & \xi & \xi^2 & \dots & \xi^{n-1} \\ 0 & 1 & 2\xi & \dots & (n-1)\xi^{n-2} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & (n-1)!\xi \end{vmatrix} \tag{3.10}$$

and the Wronskian is

$$W(\xi) = \begin{vmatrix} 1 & \xi & \xi^2 & \dots & \xi^{n-1} \\ 0 & 1 & 2\xi & \dots & (n-1)\xi^{n-2} \\ 0 & 0 & 2 & \dots & (n-1)(n-2)\xi^{n-3} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & (n-1)! \end{vmatrix} = \prod_{k=0}^{n-1} k! =: (n-1)\wr \qquad (3.11)$$

which is independent of ξ. We have used the symbol $m\wr$ defined recursively by

$$m\wr = m!(m-1)\wr, \quad 0\wr = 1 \qquad (3.12)$$

for nonnegative integers.

So $H(x, \xi)$ may be written as a polynomial of degree $n-1$ in x whose leading coefficient is

$$\frac{(-1)^{n-1}}{(n-1)\wr}\left[(-1)^{n+1}(n-2)\wr\right] = \frac{1}{(n-1)!}$$

The Green's function has the usual property

$$\frac{\partial^k}{\partial x^k} H(x, \xi) \mid_{x=\xi} = 0 \quad for \quad k = 0, 1, \dots, n-2$$

Hence ξ is a zero of multiplicity $n-1$, and therefore

$$H(x, \xi) = \frac{1}{(n-1)!}(x-\xi)^{n-1}$$

We get

$$y(x) = \frac{1}{(n-1)!} \int_c^x (x-\xi)^{n-1} f(\xi)\, d\xi$$

Since f is the nth derivative of y, we may interpret this equation as

$$y(x) = D^{-n} f(x) = \frac{1}{\Gamma(n)} \int_c^x (x-\xi)^{n-1} f(\xi)\, d\xi \tag{3.13}$$

This is meaningful even if n is not a positive integer, provided that $\Re(n) > 0$

3.1.3. Complex variables

Let $f(z)$ be a single valued analytic function on an open region A of the complex plane. On a region interior to A bounded by a closed smooth curve C, we have the Cauchy integral formula

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z} d\zeta \tag{3.14}$$

for any point z inside C. So, we can find

$$D^n f(z) = \frac{n!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta \tag{3.15}$$

Our goal now is to deduce our definition (3.3) from this formula.

Note that if n is an arbitrary number, say ν, we may replace $n!$ by $\Gamma(\nu+1)$. If ν is not an integer, the point z will now be a branch point not a pole of the integrand. So we need a different contour. We make a branch cut along the x axis from the point z to negative infinity and we assume that z is a positive real number x.

Now we suggest to define

$$\int_c^{(x+)} \frac{f(\zeta)}{(\zeta - x)^{-\nu+1}} d\zeta = \int_{\mathcal{L}} \frac{f(\zeta)}{(\zeta - x)^{-\nu+1}} d\zeta$$

Then we can write

$$\begin{aligned} {}_cD_x^{-\nu} f(x) &= \lim_{r\to 0} \frac{\Gamma(-\nu+1)}{2\pi i}\int_c^{(x+)} \frac{f(\zeta)}{(\zeta - x)^{-\nu+1}} d\zeta \\ &= \lim_{r\to 0} \frac{\Gamma(-\nu+1)}{2\pi i}\int_{\mathcal{L}} \frac{f(\zeta)}{(\zeta - x)^{-\nu+1}} d\zeta \end{aligned}$$

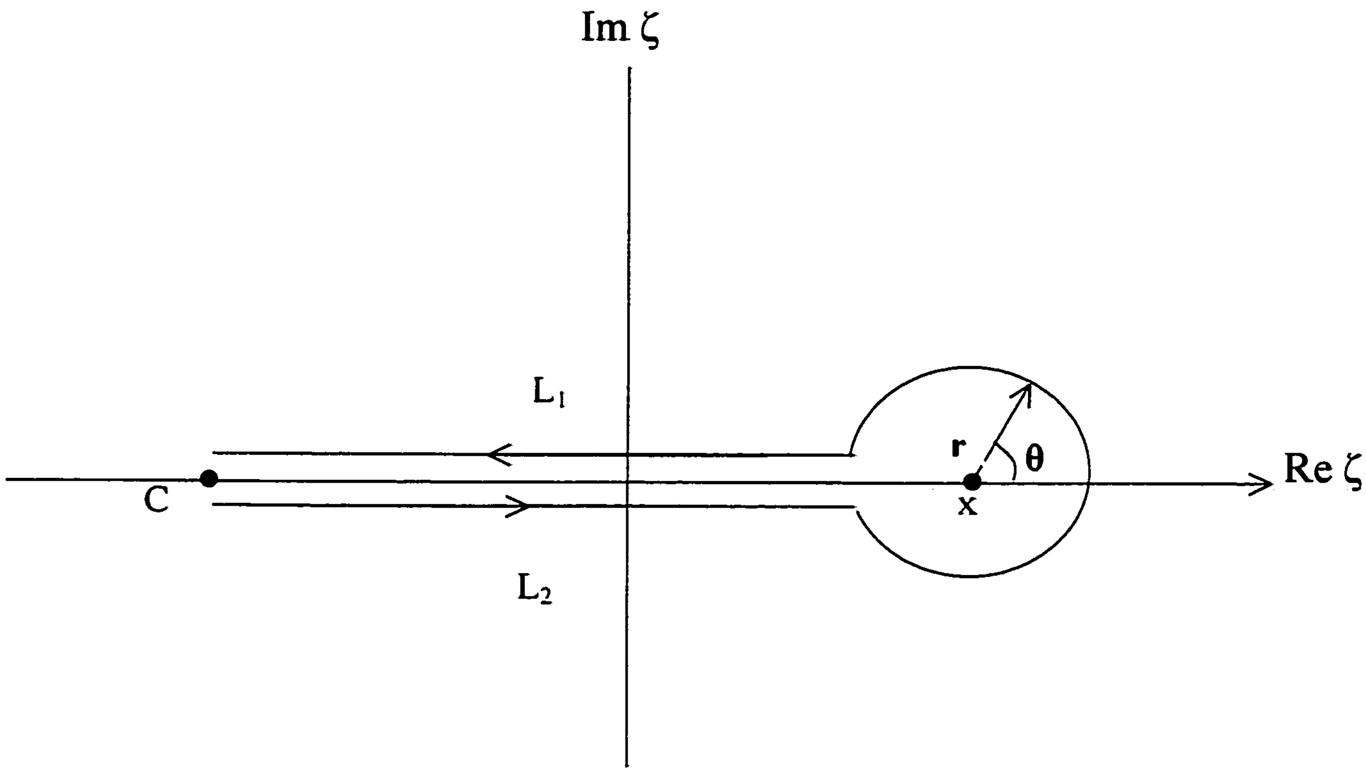


Figure (2)
The Loop $\mathcal{L}$

The loop $\mathcal{L}$, shown in Fig.2, is the union of L_1 , L_2 and γ, where γ is a circle of raduis r with center at x and L_1 , L_2 are the line segment $[c, x - r]$, these line segments coincide with a portion of the real axis in the ζ -plane but are on different sheets of the Riemann surface for $(\zeta - x)^{\nu-1}$. For purposes of visualization we have drawn them as distinct.

If $(\zeta - x)$ is a positive number, we define $\ln(x-\zeta)$ as a real number . Thus on γ

$$(\zeta - x)^{\nu-1} = \exp((\nu-1)(\ln(x-\zeta) + i\theta)$$

The argument θ is π on L_1 so, on L_1

$$(\zeta - x)^{\nu-1} = \exp((\nu-1)(\ln(x-\zeta) + i\pi))$$

and on L_2 the argument θ is $-\pi$ so, on L_2

$$(\zeta - x)^{\nu-1} = \exp((\nu-1)(\ln(x-\zeta) - i\pi))$$

Now the loop integral can be written

$$\begin{aligned}\int_c^{(x+)} \frac{f(\zeta)}{(\zeta-x)^{-\nu+1}} d\zeta &= e^{i(-\nu+1)\pi} \int_c^{x-r} \frac{f(t)}{(x-t)^{-\nu+1}} dt + \\ &\int_\gamma \frac{f(\zeta)}{(\zeta-x)^{-\nu+1}} d\zeta + e^{-i(-\nu+1)\pi} \int_{x-r}^{c} \frac{f(t)}{(x-t)^{-\nu+1}} dt\end{aligned}$$

where $t = \Re(\zeta)$. As r approaches zero we have

$$\int_\gamma \frac{f(\zeta)}{(\zeta-x)^{-\nu+1}} d\zeta \to 0$$

This is because

$$\int_\gamma \frac{f(\zeta)}{(\zeta - x)^{-\nu+1}} d\zeta = \int_{-\pi}^{\pi} r^{\nu-1} e^{i(-\nu+1)\theta} f(x + re^{i\theta})(ire^{i\theta} d\theta)$$

and

$$\left| \int_\gamma (\zeta - x)^{\nu-1} f(\zeta) d\zeta \right| \leq r^{\mathrm{Re}\,\nu} \int_{-\pi}^{\pi} \left| f(x + re^{i\theta}) \right| d\theta \to 0 \quad as \quad r \to 0$$

so we get

$$\int_c^{(x+)} \frac{f(\zeta)}{(\zeta - x)^{-\nu+1}} d\zeta \to [e^{i(-\nu+1)\pi} - e^{-i(-\nu+1)\pi}] \int_c^x \frac{f(t)}{(x-t)^{-\nu+1}} dt \quad \text{as } r \to 0$$

or

$$= \frac{\Gamma(-\nu+1)}{2\pi i} 2i \sin(-\nu+1)\pi \int_c^x \frac{f(t)}{(x-t)^{-\nu+1}} dx$$

$$= \frac{(-\nu)! \sin(-\nu+1)\pi}{\pi} \int_c^x \frac{f(t)}{(x-t)^{-\nu+1}} dt$$

and because of the reflection formula (See [8])

$$\frac{(-\nu)! \sin(-\nu+1)\pi}{\pi} = \frac{1}{(\nu-1)!} \tag{3.16}$$

we lead to the definition

$${}_cD_x^{-\nu} f(x) = \frac{1}{(\nu-1)!} \int_c^x (x-t)^{\nu-1} f(t) dt$$

See[8],[9]

3.2. Leibniz's formula for fractional integrals

A Leibniz-type formula expresses the result of operating on the product of two functions as a sum of products of operations performed on each function. The classical Leibniz rule or formula of elementary calculus is

$$D^n(f(x)g(x)) = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} [D^k g(x)][D^{n-k} f(x)] \tag{3.17}$$

where f and g are assumed to be n-fold differentiable on some interval. Now we wish to extend this to fractional operators.

Theorem 1. *Let f be continuous on $[0, X]$, and let g be analytic at $[0, X]$. Then for $\nu > 0$ and $0 < x \leq X$,*

$${}_0D_x^{-\nu}(f(x)g(x)) = \sum_{k \geq 0} \begin{pmatrix} -\nu \\ k \end{pmatrix} [{}_0D_x^k g(x)][{}_0D_x^{-\nu-k} f(x)] \tag{3.18}$$

Proof:

since $\nu > 0$, f is continuous on $[0, X]$ and g is analytic at all points of this interval, we can consider the fractional integral of their product. So, we have

$${}_0D_x^{-\nu}(f(x)\, g(x)) = \frac{1}{(\nu-1)!} \int_0^x f(\xi)g(\xi)(x-\xi)^{\nu-1} d\xi \tag{3.19}$$

We can write the series expansion for $g(\xi)$ around t as

$$g(\zeta) = \sum_{k=0}^{\infty}(-1)^k \frac{D^k g(x)}{k!}(x-\zeta)^k = g(x) + \sum_{k=1}^{\infty}(-1)^k \frac{D^k g(x)}{k!}(x-\zeta)^k \tag{3.20}$$

Now substitute (3.20) in (3.19) to get

$$_0D_x^{-\nu}(f(x)\,g(x)) = g(x)D^{-\nu}f(x) + \frac{1}{\Gamma(\nu)}\int_0^t f(\xi)(x-\xi)^{\nu}\sum_{k=1}^{\infty}(-1)^k\frac{D^k g(x)}{k!}(x-\zeta)^{k-1}d\xi$$

Since f is continuous on $[0, X]$ and $\nu > 0$,

$$(x-\xi)^{\nu} f(\xi) \tag{3.21}$$

is bounded on $[0, x]$.

Hence we may interchange the order of summation and integration to get

$$_0D_x^{-\nu}(f(x)\,g(x)) = \sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(\nu+k)}{k!\Gamma(\nu)}[_0D_x^k g(x)][_0D_x^{-\nu-k}f(x)]$$

$$= \sum_{k=0}^{\infty}\begin{pmatrix} -\nu \\ k \end{pmatrix}[_0D_x^k g(x)][_0D_x^{-\nu-k}f(x)] \tag{3.22}$$

In (3.22) we use of the identity

$$\begin{pmatrix} -\nu \\ n \end{pmatrix} = \frac{(-1)^n\Gamma(n+\nu)}{n!\Gamma(\nu)}$$

3.3. Fractional derivatives

If $D = d/dx$ is the differentiation operator, and if n is a positive integer, the meaning of $D^n f(x)$,the nth derivative of $f(x)$ (provided that it exists) is well-known. However, if n is not a positive integer, we see that while we may ascribe a meaning to $D^{-\nu}$ for $Re\ \nu > 0$, we have yet to assign a meaning to the symbol D^{ν} for $Re\ \nu > 0$. We shall undertake this task in the present section.

Definition 2. *Suppose that $\Re(\nu) > 0$ and let n be the smallest integer which is greater than $\Re(\nu)$. Define $u = n - \nu$. This means that*

$$0 < \Re(u) \leq 1$$

so, we define the fractional derivative of $f(x)$ of order ν as

$${}_cD_x^{\nu} f(x) = {}_cD_x^n[{}_cD_x^{-u} f(x)] \qquad (if\ it\ exists) \tag{3.24}$$

Then we can write

$$\begin{aligned}
{}_cD_x^{\nu} f(x) &= {}_cD_x^n[\frac{1}{(u-1)!}\int_c^x (x-t)^{u-1} f(t)dt] \\
&= \frac{1}{(u-1)!}\int_c^x \frac{d^n}{dx^n}(x-t)^{u-1} f(t)dt \\
&= \frac{1}{(-\nu-1)!}\int_c^x (x-t)^{u-1-n} f(t)dt \\
{}_cD_x^{\nu} f(x) &= \frac{1}{\Gamma(-\nu)}\int_c^x (x-t)^{-\nu-1} f(t)dt
\end{aligned} \tag{3.25}$$

Remark 3. *If ν is a positive integer,say p,and f has p continuous derivatives then $n = p + 1$ and $u = 1$, so*

$$D^p f(x) = D^{p+1} \int_0^t f(\zeta) d\zeta = D^p f(x)$$

we see that this definition agrees with the usual definition of the ordinary derivative.

For example, with $c = 0$

$${}_0D_x^{\nu} x^{\mu} = {}_0D_x^n [{}_0D_x^{-u} x^{\mu}] = \left[\frac{(\mu)!}{(\mu + u - n)!} x^{\mu+u-n} \right]$$

which means that we could write

$${}_0D_x^{u} x^{\mu} = \frac{(\mu)!}{(\mu - u)!} x^{\mu-u}$$

However, there are questions about **when the fractional derivative exists?**

The fractional integral

$${}_cD_x^{-\nu} f(x) = \frac{1}{(\nu - 1)!} \int_c^x (x - t)^{\nu-1} f(t)\, dt \tag{3.26}$$

exists if $\Re(\nu) > 0$ and f is continuous. But this is not sufficient to guarantee the existence of the fractional derivative. For example, if f is continuous but not differetiable we would be led to a contradiction.

For example, let f be continuous but not differentiable and let $\nu = 1$. Then

$$_0D_x^{-1}f(x) = \int_0^x f(x)dt$$

Now if $\nu = 1$, then $n = 2$ (since $u = n - \nu$) and formally, by(3.24)

$$_cD_x^1 f(x) =_c D_x^2[_cD_x^{-1}f(x)] =_c D_x^2 \int_c^x f(t)dt =_c D_x f(x)$$

but by hypothesis, $f(x)$ is not differentiable.

Here, we show that if f has n continuous derivatives, then the fractional derivative exists.

Theorem 2. *If f has n continuous derivatives, then the fractional derivative of order ν exists, where n is the smallest integer greater than $\Re(\nu)$.*

Proof:

We make a change of variable in the fractional integral (3.26) by writing

$$t = x - y^{\lambda}, \quad \lambda = \frac{1}{\nu}$$

Then we get

$$_cD_x^{-\nu}f(x) = \frac{1}{(\nu-1)!}\int_c^x (x-t)^{\nu-1}f(t)dt = \frac{1}{(\nu)!}\int_0^{(x-c)^{\nu}} f(x-y^{\lambda})dy$$

We can calculate ${}_cD_x^n[{}_cD_x^{-\nu}f(x)]$ as follows

$${}_cD_x^n[{}_cD_x^{-\nu}f(x)] = \sum_{k=0}^{n-1} \frac{D_x^k f(c)}{(\nu-n+k)!}(x-c)^{\nu-n+k}$$

$$+\frac{1}{(\nu)!}\int_0^{(x-c)^\nu} \frac{\partial^n}{\partial x^n} f(x-y^\lambda)dy$$

exists for positive $x > c$, since the usual derivative $D_x^n f(x)$ has been assumed to be continuous.

Remark 4. *In order to get derivatives, you cannot change $-\nu$ in the definition of fractional integral by $\mu > 0$ because the fractional integral ${}_cD_x^{-\nu}f(x)$ is defined only for $-\nu$ where $\nu > 0$. To find derivatives, you must follow the definition of fractional derivatives.*

Remark 5. *To show that the operator ${}_cD_x^0$ leaves the function unchanged, we can take the limit of ${}_cD_x^{-\nu}f(x)$ as $\nu \to 0$, but we cannot assume $\nu = 0$ because, in spite we can consider $\Gamma(0) = \infty$, the integral in the definition will diverge when.$v = 0$ as this*

$${}_cD_x^0 f(x) = \frac{1}{\Gamma(0)}\int_0^x (x-t)^{-1} f(t)\, dt$$

So we can expan

$$f(t) = f(x) + (t-x)f'(x) + ... + \frac{(t-x)^n f^{(n)}(x)}{n!} + ...$$

By change of variables in the integral we get the beta function (See [1]) and the formula as follows:

$$_aD_x^\alpha(x-a)^{-\mu} = \frac{1}{(-\alpha)!}\frac{d}{dx}\int_a^x \frac{dt}{(t-a)^\mu(x-t)^\alpha}$$

If we let $T = t - a$ then

$$_aD_x^\alpha(x-a)^{-\mu} = \frac{1}{(-\alpha)!}\frac{d}{dx}\int_0^{x-a} \frac{dT}{T^\mu(x-a-T)^\alpha}$$

Also,if we let $\tau = \frac{T}{x-a}$ then

$$_aD_x^\alpha(x-a)^{-\mu} = \frac{1}{(-\alpha)!}\frac{d}{dx}\int_0^1 (x-a)\frac{d\tau}{(x-a)^{\mu+\alpha}\tau^\mu(1-\tau)^\alpha}$$

$$\begin{aligned}
&= \frac{1}{(-\alpha)!}\frac{d}{dx}\left((x-a)^{1-\mu-\alpha}\int_0^1 \tau^{-\mu}(1-\tau)^{-\alpha}\,d\tau\right)\\
&= \frac{1}{(-\alpha)!}\frac{d}{dx}[(x-a)^{1-\mu-\alpha}]\ B(-\mu-1,-\alpha-1)\\
&= \frac{1}{(-\alpha)!}(1-\mu-\alpha)(x-a)^{-\mu-\alpha}\frac{\Gamma(-\mu+1)\Gamma(-\alpha+1)}{\Gamma(-\mu-\alpha+2)}
\end{aligned}$$

and finally

$$_aD_x^\alpha(x-a)^{-\mu} = \frac{(-\mu)!}{(-\mu-\alpha)!}\frac{1}{(x-a)^{\mu+\alpha}} \tag{3.30}$$

We notice that the derivative is zero if $\mu = 1 - \alpha$

3.4. Abel's equation

We give a discussion here of Abel's equation which leads to the result that the definition (3.24) agrees for nonnegative integer ν with the usual derivative. Rigourous proof of this, requires the idea of functions which are "absolutely continuous" on an interval. This means that for any $\epsilon > 0$ there exists a $\delta > 0$ such that for any finite set of pairwise nonintersecting intervals $[a_k, b_k]$ inside the given interval, such that the sum of the lengths of these intervals is less than δ, we have

$$\sum_{k=1}^{n} \mid f(b_k) - f(a_k) \mid < \epsilon$$

Note that the space of absolutely continuous functions contains the usual Lipschitz space as we will see in the next chapter.

Abel's integral equation can be written (see[8])

$$\frac{1}{(\alpha-1)!}\int_a^x \frac{\varphi(t)\,dt}{(x-t)^{1-\alpha}} = f(x), \quad x > a, 0 < \alpha < 1 \tag{3.31}$$

If the equation has a solution φ it may be found as follows:

Change x to t and t to s, multiply by $(x-t)^{-\alpha}$ and integrate (3.31) to get

$$\int_a^x \frac{dt}{(x-t)^\alpha}\int_a^t \frac{\varphi(s)\,ds}{(t-s)^{1-\alpha}} = (\alpha-1)!\int_a^x \frac{f(t)\,dt}{(x-t)^\alpha} \tag{3.32}$$

We can interchange the order of integration on the left side of (3.32) to get

$$\int_a^x \varphi(s)\, ds \int_s^x \frac{dt}{(x-t)^\alpha (t-s)^{1-\alpha}} = (\alpha-1)! \int_a^x \frac{f(t)\, dt}{(x-t)^\alpha} \tag{3.33}$$

The inside integral on the left can be evaluated using the change of variable:

$$t = s + \tau(x-s)$$

and the definition of the beta function to get

$$\begin{aligned} \int_s^x \frac{dt}{(x-t)^\alpha (t-s)^{1-\alpha}} &= \int_0^1 \frac{d\tau(x-s)}{(1-\tau)^\alpha (x-s)^\alpha \tau^{1-\alpha}(x-s)^{1-\alpha}} \\ \int_0^1 \tau^{\alpha-1}(1-\tau)^{-\alpha} d\tau &= B(\alpha, -\alpha+1) = \frac{\Gamma(\alpha)\Gamma(-\alpha+1)}{\Gamma(1)} \\ &= (\alpha-1)!(-\alpha)! \end{aligned}$$

so, if we substitute this in (3.33) we get

$$\int_a^x \varphi(s)\, ds (\alpha-1)!(-\alpha)! = (\alpha-1)! \int_a^x \frac{f(t)\, dt}{(x-t)^\alpha}$$

then,

$$\int_a^x \varphi(s)\, ds = \frac{1}{(-\alpha)!} \int_a^x \frac{f(t)\, dt}{(x-t)^\alpha}$$

take the derivative to get

$$\varphi(x) = \frac{1}{(-\alpha)!} \frac{d}{dx} \int_a^x \frac{f(t)\, dt}{(x-t)^\alpha} \tag{3.34}$$

By a similar way we could consider the Abel equation in the form (See[8])

$$\frac{1}{(\alpha-1)!}\int_x^b \frac{\varphi(t)\,dt}{(t-x)^{1-\alpha}} = f(x), \quad x<b, \quad 0<\alpha<1 \tag{3.35}$$

and get the solution

$$\varphi(x) = -\frac{1}{(-\alpha)!}\frac{d}{dx}\int_x^b \frac{f(t)\,dt}{(t-x)^{\alpha}} \tag{3.36}$$

The definition of functions which are absolutely continuous on an interval appears when we consider the conditions which allow a solution of Abel's equation. We define

$$f_{1-\alpha}(x) = \frac{1}{(-\alpha)!}\int_a^x \frac{f(t)\,dt}{(x-t)^{\alpha}} \tag{3.37}$$

It can be shown that the Abel equation (3.31) has a unique solution as given by (3.34) if and only if $f_{1-\alpha}$ is absolutely continuous on (a,b) and is equal to zero at $x=a$. See [9].

In fact, as a special case, if f is absolutely continuous, so is $f_{1-\alpha}$. We can substitute

$$f(t) = f(a) + \int_a^t f'(s)ds$$

into the definition of $f_{1-\alpha}$ and get

$$f_{1-\alpha}(x) = \frac{f(a)}{(1-\alpha)!}(x-a)^{1-\alpha} + \frac{1-\alpha}{(1-\alpha)!}\int_a^x \frac{dt}{(x-t)^{\alpha}}\int_a^t f'(s)\,ds \tag{3.38}$$

We can change the order of integration in (3.38) to see that

$$\int_a^x \left(\int_a^t \frac{f'(s)\,ds}{(x-t)^\alpha} \right) dt = \int_a^x \left(\int_s^x \frac{f'(s)}{(x-t)^\alpha} dt \right) ds$$

$$= \int_a^x \frac{(x-s)^{1-\alpha}}{1-\alpha} f'(s)\,ds = \int_a^x \frac{(x-t)^{1-\alpha}}{1-\alpha} f'(t)\,dt,$$

which means that

$$f_{1-\alpha}(x) = \frac{1}{(1-\alpha)!} \left[f(a)(x-a)^{1-\alpha} + \int_a^x f'(t)(x-t)^{1-\alpha} dt \right] \tag{3.39}$$

This means that we may write the solution of Abel's equation in the form

$$\varphi(x) = \frac{1}{(-\alpha)!} \left[\frac{f(a)}{(x-a)^\alpha} + \int_a^x \frac{f'(s)\,ds}{(x-s)^\alpha} \right] \tag{3.40}$$

In the same way, for the other version of Abel's equation which we gave (3.35), we get the solution in the form

$$\varphi(x) = \frac{1}{(-\alpha)!} \left[\frac{f(b)}{(b-x)^\alpha} - \int_x^b \frac{f'(s)\,ds}{(s-x)^\alpha} \right]$$

and so it is that we can write the fractional derivatives as

$$\mathcal{D}^\alpha_{a+} f = \frac{1}{(-\alpha)!} \left[\frac{f(a)}{(x-a)^\alpha} + \int_a^x \frac{f'(t)\,dt}{(x-t)^\alpha} \right] \tag{3.41}$$

$$\mathcal{D}^\alpha_{b-} f = \frac{1}{(-\alpha)!} \left[\frac{f(b)}{(b-x)^\alpha} - \int_x^b \frac{f'(t)\,dt}{(t-x)^\alpha} \right] \tag{3.42}$$

As we have introduced, from this discussion, we conclude that Abel's equation in (3.31) can be written as

$$_aD_x^{-\alpha}\varphi(t) = f(x)$$

and the solution we get is

$$\varphi(t) =_a D_x^{\alpha} f(x)$$

As a special case, when $\alpha = 1$, we get from (3.31) that

$$_aD_x^{-1}\varphi(t) = \int_a^x \varphi(t)dx = f(x)$$

and this agrees with the usual derivative.

3.5. The generalized Leibniz rule

In this section we will introduce a useful formula for ${}_aD_x^\alpha f(x)$ where f is an analytic function on (a,b) as stated in (3.23) .Then we will use this formula to reach the generalized Leibniz rule for finding ${}_aD_x^\alpha(fg)$ where f and g are analytic functions.

If the series

$$f(x) = \sum_{n\geq 0} f_n(x)$$

for f_n continuous on (a,b), is uniformly convergent on the interval, then termwise fractional integration can be done and the new series will also be uniformly convergent on the interval. The proof is done by estimates of the absolute value of the fractional integral of f and partial sums of the series and we can use the uniform convergence of the series. (See [13]).

If the fractional derivatives exist, then term by term fractional differentiation is meaningful also. To see this we reduce it to the integration case. The following result can be proved by taking the cases of differentiation and integration separately.

Theorem 4. *If $f(x)$ is an analytic function on the interval (a,b), then, for any α,*

$${}_aD_x^\alpha f(x) = \sum_{n=0}^{\infty} \begin{pmatrix} \alpha \\ n \end{pmatrix} \frac{(x-a)^{n-\alpha}}{(n-\alpha)!} f^{(n)}(x) \qquad (3.43)$$

around x in the interval.

Proof

For negative α, let $\alpha = -v$ where $v > 0$,we can use the integral definition and the power series expansion of $f(t)$ around x and the termwise integration which is possible as follows.

Since f is analytic we can write (around x in the interval)

$$f(t) = \sum_{n=0}^{\infty} \frac{(-1)^n (x-t)^n}{n!} f^{(n)}(x) \tag{3.44}$$

and

$${}_aD_x^{-\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_a^x (x-t)^{\nu-1} f(t)dt \tag{3.45}$$

If we subsitute (3.44) in (3.45) we get

$$\begin{aligned}
{}_aD_x^{-\nu} f(x) &= \frac{1}{\Gamma(\nu)} \int_a^x (x-t)^{\nu-1} \sum_{n=0}^{\infty} \frac{(-1)^n (x-t)^n}{n!} f^{(n)}(x)dt \\
&= \frac{1}{\Gamma(\nu)} \sum_{n=0}^{\infty} f^{(n)}(x) \frac{(-1)^n}{n!} \int_a^x (x-t)^{\nu-1}(x-t)^n dt \\
&= \frac{1}{\Gamma(\nu)} \sum_{n=0}^{\infty} f^{(n)}(x) \frac{(-1)^n}{n!} \int_a^x (x-t)^{\nu+n-1}\, dt \\
&= \frac{1}{\Gamma(\nu)} \sum_{n=0}^{\infty} f^{(n)}(x) \frac{(-1)^n}{n!} \frac{(x-t)^{\nu+n}}{n+\nu}
\end{aligned}$$

and use the identity (See[8])

$$\begin{pmatrix} -\nu \\ n \end{pmatrix} = \frac{(-1)^n \Gamma(n+\nu)}{n!\Gamma(\nu)} \tag{3.46}$$

to get

$$_aD_x^{-\nu}f(x) = \sum_{n=0}^{\infty} \binom{-\nu}{n} \frac{(x-t)^{n+\nu}}{(n+\nu)!} f^{(n)}(x)$$

The case $\alpha = 0$ is

$$\begin{aligned} {}_aD_x^0 f(x) &= \sum_{n=0}^{\infty} \binom{0}{n} \frac{(x-t)^n}{(n)!} f^{(n)}(x) \\ &= \sum_{n=0}^{\infty} \frac{(x-t)^n}{(n)!} f^{(n)}(x) \\ &= f(x) \end{aligned}$$

For positive α we write

$$\alpha = [\alpha] + \{\alpha\}$$

to indicate the integer and fractional parts of α with the right side, e.g.

$$\frac{7}{3} = \left[\frac{7}{3}\right] + \left\{\frac{7}{3}\right\} = 2 + \frac{1}{3}$$

Then

$$_aD_x^{\alpha} f(x) = \left(\frac{d}{dx}\right)^{[\alpha]+1} {}_aD_x^{\{\alpha\}-1} f(x) \tag{3.47}$$

and the right side of (3.47) is

$$\left(\frac{d}{dx}\right)^{[\alpha]+1} \sum_{n=0}^{\infty} \binom{\{\alpha\}-1}{n} \frac{(x-a)^{n-\{\alpha\}+1} f^{(n)}(x)}{(1-\{\alpha\}+n)!} \tag{3.48}$$

Carrying out term by term differentiation, (3.48) becomes

$$\sum_{n=0}^{\infty} \begin{pmatrix} \{\alpha\}-1 \\ n \end{pmatrix} \left(\frac{d}{dx}\right)^{[\alpha]+1} \frac{(x-a)^{n-\{\alpha\}+1} f^{(n)}(x)}{(1-\{\alpha\}+n)!}$$

then the usual Leibniz rule (3.17) gives

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \begin{pmatrix} \{\alpha\}-1 \\ n \end{pmatrix} \begin{pmatrix} [\alpha]+1 \\ k \end{pmatrix} \frac{(x-a)^{n-\alpha+k} f^{(n+k)}(x)}{(n-\alpha+k)!} \tag{3.49}$$

Now use a new variable for summation: $j = n + k$. and (3.49) becomes

$$\sum_{j=0}^{\infty} \left(\sum_{n=0}^{j} \begin{pmatrix} \{\alpha\}-1 \\ n \end{pmatrix} \begin{pmatrix} [\alpha]+1 \\ j-n \end{pmatrix} \right) \frac{(x-a)^{j-\alpha} f^{(j)}(x)}{(j-\alpha)!} \tag{3.50}$$

By "convolution" of the binomial coefficients, See [4]

$$\sum_{n=0}^{j} \binom{q}{n} \binom{Q}{j-n} = \binom{q+Q}{j} \tag{3.51}$$

we get the desired expression (3.43).

Example 1. *If the function $f(x)$ has an expansion of the form*

$$f(x) = (x-a)^{\mu} \sum_{n=0}^{\infty} c_n (x-a)^n \tag{3.52}$$

in a neighbourhood of $x = a$, then its fractional derivative is given by

$$ {}_aD_x^\alpha f(x) = (x-a)^{\mu-\alpha} g(x) $$

where

$$ g(x) = \sum_{n=0}^{\infty} \frac{c_n (n+\mu)!}{(n-\alpha+\mu)!} (x-a)^n $$

To see this, write $f(x)$ as

$$ f(x) = \sum_{n=0}^{\infty} c_n (x-a)^{n+\mu} $$

take term by term differentiation to get

$$ {}_aD_x^\alpha f(x) = \sum_{n=0}^{\infty} c_n \frac{(n+\mu)!}{(n-\alpha+\mu)!} (x-a)^{n+\mu-\alpha} $$

$$ = (x-a)^{\mu-\alpha} \sum_{n=0}^{\infty} \frac{c_n (n+\mu)!}{(n-\alpha+\mu)!} (x-a)^n \qquad (3.53) $$

The radius of convergence is not changed by differentiation.

As we will see in the next chapter, we must be careful not to take a derivative of integer order before taking a derivative of fractional order. In the case of analytic f we have

$$ {}_aD_x^\alpha \, {}_aD_x^\beta f = {}_aD_x^{\alpha+\beta} f, \quad \beta < 1 $$

Now **the generalized Leibniz rule for analytic functions** f **and** g is given by the following theorem:

Theorem 5. *For any two analytic functions f and g*

$$ {}_aD_x^{\alpha}(fg) = \sum_{k=0}^{\infty} \binom{\alpha}{k} ({}_aD_x^{\alpha-k} f) g^{(k)} \tag{3.54} $$

for all real values of α

Proof

We will use our rule for derivatives (and integrals) of analytic functions and then use the usual Leibniz rule and reverse the order of summation. The left side of (3.54) is

$$ \sum_{k=0}^{\infty} \binom{\alpha}{k} \frac{(x-a)^{k-\alpha}}{(k-\alpha)!} (fg)^{(k)} \tag{3.55} $$

by (3.17) this becomes

$$ \sum_{k=0}^{\infty} \binom{\alpha}{k} \frac{(x-a)^{k-\alpha}}{(k-\alpha)!} \sum_{j=0}^{k} \binom{k}{j} [g^{(j)}][f^{(k-j)}] $$

we can use the permutation

$$ \sum_{k=0}^{\infty} \sum_{j=0}^{k} = \sum_{j=0}^{\infty} \sum_{k=j}^{\infty} $$

to get

$$\sum_{j=0}^{\infty} g^{(j)} \sum_{k=j}^{\infty} \begin{pmatrix} \alpha \\ k \end{pmatrix} \begin{pmatrix} k \\ j \end{pmatrix} \frac{(x-a)^{k-\alpha}}{(k-\alpha)!} f^{(k-j)} \tag{3.56}$$

let $l = k - j$ to get

$$\sum_{j=0}^{\infty} g^{(j)} \sum_{l=0}^{\infty} \begin{pmatrix} \alpha \\ l+j \end{pmatrix} \begin{pmatrix} l+j \\ j \end{pmatrix} \frac{(x-a)^{l+j-\alpha}}{(l+j-\alpha)!} f^{(l)}$$

Now we can use

$$\begin{pmatrix} \alpha \\ l+j \end{pmatrix} \begin{pmatrix} l+j \\ j \end{pmatrix} = \begin{pmatrix} \alpha \\ j \end{pmatrix} \begin{pmatrix} \alpha-j \\ l \end{pmatrix} \tag{3.57}$$

to get

$$\sum_{j=0}^{\infty} g^{(j)} \sum_{l=0}^{\infty} \begin{pmatrix} \alpha \\ j \end{pmatrix} \begin{pmatrix} \alpha-j \\ l \end{pmatrix} \frac{(x-a)^{l+j-\alpha}}{(l+j-\alpha)!} f^{(l)}$$
$$= \sum_{j=0}^{\infty} g^{(j)} \begin{pmatrix} \alpha \\ j \end{pmatrix} \sum_{l=0}^{\infty} \begin{pmatrix} \alpha-j \\ l \end{pmatrix} \frac{(x-a)^{l+j-\alpha}}{(l+j-\alpha)!} f^{(l)}$$

when we apply (3.43) to the order $\alpha - j$ we get

$$\sum_{l=0}^{\infty} \begin{pmatrix} \alpha-j \\ l \end{pmatrix} \frac{(x-a)^{l+j-\alpha}}{(l+j-\alpha)!} f^{(l)} = {}_a D_x^{\alpha-j} f$$

so, we get the expression given in (3.54).

In fact, if β is non-integer and α is not a negative integer, it can be shown that

$$ {}_aD_x^\alpha(fg) = \sum_{k=-\infty}^{\infty} \begin{pmatrix} \alpha \\ k+\beta \end{pmatrix} ({}_aD_x^{\alpha-\beta-k} f)({}_aD_x^{\beta+k} g) \tag{3.58} $$

See [10], [14]

3.6. Integral of derivatives and derivatives of integrals

In this section we will show laws of exponents for :

(1) **fractional integrals.**

(2) **integral of derivatives**

(3) **derivatives of integrals**

We will state each law as a theorem as follows:

Theorem 6. *Let f be continuous on J, and let $\mu, \nu > 0$.Then for all t in J,*

$$D^{-\nu}[D^{-\mu}f(t)] = D^{-(\mu+\nu)}f(t) = D^{-\mu}[D^{-\nu}f(t)] \tag{3.59}$$

Proof

By definition of the fractional integral,

$$D^{-\nu}[D^{-\mu}f(t)] = \frac{1}{\Gamma(\nu)}\int_0^t (t-x)^{\nu-1}[\frac{1}{\Gamma(\mu)}\int_0^x (x-y)^{\mu-1}f(y)dy]dx$$

Dirichlet's Formula.(See[8]),

$$\int_0^t (t-x)^{\nu-1}dx \int_0^x (x-y)^{\mu-1}f(y)dy = B(\mu,\nu)\int_0^t (t-y)^{\nu+\mu-1}f(y)dy \tag{3.60}$$

can be used to get

$$D^{-\nu}[D^{-\mu}f(t)] = \frac{1}{\Gamma(\nu)\Gamma(\mu)}B(\mu,\nu)\int_0^t (t-y)^{\nu+\mu-1}f(y)dy$$

$$= \frac{1}{\Gamma(\mu+\nu)} \int_0^t (t-y)^{\nu+\mu-1} f(y) dy$$
$$= D^{-(\mu+\nu)} f(t)$$

In a similar way we can show that $D^{-\mu}[D^{-\nu} f(t)] = D^{-(\mu+\nu)} f(t)$

Definition 3. *We consider functions which are piecewise continuous on* $(0, \infty)$ *and integrable on any finite subinterval of* $[0, \infty)$. *Call these "functions of class* C*".*

Theorem 7. *If* f *is continuous for non-negative* x *and* Df *is of class* C*, then for* $\nu > 0$ *we have*

$$ {}_0D_x^{-\nu-1}[Df(x)] = {}_0D_x^{-\nu} f(x) - \frac{f(0)}{\nu!} x^\nu \tag{3.61}$$

Proof

Choose $\epsilon > 0$ and $\eta > 0$. Then for

$$\eta < \xi < x - \epsilon.$$

the functions

$$f(\xi), \quad (x-\xi)^{\nu-1}$$

are continuously differentiable.So, if we calculate

$$\int_\eta^{x-\epsilon} (x-\xi)^\nu (Df(\xi)) \, d\xi$$

$$= \nu \int_\eta^{x-\epsilon} (x-\xi)^{\nu-1} f(\xi) \, d\xi + \epsilon^\nu f(x-\epsilon) - (x-\eta)^\nu f(\eta) \tag{3.62}$$

and take the limits as ϵ and η go to zero and divide by $\nu!$ then we get the result in (3.61).

Theorem 8. *If Df is continuous on J, then for $t > 0$,*

$$D[{}_0D_x^{-\nu}f(x)] = {}_0D_x^{-\nu}[Df(x)] + \frac{f(0)}{\Gamma(\nu)}x^{\nu-1} \tag{3.63}$$

Proof

If we make the change of variable $\xi = x - y^\lambda$ where $\lambda = 1/\nu$ in

$${}_0D_x^{-\nu}f(x) = \frac{1}{\Gamma(\nu)}\int_0^x (x-\xi)^{\nu-1}f(\xi)\,d\xi$$

we get

$${}_0D_x^{-\nu}f(x) = \frac{1}{\nu!}\int_0^{x^\nu} f(x-y^\lambda)\,dy$$

Then for positive x we have

$$D[{}_0D_x^{-\nu}f(x)] = \frac{1}{\nu!}\left[f(0)(\nu x^{\nu-1}) + \int_0^{x^\nu}\frac{\partial}{\partial x}f(x-y^\lambda)\,dy\right]$$

Changing the variable back, $x - y^\lambda = \xi$ gives the result in (3.63).

To generalize theorem 8, let $p > 1$ be a positive integer and $\nu > 0$. If $D^p f$ is continuous on $[0,\infty)$ then we can differentiate (3.63) again to get

$$D^2[{}_0D_x^{-\nu}f(x)] = D\{{}_0D_x^{-\nu}[Df(x)]\} + \frac{f(0)}{\Gamma(\nu-1)}x^{\nu-2}$$

The expression in curly brackets is given by (3.63) when we replace f by Df, so we get

$$D^2[{}_0D_x^{-\nu}f(x)] = {}_0D_x^{-\nu}[D^2f(x)] + \frac{f(0)}{\Gamma(\nu-1)}x^{\nu-2} + \frac{Df(0)}{\Gamma(\nu)}x^{\nu-1} \tag{3.64}$$

and we can repeat this to get

$$D^p[{}_0D^{-\nu}f(x)] = {}_0D_x^{-\nu}[D^pf(x)] + Q_p(x,\nu-p) \tag{3.65}$$

where

$$Q_p(x,\mu) = \sum_{k=0}^{p-1}\frac{x^{\mu+k}}{(\mu+k)!}D^kf(0) \tag{3.66}$$

3.7. Functions of class $\mathfrak{C}$

If we compare the fractional derivative and the fractional integral of some types of functions, we will find that the fractional derivative of order μ may be obtained from the fractional integral of order ν by replacing ν by $-\mu$. That is, if $D^{-\nu}f(t)$ is the fractional integral of f of order ν, then the fractional derivative

$$D^{\mu}f(t) = [D^{-\nu}f(t)]\lfloor_{\nu=-\mu}. \tag{3.67}$$

However, this conclusion is not necessarily true for all functions of class C in definition 1. All functions that have this property are of the form

$$t^{\lambda}\mu(t)$$

or

$$t^{\lambda}(\ln t)\mu(t),$$

where $\lambda > -1$ and $\mu(t)$ is an entire function. We denote the class of these functions by $\mathfrak{C}$. We can observe that if f is of class $\mathfrak{C}$ then f has fractional derivative and fractional integral of any order. For example, t^{λ} with $\lambda > -1$, polymomials, exponentials and the sine and cosine functions all belong to $\mathfrak{C}$. For example,

By (2.6) we find

$$D^{-\nu}t^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\nu+1)}t^{\lambda+\nu} \quad , \qquad t > 0$$

and by (2.7)

$$D^{\mu}t^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\mu+1)}t^{\lambda-\mu} \quad , \qquad t>0$$

Now, for every $f \in C$, $u < \lambda+1$ and ν is arbitrary, the law of exponents holds for fractional derivative. We will show this by the following theorem:

Theorem 9. *For every $f \in C$, $u < \lambda+1$ and ν is arbitrary, the law of exponents:*

$$D^{\nu}[D^{u}f(t)] = D^{\nu+u}f(t) \tag{3.68}$$

holds for fractional derivative.

Proof: For

$$f(t) = t^{\lambda}\mu(t)$$

where

$$\mu(t) = \sum_{n=0}^{\infty} a_n t^n$$

we get from (3.53) that

$$D^{u}f(t) = t^{\lambda-u}\sum_{n=0}^{\infty} a_n \frac{\Gamma(n+\lambda+1)}{\Gamma(n+\lambda+1-u)}t^n$$

Since by hypothesis $u < \lambda+1$, it follows that $\lambda - u > -1$ and hence $D^{u}f(t) \in C$

$$D^{\nu}[D^{u}f(t)] = \sum_{n=0}^{\infty} a_n \frac{\Gamma(n+\lambda+1)}{\Gamma(n+\lambda+1-u)}[\frac{\Gamma(n+\lambda+1-u)}{\Gamma(n+\lambda+1-u-\nu)}t^{n+\lambda-u-\nu}]$$

$$= \sum_{n=0}^{\infty} a_n \frac{\Gamma(n+\lambda+1)}{\Gamma(n+\lambda+1-(u+\nu))} t^{n+\lambda-(u+\nu)}$$

$$= D^{\nu+u} f(t)$$

But for

$$f(t) = t^{\lambda}(\ln t)\mu(t)$$

we remark here that , for $t > 0$, (see [8])

$$D^{u}[t^{\lambda}(\ln t)] = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\mu+1)} t^{\lambda-\mu}[\ln t + \psi(\lambda+1) - \psi(\lambda-\mu+1)]$$

where

$$\psi(z) = \frac{D\Gamma(Z)}{\Gamma(Z)}$$

then the same arguments as above may be used to reach the law (3.68) .

CHAPTER 4

TAYLOR SERIES AND SOME GEOMETRIC CONSIDERATIONS

4. Taylor Series and Some Geometrical Considerations

4.1. A generalization of Taylor polynomials with remainder

The usual statement of Taylor's theorem with integral remainder is given here so that we can consider how we could adapt this in the case of fractional derivatives. If the derivative of order $n+1$ of f is continuous on an interval containing c and x, and if $P_n(x)$ is the nth Taylor polynomial, then the remainder

$$R_n(x) = f(x) - P_n(x) \tag{4.1}$$

is given by

$$R_n(x) = \frac{1}{n!}\int_c^x (x-t)^n f^{(n+1)}(t)\, dt \tag{4.2}$$

We can get something like this in the case of fractional calculus if we make some restrictions on the class of functions allowed. We extend the notion of absolutely continuous functions. We remind the reader that we define a Lipschitz space as follows. A function f defined on a finite interval $I = [a,b]$ satisfies a Lipschitz condition if

$$| f(x_1) - f(x_2) | \leq A \mid x_1 - x_2 \mid \tag{4.3}$$

for any x_1, x_2 in I and where A is a constant. The Lipschitz space is contained in the space of absolutely continuous functions. But the absolutely continuous functions are a larger set of functions.

For example,

$$f(x) = (x-a)^{\gamma}$$

is absolutely continuous. But it is not Lipschitz for $0 < \gamma < 1$ since the condition in (4.3) does not hold for, say, $x_2 = a$ and $x_1 = 2a$

To see this, for these two values of x_1 and x_2 we find

$$\begin{aligned} & \mid f(x_1) - f(x_2) \mid = |(2a-a)^{\gamma} - (2a-a)^{\gamma}| = a^{\gamma} = aa^{-1+\gamma} \\ = & |x_1 - x_2| \frac{1}{a^{1-\gamma}} \end{aligned}$$

but if

$$x_1 - x_2 \to 0 \text{ then } a \to 0 \quad \text{then there is no } A \text{ satisfiese (4.3).}$$

It is known that an easy characterization of absolutely continuous functions ($\mathcal{AC}$) is this:

$$f(x) \in \mathcal{AC} \Leftrightarrow f(x) = c + \int_a^x \varphi(t)\, dt, \quad where \quad \int_a^b \mid \varphi(t) \mid \, dt < \infty \tag{4.4}$$

That is, the space of indefinite integrals of the space of Lebesgue summable functions $\mathcal{L}_1$. See [13].

We denote by $\mathcal{AC}^n$ the space of functions $f(x)$ on I which have continuous derivatives up to order $n-1$ and with $f^{(n-1)}(x) \in \mathcal{AC}$. Now we require that $\Re(\alpha) > 0$

and we say that $f \in \mathcal{L}_1$ has a "summable fractional derivative" ${}_aD_x^\alpha f$ if

$$ {}_aD_x^{\alpha-n} f \in \mathcal{AC}^n, \quad where \ \ n = [\Re(\alpha)] + 1 \tag{4.5} $$

In fact, if

$$ \frac{d^n}{dx^n} {}_aD_x^{\alpha-n} f \tag{4.6} $$

exists in the usual sense, that is, the fractional integral

$$ {}_aI_x^{n-\alpha} f = {}_aD_x^{\alpha-n} f \tag{4.7} $$

is differentiable n times at every point, then f has a summable fractional derivative,

We can see this by

$$ \frac{d^n}{dx^n} {}_aD_x^{\alpha-n} f =_a D_x^{\alpha-n+n} f =_a D_x^\alpha f \in \mathcal{L}_1 $$

Now we want to show that we have

$$ {}_aD_x^\alpha \, {}_aI_x^\alpha \varphi(x) = \varphi(x) \quad for \ \ \Re(\alpha) > 0 $$

for any $\mathcal{L}_1$ function φ by the following lemma.

Lemma 1. *For any $\mathcal{L}_1$ function φ.*

$$ {}_aD_x^\alpha \, {}_aI_x^\alpha \varphi(x) = \varphi(x) \quad for \ \ \Re(\alpha) > 0 \tag{4.8} $$

Proof:

$$ {}_aD_x^\alpha \, {}_aI_x^\alpha \varphi(x) = ({}_aD_x^n \, {}_aI_x^{n-\alpha})\, {}_aI_x^\alpha \varphi(x) = $$

$$ \frac{1}{(\alpha-1)!(n-\alpha-1)!} \frac{d^n}{dx^n} \int_a^x \frac{dt}{(x-t)^{\alpha-n+1}} \int_s^x \frac{\varphi(s)\, ds}{(t-s)^{1-\alpha}} $$

Interchange the order of integration to get

$$ {}_aD_x^\alpha \, {}_aI_x^\alpha \varphi(x) = \frac{1}{(\alpha-1)!(n-\alpha-1)!} \frac{d^n}{dx^n} \int_a^x \varphi(s) \int_s^x \frac{1}{(t-s)^{1-\alpha}(x-t)^{\alpha-n+1}} dtds \tag{4.9} $$

Let us now evaluate the inner integral in (4.9)

$$ \int_s^x \frac{1}{(t-s)^{1-\alpha}(x-t)^{\alpha-n+1}} dt $$

if we make $T = t - s$ we get

$$ \int_s^x \frac{1}{(t-s)^{1-\alpha}(x-t)^{\alpha-n+1}} dt = \int_0^{x-s} \frac{dT}{T^{1-\alpha}(x-T-s)^{\alpha-n+1}} \tag{4.10} $$

now make $T = (x-s)\tau$ to get

$$ \int_s^x \frac{1}{(t-s)^{1-\alpha}(x-t)^{\alpha-n+1}} dt = \int_0^1 \frac{(x-s)d\tau}{(x-s)^{1-\alpha}\tau^{1-\alpha}(1-\tau)^{\alpha-n+1}(x-s)^{\alpha-n+1}} $$

$$ = \frac{(x-s)}{(x-s)^{2-n}} \int_0^1 \frac{d\tau}{\tau^{1-\alpha}(1-\tau)^{\alpha-n+1}} $$

$$ = (x-s)^{n-1} \int_0^1 \tau^{\alpha-1}(1-\tau)^{n-\alpha-1} d\tau $$

But in general, for f with a summable derivative, by (3.65)

$$ {}_aI_x^\alpha\, {}_aD_x^\alpha f(x) = f(x) - \sum_{k=0}^{n-1} \frac{(x-a)^{\alpha-k-1}}{(\alpha-k-1)!} f_{n-\alpha}^{(n-k-1)}(a) \tag{4.13} $$

where

$$ f_{n-\alpha}(x) = {}_aI_x^{n-\alpha} f, \;\; n = [\Re(\alpha)] + 1 $$

For $0 < \Re(\alpha) < 1$, we have

$$ {}_aI_x^\alpha\, {}_aD_x^\alpha f(x) = f(x) - \frac{f_{1-\alpha}(a)}{(\alpha-1)!}(x-a)^{\alpha-1} \tag{4.14} $$

The extra terms that appear when the order of differentiation and integration is reversed originate from the characterization we gave for $\mathcal{AC}^n$ in (4.4) and the definition in (4.5).

Example 2. *This is a simple example for which we can verify our expression for differentiation followed by integration for a function with a summable derivative. Let* $\alpha = \frac{1}{2}$, *and our function is*

$$ f(x) = \sum_{k=0}^{\infty} \frac{x^{k-\frac{1}{2}}}{(k-\frac{1}{2})!} \tag{4.15} $$

Which will have a nontrivial summable derivative because of our choice of α. *We calculate*

$$ f_{\frac{1}{2}}(x) = {}_0I_x^{\frac{1}{2}} f(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x $$

$$ {}_0D_x^{\frac{1}{2}} f(x) = \sum_{k=1}^{\infty} \frac{x^{k-1}}{(k-1)!} = e^x $$

$$ {}_0I_x^{\frac{1}{2}}\,{}_0D_x^{\frac{1}{2}}f(x) = \sum_{k=1}^{\infty} \frac{x^{k-\frac{1}{2}}}{(k-\frac{1}{2})!} = f(x) - \frac{x^{-\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)} $$

The extra term is

$$ -\frac{f_{\frac{1}{2}}(0)}{\Gamma\left(\frac{1}{2}\right)}x^{-\frac{1}{2}} $$

We can rewrite the formula (4.13) as something which resembles the Taylor expansion with integral remainder as in the following lemma:

Lemma 2. *For f with a summable derivative,*

$$ f(x) = \sum_{j=-n}^{n-1} \frac{(x-a)^{\alpha+j}}{(\alpha+j)!}({}_aD_x^{\alpha+j}f)(a) + R_n(x), \quad \Re(\alpha) > 0 \tag{4.16} $$

where

$$ R_n(x) = I_x^{\alpha+n}\, D_a^{\alpha+n} f(x) $$

Proof

Let $n = [\Re(\alpha)] + 1$, $\beta = \alpha + n$ and $m = [\Re(\beta)] + 1$ then

$$ m = [\Re(a)] + n + 1 = 2n $$

apply this value of m to (4.13) to get

$$ {}_aI_x^{\beta}\,{}_aD_x^{\beta}f(x) = f(x) - \sum_{k=0}^{m-1} \frac{(x-a)^{\beta-k-1}}{(\beta-k-1)!} f_{m-\beta}^{(m-k-1)}(a) \tag{4.17} $$

since $m = [\text{Re}\ \beta] + 1$ we get

$$R_n(x) = I_x^{\alpha+n} D_a^{\alpha+n} f(x) = f(x) - \sum_{k=0}^{2n-1} \frac{(x-a)^{\alpha+n-k-1}}{(\alpha+n-k-1)!} f_{n-\alpha}^{(2n-k-1)}(a)$$

let $j = n - k - 1$ then

$$R_n(x) = f(x) - \sum_{j=-n}^{n-1} \frac{(x-a)^{\alpha+j}}{(\alpha+j)!} f_{n-\alpha}^{n+j}(a) \tag{4.18}$$

but

$$f_{n-\alpha}(x) = [I^{n-\alpha} f(x)] \tag{4.19}$$

so

$$f_{n-\alpha}^{n+j}(a) = D^{n+j} D^{\alpha-n} f(a) = D^{\alpha+j} f(a) \tag{4.20}$$

subsitute (4.20) in (4.18) to get

$$f(x) = \sum_{j=-n}^{n-1} \frac{(x-a)^{\alpha+j}}{(\alpha+j)!} D^{\alpha+j} f(a) + R_n(x) \tag{4.21}$$

4.2. More on Taylor Expansion

In the previous section, we gave a generalization of Taylor polynomial to involve fractional derivative. The systematic method of obtaining a generalization of Taylor series for fractional derivative is to study the behavior of the remainder term (4.2). We are not going to elaborate on this point. Instead, we give an example of a formal Taylor series with fractional derivatives, proposed by Riemann in 1876, which is valid for a certain class of functions.

$$f(x+h) = \sum_{m=-\infty}^{\infty} \frac{h^{m+\alpha}}{(m+\alpha)!} {}_cD_x^{m+\alpha} f(x) \tag{4.22}$$

We illustrate this formula with the function $f(x) = 1$ and choose $c = 0$ and $\alpha = \frac{1}{2}$. But first, it may be shown by induction that

$$\left(m+\frac{1}{2}\right)!\left(-m-\frac{1}{2}\right)! = (-1)^m \pi \frac{2m+1}{2} \tag{4.23}$$

and

$$\left(-m+\frac{1}{2}\right)!\left(m-\frac{1}{2}\right)! = (-1)^{m+1} \pi \frac{2m-1}{2} \tag{4.24}$$

So we check that

$$1 \stackrel{?}{=} \sum_{-\infty}^{m=\infty} \frac{h^{m+\frac{1}{2}}}{(m+\frac{1}{2})!} D^{m+\frac{1}{2}}(1) = \sum_{-\infty}^{m=\infty} \frac{h^{m+\frac{1}{2}}}{(m+\frac{1}{2})!} \frac{x^{-m-\frac{1}{2}}}{(-m-\frac{1}{2})!}$$

$$= \sqrt{\frac{h}{x}} \frac{1}{(\frac{1}{2})!(-\frac{1}{2})!} + \sum_{m\geq 1} \left(\sqrt{\frac{h}{x}}\right)^{2m+1} \frac{1}{(m-\frac{1}{2})!(-m-\frac{1}{2})!}$$

$$+\sum_{m\geq 1}\left(\sqrt{\frac{x}{h}}\right)^{2m-1}\frac{1}{(-m+\frac{1}{2})!\,(m-\frac{1}{2})!}$$

$$=\frac{2}{\pi}\left(\sum_{m\geq 0}\left(\sqrt{\frac{h}{x}}\right)^{2m+1}\frac{(-1)^m}{2m+1}\right)$$

$$+\frac{2}{\pi}\left(\sum_{m\geq 0}\left(\sqrt{\frac{x}{h}}\right)^{2m+1}\frac{(-1)^m}{2m+1}\right)$$

$$=\frac{2}{\pi}\left(\arctan\sqrt{\frac{h}{x}}+\operatorname{arccot}\sqrt{\frac{h}{x}}\right)=1$$

CHAPTER 5

APPLICATIONS

5. Applications

In this chapter, we consider some approaches and arguments used by recent researchers in their attempts to use the fractional calculus as a tool to grapple with physical problems.

In this chapter we concentrate our work in the applications of the fractional calculus: we define the order of contact between two functions, then we apply this definition to a model for the turbulent boundary layer cross flow velocity component, to show that the non integer models "fill the gaps" between the integer models.

At the end of this chapter we give an algorithm, the functional relations, which enables us to write two functions in terms of each other, we present four examples: two well-known examples; we write $\sin t, t$ and e^t, t in terms of each other and two new: we write the relation between speed and distance and then we write the relation between voltage and current in a simple circuit.See:[3],[8].

5.1. Order of contact

In this section we will introduce the definition of the order of contact between two functions. Then we will apply this definition to a model for the turbulent boundary layer cross flow velocity component.

Suppose two curves in the plane, $y = f(x)$ and $y = g(x)$ intersect at a point χ. We say that they have "contact of order $n+1$" at this point if

$$f^{(k)}(\chi) = g^{(k)}(\chi) \quad for\ k = 0, 1, 2, \ldots, n, n+1\ \ but\ f^{(n+2)}(\chi) \neq g^{(n+2)}(\chi) \qquad (5.1)$$

This is sometimes called "$n+2$ point of contact". The idea is that an ordinary intersection of two curves means contact at one point. If they also have the same slope at this point, then they can be said to meet at two points. The infinitesimal approximations of the curves could illustrate this. For example, consider $y = x^2 - \epsilon$ and $y = 0$ near $x = 0$. At this point the curves $y = x^2$ and $y = 0$ have two point contact.

When we approximate a function by a Taylor series expansion around a point we can truncate the function at terms of degree n to get a polynomial which has $n+1$ point contact with the original function at this point. Our last definition of a Taylor type of expansion (4.16) could be used and it will easily be seen that the truncated series gives an approximating function which has an order of contact, in the generalized sense, for any number less than or equal to the highest degree of x which appears in the expansion.

5.1.1. The turbulent boundary layer flows problem

As shown in Figure(3) a typical hodograph representation of the cross flow velocity profile. For example, flow in a curved channel,

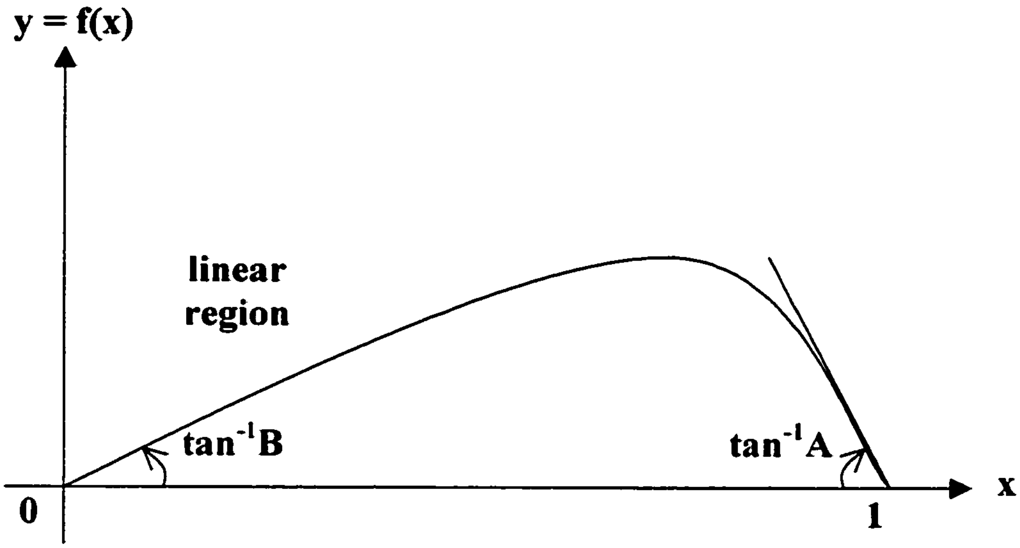


Figure (3)

The cross flow velocity profile

The following conditions are available from physical flow considerations and experimental evidence:

$$f(0) = f(1) = 0; \; f'(0) = b; \; f'(1) = -a \tag{5.2}$$

We want polynomials that satisfy these conditions and which approximate the straight line $y = bx$ near $x = 0$. Just how close the approximation is depends on a parameter n. We write

$$g(x) = f(x) - bx \tag{5.3}$$

and we add the condition that

$$g^{(k)}(0) = 0, \quad for\ k = 0, 1, \ldots, n+1 \tag{5.4}$$

So we are describing $n+2$ point contact, at least, for the curves $y = f(x)$ and $y = bx$. Such polynomials are given by

$$y = b\left(x - (n+2)x^{n+2} + (n+1)x^{n+3}\right) + a\left(x^{n+2} - x^{n+3}\right) \tag{5.5}$$

for non-negative integer n.

For $0 < x < 1$ we find that

$$\lim_{n\to\infty} g(x) = 0$$

To see this

$$\begin{aligned}\lim_{n\to\infty} g(x) &= \lim_{n\to\infty}[f(x) - bx] = \\ &\lim_{n\to\infty}[b\left(x - (n+2)x^{n+2} + (n+1)x^{n+3}\right) + a\left(x^{n+2} - x^{n+3}\right) - bx]\end{aligned}$$

since $0 < x < 1$ we have

$$\lim_{n\to\infty} x^n = 0$$

so,

$$\lim_{n\to\infty} g(x) = bx - bx = 0$$

and this could be illustrated as in this figure:

The figure illustrates the situation, for the values of $n = 2$, $n = 3$, $n = 4$ with $a = b = 1$ as an example:

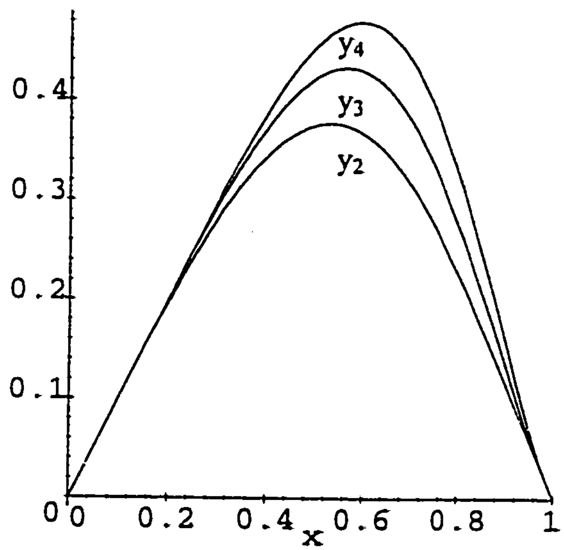


Figure (4)

Polynomials that approximate $y = bx$

$y_2 = 2x^5 - 3x^4 + x$

$y_3 = 3x^6 - 4x^5 + x$

$y_4 = 4x^7 - 5x^6 + x$

However, if we follow the usual notion of "contact" we are restricted to integer values of n. If we allow non-integer values for n, then we need to modify our function

$f(x)$ and the conditions which give the closeness of the fit to $y = bx$ become

$$ {}_0\mathcal{D}_x^{\alpha+1} g(0) = 0 \quad for\ all\ \ 0 \leq \alpha \leq n $$

Not only does fractional differentiation permit us to exactly describe the conditions on the model when n is other than integer, but also the model provides an interesting picture of the effect of the extention of differentiation to noninteger values. As shown in Figure 5, non integer models "fill the gaps" between the integer models. The figure illustrates the situation, for the values of $n = 2$, $n = 2.5$, $n = 3$ with $a = b = 1$ as an example:

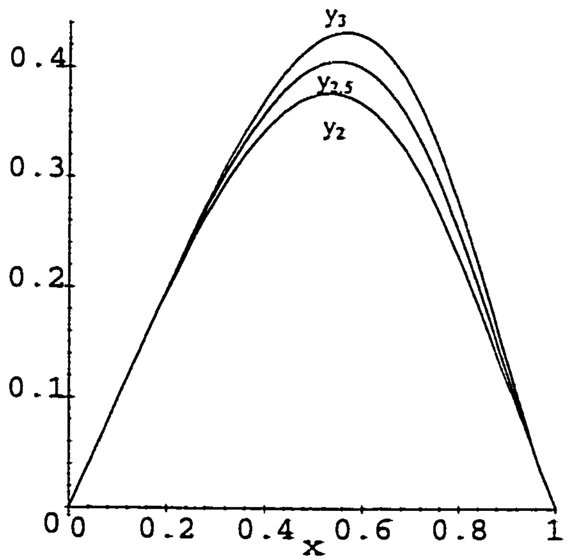


Figure (5). Generalized polynomials that approximate $y = bx$

$y_2 = 2x^5 - 3x^4 + x$

$y_{2.5} = 2.5x^{5.5} - 3.5x^{4.5} + x$

$y_3 = 3x^6 - 4x^5 + x$

5.2. A Functional Relation

By the use of the fractional derivatives we give an algorithmic method for finding a relation between two functions. This algorithm produces results that must be checked. The situation is not much different from what we do in solving an equation like

$$\sqrt{x-1} = 1 - x$$

We square both sides and solve the quadratic to get two answers, but only one of them is correct. In this algorithm there are some assumptions to be made about interchanging the order of summation and integration, but rather than justifying the assumptions we can proceed formally and then check the answer.

Now assume that the Laplace transform of $X(t)$ exists, then the Laplace transform of the fractional integral is

$$\begin{aligned} \mathcal{L}[\mathcal{D}^{-\nu}X(t)] &= \mathcal{L}[\frac{1}{\Gamma(\nu)}\int_c^x (t-\varsigma)^{\nu-1}X(\varsigma)\,d\varsigma] \\ &= \frac{1}{\Gamma(\nu)}\mathcal{L}\ \{t^{\nu-1}\}\mathcal{L}\{X(t)\} \\ &= s^{-\nu}x(s), \qquad \nu > 0 \end{aligned} \tag{5.6}$$

and the Laplace transform of the fractional derivative is

$$\begin{aligned} \mathcal{L}[\mathcal{D}^{\nu}X(t)] &= \mathcal{L}[D^m[\mathcal{D}^{-(m-\nu)}X(t)]] \\ &= s^m\mathcal{L}[\mathcal{D}^{-(m-\nu)}X(t)] - \sum_{k=0}^{m-1} s^{m-k-1}D^k[\mathcal{D}^{-(m-\nu)}X(t)]_{t=0} \end{aligned}$$

$$= s^m[s^{-(m-\nu)}x(s)] - \sum_{k=0}^{m-1} s^{m-k-1} D^{k-(m-\nu)} X(0)$$

$$= s^{\nu}x(s) - \sum_{k=0}^{m-1} s^{m-k-1} D^{k-(m-\nu)} X(0) \tag{5.7}$$

where $m-1 < \nu \leq m$, for $m = 1, 2, 3, ...$

In particular, if $X(t)$ is of class $\mathbb{C}$ then

$$D^{k-(m-\nu)} X(0) = 0, \qquad k = 1, 2, ..., m-1$$

and hence

$$\mathcal{L}[\mathcal{D}^{\nu} X(t)] = s^{\nu} x(s) \tag{5.8}$$

Example 3. *Suppose we have a function which can be written in the form*

$$X(t) = t^{\mu} \sum_{n=0}^{\infty} a_n t^n, \quad \mu > -1$$

Then its fractional derivative is given by

$$({}_0\mathcal{D}_t^{\alpha} X)(t) = t^{\mu-\alpha} \sum_{n=0}^{\infty} a_n \frac{(n+\mu)!}{(n+\mu-\alpha)!} t^n$$

If X is of exponential order then its Laplace transform given by

$$x(s) = \frac{1}{s^{\mu+1}} \sum_{n=0}^{\infty} a_n (n+\mu)! \frac{1}{s^n} \tag{5.9}$$

and

$$\begin{aligned}
\mathcal{L}[{}_0\mathcal{D}_t^\alpha X(t)] &= \sum_{n=0}^{\infty} a_n \frac{(n+\mu)!}{(n+\mu-\alpha)!} \frac{(n+\mu-\alpha)!}{s^{n+\mu-\alpha+1}} \\
&= s^\alpha \sum_{n=0}^{\infty} \frac{a_n (n+\mu)!}{s^{n+\mu+1}} \\
&= \frac{s^\alpha}{s^{\mu+1}} \sum_{n=0}^{\infty} \frac{a_n (n+\mu)!}{s^n} \\
&= s^\alpha x(s) \qquad (5.10)
\end{aligned}$$

However, for the purpose of the algorithm we use a function which is equal to X for $t > 0$ and with vanishing derivatives at $t = 0$ for the values of ν.

The problem of this section is that: we are given two functions, $X(t)$ and $Y(t)$ and we want to find a functional relation of the form

$$Y(t) = \varphi(X(t)) \qquad (5.11)$$

Now we suppose that we can write

$$Y(t) = \int_{-\infty}^{\infty} G(\alpha)\ {}_0D_t^\alpha X(t)\, d\alpha \qquad (5.12)$$

So if we could determine the kernel $G(\alpha)$ we would have the functional relationship.

Suppose that we take the Laplace transform of this equation. We get

$$y(s) = \int_{-\infty}^{\infty} s^\alpha x(s) G(\alpha)\, d\alpha \qquad (5.13)$$

which means that

$$\frac{y(s)}{x(s)} = \int_{-\infty}^{\infty} G(\alpha) s^{\alpha} \, d\alpha \tag{5.14}$$

In principle we might make the substitution $s = e^{-i\omega}$ and the integral would be a Fourier transform which we could invert. But we will not find it necessary to do this in the examples we give. The explicit form of $G(\alpha)$ will be clear.

5.3. Examples

Example 1

Consider

$$X(t) = t, \quad Y(t) = e^t$$

From (5.14) we get

$$\int_{-\infty}^{\infty} G(\alpha) s^\alpha \, d\alpha = \frac{s^2}{s-1}, \qquad s > 1 \tag{5.15}$$

If we express the right side of (5.15) by using long division, we get the series

$$s + 1 + \frac{1}{s} + \frac{1}{s^2} + \ldots$$

So we can use

$$G(\alpha) = \delta(\alpha - 1) + \delta(\alpha) + \delta(\alpha + 1) + \delta(\alpha + 2) + \ldots = \sum_{n=-1}^{\infty} \delta(\alpha + n) \tag{5.16}$$

a series with terms involving the Dirac delta function. Substituting this in the functional relation (5.12) gives

$$e^t = \int_{-\infty}^{\infty} \sum_{n=-1}^{\infty} \delta(\alpha + n) \, {}_0D_t^\alpha(t) \, d\alpha$$

$$= \sum_{n=-1}^{\infty} {}_0D_t^{-n}(t) = 1 + t + \frac{t^2}{2} + \frac{t^3}{3!} + \ldots \tag{5.17}$$

Then we calculate

$$\frac{x(s)}{y(s)} = \nu!\frac{s^2+1}{s^{\nu+1}} = \nu!(s^{1-\nu} + s^{-1-\nu}) \tag{5.22}$$

So

$$G(\alpha) = \nu!(\delta(\alpha+\nu-1) + \delta(\alpha+1+\nu)) \tag{5.23}$$

and this implies

$$t^{\nu} = \nu!({}_0D_x^{1-\nu} + {}_0D_x^{-1-\nu}) \sin t \tag{5.24}$$

If we want to extend this for $\nu > -1$ we have to take care at the integer values of ν. We can do this by using the result (3.65) for exponents.

In (5.24), if ν is an integer, say n, then we may write

$$t^{n} = n!({}_0D_x^{1-n} + {}_0D_x^{-1-n}) \sin t$$

$$t^{n} = n!({}_0D_x^{n+2}\ {}_0D_x^{-2n-1} + {}_0D_x^{n+2}\ {}_0D_x^{-3-2n}) \sin t$$

and then let $n = \nu$

$$t^{\nu} = \nu!({}_0D_x^{\nu+2}\ {}_0D_x^{-2\nu-1} + {}_0D_x^{\nu+2}\ {}_0D_x^{-3-2\nu}) \sin t \tag{5.25}$$

Example 3

The motivation and use of the functional relation algorithm is the establishment of functional relations between observed quantities. This simple example can illustrate this in a case where, again, we already know the answer. If our observations are of speed and distance as functions of time, we might have

$$X(t) = \nu_0 t^2, \quad \textit{and} \quad Y(t) = \nu_0 \frac{t^3}{3} \tag{5.26}$$

Taking the Laplace transforms we easily get

$$\frac{y(s)}{x(x)} = \frac{1}{s} = \int_{-\infty}^{\infty} G(\alpha) s^{\alpha} \, d\alpha \tag{5.27}$$

This implies that $G(\alpha) = \delta(\alpha + 1)$ which means that

$$Y(t) = \int_0^t X(\tau) \, d\tau \tag{5.28}$$

Example 4

We could model the operation of a circuit with resistance and inductance in a situation where the laboratory data is

$$\nu(t) = \nu_0 H(t), \quad I(t) = \frac{\nu_0}{R}(1 - e^{-\frac{R}{L}t})H(t) \tag{5.29}$$

where $H(t)$ is the Heaviside step function. The constants R and L are resistance and inductance. The data in (5.29) represents voltage and current measurements and we want to find voltage as a function of current.

The Laplace transforms give

$$\nu(s) = \nu_0 \frac{1}{s}, \quad i(s) = \frac{\nu_0}{R}\left[\frac{1}{s} - \frac{1}{s + \frac{R}{L}}\right]$$

Then

$$\int_{-\infty}^{\infty} G(\alpha) s^{\alpha}\, d\alpha = Ls + R \tag{5.30}$$

and we can use

$$G(\alpha) = L\delta(\alpha - 1) + R\delta(\alpha) \tag{5.31}$$

and we get the relation

$$\nu(t) = LD_t\left(\frac{\nu_0}{R}\left(1 - e^{-\frac{R}{L}t}\right)H(t)\right) + \nu_0\left(1 - e^{-\frac{R}{L}t}\right)H(t) \tag{5.32}$$

or

$$\nu(t) = L\frac{dI}{dt} + RI \tag{5.33}$$

The figure below shows a simple electrical circuit.

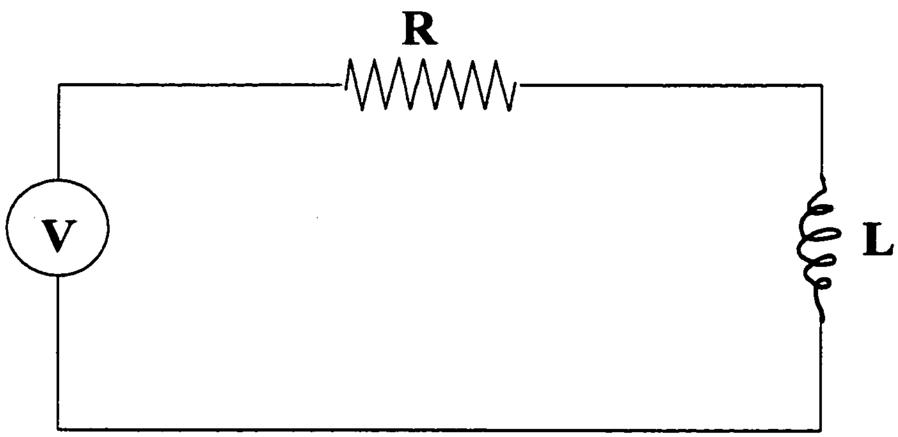


Figure (6)
Simple Circuit

APPENDIX

6. Appendix

6.1. The Gamma Function

The gamma function, denoted by $\Gamma(s)$ is defined by

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt.$$

It is the Mellin transform of e^{-t} where $s \in \mathbb{C}$. $\Gamma(s)$ is regular on $0 < Res < \infty$ or we can say $\Gamma(s)$ is analytic in the right half plane. We also have

$$\Gamma(s+1) = -\int_0^\infty t^s de^{-t} = s\,\Gamma(s)\ .$$

So we can say that $\Gamma(s)$ has meromorphic extention to the complex field $\acute{C}$, with simple poles at $s = 0, -1, -2, -3, ...$

How can we find $\Gamma(\frac{1}{2})$?

$$[\Gamma(\frac{1}{2})]^2 = \int_0^\infty e^{-s} s^{-\frac{1}{2}} ds \int_0^\infty e^{-r} r^{-\frac{1}{2}} dr = \int_0^\infty e^{-(s+r)} r^{-\frac{1}{2}} s^{-\frac{1}{2}} ds dr.$$

We replace s by x^2 and r by y^2 and change to the polar coordinates to get

$$[\Gamma(\frac{1}{2})]^2 = \int_0^\infty \int_0^\infty e^{(x^2+y^2)} 4dxdy = 4\int_0^{\frac{\pi}{2}} \int_0^\infty e^{-\rho^2} \rho d\rho d\theta = \pi$$

Therefore

$$\Gamma(\frac{1}{2}) = \sqrt{\pi}.$$

For positive integer k,

$$\Gamma(k) = (k-1)\Gamma(k-1) = (k-1)!$$

hence

$$\Gamma(\frac{3}{2}) = \frac{1}{2}\Gamma(\frac{1}{2}) = \sqrt{\pi}/2$$

and

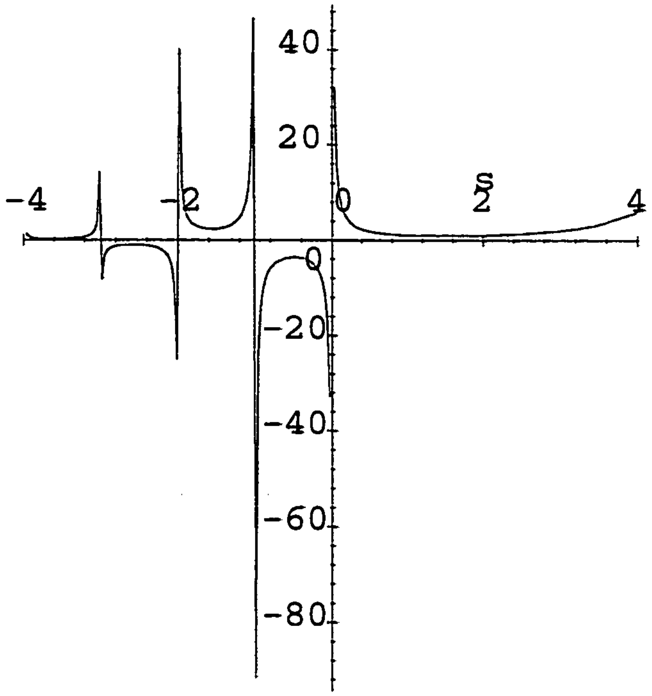
$$\Gamma(\frac{-3}{2}) = (\frac{-2}{3})\Gamma(\frac{-1}{2}) = (\frac{-2}{3})(-2)\Gamma(\frac{-1}{2}) = \frac{4}{3}\sqrt{\pi}.$$

Some usefull identities

$$\Gamma(z)\ \Gamma(1-z) = \pi/\sin z\pi$$

$$\sqrt{\pi}\ \Gamma(2z) = 2^{2z-1}\ \Gamma(z)\Gamma(z+\frac{1}{2})$$

The graph of $\Gamma(x)$



Figure(7)

The graph of $\Gamma(x)$

Few values of $\Gamma(x)$, $0 < x < 1$, can be shown in this table (See[5])

x	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$\Gamma(x)$	9.5	4.59	2.99	2.22	$\sqrt{\pi}$	1.49	1.30	1.16	1.07

6.2. The Beta Function

The Beta function, or Eulerian integral of the first kind, is defined by the Euler integral

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt, \quad Re(x) > 0, \quad Re(y) > 0,$$

and is related to The Gamma function through

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

This allows us to extend the definition of Beta function to any pair of argument where the ratio is defined.

For two equal arguments

$$\frac{4^x}{2}B(x,x) = B(\frac{1}{2},x)$$

and

$$B(x,-x) = \begin{cases} \frac{(-1)^x}{x} & x = 0, \mp 1, \mp 2, \mp 3, ... \\ 0 & otherwise \end{cases}.$$

Look at these special cases

$$B(\frac{1}{4},\frac{1}{4}) = 2\pi/U$$

$$B(\frac{1}{2},\frac{1}{2}) = \pi$$

$$B(\frac{3}{4},\frac{3}{4}) = 2U,$$

where

$$U = \frac{1}{2}(1 + \frac{1}{\sqrt{2}}) \approx 0.8472130848$$

If one (or both) of the arguments is positive integer the recipocal of the Beta function is given by

$$\frac{1}{B(x,n)} = x\binom{n+x-1}{n-1}$$

See[1].

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