On the Second Smallest Eigenvalue of The Laplacain

by

Waleed Ebrahim Al-Jasem

A Thesis Presented to the

FACULTY OF THE COLLEGE OF GRADUATE STUDIES

KING FAHD UNIVERSITY OF PETROLEUM & MINERALS

DHAHRAN, SAUDI ARABIA

In Partial Fulfillment of the
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In

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Al-Jasem, Waleed Ebrahim, M.S.

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Waleed Ebrahim Al–Jasem

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Thesis Committee

Chairman, Dr. Ahmad F. Alameddine

Member, Dr. Mohammad Z. Abu–Sbeih July 4, 93

Member, Dr. Mohammad A. Al–Bar

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الجذور المميزة للمخطط هي الجذور المميزة لمصفوفة اللابلاس . الرمز λ_{n-1} يرمز لثاني أصغر جذر مميز والرمز f يرمز لمتجه الجذر λ_{n-1} . يطلق على المخطط الشجري أنه من النوع الأول إذا وجد صفر أو أكثر في f .

في هذا البحث ندرس مجموعة خاصة من عائلة المخططات الشجرية التي تملك ثلاثة رؤوس منتهية ونثبت أنها من النوع الأول . بالاضافة إلى ذلك فإنه يوجد خواص مختلفة قد درست بالنسبة لهذه المجموعة . كذلك المراكز وأشباه المراكز ورؤوس المخططات قد درست بالنسبة لمجموعات معينة من الكاتربلر .

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THESIS ABSTRACT

Full Name of Student: Waleed Ebrahim Al-Jasem

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The eigenvalues of a graph in this thesis are the eigenvalues of its Laplacian matrix. Let λ_{n-1} and f denote the second smallest eigenvalue and its corresponding eigenvector, respectively. A tree is said to be of type I if f has one or more zeros. A certain class of family having three pendant vertices is characterized to be of type I. Various properties in this class are investigated. Furthermore, centers and centroids are defined and characterized on that family. Centers, centroids and characteristic vertices of certain classes of caterpillars are investigated.

MASTER OF SCIENCE DEGREE

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INTRODUCTION

Let G be a graph with vertex set $V = \{v_1, \ldots, v_n\}$ and edge set E. Denote by $L(G)$ the n–by–n matrix (a_{ij}), where a_{ij} is the degree of the vertex i when $j = i$; $a_{ij} = -1$ when $j \neq i$ and ij is an edge of E; and $a_{ij} = 0$, otherwise. While $L(G)$ depends on the labeling of V, its characteristic polynomial does not. If $\lambda_n \leq \lambda_{n-1} \leq \cdots \lambda_1$ are the eigenvalues of $L(G)$, then $\lambda_n = 0$ and $\lambda_{n-1} > 0$ if and only if G is connected. For connected graphs, the eigenvectors of $L(G)$ corresponding to λ_2 afford "characteristic valuation" of G, a concept introduced by M Fiedler [9].

Chapters II and III explore the "characteristic vertices" arising from characteristic valuations of trees belonging to a family with specified properties. Trees with three end–vertices togther with caterpillars are investigated. The location of a characteristic vertex is also compared with that of a center or a centroid of the tree.

Chapter 1

Basic Properties of the Laplacian Matrix of a Graph

We begin the thesis with an introductory chapter which consists of the basic definitions and concepts of graph theory and the Laplacian matrix. The origin of the problem with a historical background is given in the second section. Some of the known results about the Laplacian matrix are also presented. In the last section, we give the types of trees, introducing the characteristic edge and the characteristic vertex together with some examples.

1.1 Basic Definitions

A *graph* $G = (V, E)$ consists of two sets: a finite set V of elements called *vertices* and a finite set E of elements called *edges*. Each edge is identified with a pair of vertices. The vertices v_i and v_j associated with an edge e are called the *end vertices* of e. The edge e is then denoted as $e = v_iv_j$. If the edges of a graph

G are identified with ordered pairs of vertices, then G is called a *directed graph*. Otherwise, G is called an *undirected graph*. All edges having the same pair of end vertices are called *parallel edges*. If $e = v_iv_i$, then e is called a *self-loop* at vertex v_i. A graph is called a *simple graph* if it has no parallel edges or self loops. A graph G has *order* n if its vertex set has n elements. A graph with no edges is called an *empty graph*. A graph with no vertices (and hence no edges) is called a *null graph*.

An edge is said to be *incident on* its end vertices. Two vertices are *adjacent* if they are end vertices of some edge. If two edges have a common end vertex, then these edges are said to be *adjacent*. The number of edges incident on a vertex v_i is called the *degree (valency)* of the vertex, and it is denoted by $d(v_i)$. A vertex of degree 1 is called a *pendant vertex*. A vertex of degree 0 is called an *isolated vertex*. $\delta(G)$ and $\Delta(G)$ denote, respectively, the *minimum and maximum degrees* in G.

A graph $G' = (V', E')$ is a *subgraph* of the graph $G = (V, E)$ if V' and E' are, respectively, subsets of V and E. The graph $\overline{G} = (V, E')$ is called the *complement* of a simple graph $G = (V, E)$ if the edge v_iv_j is in E' if and only if it is not in E.

A *walk* in a graph $G = (V, E)$ is a finite alternating sequence of vertices and edges $v_0, e_1, v_1, e_2, \ldots, v_{k-1}, e_k, v_k$ beginning and ending with vertices such that v_{i-1} and v_i are the end vertices of the edge e_i, $1 \leq i \leq k$. A walk is *open* if its end vertices are distinct: otherwise it is *closed*. A walk is a *trail* if all its edges are distinct. An open trail is a *path* if all its vertices are distinct. A closed trail is a *circuit* if all its vertices except the end vertices are distinct. The number of edges in a path (circuit) is called the *length* of the path (circuit). A graph G is *connected* if there exists a path between every pair of vertices in G.

Two graphs G_1 and G_2 are said to be *isomorphic* if there exists a one–to–one correspondence between their vertex sets and a one–to–one correspondence between their edge sets so that the corresponding edges of G_1 and G_2 are incident on the corresponding vertices of G_1 and G_2. A graph is said to be *acyclic* if it has no circuits. A *tree* is a connected acyclic graph.

The *Laplacian matrix* $L(G)$ (also known as the degree matrix.) of a graph G with $V(G) = \{v_1, v_2, \ldots, v_n\}$ is the $n \times n$ matrix $L(G) = (a_{ij})$, where

$$a_{ij} = \begin{cases} \text{degree of } v_i, & \text{if } i = j, \\ -1, & \text{if there is an edge between vertex } v_i \text{ and vertex } v_j, \\ 0, & \text{otherwise}. \end{cases}$$

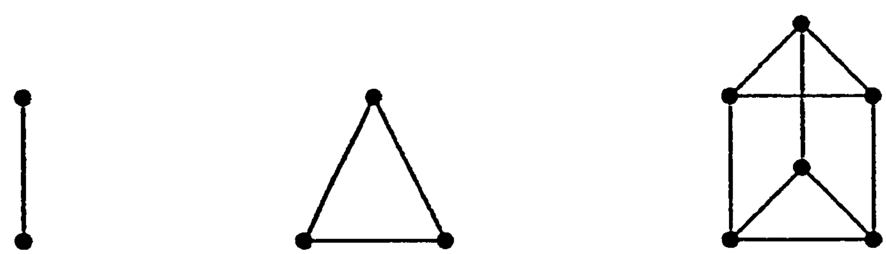
The *characteristic polynomial of* G is defined to be the characteristic polynomial of the Laplacian matrix; i.e. the characteristic polynomial of $G = \phi(G, \lambda) = |L(G) - \lambda I|$. The *eigenvalues of a graph* G of order n are defined to be the roots of the characteristic polynomial of G. Since $L(G)$ is a real symmetric matrix, the eigenvalues of G are real, and so can be ordered as follows:

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq \lambda_n.$$

The sequence of the n eigenvalues is called the *spectrum of* G. The second smallest eigenvalue λ_{n-1} is denoted by $a(G)$; and it is called the *algebraic connectivity* of G. The family of all eigenvectors that correspond to $a(G)$ is denoted by $\mathcal{E}(G)$.

The *cartesian product* $G_1 \times G_2$ of the graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is defined as $G_1 \times G_2 = (V_1 \times V_2, E)$ where $((u_1, u_2), (v_1, v_2)) \in E$ if and only if either $u_1 = v_1$ and $(u_2, v_2) \in E_2$ or $u_2 = v_2$ and $(u_1, v_1) \in E_1$.

As an example, see figure 1.1.1.



G_1 G_2 $G_1 \times G_2$

Figure 1.1.1

Let us mention two concepts related to the algebraic connectivity $a(G)$ of a graph G.

The *edge-connectivity* of G, denoted by $e(G)$ is the minimal number of edges whose removal disconnects G. Similarly, the minimal number of vertices of G

whose removal would result in losing connectivity of the graph G is the *vertex-connectivity* of G, denoted by $v(G)$.

1.2 Historical Background and Applications of the Laplace Matrix of a Graph

There are several matrices which may be associated with a finite simple graph $G = (V, E)$. If $V = \{1, \dots, n\}$, then perhaps the most commonly associated matrix is the $n \times n$ adjacency matrix, $A = A(G)$, defined by

$$a_{ij} = \begin{cases} 1, & \text{if } (i, j) \in E, \\ 0, & \text{otherwise}. \end{cases}$$

Since G is simple, A is a symmetric $(0, 1)$ matrix with zero diagonal. The term "algebraic graph theory" may be defined as the theory which relates the geometric structure of G with the spectral properties of A. Some excellent general references are the books by N. Biggs [3]; Cvetković, Doob, and Sachs [5]; and Cvetković, Doob, Gutman, and Torgasev [7]. An explosion of graph theory began in the 1950's with such people as Coollatz and Singowitz [4] and A.J. Hoffman [12].

Let the edge set of G be $\{e_1, \dots, e_m\}$. For each edge $e_j = v_i v_k$, choose one of the end vertices to be the positive end of e_j and the other to be the negative end. We refer to this procedure by saying that G has been given an orientation. The vertex–edge incidence matrix afforded by an orientation of G is the $n \times m$ matrix

$Q = (q_{ij})$, where

$$q_{ij} = \begin{cases} +1, & \text{if } v_i \text{ is the positive end of } e_j, \\ -1, & \text{if it is the negative end} \\ 0, & \text{otherwise}. \end{cases}$$

This matrix Q has been studied by Poincarré [14], among others.

The matrix that this thesis is concerned with is the matrix defined as

$$L(G) = D(G) - A(G),$$

where $D(G)$ is the diagonal matrix of vertex degrees.

This matrix is variously referred to as the Laplacian matrix, Kirchoff matrix, or matrix of Admittance. The term Laplacian comes from the fact that such matrices arise when using discretizations in looking for nontrivial solutions to $\Delta\phi = \lambda\phi$ on a region Ω.

The origin of the Laplacian matrix can be dated back to Kirchoff [13] in an 1847 paper concerned with electrical networks through the well known matrix-tree theorem: if $L = L(G)$, where G is a graph on n vertices, then:

$$k(G) = (-1)^{i+j} \det L(i|j) \qquad \text{for all } i, j = 1, \ldots, n$$

where $k(G)$ is the number of spanning trees of G and $L(i|j)$ is the matrix obtained from $L(G)$ by deleting the i-th row and j-th column.

Forsman [10] and Gutman [11] have shown how the connection between $L(G) = QQ^t$ and $K(G) = Q^tQ$ simultaneously explain the statistical and the dynamic properties of flexible branched polymer molecules. Indeed, since $L(G)$ and $K(G)$ share the same nonzero eigenvalues, it follows that for bipartite graphs the smallest eigenvalue of $A(G^*) \geq -2$, where G^* is the line graph of G. This observation, first made by Hoffman, has led to a new direction in spectral theory [6], [7]. Eichinger [8] has shown how the spectrum of $L(G)$ may be used to calculate the radius of gyration of a Gaussian molecule. Due to its importance in physical and chemical properties, the spectrum of $L(G)$ is more natural and important than the more widely studied adjacency spectrum. In [2] Bien uses the smallest positive eigenvalue of $L(G)$ to estimate the "magnifying coefficient" of G.

Another application within mathematics is in the problem of decomposition of graphs. The second smallest eigenvalue of $L(G)$ is used in characterizing reducibility. It was proved [7] that a connected graph can be decomposed into two subgraphs by the signs of the eigenvector belonging to the second smallest eigenvalue.

1.3 Known Results on the Laplacian Matrix

In this section, known results about the Laplacian matrix are given.

Theorem 1.3.1. *The number* 0 *is an eigenvalue of every tree.*

Proof: Let T be a tree of order n, and L is the Laplacian matrix of T. If $\vec{u} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$ is a vector of dimension n, then $L\vec{u} = \vec{0} = 0\vec{u}$. So, 0 is an eigenvalue of L.

Theorem 1.3.2 [16]. *Let $\lambda > 1$ be an integer. If λ is an eigenvalue of L, then λ must divide n.*

Theorem 1.3.3 [6]. *If K_n is the complete graph of order n, then $L(K_n) = nI - J$, where J is the $n \times n$ matrix all of whose entries are ones.*

Theorem 1.3.4 [6]. *If $\overline{G}$ is the complement of G, then*

$$L(G) + L(\overline{G}) = nI - J.$$

Theorem 1.3.5 [6]. *The edge connectivity $e(G)$, the vertex connectivity $v(G)$ and the algebraic connectivity $a(T)$ of any graph G of order n satisfy*

$$e(G) \geq v(G) \geq a(G).$$

Theorem 1.3.6 [14]. *If R_n is the space of all real column vectors with n coordinates, and*

$$S = \left\{ x = (x_1, \ldots, x_n)^T \in R_n; \quad \sum_{i=1}^{n} x_i = 0, \quad \sum_{i=1}^{n} x_i^2 = 1 \right\}.$$

then the algebraic connectivity of $G = (V, E)$ satisfies

$$a(G) = \min_{x \in S} \sum_{\substack{(i,k) \in E \\ i<k}} (x_i - x_k)^2, \qquad or$$

$$a(G) = \min_{x \in S} x^T L(G) x.$$

Theorem 1.3.7 [1]. *The algebraic connectivity $a(G)$ satisfies the following properties:*

1. $a(G) \geq 0$, $a(G) = 0 \Leftrightarrow G$ *is not connected.*

2. *If* $G_1 = (V, E_1)$ *and* $G_2 = (V, E_2)$ *and* $E_1 \subset E_2$, *then* $a(G_1) \leq a(G_2)$.

3. *If* $G_1 = (V, E_1)$, $G_2 = (V, E_2)$ *and* $E_1 \cap E_2 = \phi$, *then* $a(G_1) + a(G_2) \leq a(G_3)$ *where* $G_3 = (V, E_1 \cup E_2)$.

4. *If* G_1 *is obtained from G by removing k vertices (and incident edges), ,then* $a(G_1) \geq a(G) - k$.

Theorem 1.3.8 [1]. *Let* G_1 *and* G_2 *be graphs. Then*

$$a(G_1 \times G_2) = \min(a(G_1),\ a(G_2)).$$

Theorem 1.3.9. *For the complete graph* K_n, *the spectrum of* K_n *is:*

$$\begin{pmatrix} 0 & n \\ 1 & n-1 \end{pmatrix}$$

Proof: If $L(K_n)$ is the Laplacian matrix of K_n, then:

$$|L(K_n) - \lambda I| = \begin{vmatrix} n-1-\lambda & -1 & -1 & \cdots & -1 \\ -1 & n-1-\lambda & -1 & \cdots & -1 \\ -1 & -1 & n-1-\lambda & \cdots & -1 \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ -1 & -1 & -1 & \cdots & n-1-\lambda \end{vmatrix}$$

Subtract the first column of $|L(K_n)-\lambda I|$ from all the other columns of $|L(K_n)-\lambda I|$. Then we get:

$$|L(K_n) - \lambda I| = \begin{vmatrix} n-1-\lambda & -n+\lambda & -n+\lambda & \cdots & -n+\lambda \\ -1 & n-\lambda & 0 & \cdots & 0 \\ -1 & 0 & n-\lambda & \cdots & 0 \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ -1 & 0 & 0 & \cdots & n-\lambda \end{vmatrix}$$

Now adding to the first row of the above matrix every one of the other rows, we get:

$$|L(K_n) - \lambda I| = \begin{vmatrix} -\lambda & 0 & 0 & \cdots & 0 \\ -1 & n-\lambda & 0 & \cdots & 0 \\ -1 & 0 & n-\lambda & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ -1 & 0 & 0 & \cdots & n-\lambda \end{vmatrix} = -\lambda(n-\lambda)^{n-1}.$$

Therefore, the spectrum of K_n is:

$$\begin{pmatrix} 0 & n \\ 1 & n-1 \end{pmatrix}.$$

Theorem 1.3.10 [7]. *Let G be a graph of order n. Let $\overline{G}$ be its complement. If $\lambda_1 = 0 \leq \lambda_2 \leq \cdots \leq \lambda_n$ are the eigenvalues of $L(G)$, then $\lambda' = \lambda'_2 \leq \cdots \leq \lambda'_n$ are the eigenvalues of $L(\overline{G})$ where*

$$\lambda'_k = n - \lambda_{n+2-k}, \quad k = 2, \ldots, n.$$

In addition, the eigenvectors of $L(\overline{G})$ corresponding to λ'_k and those of $L(G)$ corresponding to λ_{n+2-k} coincide.

Theorem 1.3.11. *If m is the minimum valency of a noncomplete graph G, then $a(G) \leq m$.*

The *star* S_n is a tree of order n, having $n-1$ pendant vertices and one vertex of degree $n-1$. Figure 1.2.1 shows the general shape of S_n.

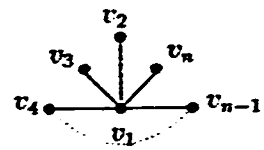


Figure 1.2.1

Theorem 1.3.12. *If S_n is a star of order n, then the spectrum of S_n is:*

$$\begin{pmatrix} 0 & n & 1 \\ 1 & 1 & n-2 \end{pmatrix}.$$

Proof: If $L(S_n)$ is the Laplacian matrix of S_n, then:

$$|L(S_n)-\lambda I| = \begin{vmatrix} 1-\lambda & 0 & 0 & \cdots & -1 \\ 0 & 1-\lambda & 0 & \cdots & -1 \\ 0 & 0 & 1-\lambda & \cdots & -1 \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ -1 & -1 & -1 & \cdots & n-1-\lambda \end{vmatrix}$$

Add to the last row of $|L(S_n)-\lambda I|$ every other row multiplied by $\left(\frac{1}{1-\lambda}\right)$. Then we get:

$$|L(S_n)-\lambda I| = \begin{vmatrix} 1-\lambda & 0 & 0 & \cdots & -1 \\ 0 & 1-\lambda & 0 & \cdots & -1 \\ 0 & 0 & 1-\lambda & \cdots & -1 \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ 0 & 0 & 0 & \cdots & n-1-\lambda-\frac{(n-1)}{1-\lambda} \end{vmatrix}$$

$$= \left[n - 1 - \lambda - \frac{(n-1)}{1-\lambda}\right](1-\lambda)^{n-1}$$

$$= \lambda(\lambda - n)(1-\lambda)^{n-2}$$

Therefore, the spectrum of S_n is:

$$\begin{pmatrix} 0 & n & 1 \\ 1 & 1 & n-2 \end{pmatrix}$$

Theorem 1.3.13 [1]. *If C_n is a circuit of order n, then*

$$a(C_n) = 2\left(1 - \cos\frac{2\pi}{n}\right)$$

Theorem 1.3.14 [5]. (Interlacing Theorem). *Let G be a graph with spectrum $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p$ and let the spectrum of $G - v_1$ be $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{p-1}$. Then the spectrum of $G - v_1$ is "interlaced" with the spectrum of G; that is,*

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \mu_{p-1} \geq \lambda_p.$$

Theorem 1.3.15. *Let T be a tree of order n, and $\vec{f} = (x_1, \ldots, x_n)^t \in \mathcal{E}(T)$, then:*

$$\sum_{i=1}^{n} x_i = 0.$$

Proof. Since the vector $\vec{u} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$ is an eigenvector of L corresponding to zero, i.e., $L\vec{u} = 0\vec{u}$, then each $\vec{x} \in \mathcal{E}(T)$ must be perpendicular to it, i.e.

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad \vec{x} \cdot \vec{u} = 0 \quad \text{or} \quad \sum_{i=1}^{n} x_i = 0.$$

Theorem 1.3.16. *Let T be a tree of order n. Then, $a(T)$ is bounded as follows:*

$$0 = \lambda_n < \lambda_{n-1} = a(T) \leq 1.$$

Proof. Since T is connected, $a(G) = \lambda_{n-1} \neq 0$ (Theorem 1.3.7(1)). Also, using Theorem 1.3.11, $a(T) \leq 1$.

1.4 Types of Trees

We will discuss two types of trees according to the corresponding eigenvectors of $a(T)$. Type I and type II trees will be defined and some results will be obtained based on type I trees.

Theorem 1.4.1 [9]. *Let T be a tree. Suppose $\vec{z} = (z_1, \ldots, z_n)^t$ is an eigenvector of $L(T)$ corresponding to $a(T)$. Then two cases can occur:*

1. *$\tilde{V} = \{i \in V | z_i = 0\} \neq \phi$, then the graph $\tilde{T} = (\tilde{V}, \tilde{E})$ induced by T on $\tilde{V}$ is connected and there exists exactly one vertex $j \in \tilde{V}$ which is adjacent (in*

T) to a vertex not belonging to $\tilde{V}$. Moreover, the values of $\vec{z}$ along any path starting at j are increasing, decreasing, or identically zero.

2. *If $z_i \neq 0$ for all $i \in V$, then T contains exactly one edge jk such that z_j and z_k have different signs, say $z_j > 0$ and $z_k < 0$. Moreover, the values of $\vec{z}$ along any path that starts at j and does not contain k increase while the values of $\vec{z}$ along any path that starts at k and does not contain j decrease.*

If a tree satisfies the first case of the previous theorem. then it is called a *type I tree*, and if it satisfies the second case, then it is a *type II tree*.

The vertex described in the first part of the above theorem is referred to as the *characteristic vertex* of T, and the edge described in the second part of that theorem is called the *characteristic edge* of T.

Theorem 1.4.2. *Let $T = (V, E)$ be a tree. Let $g, h \in \mathcal{E}(T)$. Then $u \in V$ is a characteristic vertex of T afforded by g, if and only if u is a characteristic vertex of T afforded by h.*

Proof: If $a(T)$ is a simple eigenvalue, then g is a nonzero multiple of h and the result is immediate from the definitions. So, we assume $a(T)$ is a multiple root. Let $V_0 = \{v \in V | z(v) = 0$ for all eigenvectors z corresponding to $a(T)\}$. If V_0 were empty, we could find some z such that $z(v) \neq 0$ for all $v \in V$. This contradicts Theorem 1.4.1.

Theorem 1.4.3 [1]. *If T is a tree, and $a(T)$ is multiple, then T is of type I.*

Theorem 1.4.4 [16]. (A Reduction Theorem for Type I Trees). *Let $T = (V, E)$ be a tree on $n \geq 4$ vertices. Suppose there is an eigenvector $\vec{X} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ belonging to $a(T)$ and a pendant vertex $v \in V$ such that $x_v = 0$. Let u be the vertex adjacent to v. Denote by $T_1 = (V_1, E_1)$ the subgraph obtained from T by deleting v from V and $\{u, v\}$ from E. Then*

(i) $x_u = 0$

(ii) $a(T_1) = a(T_2)$

(iii) $\vec{X}|_1$ *is an eigenvector belonging to* $a(T_1)$, *where* $\vec{X}|_1$ *is the restriction of* $\vec{X}$ *to* V_1.

(iv) $F(T_1) = F(T_2)$ *where* $F(T)$ *denotes the set of characteristic vertices of* T.

Example 1.4.1. Let T be the tree shown below in Figure 1.4.1.

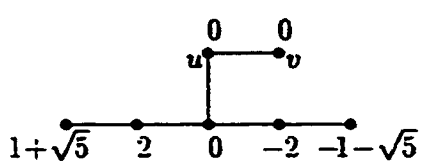


Figure 1.4.1

Then the characteristic polynomial of T is

$$x(x^2 - 3x + 1)^2(x^2 - 6x + 7)$$

and $a(T) = \dfrac{3-\sqrt{5}}{2} \cong 0.382$ has multiplicity 2. One eigenvector belonging to $a(T)$ is shown in Figure 1.4.1.

If vertex v and its edge are erased, the result is T_1, as shown in Figure 1.4.2.

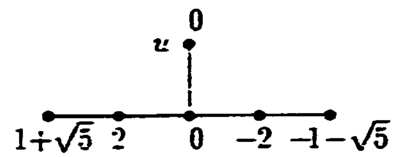


Figure 1.4.2

Not only that, but the numbers which remain constitute an eigenvector belonging to $a(T_1)$. Indeed, the characteristic polynomial of T_1 is

$$x(x-2)(x^2-3x+1)(x^2-5x+3)$$

and one can check that $a(T_1) = a(T)$. (In the case of T_1, $\frac{3-\sqrt{5}}{2}$ is a simple eigenvalue). We may apply the theorem again by removing vertex u from T_1, obtaining the eigenvectors of $a(T_2)$ as shown in Figure 1.4.3. The characteristic polynomial of T_2 is

$$x(x^2-3x+1)(x^2-5x+5).$$

One can check that $a(T_2) = a(T)$, too. Since there is no longer a pendant vertex of value 0, the reduction process stops.

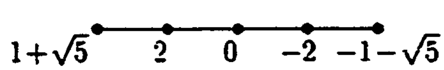


Figure 1.4.3

Example 1.4.2. The reduction process described in Theorem 1.4.4 and in Example 1.4.1 is not entirely reversible. If we increase the degree of vertex v in Figure 1.4.1 by attaching a new pendant vertex to it, we obtain a tree T' of type II.

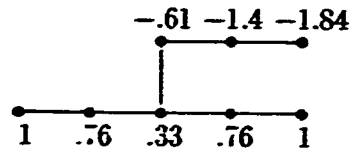


Figure 1.4.4

The characteristic polynomial for T' is

$$q(x) = x(x-2)(x^2-3x+1)(x^4-4x^3+25x^2-22x+4).$$

While it is true that $\lambda = \dfrac{3-\sqrt{5}}{2}$ is an eigenvalue of $L(T')$, λ is greater than the simple eigenvalue of $a(T') = 0.2434$. An approximate valuation of T' is shown in Figure 1.4.4.

Theorem 1.4.5. **(A Partial Converse to the Reduction Theorem).** *Let $T = (V, E)$ be a type I tree. Choose an eigenvector f belonging to $a(T)$ and choose $w \in V$ such that $f(w) = 0$. Let $T' = (V', E')$ be a tree obtained from T by adjoining a new pendant vertex u at w. (So, $V' = V \cup \{u\}$ and $E' = E \cup \{(u, w)\}$). Then $a(T)$ is an eigenvalue of $L(T')$. If $a(T) = a(T')$, then $F(T) = F(T')$ and f' is an eigenvector belonging to $a(T')$, where $f'(v) = f(v)$, $v \in V$, and $f'(u) = 0$.*

Chapter 2

Trees With Three End–Vertices

Let $\mathcal{T}$ be the family of trees with three end vertices. If $T \in \mathcal{T}$, and T is a type I tree, then what is the general shape of T? In this chapter we will partially answer this question by introducing the basic definitions and concepts of passive and active branches. Furthermore, automorphism of graphs is given together with two important theorems. Also, some basic and important theorems for the tree $T \in \mathcal{T}$ are given together with the proofs.

In the last section, we give a list of theorems characterizing a class of type I trees $T \in \mathcal{T}$, together with the general form of the characteristic eigenvector of $a(T)$. Furthermore, the characteristic vertices of these trees are determined and the values of $a(T)$ are given.

2.1 Passive and Active Branches; Automorphism

Suppose v is a vertex of a tree T. Denote by T_v the subgraph of T obtained by deleting v and all edges incident with it. A *branch* (of T) at v is a connected component of T_v. If T is a type I tree, $v = w_T$, the characteristic vertex of T, $\vec{f} \in \mathcal{E}(T)$, and B is a branch at w_T, then f is uniformly $+$, uniformly $-$, or identically zero on the vertices of B. Of course, every $\vec{f} \in \mathcal{E}(T)$ is orthogonal to the vector each of whose component is 1, i.e. an eigenvector afforded by 0. Thus there will always be a positive branch and a negative branch at w_T for any characteristic valuation.

If B is a branch at v, we denote by $r(B)$ the vertex of B which is adjacent (in T) to v. It will frequently be convenient to view B as a rooted tree. In such a situation, we always take $r(B)$ as the root. If $v = w_T$ and $f \in \mathcal{E}(T)$, then $f(r(B))$ determines the sign of f throughout B.

Let T be a type I tree. Let B be a branch at w_T. We call B *passive* if $f(r(B)) = 0$ for every $f \in \mathcal{E}(T)$. A branch at w_T is *active* if it is not passive.

Theorem 2.1.1 (15) *Let T be a type I tree with characteristic vertex w and algebraic connectivity $a(T)$. Let m be the multiplicity of $a(T)$ as an eigenvalue of $L(T)$. Then exactly $m+1$ of the branches at w are active.*

Theorem 2.1.2 (15) *Let w be a vertex of the tree $T = (V, E)$. Suppose $B_1 = (V_1, E_1)$ and $B_2 = (V_2, E_2)$ are two (different) branches of T rooted at w. Let $\alpha : V_1 \to V_2$ be an isomorphism of the rooted trees B_1 and B_2. (Then, in particular, $\alpha(w) = w$.) Let $f \in \mathcal{E}(T)$ be fixed but arbitrary, then either*

(i) $f(\alpha(v)) = f(v), \quad v \in V_1$, *or*

(ii) $F(T) = \{w\}$ *and there is a* $g \in \mathcal{E}(T)$ *such that* $g(v) = -g(\alpha(v)) > 0$, $w \neq v \in V_1$; *and* $g(v) = 0, \quad v \notin V_1 \cup V_2$.

Let w be a vertex of the tree $T = (V, E)$. Suppose $B_1 = (V_1, E_1)$ and $B_2 = (V_2, E_2)$ are two branches of T rooted at w. Assume the rooted trees B_1 and B_2 are isomorphic. We say that $f \in \mathcal{E}(T)$ *distinguishes* B_1 from B_2 if there is an isomorphism

$$\alpha : V_1 \to V_2$$

such that $f(\alpha(v)) \neq f(v)$ for some $v \in V_1$.

Theorem 2.1.3 (15) *Let $T = (V, E)$ be a tree. Choose $w \in V$. Suppose there are $k > 2$ branches $B_1 = (V_1, E_1), \ldots, \ B_k = (V_k, E_k)$ rooted at w. If these branches are all isomorphic as rooted trees, and if there is a characteristic valuation of T which distinguishes B_1 from B_2, then the multiplicity of $a(T)$ is at least $k - 1$.*

Theorem 2.1.4 (15) *Let $T = (V, E)$ be a type I tree with characteristic vertex w and algebraic connectivity $a(T)$. Let $f \in \mathcal{E}(T)$. Suppose $T' = (V', E')$ is the* *tree obtained from T by adjoining a new pendant vertex, p, to w. [So, $V' = V \cup \{p\}$, $E' = E \cup \{\{p,w\}\}$.] Extend f to a function f' on V' by defining $f'(p) = 0$. Then $a(T') = a(T)$ and $f' \in \mathcal{E}(T')$. [In particular, T' is a type I tree with the characteristic vertex w.]*

2.2 Basic Results on the Laplacian Matrix of a Tree

The next few theorems are just basic results on arbitrary trees of type I. Some of the proofs are presented to simplify the restriction to T. They will be used in the next section.

Theorem 2.2.1 *Let $\{i,j\} \in E$ and $z_i > 0$, $z_j > 0$. Suppose also that vertex i is on the unique path from vertex j to either the characteristic vertex of T (Case 1) or the characteristic edge of T (Case 2). If the degree of vertex j is $k+1$, let vertices $i_1, i_2, \ldots i_k$ be the other neighbors of j besides i. Then*

1. $z_i < z_j < z_{i_s}$, $s = 1, \ldots, k$

and

2. $z_j - z_i = a \cdot z_j + \sum_{s=1}^{k} (z_{i_s} - z_j)$ *where* $a = a(T)$.

Proof:

1. Assume that $\tilde{V} = \{i \in V | z_i = 0\} \neq \phi$, then call the characteristic vertex of T, c. Then any path from c to i_s will be either increasing or decreasing. (The values of z along this path cannot be identically zero, since otherwise z_i and z_j will be zero.) If the values of z along this path are decreasing, then $z_i < 0$, $z_j < 0$ (contradiction). So the values of z along this path is increasing, and so $z_i < z_j < z_{i_s}$.

   Assume that $\tilde{V} = \phi$, then the characteristic edge of T is say ℓ, k such that $z_\ell > 0$, $z_k < 0$. Consider the path from ℓ to i_s, if the path contains k, then the path from k to i_s is decreasing, but $z_k < 0$; so $z_{i_s}, z_j, z_i < 0$ (contradiction). So the path does not contain k, so the path must be increasing for the values of z. Therefore, $z_i < z_j < z_{i_s}$.

2. Since a is the smallest eigenvalue for L, then

$$L\vec{z} = a\vec{z} \quad \text{where } \vec{z} \text{ is the corresponding eigenvector.}$$

   Then by considering the j-th element of this equality we have:

$$-z_i + (k+1)z_j - z_{i_1} - z_{i_2} - \cdots - z_{i_s} = az_j$$

$$\Rightarrow -z_i + kz_j + z_j - \sum_{s=1}^{k} z_{i_s} = az_j$$

$$\Rightarrow z_j - z_i = az_j + \sum_{s=1}^{k} (z_{i_s} - z_j)$$

.

Remark 2.2.1 *Let T be a tree with $\lambda_{n-1} = a$. If i is a vertex with degree d, then for any eigenvector $\vec{X} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ belonging to a, we have:*

$$\sum_{(j,i)\in E} x_j = (d-a)x_i.$$

Proof: By the definition of the eigenvalue a, we have:

$$L\vec{x} = a\vec{x}.$$

By the definition of the Laplacian matrix, we will get:

$$-\sum_{(j,i)\in E} x_j + dx_i = ax_i \quad \text{for the } x_i \text{ element in } x$$
$$\Rightarrow \sum_{(j,i)\in E} x_j = (d-a)x_i$$

Remark 2.2.2 *Suppose v is a pendant vertex with $(u,v) \in E$. If $\vec{X} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, then $x_u = (1-a)x_v$. In particular, if $a = 1$, then $x_u = 0$ for every u adjacent to a pendant vertex. In addition, if $x_v = 0$, then $x_u = 0$, i.e., a pendant vertex is never an* isolated zero *of an eigenvector belonging to a.*

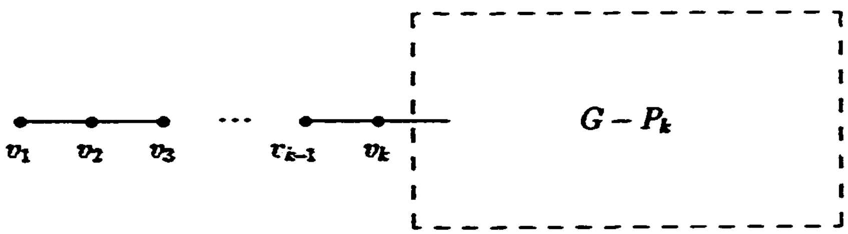


Figure 2.2.1

Theorem 2.2.2 *Let T be a tree as shown in Figure 2.2.1 which has a path of order k as a subtree of T. If $g = (a_1, \ldots, a_n)^t \in \mathcal{E}(T)$, then:*

$$a_i = a_1 f_i(a); \qquad i = 2, \ldots, k$$

where $f_i(a)$ is a polynomial in a.

Proof: By Remark 2.2.2

$$\Rightarrow a_2 = (1-a)a_1 = a_1 f_2(a) \text{ where } f_2(a) = (1-a).$$

Again,

$$a_1 + a_3 = (2-a)a_2$$

$$\Rightarrow a_3 = (2-a)a_2 - a_1$$

$$= (2-a)(1-a)a_1 - a_1 = [(2-a)(1-a) - 1]\, a_1$$

$$= a_1 f_3(a) \text{ where } f_3(a) = (2-a)(1-a) - 1$$

$$a_i + a_{i-2} = (2-a)a_{i-1}$$

$$\Rightarrow a_i = (2-a)a_{i-1} - a_{i-2}$$

Assume that the statement is true for $a_1, a_2, \ldots, a_{i-1}$.

$$a_i = (2-a)a_1 f_{i-1}(a) - a_1 f_{i-2}(a)$$

$$= a_1 \left[(2-a) f_{i-1}(a) - f_{i-2}(a)\right]$$

$$a_i = a_1 f_i(a).$$

Theorem 2.2.3 *If T is a tree of order n, and the diameter of T is d, then $a(T) \leq 2\left(1 - \cos\frac{\pi}{d+1}\right)$.*

Proof: Let $n = d+k+1$ and let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} > \lambda_n = 0$ be the eigenvalues of T. If we have more than one longest path, then specify one of them, say P_1. Remove one pendant vertex which is not belonging to P_1 (say v_1). Let T_1 be the tree obtained after removing v_1, having the spectrum:

$$\mu_1^{(1)} \geq \mu_2^{(1)} \geq \cdots \geq \mu_{n-2}^{(1)} > \mu_{n-1}^{(1)} = 0.$$

By using the Interlacing Theorem 1.3.14, we have:

$$\lambda_1 \geq \mu_1^{(1)} \geq \lambda_2 \geq \mu_2^{(1)} \geq \cdots \geq \mu_{n-2}^{(1)} \geq \lambda_{n-1} > \mu_{n-1}^{(1)} = \lambda_n = 0.$$

Again, remove one pendant vertex v_2 which is not belonging to P_1 to get the tree T_2. Let the spectrum of T_2 be:

$$\mu_1^{(2)} \geq \mu_2^{(2)} \geq \cdots \geq \mu_{n-3}^{(2)} > \mu_{n-2}^{(2)} = 0.$$

Using the Interlacing Theorem, we get:

$$\mu_1^{(1)} \geq \mu_1^{(2)} \geq \mu_2^{(1)} \geq \mu_2^{(2)} \geq \cdots \geq \mu_{n-3}^{(2)} \geq \mu_{n-2}^{(1)} > \mu_{n-2}^{(2)} = \mu_{n-1}^{(1)} = 0.$$

We remove pendant vertices which are not belonging to P_1 k-times so that we get a path of order $d+1$. Let $\mu_1^{(i)} \geq \mu_2^{(i)} \geq \cdots \geq \mu_{n-i}^{(i)} = 0$ be the eigenvalues of the tree T_i after removing the vertex v_i. Again, applying the Interlacing Theorem we get:

$$\mu_{n-k-1}^{(k)} \geq \mu_{n-k}^{(k-1)} \geq \mu_{n-k-1}^{(k-2)} \geq \cdots \geq \mu_{n-2}^{(1)} \geq \lambda_{n-1}$$

but by Theorem 2.3.3 $\mu_{n-k-1}^{(k)} = a(P_{d+1}) = 2\left(1 - \cos\frac{\pi}{d+1}\right)$, $\lambda_{n-1} = a(T) \Rightarrow a(T) \leq 2\left(1 - \cos\frac{\pi}{d+1}\right)$.

2.3 Towards a Characterization of Type I Trees with Three End–Vertices

In this section, we will study all trees $T \in \mathcal{T}$ which are of type I, giving their general shape. We will also locate the characteristic vertices of these trees. Furthermore, we will give the general formulas for $a(T)$ with the corresponding eigenvector(s).

Theorem 2.3.1 *If $T \in \mathcal{T}$ is a tree of type I, then the multiplicity of $a(T)$ is not more than 2.*

Proof: If the multiplicity is more than 2, then by Theorem 2.1.1, the active branches are at least 4. But we could only have 2 or 3 branches.

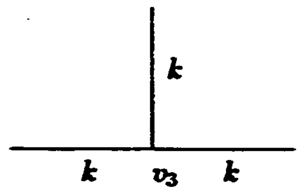


Figure 2.3.1

Theorem 2.3.2 *If T is of order $n = 3k + 1$, having the shape shown in Figure 2.3.1, then:*

1. *T is of type I.*

2. *The only characteristic vertex is v_3.*

3. *There exists $f \in \mathcal{E}(T)$ such that $|\tilde{V}| = 1$ where $\tilde{V} = \{v_i : \vec{f}(v_i) = 0\}$.*

Proof: The three branches rooted at v_3 are all isomorphic as rooted trees. Call them B_1, B_2 and B_3. Assume that there is no characteristic valuation of T which distinguishes B_1 from B_2, B_2 from B_3, or B_1 from B_3. If we have $f(v)$, $v \in B_1$ is somewhere positive and somewhere negative, then according to our first assumption $f(B_2)$ and $f(B_3)$ will be so, and this contradicts Theorem 1.4.1. So, $f(B_1), f(B_2)$ and $f(B_3)$ are either all positive or all negative.

By Theorem 1.3.15, $\vec{f}(v_3) = -\sum_{\substack{v_i \in V \\ i \neq 3}} \vec{f}(v_i)$. This result again contradicts Theorem 1.4.1. Therefore, this is a contradiction to our first assumption. As a result, there exists a characteristic valuation of T which distinguishes two of the branches. According to Theorem 2.1.3, the multiplicity of $a(T)$ is at least 2. By Theorem 1.4.3 the graph is of type I. Also, by Theorem 2.3.1, the multiplicity of $a(T)$ is 2.

Now, if the characteristic vertex $w_T \neq v_3$, then we will have at most 2 active branches, but by Theorem 2.1.1, we should have 3 active branches at w_T, and we have a contradiction. Therefore, $w_T = v_3$. Now, since we have 3 active branches at $w_T = v_3$, $\tilde{V} = \{v_3\}$.

Theorem 2.3.3 *[6] If P_n is a path of order n, then the eigenvalues of $L(P_n)$ are* $\lambda_1 = 4\sin^2\left(\frac{\pi k}{2n}\right)$ $\quad k = 0, 1, 2, \ldots, n-1$. $a(P_n) = \lambda_1 = 4\sin^2\left(\frac{\pi}{2n}\right)$.

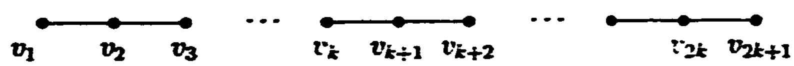


Figure 2.3.2

Theorem 2.3.4 *If P_{2k+1} is a path of order $2k+1$, labelled as shown in figure (2.3.2), then:*

1. *The path is of type I.*

2. *The only characteristic vertex is $w_T = v_{k+1}$.*

3. The multiplicity of $a(T)$ is one, and if $\vec{f} \in \mathcal{E}(P_{2k+1})$, then

$$\vec{f} = \begin{pmatrix} a_1 \\ a_1 f_2(a) \\ a_1 f_3(a) \\ \vdots \\ a_1 f_k(a) \\ o \\ -a_1 f_k(a) \\ -a_1 f_{k-1}(a) \\ \vdots \\ -a_1 f_3(a) \\ -a_1 f_2(a) \\ -a_1 \end{pmatrix}$$

where a_1 is a nonzero real number, and $f_i(a)$ are polynomials in a.

Proof: Let $\vec{f} \in \mathcal{E}(P_{2k+1})$ and $\vec{f}(v_i) = a_i$. By using Theorem 2.2.4,

$$\vec{f}(v_{k+1}) = a_1 f_{k+1}(a) = a_{2k+1} f_{k+1}(a)$$

$$\Rightarrow (a_1 - a_{2k+1}) f_{k+1}(a) = 0.$$

If $a_1 = a_{2k+1}$ then either $\sum_{i=1}^{2k+1} \vec{f}(v_i) \neq 0$ which contradicts Theorem 1.3.15, or we will have more than 2 characteristic vertices which contradicts Theorem 1.4.1. Therefore, $f_{k+1}(a) = 0$ and $f(v_{k+1}) = 0$. So, the path is a type I tree. Since we don't have more than 2 branches, then by using Theorem 2.1.1, the multiplicity of $a(T)$ is one. Again, by Theorem 2.2.4,

$$\vec{f} = \begin{pmatrix} a_1 \\ a_1 f_2(a) \\ a_1 f_3(a) \\ \vdots \\ a_k f_k(a) \\ o \\ a_{2k+1} f_k(a) \\ \vdots \\ a_{2k+1} f_2(a) \\ a_{2k+1} \end{pmatrix}$$

By using Remark 2.2.2:

$$a_k + a_{k+2} = 0 \Rightarrow a_k = -a_{k+2}.$$

So $a_1 f_k(a) = -a_{2k+1} f_k(a)$. Since $f_k(a) \neq 0$, $a_1 = -a_{2k+1}$, and so

$$\vec{f} = \begin{pmatrix} a_1 \\ a_1 f_2(a) \\ a_1 f_3(a) \\ \vdots \\ a_1 f_k(a) \\ o \\ -a_1 f_k(a) \\ \vdots \\ -a_1 f_3(a) \\ -a_1 f_2(a) \\ -a_1 \end{pmatrix}.$$

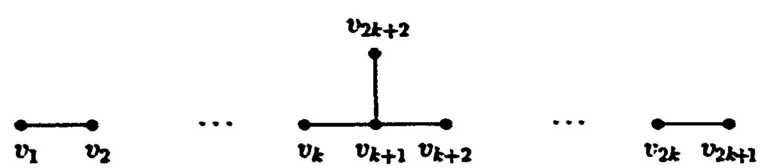


Figure 2.3.3

Theorem 2.3.5 *If T is of order $n = 2k + 2$, having the shape shown in figure 2.3.3, then:*

1. *T is of type I.*

2. *The only characteristic vertex is* $w_T = v_{k+1}$.

3. $\tilde{V} = \{v_{2k+1}, v_{2k+2}\}$.

4. $a(T) = 4\sin^2\left(\frac{\pi}{2(2k+1)}\right)$.

5. *If* $B_1 = (V_1, E_1)$ *and* $B_2 = (V_2, E_2)$ *are the two (different) isomorphic branches of* T *rooted at* v_{k+1}, *and* $\alpha : V_1 \to V_2$ *is an isomorphism of the rooted trees* B_1 *and* B_2, *then there exists* $\vec{f} \in \mathcal{E}(T)$ *such that* $\vec{f}(v) = -\vec{f}(\alpha(v)) > 0, \quad v \notin \{v_{2k+1}, v_{2k+3}\}$, *and* $f(v_{k+1}) = f(v_{2k+2}) = 0$.

Proof: Let P_{2k+1} be a path of order $2k+1$, then adjoin the vertex v_{2k+2} to the vertex v_{k+1}, obtaining the tree T shown in figure 2.3.3. By Theorems 2.1.4 and 2.3.4 $a(T) = a(P_{2k+1}) = 4\sin^2\left(\dfrac{\pi}{2(2k+1)}\right)$ and if $\vec{g} \in \mathcal{E}(P_{2k+1})$ such that $\vec{g}(v) = -\vec{g}(\alpha(v)) > 0, \quad v_{k+1} \neq v \in V_1$, and $\vec{g}(v_{k+1}) = 0$, and $\vec{f}$ is the extension of $\vec{g}$ by defining $\vec{f}(v_{2k+2}) = 0$, then $\vec{f} \in \mathcal{E}(T)$ and so T is type I tree with characteristic vertex $w_T = v_{k+1}$. Therefore all parts of the theorem are proved.

Theorem 2.3.6 *Let* T *be as shown in figure 2.3.1. Then* $a(T) = 4\sin^2\left(\frac{\pi}{2(2k+1)}\right)$.

Proof: By Theorem 2.1.2, we will have two cases:

1. there exists $\vec{f} \in \mathcal{E}(T)$ s.t. $\vec{f}(\alpha(v)) = \vec{f}(v), \quad v \in V_1$, or

2. $F(T) = \{w\}$ and there is a $\vec{g} \in \mathcal{E}(T)$ such that $\vec{g}(v) = \vec{g}(\alpha(v)) > 0$, $w \neq v \in V_1$, and $\vec{g}(v) = 0$, $v \notin V_1 \cup V_2$.

The first case is illustrated in figure 2.3.4 and the second one is illustrated in figure 2.3.5. Both cases could be proved by the same technique used in proving Theorem 2.3.5.

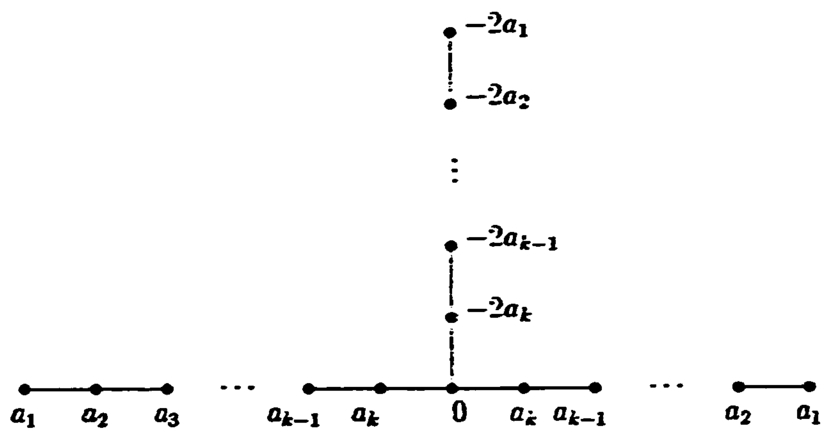


Figure 2.3.4

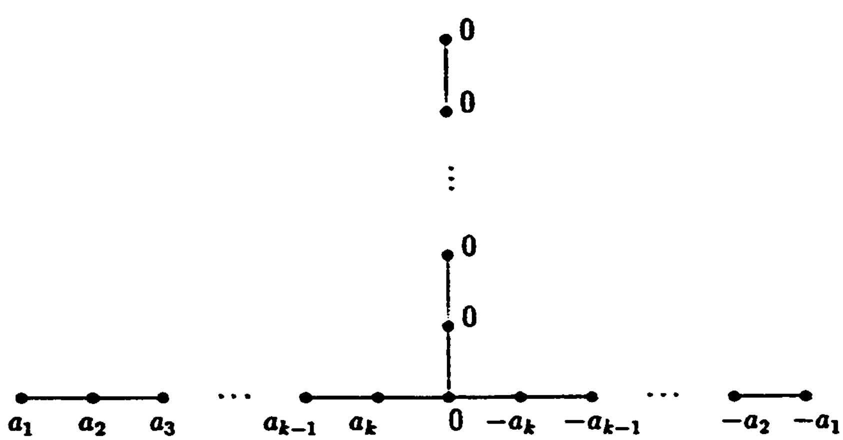


Figure 2.3.5

In Theorem 2.3.3, we have proved that there exists $\vec{f} \in \mathcal{E}(T)$ such that the first case is satisfied. Since the multiplicity of $a(T)$ is 2, there exists two linearly independent vectors $\vec{g}_1, \vec{g}_2 \in \mathcal{E}(T)$. If $\vec{g}_1 \in \mathcal{E}(T)$ satisfies the first case, then

$$\vec{g}_1 = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_k \\ 0 \\ b_k \\ \vdots \\ b_1 \\ -2b_k \\ \vdots \\ -2b_1 \end{pmatrix}$$

Now, if $\vec{g}_2$ also satisfies the first case, then

$$\vec{g}_2 = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \\ 0 \\ a_k \\ \vdots \\ a_1 \\ -2a_k \\ \vdots \\ -2a_1 \end{pmatrix}$$

and by Theorem 2.2.2 we have

$$g_2 = \begin{pmatrix} a_1 \\ a_1 f_2(a) \\ a_1 f_3(a) \\ \vdots \\ a_1 f_k(a) \\ 0 \\ a_1 f_k(a) \\ \vdots \\ a_1 f_1(a) \\ a_1 \\ -2a_1 \\ -2a_1 f_2(a) \\ \vdots \\ -2a_1 f_k(a) \end{pmatrix} = a_1 \begin{pmatrix} 1 \\ f_2(a) \\ f_3(a) \\ \vdots \\ f_k(a) \\ f_k(a) \\ \vdots \\ f_2(a) \\ 1 \\ -2 \\ -2f_2(a) \\ \vdots \\ -2f_k(a) \end{pmatrix}$$

So,

$$\Rightarrow \frac{1}{a_1} g_2 - \frac{1}{b_1} g_1 = 0$$

This implies that

g_1 and g_2 are linearly dependent (contradiction).

Therefore,

g_2 cannot satisfy the first condition

So,

g_2 satisfies the second condition.

Remove the pendant vertex v with $\vec{g}_2(v) = 0$. By Theorem 1.4.4 (Reduction Theorem), we have the tree T_{k-1} with $a(T_{k-1}) = a(T)$ and $\vec{g}_2|_{k-1}$ is the eigenvector belonging to $a(T_{k-1})$, where $\vec{g}_2|_{k-1}$ is the restriction of $\vec{g}_2$ to $V_{k-1} = V - \{v\}$. Repeat this technique $(k-1)$ times until you get the path P_{2k+1} with $a(P_{2k+1}) = a(T)$ and $\vec{g}_{2|V_0}$ is the restriction of $\vec{g}_2$ to $V_0 = V - \bigcup_{i=1}^{k}\{v_i\}$ (where $\vec{g}_2(v_i) = 0$). Therefore, by Theorem 2.3.4, $a(T) = 4\sin^2\left(\frac{\pi}{2(2k+1)}\right)$.

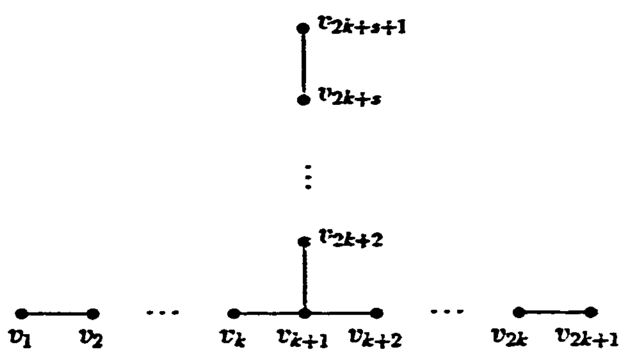


Figure 2.3.6

Theorem 2.3.7 *If $T \in \mathcal{T}$ is of order $n = 2k + s + 1$, having the shape shown in figure 2.3.6, where $1 \leq s < k$, then*

1. *T is of type I.*

2. *The characteristic vertex $w_T = v_{k+1}$.*

3. $a(T) = 4\sin^2\left(\frac{\pi}{2(2k+1)}\right)$.

4. *If $B_1 = (V_1, E_1)$ and $B_2 = (V_2, E_2)$ are two (different) isomorphic branches of T rooted at v_{k+1} and $\alpha : V_1 \rightarrow V_2$ is the isomorphism of the rooted trees B_1 and B_2, then $\exists\ g \in \mathcal{E}(T)$ such that $\vec{g}(v) = -\vec{g}(\alpha(v)) > 0, \quad v_{k+1} \neq v \in V_1$ and $\vec{g}(v) = 0, \quad v \notin V_1 \cup V_2$.*

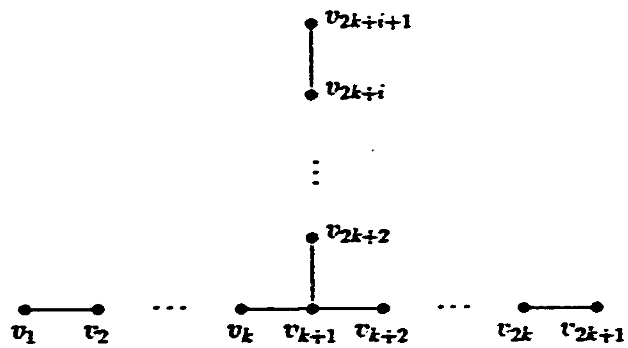


Figure 2.3.7

Proof: Consider all the trees T_i such that $1 \leq i \leq k$ as shown in Figure 2.3.7. We have, $T = T_s$, and by The Interlacing Theorem 1.3.14. we have:

$$a(T_1) \geq a(T_2) \geq \cdots \geq a(T_s) \geq \cdots \geq a(T_k).$$

Since $a(T_1) = a(T_k)$, then:

$$a(T_1) = a(T_2) = \cdots = a(T_s) = \cdots = a(T_k).$$

Now, for the tree T_1, there exists $\vec{g}_1 \in \mathcal{E}(T_1)$ such that $\vec{g}_1(v) = -\vec{g}_1(\alpha(v)) > 0$, $v_{k+1} \neq v \in V_1$, $\vec{g}_1(v_{k+1}) = \vec{g}_1(v_{2k+2}) = 0$ where $\alpha : V_1 \to V_2$ is the isomorphism of the two different isomorphic branches of T, $B_1 = (V_1, E_1)$ and $B_2 = (V_2, E_2)$ rooted at v_{k+1}. Adjoin a vertex v_{2k+3} to the vertex v_{2k+2}. Since $a(T_1) = a(T_2)$, then by Theorem 1.4.5, $F(T_1) = F(T_2)$ and $\vec{g}_2 \in \mathcal{E}(T_2)$, $\vec{g}_2(v) = g_1(v)$, $v \in V(T_1)$ and $\vec{g}_2(v_{2k+2}) = 0$. If $S = 2$, then it's done. Otherwise, in the same way we can prove that, for the tree T_3, $\exists$ $\vec{g}_3 \in \mathcal{E}(T_3)$ such that

$\vec{g}_3(v) = \vec{g}_2(v)$, $v \in V(T_2)$ and $\vec{g}_3(v_{2k+3}) = 0$. Continuing in the same manner, we conclude the existence of the $\vec{g}_s \in \mathcal{E}(T_s)$ such that $\vec{g}(v) = -\vec{g}(\alpha(v)) > 0$ $v_{k+1} \neq v \in V_1$ and $\vec{g}(v) = 0$, $v \notin V_1 \cup V_2$. Therefore, all parts of the theorem have been proved.

Chapter 3

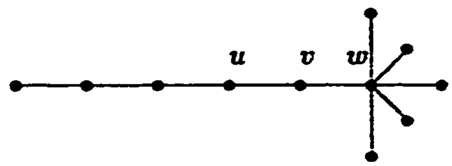
Centers, Centroids and Characteristic Vertices of A Caterpillar

In this chapter, we will introduce centers and centroids of a graph G. If $T \in \mathcal{T}$ and $F(T)$ is the set of characteristic vertices of T, then should the center and centroid overlap the characteristic vertices of T? We will see that the answer is no. Furthermore, we study type I caterpillars together with their centers and centroids.

3.1 Center and Centroid

Let T be a tree. A *branch* rooted at $v \in V$ is a maximal subtree containing v as a pendant vertex. (The number of branches of v is $d(v)$). The *weight*, $w(v)$, of v is the maximum number of edges in any branch at v. A vertex v is a *centroid* *point* of T if

$$w(v) = \min_{u \in V} w(u).$$



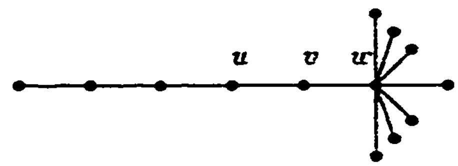
uv: characteristic edge
w: centroid

Figure 3.1.1

Example 3.1.1 *Let T be the tree shown in Figure 3.1.1. Then the characteristic polynomial is $x(x-1)^4(x^6 - 16x^5 + 91x^4 - 232x^3 + 266x^2 - 116x + 11)$ and $a(T) =$ 0.1288129 is a simple eigenvalue. Moreover, T is of type II and $F(T) = \{u, v\}$. But w is the unique centroid point of T.*

The *eccentricity* $e(v)$ of a vertex v of a connected graph G is the number $\max_{u \in V(G)} d(u, v)$. The *radius* rad G is defined as $\min_{v \in V(G)} e(v)$ while the *diameter* diam G is $\max_{v \in V(G)} e(v)$. A vertex v is a *center point* of T if

$$e(v) = \min_{u \in V} e(u)$$



vw: characteristic edge

u: center

Figure 3.1.2

Example 3.1.2 *Let T be the tree shown in Figure 3.1.2. Its characteristic polynomial is $x(x-1)^6(x^6-18x^5+109x^4-288x^3+336x^2-146x+13)$ and $a(T) =$ 0.1179743. In this case, $F(T)=\{v, w\}$, but u is the unique center point.*

If $T \in \mathcal{T}$, then we label the vertices as shown in Figure 3.1.3

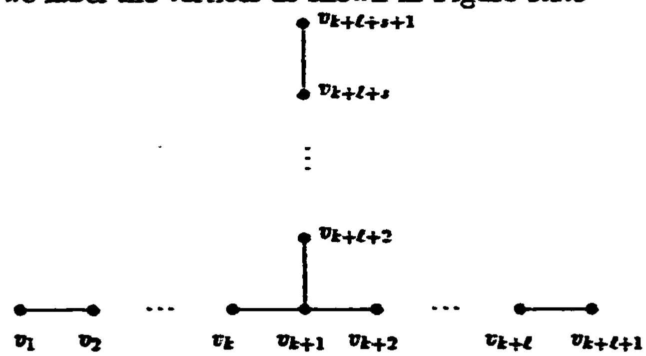


Figure 3.1.3

Theorem 3.1.1 *If $T \in \mathcal{T}$ such that T is a type I tree then the only center (centroid) in the tree T is the characteristic vertex v_{k+1}.*

Proof: If the centroid o is not v_{k+1}, then $w(o) \geq k+s+1 > w(v_{k+1}) = k$ which contradicts the definition of a centroid, so v_{k+1} is the centroid. If the center C is not v_{k+1}, then

$$e(c) \geq k+1 > e(v_{k+1}) = k$$

which also contradicts the definition of a center of T, so v_{k+1} is the center.

Theorem 3.1.2 *If $T \in \mathcal{T}$ such that T is of order $n = k + \ell + s + 1$ where $k > \ell \geq s \geq 1$, k, ℓ and s are the orders of the branches of T at v_{k+1}, then if $k + \ell$ is even then the only center is $\frac{k+\ell+2}{2}$, and if $k + \ell$ is odd, then the only centers are $\frac{k+\ell+1}{2}$ and $\frac{k+\ell+3}{2}$.*

Proof: Assume that $k + \ell$ is even

$$e\left(\frac{k+\ell+2}{2}\right) = \frac{k+\ell}{2}$$

$$\begin{aligned} e(i) &= i - 1 \qquad \left(\frac{k+\ell+2}{2} < i \leq k + \ell + 1\right) \\ &> \frac{k+\ell}{2} \end{aligned}$$

$$\begin{aligned} e(j) &= k + \ell - j + 1 \qquad \left(1 \leq j < \frac{k+\ell+2}{2}\right) \\ &\geq k + \ell - \frac{k+\ell}{2} + 1 \\ &= \frac{k+\ell}{2} + 1 > \frac{k+\ell}{2} \end{aligned}$$

$$\begin{aligned} e(m) &= m - 1 \qquad (k + \ell + 2 \leq m \leq k + \ell + s + 1) \\ &\geq (k + \ell + 2) - 1 = k + \ell + 1 > \frac{k+\ell}{2} \end{aligned}$$

$\Rightarrow$ The only center is $\frac{k+\ell+2}{2}$

Assume that $k + \ell$ is odd

$$e\left(\frac{k+\ell+1}{2}\right) = \max\left\{\frac{k+\ell-1}{2},\ k + \ell - \frac{k+\ell-1}{2}\right\}$$

$$= \max\left\{\frac{k+\ell-1}{2}, \frac{k+\ell+1}{2}\right\}$$

$$= \frac{k+\ell+1}{2}$$

$$e\left(\frac{k+\ell+3}{2}\right) = \max\left\{\frac{k+\ell+1}{2},\ k+\ell-\frac{k+\ell+1}{2}\right\}$$

$$= \max\left\{\frac{k+\ell+1}{2}, \frac{k+\ell-1}{2}\right\}$$

$$= \frac{k+\ell+1}{2}$$

$$e(i) = i-1 \qquad \left(\frac{k+\ell+3}{2} < i < k+\ell+1\right)$$

$$> \frac{k+\ell+3}{2} - 1 = \frac{k+\ell+1}{2}$$

$$e(j) = k+\ell-(j-1) \qquad \left(1 \le j \le \frac{k+\ell+1}{2}\right)$$

$$= k+\ell-j+1$$

$$> k+\ell-\frac{k+\ell+1}{2}+1 = \frac{k+\ell+1}{2}$$

$$e(m) = m-1 \qquad (k+\ell+2 \le m \le k+\ell+s+1)$$

$$\ge (k+\ell+2)-1 = k+\ell+1 > \frac{k+\ell+1}{2}$$

$\Rightarrow$ The only centers are $\frac{k+\ell+1}{2}$ and $\frac{k+\ell+3}{2}$

Theorem 3.1.3 *If $T \in \mathcal{T}$ is a tree of order $n = k+\ell+s+1$ where $k > \ell \ge s \ge 1$, k,ℓ and s are the orders of the branches of T at v_{k+1}. Let $i = \frac{k-\ell-s+2}{2}$. Then:*

1. If $i < 1$ then the only centroid is $\ell + 1$.

2. If $i \geq 1$ then if n is odd, then the only centroid is $\ell + i$, and if n is even, then the only centroids are $\ell + [i]$ and $\ell + [i] + 1$.

Proof:

1. Assume that $i = \frac{k-\ell-s+2}{2} < 1$:

$$
\begin{aligned}
&\Rightarrow k < \ell + s \\
w(\ell+1) &= k \\
w(\ell+j) &= \ell + s + (j-1) \qquad (2 \leq j \leq k+1) \\
&\geq \ell + s + 1 > k + 1 > k = w(\ell+1) \\
w(n) &= k + \ell - n + 1 + s \qquad (1 \leq n \leq \ell) \\
&\geq k + \ell - \ell + 1 + s = k + 1 + s > k = w(\ell+1) \\
w(m) &= m - 1 \qquad (k + \ell + 2 \leq m \leq k + \ell + s + 1) \\
&\geq k + \ell + 2 - 1 = k + \ell + 1 > k = w(\ell+1) \\
&\Rightarrow \ell + 1 \text{ is the centroid .}
\end{aligned}
$$

2. Assume that $i \geq 1$ and n is odd:

If $i = 1$ then $k = \ell + s$ and

$$
\begin{aligned}
w(\ell+i) &= \ell + s = k \\
w(\ell+j) &= \ell + s + (j-1) \qquad (2 \leq j \leq k+1)
\end{aligned}
$$

$$\geq \ell + s + (2-1) = \ell + s + 1 > k = w(\ell+1)$$

$$w(n) = k + \ell - n + s + 1 \qquad (1 \leq n \leq \ell)$$

$$\geq k + \ell - \ell + 1 + s = k + 1 + s > k = w(\ell+1)$$

$$w(m) = m - 1 \qquad (k + \ell + 2 \leq m \leq k + \ell + s + 1)$$

$$\geq (k + \ell + 2) - 1 = k + \ell + 1 > k = w(\ell+1)$$

$\Rightarrow \ell + 1$ is the centroid .

If $i > 1$ then $k > \ell + s$, and

$$\begin{aligned}
w(\ell+i) &= w\left(\frac{k+\ell-s+2}{2}\right) \\
&= \max\left\{\frac{k+\ell-s+2}{2} - 1 + s,\ k + \ell - \frac{k+\ell-s+2}{2} + 1\right\} \\
&= \frac{k+\ell+s}{2} \\
w(\ell+i+j) &= w\left(\frac{k+\ell-s+2}{2} + j\right) \qquad \left(1 \leq j \leq \frac{k+\ell+s}{2}\right) \\
&= \frac{k+\ell+s}{2} + j \\
&\geq \frac{k+\ell+s}{2} + 1 > \frac{k+\ell+s}{2} = w(\ell+i) \\
w(n) &= k + \ell - n + 1 + s \qquad (1 \leq n \leq \ell) \\
&\geq k + \ell - \ell + 1 + s = k + 1 + s > \frac{k+\ell+s}{2} = w(\ell+i) \\
w(m) &= m - 1 \qquad (k + \ell + 2 \leq m \leq k + \ell + s + 1) \\
&\geq (k + \ell + 2) - 1 = k + \ell + 1 > \frac{k+\ell+s}{2} = w(\ell+i) \\
w(\ell+1) &= k \quad \text{but} \quad k > \ell + s \Rightarrow 2k > k + \ell + s \\
&\Rightarrow k > \frac{k+\ell+s}{2} \Rightarrow w(\ell+1) > w(\ell+i)
\end{aligned}$$

.

$$w(\ell+f) = \max\{\ell+f-1+s,\ k+\ell-(\ell+f)+1\} \quad (\ell+2 \leq \ell+f \leq \frac{k+\ell-s}{2})$$

If $\ell+f = \frac{k+\ell-s}{2}$, then:

$$\begin{aligned} w(\ell+f) &= \max\left\{\frac{k+\ell-s}{2}-1+s,\ k+\ell-\frac{k+\ell-s}{2}+1\right\} \\ &= \max\left\{\frac{k+\ell+s}{2}-1,\ \frac{k+\ell+s}{2}+1\right\} \\ &= \frac{k+\ell+s}{2}+1 \end{aligned}$$

So, if $\ell+2 \leq \ell+f \leq \dfrac{k+\ell-s}{2}$, then

$$\begin{aligned} w(\ell+f) &= k+\ell-(\ell+f)+1 \\ &\geq k+\ell-\frac{k+\ell-s}{2}+1 = \frac{k+\ell+s}{2}+1 \\ &> \frac{k+\ell+s}{2} = w(\ell+i) \end{aligned}$$

Assume that $i \geq 1$ and n is even. Since n is even, $i = \dfrac{k-\ell-s+2}{2}$ is not an integer. So, if

$$i = \frac{3}{2} \Rightarrow k = \ell+s+1$$

$$\begin{aligned} w(\ell+1) &= k \\ w(\ell+2) &= \ell+s+1 = k \\ w(\ell+j) &= \ell+s+(j-1) \qquad (3 \leq j \leq k+1) \\ &\geq \ell+s+(3-1) = \ell+s+2 = k+1 > k \\ w(n) &= k+\ell-n+1+s \qquad (1 \leq n \leq \ell) \\ &\geq k+\ell-\ell+1+s = k+1+s > k \end{aligned}$$

$$\begin{aligned} w(m) &= m-1 \qquad (k+\ell+2 \le m \le k+\ell+s+1) \\ &\ge (k+\ell+2)-1 = k+\ell+1 > k \end{aligned}$$

$\Rightarrow$ $\ell+1$ and $\ell+2$ are the only centroids in this case.

Assume that $i > \frac{3}{2}$, then

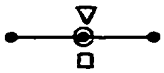
$$\begin{aligned} k &> \ell+s+1 \\ w(\ell+[i]) &= w\left(\frac{k+\ell-s+1}{2}\right) \\ &= \max\left\{\frac{k+\ell-s+1}{2}-1+s,\ k+\ell-\frac{k+\ell-s+1}{2}+1\right\} \\ &= \max\left\{\frac{k+\ell+s-1}{2},\ \frac{k+\ell+s+1}{2}\right\} = \frac{k+\ell+s+1}{2} \\ w(\ell+[i]+1) &= w\left(\frac{k+\ell-s+3}{2}\right) \\ &= \max\left\{\frac{k+\ell-s+3}{2}-1+s,\ k+\ell-\frac{k+\ell-s+3}{2}+1\right\} \\ &= \max\left\{\frac{k+\ell+s+1}{2},\ \frac{k+\ell+s-1}{2}\right\} = \frac{k+\ell+s+1}{2} \\ w(q) &= k+\ell-q+1+s \qquad (1 \le q \le \ell) \\ &\ge k+\ell-\ell+1+s = k+1+s > \frac{k+\ell+s+1}{2} \\ w(m) &= m-1 \qquad (k+\ell+2 \le m \le k+\ell+s+1) \\ &\ge (k+\ell+2)-1 = k+\ell+1 > \frac{k+\ell+s+1}{2} \\ w(\ell+[i]+j) &= \frac{k+\ell-s+1}{2}+j-1+s \qquad \left(2 \le j \le \frac{k+\ell+s+1}{2}\right) \\ &= \frac{k+\ell+s-1}{2}+j \ge \frac{k+\ell+s-1}{2}+2 \\ &= \frac{k+\ell+s+3}{2} > \frac{k+\ell+s+1}{2} \end{aligned}$$

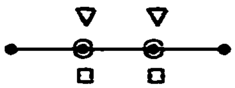
$\Rightarrow$ $\ell+[i]$ and $\ell+[i]+1$ are the only centroids.

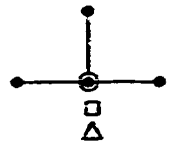
3.2 Center and Centroid of a Caterpillar

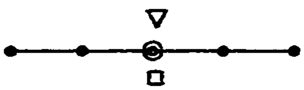
A *caterpillar* C is a tree such that if we remove all pendant vertices, we get a path. A list of caterpillars showing the location of the characteristic vertices, centers and centroids is given in the next page.

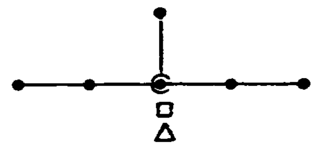
The circle ○ indicates that the vertex is a center. The square □ indicates that the vertex is a centroid. The triangle △ indicates that the vertex is a characteristic vertex.

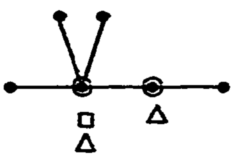


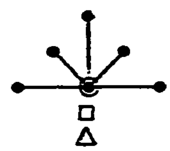


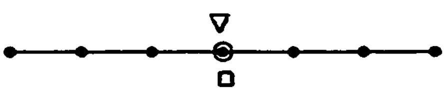


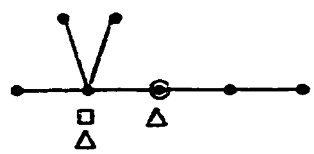


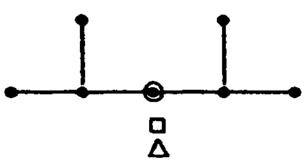


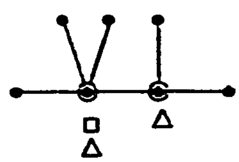


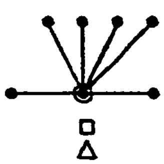


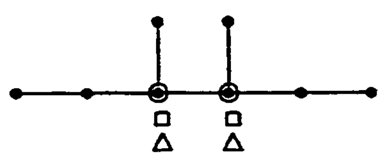












A list of caterpillars of order n, $3 \le n \le 9$ is given in Appendix 4.

Theorem 3.2.4 *If a caterpillar is of the shape shown below, where* $n = i+j+\ell+1$ *and* $j \ge i$, *then:*

1. *If* $j < i+\ell+1$ *then the only centroid is* $i+1$.
2. *If* $j = i+\ell+2s$ *(where* s *is a natural number) then the only centroid is* $i+s+1$.
3. *If* $j = i+\ell+2s+1$ *(where* s *is a whole number), then the only centroids are* $i+s+1$ *and* $i+s+2$.

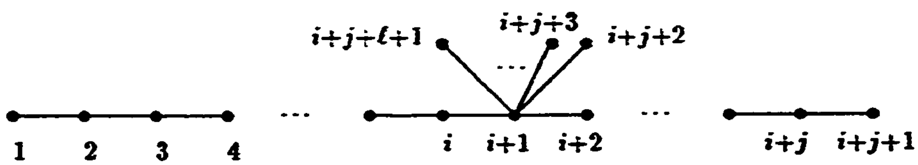


Figure 3.2.1

Proof:

1. Assume that $j < i+\ell+1$, then:

$$w(i+1) = j$$

$$w(n) = (i-n+1)+j+\ell \qquad (1 \le n \le i)$$

$$\ge (i-i)+1+j+\ell = 1+j+\ell > w(i+1)$$

$$w(m) = n - 1 > w(i+1) \qquad (i+j+2 \leq m \leq i+j+\ell+1)$$

$$w(p) = (p-1) + \ell \qquad (i+2 \leq p \leq i+j+1)$$

$$\geq ((i+2)-1) + \ell = i + \ell + 1 > w(i+1)$$

So $i+1$ is the only centroid.

2. Assume that $j = i + \ell + 2s$

$$w(i+s+1) = \max\{i+s+\ell, j-s\}$$

$$= \max\{i+s+\ell, (i+\ell+2s)-s\}$$

$$= \max\{i+s+\ell, i+\ell+s\}$$

$$= i+s+\ell$$

$$w(m) = (i+j+1) - m + \ell \qquad (1 \leq m \leq i)$$

$$\geq (i+j+1) - i + \ell = j + \ell + 1$$

$$= i + \ell + 2s + \ell + 1 = i + 2\ell + 2s + 1 > w(i+s+1)$$

$$w(i+1) = j = i + \ell + 2s > w(i+s+1)$$

$$w(n) = i + j + 1 - n \qquad (i+2 \leq n \leq i+s)$$

$$= 2i + \ell + 2s + 1 - n$$

$$\geq 2i + \ell + 2s + 1 - (i+s) = i + \ell + s + 1 > w(i+s+1)$$

$$w(p) = p - 1 + \ell \qquad (i+s+2 \leq p \leq i+j+1)$$

$$\geq (i+s+2) - 1 + \ell = i + s + \ell + 1 > w(i+s+1)$$

$$w(q) = i + j + \ell \qquad (i+j+2 \leq q \leq i+j+\ell+1)$$

$$= i + (i + \ell + 2s) + \ell = 2i + 2\ell + 2s > w(i + s + 1)$$

So, $i + s + 1$ is the only centroid.

3. Assume that $j = i + \ell + 2s + 1$

$$w(i + s + 1) = \max\{i + s + \ell, j - s\} = \max\{i + s + \ell, i + \ell + 2s + 1 - s\}$$

$$= \max\{i + s + \ell, i + s + \ell + 1\} = i + s + \ell + 1$$

$$w(i + s + 2) = \max\{i + s + 1 + \ell, j - s - 1\}$$

$$= \max\{i+s+1+\ell, i+\ell+2s+1-s-1\} = \max\{i+s+1+\ell, i+\ell+s\}$$

$$= i + s + \ell + 1$$

$$w(m) = (i + j + 1) - m + \ell \qquad (1 \leq m \leq i)$$

$$= (i + i + \ell + 2s + 1) - m + \ell$$

$$= 2i + 2\ell + 2s + 1 - m \geq 2i + 2\ell + 2s + 1 - i$$

$$= i + 2\ell + 2s + 1 > w(i + s + 1) = w(i + s + 2)$$

$$w(i + 1) = i + \ell + 2s > w(i + s + 1) = w(i + s + 2)$$

$$w(r) = i + j + 1 - r \qquad (i + 2 \leq r \leq i + s)$$

$$= 2i + \ell + 2s + 2 - r \geq 2i + \ell + 2s + 2 - (i + s)$$

$$= i + \ell + s + 2 > w(i + s + 1) = w(i + s + 2)$$

$$w(p) = p - 1 + \ell \geq i + s + 3 - 1 + \ell \qquad (i + s + 3 \leq p \leq i + j + 1)$$

$$= i + s + \ell + 2 > w(i + s + 1) = w(i + s + 2)$$

$$w(q) = i + j + \ell \qquad (i + j + 2 \leq q \leq i + j + \ell + 1)$$

$$= i+(i+\ell+2s+1)+\ell=2i+2\ell-2s-1 > w(i+s+1)=w(i+s+2)$$

So, $i+s+1$ and $i+s+2$ are the only centroids.

Corollary 3.2.1 *If the caterpillar is of the shape shown below, where $n = k+\ell$, then:*

1. *If $k < \ell + 4$, then the only centroid is 2.*
2. *If $k = \ell + 2s + 1$ (s is a natural number), then the only centroid is $s+1$.*
3. *If $k = \ell + 2s$ (s is a natural number), then the only centroids are s and $s+1$.*

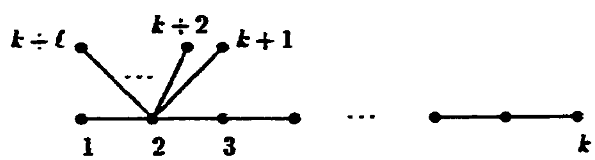


Figure 3.2.2

Proof: This is a direct consequence of the last theorem by putting $i = 1$ and $j+2 = k$.

Theorem 3.2.5 *Let $C = (V, E)$ be a caterpillar of order n, and let $B_1 = (V_1, E_1)$ and $B_2 = (V_2, E_2)$, $V_1 \cap V_2 = \phi$, be two isomorphic (but different) branches rooted at $v \in V$ such that any vertex of a longest path of C belongs to $V_1 \cup V_2$, then v is the centroid of C.*

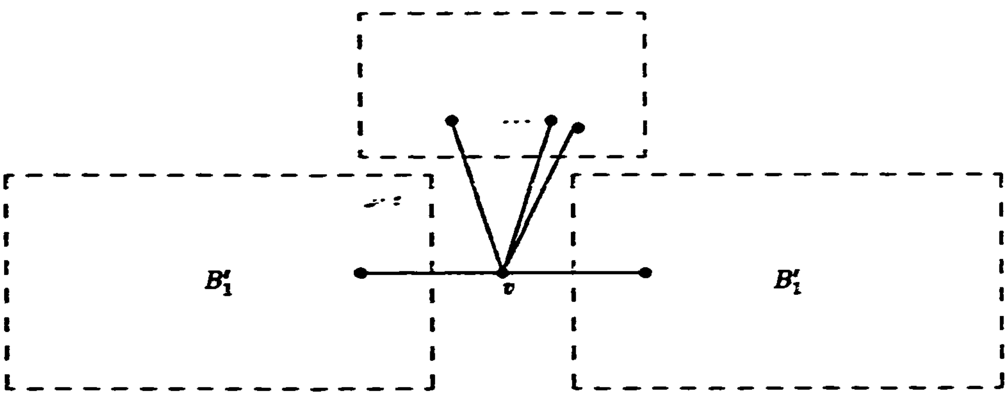


Figure 3.2.3

Proof: Let B_1' arise from B_1 by removing the vertex v and the adjacent edge. Also, let B_2' arise from B_2 by removing the vertex v and the adjacent edge. We have:

$$V_1' = V_1 - \{v\} \quad \text{and} \quad V_2' = V_2 - \{v\}$$

where V_1' and V_2' are the sets of vertices of B_1' and B_2' respectively. Let V_3 be the set of vertices which are adjacent to v and not included in $V_1 \cup V_2$.

$$w(v) = |V_1| - 1 = |V_2| - 1$$

$$w(u) = (|V_1| - 1) + |V_3| + r \qquad u \in V_1' \cup V_2'$$

where r is the length of the path between v and u. Since $|V_3| \geq 0$ and $r \geq 1$, $w(u) > w(v)$. Also, if $p \in V_2$, then:

$$w(p) = n - 1 > w(v).$$

So, v is the only centroid of C.

Theorem 3.2.6 *Let $C = (V, E)$ be a caterpillar of order n and let $B_1 = (V_1, E_1)$ and $B_2 = (V_2, E_2)$ be two isomorphic branches rooted at v_1 and v_2 respectively, where v_1 and v_2 are adjacent and $deg(v_1) = deg(v_2)$ in C. Suppose also, that any vertex belonging to a longest path in C must also belong to $V_1 \cup V_2$. Then, both v_1 and v_2 are the only centroids of C.*

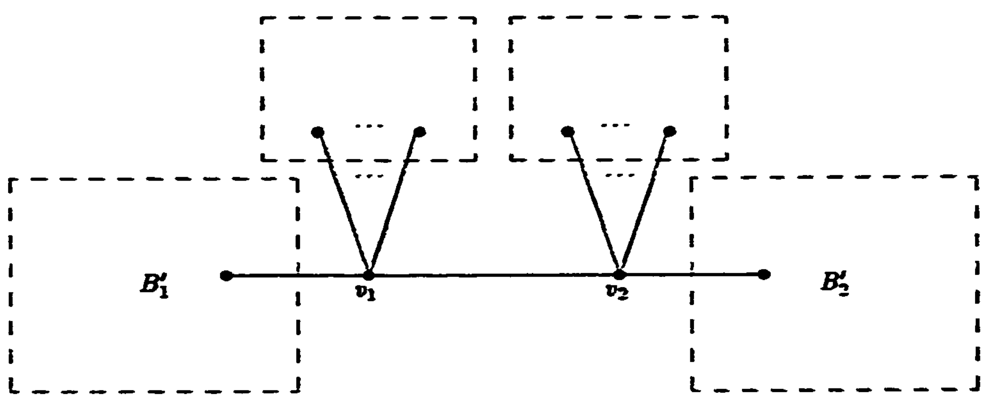


Figure 3.2.4

Proof: Let B_i' arise from B_i by removing the vertex v_i and the adjacent edges. We have:

$$V_i' = V_i - \{v_i\}$$

where V_i' is the set of vertices of B_i', $(i = 1, 2)$. Let V_i^- be the set of vertices which are adjacent to v_i and not included in V_1 or V_2.

$$w(v_1) = |V_2^-| + |V_2'| + 1$$

$$w(v_2) = |V_1^-| + |V_1'| + 1$$

since $|V_1^*| = |V_2^*|$ and $|V_1'| = |V_2'|$, we have:

$$w(v_1) = w(v_2) = |V_1^*| + |V_1'| + 1.$$

If $p \in V_1' \cup V_2'$, then:

$$w(p) = |V_1^*| + |V_1'| + 1 + |V_2^*| + r$$

where $r = \min_{i=1,2}\{$ length of the path between v_i and $p\}$. Since $|V_1^*| \geq 0$, $|V_2^*| \geq 0$ and $r \geq 1$, then:

$$w(p) > w(v_1) = w(v_2).$$

If $q \in V_1^* \cup V_2^*$, then

$$w(q) = n - 1 > w(v_1) = w(v_2).$$

So, v_1 and v_2 are the only centroids of C.

3.3 Type I Caterpillars

In this section, a characterization of a class of caterpillars of type I is given.

Theorem 3.3.7 *Let $C = (V, E)$ be a caterpillar of order n, and let $B_1 = (V_1, E_1)$ and $B_2 = (V_2, E_2)$, $V_1 \cap V_2 = \phi$ be two isomorphic (but different) branches rooted at $w \in V$ such that any vertex of a longest path of C belongs to $V_1 \cup V_2$, then $F(C) = \{w\}$ and there is a $g \in \mathcal{E}(C)$ such that $g(v) = -g(\alpha(v)) > 0, w \neq v \in V_1$ and $g(v) = 0, \quad v \notin V_1 \cup V_2$.*

Proof: Let $C^* = (V^*, E^*)$ be the caterpillar obtained from $C = (V, E)$ by removing all pendant vertices incident with w and all corresponding edges (see figure 3.2.3). Let $\alpha : V_1 \rightarrow V_2$ be an isomorphism of the rooted trees B_1 and B_2. Let $f \in \mathcal{E}(C^*)$ be fixed but arbitrary. We need to prove that $f(\alpha(v)) = f(v), \quad v \in V_1$ is not satisfied. Assume that $f(\alpha(v)) = f(v), \quad v \in V_1$. If C^* is a type II caterpillar, then one of the following conditions should happen:

1. $f(v) > 0, w \neq v \in C^*$ and $f(w) < 0$.

2. $f(v) < 0, w \neq v \in C^*$ and $f(w) > 0$.

3. $\exists\ v_1, v_2 \in V_1$ and $v_3, v_4 \in V_2$ such that $f(v_3) = f(v_1) > 0$ and $f(v_4) = f(v_2) < 0$.

In (1) or (2) we will get 3 characteristic vertices (2 characteristic edges) which contradicts Theorem 1.4.1. In (3), we will get either 2 or 4 characteristic edges which again contradicts Theorem 1.4.1.

If C^* is a type I caterpillar, then:

If $v_1 \in V_1(V_2)$ is the characteristic vertex of C^* ($f(v_1) = 0$) then $\exists\ v_2 \in V_2(V_1)$ such that $f(v_2) = 0$. In this case, we will have two characteristic vertices which contradicts Theorem 1.4.1. If w is the characteristic vertex ($f(w) = 0$), then $\sum_{\substack{v_i \in C^* \\ v_i \neq w}} f(v_i) < 0$ or $\sum_{\substack{v_i \in C^* \\ v_i \neq w}} f(v_i) > 0$ which contradicts Theorem 1.3.15 So, we have proved that $f(\alpha(v)) = f(v), \quad v \in V_1$ is not satisfied for C^*. By Theorem 2.1.2, applied to w. we have $F(C^{*}) = \{w\}$ and there is a $g \in \mathcal{E}(C^{*})$ such that

$$g(v) = -g(a(v)) > 0, \quad w \neq v \in V_1 \text{ and } g(v) = 0, \quad v \notin V_1 \cup V_2.$$

Now, adjoin a new pendant vertex p_1 to w, so $V^{(1)} = V^{*} \cup \{p_1\} E^{(1)} = E^{*} \cup \{\{p_1, w\}\}$. Extend f to a function $f^{(1)}$ on $V^{(1)}$ by defining $f^{(1)}(p_1) = 0$, and then applying Theorem 2.1.4, the new caterpillar $C^{(1)}$ is a type I tree. $f' \in \mathcal{E}(C^{(1)})$ and $a(C^{(1)}) = aC^{(*)}$). Apply the same technique to $C^{(1)}$ to get $C^{(2)}$ with $f'' \in \mathcal{E}(C^{(2)})$ where $f^{(2)}$ is the extension of $f^{(1)}$ on $V^{(2)}$ by defining $f^{(2)}(p_2) = 0$ where $V^{(2)} = V^{(1)} \cup \{p_2\}$. We continue applying the same technique until we get $C = C^{(q)}$ where q is the number of times the technique is used.

Remark: Actually, it is found that this theorem is true not only for caterpillars, but also for any tree T. The same proof is used.

Appendices

The results below were obtained with the help of Mathematica.

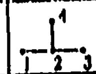
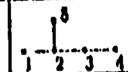
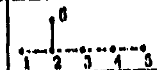
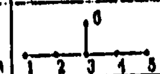
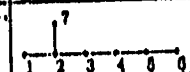
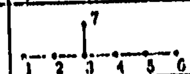
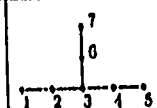
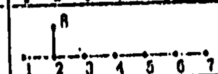
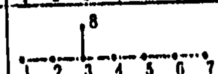
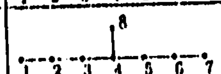
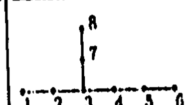
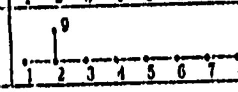
Appendix 1: The eigenvalues of all trees with three end vertices; $4 \leq n \leq 14$.

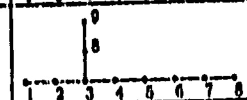
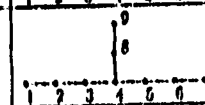
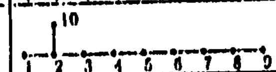
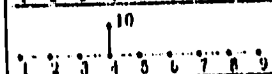
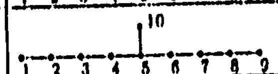
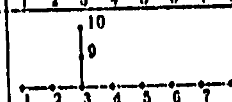
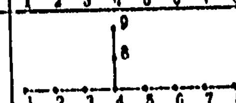
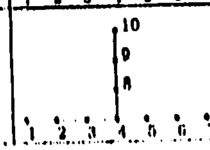
Appendix 2: The second smallest eigenvalue with the corresponding eigenvector for all trees with three end vertices; $4 \leq n \leq 14$.

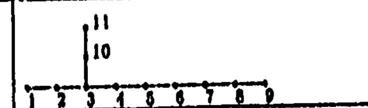
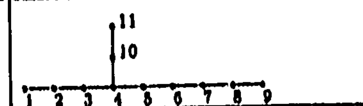
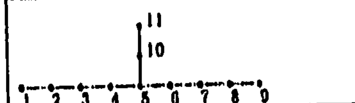
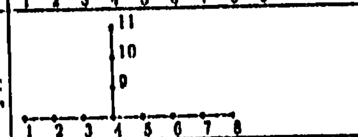
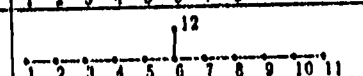
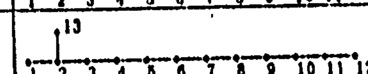
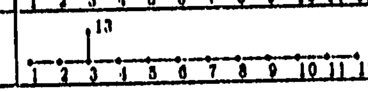
Appendix 3: The eigenvalues of all paths of order n with the eigenvector for the second smallest eigenvalue; $1 \leq n \leq 14$.

Appendix 4: The second smallest eigenvalue with the corresponding eigenvector for all Caterpillars of order n; $3 \leq n \leq 8$.

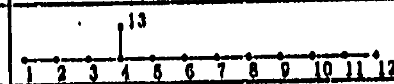
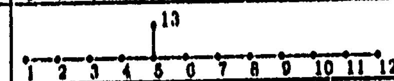
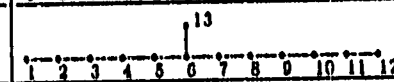
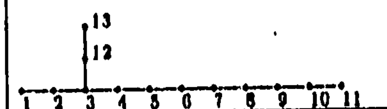
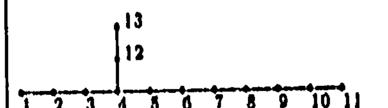
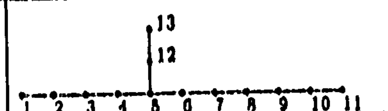
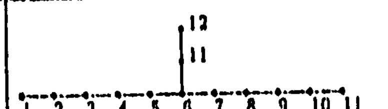
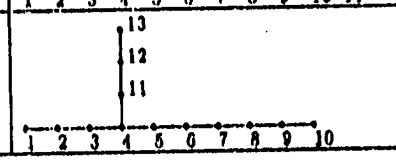
Appendix 1: The eigenvalues of all trees with three end vertices; $4 \leq n \leq 14$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	Graph
T41	0	1	1	4											
T51	0														
T61	0	0.324869	1	1.46081	3	4.21432									
T62	0	0.381966	0.697224	2	2.61803	4.30278									
T71	0	0.225377	1	1	2.18589	3.36041	4.22833								
T72	0	0.260323	0.62621	1.40548	2.27421	3.09958	4.3342								
T73	0	0.381966	0.381966	1.58579	2.61803	2.61803	4.41421								
T81	0	0.166717	0.727601	1	1.63527	2.67201	3.56429	4.23321							
T82	0	0.186393	0.585786	1	2	2.47068	3.41421	4.324292							
T83	0	0.198062	0.491519	1.32036	1.55496	2.82578	3.24698	4.36234							
T84	0	0.243402	0.381966	1.17975	2	2.61803	3.13859	4.43828							
T91	0	0.128875	0.554046	1	1.26135	2.13263	3	3.68811	4.23499						

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	Graph
T92	0	0.140419	0.536221	0.775431	1.5803	2.24487	2.7784	3.59883	4.34552						
T93	0	0.0994218	0.425424	0.753275	1.39972	2.07259	2.95098	3.21331	4.34154						
T94	0	0.170839	0.381966	0.850274	1.67608	2.41647	2.61803	3.4421	4.44423						
T95	0	0.198062	0.300372	1	1.55496	2.23912	3	3.24698	4.4605						
T101	0	0.10288	0.436731	1	1	1.72504	2.50638	3.22554	3.76770	4.23566					
T102	0	0.109988	0.401579	0.669715	1.24147	2	2.40102	3.057014	3.71197	4.34632					
T103	0	0.117246	0.381966	0.758554	1.38197	1.66743	2.61803	3.08404	3.61803	4.37213					
T104	0	0.120615	0.3489107	1	1	2	2.3473	3.27389	3.53209	4.3772					
T105	0	0.127724	0.381966	0.624674	1.38197	2	2.61803	2.79682	3.61803	4.44579					
T106	0	0.147892	0.281431	0.787299	1.29311	2	2.4631	3.09257	3.46873	4.40588					
T107	0	0.198062	0.198062	0.829914	1.55496	1.55496	2.68889	3.24698	3.24698	4.48119					

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	Graph
T111	0	0.0841755	0.353655	0.811602	1	1.41756	2.10311	2.78096	3.3658	3.82143	4.23601				
T112	0	0.0887437	0.381966	0.620874	1	1.67451	2.22011	2.61803	3.26428	3.78492	4.34050				
T113	0	0.0937669	0.342137	0.609602	1.22705	1.44093	2.12562	2.87624	3.18321	3.72267	4.37278				
T114	0	0.0973616	0.302002	0.745726	1	1.59526	2.10104	2.70718	3.37152	3.64065	4.37027				
T115	0	0.997689	0.381966	0.482937	1.11386	1.73162	2.3109	2.61803	3.08025	3.72540	4.4402				
T116	0	0.112949	0.273718	0.604027	1.14332	1.65576	2.21289	2.7721	3.12817	3.62082	4.40720				
T117	0	0.120615	0.239231	0.707909	1	1.73655	2.3473	2.50304	3.34218	3.53209	4.47109				
T118	0	0.141043	0.198062	0.677314	1.1933	1.55496	2.15883	2.86161	3.24698	3.48181	4.4861				
T125	0	0.0810141	0.254543	0.690279	0.798660	1.5089	1.71537	2.53656	2.83083	3.52003	3.68251	4.38131			
T131	0	0.0595297	0.24635	0.565009	1	1	1.52001	2.08434	2.6477	3.16422	3.60050	3.88021	4.23008		
T132	0	0.0616214	0.264326	0.545393	0.72038	1.17202	1.73346	2.19878	2.52299	3.04093	3.52482	3.86802	4.34607		

P3C

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	Graph
T133	0	0.064095	0.263646	0.457992	0.825863	1.33841	1.49225	2.09882	2.73386	3.07411	3.43927	3.83867	4.37303		
T134	0	0.0664462	0.239083	0.48073	1	1	1.6869	2.15107	2.58043	3.19743	3.48069	3.79050	4.38006		
T135	0	0.0679419	0.222391	0.558651	0.764316	1.23845	1.60806	2.11955	2.60412	3.05216	3.57715	3.74528	4.3819		
T136	0	0.065014	0.309889	0.38966	0.737267	1.27434	1.7703	2.20162	2.01803	2.86316	3.42933	3.84125	4.44634		
T137	0	0.0724564	0.2611	0.381966	0.847568	1.2376	1.72091	2.19178	2.01803	3.08573	3.31641	3.79876	4.40769		
T138	0	0.0782928	0.207598	0.469418	0.809207	1.19304	1.7756	2.28105	2.47003	3.05870	3.43092	3.73821	4.47275		
T139	0	0.0810141	0.191007	0.548898	0.690279	1.2976	1.712537	2.18905	2.74303	2.83083	3.55004	3.68251	4.47371		
T1310	0	0.083532	0.198062	0.426787	0.807819	1.354	1.55496	2.11852	2.76060	3.16417	3.24698	3.7369	4.48762		

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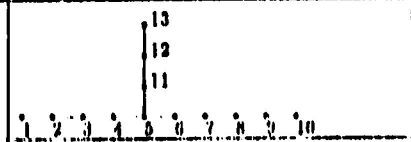
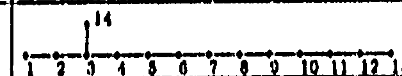
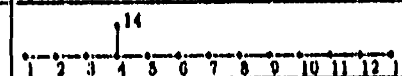
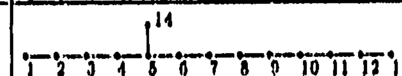
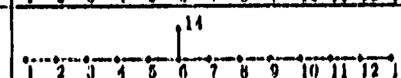
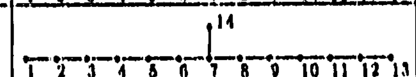
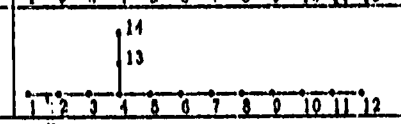
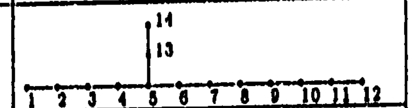
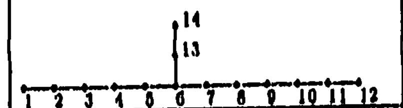
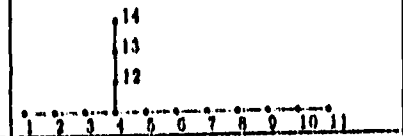
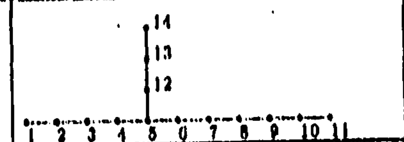
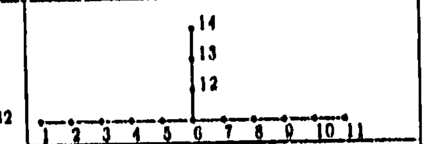
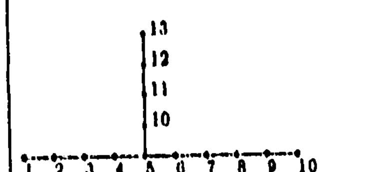
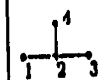
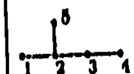
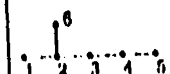
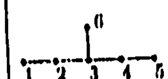
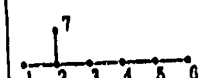
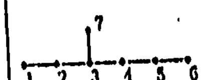
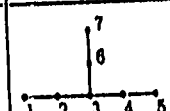
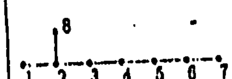
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	Graph
T1311	0	0.0953839	0.158104	0.491698	0.835581	1.20299	1.65799	2.14774	2.67509	3.07858	3.48511	3.6491	4.40204		
T141	0	0.0511313	0.210436	0.482064	0.8561	1	1.31061	1.81602	2.33841	2.84225	3.20277	3.65773	3.90042	4.24606	
T142	0	0.0526071	0.224139	0.492413	0.655536	1	1.49493	2	2.32519	2.71234	3.1045	3.60807	3.8930	4.32667	
T143	0	0.0543892	0.228402	0.416011	0.697491	1.16730	1.41983	1.78042	2.380137	2.70107	3.15137	3.53978	3.87184	4.37301	
T144	0	0.0561887	0.214386	0.409244	0.819655	1	1.40137	2	2.25836	2.80208	3.3166	3.50036	3.84163	4.38011	
T145	0	0.0575821	0.198062	0.455439	0.75302	1	1.55469	1.76379	2.44504	2.73003	3.24698	3.60118	3.80194	4.3820	
T146	0	0.0581164	0.191894	0.502979	0.648384	1.20465	1.29079	2.24107	2.90817	3.13013	3.60442	3.77091	4.38249		
T147	0	0.056218	0.256726	0.381966	0.613088	1.08297	1.56008	2	2.50729	2.61803	3.05657	3.53705	3.87390	4.4403	
T148	0	0.0602012	0.249058	0.323162	0.709967	1.11842	1.48410	2	2.33902	2.83878	3.12030	3.44511	3.84404	4.4077	

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	Graph
T149	0	0.0645677	0.198062	0.381966	0.75302	1	1.55496	2	2.44504	2.61803	3.24698	3.4626	3.80194	4.4728	
T1410	0	0.067881	0.173913	0.465857	0.624487	1.0902	1.51839	2	2.37475	2.77547	3.07641	3.60107	3.74783	4.4740	
T1411	0	0.0677032	0.198062	0.343328	0.75302	1.14256	1.55496	1.74909	2.44504	2.88515	3.24698	3.32357	3.80194	4.4876	
T1412	0	0.07619	0.155327	0.403342	0.732505	1.1558	1.40906	2	2.20002	2.89049	3.1816	3.40072	3.74161	4.4923	
T1413	0	0.0810141	0.139977	0.441657	0.690279	1.11703	1.62997	1.71537	2.47041	2.83083	3.11719	3.68251	3.58180	4.4932	
T1414	0	0.0919248	0.120615	0.432248	0.791513	1	1.50437	2	2.3473	2.72014	3.30194	3.53209	3.65521	4.4900	
T1312	0	0.120615	0.120615	0.504492	1	1.78031	2.3473	2.3473	3.21069	3.53209	3.53209	4.40551			

Appendix 2: The second smallest eigenvalue with the corresponding eigenvector for all trees with three end vertices; $4 \leq n \leq 14$.

n	Graph	Min Eigenvalue		Eigenvector														Drawing
		λ_{n-1}	Mult.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	
4	T41	1		-1	0	0	1											
5	T51	0.518806		0.70242	0.33800	-0.20177	-0.41932	-0.41932										
6	T61	0.324869		0.369599	0.249528	-0.0716788	-0.369599	0.547448	0.369599									
	T62	0.381966		-0.4637	0.286582	0	0.286582	0.4637	0									
7	T71	0.225377		-0.530921	-0.415911	-0.080155	0.273666	0.50581	0.730432	-0.530921								
	T72	0.260323		0.419365	0.310194	0.120274	-0.143286	-0.369546	-0.499004	0.102003								
	T73	0.381966		-0.802001	-0.495664	0	0.247832	0.401001	0.247832	0.401001								
8	T81	0.166717		0.392963	0.327449	0.141831	-0.0674334	-0.205455	-0.419221	-0.503095	0.392963							

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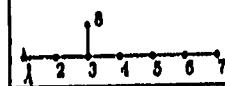
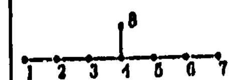
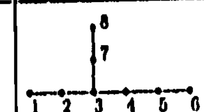
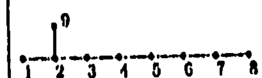
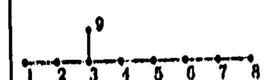
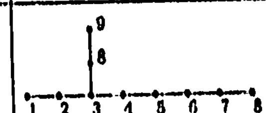
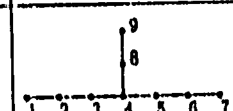
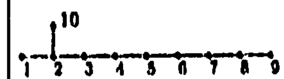
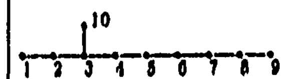
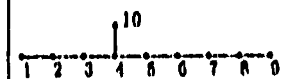
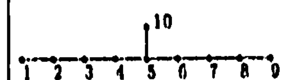
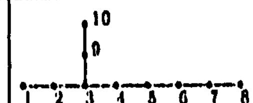
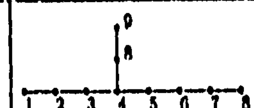
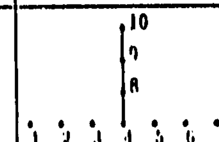
n	Graph	Min Eigenvalue		Eigenvector														Drawing
		λ_{n-1}	Mult.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	
	T82	0.186393		-0.497968	-0.40515	-0.236814	0.0299145	0.291068	0.497968	0.01205	-0.291068							
	T83	0.198062		-0.447868	-0.359162	-0.19932	0	0.19932	0.359162	0.447868	0							
	T84	0.243402		-0.194570	-0.147218	-0.0640242	0.117948	0.271211	0.358461	-0.147218	-0.194570							
9	T91	0.128675		0.392535	0.341947	0.196702	0.0261081	-0.147851	0.302750	0.418643	-0.480577	0.392535						
	T92	0.140419		0.384585	0.330582	0.230159	0.0598194	-0.11802	-0.280961	-0.40355	0.409472	0.267757						
	T93	0.0994218		0.596051	0.536791	0.424162	0.260361	0.058044	-0.159044	-0.201276	0.223496	0.290098						
	T94	0.170839		0.237997	-0.197338	-0.122966	0.0467855	0.208544	0.334676	0.403032	0.197338	-0.237997						
	T95	0.198062		0.372209	0.298488	0.165648	0	-0.165648	-0.298488	-0.372209	0	0						

n	Graph	Min Eigenvalue		Eigenvector														Drawing
		λ_{n-1}	Mult.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	
1)	T101	0.10288		-0.480968	-0.431480	-0.288131	-0.115133	0.0697105	0.247382	0.399002	0.510712	0.509270	-0.480908					
	T102	0.109988		0.377575	0.330046	0.257550	0.118909	-0.032817	-0.180933	-0.309149	-0.403362	-0.45321	0.289388					
	T103	0.117246		0.387325	0.341912	0.256412	0.140849	-0.00993014	-0.159550	-0.290408	0.387325	-0.438769	0.159550					
	T104	0.120615		0.464243	0.408248	0.303013	0.16123	0	-0.16123	-0.303013	-0.408248	-0.464243	0					
	T105	0.127724		0.269825	0.235362	0.170837	0.0199087	-0.13345	-0.269825	-0.371736	0.420108	0.235302	0.269825					
	T106	0.147892		-0.306943	-0.261549	-0.177473	-0.0671507	0.0840146	0.224422	0.330739	0.388142	-0.0989024	-0.116138					
	T107	0.198062		0.694486	0.556934	0.309075	0	-0.154538	0.278467	0.347243	0.154538	0.278467	0.347243					

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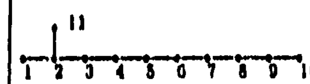
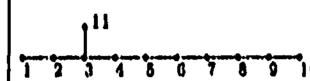
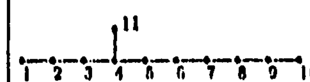
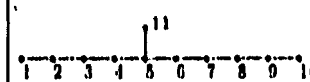
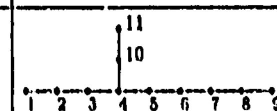
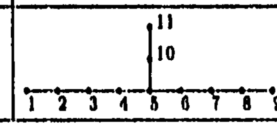
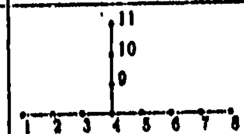
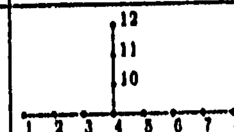
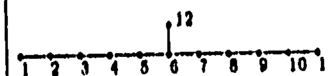
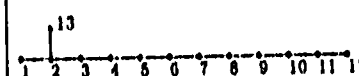
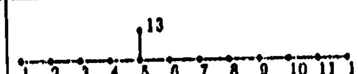
r	Graph	Min Eigenvalue		Eigenvector														Drawing
		λ_{n-1}	Mult.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	
11	T111	0.0841755		0.375961	0.344314	0.252038	0.138546	0.0133923	-0.112889	-0.229007	-0.327114	-0.397028	-0.433516	0.375961				
	T112	0.0887437		-0.401936	-0.360267	-0.298094	-0.174430	-0.0352988	0.106971	0.239748	0.351240	0.431879	0.473009	-0.327124				
	T113	0.0937669		0.372699	0.337753	0.271136	0.179095	0.0517308	-0.0804842	-0.205153	-0.310584	-0.386894	-0.420925	0.197026				
	T114	0.0973616		0.389506	-0.351584	-0.27943	-0.18007	-0.0031791	0.0666782	0.190044	0.294900	0.371050	0.411079	0.0699038				
	T115	0.0997684		-0.318964	-0.287142	-0.226672	-0.0831164	0.0687312	0.213722	0.337389	0.427396	0.474763	-0.287142	-0.318904				
	T116	0.112949		0.31931	0.283244	0.215186	0.122823	-0.0222592	-0.164827	-0.288778	-0.380111	-0.428511	0.161060	0.182284				
	T117	0.120615		0.370095	0.325456	0.241562	0.128533	0	-0.128533	0.241562	-0.325456	-0.370095	0	0				

n	Graph	Min Eigenvalue		Eigenvector														Drawing
		λ_{n-1}	Mult.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	
	T118	0.141043		0.177923	0.152828	0.106178	0.0445526	-0.0849820	-0.202532	-0.291515	-0.339387	0.106178	0.152828	0.177923				
12	T1210	0.135056		0.415076	0.359017	0.254472	0.115558	0.0239207	-0.0709476	-0.150234	-0.22042	-0.254637	0.0520759	0.074317	0.0859211			
	T125	0.0810141		-0.392299	-0.360517	-0.299528	-0.214274	-0.11166,	0	0.11166	0.214274	0.299528	0.360517	0.392209	0			
13	T131	0.0595297		-0.431669	-0.405972	-0.33041	-0.235179	-0.125948	-0.00021071	0.108058	0.218003	0.310716	0.395076	0.451081	0.470631	-0.431669		
	T132	0.0616214		0.361538	0.339259	0.296075	0.215204	0.121071	0.0104781	-0.0833152	-0.180075	-0.267482	-0.337507	0.380734	-0.41213	0.315518		
	T133	0.064095		-0.351091	-0.328588	-0.285024	-0.223101	-0.131768	-0.031899	0.0700145	0.167441	0.254134	0.32454	0.374143	0.309706	-0.238477		
	T134	0.0664462		-0.385899	-0.360257	-0.310678	-0.240456	-0.154256	-0.0468207	0.0037137	0.17002	0.20503	0.342429	0.397070	0.425338	-0.105235		
	T135	0.0679419		0.368701	0.34365	0.295252	0.226794	0.142926	0.0493183	-0.0511797	-0.14823	-0.23521	-0.300209	-0.356404	-0.382384	0.0529485		

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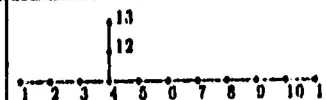
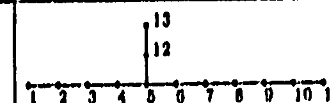
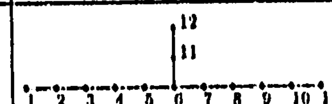
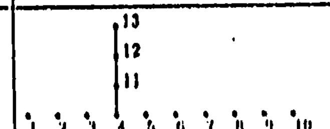
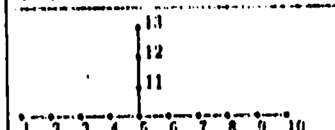
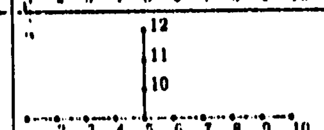
n	Graph	Min Eigenvalue		Eigenvector														Drawing
		λ_{n-1}	Mult.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	
	T136	0.0665014		0.331973	-0.309896	-0.267211	-0.164071	-0.0500195	0.0073582	0.180250	0.281167	0.36338	0.421426	0.45145	-0.309890	0.331073		
	T137	0.0724564		-0.359432	-0.333389	-0.283189	-0.212471	-0.0886946	0.0415085	0.108704	0.283676	0.378003	0.448116	0.470887	-0.280135	-0.209074		
	T138	0.0782928		0.346227	0.31912	0.267028	0.19403	0.10584	-0.0112829	-0.127523	-0.233779	-0.321731	0.384405	-0.417186	0.126488	0.137232		
	T139	0.0810141		0.349193	0.320903	0.266616	0.190729	0.0993906	0	-0.0993906	-0.190729	0.266616	-0.320903	-0.349193	0	0		
	T1310	0.083532		0.200042	0.183332	0.151308	0.106645	0.00841063	0.0905263	0.181901	0.256082	0.312704	0.341206	0.161308	0.181132	0.200042		
	T1311	0.0953839		0.240335	0.217411	0.173749	0.113516	0.0424520	-0.055185	-0.147580	0.225850	0.282616	0.312414	0.0649707	0.0813084	0.0898817		
	T1312	0.120615	2	0.612813	0.5388099	0.399986	0.212828	0	0.100414	0.199093	0.26945	0.300407	0.100414	0.199093	0.26945	0.300407		

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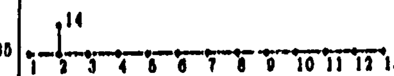
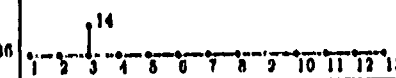
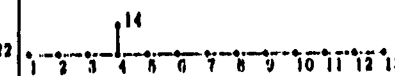
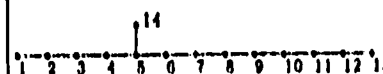
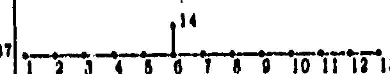
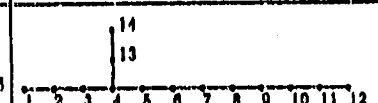
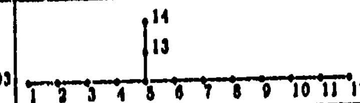
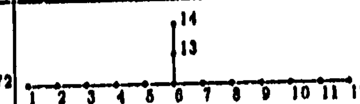
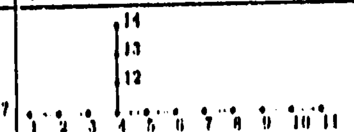
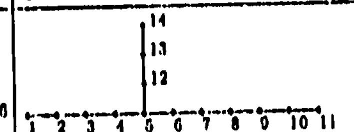
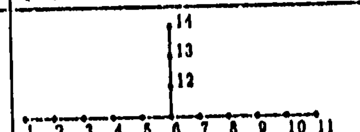
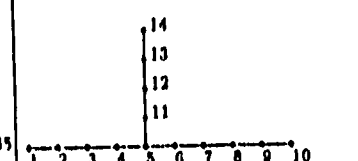
m	Graph	Min Eigenvalue		Eigenvector														Drawing
		λ_{n-1}	Mult.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	
14	T141	0.0511313		-0.352335	-0.33432	-0.281195	-0.213692	-0.135263	-0.0409175	0.0379803	0.123930	0.203886	0.272766	0.328029	0.366821	0.380271	-0.352335	
	T142	0.0526071		-0.370893	-0.351381	0.313384	-0.2415	-0.156911	0.0040068	0.0321474	0.12067	0.21453	0.291103	0.352302	0.395085	0.417023	-0.330786	
	T143	0.0543892		-0.3338545	-0.320132	0.284307	-0.233019	-0.155685	-0.069824	0.0198041	0.108355	0.191013	0.263281	0.32123	0.361708	0.382512	-0.246422	
	T144	0.0561857		0.345861	0.326427	0.288652	0.234658	0.107479	0.0809191	-0.0101878	-0.100722	0.185597	0.260044	0.319879	0.361741	0.383276	0.17745	
	T145	0.0575821		-0.37861	-0.356809	-0.314462	-0.254008	-0.178927	-0.0935432	0.00204248	0.0992587	0.180859	0.269528	0.333076	0.37861	0.401743	0.0992587	
	T146	0.0581164		-0.364467	-0.343286	-0.302154	-0.243462	-0.17062	-0.0878634	0	0.0878634	0.17062	0.243462	0.302154	0.343286	0.364467	0	
	T147	0.0560218		0.296796	0.280169	0.247847	0.169316	0.0813008	-0.0112694	-0.103208	-0.189305	0.264914	0.325621	0.368087	0.389931	0.280169	0.296796	
	T148	0.0602012		-0.330433	-0.310541	-0.271953	-0.216994	-0.118182	-0.0122554	0.094409	0.19539	0.284008	0.350692	0.407304	0.433394	-0.247783	-0.263656	

n	Graph	Min Eigenvalue		Eigenvector														Drawing
		λ_{n-1}	Mult.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	
	T149	0.0645677		-0.361501	-0.33816	-0.292984	-0.228891	-0.15002	-0.0383299	0.0758348	0.185103	0.282419	0.361501	0.417241	0.446041	-0.173181	-0.185103	
	T1410	0.0676881		0.351312	0.327533	0.281583	0.216573	0.136904	0.0479686	-0.0520418	-0.14853	0.234964	-0.305494	-0.358348	-0.381144	0.0587063	0.0598472	
	T1411	0.0677032		0.222907	0.207816	0.178654	0.137398	0.0455818	-0.04932	-0.140883	0.222007	0.28981	-0.33715	-0.361634	0.178654	0.207816	0.222007	
	T1412	0.07619		0.235445	0.217507	0.182996	0.134543	0.0758396	-0.0159543	-0.106533	0.188094	0.257056	0.305534	-0.330732	0.103182	0.122008	0.132716	
	T1413	0.0810141		0.316669	0.291015	0.241784	0.172965	0.0901334	0	-0.0901334	0.172965	0.241784	-0.291015	-0.316669	0	0	0	
	T1414	0.0919248		-0.157765	-0.143263	-0.115591	-0.0772931	-0.0318904	0.0618406	0.149898	0.224171	0.277836	0.305962	-0.0772931	-0.115591	-0.143263	-0.157765	

Appendix 3:

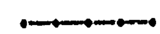
The smallest eigenvalue with the coreponding eigenvector of $L(Pn)$ where Pn is a path on n vertices:

n	λ_{n-1}	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1		1													
2	2	0.707107	-0.707107												
3	1	-0.707107	0	0.707107											
4	0.585786	-0.653281	-0.270598	0.270598	0.653281										
5	0.381966	-0.601501	-0.371748	0	0.371748	0.601501									
6	0.267949	-0.557678	-0.408248	-0.149429	0.149429	0.408248	0.557678								
7	0.198062	0.521121	0.417907	0.231921	0	-0.231921	-0.417907	-0.521121							
8	0.152241	-0.490393	-0.415735	-0.277785	-0.0975452	0.0975452	0.277785	0.415735	0.490393						
9	0.120615	0.464243	0.408248	0.303013	0.16123	0	-0.16123	-0.303013	-0.408248	0.464243					
10	0.097887	-0.441708	-0.39847	-0.310228	-0.203031	-0.0699500	0.0699500	0.203031	0.310228	0.39847	0.441708				
11	0.0810141	-0.422061	0.387868	-0.322253	0.23053	-0.120131	0	0.120131	0.23053	0.322252	0.387868	0.422061			
12	0.0681483	-0.404756	-0.323885	-0.323885	-0.248520	-0.15623	0.0532871	0.0532871	0.15623	0.248520	0.121885	0.377172	0.404756		
13	0.0581164	0.389372	0.366744	0.322801	0.260098	0.182270	0.0938073	0	0.0938073	0.182270	-0.260098	-0.322801	-0.366744	-0.389372	
14	0.0501442	0.375588	-0.356754	-0.320032	-0.267261	-0.201089	-0.124834	-0.0423186	0.0423186	0.124834	0.201080	0.2070201	0.320032	0.356754	0.375588

The eigenvalues of $L(P_n)$ where P_n is a path on n vertices:

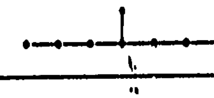
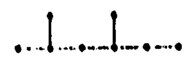
n	λ_0	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	λ_9	λ_{10}	λ_{11}	λ_{12}	λ_{13}
1	0													
2	0	2												
3	0	1	3											
4	0	0.585786	2	3.41421										
5	0	0.381966	1.38197	2.61803	3.61803									
6	0	0.267949	1	2	3	3.73205								
7	0	0.198062	0.75302	1.55496	2.44504	3.24698	3.80194							
8	0	0.152241	0.585786	1.23463	2	2.76537	3.41421	3.84776						
9	0	0.120615	0.467911	1	1.6527	2.3473	3	3.53209	3.87939					
10	0	0.97887	3.81966	0.824429	1.38197	2	2.61803	3.17557	3.61803	3.90211				
11	0	0.0810141	0.371493	0.600279	1.16917	1.171637	2.28463	2.83083	3.30972	3.08251	3.91899			
12	0	0.0681483	0.267949	0.585786	1	1.48236	2	2.51764	3	3.41421	3.73205	3.93185		
13	0	0.0581164	0.229088	0.502979	0.863871	1.29079	1.75893	2.24107	2.70921	3.11613	3.49702	3.77091	3.94188	
14	0	0.051442	0.198062	0.436337	0.75302	1.13223	1.55496	2	2.44504	2.86777	3.24698	3.56366	3.80194	3.94986

Appendix 4: The second smallest eigenvalue with the corresponding eigenvector for all Caterpillars of order n; $3 \leq n \leq 8$.

	a_T	1	2	3	4	5	6	7	8	Graph
$C31$	1	−0.707107	0	0.707107						
$C41$	0.585786	−0.653281	−0.270598	−0.270598	0.653281					
$C42$	1	−1	0	0	1					
$C51$	0.381966	−0.601501	−0.371748	0	0.371748	0.601501				
$C52$	0.518806	0.70242	0.33800	−0.20177	−0.41932	−0.41932				
$C53$	1	−0.948683	0	0.316228	0.316228	0.316228				
$C61$	0.267949	−0.557678	−0.408248	−0.149429	0.149429	0.408248	0.557678			
$C62$	0.324869	0.369599	0.249528	−0.0716788	−0.369599	−0.547448	0.369599			

	a_T	1	2	3	4	5	6	7	8	Graph
C63	0.381966	−0.4637	−0.286582	0	0.286582	0.4637	0			
C64	0.485863	−0.232066	−0.119314	0.270913	0.538598	−0.232066	−0.232066			
C65	0438477	0.501939	0.281865	−0.281865	−0.501939	−0.501939	0.501939			
C66	1	1	0	−1	0	0	0			
C71	0.198062	0.52111	0.417907	0.231921	0	−0.231921	−0.417907	−0.521121		
C72	0.225377	−0.530921	−0.415911	−0.080155	0.273666	0.56581	0.730432	−0.530921		
C73	0.260323	0.410365	0.310194	0.120274	−0.143286	−0.369546	−0.499604	0.162003		
C74	0.295532	−0.316382	0.222881	0.123491	0.433307	0.01517	−0.316382	−0.316382		

	a_T	1	2	3	4	5	6	7	8	Graph
C75	0.381966	0.459049	0.283708	0	−0.283708	0	0	0		
C76	0.32172	−0.408121	−0.27682	0.0748301	0.300024	0.540062	0.110337	−0.408121		
C77	0.267949	−0.422573	−0.300345	0	0.300345	0.422573	0.422573	−0.422573		
C78	0.465593	−0.190041	−0.10502	0.310300	0.58119	−0.190041	−0.190041	−0.190041		
C79	0.398321	0.412022	0.247905	−0.343192	−0.57039	−0.57039	0.412022	0.412022		
C710	1	1	0	−1	0	0	0	0		
C81	0.152241	−0.490393	−0.415735	−0.277785	−0.097545	0.097545	0.277785	0.415735	0.490393	
C82	0.166717	0.392063	0.327449	0.141831	−0.0074334	−0.205455	−0.419221	−0.503095	0.392063	

	a_T	1	2	3	4	5	6	7	8	Graph
C83	0.186393	−0.497068	−0.40515	−0.236814	0.0299145	0.291008	0.497067	0.61205	−0.291008	
C84	0.198062	−0.447868	−0.359162	−0.19932	0	0.19932	0.359162	0.447868	0	
C85										
C86	0.253787	0.38079	0.284151	0.111115397	−0.161135	−0.390774	−0.531717	0.154044	0.154044	
C87	0.224287	0.448372	0.347808	0.0686709	−0.245723	−0.505005	−0.05102	0.0885201	0.448372	
C88	0.213682	0.466949	0.367171	0.0891551	−0.207011	−0.404051	−0.513852	−0.204411	0.466949	
C89	0.18639	−0.6089	−0.544222	−0.193425	0.193425	0.544222	0.6089	0.6089	−0.6089	
C810	0.25082	0.503091	0.376874	0.156106	−0.156106	−0.376874	−0.503091	−0.208387	0.208387	

	a_T	1	2	3	4	5	6	7	8	Graph
C811	0.277407	−0.277805	−0.200783	0.103242	0.481083	0.667018	−0.277805	−0.277805	−0.277805	
C812	0.381966	0.458955	0.283032	0	−0.283032	−0.457055	0	0	0	
C813	0.288801	−0.35030	−0.253443	0.128504	0.421150	0.592178	0.180080	−0.35030	−0.35030	
C814	0.238443	−0.359138	−0.273504	0.0480120	0.359138	0.471584	0.471584	−0.359138	−0.359138	
C815	0.188669	−0.461535	−0.314458	0.0799043	0.374058	0.549012	0.117277	0.117277	−0.461535	
C816	0.267949	0.393927	0.288375	0	−0.288375	−0.393927	−0.393927	0	0.393927	
C817	0.452493	0.171206	−0.0937363		0.336026	0.613739	−0.171206	−0.171206	−0.171206	
C818	0.373802	0.353219	0.221185	−0.389631	−0.62216	−0.62216	0.353219	0.353219	0.353219	

	a_T	1	2	3	4	5	6	7	8	Graph
$C819$	0.354249	0.520227	0.335938	−0.335938	−0.520027	−0.520027	−0.520027	−0.520027	−0.520027	
$C820$	1	1	0	−1	0	0	0	0	0	

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