

ON SOME SEISMIC INVERSE PROBLEMS

BY

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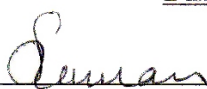
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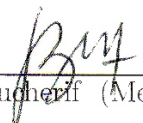
DEANSHIP OF GRADUATE STUDIES

This Dissertation, written by **ZEID AL-MUHIAMEED** under the direction of his Dissertation advisor and approved by his dissertation committee, has been presented to and accepted by the Dean of Graduate Studies, in partial fulfillment of the requirements for the degree of **DOCTOR OF PHILOSOPHY IN MATHEMATICAL SCIENCES.**

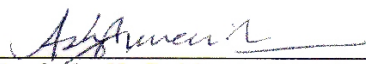
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DEDICATION

This Thesis is Dedicated to

My Father for his care and continuous prayers for my success;

My Wife for her care, patience and support to complete this work;

My lovely and intelligent daughters and sons:

Fatemah, Lolo, Nassibah, Marya, Osamah and Abdallah.

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ABSTRACT

Name: Zeid Al-Muhiameed
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The problem of mapping the interior of earth, or gaining information about in-accessible part of a body is an interesting and important variety of what is generally referred to an inverse problem. The methods presented in this dissertation are based upon sending one or more signals from the surface of the earth, receiving the response of these after interaction with the obstacles, inhomogenities or inclusions present in the interior and then trying to obtain an inversion formula giving the desired information. Most frequently used model is based upon the assumption that the medium under investigation is homogenous and isotropic.

We consider a more general model of damped medium and formulate and study the inverse problems arising from variations in speed, density, bulk modulus or/and the damping parameter. These situations may arise in practical problems in which information about the variation in wave speed due to an inhomogeneity may not be enough.

The methods are based upon solving a direct problem to obtain the scattered field in term's of Green's function and the small change or perturbation of

the material parameter. This non-linear integral equation is then linearized using the Born approximation. Then an asymptotic approximation of the Green's function is used to enable us to use the Fourier inversion formula. An improvement to the final result is presented and some numerical examples in one-dimension have been presented to implement the inversion procedure.

An inverse problem in one and higher dimensions is used to recover the variation Bulk modulus as well as the density in a damped medium. In order to allow radial a cylindrical symmetry of the problem, we also study the inverse problem involving the Bessel operator. The asymptotic behavior of the Green's function and the WKB approximation is used to obtain the solution to the inverse problem.

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الخلاصة

الإسم: زيد بن إبراهيم المحيميد
العنوان: بعض المسائل العكسية للاهتزازات الزلزالية
التخصص الرئيسي: رياضيات تطبيقية
تاريخ نيل الدرجة: يونيو 2005

إن مسألة تخطيط جوف الأرض أو الحصول على معلومات عن مناطق لا يمكن الوصول إليها هي جسم ما تعتبر صورة ممتعة و مهمة في نطاق مسائل تعرفه بالمسائل العكسية. إن الطرق المعروضة في هذه الرسالة تعتمد على إرسال إشارات من على سطح الأرض واستقبالها ثانية بعد تحايلها وانعكاسها من المكونات الداخلية الغير متجانسة للأرض والتي من خلالها يمكن الحصول على معلومات مفيدة. النموذج الأكثر استخداماً هي ذلك يفترض أن الوسط متجانس ومتناهي (ثابت في الإتجاهات). نستخدم في دراستنا هنا على نموذج الوسط المخمد (المجهد) حيث نقوم ببناء ودراسة المسائل العكسية الناتجة عن التغيرات في السرعة، الكثافة، معامل التكتل (البلك)، والتغيرات الوسطية للإخماد.

الطريقة تعتمد على حل المسألة المباشرة للحصول على حقل التشتت بدلالة دالة "جرين" و معدلات التشويش في المتغيرات الوسطية. وتحول معادلات التكامل الغير خطي إلى معادلات خطية باستخدام طريقة "بورن". ولذي نتمكن من استخدام صيغة "فوربير" العكسية نستخدم صيغة التقاربية لدالة "جرين". ونسرد هنا بعض النتائج العددية للنموذج العكسي.

نستطيع إستقراء معامل التكتل (البلك) والكثافة في وسط مخمد إحدادي أو متعدد الأبعاد بحل المسألة العكسية. وبمعالجة المسألة العكسية المتضمنة على مؤثر "بيسل" نسمح بمعالجة التماثل الإسطواني في المسائل. يمكن السلوك التقاربي لدالة "جرين" بالإضافة الي تقريبه "WKB" من حل المسألة العكسية.

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Chapter 1

Introduction

The inverse problem received its baptism at the hands of Abel's in connection with his solution of the tautochrone problem, published in 1826 [1]. Abel's solution to this problem in classical mechanics is of importance in many areas of inverse scattering. The next major exposition of inverse scattering was instigated by the investigation into the structure of the atom by Rutherford in 1911 [73] and culminated in the discovery of the atomic nucleus. It was with the foundation of the modern quantum theory and particularly the wave mechanical formulation of Schrodinger in 1926 [18], that inverse scattering became a subject of paramount importance. Therefore, the study of quantum mechanics provided both the means and the motivation to further develop the inverse scattering problem. Born eventually showed in 1926 that provided the scattering interaction was sufficiently weak, a particularly simple relationship existed between the scattered field and the scattering potential. Inversion is not a new process in geophysical methods. During the pioneering years, the inversion process was commonly used, in forward and inverse problem of scalar, acoustic and elastic waves (e.g., Claerbout, 1970 [21]; Claerbout, 1972 [23]; Flatte and Tappert, 1975 [43];

McCoy, 1977 [60]; Wales and McCoy, 1983 [81]; Bleistein, 1984 [10]; Weglein and Gasparotto, 1989 [83]; Wu and Hung, 1992 [85]; Idemen, M. and Akduman, 1996 [53] Bleistein, Cohen and Stockwell, *at al* 2001 [13]).

The mapping of the interior of the earth from observation on surface of the earth is also an inverse problem [3]. Despite the mathematical elegance of the exact nonlinear inversion schemes, they are of limited applicability. In order to increase applicability to a wide variety of problems, many researchers used approximation techniques such as linearization and perturbation methods [8, 30, 38, 39, 40, 42, 49, 50, 51]. In earlier work, a perturbation in reference sound speed velocity was determined by assuming that the reference velocity was constant. Bleistein and Gray allowed the reference velocity to be a function of the depth variable. The output of this method is a high frequency bandlimited reflectivity function of the subsurface. Bleistein introduced the constant-background zero-offset equation to solve some inverse problems. The variations in density is described by small perturbation terms and the Born approximation is used to linearize the inversion formula, see for example Gerver [46], Cohen and Bleistein [25, 26], Clayton [24], Bleistein and Gray [14], among others. Zaman and Masood have introduced the damping effects in the model through damped wave equation and used high frequency estimates to recover the velocity profiles and damping parameters of the medium with constant density.

The inverse scattering problems constitute an important class of inverse problems in which we aim at obtaining information about inclusions, defects or inhomogeneities in the interior of a body without being able to gain access to the location of interest. This situation arises in several interesting applications. In seismic exploration, the focus of study is to obtain the mapping of the interior of earth for exploration,

detection of geological faults or earthquake prediction. In material sciences, we aim to detect cracks or inhomogeneities within a structure. In medical imaging, detection and monitoring of tumors may be carried out by this process. The main focus of our study is the seismic inversion in a suitable model of earth. The seismic modeling used as a mean is instrumental in studying such basic problems of seismology as the origin of earthquakes, the wave pattern and its relation to the structure of the earth and its surface. The seismic methods of geophysical exploration utilize the fact that elastic waves travel with different velocities in different rocks [20].

The principle is to initiate such waves at a point referred to as source and determine at a number of other points called receivers, the time of arrival of the energy that is refracted or reflected by the discontinuities between different rock formations. Then this enables the position of the discontinuities to be deduced [68]. This constitutes the simplest inverse seismic problem. In more interesting cases, we attempt to determine the nature of this discontinuously by recovering the material properties of the rocks around it. There is, in most cases, quite a natural distinction between the direct and the inverse problem. For instance, predicting the future behavior of a physical system from the knowledge of its present state is regarded as a direct problem. However, determining the past state of a system from present observations, or the identification of physical parameters from observations of probed data constitute inverse problems [57]. Also, during the last two decades, the struggle to find oil under more and more difficult conditions has led to very sophisticated acquisition, processing, and interpretation procedure to filter the unwanted data and to "invert" the signals in order to reconstruct the underground lithology in great detail. For such problems, the propagation of acoustic, elastic, or electromagnetic signals into the earth is modeled by the

appropriate equation or system of equations in which one or more functions characterizing the interior of the earth (soundspeed, elastic, or electromagnetic coefficients) are left free. One or more signals consistent with the model are introduced at or near the surface of the earth in a region of interest [57, 68, 70]. The "irregularities" of the interior of the earth produce a response to those signals. Observations of those responses are recorded. This response, frequently known as the scattered field is used to find the material properties (sound speed, elastic properties for example). The direct problem consists of finding the scattered field in terms of the material parameters of the inhomogeneity. The inverse problem uses the scattered field as data and produces the information about the material properties of the inhomogeneity [13, 19, 42, 50]. This type of the inverse problem is known by the title *inverse scattering problem* [25]. In inversion, the objective is to obtain an equation or equations in which one or more variable characterize the unknowns to be determined. The solution of the inverse problem then quantifies the unknowns (e.g., velocity variation) in terms of the known observations. Of course, full solution of an inverse problem is usually difficult, and approximations are made as part of the solution technique. However, these approximations have analytical form and can therefore be analyzed. So, the objective of the inverse problem in geophysics is to understand the structure and properties of the interior of the earth [34]. The problems that are similar in nature and methodology for seismic inversion can also be studied using similar techniques for example, problems in ocean acoustics, medical imaging, migration problems and atmospheric physics.

In all of these situations, a signal is introduced either from a transmitter or from a natural source such as explosions or earthquakes and the response is then recorded. Based upon the response or received signal (also called a scattered wave), we attempt

to determine the properties of the medium (parameter identification problem) [74], or the nature of the initial signal (determination of initial profile problem). The main difficulty in the problem is the non-linearity and non well-posedness, [69, 70]. Raz [71] discussed some theoretical issues such as existence and uniqueness and in discussed some inverse scattering problems arising from geophysics. The non-well posedness Devancy [36, 37] and the relevant regularization techniques have been discussed by Engle and Neubauer [40, 42]. There have been many approaches to study the nonlinearity and different approximate solutions have been suggested over the years [2, 16, 36, 37]. It must be emphasized that the construction of these approximate solutions involve (explicitly or implicitly) two major, separate steps: the first step is a linearization of the inverse problem and the second is the solution of the inverse problem [17]. One basic assumption which is made in such cases [13, 22, 23] is that the irregularity or the inhomogeneity presents a small variation in the material properties of the background medium (medium around the inhomogeneity) and thus we can write it as a small perturbation of the background medium property. This assumption enables us to expect that the scattered field is small as compared to the incident one and the linearization can be performed. This is usually known as the Born approximations [13, 14, 18, 36, 37, 49, 55, 59]. Among various forms of scattering integrals, the so-called Born's integrals, representing the single-scattering approximation, are very popular. The Born approximation plays a very important role both in direct and inverse problems of seismology, particularly in seismic exploration for oil [49]. The Born approximation has also been generalized and used to increase the accuracy of the zeroth-order approximation of the ray method [39, 40, 47]. Cohen and Hagin [28] among others used the Born approximation [59] and high frequency assumptions to

study the inverse problem in one and higher dimensions. The linearization used in the derivation procedure is often referred to as the Born approximation. An excellent account of the seismic inversion based upon these techniques can be found in Bleistein, Cohen and Stokwell [13]. In reference, a theory is developed to extract information from high frequency band-limited Fourier transform of a piecewise constant function. In this context, recovery of velocity profile has been used as a mean of estimating medium properties [12].

In order to ignore the "noise" produced by still smaller, negligible variation, the assumption of high frequency is used in seismic inversion problems. Other approaches are based upon mostly numerical computations. Clayton and Stolt [24] presented a procedure under the assumption of variable background in 1981, a phase shift migration is used. For a laterally variable background, the WKBJ Green's operators have to be constructed numerically [84]. Miller [58] recognized that the seismic imaging for the general case of variable background and irregular source-receiver geometry could be cast as the problem of inverting a generalized Radon transform (GRT). Previous workers, notably Norton and Linser [67] and Roth [72], have made the connection between the radon transform and the linearized inverse scattering problem. Norton and Linser derived explicit inversion formulas for a constant background velocity and coincident sources and receivers for plane and cylindrical apertures.

The applicability range of the Born approximation has been most thoroughly studied in the context of propagation and scattering by random media. References such as Tatarskii [80] and Ishimaru [54] may be consulted. Much of their work and methodology has been given in [76]. This reference provides an excellent exposition of one and higher dimensional problems for constant and variable background speed,

approximations for higher frequency or large wave numbers. With its higher dimensional Zaman and Masood [87] has utilized the method presented in [13] to recover the damping of the medium and the wave velocity in an elastic medium with damping effects. The model has been further studied by us [89] to include the variation in the Bulk modulus and density (Section 4.2). Nolet [66] showed how the inversion problem for teleseismic waves can be linearized and another class of inversion problems are modeled by using the formulation of Gelfand and Levitan [45]. In this method, see Kay [55], Keller [56], the Helmholtz equation is transformed to Schrodinger equation and the potential is recovered using the scattering data. In yet another different approach Aktosun [4] has utilized the method presented in [5 – 7] and formulated the inverse scattering problem as a Riemann-Hilbert boundary value problem, to transform this Riemann-Hilbert problem into a nonhomogeneous integral equation where the kernel and the nonhomogeneous term contain the Fourier transform of the scattering data and to obtain the potential from the solution of the resulting integral equation. Recently Zaman and Masood [88] have used the Gelfand and Levitan approach to invert shear velocity in a layered model of earth using the Love waves propagating in the upper layer. A number of authors have worked on seismic velocity inversion problems [11, 21, 37]. One of the works is by Bleistain [10] which deals with the inverse scattering problem to determine small variation scattering in propagation speed. Subsequent work (Bleistein, Cohen, Hagin, [11] and Cohen and Bleistein [28] covered this worked further. In case of nonlinear inversion, Snieder and Aldridge [75] used the Born approximation and assumption that the near surface is inhomogeneous. Colten and Kress [32], Colten and Monk [33] considered inhomogeneous nature of the medium and different boundary conditions to deal with inverse scattering problem for acoustic and

electromagnetic waves.

In many situations of interest, the wave equation involving the Bessel operator arises [86]. The problems of wave propagation excited by a line source, scattering of waves by cylindrical objects or inversion of variation in parameters in bodies with cylindrical geometry lead to the wave equation in terms of the Bessel operator. Some interesting situations arise from seismic inversion in objects with cylindrical symmetry, detection and monitoring of tumor in human body and non destructive testing. The interpretation of seismic reflection data, ultrasound reflectivity imaging in medical applications, and various other methods of nondestructive evaluation require a solution to the inverse scattering problem of this nature.

1.1 Objective of the Dissertation

The objective for this dissertation is five-fold. Firstly, to consider some mathematics of basic seismic imaging and inversion . Secondly, to consider a one-dimensional model with constant-background, single layer and multilayer. Also to consider a one-dimensional model which takes into account the variable density and bulk modulus of the background medium and derive an inversion formula in the presence of damping. Thirdly, to consider the velocity\density inversion procedure based on the Born approximation applied to seismic inverse problem in 3-dimensional. Fourthly, to consider an inverse problem in wave equation with Bessel operator and also recover a more accurate reconstruction of the velocity. Fifthly, to consider high seismic frequency inversion involving Bessel equation.

1.2 Organization of the Dissertation

In Chapter 2, we present some basic notions and mathematical techniques that is the basis of our approach. In section 1, we introduce the basic problem while in section 2, the seismic experiment to called the data has been described. In section 3, the bandlimiting, its need and examples are given. Section 4 deals with the asymptotic approximations of WKB and WKBJ type. In section 5, Bleistein model in one dimension has been described and the inverse formula as well as the inversion of the Fourier integral is introduced.

The chapter 3 deals with the damped medium one dimensional model and the recovery of velocity and damping in case of constant density. The numerical examples are considered for one layer and multi-layer models to implement the results derived here.

In chapter 4, we discuss the model under a more general setting of variable density and bulk modulus with or without damping. The recovery procedure for variation in density, bulk modulus or the damping is presented. Higher dimensional model is also introduced with one dimensional variability of velocity/density profile.

Chapter 5 deals with the seismic inverse problems with radial symmetry and discuss the wave equation containing the Bessel operator. The inverse formula is presented using the asymptotic form of Green's function and the WKB approximation for higher frequency assumption. Finally, some recommendations and conclusions are made.

The graphs related to chapter 2 are given in appendix "A" and a glossary of geophysical terminology is included in the appendix "B".

Chapter 2

Mathematics of Seismic Imaging and Inversion

2.1 Introduction

The seismic imaging and inversion is based upon the idea of sending a wave or signal (usually referred to as the incident field) and recording the response from the interior, which in the presence of a change or variation in the properties of the medium is termed as the scattered field. The total field in the body then consists of the sum of the incident field and the scattered field. The variation in the material property or the wave speed is assumed to be in the form that can be written as the background medium property (or wave speed) corresponding to the situation if the in homogeneity causing the variation was not present, plus a small perturbation due to the presence of the inhomogeneity. This enables us to write an integral equation for the scattered field containing the parameter describing the variation. The focus of the inversion procedure is to obtain an expression for this variation by solving the integral equation.

The integral equation giving the scattered field is non-linear as it contains non-linearity in terms of the product of the incident field. The Born approximation is then used to linearize this problem. The next step is to use an approximation Green's function which appears as the kernel of this equation. In case of constant background medium, this Green's function is of convolution type and thus the Fourier inversion formula could be employed to recover the sought after parameter giving the variation. However, if the background medium possesses properties depending upon position, this approach fails. In such a case an approximation to Green's function is used to simplify the integral equation. For seismic inversion, the assumption that the field consists of high frequency waves proves to be appropriate one and enables us to replace the Green's function by its WKBJ approximation.

In this chapter, we present the essentials of the so-called seismic experiment which is used to gather the scattered field and describe why and how the "band limiting" is introduced. The Born approximation, WKB approximation, WKBJ approximations to the Green's function and the Fourier inversion have then been presented for their latter use in our theory. One dimensional Bleistein model [13] illustrate some of these ideas that have been presented in this chapter.

2.2 The Seismic Experiment

The seismic experiment is used to map the internal geological structure with the aim of locating inhomogeneities indicating presence of mineral deposits. It is based upon sending one or more signals from the source and detecting the response of the interior by using receivers placed at different locations usually called geophones. The source may use a controlled source of seismic energy such as an explosion, weight drops or

vibrators.

The essential feature of an exploration seismic experiment are:

- Using controlled source of seismic energy,
- Illumination of a subsurface target area with the downward propagation waves,
- Reflection, Refraction, and diffraction of the seismic waves by subsurface heterogeneities.

- Detection of the back-scattered seismic energy on seismometers spread out on the Earth's surface.

On land, seismometers are called geophone. Generally they work by measuring the motion of a magnet relative to a coil attached to the housing and implanted in the earth. The motion of the magnet relative to the coil produces a voltage which is proportional to the velocity of the displacement on the earth's surface.

2.2.1 Source-Receiver Configurations

In Seismic experiments, the position of source and receivers may be different and such a configuration is called a non zero offset. The configuration may consist of one source and several receivers located at different position called shot profiles.

1. **Zero-Offset Data:** It consists of seismograms recorded when source and receiver are placed next to each other on the surface of the earth; so close in fact that they are effectively at the same location. With this arrangement we can record the seismic echoes from the same point whence they originated. The geometry of such an experiment is shown in figure 1 (appendix A).

2. **Common Source Profile:** It consists of seismograms recorded at positions of increasing range from a single source. The geometry of such an experiment is shown

in figure 2 (appendix A).

3. Common receiver Profile: It consist of seismograms recorded at positions along the surface of the earth with many sources but a single receiver. Its scheme is shown in figure 3 (appendix A).

The most often used arrangement in practice is the "common midpoint" geometry, which can be used to approximate zero-offeset profiles after proper processing steps. For this reason, the zero offset profile will be used in the development.

2.3 Bandlimited Data and its Causes

In reflection seismology, the assumption of high frequency plays an important part. For this purpose, it is important to use a "band filter" which only let signal of the desired frequency pass. Use of frequency response of seismic detectors is therefore influenced by derive to limit ambient noise. In mathematical terms, it is satisfactory to introduce a single function $F(\omega)$, called band filter which posses some nice properties. The band filter $F(\omega)$ is assumed to have symmetric real part and anti-symmetric imaginary part so as to produce a real valued output. We also make sure that the area under the curve describing the real part of $F(\omega)$ contains important piece of information to enable us to recover the required information.

Thus bandlimiting function will include the range frequencies over which a given device is designed to operator within specified limits. Outsides these limits, the Fourier transform of this function vanishes. Bandlimited step functions and bandlimited delta function occurs as the output of the seismic inversion formula we derive.

Bandlimiting of observed data has a variety of causes. We list a few major causes below:

- The frequency of the seismic source is related to the finite nonzero action time and physical geometry of the source mechanism. Equally important is the coupling between the source and the propagating medium.

- The presence of small heterogeneities, randomly distributed in the interior of the earth, scatter the high frequency energy in an incoherent fashion, preventing an image of gross structure being constructed with waves of too high a frequency. Thus higher-frequency signals do not always guarantee better resolution.

- The geophone is not rigidly attached to the earth, bandlimiting associated with natural resonance of the soil-geophone system exists.

- The seismic detectors and recorders contain built-in-filter to limit ambient noise level.

Consider the set of points x where $f(x) \neq 0$; the **support** of f is the closure of this set. A function whose Fourier transform has a bounded support is said to be **bandlimited** and in this case the support is called band of the function.

Example 2.1

The function defined by

$$f(x) = \sin(\pi x) / \pi x,$$

it is bandlimited because it is the inverse Fourier transform of the characteristic function on the interval $[-\pi, \pi]$. In this case we call π the bandwidth of the function f . A basic property of a bandlimited function is the possibility of representing such a function, without any loss of information, by means of its sample taken at equidistance points. The purpose of the next examples are to provide us with some insight into the nature of bandlimited delta function

Example 2.2 " The Case of Constant Back-ground Single Layer"

Let us consider a one dimensional model consisting of one layer in which there is a step-like wave speed change. The Fourier transforms of this delta function type variation is given by

$$I(t) = \frac{1}{2\pi} \int_{\Omega} e^{i\omega t} d\omega, \quad (2.1)$$

where Ω_0 is the symmetric domain, $\omega_- \leq |\omega| \leq -\omega_+$. We introduce the filter(see figure 2.1). The filter can be described by

$$F(\omega) = \begin{cases} 1, & \omega_- \leq |\omega| \leq -\omega_+ \\ 0, & \text{otherwise} \end{cases}.$$

The bandlimited version of equation (2.1) becomes

$$I(t) = \frac{1}{2\pi} \int_{\Omega} F(\omega) e^{i\omega t} d\omega.$$

It is clear that the function $F(\omega)$ has the Fourier transform as in figure 2.1.

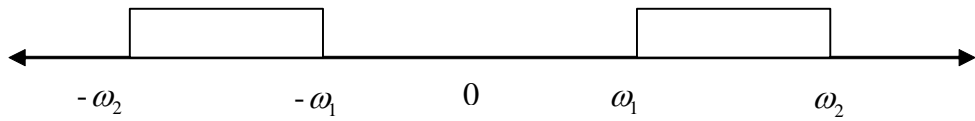


Figure 2.1: The band-limiting for the band $F(\omega)$ for two frequencies.

Let $H(t - h)$ be Heaviside step function, equal to 0 for $t < h$ and equal to 1 for

$t > h$. If the Fourier transform of Heaviside step function is written as

$$F(H(t-h)) = \chi(\omega),$$

then the band-limited Fourier transform of $H(t)$ becomes

$$\chi(\omega)_{\text{band-limited}} = \chi(\omega) \cdot F(\omega). \quad (2.2)$$

The band-limited inversion of $H(t-h)$ is obtained from equation (2.2) by taking inverse Fourier transform and applying the convolution theorem:

$$\begin{aligned} H(t-h)_{\text{band-limited}} &= F^{-1}[\chi(\omega) \cdot F(\omega)] \\ &= F^{-1}(\chi(\omega)) * F^{-1}(F(\omega)) \\ &= H(t-h) * F^{-1}(F(\omega)), \end{aligned} \quad (2.3)$$

where $*$ is the convolution sign and the inverse Fourier transform of $F(\omega)$ can be computed as follows

$$F^{-1}(F(\omega)) = \frac{1}{2\pi} \int F(\omega) e^{i\omega t} d\omega \quad (2.4)$$

$$\begin{aligned} &= \frac{1}{2\pi} \int_{-\omega_2}^{-\omega_1} e^{i\omega t} d\omega + \frac{1}{2\pi} \int_{\omega_1}^{\omega_2} e^{i\omega t} d\omega \\ &= \frac{1}{2\pi i t} [e^{-i\omega_1 t} - e^{-i\omega_2 t} + e^{i\omega_2 t} - e^{i\omega_1 t}] \\ &= \frac{\sin \omega_2 t - \sin \omega_1 t}{\pi t} \end{aligned} \quad (2.5)$$

$$r(t) = \frac{2}{\pi t} \sin \left[\left(\frac{\omega_2 - \omega_1}{2} \right) t \right] \cos \left[\left(\frac{\omega_2 + \omega_1}{2} \right) t \right]. \quad (2.6)$$

Where $r(t)$ is the response function. Since $\omega = 2\pi f$, above equation takes the form,

$$\begin{aligned} H(t - t_0)_{\text{band-limited}} &= \int_0^\infty H(\tau - t_0) \frac{\sin \omega_2 (t - \tau) - \sin \omega_1 (t - \tau)}{\pi (t - \tau)} d\tau \\ &= \int_0^\infty H(\tau - t_0) \frac{\sin (2\pi f_2 (t - \tau)) - \sin (2\pi f_1 (t - \tau))}{\pi (t - \tau)} d\tau. \end{aligned}$$

The above formula given bandlimited version of the step function. we can find the percentage bandwidth of the $F(\omega)$ and it is defined by

$$\begin{aligned} \text{percentage bandwidth} &= 100 * \frac{\omega_2 - \omega_1}{\omega_2 + \omega_1}. \\ &= 100 * \frac{f_2 - f_1}{f_2 + f_1}. \end{aligned} \tag{2.7}$$

Example 2.3 *"The Case of Constant Back-ground Multilayer model"*

The band-filter for the above case

$$\Omega_1 = [-\omega_4, -\omega_3] \cup [-\omega_3, -\omega_2] \cup [-\omega_2, -\omega_1] \cup [\omega_1, \omega_2] \cup [\omega_2, \omega_3] \cup [\omega_3, \omega_4].$$

Let us now consider a medium which undergoes change in the material proprieties at two points so that it can be regarded as one having two layers. In such a case we could use the band limiting function $F(\omega)$ defined over Ω given in figure (2.2). As is evident from the compact support of $F(\omega)$, the Fourier transform of $F(\omega)$ exists.

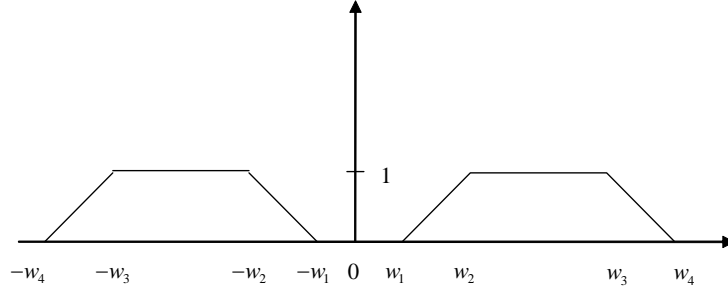


Figure 2.2: The band-limiting for the band $F(\omega)$ for four frequencies.

From example (2.2), we find the inverse Fourier transform of $F(\omega)$ can be computed as follows

$$\begin{aligned}
 F^{-1}(F(\omega)) &= \frac{1}{2\pi} \int F(\omega) e^{i\omega t} d\omega \\
 &= \frac{1}{2\pi} \int_{-w_4}^{-w_3} (a_1\omega + b_1) e^{i\omega t} d\omega + \frac{1}{2\pi} \int_{-w_2}^{-w_3} e^{i\omega t} d\omega \\
 &\quad + \frac{1}{2\pi} \int_{-w_2}^{-w_1} (a_2\omega + b_2) e^{i\omega t} d\omega + \frac{1}{2\pi} \int_{w_1}^{w_2} (a_3\omega + b_3) e^{i\omega t} d\omega \\
 &\quad + \frac{1}{2\pi} \int_{w_2}^{w_3} e^{i\omega t} d\omega + \frac{1}{2\pi} \int_{w_3}^{w_4} (a_4\omega + b_4) e^{i\omega t} d\omega. \tag{2.8}
 \end{aligned}$$

After some simplification, the resulting equation gives the response function:

$$r(t) = \frac{\cos \omega_3 t - \cos \omega_4 t}{\pi (\Delta_1 \omega) t^2} + \frac{\cos \omega_2 t - \cos \omega_1 t}{\pi (\Delta_2 \omega) t^2},$$

where

$$\Delta_1 \omega = \omega_4 - \omega_3$$

$$\Delta_2 \omega = \omega_2 - \omega_1.$$

2.3.1 Reflectivity Function

The reflectivity or reflectivity coefficient of the reflector is the ratio of the amplitude of the field displacement of the reflected wave to that of the incident wave. In case of normal incidence, the reflectivity function gives reflection coefficient of the reflected wave at different positions. The role of the reflectivity function is to locate the boundary of the scattering object and it characterizes the change in medium properties through the normal reflection coefficient.

2.4 Asymptotic Methods

2.4.1 WKB Approximation

This subsection is concerned with asymptotic methods for ordinary differential equations. This subject is large and very much of current interest under the title of singular perturbation theory. The propose of this technique is to derive the leading-order of the WKB solutions to the homogeneous form of the equation[13, 62, 75, 76, 86]

$$Lu = \frac{d^2u}{dx^2} + \frac{1}{x} \frac{du}{dx} + \frac{\omega^2}{v^2(x)}u = -\delta(x - x_S). \quad (2.9a)$$

This will be an asymptotic solution for "large" values of ω .

Thus, we consider linear second-order homogeneous differential equation for $u(x)$ of the form

$$u''(x, \omega) + pu'(x, \omega) + qu(x, \omega) = 0, \quad (2.10)$$

where x is in general the space variable and $p(x)$ and $q(x)$ are analytic functions of all x and any parameters of the problem. A more convenient form of (2.10) for study

is obtained by setting

$$u(x, \omega) = W(x, \omega) e^{-\frac{1}{2} \int p(x) dx}, \quad (2.11)$$

which reduces (2.10) to

$$W'' + \left(q - \frac{1}{2} p' - \frac{1}{4} p^2 \right) W = 0. \quad (2.12)$$

Thus, with out loss of generality, we may consider at the outset

$$W'' + f(x; \omega) W = 0. \quad (2.13)$$

We discuss the asymptotic method in which $f(x; \omega)$ possesses an asymptotic expansion as $\omega \rightarrow \infty$. Let us suppose that $f(x; \omega)$ in (2.13) has an asymptotic expansion of the form

$$\begin{aligned} f(x; \omega) &\sim \omega^2 \phi_0(x) + \omega \phi_1(x) + \phi_2(x) + \dots \quad (2.14) \\ &= \sum_{n=0}^{\infty} \omega^{2-n} \phi_n(x) \quad \text{as } \omega \rightarrow \infty, \end{aligned}$$

where $\phi_n(x)$, $n = 0, 1, 2, \dots$, are continuous twice-differentiable functions of x . Let

$$\begin{aligned} W(x; \omega) &\sim \exp \{ g_0(\omega) \psi_0(x) + g_1(\omega) \psi_1(x) + \dots \} \quad (2.15) \\ &= \exp \left\{ \sum_{n=0}^{\infty} g_n(\omega) \psi_n(x) \right\} \quad \text{as } \lambda \rightarrow \infty, \end{aligned}$$

where $\{\psi_n(x), n = 0, 1, 2, \dots\}$ and $\{g_n(\omega), n = 0, 1, 2, \dots\}$ are asymptotic as $\omega \rightarrow \infty$, and are to be found.

Substituting (2.15), with $f(x; \omega)$ given by (2.14) into (2.13) and comparing the

like asymptotic terms. Equation (2.13) becomes. on canceling out the exponential in each term gives,

$$g_0(\omega) \psi_0'' + g_1(\omega) \psi_1'' + \dots + \{g_0(\omega) \psi_0' + g_1(\omega) \psi_1' + \dots\} + \omega^2 \phi_0(x) + \omega \phi_1(x) + \phi_2(x) + \dots \sim 0. \quad (2.16)$$

First compare terms in $O(\omega^2)$ in (2.16) to get

$$g_0^2(\omega) \psi_0'^2(x) + \omega^2 \phi_0(x) \dots \sim 0. \quad (2.17)$$

From above it is immediately clear that

$$g_0(\omega) = \omega, \quad \psi_0(x) = \pm i \int^x \sqrt{\phi_0(t)} dt. \quad (2.18)$$

Secondly comparing terms $O(\omega)$ (2.16) and using (2.18) leads

$$g_0(\omega) \psi_0''(x) + 2g_0(\omega) g_1(\omega) \psi_0'(x) \psi_1'(x) + \omega \phi_1(x) \dots \sim 0, \quad (2.19)$$

which gives

$$g_1(\omega) = 1, \quad \psi_1(x) = \log(\phi_0(x))^{\frac{1}{4}} \pm \frac{1}{2i} \int^x \frac{\phi_1(t)}{\sqrt{\phi_0(t)}} dt, \quad \phi_0(t) \neq 0. \quad (2.20)$$

Lastly we compare terms of $O(1)$ to obtain,

$$g_0(\omega) \psi_1''(x) + 2g_1^2(\omega) \psi_0'^2(x) + 2g_0(\omega) g_2(\omega) \psi_0'(x) \psi_2'(x) \sim 0, \quad (2.21)$$

which gives $g_2(\omega) = \frac{1}{\omega}$. With the specific form (2.14) it is clear that

$$g_n(\omega) = \{\omega^{1-n}\}, \quad n = 0, 1, \dots \quad (2.22)$$

Using the results from (2.18), (2.20), and (2.21), we finally get the asymptotic solutions to (2.13), with f as in (2.14), and the solution (2.15) is given asymptotically by

$$W(x; \omega) \sim \frac{1}{\phi_0^{\frac{1}{4}}(x)} \exp\left(\pm i \int^x \left\{ \omega \sqrt{\phi_0(t)} + \frac{1}{2} \frac{\phi_1(t)}{\sqrt{\phi_0(t)}} \right\} dt\right) \cdot \left\{ 1 + O\left(\frac{1}{\omega}\right) \right\}. \quad (2.23)$$

2.4.2 WKBJ Approximation for Wave Equation

The objective of this subsection is to derive the leading-order term of the WKBJ solutions of the homogeneous form of equation

$$L_0 u = \frac{d^2 u}{dx^2} + \frac{\omega^2}{v_0^2(x)} u = -\delta(x - x_S), \quad (2.24)$$

with radiation condition

$$\frac{du}{dx} \mp i \frac{\omega}{v_0(x)} u \rightarrow 0, \quad \text{as } x \rightarrow \pm\infty. \quad (2.25)$$

We seek an asymptotic solution for "large" values of ω and, hence, use as series in the inverse power of $i\omega$, while assuming a solution of (2.24) the form

$$u(x, \omega) \sim (i\omega)^p e^{i\omega\Phi(x)} \sum_{n=0}^{\infty} \frac{A_n(x)}{(i\omega)^n}. \quad (2.26)$$

Substituting this series solution into equation (2.24) to obtain the following equation

$$\begin{aligned} L_0 u(x, \omega) = & (i\omega)^p e^{i\omega\Phi(x)} \sum_{n=0}^{\infty} \left\{ (i\omega)^{2-n} A_n(x) \left[\Phi'^2(x) - \frac{1}{v_0^2} \right] + \right. \\ & \left. + (i\omega)^{1-n} [2A'_n \Phi'(x) + A_n \Phi''(x)] + \frac{A''_n(x)}{(i\omega)^n} \right\}. \end{aligned} \quad (2.27)$$

To determine $\Phi(x)$ and the values of A_n in (2.27) for $n = 0, 1, 2, \dots$, we set the coefficients of each power of $i\omega$ equal to zero, starting with $(i\omega)^{p+2}$ and proceeding to lower powers. Setting the coefficient of highest power equal to zero leads to the equation,

$$\Phi'^2(x) = \frac{1}{v_0^2}, \quad (2.28)$$

from which it follows that

$$\begin{aligned} \Phi'(x) &= \pm \frac{1}{v_0}, \\ \Phi(x) &= \pm \int^x \frac{dx'}{v_0(x')}. \end{aligned} \quad (2.29)$$

The x' in the above is as a dummy variable of integration to distinguish it from the endpoint x . There are two possible solutions, which we denote by u_{\pm} , below. Now,

the coefficient in $(i\omega)$ in the next power becomes

$$2A_0'\Phi' + A_0\Phi'' = 0. \quad (2.30)$$

Multiplication by A_0 above equation because can be easily solved to obtain,

$$A_0^2 = v_0(x). \quad (2.31)$$

In order that A_0 remains real, we must choose the constant positive or negative in accordance with the sign of Φ' . Thus

$$A_0 = K\sqrt{c(x)}. \quad (2.32)$$

The two leading-order solutions therefore are of the form

$$u_{\pm}(x, \omega) = (i\omega)^p K_{\pm} \sqrt{c(x)} \exp \left\{ \pm i\omega \int_{K_{\pm}}^x \frac{dx'}{v_0(x')} \right\}, \quad (2.33)$$

with constants, K_{\pm} as the lower limits of integration, and the choice of p undetermined without further information about the solutions we seek. The Wronskian W is given by of the two linearly independent solutions

$$W = \det \begin{bmatrix} u_+(x, \omega) & u_-(x, \omega) \\ u'_+(x, \omega) & u'_-(x, \omega) \end{bmatrix}, \quad (2.34)$$

the leading order terms gives us

$$W = -2K_+K_-(i\omega)^{2p+1}, \quad (2.35)$$

In $K_{\pm} \neq 0$, the solutions in (2.24) are linearly independent. This insures that the two solutions are *linearly independent* and we can build solutions of the hetrogenous equations in terms of them.

We begin by assuming that for $x > \xi$ and for $x < \xi$, $g(x, \xi, \omega)$ is a linear combination of the obove two solutions, u_{\pm} , derived in equation (2.24). For convenience, we take all the lower limits of integration in the integrals that appear in the phase to be ξ . We use the radiation condition, saving only terms of leading order in $i\omega$, to eliminate one term on each side, $x > \xi$ and $x < \xi$, and conclude that

$$g(x, \xi, \omega) = \left\{ \begin{array}{l} V_+ (i\omega)^p \frac{\sqrt{v_0(x)}}{2i\omega} \exp \left\{ i\omega \int_{\xi}^x \frac{dx'}{v_0(x')} \right\}, \quad x > \xi \\ V_- (i\omega)^p \frac{\sqrt{v_0(x)}}{2i\omega} \exp \left\{ i\omega \int_x^{\xi} \frac{dx'}{v_0(x')} \right\}, \quad x < \xi, \end{array} \right\}. \quad (2.36)$$

Now, to determine V_{\pm} and p , we must apply the correct conditions at $x = \xi$. First, we must require that g itself be continuous at ξ . The continuity at $x = \xi$ leads to the requirement that $V_+ = V_-$. The first derivative will be discontinuous at $x = \xi$ and then jump to be equal to -1 . Thus, we determine the power p as well as the common constant V_{\pm} , and that the WKBJ approximate Green's function is

$$g(x, \xi, \omega) = \left\{ \begin{array}{l} -\frac{\sqrt{v(\xi)v_0(x)}}{2i\omega} \exp \left\{ i\omega \int_{\xi}^x \frac{dx'}{v_0(x')} \right\}, \quad x > \xi \\ -\frac{\sqrt{v(\xi)v_0(x)}}{2i\omega} \exp \left\{ i\omega \int_x^{\xi} \frac{dx'}{v_0(x')} \right\}, \quad x < \xi, \end{array} \right\}. \quad (2.37)$$

2.5 Inversion in One Dimension- Bleistein Model

In this section we briefly present the one dimensional model for recovery of variation in the velocity profile for the inhomogeneity in the interior of the earth. Claerbout [22] first introduced the idea of recovery of the velocity profile under the assumption of

constant background. Gerver [46, 47] showed that the wave velocity can be determined from observations at one point. Cohen and Bleistein [25, 26], Cohen and Hagin [28] further developed the model to include the case of variable density. A good account of the main results for such a model is found in Bleistein, Cohen and Stackgold [13].

Let us assume that the propagating field u satisfies

$$Lu(x, \omega) = \frac{d^2 u(x, \omega)}{dx^2} + \frac{\omega^2}{v^2(x)} u(x, \omega) = -\delta(x), \quad (2.38)$$

where ω is the angular frequency and $v(x)$ is the wavespeed profile, with $u(x, \omega)$ satisfying the Sommerfeld radiation condition

$$\frac{du}{dx} \mp \frac{i\omega}{v(x)} u \rightarrow 0, \text{ as } x \rightarrow \pm\infty. \quad (2.39)$$

We assume that the field velocity $v(x)$, can be approximated by a background or reference speed $v_0(x)$ plus a perturbation term as follows

$$\frac{1}{v^2(x)} \approx \frac{1}{v_0^2(x)} [1 + \alpha(x)] \quad , \quad \alpha(x) \ll 1. \quad (2.40)$$

The total field in the medium consists of the incident field $u_I(x, \omega)$ which would be present if there were no variation in the velocity and the scattered field $u_S(x, \omega)$ which is produced due to the perturbation or change in the velocity. It is thus reasonable to write

$$u(x, \omega) = u_I(x, \omega) + u_S(x, \omega) \quad (2.41)$$

and require that $u_I(x, \omega)$ and $u_S(x, \omega)$ are solutions of the following problems:

$$L_0 u_I = \frac{d^2 u_I}{dx^2} + \frac{\omega^2}{v^2(x)} u_I = -\delta(x), \quad (2.42)$$

$$L_0 u_S = \frac{-\omega^2 \alpha(x)}{v_0^2(x)} [u_I(x, \omega) + u_S(x, \omega)]. \quad (2.43)$$

Next, we write the solution to (2.43) in terms of the Green's function as

$$u_S(\xi, \omega) = \omega^2 \int_0^\infty \frac{\alpha(x)}{v_0^2(x)} [u_I(x, \omega) + u_S(x, \omega)] g(x, \xi, \omega) dx. \quad (2.44)$$

2.5.1 The Born Approximation

The integral equation (2.44) is non linear as it contains product of $u_S(x, \omega)$ and $g(x, \xi, \omega)$ because $u_S(x, \omega)$ is also determined in terms of the Green's function $g(x, \xi, \omega)$. To overcome this problem, we attempt to linearize it under the assumption that variation $\alpha(x)$ of the velocity from the background velocity is small. It is reasonable to expect that the resulting scattered field $u_S(x, \omega)$ is also small, and therefore the product $\alpha(x) u_S(x, \omega)$ appearing under the integral in (2.44) is significantly smaller than the product $\alpha(x) u_I(x, \omega)$. This enables us to retain the first term and neglect the second term to obtain the so called *Born approximation* [13, 14, 18, 36, 37, 49, 55, 59]

$$u_S(\xi, \omega) = \omega^2 \int_0^\infty \frac{\alpha(x)}{v_0^2(x)} u_I(x, \omega) g(x, \xi, \omega) dx. \quad (2.45)$$

This approximation will be used throughout to recover the variation in the velocity or material parameter of the medium.

2.5.2 The Inverse Scattering Integral Equation

Consider the case when the source and receiver are located at the same place (for simplicity we choose $x_S = x_g = \xi = 0$). Using band limiting filter introduced in section (2.3), we write

$$u_I(x, \omega) = F(\omega)g(x, 0, \omega),$$

where $F(\omega)$ is some frequency domain filter. The equation (2.45) becomes

$$u_S(0, \omega) = \omega^2 \int_0^\infty F(\omega) \frac{\alpha(x)}{v_0^2(x)} g^2(x, 0, \omega) dx. \quad (2.46)$$

Thus have formally reduced the inverse problem to the problem of solving the integral equation for $\alpha(x)$ using the observed data $u_S(0, \omega)$. In the formal theory of integral equations, equation (2.46) is called a *Fredholm integral equation* of the first kind [20, 35, 54, 77]. That part of the integral excluding $\alpha(x)$ is called the *kernel* of this integral equation.

In case of a variable-background medium, we will seek only a "high-frequency" solution to the inverse scattering integral equation (2.46). This means that we are free to use a high frequency approximation of the Green's function. Therefore, we shall use the approximate Green's function as WKBJ Green's function :

$$g(x, 0, \omega) = -\frac{A(x)}{2i\omega} e^{[i\omega\phi(x,0)]}, \quad \phi(x, y) = \int_x^y \frac{dt}{v_0(t)}. \quad (2.47)$$

In the simplest case, when $v(x)$ is continuous, the WKBJ amplitude $A(x)$ is given by

$$A(x) = \sqrt{v_0(0)v_0(x)}.$$

Using $g(x, 0, \omega)$ in (2.46) leads to the integral equation,

$$u_S(0, \omega) = - \int_0^{\infty} F(\omega) \frac{\alpha(x) A^2(x)}{4v_0^2(x)} e^{[2i\omega\phi(x,0)]} dx. \quad (2.48)$$

Since $\alpha(x) = 0$ for $x < 0$, this is the Fourier -type integral because the lower limit can be extended to $-\infty$. However, the amplitude in this more general Fourier integral should be calculated separately.

2.5.3 Amplitude Calculation of the Inverse Fourier Integral

The inversion operator (2.48) has the form

$$\alpha(y) = \int_{-\infty}^{\infty} b(y, \omega) u_S(0, \omega) e^{-2i\omega\phi(y,0)} d\omega, \quad (2.49)$$

where $b(y, \omega)$ is to be determined. To see the form of $b(y, \omega)$, substitute values of $u_S(0, \omega)$ from (2.48) in the above expression to get,

$$\begin{aligned} \alpha(y) = & - \int_0^{\infty} dx \alpha(x) \frac{A^2(x)}{4v_0^2(x)} \\ & \cdot \int_{-\infty}^{\infty} F(\omega) b(y, \omega) e^{2i\omega\phi(x,y)} d\omega. \end{aligned} \quad (2.50)$$

The above results in an equation of the form,

$$\alpha(y) = \int_0^{\infty} \alpha(x) f(x, y) dx, \quad (2.51)$$

with

$$f(x, y) = - \frac{A^2(x)}{4v_0^2(x)} \int_{-\infty}^{\infty} F(\omega) b(y, \omega) e^{2i\omega\phi(x,y)} d\omega. \quad (2.52)$$

For x and y greater than zero, the equation (2.51) is asymptotically if we set [13, 24, 28]

$$f(x, y) = \delta_B(x - y). \quad (2.53)$$

Definition 2.4 *Bandlimited Delta Function*

The one-dimensional *bandlimited delta function*, $\delta_B(x - y)$, is the one that is defined only through its integral

$$\delta_B(x - y) = \frac{1}{\pi} \int_0^\infty F(\omega) e^{2i\omega\phi(x,y)} d\omega. \quad (2.54)$$

Definition 2.5 *Transformation Property*

One can represent a finite number of Dirac delta functions by one whose argument is a function. Consider $\delta(f(x))$ where $f(x)$ has a non-repeated null at x_0 and whose derivative does not vanish at x_0 , then

$$\delta(f(x)) = \frac{\delta(x - x_0)}{|f'(x_0)|}. \quad (2.55)$$

Now in (2.52), it is possible to construct that result with a $b(y, \omega)$ that is independent of ω ; that is $b(y, \omega) = b(y)$. Then,

$$f(x, y) = -\frac{A^2(x)}{4v_0^2(x)} \int_{-\infty}^\infty F(\omega) e^{2i\omega\phi(x,y)} d\omega = -\frac{\pi A^2(x) b(y)}{4v_0^2(y)} \delta_B(\phi(x, y)), \quad (2.56)$$

where we have applied the definition of the bandlimited delta function. Applying the

transformation property to the bandlimited delta function, recognizing that

$$\phi'(x) = 1/v_0(x),$$

and noting that the action of the delta function is at $x = y$ yields,

$$\begin{aligned} f(x, y) &= -\frac{\pi A^2(x) b(y)}{4v_0(y)} \delta_B(x - y) \\ &= -\frac{\pi A^2(y) b(y)}{4v_0(y)} \delta_B(x - y). \end{aligned} \quad (2.57)$$

These are the asymptotic equalities depending upon sufficiently high frequencies. The choice of $b(y)$ needed to satisfy (2.53) true is apparent from (2.56)

$$b(y) = -\frac{4v_0(y)}{\pi A^2(y)}, \quad (2.58)$$

and the inversion operator, (2.49) becomes

$$\alpha(y) = -\frac{4v_0(y)}{\pi A^2(y)} \int_{-\infty}^{\infty} u_S(0, \omega) e^{-2i\omega\phi(y,0)} d\omega. \quad (2.59)$$

The reflectivity function of the surface is defined to be normal reflection strength multiplied by the singular function. The reflectivity function locates the boundary of the scattering object and characterizes the changes in medium properties through the normal reflection coefficient. If the discontinuities are our primary interest, the the band-limited delta-functions are easier to identify as compared to band-limited step functions. Some examples applied to layered media, where the artifacts are produced by the reflectivity function, are nicely demonstrated in Bleistein [13] et al. Following

[13], we can obtain the reflectivity function $\beta(y)$ from (2.59) by differentiating it with respect to y and dividing by the scaling factor -4 , keeping only leading-order terms in ω . This gives

$$\beta(y) = -\frac{2}{\pi A^2(y)} \int_{-\infty}^{\infty} i\omega u_S(0, \omega) e^{[-2i\omega\phi(y,0)]} d\omega. \quad (2.60)$$

Chapter 3

Computational and Asymptotic Aspects of Velocity Inversion in the Presence of Damping in Single and Multilayer Model

Abstract

The inverse problem plays an important role in mapping the interior of earth, geological prospective and governing information about an accessible part. The problem of recovery of velocity profile in a one dimension model of damped wave in a medium with constant background wave velocity is considered. We assume that the wave velocity is well approximated by the background velocity plus a perturbation term. The Born approximation, Zero offset assumption and the method of approximation to the Green's

function are used to derive the inversion formula.

The application of the method leads to a linear integral equation involving variations in sound speed and damping. Our aim is to recover these variations in velocity and damping, which in turn yields a map of the interfaces in the interior of the earth. We consider a model that incorporates the effects of damping in the medium and develop an inversion procedure in this case. It is hoped that the results based upon this model will prove to be more realistic in situations of interest. The inversion formula is numerically implemented to the two cases of interest consisting of one layer and multi layer in the velocity.

3.1 Introduction

The problem frequently studied in seismology is that of impact of earthquakes on natural and man-made structures. The tremors caused by earthquakes travel through the earth and are recorded at distinct locations using sensitive seismographs. For structure of the earth, the major source of information comes from the records of ground motion produced by the waves from earthquakes at seismic stations across the globe [4]. Most man-made sources of seismic energy such as chemical explosions, surface vibrations or weight-dropping devices have a limited range over which they give detectable arrivals [31, 32]. Inverse scattering problems obtain information about an inaccessible region of space from measurement made far away, in the accessible region [69]. These problems occur while determining the properties of elementary particles [43] in seismic prospecting [51], remote sensing of the earth [57], nondestructive evaluation of materials [61], and in medical imaging [31].

The study of inverse problem in one dimension provides an important model. This

model is applicable in many situations when the medium has its properties varying in one dimension only. In case of earth or an ocean, for example, the variability in the properties with depth describes some situations effectively.

One focus of study in this chapter is the recovery of the velocity profile of an inclusion which has variable wave velocity when velocity in the background medium remains constant. An approximate solution of the inverse problem was obtained by Clearbout [21, 22] assuming the density constant. Gerver [46] discussed this problem in detail and showed that the velocity of propagation can be determined uniquely from the observations at one point. Cohen and Bleistein [25, 26] study the inverse problem of determining small variations in wave velocity through the medium of interest. showing that a solution of wide variety of inverse problem could be given in closed form. A good presentation of this approach can be found in Bleistein, Cohen and Stockwell [13] which Pakesh [69] describe the problem of linearization of inverse problem in such cases.

One dimensional inversion results presented in this chapter are useful in situations when the material parameters have only one dimension of variability. Our primary interest is to improve the model to be applicable to a wider class of problems. For this purpose we consider the wave equation to have a damping parameter. Stakgold [77, 78] giving rise to one additional parameter in the wave equation. The damping may be caused by the third situation of porous medium or presence of impurities. The damped equation model was considered by Zaman and Masood [87] in which they presented an approximate inversion formula which exhibited the role played by the damping parameter in a simple one-dimensional model with constant density.

We present implementation of the inversion formula in two important cases of

interest. The first example which we present is that in which there is a single step variation in material properties (called the single layer model) and the second case studied is that of two step variability known as a two layer model. By introducing appropriate band limiting functions, the inversion formula has been applied to these models.

In section (2), we present a model of one dimensional wave with damping in the medium. Using the Born approximation, the Green's function and Fourier inversion formula, and following Bleistein et al [13] and Zaman and Masood [87] obtained an inversion formula for the velocity profile. In section (3), a model is introduced in which we have one step variability in the velocity and damping parameter to implement the inversion formula and also compute the reflectivity function. The results show good agreement with Bleistein et al [13] in the absence of damping which validates our inversion formula. In section (4), the numerical results and graphs of the model are presented. In section (5), another model of interest is considered which has a two-layer variability in the velocity and damping parameter. Numerical results and graphs showing implementation of our formula to this case are presented in section (6).

3.2 The One Dimensional Inverse Problem

We formulate the direct forward problem in the frequency domain for some observable parameter, $u(x, x_s, \omega)$, called the "field". Here, x represents the general field, while x_s represents the location of the source, and ω the frequency. The field may represent plane acoustic pressure waves (propagating parallel to the x -axis), the transverse displacement of a string in one dimension, or some other equally appropriate parameter that may be represented as a one-dimensional wave. The only important condition

is that propagation of $u(x, x_s, \omega)$ be governed by the scalar Helmholtz equation with damping.

$$Lu(x, x_s, \omega) = \frac{d^2 u(x, x_s, \omega)}{dx^2} + \left[\frac{\omega^2 + i\omega\gamma(x)}{v^2(x)} \right] u(x, x_s, \omega) = -\delta(x - x_s), \quad (3.1)$$

together with the radiation condition

$$\frac{du}{dx} \mp i \frac{\omega}{v(x)} u \rightarrow 0, \quad \text{as } x \rightarrow \pm\infty \quad (3.2)$$

in which $\gamma(x)$ represents damping with source placed at point x_s . It is assumed that the source point is to the left of the region where $v(x)$ and $\gamma(x)$ are unknown. We assume that the velocity of the medium and the damping parameter can be written as a constant plus a perturbation as follows

$$\begin{aligned} v(x) &= v_0(x) + \delta v(x), \\ \frac{1}{v^2(x)} &\approx \frac{1}{v_0^2(x)} [1 + \alpha(x)] \\ \gamma(x) &= \gamma_0(x) + \delta\gamma(x). \end{aligned} \quad (3.3)$$

Rewriting (3.1), using the perturbation representation (3.3) yields an equivalent expression for the Helmholtz equation

$$\begin{aligned} L_0 u(x, x_s, \omega) &= \frac{d^2 u(x, x_s, \omega)}{dx^2} + \left[\frac{\omega^2 + i\omega\gamma_0(x)}{v_0^2(x)} \right] u = -\delta(x - x_s) \\ &+ [-i\omega\delta\gamma(x) - \omega^2\alpha(x) - i\omega\gamma_0(x)\alpha(x)] \frac{u(x, x_s, \omega)}{v_0^2(x)}. \end{aligned} \quad (3.4)$$

Here, the term involving $\alpha(x)$ has been moved to the right side of the equation.

Equation (3.4) is posed in terms of the total field $u(x, x_s, \omega)$ generated by the impulsive source $-\delta(x - x_s)$ plus the more complicated "scattering source" represented by the term on the far right in (3.4). The scattered waves generated by this new "source" interacts with regions at greater depth than x_s and x_g and contains information about the wave speed profile at these greater depths. The total field $u(x, x_s, \omega)$ can be separated into the incident part $u_I(x, x_s, \omega)$ in the absence of perturbations and $u_S(x, x_s, \omega)$ in the presence of perturbations. Thus we set

$$u(x, x_s, \omega) = u_S(x, x_s, \omega) + u_I(x, x_s, \omega). \quad (3.5)$$

As $u_I(x, x_s, \omega)$ satisfies the field equation in the absence of variations assumed in (3.3), we have

$$L_0 u_I(x, x_s, \omega) = \frac{d^2 u_I(x, x_s, \omega)}{dx^2} + \left[\frac{\omega^2 + i\omega\gamma_0(x)}{v_0^2(x)} \right] u_I = -\delta(x - x_s), \quad (3.6)$$

so that the equation to be satisfied by $u_S(x, x_s, \omega)$ can be seen to be

$$L_0 u_S(x, x_s, \omega) = \left[-i\omega\delta\gamma(x) - \omega^2\alpha(x) - i\omega\gamma_0(x)\alpha(x) \right] \frac{[u_I(x, x_s, \omega) + u_S(x, x_s, \omega)]}{v_0^2(x)}. \quad (3.7)$$

We now construct a Green's function representation of the solution of (3.7) given by

$$\frac{d^2 g(x, x_g, \omega)}{dx^2} + \left[\frac{\omega^2 + i\omega\gamma_0(x)}{v_0^2(x)} \right] g(x, x_g, \omega) = -\delta(x - x_g). \quad (3.8)$$

Consider the case where the source and receiver are located at the same place (for

simplicity we may choose $x_s = x_g \equiv 0$, $v_0(x) = v_0$ and $\gamma_0(x) = \gamma_0$. Let

$$u_I(x, 0, \omega) = g(x, 0, \omega).$$

This is the "zero-offset problem" see figure 1 (appendix A). Following Wu [84] we assume that

$$\sqrt{\frac{\omega^2 + i\omega\gamma_0}{v_0^2}} \approx \frac{\omega}{v_0} \left(1 + \frac{i\gamma_0}{2\omega} \right). \quad (3.9)$$

The Green's function may then be written as

$$g(x, 0, \omega) \approx -\frac{v_0}{2i\omega \left(1 + \frac{i\gamma_0}{2\omega} \right)} \exp \left(i\omega \left(1 + \frac{i\gamma_0}{2\omega} \right) x/v_0 \right). \quad (3.10)$$

The solution of (3.7) , after the Born approximation equation (2.2) [13, 14, 18, 36, 49, 59, 69], written in terms of Green's function can be written as

$$u_S(0, \omega) = -\int_0^\infty \left[-i\omega\delta\gamma(x) - \omega^2\alpha(x) - i\omega\gamma_0(x)\alpha(x) \right] \frac{g^2(x, 0, \omega)}{v_0^2(x)} dx. \quad (3.11)$$

Now using the Green's function representation (3.10) in (3.11) gives,

$$u_S(0, \omega) = \int_0^\infty \left[\frac{-i\delta\gamma(x)}{4\omega \left(1 + \frac{i\gamma_0}{2\omega} \right)^2} - \frac{\alpha(x)}{4 \left(1 + \frac{i\gamma_0}{2\omega} \right)^2} - \frac{i\gamma_0\alpha(x)}{4\omega \left(1 + \frac{i\gamma_0}{2\omega} \right)^2} \right] e^{2i\omega \left(1 + \frac{i\gamma_0}{2\omega} \right) x/v_0} dx, \quad (3.12)$$

and retaining only the leading order terms in ω , we get,

$$u_S(0, \omega) = - \int_0^{\infty} \frac{\alpha(x)}{4 \left(1 + \frac{i\gamma_0}{2\omega}\right)^2} e^{2i\omega \left(1 + \frac{i\gamma_0}{2\omega}\right)x/v_0} dx. \quad (3.13)$$

Since $\alpha(x) = 0$ for $x < 0$, we can extend integration from ∞ to $-\infty$. Due to this extension, (3.13) can be regarded as a Fourier type integral. This can be treated as a Fourier transform of $\alpha(x)$ and inversion can be performed as

$$\alpha(x) = \frac{-4e^{-\gamma_0 x/v_0}}{\pi v_0} \int_{-\infty}^{\infty} \left(1 + \frac{i\gamma_0}{2\omega}\right)^2 u_S(0, \omega) e^{-2i\omega x/v_0} d\omega. \quad (3.14)$$

As a simple check on this result, note that when $\gamma_0 = 0$, this results reduces to the constant-background inversion formula given by Bleistein et al [13].

3.3 Computational and Asymptotic Aspects of Velocity Inversion in a Single Layer

The zero offset, constant background inversion formula (3.14) derived above is the first example of the kind of formula that is the goal of our investigations. An advantage in the one-dimension problem is that exact scattered-field data can be generated analytically for a variety of wavespeed profiles. The simplest of these profiles is a perturbation of size ε . This step like wavespeed change, located at the position $x = h$, defines the boundary between two constant-wave speed and damping media and is represented mathematically as:

$$\alpha(x) = \varepsilon H(x - h) = \begin{cases} 0 & \text{for } x < h \\ \varepsilon & \text{for } x > h \end{cases} \quad (3.15)$$

$$\begin{aligned}
v(x) &= \begin{cases} v_0, & \text{for } x < h \\ v_1 = \frac{v_0}{\sqrt{1+\varepsilon}} & \text{for } x > h \end{cases} \\
\gamma(x) &= \begin{cases} \gamma_0, & \text{for } x < h \\ \gamma_1, & \text{for } x > h \end{cases} .
\end{aligned} \tag{3.16}$$

Equation(3.16) follows from the definition of $\alpha(x)$ and $\gamma(x)$ in (3.3). The objective here is solve this problem by writing down fairly general solutions in each of the two regions, with constants to be determined by interface and radiation conditions. Let us consider now a plane wave incident from $x < 0$ on the plane $x = 0$. This requires that $\text{sign}k = \text{sign}\omega$. Let us suppose that v in equation (3.1) is replaced by the values v_1 in $x > 0$ and the total solution and its normal derivative must be continuous across $x = 0$. Then exact solution to the problem (3.1) for this wave speed profile is

$$u(x, \omega) = \begin{cases} u_I(x, \omega) + u_R(x, \omega), & x < h \\ u_T(x, \omega), & x > h \end{cases} . \tag{3.17}$$

Here $u_R(x, \omega)$ is the reflected and $u_T(x, \omega)$ is the transmitted wave after the interaction of $u_I(x, \omega)$ with the inhomogeneity. Keeping in mind the condition at ∞ , we can expect the solution to be of the form (see for example Hopcraft and Smith [49],)

$$u = \begin{cases} -\frac{v_0}{2i\omega(1+\frac{i\gamma_0}{2\omega})} \left[e^{i\omega(1+\frac{i\gamma_0}{2\omega})x/v_0} + A_1 e^{i\omega(1+\frac{i\gamma_0}{2\omega})(2h-x)/v_0} \right], & x < h \\ -\frac{v_0}{2i\omega(1+\frac{i\gamma_0}{2\omega})} \left[A_2 e^{i\omega[(1+\frac{i\gamma_0}{2\omega})h/v_0 + (1+\frac{i\gamma_1}{2\omega})(x-h)/v_1]} \right], & x > h \end{cases} , \tag{3.18}$$

we require that u and its first derivative be continuous at $x = h$ to obtain the following

system of equations:

$$\begin{aligned} A_1 - A_2 &= -1 \\ \frac{\left(1 + \frac{i\gamma_0}{2\omega}\right) A_1}{v_0} - \frac{\left(1 + \frac{i\gamma_1}{2\omega}\right) A_2}{v_1} &= \left(1 + \frac{i\gamma_0}{2\omega}\right). \end{aligned} \quad (3.19)$$

Solving (3.19) for A_1 and A_2 , we find that

$$\begin{aligned} A_1 &= \frac{\left(1 + \frac{i\gamma_0}{2\omega}\right) v_1 - \left(1 + \frac{i\gamma_1}{2\omega}\right) v_0}{\left(1 + \frac{i\gamma_0}{2\omega}\right) v_1 + \left(1 + \frac{i\gamma_1}{2\omega}\right) v_0} = R \\ A_2 &= \frac{2\left(1 + \frac{i\gamma_0}{2\omega}\right) v_1}{\left(1 + \frac{i\gamma_0}{2\omega}\right) v_1 + \left(1 + \frac{i\gamma_1}{2\omega}\right) v_0} = T, \end{aligned} \quad (3.20)$$

Thus

$$u_I(x, \omega) = -\frac{v_0 e^{i\omega\left(1 + \frac{i\gamma_0}{2\omega}\right)x/v_0}}{2i\omega\left(1 + \frac{i\gamma_0}{2\omega}\right)}, \quad (3.21)$$

$$u_R(x, \omega) = -\frac{v_0 R e^{-i\omega\left(1 + \frac{i\gamma_0}{2\omega}\right)(x-2h)/v_0}}{2i\omega\left(1 + \frac{i\gamma_0}{2\omega}\right)}, \quad (3.22)$$

$$u_T(x, \omega) = -\frac{v_0 T e^{i\omega\left[\left(1 + \frac{i\gamma_0}{2\omega}\right)h/v_0 + \left(1 + \frac{i\gamma_1}{2\omega}\right)(x-h)/v_1\right]}}{2i\omega\left(1 + \frac{i\gamma_0}{2\omega}\right)},$$

where the "reflection coefficient" R and the "transmission coefficient" T have the

usual definitions [4, 13, 22, 29, 43, 51],

$$\begin{aligned} R &= \frac{\left(1 + \frac{i\gamma_0}{2\omega}\right) - \left(1 + \frac{i\gamma_1}{2\omega}\right) \sqrt{1 + \varepsilon}}{\left(1 + \frac{i\gamma_0}{2\omega}\right) + \left(1 + \frac{i\gamma_1}{2\omega}\right) \sqrt{1 + \varepsilon}} \\ T &= \frac{2 \left(1 + \frac{i\gamma_0}{2\omega}\right) \sqrt{1 + \varepsilon}}{\left(1 + \frac{i\gamma_0}{2\omega}\right) + \left(1 + \frac{i\gamma_1}{2\omega}\right) \sqrt{1 + \varepsilon}}. \end{aligned} \quad (3.23)$$

The scattered wave $u_S(0, \omega)$ needed for (3.14) is then the expression for the reflected wave $u_R(x, \omega)$ in (3.22), evaluated at $x = 0$.

$$u_S(0, \omega) = -\frac{v_0 R e^{2i\omega(1 + \frac{i\gamma_0}{2\omega})h/v_0}}{2i\omega \left(1 + \frac{i\gamma_0}{2\omega}\right)}. \quad (3.24)$$

Thus, the integral representation of the wave speed perturbation in (3.14) becomes

$$\alpha(x) = \frac{-4e^{-\gamma_0(h-x)/v_0}}{\pi} \int_{-\infty}^{\infty} F(\omega) \tilde{R} \frac{e^{2i\omega(h-x)/v_0}}{2i\omega} d\omega, \quad (3.25)$$

where

$$\tilde{R} = \left[\left(1 + \frac{i\gamma_0}{2\omega}\right) R \right], \quad (3.26)$$

since we are dealing with high frequency, the expression \tilde{R} can be written as

$$\tilde{R} \approx R = \frac{\left(1 + \frac{i\gamma_0}{2\omega}\right) - \left(1 + \frac{i\gamma_1}{2\omega}\right) \sqrt{1 + \varepsilon}}{\left(1 + \frac{i\gamma_0}{2\omega}\right) + \left(1 + \frac{i\gamma_1}{2\omega}\right) \sqrt{1 + \varepsilon}} \approx \frac{1 - \sqrt{1 + \varepsilon}}{1 + \sqrt{1 + \varepsilon}},$$

and can be moved outside the integral (3.25) [27 – 29],

$$\alpha(x) = \frac{4R e^{-\gamma_0(h-x)/v_0}}{\pi} \int_{-\infty}^{\infty} \frac{e^{2i\omega(h-x)/v_0}}{2i\omega} d\omega. \quad (3.27)$$

For this analytic representation of the data, the caveat about passing above singularities on the real ω -axis becomes important [13]. For the path of integration is interpreted as passing over the pole at $\omega = 0$, we find

$$\alpha(x) = -4Re^{-\gamma_0(h-x)/v_0}H(x-h). \quad (3.28)$$

If we compare the result in equation (3.28) with that in equation (3.15), we see that the discontinuity is in the right location but the magnitude of the jump is not exact. However, for small ε ,

$$R = -\frac{\varepsilon}{4} + O(\varepsilon^2). \quad (3.29)$$

This result follows by using the binomial theorem to expand the square roots in the numerator and denominator of the expression for \tilde{R} . As discussed in chapter 2, we apply a real-valued filter $F(\omega)$ to the data in the ω -domain [17, 23, 25, 26]. The scattered field in (3.22) is then replaced by

$$u_S(x, \omega) = -F(\omega) \frac{v_0 R e^{-i\omega(1+\frac{i\gamma_0}{2\omega})(x-2h)/v_0}}{2i\omega(1+\frac{i\gamma_0}{2\omega})}. \quad (3.30)$$

Substituting this function into (3.27), the output will be some bandlimited version of $\alpha(x)$, represented here as $\alpha_B(x)$, given by

$$\alpha_B(x) = \frac{4Re^{-\gamma_0(h-x)/v_0}}{\pi} \int_{-\infty}^{\infty} F(\omega) \frac{e^{2i\omega(h-x)/v_0}}{2i\omega} d\omega. \quad (3.31)$$

The reflectivity function $\beta(y)$ [13] can be obtained by differentiating (3.14) with re-

spect to x and dividing by -4 . Thus, we get

$$\beta(x) = -\frac{2e^{-\gamma_0(h-x)/v_0}}{\pi v_0^2} \int_{-\infty}^{\infty} i\omega \left(1 + \frac{i\gamma_0}{2\omega}\right)^2 u_S(0, \omega) e^{-2i\omega x/v_0} d\omega \quad (3.32)$$

$$-\frac{\gamma_0 e^{-\gamma_0(h-x)/v_0}}{\pi v_0^2} \int_{-\infty}^{\infty} \left(1 + \frac{i\gamma_0}{2\omega}\right)^2 u_S(0, \omega) e^{-2i\omega x/v_0} d\omega.$$

As a simple check on this result, note that when $\gamma_0 = 0$, this result reduces to the constant-background inversion formula without damping given by Zaman et al [87].

3.4 Numerical Results and Graphs

There are two important questions that can be raised:

1. How would numerical integration deal with the singular integral in (3.27) ?
2. How would the bandlimiting present in any real-world experiment changes the result?

We assume the band-filter for the band $\Omega_0 = [-\omega_2, -\omega_1] \cup [\omega_1, \omega_2]$ has the transform functions as in figure (2.1). The band limiting of $\alpha(x)$ can be obtained by the convolution theorem and since the filter are time-invariant integral operators designed to remove unwanted harmonic components from an observed signal. The filtering in time domain is realized by a convolution intertgal.

Since we have the equation (3.28)

$$\alpha(x) = -4Re^{-\gamma_0(h-x)/v_0} H(x-h),$$

then let

$$x = v_0 t, \quad h = v_0 t_0,$$

we find that

$$\alpha(v_0 t) = \psi(t) = -4R e^{-\gamma_0(h-x)/v_0} H([t - t_0] v_0), \quad (3.33)$$

and by convolution theorem

$$\psi(t)_{\text{bandlimited}} = \psi(t) * F(\omega), \quad (3.34)$$

where $F(\omega)$ is the filter which must be symmetric and nonnegative in the ω -domain.

From example (2.1), the filter gives,

$$r(t) = \frac{\sin \omega_2 t - \sin \omega_1 t}{\pi t}. \quad (3.35)$$

From equation (3.33) and (3.34), the band limiting of $\alpha(x)$ is

$$\psi(t)_{\text{bandlimited}} = \frac{-4R}{\pi} \int_0^\infty e^{-\gamma_0(t_0-\tau)/v_0} H([\tau - t_0] v_0) \frac{\sin \omega_2(t - \tau) - \sin \omega_1(t - \tau)}{\pi(t - \tau)} d\tau. \quad (3.36)$$

For a step in wave speed like in figure (3.1), numerical inversion in the absence of zero-frequency information produces the output in figure (3.2). In figure (3.3) and (3.4), we show this, with 4 – 50 Hz and 10 – 50 Hz outputs, respectively. We conclude that even a moderate loss of low frequency energy already makes the step unrecognizable and indistinguishable from an $\alpha(x)$ that is slowly varying except in the neighborhood of $x = h$, where it exhibits a rapid, doublet-like behavior. From figure (3.7), an even further degeneration of the output is apparent, although the region of the discontinuity of the propagation speed is certainly still recognizable.

We retain now consideration of the bandlimited solution in (3.31). We will take the x-derivative of α_B by multiplying the integral in (3.31) by the factor, $-2i\omega/v_0$.

The result is the reflectivity function

$$\beta(x) = -\frac{2R.e^{-\gamma_0(h-x)/v_0}}{\pi v_0} \int_{-\infty}^{\infty} F(\omega) e^{2i\omega(h-x)/v_0} d\omega \quad (3.37)$$

$$-\frac{R\gamma_0 e^{-\gamma_0(h-x)/v_0}}{\pi v_0} \int_{-\infty}^{\infty} F(\omega) \frac{e^{2i\omega(h-x)/v_0}}{2i\omega} d\omega.$$

If $F(\omega)$ were replaced by unity here, the first integral would be proportional to the Dirac delta function:

$$\beta(x) = -\frac{2R e^{-\gamma_0(h-x)/v_0}}{\pi v_0} \delta_B(x-h) \quad (3.38)$$

$$+\frac{R\gamma_0 e^{-\gamma_0(h-x)/v_0}}{v_0} H(x-h).$$

The result of applying this multiplier to the 10 – 50 Hz step function Fourier data is shown in figure(3.5). As a simple check on this result, note that when $\gamma_0 = 0$, this result reduces to the constant-background inversion formula without damping [13]. We can see the effect of bandlimiting of step function with damping when we fix values of $f1 = 0Hz$, $f2 = 50Hz$ with change of values of damping as in figures (3.7 – 3.13).

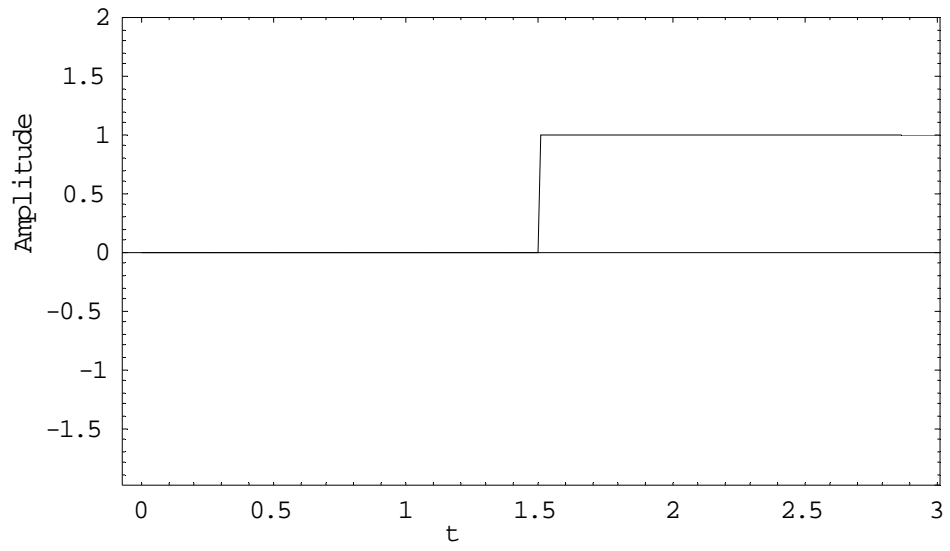


Figure 3.1: A full bandwidth representation of a step function.

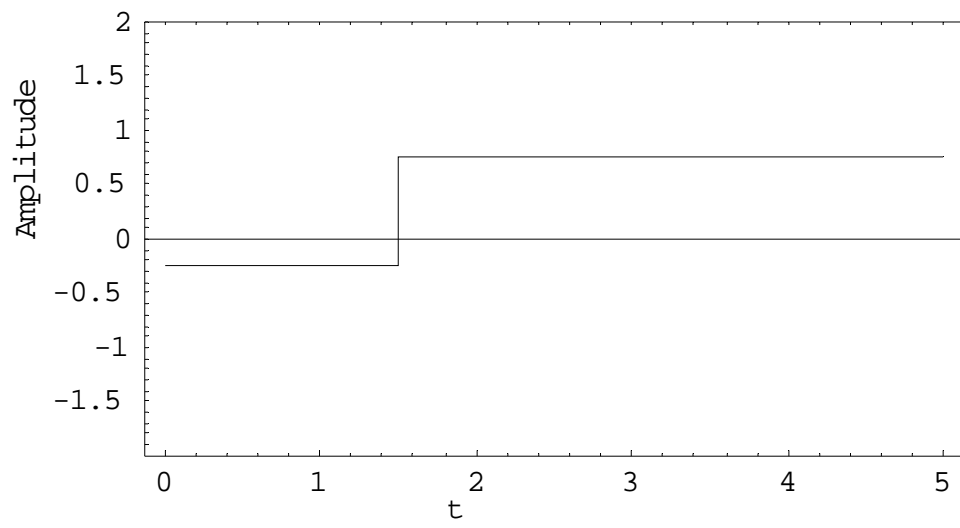


Figure 3.2: A step function lacking only zero-frequency information.

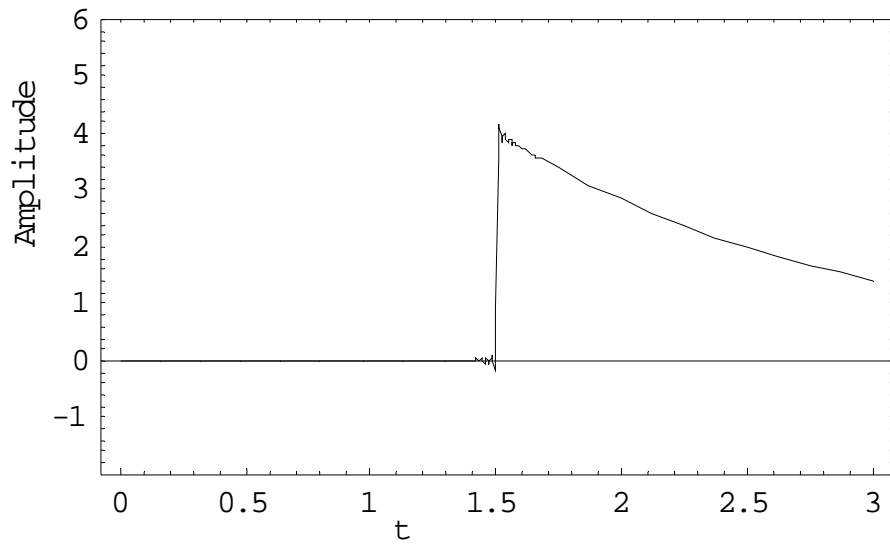


Figure 3.3: A $0 - 50 \text{ Hz}$ bandwidth representation of a step function and $\gamma_0 = 0.5$

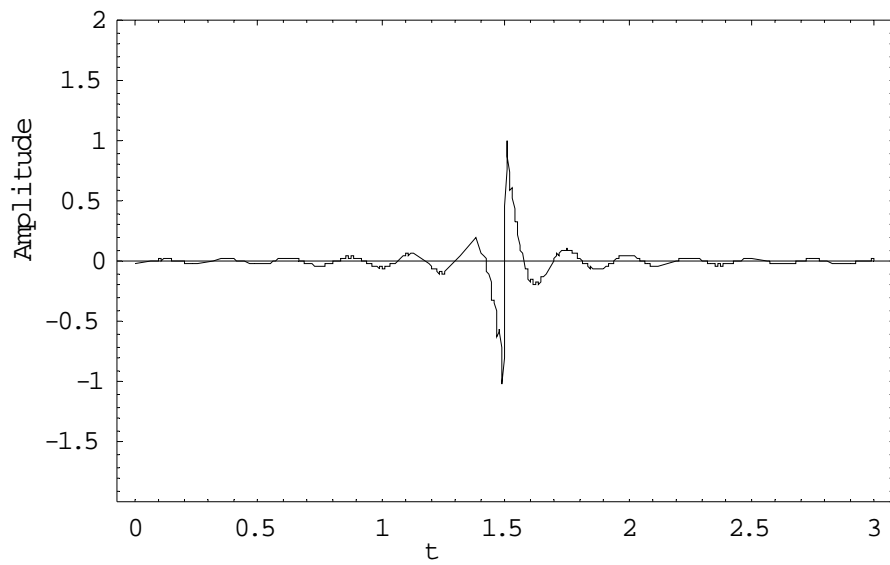


Figure 3.4: A $4 - 50 \text{ Hz}$ bandwidth representation of a step function and $\gamma_0 = 0.5$

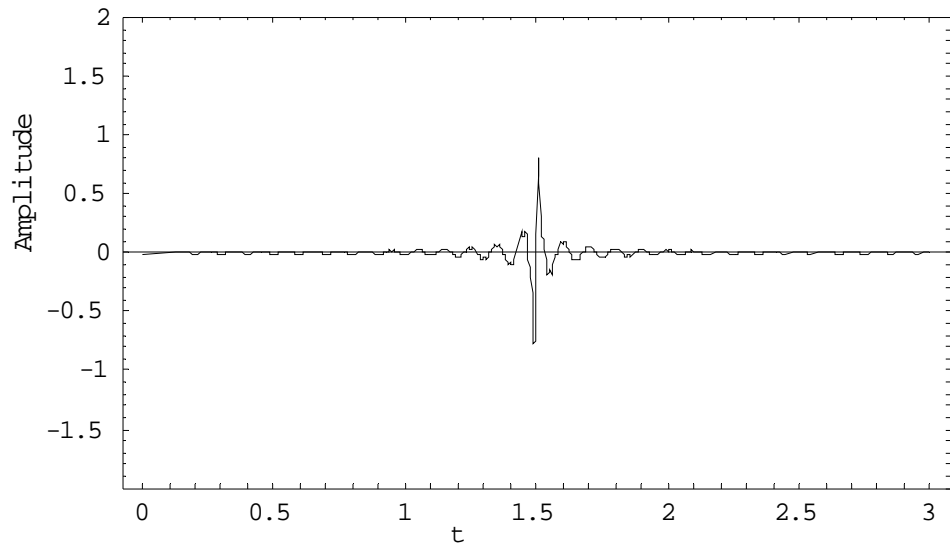


Figure 3.5: A $10 - 50 \text{ Hz}$ bandwidth representation of a step function and $\gamma_0 = 0.5$

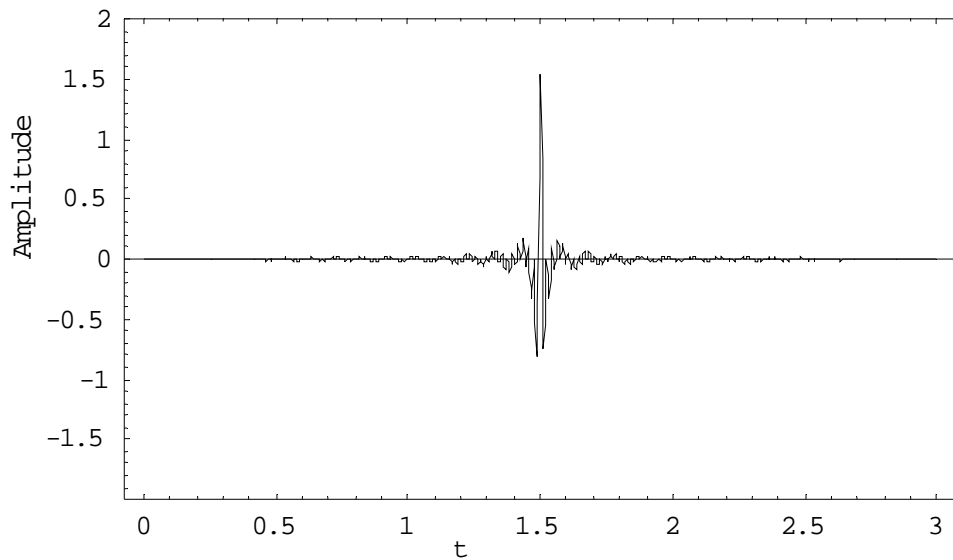


Figure 3.6: A $10 - 50 \text{ Hz}$ bandwidth representation of a step function with derivative operator applied and $\gamma_0 = 0.5$

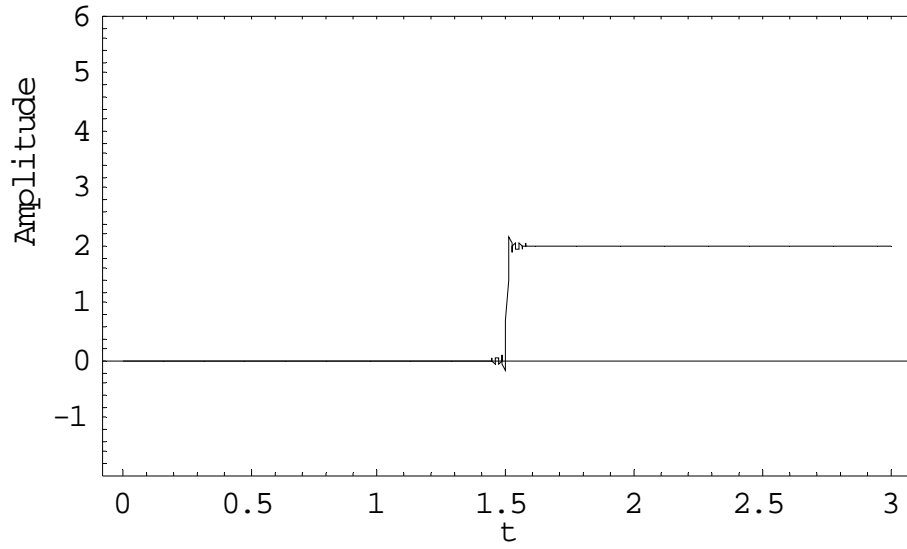


Figure 3.7: A $0 - 50$ Hz bandwidth representation of a step function and $\gamma_0 = 0.0$.

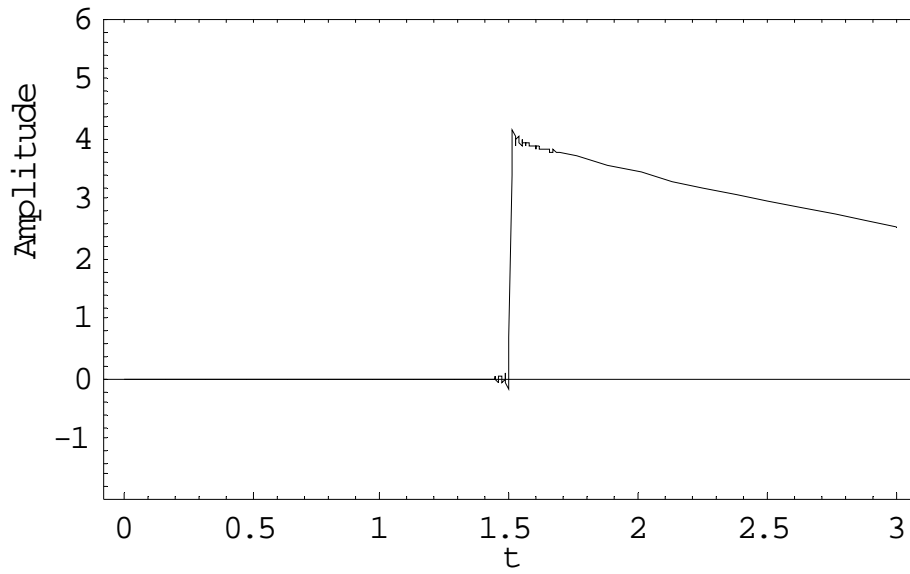


Figure 3.8: A $0 - 50$ Hz bandwidth representation of a step function and $\gamma_0 = 0.3$.

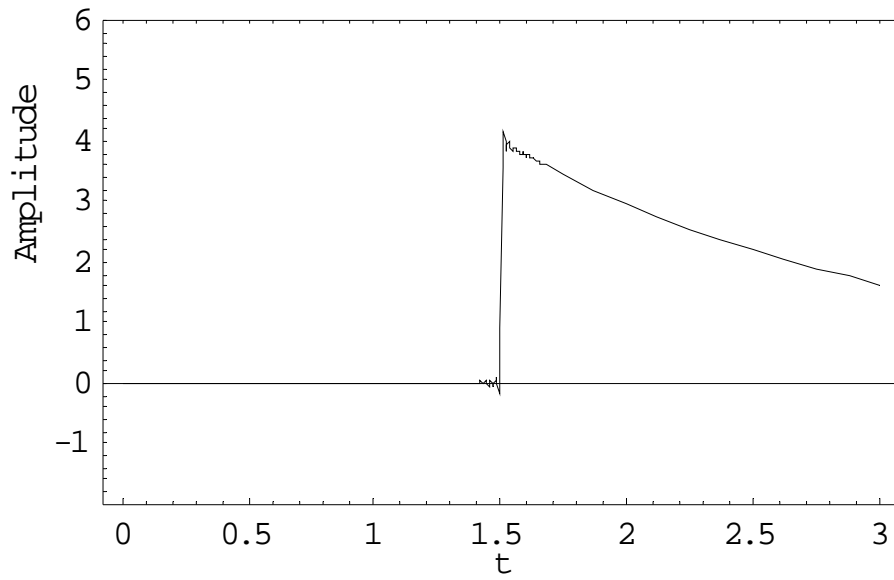


Figure 3.9: A 0 – 50 Hz bandwidth representation of a step function and $\gamma_0 = 0.6$.

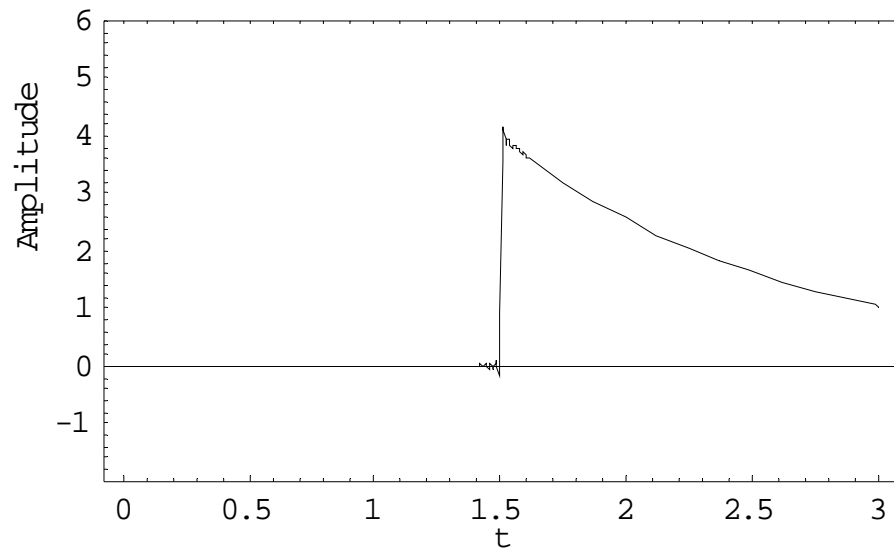


Figure 3.10: A 0 – 50 Hz bandwidth representation of a step function and $\gamma_0 = 0.9$.

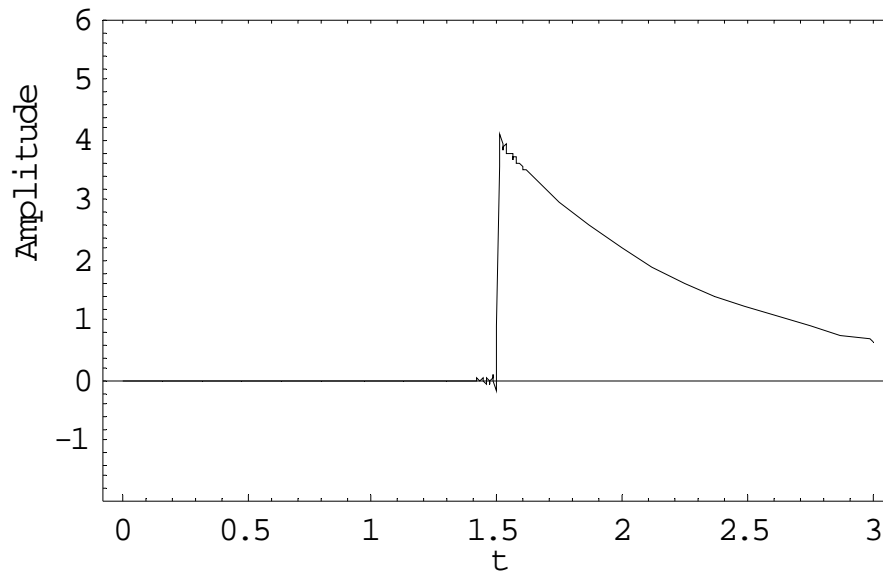


Figure 3.11: A $0 - 50 \text{ Hz}$ bandwidth representation of a step function and $\gamma_0 = 1.2$.

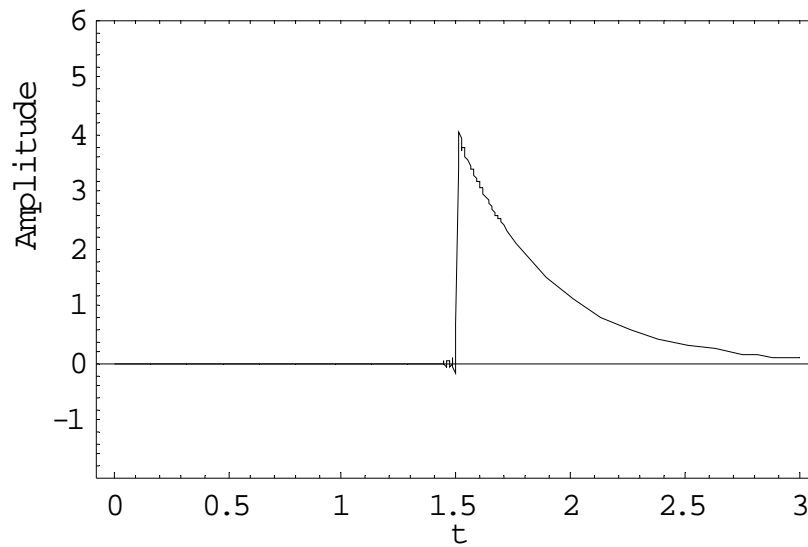


Figure 3.12: A $0 - 50 \text{ Hz}$ bandwidth representation of a step function and $\gamma_0 = 2.5$.

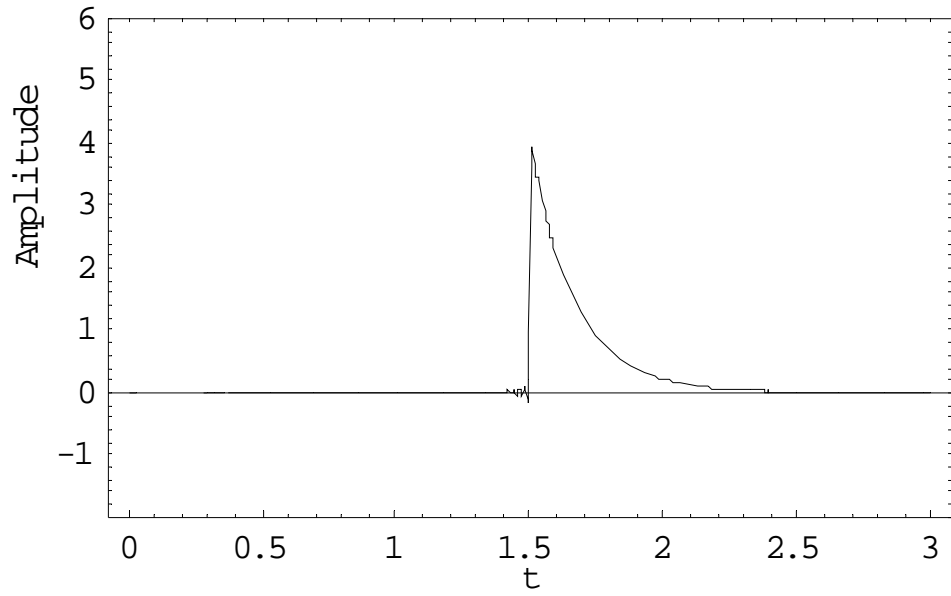


Figure 3.13: A 0 – 50 Hz bandwidth representation of a step function and $\gamma_0 = 5.9$.

3.5 Computational Inverse Scattering in Multilayer Inverse Problem in the Presence of Damping

We will now apply our constant-background inversion formula (3.32) to the data gathered in the two-layer model. Let $u(x, \omega)$ be a solution of the problem (3.1). Assume

$$v(x) = \begin{cases} v_0, & \text{for } x < h_1 \\ v_1, & \text{for } h_1 < x < h_2 \\ v_2, & \text{for } h_2 < x \end{cases} \quad (3.39)$$

$$\gamma(x) = \begin{cases} \gamma_0, & \text{for } x < h_1 \\ \gamma_1, & \text{for } h_1 < x < h_2 \\ \gamma_2, & \text{for } h_2 < x \end{cases} .$$

We begin to solve this problem by writing down fairly general solutions in each of the three regions, with constants to be determined by interface and radiation conditions. As before, we now have the following suitable form of the solution:

$$u = \begin{cases} -\frac{v_0}{2i\omega(1+\frac{i\gamma_0}{2\omega})} \left[e^{i\omega(1+\frac{i\gamma_0}{2\omega})x/v_0} + A_1 e^{i\omega(1+\frac{i\gamma_0}{2\omega})(2h_1-x)/v_0} \right], & x < h_1 \\ -\frac{v_0}{2i\omega(1+\frac{i\gamma_0}{2\omega})} \left\{ A_2 e^{i\omega[(1+\frac{i\gamma_0}{2\omega})h_1/v_0+(1+\frac{i\gamma_1}{2\omega})(x-h_1)/v_1]} \right. \\ \left. + A_3 e^{i\omega[(1+\frac{i\gamma_0}{2\omega})h_1/v_0+(1+\frac{i\gamma_1}{2\omega})(h_1-x)/v_1]} \right\}, & h_2 < x < h_1 \\ -\frac{v_0}{2i\omega(1+\frac{i\gamma_0}{2\omega})} A_4 \left[e^{i\omega[(1+\frac{i\gamma_0}{2\omega})h_1/v_0+(1+\frac{i\gamma_1}{2\omega})(h_2-h_1)/v_1+(1+\frac{i\gamma_2}{2\omega})(x-h_2)/v_2]} \right], & h_2 < x \end{cases} \quad (3.40)$$

We require that u and its first derivative be continuous at $x = h_1$ and $x = h_2$. This leads to the system of equations,

$$A_1 - A_2 - A_3 = -1$$

$$-\frac{\left(1 + \frac{i\gamma_0}{2\omega}\right) A_1}{v_0} - \frac{\left(1 + \frac{i\gamma_1}{2\omega}\right) A_2}{v_1} + \frac{\left(1 + \frac{i\gamma_1}{2\omega}\right) A_3}{v_1} = -\frac{\left(1 + \frac{i\gamma_0}{2\omega}\right)}{v_0} \quad (3.41)$$

$$\left(1 + \frac{i\gamma_1}{2\omega}\right) A_2 e^{i\omega\left(1 + \frac{i\gamma_1}{2\omega}\right)\tau/2v_1} + \left(1 + \frac{i\gamma_1}{2\omega}\right) A_3 e^{-i\omega\left(1 + \frac{i\gamma_1}{2\omega}\right)\tau/2v_1} - \left(1 + \frac{i\gamma_1}{2\omega}\right) A_4 e^{i\omega\left(1 + \frac{i\gamma_2}{2\omega}\right)\tau/2v_1} = 0$$

$$\frac{\left(1 + \frac{i\gamma_1}{2\omega}\right) A_2 e^{i\omega\left(1 + \frac{i\gamma_1}{2\omega}\right)\tau/2v_1}}{v_1} - \frac{\left(1 + \frac{i\gamma_1}{2\omega}\right) A_3 e^{-i\omega\left(1 + \frac{i\gamma_1}{2\omega}\right)\tau/2v_1}}{v_1} - \frac{\left(1 + \frac{i\gamma_2}{2\omega}\right) A_2 e^{i\omega\left(1 + \frac{i\gamma_2}{2\omega}\right)\tau/2v_1}}{v_2} = 0,$$

where $\tau = 2(h_2 - h_1)$. After solving for these constants, we obtain

$$u_S(0, \omega) = -F(\omega) \frac{v_0}{2i\omega \left(1 + \frac{i\gamma_0}{2\omega}\right)} \left[1 + \frac{R_1 + R_2 e^{i\omega\left(1 + \frac{i\gamma_1}{2\omega}\right)\tau/v_1}}{1 + R_1 R_2 e^{i\omega\left(1 + \frac{i\gamma_1}{2\omega}\right)\tau/v_1}} e^{2i\omega\left(1 + \frac{i\gamma_0}{2\omega}\right)h_1/v_0} \right], \quad (3.42)$$

where

$$R_1 = \frac{\left(1 + \frac{i\gamma_0}{2\omega}\right) v_1 - \left(1 + \frac{i\gamma_1}{2\omega}\right) v_0}{\left(1 + \frac{i\gamma_0}{2\omega}\right) v_1 + \left(1 + \frac{i\gamma_1}{2\omega}\right) v_0}, \quad (3.43)$$

$$R_2 = \frac{\left(1 + \frac{i\gamma_1}{2\omega}\right) v_2 - \left(1 + \frac{i\gamma_2}{2\omega}\right) v_1}{\left(1 + \frac{i\gamma_1}{2\omega}\right) v_2 + \left(1 + \frac{i\gamma_2}{2\omega}\right) v_1}.$$

By expanding the denominator in (3.42) in geometric series $\left(\left|R_1 R_2 e^{i\omega\left(1 + \frac{i\gamma_1}{2\omega}\right)\tau/v_1}\right| < 1\right)$,

the response can be written as

$$u_S(0, \omega) = -F(\omega) \frac{v_0}{2i\omega \left(1 + \frac{i\gamma_0}{2\omega}\right)} \left\{ 1 + R_1 e^{2i\omega \left(1 + \frac{i\gamma_0}{2\omega}\right) h_1/v_0} + R_2 [1 - R_1^2] \right. \quad (3.44)$$

$$\left. \sum_{n=2}^{\infty} [-R_1 R_2]^{n-1} e^{in\omega \left(1 + \frac{i\gamma_1}{2\omega}\right) \tau/v_1 + 2i\omega \left(1 + \frac{i\gamma_0}{2\omega}\right) h_1/v_0} \right\}.$$

Let us consider the case $F(\omega) = 1$. The Fourier inversion of each term in this series is exactly the inversion carried out above for the case of a single layer. However, the only difference here similar to the step at $x = h$ in (3.28) will have to be replaced by a step at appropriate position determined by the phase of the particular term in the series.

$$\alpha(x) = \frac{-4e^{-\gamma_0 x/v_0}}{\pi} \int_{-\infty}^{\infty} \frac{\left(1 + \frac{i\gamma_0}{2\omega}\right)}{2i\omega} R_1 e^{-\gamma_0 h_1/v_0} e^{2i\omega(h_1-x)/v_0} + R_2 [1 - R_1^2] \quad (3.45)$$

$$\sum_{n=2}^{\infty} [-R_1 R_2]^{n-1} e^{-n\gamma_1(h_2-h_1)/v_1 - \gamma_0 h_1/v_0} e^{2in\omega(h_2-h_1)/v_1 + 2i\omega(h_1-x)/v_0} \Bigg\} d\omega.$$

Since we are dealing with high frequency, the expression R_1 and R_2 can be written as

$$R_1 \approx \frac{v_1 - v_0}{v_1 + v_0} \quad (3.46)$$

$$R_2 \approx \frac{v_2 - v_1}{v_2 + v_1},$$

thus

$$\alpha(x) \approx \frac{-4e^{-\gamma_0 x/v_0}}{\pi} \int_{-\infty}^{\infty} \frac{1}{2i\omega} \left\{ R_1 e^{-\gamma_0 h_1/v_0} e^{2i\omega(h_1-x)/v_0} + R_2 [1 - R_1^2] \right. \quad (3.47)$$

$$\left. \sum_{n=2}^{\infty} [-R_1 R_2]^{n-1} e^{-2n\gamma_1(h_2-h_1)/v_1 - \gamma_0 h_1/v_0} e^{2i\omega[n(h_2-h_1)/v_1 + (h_1-x)]/v_0} \right\} d\omega.$$

leads to

$$\alpha(x) \approx 4 \left\{ [R_1 e^{-\gamma_0(h-x)/v_0} H(x-h)] + R_2 [1 - R_1^2] e^{-2\gamma_1(h_2-h_1)/v_1 - \gamma_0(h-x)/v_0} H((h_2-h_1)v_0/v_1 + h_1 - x) + e^{-2n\gamma_1(h_2-h_1)/v_1 - \gamma_0(h-x)/v_0} R_2 [1 - R_1^2] \sum_{n=2}^{\infty} [-R_1 R_2]^{n-1} H(n(h_2-h_1)v_0/v_1 + h_1 - x) \right\}. \quad (3.48)$$

3.6 Numerical Results and Graphs

We assume the band-filter for the band

$$\Omega_1 = [-\omega_4, -\omega_3] \cup [-\omega_3, -\omega_2] \cup [-\omega_2, -\omega_1] \cup [\omega_1, \omega_2] \cup [\omega_2, \omega_3] \cup [\omega_3, \omega_4]$$

has the transform functions as in figure (2.2). The band limiting of $\alpha(x)$ can be obtained by the convolution theorem. We consider the equation (3.48) and substitute

$$x = v_0 t, \quad h = v_0 t_0,$$

by convolution theorem,

$$\alpha(v_0 t)_{\text{bandlimited}} = \psi(t)_{\text{bandlimited}} = \psi(t) * F(\omega), \quad (3.49)$$

where $F(\omega)$ is the filter which must be symmetric and nonnegative in the ω -domain.

By example (2.2), the inverse Fourier transform of $F(\omega)$ gives,

$$r(t) = \frac{\cos \omega_4 t - \cos \omega_3 t}{\pi \Delta_1 \omega t^2} + \frac{\cos \omega_2 t - \cos \omega_1 t}{\pi \Delta_2 \omega t^2}, \quad (3.50)$$

where

$$\begin{aligned}\Delta_1 &= \omega_4 - \omega_3, \\ \Delta_2 &= \omega_2 - \omega_1.\end{aligned}$$

From equation (3.48), the band limiting of $\alpha(x)$ for two layers is

$$\begin{aligned}\psi(t)_{\text{bandlimited}} &= -\frac{4}{\pi} \int_0^\infty \left\{ R_1 e^{-\gamma_0(t_0-\tau)/v_0} H([\tau - t_0] v_0) + R_2 [1 - R_1^2] \right. \\ &\quad \left. e^{-2\gamma_1(h_2-h_1)/v_1 - \gamma_0(h-x)/v_0} H((h_2 - h_1) v_0/v_1 + h_1 - x) \right\} \\ &\quad \cdot \left[\frac{\cos \omega_4(t - \tau) - \cos \omega_3(t - \tau)}{\pi \Delta_1 \omega (t - \tau)^2} + \frac{\cos \omega_2(t - \tau) - \cos \omega_1(t - \tau)}{\pi \Delta_2 \omega (t - \tau)^2} \right] d\tau.\end{aligned}\quad (3.51)$$

The first term in this expression is just what was obtained for the single layer, $-4R_1$, where R_1 is the reflection coefficient of the first boundary. For small perturbations $\alpha = O(\varepsilon)$, this term produces the step at $x = h_1$. The second term produces a step at $x = h_1 + (h_2 - h_1)c_0/c_1$, instead of a step at $x = h_2$ as shown in figure (3.14) for frequencies of 10, 20, 50, and 60 *Hz*, respectively. The amplitude, $R_2 [1 - R_1^2]$ is correct to order ε .

Let us now return to the bandlimited data, i.e. $F(\omega)$ is no longer identically equal to 1. We compute the reflectivity function $\beta(x)$, to the solution representation (3.44).

$$\begin{aligned}\beta(x) &= -\frac{R_1 e^{-\gamma_0(h-x)/v_0}}{\pi v_0} \delta_B(x - h_1) \\ &\quad + \frac{R_1 \gamma_0 e^{-\gamma_0(h-x)/v_0}}{v_0} H(x - h_1) \\ &\quad - \frac{R_2 [1 - R_1^2] e^{-2\gamma_1(h_2-h_1)/v_1 - \gamma_0(h-x)/v_0}}{\pi v_0} \delta_B((h_2 - h_1) v_0/v_1 + h_1 - x) \\ &\quad + \frac{R_2 [1 - R_1^2] \gamma_0 e^{-2\gamma_1(h_2-h_1)/v_1 - \gamma_0(h-x)/v_0}}{\pi v_0} H((h_2 - h_1) v_0/v_1 + h_1 - x).\end{aligned}\quad (3.52)$$

The band limited recovery of the variation in velocity for $\omega_i = 10, 20, 50, 60$, for $i = 1, 2, 3, 4$ and the corresponding reflectivity function are given graphically in figures (3.14) and (3.15) respectively.

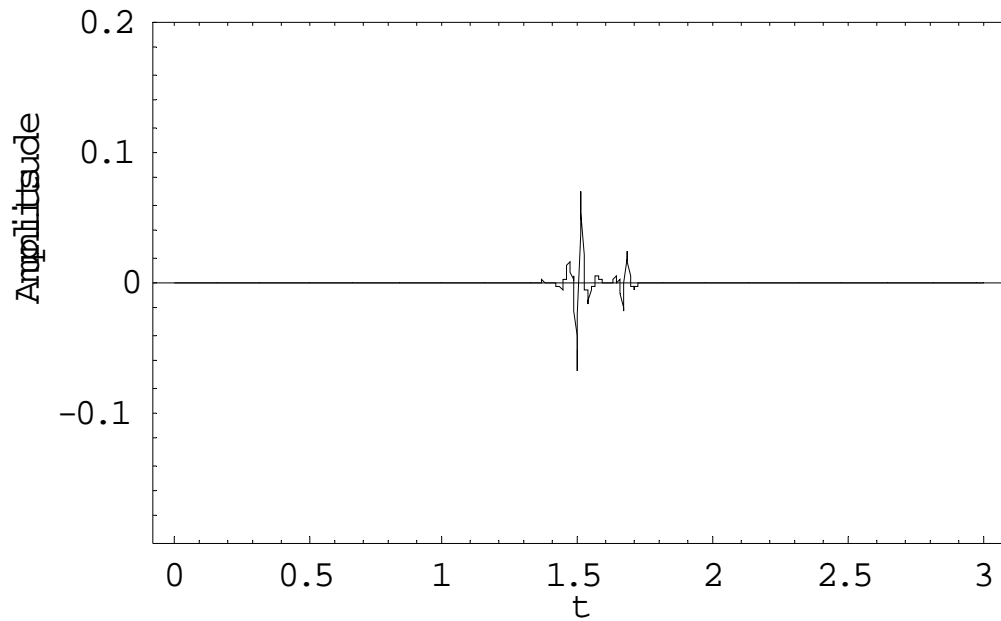


Figure 3.14: The bandwidth of the data with frequencies of 10, 20, 50, 60 Hz , respectively

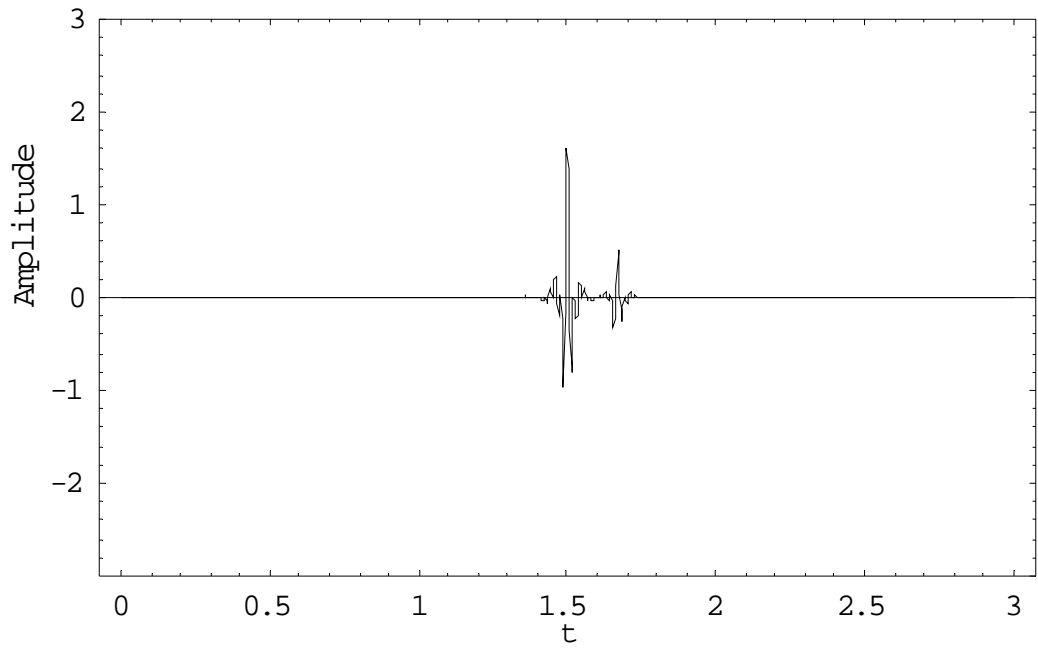


Figure 3.15: Step Data, 10, 20, 50, 60 Hz , (with derivative operator).

3.7 Conclusions

The theory developed in chapter 2 has been applied to data that would be obtained from piecewise constant wave speed with damping effect consisting of a single layer. A constant-background wavespeed equal to the first wavespeed that waves would encounter was assumed for the inversion. When the data are bandlimited to a range of frequencies that might be typical of a seismic experiment, the inversion output suggests that recovery of information about discontinuities of variation wave speed from the data should be expected.

For more than one layer, we solve this problem by writing down fairly general solutions in multilayer regions, with constants to be determined by interface and radiation conditions. We have derived approximate solutions to the inverse problem of finding the velocity and damping from the observed wave-field. We deal with numerical integration of the singular integral by applying the bandwidth to the step function, which enables us to recover the velocity profile in a medium with one or more layers in the presence of damping. The results apply to a more general model as compared with Bleistein et. al [13] and Zaman and Masood [87].

Chapter 4

Velocity Inversion In the Presence of Variable Density in One and Higher Dimension

4.1 Recovery of Variation in Density and Bulk Modulus in the Presence of Damping

Abstract

An inverse problem arising from a one dimensional problem in a medium with variable density and bulk modulus in the presence of damping in the medium is discussed. The velocity profile of a medium may change due to the change in the bulk modulus and/ or change in the density. In this section we formulate an inverse problem which recovers these variations in a medium with damping. Another interesting feature is that we let the material properties of the background medium to be variable also. Our

method is based upon the Born approximation and the assumption that the wave speed and damping are well approximated by the back ground speed plus a perturbation term. The WKBJ approximation of Green's function for high frequency seismic data is used to derive the inversion formula. Also, an inverse problem arising from a three dimensional problem in a medium with wave speed and density in the medium is discussed. The source and receiver are coincident on a flat horizontal surface and the background speed and density are constant (constant-background zero-offset equation).

4.1.1 Introduction

The inverse problems used to map the properties of the interior of earth, geological prospecting and governing information about inaccessible parts can be further improved by considering more interesting models. As the wave velocity is dependent upon the bulk modulus and the density of the medium, we consider separate variations in wave velocity, density and the bulk modulus of the medium in the presence of damping. This is a considerable improvement of the model considered, [6 – 14, 25, 28, 63 – 65] and Zaman and Masood [87]. It is further assumed that the background medium also has material properties that are dependent upon position. In this regard one may refer to Engle Louiss and Rundell [41] who discuss inverse problems in other physical phenomena, Engle and Rundell [42] a regularization methods to deal with the ill-posedness of the integral equation and some Colton and Kress [32] who discuss inverse obstacle and inverse medium problems, interesting approaches to these problems.

Under the assumption of constant density, Claerbout presented an approximation method to the inverse problem for velocity inversion. Gerver demonstrated that the

velocity of propagation can be determined from observations at one point. Cohen and Bleistein [26, 27], Cohen and Hagin [28] among others used the Born approximation [13, 14] and high frequency assumptions to study the inverse problem in one and higher dimensions. The linearization used in the derivation procedure is often referred to as the Born's approximation. An excellent account of the seismic inversion based upon these techniques can be found in Bleistein, Cohen and Stokwell [13]. In reference, a theory is developed to extract information from high frequency band-limited Fourier transform of a piecewise constant function. In this context, recovery of velocity profile has been used as a mean of estimating medium properties. To account for such behavior, Zaman and Masood introduced the damping effects in the model through damped wave equation and used high frequency estimates to recover the velocity profiles and damping parameters of the medium [77, 78]. Stolt [79] determined the rapid variation in bulk modulus and density from the amplitude versus offset information present in a seismic reflection survey. We should note there are full waveform inversion methods that address the nonlinearity of the inverse problem through optimization; see, for example Mora [61].

In this section we presented a one-dimensional model which takes into account the variable density and bulk modulus of the background medium and derive an inversion formula in the presence of damping. The variations in density and bulk modulus is described by small perturbation terms and the Born approximation is used to linearize the inversion formula [33]. WKBJ method is used to find Green's function under the high frequency assumptions [13, 17, 24, 39, 51, 66]. This is needed in order to let the background material parameter to vary with position, the variable density and bulk modulus with damping parameter is recovered and an iteration procedure to improve

the results is also presented. These results present an important improvement to the earlier models and can be help to get a better mapping of the interior of the earth.

In subsection (2) of this section (1), we first formulate the direct problem of a one dimensional model in which the background medium has variable wave velocity as well as variable density. The inverse formula is derived by writing these parameters of the inhomogeneity as small perturbations of respective background medium parameters. In subsection (3), the model is further improved to include the damping of the medium which could account for small inhomogeneous or viscous effects. The damping parameter is recovered in subsection (4) for such a medium. In order to have a more general model, we consider the inverse problem of recovering the variation in the bulk modulus and density. This model could prove to be more useful when either of these two parameters vary more significantly than the other.

In section (2) of this chapter, a higher dimension model is considered to study inverse problems of the kind studied in section (1) to account for wave field depending upon more than one space variables. In subsection (2), we formulate inverse problem in higher dimensions and for zero offset experiment in subsection (3) with constant background wave velocity and density. The results are then derived for variation of material parameters with depth only in subsection (4). This could provide with a useful model for further studies in higher dimension.

4.1.2 Variable density without damping

The one-dimensional problem for both variable density and variable wave speed will be consider here. The main result will be that inversion will now yield an estimate of the *impedance* of the medium, involving the product ρv , with ρ and v being the

density and the propagation speed, respectively. We begin by considering the acoustic equation for pressure in a variable sound speed, variable-density medium with source at a point x_s is given by,

$$Lu(x, x_s, \omega) = \rho(x) \frac{d}{dx} \left[\frac{1}{\rho(x)} \frac{du(x, x_s, \omega)}{dx} \right] + \frac{\omega^2}{v^2(x)} u(x, x_s, \omega) = -\delta(x - x_s), \quad (4.1)$$

together with the radiation condition (3.2). We assume that variations in density and sound speed have the following parallel form:

$$\rho(x) = \rho_0(x) + \delta\rho(x), \quad (4.2)$$

$$v(x) = v_0(x) + \delta v(x). \quad (4.3)$$

Thus

$$\frac{1}{\rho(x)} \approx \frac{1}{\rho_0(x)} \left[1 - \frac{\delta\rho(x)}{\rho_0(x)} \right]. \quad (4.4)$$

$$\frac{1}{v^2(x)} \approx \frac{1}{v_0^2(x)} \left[1 - \frac{2\delta v(x)}{v_0(x)} \right]. \quad (4.5)$$

Substituted above into (4.1), and only linear terms in $\delta v(x)$ and $\delta\rho(x)$ are retained.

After some algebra, the resulting equation becomes,

$$L_0 u = \rho_0 \frac{d}{dx} \left[\frac{1}{\rho_0} \frac{du}{dx} \right] + \frac{\omega^2}{v_0^2(x)} u = -\delta(x - x_s) + \frac{\omega^2}{v_0^2(x)} \frac{2\delta v(x)}{v_0(x)} u + \frac{d}{dx} \left[\frac{\delta\rho(x)}{\rho_0(x)} \right] \frac{du}{dx}. \quad (4.6)$$

The total field $u(x, x_s, \omega)$ can be separated into the incident part $u_I(x, x_s, \omega)$ in the absence of the perturbations and $u_S(x, x_s, \omega)$ in the presence of perturbations. Thus

we set

$$u(x, x_s, \omega) = u_S(x, x_s, \omega) + u_I(x, x_s, \omega), \quad (4.7)$$

Substituting (4.7) in (4.6) we arrive at the following equations:

$$L_0 u_I(x, x_s, \omega) = \rho_0 \frac{d}{dx} \left[\frac{1}{\rho_0} \frac{du_I}{dx} \right] + \left[\frac{\omega^2}{v_0^2(x)} \right] u_I = -\delta(x - x_s), \quad (4.8)$$

$$\begin{aligned} L_0 u_S(x, x_s, \omega) &= \frac{\omega^2}{v_0^2(x)} \frac{2\delta v(x)}{v_0(x)} [u_I(x, x_s, \omega) + u_S(x, x_s, \omega)] \\ &+ \frac{d}{dx} \left[\frac{\delta\rho(x)}{\rho_0(x)} \right] \frac{d}{dx} [u_I(x, x_s, \omega) + u_S(x, x_s, \omega)]. \end{aligned} \quad (4.9)$$

The solution of (4.9), after the Born approximation, written in terms of Green's function takes the form,

$$\begin{aligned} u_S(x_g, x_s, \omega) &= - \int_0^\infty \left\{ \frac{\omega^2}{v_0^2(x)} \frac{\delta v(x)}{v_0(x)} u_I(x, x_s, \omega) g(x, x_g, \omega) \right. \\ &\quad \left. - \frac{\delta\rho(x)}{\rho_0(x)} \frac{d}{dx} \left[g(x, x_g, \omega) \frac{du_I(x, x_s, \omega)}{dx} \right] \right\} dx. \end{aligned} \quad (4.10)$$

We are concerned only with high-frequency inversion of this equation. To that end, we will use WKBJ Green's function approximation for the variable-density using the same

as in Chapter 2— for $g(x, x_g, \omega)$ and then $u_I(x, x_s, \omega)$,

$$u_I(x, x_s, \omega) = -\frac{1}{2i\omega} F(\omega) A(x, x_s) e^{i\omega\phi(x, x_s)}, \quad (4.11)$$

$$g(x_g, x, \omega) = -\frac{1}{2i\omega} A(x_g, x) e^{i\omega\phi(x_g, x)}, \quad (4.12)$$

$$\text{where, } \phi(x, \xi) = \int_{x_<}^{x_>} \frac{dt}{v_0(t)}, \quad \text{and,} \quad (4.13)$$

$$x_> = \max(x, \xi), \quad x_< = \min(x, \xi).$$

We note that, for the case of continuous v_0 and ρ_0 [13],

$$A(x, x_s) = \sqrt{\frac{v(x)v(x_s)\rho(x)}{\rho(x_s)}}, \quad (4.14)$$

$$A(x_g, x) = \sqrt{\frac{v(x)v(x_g)\rho(x_g)}{\rho(x)}}. \quad (4.15)$$

Substitute these WKBJ approximations (4.11), (4.12) into (4.10) and retaining only the leading-order terms in ω [$O(\omega^2)$], we obtain,

$$u_S(x_g, x_s, \omega) = \int_0^\infty F(\omega) \frac{1}{2} \left[\frac{\delta v(x)}{v_0(x)} + \frac{\delta \rho(x)}{\rho_0(x)} \right] \frac{A(x_g, x) A(x_s, x)}{v_0^2(x)} e^{i\omega\{\phi(x_g, x) + \phi(x_s, x)\}} dx. \quad (4.16)$$

Remark 4.1 *This equation should be compared with (2.48). Here, the source and receiver are separated along the line; where as, they were coincident in (2.48). As a consequence, the observed field is $u_S(x_g, x_s, \omega)$ as opposed to $u_S(0, \omega)$, and we have two different amplitudes arising from u_I and g , whereas in (2.48) the square of a single amplitude appeared. Here, phase involves a sum of travel time from the source and receiver point to the scattering point at depth; there, the phases were the same and*

a two-way travelttime appeared in the exponent. The unknown $-\alpha/4$ of the previous expression is replaced by $\frac{1}{2} \left[\frac{\delta v(y)}{v_0(y)} + \frac{\delta \rho(y)}{\rho_0(y)} \right]$.

Remark 4.2 *The derivation of the inversion formula carried out using (2.48) can be repeated here step by step with these modifications in place. With some thought, it becomes apparent that the result of inversion for this equation can be deduced from (2.59) by making the same changes in the inversion formula as were observed in the comparison of (2.48) and (4.16). That is, we*

1. rewrite (2.59) as an equation for $-\alpha/4$;
2. replace $v_0(y)/A^2(y)$ by $v_0(y)/A(x, x_s)A(x_g, x)$;
3. replace $2\phi(y, 0)$ by $[\phi(x_g, y) + \phi(x_s, y)]$.

After these changes (4.16) take the form,

$$\frac{1}{2} \left[\frac{\delta v(y)}{v_0(y)} + \frac{\delta \rho(y)}{\rho_0(y)} \right] = \frac{v_0(y)}{\pi A(x_g, y) A(x_s, y)} \int_{-\infty}^{\infty} u_S(x_g, x_s, \omega) e^{-i\omega\{\phi(x_g, y) + \phi(x_s, y)\}} d\omega. \quad (4.17)$$

The reflectivity function $\beta(y)$ [13] can be obtained by differentiating (4.17) with respect to y and dividing by -4 . Thus multiplying (4.17) by the factor $\frac{i\omega}{v_0(y)}$ we obtain the reflectively function

$$\beta(y) = \frac{-2}{\pi A(x_g, y) A(x_s, y)} \int_{-\infty}^{\infty} i\omega u_S(x_g, x_s, \omega) e^{-i\omega\{\phi(x_g, y) + \phi(x_s, y)\}} d\omega. \quad (4.18)$$

4.1.3 Variable Density in the Presence of Damping

In this section, the one-dimensional problem for variable density and velocity in the presence of damping will be considered. The main difference of the present problem

with that considered in chapter (2) is that we assume the non-zero offset condition i.e. the source and the receiver as no longer placed at the same location. the propagation of the field governed by the scalar Helmholtz equation with damping $\gamma(x)$ and source placed at point x_s is;

$$Lu(x, x_s, \omega) = \rho(x) \frac{d}{dx} \left[\frac{1}{\rho(x)} \frac{du(x, x_s, \omega)}{dx} \right] + \left[\frac{\omega^2 + i\omega\gamma(x)}{v^2(x)} \right] u(x, x_s, \omega) = -\delta(x - x_s), \quad (4.19)$$

It is assumed that the source point is to the left of the region where $v(x)$, $\rho(x)$ and $\gamma(x)$ are unknown. The impulse response will be observed at point x_g as well as to the left of the region of unknown $v(x)$, $\rho(x)$ and $\gamma(x)$. The objective will be to see what can be determined about the perturbations from the observed response. We assume that as in (4.2) and (4.3) where we expressed velocity and density as perturbation of the background medium, we can write a similar expression for the damping as

$$\gamma(x) = \gamma_0(x) + \delta\gamma(x). \quad (4.20)$$

Substituting (4.4), (4.5), (4.7) and (4.20) into (4.19) and separating the unperturbed field as

$$L_0 u_I(x, x_s, \omega) = \rho_0 \frac{d}{dx} \left[\frac{1}{\rho_0} \frac{du_I}{dx} \right] + \left[\frac{\omega^2 + i\omega\gamma_0(x)}{v_0^2(x)} \right] u_I = -\delta(x - x_s), \quad (4.21)$$

while retaining only linear terms of perturbations in velocity, density and damping, we get

$$\begin{aligned}
L_0 u_S(x, x_s, \omega) &= \left[-i\omega \delta\gamma(x) + \frac{2\omega^2 \delta v(x)}{v_0(x)} + \frac{2i\omega \gamma_0(x) \delta v(x)}{v_0(x)} \right] \frac{u_I(x, x_s, \omega)}{v_0^2(x)} \\
&+ \frac{d}{dx} \left[\frac{\delta\rho(x)}{\rho_0(x)} \right] \frac{du_I(x, x_s, \omega)}{dx}.
\end{aligned} \tag{4.22}$$

We now construct a Green's function representation of solution of (4.22). As the operator L_0 in (4.22) is not self-adjoint. Therefore, we use the Green's function $g^*(x, x_g, \omega)$ which is the impulse response for the "adjoint equation"

$$\begin{aligned}
L_0^* g^*(x, x_g, \omega) &= \frac{d}{dx} \left[\frac{1}{\rho_0(x)} \frac{d}{dx} [\rho_0(x) g^*(x, x_g, \omega)] \right] + \left[\frac{\omega^2 + i\omega \gamma_0(x)}{v_0^2(x)} \right] g^*(x, x_g, \omega) \\
&= -\delta(x - x_g),
\end{aligned} \tag{4.23}$$

Theorem 4.3 *Reciprocity Theorem*

Assume that we have unbounded media problem

$$\begin{aligned}
Lg(x, x_s) &= -\delta(x - x_s) \\
L^* g^*(x, x_g) &= -\delta(x - x_g),
\end{aligned}$$

with L and L^* , being defined as in equation with volume integral in Green's theorem. Hence, above equation can then be written as (Beliestein [13]):

$$g^*(x_s, x_g) = g(x_g, x_s).$$

Thus by the reciprocity theorem it can be utilized to obtain

$$g^*(x, x_g, \omega) = g(x_g, x, \omega).$$

Since we are concerned only with high-frequency inversion of this equation, we will use WKBJ approximation for $u_I(x, x_s, \omega)$ and $g(x_g, x, \omega)$. It can be found using the procedure outlined in section (2.4) as

$$u_I(x, x_s, \omega) = -\frac{1}{2i\omega} F(\omega) A(x, x_s) e^{[i\omega\phi(x_s, x) - \frac{1}{2}\psi(x_s, x)]}, \quad (4.24)$$

$$g(x_g, x, \omega) = -\frac{1}{2i\omega} A(x_g, x) e^{[i\omega\phi(x_g, x) - \frac{1}{2}\psi(x_g, x)]}, \quad (4.25)$$

$$\phi(x_g, x) = \int_{x_g}^x \frac{dt}{v_0(t)}, \quad (4.26)$$

$$\psi(x_g, x) = \int_{x_g}^x \frac{\gamma_0(t) dt}{v_0(t)}, \quad (4.27)$$

where $A(x, x_s)$ and $A(x_g, x)$ are as given by (4.14) and (4.15). The solution of (4.22) in terms of Green's function is given by

$$u_S(x_g, x_s, \omega) = -\int_0^\infty F(\omega) \left\{ \left[-i\omega\delta\gamma(x) + \frac{2\omega^2\delta v(x)}{v_0(x)} + \frac{2i\omega\gamma_0(x)\delta v(x)}{v_0(x)} \right] \frac{u_I(x, x_s, \omega)g(x_g, x, \omega)}{v_0^2(x)} - \frac{\delta\rho(x)}{\rho_0(x)} \frac{d}{dx} \left[\frac{du_I(x, x_s, \omega)}{dx} g(x_g, x, \omega) \right] \right\} dx. \quad (4.28)$$

Now using the WKBJ representations (4.24) and (4.25) in (4.28), and retaining only the leading order terms in ω , we have

$$u_S(x_g, x_s, \omega) = \int_0^\infty F(\omega) \frac{1}{2} \left[\frac{\delta v(x)}{v_0(x)} + \frac{\delta\rho(x)}{\rho_0(x)} \right] \frac{A(x_g, x) A(x_s, x)}{v_0^2(x)} e^{\{i\omega[\phi(x_g, x) + \phi(x_s, x)]\}} e^{\{-\frac{1}{2}[\psi(x_g, x) + \psi(x_s, x)]\}} dx. \quad (4.29)$$

This can be treated as a Fourier transform of $[\delta v(x) + \delta \rho(x)]$ by extending the integration limited from $-\infty$ to ∞ . Since inversion can be performed in the same way as (4.16), we have to make the following replacement:

- replace $v_0(y)/A(x, x_S)A(x_g, x)$ by $v_0(y)e^{\frac{1}{2}[\psi(x_g, y)+\psi(x_s, y)]}/A(x, x_S)A(x_g, x)$;
- the result of these replacements is

$$\frac{1}{2} \left[\frac{\delta v(y)}{v_0(y)} + \frac{\delta \rho(y)}{\rho_0(y)} \right] = \frac{v_0(y) e^{\frac{1}{2}[\psi(x_g, y)+\psi(x_s, y)]}}{\pi A(x_g, y) A(x_s, y)} \int_{-\infty}^{\infty} u_S(x_g, x_s, \omega) e^{-i\omega\{\phi(x_g, y)+\phi(x_s, y)\}} d\omega. \quad (4.30)$$

This result gives a combination of variations in the wave speed and density. Unfortunately, we cannot separate the jumps in soundspeed from the jumps in density in this result. The reflectivity function $\beta(y)$ can be obtained in the same way as in (4.18) and is given by

$$\beta(y) = \frac{-2e^{\frac{1}{2}[\psi(x_g, y)+\psi(x_s, y)]}}{\pi A(x_g, y) A(x_s, y)} \int_{-\infty}^{\infty} i\omega u_S(x_g, x_s, \omega) e^{-i\omega\{\phi(x_g, y)+\phi(x_s, y)\}} d\omega. \quad (4.31)$$

If we take $\gamma = 0$ in (4.30) and (4.31), then these results agree with the previous results (4.17) and (4.18) obtained for Bleistein's model in which there is no damping present.

4.1.4 Recovery of the Damping Effect with Wavespeed.

Consider equation (4.28) again and retain terms of order $\frac{1}{\omega}$ to get,

$$u_S(x_g, x_s, \omega) = - \int_0^{\infty} F(\omega) \left[\frac{i}{4\omega} \left(\delta\gamma(x) - \frac{2\gamma_0(x) \delta v(x)}{v_0(x)} \right) - \frac{2\delta v(x)}{4v_0(x)} - \frac{\delta \rho(x)}{2\rho_0(x)} \right] \frac{A(x_g, x) A(x_s, x)}{v_0^2(x)} e^{i\omega\{\phi(x_g, x)+\phi(x_s, x)\}} e^{-\frac{1}{2}[\psi(x_g, x)+\psi(x_s, x)]} dx. \quad (4.32)$$

Since $\frac{1}{2} \left[\frac{\delta v(x)}{v_0(x)} + \frac{\delta \rho(x)}{\rho_0(x)} \right]$ is known from (4.30), therefore set

$$V(x_g, x_s, \omega) = \frac{1}{2} \int_0^\infty F(\omega) \left[\frac{\delta v(x)}{v_0(x)} + \frac{\delta \rho(x)}{\rho_0(x)} \right] \frac{A(x_g, x) A(x_s, x)}{v_0^2(x)} e^{\{i\omega[\phi(x_g, x) + \phi(x_s, x)]\}} e^{\{-\frac{1}{2}[\psi(x_g, x) + \psi(x_s, x)]\}} dx. \quad (4.33)$$

Thus we can write (4.32) as

$$W(x_g, x_s, \omega) = \int_0^\infty F(\omega) \left[\delta\gamma(x) - \frac{2\gamma_0(x) \delta v(x)}{v_0(x)} \right] \frac{A(x_g, x) A(x_s, x)}{v_0^2(x)} e^{\{i\omega[\phi(x_g, x) + \phi(x_s, x)]\}} e^{\{-\frac{1}{2}[\psi(x_g, x) + \psi(x_s, x)]\}} dx, \quad (4.34)$$

where

$$W(x_g, x_s, \omega) = 4i\omega [u_S(x_g, x_s, \omega) - V(x_g, x_s, \omega)]. \quad (4.35)$$

Since $\delta\gamma(x) - \frac{2\gamma_0(x) \delta v(x)}{v_0(x)} = 0$, for $x < 0$, therefore lower integral limit in (4.34) can be extended to $-\infty$. Thus it becomes the Fourier type integral and can be inverted in the same way as in (4.16). The result is

$$\delta\gamma(y) - \frac{2\gamma_0(y) \delta v(y)}{v_0(y)} = \frac{v_0(y) e^{\{\frac{1}{2}[\psi(x_g, y) + \psi(x_s, y)]\}}}{\pi A(x_g, y) A(x_s, y)} \int_{-\infty}^\infty W(x_g, x_s, \omega) \cdot e^{-i\omega\{\phi(x_g, y) + \phi(x_s, y)\}} d\omega. \quad (4.36)$$

We write it in a more suitable form as follows

$$\left[\frac{\delta\gamma(y)}{\gamma_0(y)} - \frac{2\delta v(y)}{v_0(y)} \right] = \frac{v_0(y) e^{\{\frac{1}{2}[\psi(x_g, y) + \psi(x_s, y)]\}}}{\gamma_0(y) \pi A(x_g, y) A(x_s, y)} \int_{-\infty}^\infty W(x_g, x_s, \omega) \cdot e^{-i\omega\{\phi(x_g, y) + \phi(x_s, y)\}} d\omega. \quad (4.37)$$

This result gives a combination of variations in damping and sound speed. Now we present an iterative procedure to improve the already obtained velocity and damping profiles. For this we substitute the approximation obtained for $\delta\gamma(x) - \frac{2\gamma_0(y)\delta v(x)}{v_0(x)}$ from (4.37), in expression(4.32) by setting,

$$H(x_g, x_s, \omega) = \int_0^\infty F(\omega) \left[\delta\gamma(x) - \frac{2\gamma_0(x)\delta v(x)}{v_0(x)} \right] \frac{A(x_g, x)A(x_s, x)}{v_0^2(x)} e^{\{i\omega[\phi(x_g, x) + \phi(x_s, x)]\}} e^{\{-\frac{1}{2}[\psi(x_g, x) + \psi(x_s, x)]\}} dx. \quad (4.38)$$

Thus the equation (4.32) becomes,

$$X(x_g, x_s, \omega) = \int_0^\infty \left[\frac{\delta v(x)}{2v_0(x)} + \frac{\delta\rho(x)}{2\rho_0(x)} \right] \frac{A(x_g, x)A(x_s, x)}{v_0^2(x)} e^{\{i\omega[\phi(x_g, x) + \phi(x_s, x)]\}} e^{\{-\frac{1}{2}[\psi(x_g, x) + \psi(x_s, x)]\}} dx, \quad (4.39)$$

where

$$X(x_g, x_s, \omega) = \left[u_S(x_g, x_s, \omega) + \frac{i}{4\omega} H(x_g, x_s, \omega) \right]. \quad (4.40)$$

The integral appearing in (4.39) can again be regarded as of the Fourier type and can be inverted to give,

$$\frac{1}{2} \left[\frac{\delta v(y)}{v_0(y)} + \frac{\delta\rho(y)}{\rho_0(y)} \right] = \frac{v_0(y) e^{\{\frac{1}{2}[\psi(x_g, y) + \psi(x_s, y)]\}}}{\pi A(x_g, y)A(x_s, y)} \int_{-\infty}^\infty X(x_g, x_s, \omega) \cdot e^{-i\omega[\phi(x_g, y) + \phi(x_s, y)]} d\omega. \quad (4.41)$$

Now from (4.41), $\frac{1}{2} \left[\frac{\delta v(y)}{v_0(y)} + \frac{\delta\rho(y)}{\rho_0(y)} \right]$ can be calculated and the above procedure from (4.33) to (4.41) can be repeated to get the next approximations for $\frac{1}{2} \left[\frac{\delta v(y)}{v_0(y)} + \frac{\delta\rho(y)}{\rho_0(y)} \right]$ and $\left[\frac{\delta\gamma(y)}{\gamma_0(y)} - \frac{2\delta v(y)}{v_0(y)} \right]$. Thus we have derived an approximate solution to the inverse problem for the density and wave speed in the presence of damping.

4.1.5 Variable Density and Bulk Modulus without Damping

Let us now study the case of recovery of variation in bulk modulus. For this, we now assume that the bulk modulus κ has the following form:

$$\kappa(x) = \kappa_0(x) + \delta\kappa(x). \quad (4.42)$$

The relation $v(x) = \sqrt{\frac{\kappa(x)}{\rho(x)}}$ together with (4.2) and (4.42) gives us

$$\frac{1}{v^2(x)} \approx \frac{1}{v_0^2(x)} \left[1 + \frac{\delta\rho(x)}{\rho_0(x)} - \frac{\delta\kappa(x)}{\kappa_0(x)} \right]. \quad (4.43)$$

We use (4.43) and (4.7) in equation (4.1) to get the following expressions

$$L_0 u_I(x, x_s, \omega) = \rho_0 \frac{d}{dx} \left[\frac{1}{\rho_0} \frac{du_I}{dx} \right] + \frac{\omega^2}{v_0^2(x)} u_I = -\delta(x - x_s), \quad (4.44)$$

$$L_0 u_S(x, x_s, \omega) = \frac{\omega^2}{v_0^2(x)} \left[\frac{\delta\kappa(x)}{\kappa_0(x)} - \frac{\delta\rho(x)}{\rho_0(x)} \right] u_I + \frac{d}{dx} \left[\frac{\delta\rho(x)}{\rho_0(x)} \right] \frac{du_I}{dx}. \quad (4.45)$$

The solution of (4.45), written in terms of Green's function is

$$u_S(x_g, x_s, \omega) = - \int_0^\infty \left\{ \frac{\omega^2}{v_0^2(x)} \left[\frac{\delta\kappa(x)}{\kappa_0(x)} - \frac{\delta\rho(x)}{\rho_0(x)} \right] u_I(x, x_s, \omega) g(x_g, x, \omega) - \frac{\delta\rho(x)}{\rho_0(x)} \frac{d}{dx} \left[g(x_g, x, \omega) \frac{du_I(x, x_s, \omega)}{dx} \right] \right\} dx. \quad (4.46)$$

Using (4.11) and (4.12) in (4.46) to get the following expression

$$u_S(x_g, x_s, \omega) = \int_0^\infty F(\omega) \frac{1}{4} \left[\frac{\delta\kappa(x)}{\kappa_0(x)} + \frac{\delta\rho(x)}{\rho_0(x)} \right] \frac{A(x_g, x) A(x_s, x)}{v_0^2(x)} \cdot e^{i\omega\{\phi(x_g, x) + \phi(x_s, x)\}} dx, \quad (4.47)$$

which on inversion gives,

$$\frac{1}{4} \left[\frac{\delta\kappa(x)}{\kappa_0(x)} + \frac{\delta\rho(x)}{\rho_0(x)} \right] = \frac{v_0(y)}{\pi A(x_g, y) A(x_s, y)} \int_{-\infty}^\infty u_S(x_g, x_s, \omega) \cdot e^{-i\omega\{\phi(x_g, y) + \phi(x_s, y)\}} d\omega. \quad (4.48)$$

The reflectivity function $\beta(y)$ can be obtained by differentiating (4.48) with respect to y and dividing by -4 . Thus multiplying (44) by the factor $\frac{i\omega}{v_0(y)}$, we obtain

$$\beta(y) = \frac{-2}{\pi A(x_g, y) A(x_s, y)} \int_{-\infty}^\infty i\omega u_S(x_g, x_s, \omega) e^{-i\omega\{\phi(x_g, y) + \phi(x_s, y)\}} d\omega. \quad (4.49)$$

4.1.6 Variable Density and Bulk Modulus with Damping.

We are now in a position to introduce a more general model than those considered by Bleistein et al [13] or Zaman and Masood [87]. The one dimensional problem for variable density and bulk modulus in the presence of damping is considered. This would mean that a new term containing the damping parameter will be introduced. Now we substitute (4.7), (4.20) and (4.43) in (4.19) to get the following expressions:

$$L_0 u_I(x, x_s, \omega) = \rho_0 \frac{d}{dx} \left[\frac{1}{\rho_0} \frac{du_I}{dx} \right] + \left[\frac{\omega^2 + i\omega\gamma_0(x)}{v_0^2(x)} \right] u_I = -\delta(x - x_s), \quad (4.50)$$

$$L_0 u_S(x, x_s, \omega) = \left[-i\omega\delta\gamma(x) - \frac{\omega^2\delta\rho(x)}{\rho_0(x)} + \frac{\omega^2\delta\kappa(x)}{\kappa_0(x)} - \frac{i\omega\gamma_0(x)\delta\rho(x)}{\rho_0(x)} + \frac{i\omega\gamma_0(x)\delta\kappa(x)}{\kappa_0(x)} \right] \frac{u_I(x, x_s, \omega)g(x_g, x, \omega)}{v_0^2(x)} + \frac{d}{dx} \left[\frac{\delta\rho(x)}{\rho_0(x)} \right] \frac{du_I(x, x_s, \omega)}{dx}. \quad (4.51)$$

The solution of (4.51) in terms of Green's function is given by,

$$u_S(x_g, x_s, \omega) = - \int_0^\infty \left\{ \left[-i\omega\delta\gamma(x) - \frac{\omega^2\delta\rho(x)}{\rho_0(x)} + \frac{\omega^2\delta\kappa(x)}{\kappa_0(x)} - \frac{i\omega\gamma_0(x)\delta\rho(x)}{\rho_0(x)} + \frac{i\omega\gamma_0(x)\delta\kappa(x)}{\kappa_0(x)} \right] \frac{u_I(x, x_s, \omega)g(x, x_g, \omega)}{v_0^2(x)} - \frac{\delta\rho(x)}{\rho_0(x)} \frac{d}{dx} \left[\frac{du_I(x, x_s, \omega)}{dx} g(x_g, x, \omega) \right] \right\} dx. \quad (4.52)$$

Now using the WKBJ representations (4.24) and (4.25) in (4.52), we obtain

$$u_S(x_g, x_s, \omega) = - \int_0^\infty F(\omega) \left[\frac{i\delta\gamma(x)}{4\omega} - \frac{\delta\rho(x)}{4\rho_0(x)} - \frac{\delta\kappa(x)}{4\kappa_0(x)} + \frac{i\gamma_0(x)\delta\rho(x)}{4\omega\rho_0(x)} - \frac{i\gamma_0(x)\delta\kappa(x)}{4\omega\kappa_0(x)} \right] \frac{A(x_g, x)A(x_s, x)}{v_0^2(x)} e^{\{i\omega[\phi(x_g, x) + \phi(x_s, x)]\}} e^{\{-\frac{1}{2}[\psi(x_g, x) + \psi(x_s, x)]\}} dx. \quad (4.53)$$

Under the assumption of high frequency, the terms containing powers of $\frac{1}{\omega}$ will be compared in (4.53) to get,

$$u_S(x_g, x_s, \omega) = \int_0^\infty F(\omega) \frac{1}{4} \left[\frac{\delta\rho(x)}{\rho_0(x)} + \frac{\delta\kappa(x)}{\kappa_0(x)} \right] \frac{A(x_g, x)A(x_s, x)}{v_0^2(x)} e^{\{i\omega[\phi(x_g, x) + \phi(x_s, x)]\}} e^{\{-\frac{1}{2}[\psi(x_g, x) + \psi(x_s, x)]\}} dx. \quad (4.54)$$

This can be treated as a Fourier transform of $\frac{1}{4} \left[\frac{\delta\rho(x)}{\rho_0(x)} + \frac{\delta\kappa(x)}{\kappa_0(x)} \right]$ and inversion can be

performed in the same way as in the last remark. The result is

$$\frac{1}{4} \left[\frac{\delta\rho(x)}{\rho_0(x)} + \frac{\delta\kappa(x)}{\kappa_0(x)} \right] = \frac{v_0(y) e^{\frac{1}{2}[\psi(x_g,y)+\psi(x_s,y)]}}{\pi A(x_g,y) A(x_s,y)} \int_{-\infty}^{\infty} u_S(x_g, x_s, \omega) e^{-i\omega\{\phi(x_g,y)+\phi(x_s,y)\}} d\omega. \quad (4.55)$$

The reflectivity function $\beta(y)$ can be written as

$$\beta(y) = -\frac{2e^{\frac{1}{2}[\psi(x_g,y)+\psi(x_s,y)]}}{\pi A(x_g,y) A(x_s,y)} \int_{-\infty}^{\infty} i\omega \cdot u_S(x_g, x_s, \omega) \cdot e^{-i\omega\{\phi(x_g,y)+\phi(x_s,y)\}} d\omega. \quad (4.56)$$

If we take $\gamma = 0$ in (4.55) and (4.56), then these results agree with the previous results (4.48) and (4.49) respectively.

4.1.7 Recovery of the Damping Effect with Bulk Modulus

Consider (4.53) again and retain terms of order $\frac{1}{\omega}$ to get

$$\begin{aligned} u_S(x_g, x_s, \omega) &= -\int_0^\infty F(\omega) \frac{i}{4\omega} \left[\delta\gamma(x) + \frac{\gamma_0(x) \delta\rho(x)}{\rho_0(x)} - \frac{\gamma_0(x) \delta\kappa(x)}{\kappa_0(x)} \right] \\ &\quad \frac{A(x_g,x)A(x_s,x)}{v_0^2(x)} \cdot e^{i\omega[\phi(x_g,x)+\phi(x_s,x)]} e^{\{-\frac{1}{2}[\psi(x_g,x)+\psi(x_s,x)]\}} dx \\ &\quad + \frac{1}{4} \int_0^\infty F(\omega) \left[\frac{\delta\rho(x)}{\rho_0(x)} + \frac{\delta\kappa(x)}{\kappa_0(x)} \right] \\ &\quad \frac{A(x_g,x)A(x_s,x)}{v_0^2(x)} \cdot e^{i\omega[\phi(x_g,x)+\phi(x_s,x)]} e^{\{-\frac{1}{2}[\psi(x_g,x)+\psi(x_s,x)]\}} dx. \end{aligned} \quad (4.57)$$

Since $\frac{1}{4} \left[\frac{\delta\rho(x)}{\rho_0(x)} + \frac{\delta\kappa(x)}{\kappa_0(x)} \right]$ is known from (4.55), therefore we set

$$\begin{aligned} T(x_g, x_s, \omega) &= \frac{1}{4} \int_0^\infty F(\omega) \left[\frac{\delta\rho(x)}{\rho_0(x)} + \frac{\delta\kappa(x)}{\kappa_0(x)} \right] \\ &\quad \frac{A(x_g,x)A(x_s,x)}{v_0^2(x)} e^{i\omega[\phi(x_g,x)+\phi(x_s,x)]} \cdot e^{\{-\frac{1}{2}[\psi(x_g,x)+\psi(x_s,x)]\}} dx \end{aligned} \quad (4.58)$$

Using (4.58) in (4.57) we get the following expression

$$Z(x_g, x_s, \omega) = \int_0^\infty F(\omega) \left[\delta\gamma(x) + \gamma_0(x) \left(\frac{\delta\rho(x)}{\rho_0(x)} - \frac{\delta\kappa(x)}{\kappa_0(x)} \right) \right] \frac{A(x_g, x) A(x_s, x)}{v_0^2(x)} \cdot e^{\{i\omega[\phi(x_g, x) + \phi(x_s, x)]\}} \cdot e^{\{-\frac{1}{2}[\psi(x_g, x) + \psi(x_s, x)]\}} dx, \quad (4.59)$$

where

$$Z(x_g, x_s, \omega) = 4i\omega [u_S(x_g, x_s, \omega) - T(x_g, x_s, \omega)]. \quad (4.60)$$

Since $\delta\gamma(x) + \gamma_0(x) \left(\frac{\delta\rho(x)}{\rho_0(x)} - \frac{\delta\kappa(x)}{\kappa_0(x)} \right) = 0$ for $x < 0$, the lower integral limit can be extended to $-\infty$. This change of limit changes (4.59) into a Fourier-type integral and so it can be inverted in the same way as in earlier analysis to give,

$$\delta\gamma(y) + \gamma_0(y) \left(\frac{\delta\rho(y)}{\rho_0(y)} - \frac{\delta\kappa(y)}{\kappa_0(y)} \right) = \frac{v_0(y) e^{\{\frac{1}{2}[\psi(x_g, y) + \psi(x_s, y)]\}}}{\pi A(x_g, y) A(x_s, y)} \int_{-\infty}^\infty Z(x_g, x_s, \omega) e^{-i\omega\{\phi(x_g, y) + \phi(x_s, y)\}} d\omega, \quad (4.61)$$

or

$$\frac{\delta\gamma(y)}{\gamma_0(y)} + \frac{\delta\rho(y)}{\rho_0(y)} - \frac{\delta\kappa(y)}{\kappa_0(y)} = \frac{v_0(y) e^{\{\frac{1}{2}[\psi(x_g, y) + \psi(x_s, y)]\}}}{\gamma_0(y) \pi A(x_g, y) A(x_s, y)} \int_{-\infty}^\infty Z(x_g, x_s, \omega) e^{-i\omega\{\phi(x_g, y) + \phi(x_s, y)\}} d\omega. \quad (4.62)$$

Substituting the approximation obtained for $\delta\gamma(x) + \frac{\gamma_0(y) \delta\rho(x)}{\rho_0(x)} - \frac{\gamma_0(y) \delta\kappa(x)}{\kappa_0(x)}$ from (4.61) in (4.57) and setting

$$M(x_g, x_s, \omega) = \int_0^\infty F(\omega) \left[\delta\gamma(x) + \frac{\delta\gamma_0(x)\rho(x)}{\rho_0(x)} - \frac{\delta\gamma_0(x)\kappa(x)}{\kappa_0(x)} \right] \cdot \frac{A(x_g, x) A(x_s, x)}{v_0^2(x)} \cdot e^{\{i\omega[\phi(x_g, x) + \phi(x_s, x)]\}} \cdot e^{\{-\frac{1}{2}[\psi(x_g, x) + \psi(x_s, x)]\}} dx \quad (4.63)$$

(4.57) can be written as

$$N(x_g, x_s, \omega) = \frac{1}{4} \int_0^\infty \left[\frac{\delta\rho(x)}{\rho_0(x)} + \frac{\delta\kappa(x)}{\kappa_0(x)} \right] \cdot \frac{A(x_g, x) A(x_s, x)}{v_0^2(x)} e^{\{i\omega[\phi(x_g, x) + \phi(x_s, x)]\}} \cdot e^{\{-\frac{1}{2}[\psi(x_g, x) + \psi(x_s, x)]\}} dx, \quad (4.64)$$

where

$$N(x_g, x_s, \omega) = \left[u_S(x_g, x_s, \omega) + \frac{i}{4\omega} M(x_g, x_s, \omega) \right]. \quad (4.65)$$

The expression (4.64) can be inverted to get the improved value of $\frac{1}{4} \left[\frac{\delta\rho(y)}{\rho_0(y)} + \frac{\delta\kappa(y)}{\kappa_0(y)} \right]$ as follows

$$\frac{1}{4} \left[\frac{\delta\rho(y)}{\rho_0(y)} + \frac{\delta\kappa(y)}{\kappa_0(y)} \right] = \frac{v_0(y) e^{\{\frac{1}{2}[\psi(x_g, y) + \psi(x_s, y)]\}}}{\pi A(x_g, y) A(x_s, y)} \int_\infty^{-\infty} N(x_g, x_s, \omega) \cdot e^{-i\omega[\phi(x_g, x) + \phi(x_s, x)]} d\omega. \quad (4.66)$$

Thus we have derived an approximate solution to the inverse problem for the recovery of bulk modulus and damping of the medium. The process presented in this section from (4.57) to (4.66) describe an iterative procedure to get increasingly better approximations.

4.1.8 Conclusions

We have derived approximate solutions to the inverse problem of finding the velocity and damping from the observed wave-field. The recovery of variations in velocity, bulk modulus and damping in the presence of variable density is considered. An iterative procedure to improve velocity and damping profiles is also presented. These results are

an improvement to earlier known results as they involve variations of bulk modulus as well as the damping parameter. We expect that these results provide a more accurate map of the interior of the earth.

4.2 Determination of Inversion Velocity/Density Variation in Constant Background Medium in Higher Dimensions

4.2.1 Inversion in Higher Dimensions

In this section we study the three-dimensional wave equation. Our aim is to derive an inversion formula assuming a constant-background wave speed and density. To simplify the problem, we introduce the added restriction that the source and receiver be located at the same place. The model studied here is different from earlier models in that it is based upon a three dimensional wave equation.

4.2.2 The Scattering Problem

We introduce a three-dimensional coordinate system, (x_1, x_2, x_3) , with x_3 positive in the downward direction. The propagation speed and density is assumed to be known in some portion of the region $x_3 \geq 0$ and unknown outside the portion of the region. We consider a bandlimited impulse point source at x_s and response of this source will be observed at one or more geophone x_g . The objective is to obtain information about the propagation speed, $v(x)$, variation of density, $\rho(x)$, from observation of the wavefield. For a linear isotropic acoustic medium, the wave equation in frequency

domain [19, 83, 84] at a point x_s is

$$Lu(x, x_s, \omega) = \nabla \cdot \left[\frac{1}{\rho} \nabla u(x, x_s, \omega) \right] + \frac{\omega^2}{\kappa} u(x, x_s, \omega) = -\delta(x - x_s), \quad (4.67)$$

where ρ and κ are the density and bulk module of the medium respectively. Since $\kappa = v^2(x) \rho$, then $u(x, x_s, \omega)$ be a solution of the equation

$$Lu(x, x_s, \omega) = \nabla \cdot \left[\frac{1}{\rho} \nabla u(x, x_s, \omega) \right] + \frac{\omega^2}{v^2(x) \rho} u(x, x_s, \omega) = -\delta(x - x_s), \quad (4.68)$$

where the function $u(x, x_s, \omega)$ satisfies the Sommerfeld radiation condition

$$ru \quad \text{bounded}, \quad r \left(\frac{\partial u}{\partial r} - \frac{i\omega}{v} u \right) \rightarrow 0 \quad \text{as } r \rightarrow \infty, \quad r = |x|, \quad (4.69)$$

We introduce variations in density and propagation speed to have parallel form, that is the background plus the perturbation, such that these profiles have the following representations

$$\rho(x) = \rho_0(x) + \delta\rho(x),$$

and

$$v(x) = v_0(x) + \delta v(x),$$

with

$$\frac{1}{\rho(x)} \approx \frac{1}{\rho_0(x)} \left[1 - \frac{\delta\rho(x)}{\rho_0(x)} \right].$$

$$\frac{1}{v^2(x)} \approx \frac{1}{v_0^2(x)} \left[1 - \frac{2\delta v(x)}{v_0(x)} \right].$$

respectively. These representations are substituted into (4.68), and only linear terms in $\delta v(x)$ and $\delta\rho(x)$ are retained. After some algebra, the resulting equation takes the form,

$$\begin{aligned} L_0 u &= \nabla \cdot \left[\frac{1}{\rho_0} \nabla u \right] + \frac{\omega^2}{v_0^2(x) \rho_0} u = -\delta(x - x_s) + \frac{2\omega^2}{\rho_0 v_0^2(x)} \frac{\delta v(x)}{v_0(x)} u \\ &+ \frac{\omega^2}{\rho_0^2(x)} \frac{\delta\rho(x)}{v_0^2(x)} + \nabla \cdot \left[\frac{\delta\rho(x)}{\rho_0^2(x)} \right] \nabla u. \end{aligned} \quad (4.70)$$

The wavefield can be decomposed into a reference field $u_I(x, x_s, \omega)$ and a scattered field $u_S(x, x_s, \omega)$ as in equation (4.7)

Now we substitute (4.7) into (4.70) and require that $u_I(x, x_s, \omega)$ is a solution of the unperturbed equation,

$$L_0 u_I(x, x_s, \omega) = \nabla \cdot \left[\frac{1}{\rho_0(x)} \nabla u \right] + \frac{\omega^2}{v_0^2(x) \rho_0(x)} u = -F(\omega) \delta(x - x_s), \quad (4.71)$$

subject to the radiation condition (4.69). It now follows that $u_S(x, x_s, \omega)$ satisfies the following equation:

$$\begin{aligned}
L_0 u_S(x, x_s, \omega) &= \left(\frac{2\omega^2}{\rho_0 v_0^2(x)} \frac{\delta v(x)}{v_0(x)} + \frac{\omega^2}{\rho_0^2(x)} \frac{\delta \rho(x)}{v_0^2(x)} \right) (u_I + u_S) \\
&\quad + \nabla \left[\frac{\delta \rho(x)}{\rho_0^2(x)} \right] \nabla (u_I(x, x_s, \omega) + u_S(x_g, x_s, \omega)).
\end{aligned} \tag{4.72}$$

The solution of (4.72), written in terms of Green's function becomes,

$$\begin{aligned}
u_S(x_g, x_s, \omega) &= - \int_D \left\{ \left(\frac{2\omega^2}{\rho_0 v_0^2(x)} \frac{\delta v(x)}{v_0(x)} + \frac{\omega^2}{\rho_0^2(x)} \frac{\delta \rho(x)}{v_0^2(x)} \right) (u_I(x, x_s, \omega) + u_S(x_g, x_s, \omega)) g(x, x_g, \omega) \right. \\
&\quad \left. + \nabla \left[\frac{\delta \rho(x)}{\rho_0^2(x)} \right] \nabla (u_I(x, x_s, \omega) + u_S(x_g, x_s, \omega)) g(x, x_g, \omega) \right\} dx,
\end{aligned} \tag{4.73}$$

where, the domain D of integration must contain the support of $\delta v(x)$ and $\delta \rho(x)$ assumed to be some finite subdomain of $x_3 > 0$. Therefore, for convenience, take D to be the semi-infinite domain, $x_3 > 0$.

4.2.3 The Born Approximation In Higher Dimensions

As the one-dimension case for small perturbations, we would like to argue that the scattered field $u_S(x, x_s, \omega)$ is small, so that we can neglect the product of $\delta v(x)$ and $\delta \rho(x)$ with $u_S(x, x_s, \omega)$ as compared with that of $u_I(x, x_s, \omega)$ under the integral sign (4.73). Unfortunately, this is not always true in three dimensions if the reflected field is observed beyond the critical angle of reflection. In this case the reflection coefficient has unit magnitude and the scattered field is comparable with the incident field, at least in a subdomain of D . Thus we cannot neglect the former product as compared with the latter in three dimensions, as it was in the one-dimensional problem. However, for near zero-offset or backscattered observations, it is true that small $\delta v(x)$ and $\delta \rho(x)$

imply small $u_S(x, x_S, \omega)$. Since this is the problem to be considered, we proceed to make the Born approximation to obtain the integral equation

$$u_S(x_g, x_s, \omega) = - \int_D \left\{ \left(\frac{2\omega^2}{\rho_0 v_0^2(x)} \frac{\delta v(x)}{v_0(x)} + \frac{\omega^2}{\rho_0^2(x)} \frac{\delta \rho(x)}{v_0^2(x)} \right) u_I(x, x_S, \omega) g(x, x_g, \omega) \right. \\ \left. + \nabla \left[\left(\frac{\delta \rho(x)}{\rho_0^2(x)} \right) \nabla u_I(x, x_s, \omega) g(x, x_g, \omega) \right] \right\} dx. \quad (4.74)$$

Leading to

$$u_S(x_g, x_s, \omega) = - \int_D \left\{ \left(\frac{2\omega^2}{\rho_0 v_0^2(x)} \frac{\delta v(x)}{v_0(x)} + \frac{\omega^2}{\rho_0^2(x)} \frac{\delta \rho(x)}{v_0^2(x)} \right) u_I(x, x_S, \omega) g(x, x_g, \omega) \right. \\ \left. - \frac{\delta \rho(x)}{\rho_0^2(x)} \nabla [\nabla u_I(x, x_s, \omega) g(x, x_g, \omega)] \right\} dx. \quad (4.75)$$

In the second form of the result, we have integrated by parts to eliminate the differentiation of $\delta\rho$.

4.2.4 The Constant Background Zero-Offset Equation

We shall derive the expression for the Green's function in the frequency domain with constant density. In this case, equation (5.5) takes the form

$$L_0 g(x, x_g, \omega) = \frac{1}{\rho_0} \nabla^2 g + \frac{\omega^2}{v_0^2(x) \rho_0} g = -\delta(x - x_s). \quad (4.76)$$

The simplest problem to deal with is one in which the source and receiver are coincident, that is $x_s = x_g$, one a flat horizontal surface, $x_3 = 0$, and the background speed/density are constant $v_0(x) = v_0$. In this case it is convenient to introduce

$$\zeta = (\zeta_1, \zeta_2, 0) = x_s = x_g, \quad (4.77)$$

Following [19, 35, 49], the Green's function takes the form,

$$g(\zeta, x, \omega) = \frac{\rho_0 \exp\left(\frac{i\omega}{v_0} r\right)}{4\pi r}, \quad r = |x - \zeta|, \quad (4.78)$$

and the incident field [13] is,

$$u_I(x, \zeta, \omega) = F(\omega) \frac{\rho_0 \exp\left(\frac{i\omega}{v_0} r\right)}{4\pi r}, \quad r = |x - \zeta|. \quad (4.79)$$

Therefore we can write (4.75) as,

$$u_S(x_g, x_s, \omega) = - \int_D \left\{ F(\omega) \left(\frac{2\omega^2}{\rho_0 v_0^2} \frac{\delta v(x)}{v_0} + \frac{\omega^2}{\rho_0^2} \frac{\delta \rho(x)}{v_0^2} \right) \frac{\rho_0^2 e^{2i\frac{\omega}{v_0} r}}{(4\pi)^2 r^2} - \frac{\delta \rho(x)}{\rho_0^2} \frac{2\rho_0^2 \left(\frac{i\omega}{v_0}\right)^2 e^{2i\frac{\omega}{v_0} r}}{(4\pi)^2 r^2} \right\} dx, \quad (4.80)$$

after some algebra, the result takes the form,

$$u_S(x_g, x_s, \omega) = - \frac{\omega^2 F(\omega)}{(4\pi v_0)^2} \int_D \left(\frac{2\rho_0 \delta v(x)}{v_0} + 3\delta \rho(x) \right) \frac{e^{2i\frac{\omega}{v_0} r}}{r^2} dx, \quad (4.81)$$

where $\alpha^{vd}(x)$ represents the "perturbation" in wave speed and density [13, 64, 65]. Let us introduce

$$\alpha^{vd}(x) = \frac{2\rho_0 \delta v(x)}{v_0} + 3\delta \rho(x). \quad (4.82)$$

4.2.5 The One Dimensional Variation in $\alpha^{vd}(x)$

Suppose that data is collected for one zero-offset experiment. In this case, we seek an inversion only for $\alpha^{vd}(x_3)$, i.e. the wave speed and density vary from constant

backgrounds as functions of depth only. Furthermore, the coordinates of that single experiment might as well be taken to be $(0, 0, 0)$, so that (4.81) becomes

$$u_S(0, 0, 0, \omega) = \frac{\omega^2 F(\omega)}{(4\pi v_0)^2} \int_{x_3 > 0} \alpha^{vd}(x_3) \frac{\exp\left(\frac{2i\omega}{v_0} r\right)}{r^2} d^3x, \quad r = \sqrt{x_1^2 + x_2^2 + x_3^2}. \quad (4.83)$$

The dependence of the integral on x_1 and x_2 is only through r . Therefore, integration in these variables can be carried out by introducing polar coordinates (σ, θ) in place of (x_1, x_2) . Thus, the integral equation(4.83) takes the form

$$\begin{aligned} u_S(0, 0, 0, \omega) &= F(\omega) \frac{\omega^2}{(4\pi v_0)^2} \int_{x_3 > 0} dx_3 \alpha^{vd}(x_3) \int_0^{2\pi} d\theta \int_0^\infty d\sigma \cdot \sigma \frac{\exp\left(\frac{2i\omega}{v_0} r\right)}{r^2}, \\ r &= \sqrt{\sigma^2 + x_3^2}. \end{aligned} \quad (4.84)$$

Integrating with respect to θ the above equation yields a multiplier of 2π , and reduces the integral to

$$u_S(0, 0, 0, \omega) = F(\omega) \frac{\omega^2}{8\pi v_0^2} \int_{x_3 > 0} dx_3 \alpha^{vd}(x_3) \int_0^\infty d\sigma \cdot \sigma \frac{\exp\left(\frac{2i\omega}{v_0} r\right)}{r^2}. \quad (4.85)$$

While it may not be possible to evaluate the integral over σ exactly, it is possible to find an approximation to this integral that is consistent with the high frequency assumption. To do this we require that $\text{Im } \omega > 0$, and integrate by parts with respect to σ . The term of integrand is $\exp\left(\frac{2i\omega}{v_0} r\right) \frac{\sigma}{r}$, and noting that $\frac{d\sigma}{dr} = \frac{\sigma}{r}$, we

obtain

$$\begin{aligned}
u_S(0, 0, 0, \omega) &= F(\omega) \frac{\omega^2}{8\pi v_0^2} \int_{x_3 > 0} \alpha^{vd}(x_3) \frac{v_0}{2i\omega r} \exp\left(\frac{2i\omega}{v_0} r\right) \Big|_0^\infty dx_3 \quad (4.86) \\
&+ F(\omega) \frac{\omega^2}{8\pi v_0^2} \int_{x_3 > 0} \alpha^{vd}(x_3) \frac{v_0}{2i\omega r^2} \exp\left(\frac{2i\omega}{v_0} r\right) \sigma d\sigma dx_3.
\end{aligned}$$

Keeping only the leading terms in ω and setting $F(\omega) = 1$, we obtain

$$u_S(0, 0, 0, \omega) \sim \frac{i\omega}{16v_0\pi} \int_{x_3 > 0} \frac{\alpha^{vd}(x_3)}{x_3} \exp\left(\frac{2i\omega}{v_0} x_3\right) dx_3.$$

This equation represents the observed data as some multiple of the Fourier transform of

$$\alpha^{vd}(x_3) \frac{1}{x_3},$$

and the solution is obtained by Fourier inversion so the transform variable is $\frac{2\omega}{v_0}$, here the Fourier inversion formula must be with respect to this transform variable. Thus

$$\alpha^{vd}(x_3) = 16 x_3 \int_{-\infty}^{\infty} \frac{u_S(0, 0, 0, \omega)}{i\omega} \exp\left(\frac{-2i\omega}{v_0} x_3\right) d\omega, \quad (4.87)$$

which can be written in a more suitable form as

$$\left[\frac{2\delta v(x_3)}{v_0} + \frac{3\delta\rho(x_3)}{\rho_0} \right] = \frac{16 x_3}{\rho_0} \int_{-\infty}^{\infty} \frac{u_S(0, 0, 0, \omega)}{i\omega} \exp\left(\frac{-2i\omega}{v_0} x_3\right) d\omega. \quad (4.88)$$

4.2.6 Conclusions

We have derived approximate solutions with variation in speed and density and consider the case of constant background. The medium has variability from the back-

ground wave speed and density in the vertical direction only. In this part of the chapter we have studied a three dimensional model in which we recover the variation in speed and density variations written an inclusion or inhomogeneity with background medium having constant properties. The variability in speed and density is assumed to be in the vertical direction only. Further studies may consider a more general three dimensional variability with damping in the medium.

Chapter 5

Seismic Inverse Problem With Bessel Operator

5.1 An Inverse Problem Involving Bessel Equation

Abstract

An inverse problem arising from inclusions or inhomogeneties having radial or cylindrical symmetry has been considered. The field equation, which involves Bessel operator is used to find the scattered field under the Born approximation assumption. The constant and variable background medium properties have been assumed. In the later case WKB approximation to the Green's function is used to arrive at the inversion formula.

5.1.1 Introduction

In several situations of interest, the wave equation involving the Bessel operator arises [62, 86]. The problems of wave propagation excited by a line source [3], scattering of waves by cylindrical objects [52], some seismic inversion problem [57], ultrasound reflective imaging in medical application [31] and various methods of non destructive testing [44, 60] require the field equation to be written in terms of cylindrical coordinates. Colton [29 – 33] studied important features of an inverse problem involving the Bessel operator arising from leukemia in human body. Some interesting situations also arise in detection and monitoring of tumor in the human body. In all such cases, one or more signals are introduced in the region of interest and responses from the inhomogeneities or in the region are recorded. These recorded signals constitute the total field consisting of the incident and the scattered field. The direct scattering problem then enables us to obtain the integral equation in terms of variation of the field parameters from the background medium parameters [4 – 8, 13, 16, 43, 55]. This non-linear problem is linearized using the Born approximation [14, 48, 69]. In case of constant background medium, Green's function for the Bessel operator [10, 59, 78] can be used to enable Fourier inversion to be carried out. However, in case of variable background medium, the WKB approximation of the Green's function [10, 15, 19, 24, 38, 62, 74] is utilized to derive the inversion formula. This approximation is based upon the high frequency nature of the wave is suitable for seismic problems [13]. Our method is thus based upon analytic and asymptotic techniques. Some numerical procedures have been described by Zhang and Shum [91] who using fast Hankel transform for the direct numerical evaluation, while Jacobs and Stolt [51] used Gelfand-Levitan algorithm for seismic inversion. In these cases, one or more signals are introduced in

the region of interest and responses from the irregularities or inclusions in the interior are recorded [9, 46]. Under the assumption that the background medium has constant material properties variations from the material parameters are recorded as variation. Under the assumption of constant speed, Claerbout [23] presented an approximation method to the inverse problem for velocity inversion.

In subsection (2) of this section (1), we set up the direct problem in terms of the Bessel operator and derive an inversion formula for velocity variation. This formula is implemented in subsection (3) of this section (1), through finding the Green's function for a constant background medium case. Asymptotic approximation to the Green's function is obtained to write the solution of the inverse problem in subsection (4). An importance to this estimate is introduced in subsection (5). We then extend our model to that having variable background medium parameters in section (2). We use the WKB approximation to the Green's function to find the velocity variation and the reflectivity function for this important case. The results presented in subsection (1) and (2) are interesting in that seismic inversion problem has not been well studied for radially symmetric inclusions.

5.1.2 The Inverse Problem

Consider the wave equation with Bessel operator

$$\frac{\partial^2 u}{\partial x^2} + \frac{1}{x} \frac{\partial u}{\partial x} - \frac{1}{v^2(x)} \frac{\partial^2 u}{\partial t^2} = -\delta(x) \delta(t), \quad (5.1)$$

with Sommerfield radiation condition

$$\frac{du}{dx} - i \frac{\omega}{v(x)} u \rightarrow 0, \quad \text{as } x \rightarrow \infty. \quad (5.2)$$

We take the Fourier transform of (5.1) with respect to time, to obtain

$$Lu = \frac{d^2u}{dx^2} + \frac{1}{x} \frac{du}{dx} + \frac{\omega^2}{v^2(x)} u = -\delta(x), \quad (5.3)$$

where the background speed $c(x)$ is perturbed as follows

$$\frac{1}{v^2(x)} \approx \frac{1}{c^2(x)} [1 + \alpha(x)], \quad (5.4)$$

Using (5.4) with the term involving $\alpha(x)$ in (5.3), we obtain

$$L_0 u(x, \omega) = \frac{d^2u(x, \omega)}{dx^2} + \frac{1}{x} \frac{du(x, \omega)}{dx} + \frac{\omega^2}{c^2(x)} u(x, \omega) = -\delta(x) - \frac{\omega^2}{c^2(x)} \alpha(x) u(x, \omega). \quad (5.5)$$

The total field $u(x, x_s, \omega)$ can be separated into the incident part $u_I(x, x_s, \omega)$ in the absence of perturbation and $u_S(x, x_s, \omega)$ in the presence of perturbation. Thus we set

$$u(x, \omega) = u_I(x, \omega) + u_S(x, \omega). \quad (5.6)$$

Substituting (5.6) in (5.5) we reach at the following equations requiring that $u_I(x, \omega)$ be a solution of the following problem:

$$L_0 u_I(x, \omega) = \frac{d^2u_I(x, \omega)}{dx^2} + \frac{1}{x} \frac{du_I(x, \omega)}{dx} + \frac{\omega^2}{c^2(x)} u_I(x, \omega) = -\delta(x), \quad (5.7)$$

$$L_0 u_S(x, \omega) = -\frac{\omega^2}{c^2(x)} \alpha(x) [u_I(x, \omega) + u_S(x, \omega)]. \quad (5.8)$$

We now find Green's function of (5.8) satisfying,

$$L_0 g(x, \xi, \omega) = \frac{d^2 g(x, \xi, \omega)}{dx^2} + \frac{1}{x} \frac{dg(x, \xi, \omega)}{dx} + \frac{\omega^2}{c^2(x)} g(x, \xi, \omega) = -\delta(x - \xi). \quad (5.9)$$

The solution of (5.8), in terms of Green's function given by(5.9) can be written as

$$u_S(\xi, \omega) = \omega^2 \int_0^\infty \frac{\alpha(x)}{c^2(x)} [u_I(x, \omega) + u_S(x, \omega)] g(x, \xi, \omega) dx. \quad (5.10)$$

The Born approximation of (5.10) yields

$$u_S(\xi, \omega) = \omega^2 \int_0^\infty \frac{\alpha(x)}{c^2(x)} u_I(x, \omega) g(x, \xi, \omega) dx. \quad (5.11)$$

5.1.3 The Case of Constant Background

We apply the Born approximation integral equation (5.11) for the scattered field to the problem of constant background speed case while assuming

$$c(x) = c_0,$$

equation (5.9) can then be written as (Stakgold [77 – 78])

$$\begin{aligned} -\left(xg'\right)' - \frac{\omega^2}{c_0^2}xg &= \delta(x - \xi), \text{ together with} \\ (xg)'_{x=0^+} &= 0, \quad \int_0^\infty x|g|^2 dx < 0. \end{aligned} \tag{5.12}$$

Following Stakgold [77 – 78], we find that

$$\begin{aligned} g(x, \xi, \omega) &= \frac{\pi i}{2} J_0\left(\frac{\omega}{c_0}x_{<}\right) H_0^{(1)}\left(\frac{\omega}{c_0}x_{>}\right), \\ \text{where } x_{<} &= \min(x, \xi), \quad x_{>} = \max(x, \xi). \end{aligned} \tag{5.13}$$

We assume that the incident wave has the form (Jones [52]),

$$u_I(x, \omega) = -\exp\left(i\frac{\omega}{c_0}\cos\theta x\right),$$

where $J_0(\lambda x)$ and $H_0^{(1)}(\lambda x)$ denote the Bessel function and the Hankel function of first kind of order zero, respectively. If we are measuring the scattered field in the region $x < \xi$, $g(x, \xi, \omega)$ [49] can be written as,

$$g(x, \xi, \omega) = \frac{\pi i}{2} J_0\left(\frac{\omega}{c_0}x\right) H_0^{(1)}\left(\frac{\omega}{c_0}\xi\right). \tag{5.14}$$

Substituting (5.14) into (5.11), we get

$$u_S(\xi, \omega) = -\frac{i\pi\omega^2}{2} \int_0^\infty \frac{\alpha(x)}{c_0^2} J_0\left(\frac{\omega}{c_0}x\right) H_0^{(1)}\left(\frac{\omega}{c_0}\xi\right) \exp\left(i\frac{\omega}{c_0}\cos\theta x\right) dx. \quad (5.15)$$

This gives the scattered field due to the perturbation $\alpha(x)$

5.1.4 Asymptotic Approximation to the Bessel-Hankel Functions

We solve (5.15) using the asymptotic approximation of $J_0\left(\frac{\omega}{c_0}x\right)$ and $H_0^{(1)}\left(\frac{\omega}{c_0}x\right)$ for $x \rightarrow \infty$ given by [3, 15, 20, 31, 35, 86]

$$J_0(x) \approx \sqrt{\frac{2}{\pi x}} \cos\left[x - \frac{\pi}{4}\right], \quad x \rightarrow \infty. \quad (5.16)$$

$$H_0^{(1)}(x) \approx \sqrt{\frac{2}{\pi\xi}} \exp\left[i\left(\xi - \frac{\pi}{4}\right)\right], \quad x \rightarrow \infty. \quad (5.17)$$

Following Fuchs and Muller [44], we rewrite equation (5.16) as the sum of two exponential terms,

$$J_0(x) = \frac{1}{\sqrt{2\pi x}} \left\{ \exp\left[i\left(x - \frac{\pi}{4}\right)\right] + \exp\left[-i\left(x - \frac{\pi}{4}\right)\right] \right\}. \quad (5.18)$$

Substituting the asymptotic expressions equation (5.17) and (5.18) in equation (5.15), we obtain,

$$\begin{aligned} u_S(\xi, \omega) \sim & -\frac{\omega}{2} \int_0^\infty \frac{\alpha(x)}{c_0} \sqrt{\frac{1}{\xi x}} \exp\left[i\frac{\omega}{c_0}(1 + \cos\theta)x\right] \exp\left[i\left(\frac{\omega}{c_0}\xi\right)\right] dx \\ & -\frac{\omega}{2} \int_0^\infty \frac{\alpha(x)}{c_0} \sqrt{\frac{1}{\xi x}} \exp\left[\left(-i\frac{\omega}{c_0}(1 - \cos\theta)x\right)\right] \cdot \exp\left[i\left(\frac{\omega}{c_0}\xi\right)\right] dx. \end{aligned} \quad (5.19)$$

The first exponential term in (5.19) represents the waves with positive wave number $k \left(= \frac{\omega}{c_0} \right)$, corresponding to waves propagating in the positive direction (away from the source), whereas the second term describes waves traveling towards the source. As the scattered field consisting of waves traveling towards the source is assumed to be small [81 – 82], we neglect the second term. This gives

$$\begin{aligned} u_S(\xi, \omega) &= \frac{-\omega}{\sqrt{\xi}} \int_0^{\infty} \frac{\alpha(x)}{2c_0\sqrt{x}} \cdot e^{i\omega(\eta x + \xi)/c_0} dx, \\ \eta &= 1 + \cos \theta. \end{aligned} \quad (5.20)$$

Since $\alpha(x) = 0$ for $x < 0$, the lower limit in equation (5.20) can be extended to $-\infty$ so that it becomes a Fourier type integral. The inversion gives,

$$\frac{\alpha(x)}{\sqrt{x}} = \frac{-\eta\sqrt{\xi}}{\pi} \int_{-\infty}^{\infty} \frac{u_S(\xi, \omega)}{\omega} e^{-i\omega(\eta x + \xi)/c_0} d\omega, \quad (5.21)$$

which can be written in a more suitable form as

$$\alpha(x) = \frac{-\eta\sqrt{x\xi}}{\pi} \int_{-\infty}^{\infty} \frac{u_S(\xi, \omega)}{\omega} e^{-i\omega(\eta x + \xi)/c_0} d\omega. \quad (5.22)$$

5.1.5 More Accurate Reconstruction of Variation in velocity

Let us introduce a new independent variable τ defined by

$$\tau = \int_0^x \frac{dt}{c(t)}, \quad (5.23)$$

and set

$$u(x(\tau), \omega) = \Phi(\tau, \omega), \quad (5.24)$$

$$c(x(\tau)) = c(\tau). \quad (5.25)$$

In terms of new variables introduced in (5.23) – (5.25), the equation (5.3) takes the form

$$\frac{d^2\Phi(\tau, \omega)}{d\tau^2} + \Gamma_1(\tau) \frac{d\Phi(\tau, \omega)}{d\tau} - \Gamma_2(\tau) \frac{d\Phi(\tau, \omega)}{d\tau} + \omega^2\Phi(\tau, \omega) = -c(0)\delta(\tau).$$

We can write the above equation as

$$\frac{d^2\Phi(\tau, \omega)}{d\tau^2} + [\Lambda(\tau)] \frac{d\Phi(\tau, \omega)}{d\tau} + \omega^2\Phi(\tau, \omega) = -c(0)\delta(\tau), \quad (5.26)$$

where

$$\begin{aligned} \Gamma_1(\tau) &= \frac{c(\tau)}{\int_0^\tau c(t) dt}, \quad \Gamma_2(\tau) = \frac{c'(\tau)}{c(\tau)}, \\ \Lambda(\tau) &= \Gamma_1(\tau) - \Gamma_2(\tau). \end{aligned}$$

Setting

$$\Phi(\tau, \omega) = \Phi_I(\tau, \omega) + \Phi_S(\tau, \omega), \quad (5.27)$$

we get

$$L_0\Phi_I(\tau, \omega) = \frac{d^2\Phi_I(\tau, \omega)}{d\tau^2} + \omega^2\Phi_I(\tau, \omega) = -c(0)\delta(\tau), \quad (5.28)$$

$$L_0\Phi_S(\tau, \omega) = -[\Lambda(\tau)] \frac{d}{d\tau}(\Phi_I(\tau, \omega) + \Phi_S(\tau, \omega)). \quad (5.29)$$

The solution of (5.29), written in terms of Green's function using the Born approximation, is given by

$$u_S(\xi, \omega) = - \int_0^\infty \Lambda(\tau) \frac{d}{d\tau} (\Phi_I(\tau, \omega)) g(\tau, \xi, \omega) d\tau, \quad (5.30)$$

where

$$g(\tau, \xi, \omega) = \frac{c(0)}{2i\omega} \exp(i\omega(\tau - \xi)), \quad (5.31)$$

$$u_I(\tau, \omega) = - \exp(i\omega \cos \theta \tau). \quad (5.32)$$

Substituting above equations in (5.30), we obtain

$$\Phi_S(\xi, \omega) = \frac{i\omega c(0)}{2} \int_0^\infty \Lambda(\tau) e^{i\omega(\eta\tau - \xi)} d\tau. \quad (5.33)$$

Since $\Lambda(\tau) = 0$ for $x < 0$, (5.33) is a Fourier type integral because lower limit can be extended to $-\infty$. Hence inversion integral gives,

$$\Lambda(\tau) = -\frac{\eta i}{\pi c(0)} \int_{-\infty}^\infty \frac{\Phi_S(\xi, \omega)}{\omega} e^{-i\omega(\eta\tau - \xi)} d\omega. \quad (5.34)$$

As remarked by Bleistein [10], the expression of $\Lambda(\tau)$ given by (5.34) gives a better reconstruction of the velocity profiles as compared to that given by equation (5.22).

5.2 High Frequency Seismic Inversion Involving Bessel Equation

5.2.1 The Case of High Frequency Inversion in a Variable Background Medium.

We consider the inverse scattering integral equation (5.11) for a high frequency [16, 90]. Thus we replace the Green's function by its WKB approximation [10, 48, 62]. The propose of this technique is to derive the leading-order WKB approximation of the Green's function, $g(x, x_g, \omega)$, that is a solution the equation (5.7) with radiation condition [5.2]. This means that we are free to use a high frequency approximation of the Green's function [48, 62]. Therefore, the approximate Green's function may then be written as

$$\begin{aligned} g(x, x_g, \omega) &\sim \frac{A}{\sqrt{x}} e^{\pm i\omega\varphi(x, x_g)}, \\ \varphi(x, \xi) &= \int_{x_<}^{x_>} \frac{1}{v_0(z)} dz, \\ x_> &= \max(x, \xi) \\ x_< &= \min(x, \xi). \end{aligned} \tag{5.35}$$

Because this discussion deals with high frequency solutions [13, 47, 52],

$$\begin{aligned} u_I(x, x_S, \omega) &= -F(\omega) \exp(i\omega \cos \theta \varphi(x, x_S)), \\ \varphi(x, x_S) &= \int_{x_S}^x \frac{1}{v_0(z)} dz, \end{aligned} \quad (5.36)$$

above equation, will be taken to be the representation of the incident field. To find A the normal component of the particle velocity at the surface must vanish [3, 13, 52].

This results gives,

$$\left. \frac{\partial u}{\partial x} \right|_{x=a} = \left[\frac{\partial g}{\partial x} + \frac{\partial u_I}{\partial x} \right]_{x=a} = 0, \quad (5.37)$$

Substituting for the Green's function and the incident field, yields in the case of high frequency [24]:

$$A = \frac{(a^{-1}c(a) + 2i\omega)}{2i\sqrt{a}\omega \cos \theta} e^{i\omega(\varphi(a, x_g) - \cos \theta \varphi(a, x_S))} \sim \sigma(a). \quad (5.38)$$

Substituting equations (5.35) and (5.36) in equation (5.11) yields

$$u_S(x_g, x_S, \omega) = -\omega^2 \int_0^\infty F(\omega) \frac{\sigma(a) \alpha(x)}{\sqrt{x} v_0^2(x)} e^{i\omega(\varphi(x, x_g) + \cos \theta \varphi(x, x_S))} dx. \quad (5.39)$$

The form of the amplitude factor in this more general Fourier inversion remains to be deduced and will be left as an unknown for the time being. The general form of the inversion can be written as

$$\frac{\alpha(y)}{\sqrt{y}} = \int_{-\infty}^\infty b(y, \omega) u_S(x_g, x_S, \omega) e^{-i\omega(\varphi(y, x_g) + \cos \theta \varphi(y, x_S))} d\omega, \quad (5.40)$$

with the amplitude factor $b(y, \omega)$ remaining to be determined. Substituting (5.39) for $u_S(x_g, x_S, \omega)$ into this equation, gives

$$\frac{\alpha(y)}{\sqrt{y}} = - \int_0^\infty dx \frac{\sigma(a) \alpha(x)}{\sqrt{x} v_0^2(x)} \int_{-\infty}^\infty d\omega F(\omega) b(y, \omega) e^{i\omega(\varphi(x,y) + \cos\theta\varphi(x,y))}. \quad (5.41)$$

The equation (5.41) is of the form

$$\frac{\alpha(y)}{\sqrt{y}} = \int_0^\infty \frac{\alpha(x)}{\sqrt{x}} f(x, y) dx, \quad (5.42)$$

where

$$\begin{aligned} f(x, y) &= - \frac{\sigma(a)}{v_0^2(x)} \int_{-\infty}^\infty F(\omega) b(y, \omega) e^{i\omega(\varphi(x,y) + \cos\theta\varphi(x,y))} d\omega, \\ &= - \frac{\sigma(a)}{v_0^2(x)} \int_{-\infty}^\infty F(\omega) b(y, \omega) e^{i\omega\varphi(x,y)[1 + \cos\theta]} d\omega. \end{aligned} \quad (5.43)$$

For x and y greater than zero, the equation (5.42) will be satisfied asymptotically if we set

$$\begin{aligned} f(x, y) &= \delta_B(x - y) \\ &= \frac{1}{\pi} \int_{-\infty}^\infty F(\omega) e^{i\omega\varphi(x,y)} d\omega. \end{aligned} \quad (5.44)$$

In (5.43), it is possible to construct that result with a $b(y, \omega)$ that is independent of ω ; that is,

$$b(y, \omega) = b(y).$$

Since the angle of incident is zero for normal incident and $\frac{\pi}{2}$ for tangential incident so

$$0 \leq \cos \theta < 1. \quad (5.45)$$

Let us consider the normal incident only and put $\theta = 0$.

Then (5.43) gives

$$\begin{aligned} f(x, y) &= -\frac{\sigma(a) b(y)}{v_0^2(x)} \int_{-\infty}^{\infty} F(\omega) e^{2i\omega(\varphi(x, y))} d\omega, \\ &= -\frac{\pi \sigma(a) b(y)}{c^2(x)} \delta_B(\varphi(x, y)). \end{aligned} \quad (5.46)$$

Applying the property

$$|f'(y)| \delta(f(x)) = \delta(x - y),$$

we get

$$\begin{aligned} f(x, y) &= -\frac{\pi \sigma(a) b(y)}{v_0(y)} \delta_B(x - y) \\ &= -\frac{\pi \sigma(a) b(y)}{v_0(y)} \delta_B(x - y). \end{aligned} \quad (5.47)$$

The choice of $b(y)$ needed to make (5.44) true is apparent from (5.47), because

$$b(y) = -\frac{v_0(y)}{\pi \sigma(a)}, \quad (5.48)$$

and the inversion formula, (5.40) becomes

$$\frac{\alpha(y)}{\sqrt{y}} = -\frac{v_0(y)}{\pi \sigma(a)} \int_{-\infty}^{\infty} u_S(x_g, x_S, \omega) e^{-i\omega(\varphi(y, x_g) + \varphi(y, x_S))} d\omega, \quad (5.49)$$

Writing it in a more suitable form we get,

$$\alpha(y) = -\frac{\sqrt{y}v_0(y)}{\pi\sigma(a)} \int_{-\infty}^{\infty} u_S(x_g, x_S, \omega) e^{-i\omega(\varphi(y, x_g) + \varphi(y, x_S))} d\omega, \quad (5.50)$$

The reflectivity function $\beta(y)$ [3] can be obtained as before. Thus multiplying (5.50) by the factor $\frac{i\omega}{2v_0(y)}$. Thus, the reflectivity function is given by

$$\beta(y) = -\frac{2\sqrt{y}}{\pi\sigma(a)} \int_{-\infty}^{\infty} i\omega u_S(x_g, x_S, \omega) e^{-i\omega(\varphi(y, x_g) + \varphi(y, x_S))} d\omega. \quad (5.51)$$

5.2.2 Conclusions.

We have studied the inverse problem arising for inclusions or irregularities having radial symmetry. The wave equation involves the Bessel operator and an inversion formula is derived to recover the variation in wave speed for constant/variable background medium. The results may be of interest in other situations such as tumor detection or non-destructive testing.

5.3 Summary and Conclusion of the Dissertation

The variational properties in the interior of earth due to inhomogeneities or damping effects can be recovered using Born's approximation and suitable Green's function. In case of variable background medium, high frequency approximation is utilized in a number of cases of interest in one and higher dimensions. The inverse problem in which the properties of interior of earth modelled as a half space having constant or variable background is considered. A one dimensional model is developed in which the velocity profile, damping of the medium, Density and/or bulk modulus is recovered from the observed data using Born's approximation and WKBJ Green's function. We derive approximate solutions to the inverse problem of finding the velocity and damping from the observed wave-field. The recovery of variations in velocity, bulk modulus and damping in the presence of variable density is considered. An iterative procedure to improve velocity and damping profiles is also presented. These results may provide more accurate map of the interior of earth. The recovery procedure is then extended to two and three dimensions. In order to study the problems in geometries with cylindrical symmetry, the inverse problem involving the Bessel function is formulated and velocity profile re-constructed for high frequency approximation using the WKB Green's function for Bessel operator is also presented. The inverse formula is implemented to a medium consisting of single\multilayer and numerical results are presented.

5.4 Open problems and Further Research

The inverse problems studied in this dissertation focused on one or more parameter variation in one or higher dimensions of a model of earth which has damping in the medium. The implementation of these results to real situations using geophysical data is to be undertaken by collaborating with earth scientists.

The numerical results, geophysical representations of the parameter profiles of different structures of geophysical interest in variety of source-receiver geometries for variable background density and bulk modulus can be focus of further study which could provide mapping of interior of earth using the more general model presented in the dissertation.

The problems arising from biological situation such as tumor detection and monitoring are also similar in nature to those considered here. We hope that results derived here could be adopted for applications to the study of tumor detection and monitoring problems.

As a further step in moving towards the real life problems, the anisotropy of the medium, giving rise to a more complex model may be introduced in the inverse problems.

Appendix A

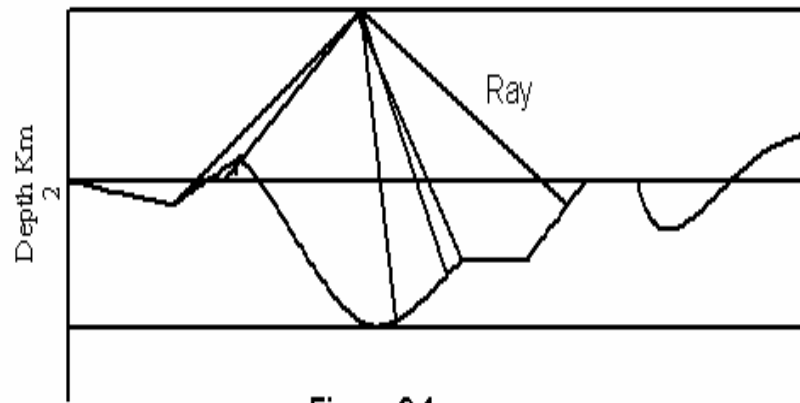


Figure 2.1

Figure 1. A synthetic Zero-offset section and earth model

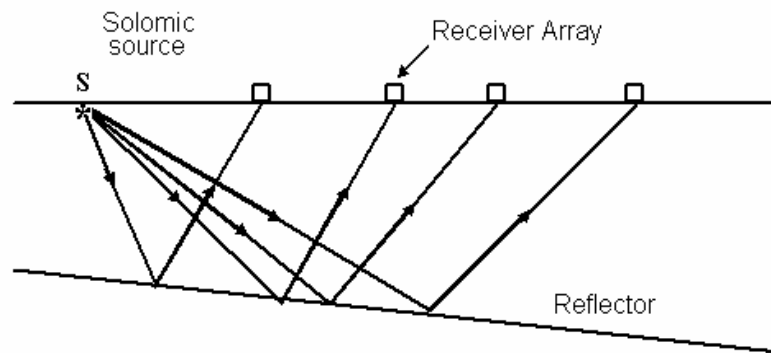


Figure 2.3

Figure 2. Common source profile.

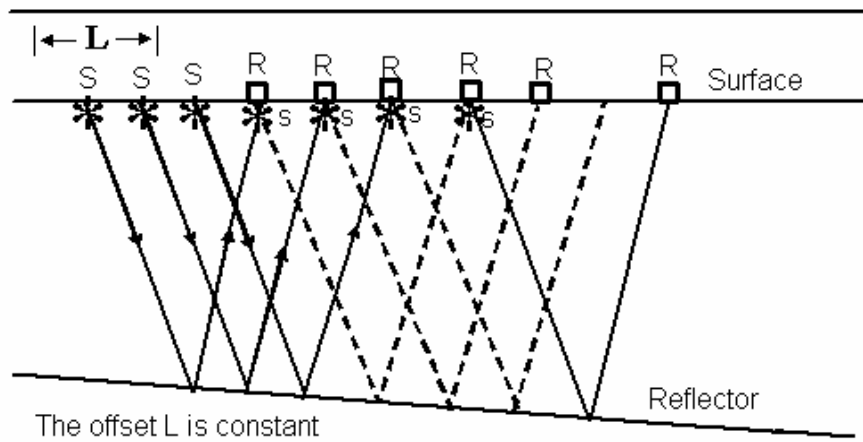


Figure 2.2

Figure 3. Common offset profile.

Appendix B

Glossary of Geophysical Terminology¹

¹ The help of Prof. Gabor Korvin (KFUPM, Earth Sci. Dept.) in compiling this Glossary is acknowledged with thanks.

(1) Band or frequency band:

Is a finite interval $[f_1, f_2]$ or $[\omega_1, \omega_2] = [2\pi f_1, 2\pi f_2]$.

(2) Bandlimited function :

Is a function whose Fourier transform is identically zero outside the union of two intervals $[-\omega_1, -\omega_2] \cup [\omega_1, \omega_2]$, $0 \leq \omega_1 \leq \omega_2 < \infty$.

(3) Bandwidth:

If the Fourier transform of $f(t)$ is identically zero outside $[-\omega_1, -\omega_2] \cup [\omega_1, \omega_2]$, $0 \leq \omega_1 \leq \omega_2 < \infty$, then the bandwidth of $f(t)$ is $\omega_2 - \omega_1$.

(4) Bulk modulus:

Is one of the elastic parameters occurring in Hooke's law which connects stress and strain in an elastic body.

(5) Direct Problem:

In Geophysics, the direct problem is defined as determination of the acoustic wave field, gravity potential field, magnetic potential field, static electric potential field, or electromagnetic field on the surface of the earth, if these field are due to the presence of a deep-lying body of known size, known depth, known location and known material properties. The material properties of the medium in which the body is embedded also should be known.

(6) Earth layering:

With a good approximation, the near-surface section of the earth can be modeled as if it consisted of homogeneous isotropic layers with different physical properties, separated with horizontal planar boundaries. In *earthquake seismology*, it is more appropriate to consider the layers as spherical shells.

(7) Filter:

Filters are time-invariant integral operators designed to remove unwanted harmonic components from an observed signal. The filtering in time domain is realized by a convolution integral $y(t) = \int_0^\infty f(\tau) x(t - \tau) dt$ where $x(t)$, $x(t) \equiv 0$ for $t \leq 0$ is the observed signal, $f(t)$, $\int_0^\infty |f(t)|^2 dt < \infty$ is the filter, and $y(t)$ is the filtered signal.

(8) High Frequency:

Elastic waves of frequencies higher than $\sim 100Hz$ are rapidly absorbed in Earth materials. Such frequencies are considered as "high" in seismic practice.

(9) Ill-conditioned problem:

Is an inverse problem of the type $Af = g$, where A is a linear or nonlinear operator, g is a measured function, f from g is ill-conditioned if small measurement errors in g could lead to arbitrary large errors in f .

(10) Impulse response:

The impulse response of a linear time-invariant system ("operator") L is $L\delta(t)$ where $\delta(t)$ is the Dirac-delta function.

(11) Incident wave:

An elastic wave generated by a distinct source propagating toward a layer or an object from its outside.

(12) Interface:

Planar or irregular, continuous, possibly non-differentiable surface separating parts of the earth with different physical properties.

(13) Inverse Problem:

In Geophysics, the inverse problem consists of finding the shape, size, location, depth and material properties of a body hidden deep inside the Earth, by using the measured acoustic, gravity, magnetic, static electric, or electromagnetic fields caused by the body, and measured on the surface of the earth.

(14) Offset:

When both the source and receiver are on the surface of the earth, their distance is called *offset*.

(15) Reflection Coefficient:

The reflection coefficient for elastic waves at the boundary of two media separated by a planar layer expresses the amplitude of the wave reflected from the boundary relative to the amplitude of the incident wave coming from the first layer. If the two media have the respective wave-velocities v_1 and v_2 and densities ρ_1 and ρ_2 , the reflection coefficient is

$$R = \frac{v_2 \rho_2 - v_1 \rho_1}{v_2 \rho_2 + v_1 \rho_1}.$$

(16) Reflectivity function:

If the elastic wave velocity $v(z)$ and density $\rho(z)$ are continuously differentiable functions of depth z , the reflectivity function is obtained by dividing the medium to infinitesimally thin layer, as:

$$\begin{aligned} \beta(z) &= \lim_{\Delta z \rightarrow 0} \frac{v(z + \Delta z) \rho(z + \Delta z) - v(z) \rho(z)}{v(z + \Delta z) \rho(z + \Delta z) + v(z) \rho(z)} \\ &\approx \frac{\Delta z}{2} \frac{d}{dz} \{ \log v(z) \rho(z) \}. \end{aligned}$$

(17) Scattered wave field:

It is the field due to waves generated at a distant source, which are incident on an irregular object. The scattered wave-field is observed in the same medium where the source is, at large distance from the object.

(18) Scattering:

Is the interaction of an incident elastic wave with a body of irregular shape whose characteristic size is much smaller than the wavelength of the wave.

(19) Seismic waves:

These are the elastic longitudinal (P -) or shear (S -) waves propagating inside the Earth (body waves) or on its surface (surface waves). Seismic waves can be due to natural sources (earthquake) or artificial sources (explosion, vibrator, etc.).

(20) Time Domain, and Frequency Domain:

If $f(t)$ and $F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}d\omega$ are a Fourier transform pair, t is time, $\omega = 2\pi f$ where f is frequency, we say that $f(t)$ and $F(\omega)$ are time domain-resp. Frequency domain representations of the same function.

(21) Transmission Coefficient:

The transmission coefficient for elastic waves at the boundary of two physically different media by a planar layer expresses the amplitude of the wave transmitted through the boundary into the second layer relative to the amplitude of wave incident from the first layer. If the two media have the respective wave-velocities v_1 and v_2 and densities ρ_1 and ρ_2 , the transmission coefficient is

$$T = \frac{2v_1\rho_1}{v_2\rho_2 + v_1\rho_1}.$$

(22) Variable density:

Is a method to display oscillatory seismic signals using a special black-and-white plotter. In this display mode, the local intensity of the image is controlled by the changes in the shape of the signal.

(23) Variation in density:

The density of rocks in the earth has a systematic variation with depth, it is generally increasing with depth. Also, it has a small random fluctuation in vertical direction, and a statistically independent random variation in horizontal direction. The density generally increases with depth from about $2.2g/cm^3$ at the surface to about $2.8g/cm^3$ at a few km depth. The random density fluctuation is less than $\pm 5\%$.

(24) Variations in wave speed:

The wave speed in the Earth has a systematic variation with depth, it is generally increasing with depth. Also, it has a small random fluctuation in vertical direction, and a statistically independent random variation in horizontal direction. The P -wave velocity generally increases with depth from about $2000 m/s$ at the surface to about $500 m/s$ at a few km depth. The S -wave velocity is about 60% of the P -wave velocity. The random velocity fluctuation is less than 10%.

(25) Zero-offset experiment:

Is a scattering experiment where the source of the wave and the receiver are at the same point on the surface of the earth, and the scattering object is at a great depth inside the earth.

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