# A Solution of the Inverse Problem in Quantum Mechanical Scattering Theory using the Distorted Born and Other Approximations

by

Saleh Bedawi Al-Ruwaily

A Thesis Presented to the

FACULTY OF THE COLLEGE OF GRADUATE STUDIES

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DHAHRAN, SAUDI ARABIA

In Partial Fulfillment of the
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This thesis, written by Saleh Bedawi Al-Ruwaili (Student I.D. No. 842651) under the direction of his Thesis Advisor and approved by his Thesis Committee, has been presented to and accepted by the Dean of the College of Graduate Studies, in partial fulfillment of the requirements for the degree of MASTER OF SCIENCE.

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To The Memory of My Father,

**Bedawi Hassan AL-Ruwaili**

إهـــداء

إلى ذكرى أبي العزيز،

بـديـوي حسـن الـرويلـي

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# ABSTRACT

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TITLE: A SOLUTION OF THE INVERSE PROBLEM IN QUANTUM MECHANICAL SCATTERING THEORY USING THE DISTORTED BORN AND OTHER APPROXIMATIONS

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In this thesis, we extend previous work [1] where it was shown how one can use the *Simple Born Approximation* and special mathematical techniques in order to extract the interaction from the phase shifts it produces. In particular we use the *Distorted Born Approximation* and *Auxiliary Potential* to extract the potential from the phase shifts it produces. Our approach includes, as a special case, the previous work [1]. Our results are much more accurate than those in [1]. Moreover, we obtained very good results for strong potentials which the previous work could not. Finally, in the course of this research, we obtained some useful and relevant mathematical expressions. Some of these involve new results.

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# INTRODUCTION

The inverse problem in quantum mechanical scattering theory is one of the interesting but not yet exhaustively treated problems in physics. It involves going from the phase shifts (and cross-sections) produced by a certain interaction (potential) to the interaction itself, as opposed to the well-known direct problem of obtaining the phase shifts (and thus cross-sections) produced by a given potential. The simplest example of the direct problem is the Simple Born Approximation which gives the phase shifts for a given potential. If one could invert this approximation one would have the potential as a function of the phase shifts i.e. an instance of the inverse problem in quantum mechanical scattering theory.

In some recent research [1], the Simple Born Approximation was used together with special mathematical techniques to obtain the potential (interaction) as a function of the phase shifts it produces. The results obtained were not satisfactory for strong potentials. However, this should be expected since the simple Born approximation (and thereby all the subsequent relations) are not applicable for strong potentials.

As an *improvement* and *generalized extension* of that research, we use the *Distorted Born Approximation and Auxiliary Potentials* together with certain math-

ematical techniques to extract the unknown potential as a function of its phase shifts the auxiliary potential used and the phase shift produced by the latter. In fact, this is a more general approach than that in [1] since the results obtained in [1] are no more than a special case of this general approach. Thus the auxiliary potentials used in this technique, if chosen properly, are an aid in improving the results. This is explained in chapters three and four.

In this thesis we also obtained very powerful, useful and relevant mathematical expressions. Some of them involve the inverse functions of the square of Bessel functions ($g_\ell(\rho)$), which were obtained in [3]. Others involve the spherical Bessel ($j_\ell(\rho)$) and Neumann ($n_\ell(\rho)$) functions. Moreover, a few summation relations were obtained in the course of this research. In fact, some of the expressions obtained in this thesis are *new*. Chapter two is devoted to these mathematical expressions.

# خلاصة الرسالة


**الإسم** : صالح بديوي الرويلي

**العنوان** :طريقة لحل المسئلة العكسية في نظرية التصادمات في ميكانيكا الكم بإستخدام علاقة بورن التقريبية وغيرها .

**التخصص** : فيزياء

**التاريخ** :ذو الحجة ١٤١٢هـ

في هذه الرسالة ، نقوم بتطوير عمل سابق أستخدمت فيه علاقة بورن التقريبية البسيطة وبعض الطرق الرياضية لإستخراج المجال التفاعلي للجسيمات من الإنحراف الموجي الذي تسببه .

على وجه الخصوص نستخدم علاقة بورن الإنحرافية التقريبية بالإضافة إلى مجال تفاعلي مساعد لإستخراج المجال التفاعلي المطلوب من الإنحراف الموجي الذي يسببه .

طريقتنا تشتمل على البحث السابق كحالة خاصة ، كما أن نتائجنا أكثر دقة من نتائج البحث السابق بكثير ، بالإضافة إلى ذلك نحن حصلنا على نتائج جيدة جداً للمجالات القوية بينما البحث السابق لم يستطع .

وأخيراً . من خلال العمل في هذا البحث ، حصلنا على علاقات رياضية مفيدة وذات صله بالموضوع . في الحقيقة بعض هذه العلاقات جديدة .


درجة الماجستير في العلوم

جامعة الملك فهد للبترول والمعادن

الظهران - المملكة العربية السعودية

ذو الحجة ١٤١٢هـ

# CHAPTER 1

## Preliminary Discussion

The partial wave Born approximation is one of the simplest in quantum mechanical scattering theory. It relates a particular integral of the scattering potential $V_\ell(r)$ to the phase shifts $\delta_\ell(k)$ produced by this potential;

$$\frac{-\hbar^2}{2m}\,\frac{\delta_\ell(k)}{k} \approx \int_0^\infty j_\ell^2(kr)V_\ell(r)r^2\,dr \tag{1.1}$$

where: $k = \sqrt{2mE/\hbar^2}$; $m$ and $E$ are the mass and energy of the incident particle; $V_\ell(r)$ is the scattering potential in the angular momentum $\ell$ channel; and $j_\ell(kr)$ is the spherical Bessel function; [1]. Eqn. (1.1) is applicable for weak potentials. It becomes very poor for strong potentials.

However, this relation (eqn. (1.1)) involves going from the scattering potential (interaction) to the phase shifts (and cross-sections) produced by this potential, which is a direct and not very useful relation in its simplest form shown above. In fact, the most useful and usually required relation in scattering theory is the inverse of relation (1.1) – shown above – hence the topic of this thesis is: "Inverse Problem in Quantum Mechanical Scattering Theory". This inverse problem, which is one of the most interesting and not yet exhaustively treated problems in scattering theory

[2], involves going from the phase shifts and cross-sections produced by a certain interaction (or potential) to the interaction (potential) itself. The direct relation (eqn. (1.1)) shown above has the *important unknown* quantity – $V_\ell(r)$ – under the integral sign. On the other hand, the phase shifts – $\delta_\ell(k)$ – which can be obtained from experiment as a function of energy or ($k$), are on the left hand side of the relation. To invert this relation, some technique has to be developed or derived in order to extract $V_\ell(r)$ – the interaction – from the integral and have it on its own on the left hand side (LHS) of the relation as a function of the phase shifts, which can be – the phase shifts – used as an experimental input to the problem.

Recently [1], a certain inversion technique was developed, for use with the Born approximation, and applied using two parallel approaches:

The first made use of a set of functions $g_\ell(kr)$ that was obtained in the literature [3]. These functions satisfy this relation

$$\int_0^\infty j_\ell^2(kr) g_\ell(kr') dk = \frac{r'^2}{r^2} \left\{ \delta(r - r') + \delta(r + r') - (-1)^\ell 2\delta(r') \right\}. \qquad (1.2)$$

These $g_\ell(kr)$ functions are further discussed in the next chapter.

To begin with, we will concentrate on the first inversion approach [1];

Premultiplying Eqn. (1.1) by $g_\ell(kr')$ and then taking the integral over $k(0 \to \infty)$;

$$\int_0^\infty \left(\frac{-\hbar^2}{2m}\frac{\delta_\ell(k)}{k}\right) g_\ell(kr')dk = \int_0^\infty \int_0^\infty \left[j_\ell^2(kr)g_\ell(kr')\right] dk \cdot V_\ell(r)r^2dr.$$

Making use of eqn. (1.2) the above eqn. gives

$$\int_0^\infty \left(\frac{-\hbar^2}{2m}\frac{\delta_\ell(k)}{k}\right) g_\ell(kr')dk = \int_0^\infty \frac{r'^2}{r^2}\left\{\delta(r-r') + \delta(r+r') - (-1)^\ell 2\delta(r')\right\} V_\ell(r)r^2dr$$

Hence

$$\text{LHS} = r'^2V_\ell(r') + -2(-1)^\ell r'^2 \int_0^\infty V_\ell(r)dr$$

Notice that

$$\int_0^\infty \frac{r'^2}{r^2}\delta(r+r')r^2V_\ell(r)dr = 0,$$

Since $(r + r')$ cannot be zero unless $r = r' = 0$, this results in a zero contribution from this integral. Hence,

$$\int_0^\infty \left(\frac{-\hbar^2}{2m}\frac{\delta_\ell(k)}{k}\right) g_\ell(kr')dk = r'^2V_\ell(r') - 2(-1)^\ell r'^2\delta_\ell(r') \int_0^\infty V_\ell(r)dr$$

Hence;

$$V_\ell(r') = \frac{1}{r'^2}\int_0^\infty \left(\frac{-\hbar^2}{2m}\frac{\delta_\ell(k)}{k}\right) g_\ell(kr')dk + 2(-1)^\ell\delta_\ell(r') \int_0^\infty V_\ell(r)dr. \qquad (1.3)$$

Notice that, even though the potential $V_\ell(r')$ is on the LHS of eqn. (1.3), the second term on the RHS involves an integral over the potential $V_\ell(r)$, which renders this eqn. (eqn. (1.3)) not useful or helpful practically. Nevertheless, by examining eqn. (1.1) for large $k$ (corresponding to a large energy for the scattering particle); one

obtains

$$\left\{\frac{-\hbar^2}{2m}\frac{\delta_\ell(k)}{k}\right\} \underset{k\to\infty}{\longrightarrow} \quad \int_0^\infty \lim_{k\to\infty}\left(j_\ell^2(kr)\right) V_\ell(r) r^2 dr.$$

But,

$$\lim_{k\to\infty} j_\ell(kr) \to \frac{\sin(kr-\ell\pi/2)}{kr}$$

hence;

$$\left(\frac{-\hbar^2}{2m}\frac{\delta_\ell(k)}{k}\right)_{k\to\infty} \quad \int_0^\infty \frac{\sin^2(kr-\ell\pi/2)}{k^2}\cdot V_\ell(r)dr.$$

Now,

$$\begin{aligned}
\sin^2(kr-\ell\pi/2) &= \frac{1}{2}-\frac{1}{2}\cos\left[2kr-\ell\pi\right] \\
&= \frac{1}{2}-\frac{1}{2}\left\{\cos(2kr)\underbrace{\cos(\ell\pi)}_{(-1)^\ell}+\sin(2kr)\underbrace{\sin(\ell\pi)}_{0}\right\} \\
&= \frac{1}{2}-\frac{1}{2}(-1)^\ell\cos(2kr)
\end{aligned}$$

Thus

$$\left(\frac{-\hbar^2}{2m}\frac{\delta_\ell(k)}{k}\right)_{k\to\infty} \quad \frac{1}{2k^2}\int_0^\infty \left[1+(-1)^{\ell+1}\cos(2kr)\right] V_\ell(r)dr.$$

However, $\int_0^\infty \cos(2kr)V_\ell(r)dr \to 0$ in the limit as $k$ tends to infinity. To see this, write

$$\int_0^\infty \cos(2kr)V_\ell(r)dr = V_\ell(r)\frac{\sin(2kr)}{2k}\Big|_{r=0}^{r=\infty} - \int_0^\infty \frac{\sin(2kr)}{2k}\frac{d}{dr}V_\ell(r)\cdot dr.$$

Now, *the first term on the RHS* is zero as $k$ tends to infinity, since the potential and $\sin(2kr)$ are finite at the upper limit ($r=\infty$), while the denominator ($2k$) goes to

infinity. Moreover, at the lower limit ($r = 0$), $\sin(2kr)$ makes the whole term zero. As to the second term;

$$-\frac{1}{2k}\int_0^\infty \sin(2kr)\frac{dV_\ell(r)}{dr}\cdot dr$$

one can apply a similar procedure and observe that at the lower limit ($r = 0$) and the upper limit ($r = \infty$) the factor $\frac{1}{(2k)^2}$ dominates the other quantities ( as $k \to \infty$) and therefore, this integral goes to zero. Hence,

$$\lim_{k\to\infty}\left(k^2\left[\frac{-\hbar^2}{2m}\frac{\delta_\ell(k)}{k}\right]\right) \to \frac{1}{2}\int_0^\infty V_\ell(r)dr. \tag{1.4}$$

Thus, one can replace the second term on the RHS of eqn. (1.3) by the equivalent term from eqn. (1.4). Thereafter; (also replacing $r'$ by $r$, for simplicity):

$$V_\ell(r) = \frac{1}{r^2}\int_0^\infty \left(\frac{-\hbar^2}{2m}\frac{\delta_\ell(k)}{k}\right) g_\ell(kr)dk + 4(-1)^\ell\delta(r) \times$$

$$\lim_{k'\to\infty}\left\{\frac{-\hbar^2}{2m}k'\delta_\ell(k')\right\}$$

However, it is shown in the next chapter that;

$$\int_0^\infty \frac{g_\ell(kr)}{k^2r^2}dk = 4(-1)^{\ell+1}\delta_\ell(r)$$

Hence

$$V_\ell(r) = \frac{1}{r^2}\int_0^\infty \left(\frac{-\hbar^2}{2m}\frac{\delta_\ell(k)}{k}\right) g_\ell(kr)dk +$$

$$\left(\frac{1}{r^2}\int_0^\infty \frac{g_\ell(kr)}{k^2}dk\right)\lim_{k\to\infty}\left(\frac{-\hbar^2}{2m}k'\delta_\ell(k')\right)$$

or,

$$V_\ell(r) = \frac{1}{r^2}\int_0^\infty \left(\frac{-\hbar^2}{2m}\frac{\delta_\ell(k)}{k} - \frac{1}{k^2}\lim_{k'\to\infty}\left\{\frac{-\hbar^2}{2m}k'\delta_\ell(k')\right\}\right) g_\ell(kr)dk \qquad (1.5)$$

Eqn. (1.5) represents an operationally useful outcome of the first inversion approach used by [1].

The second inversion procedure involves using a formula derived in [1]; namely:

$$\left(\frac{d}{d\rho^2}\right)^\ell \frac{d}{d\rho}\left(\frac{d}{d\rho^2}\right)^\ell \left(\rho^{\ell+1} j_\ell(\rho)\right)^2 = \sin(2\rho) \qquad (1.6)$$

or,

$$\hat{\Theta}_\ell(\rho)\left[j_\ell^2(\rho)\right] = \sin(2\rho)$$

where;

$$(d/d\rho^2) \equiv \frac{1}{2\rho}\frac{d}{d\rho}$$

$$\hat{\Theta}_\ell(\rho) \equiv (d/d\rho^2)^\ell\,(d/d\rho)\,(d/d\rho^2)^\ell\,\rho^{2\ell+2}$$

Eqn. (1.6) can easily be verified for $\ell = 0, 1, 2, 3$. Operating on eqn. (1.1) with the operator in eqn. (1.6)

$$\underbrace{(d/dk^2)^\ell \frac{d}{dk}(d/dk^2)^\ell}_{\equiv \hat{O}_\ell(k)}\left(k^{2\ell+2}\left\{\frac{-\hbar^2}{2m}\frac{\delta_\ell(k)}{k}\right\}\right) = \int_0^\infty \hat{O}_\ell(k)\left[k^{2\ell+2} j_\ell^2(kr)\right] V_\ell(r) r^2 dr$$

or;

$$\text{LHS} = \int_0^\infty \left\{\left(\frac{d}{d(kr)^2}\right)^\ell \frac{d}{d(kr)}\left(\frac{d}{d(kr)^2}\right)^\ell \left((kr)^{2\ell+2} j_\ell^2(kr)\right)\right\} V_\ell(r) r^{2\ell+1} dr$$

Now, from eqn. (1.6), the above relation becomes;

$$\text{LHS} = \int_0^\infty \sin(2kr) V_\ell(r) r^{2\ell+1}\, dr$$

which is the Fourier sine transform of $(V_\ell(r)/2)\, r^{2\ell+1}$ [4].

Multiplying this eqn. by $\frac{4}{\pi}\sin(2kr')$ and integrating over $k$; one obtains;

$$\frac{4}{\pi}\int_0^\infty \left\{ \left(\frac{d}{dk^2}\right)^\ell \frac{d}{dk} \left(\frac{d}{dk^2}\right)^\ell \left( k^{2\ell+2} \left\{ \frac{-\hbar^2}{2m} \frac{\delta_\ell(k)}{k} \right\} \right) \right\} \sin(2kr')\, dk$$

$$= \frac{4}{\pi}\int_0^\infty \int_0^\infty \sin(2kr)\sin(2kr') V_\ell(r) r^{2\ell+1}\, dk\, dr$$

But,

$$\frac{4}{\pi}\int_0^\infty \sin(2kr)\sin(2kr') dk = \delta(r-r') - \delta(r+r')$$

$$\text{LHS} = \int_0^\infty \{\delta(r-r') - \delta(r+r')\}\, V_\ell(r) r^{2\ell+1}\, dr$$

the second term on RHS gives a zero contribution, while the first gives; $V_\ell(r')\, r'^{2\ell+1}$.

Hence;

$$V_\ell(r') = \frac{4}{\pi r'^{2\ell+1}} \int_0^\infty \left\{ \left(\frac{d}{dk^2}\right)^\ell \frac{d}{dk} \left(\frac{d}{dk^2}\right)^\ell \left( k^{2\ell+2} \left\{ \frac{-\hbar^2}{2m} \frac{\delta_\ell(k)}{k} \right\} \right) \right\} \sin(2kr') dk \tag{1.7}$$

Eqn. (1.7) represents the useful outcome of the second inversion approach used in [1].

The above two eqn.s ((1.5), (1.7)) have been used in conjunction with a set of

"experimental" phase shifts produced by a hypothetical potential to rederive the interaction involved.

It has been found that these techniques are of limited utility, especially for strong potentials (or interactions). This is expected since the simple Born Approximation is poor for strong potentials [5].

Hence, in this thesis we develop two approaches which are much more general than the ones discussed above and much more powerful. In fact, the above approaches are shown to be no more than special cases of the more general inversion approaches which will be derived and used in this thesis. Before discussing these general techniques, I will devote one chapter to some mathematical expressions that are useful and relevant. Some of these involve new results.

# CHAPTER 2

## Some Useful Mathematical Relations

**The $g_\ell(kr)$ Functions:**

In [3], it is shown that

$$g_\ell(\rho) = \frac{-8\rho^2}{\pi(2\ell+1)}\,{}_1F_2\left(\frac{3}{2};\frac{1}{2}-\ell,\ \ell+\frac{3}{2};-\rho^2\right) \tag{2.1}$$

and,

$$g_\ell(\rho) = \frac{(-1)^\ell 16\rho^{3+\ell}\Gamma(\frac{1}{2}-\ell)}{\pi^{3/2}(\ell+\frac{1}{2})B(\frac{3}{2},\ell)}\int_0^\infty y^{\ell+3}(1-y^2)^{\ell-1}\,n_\ell(2\rho y)dy \tag{2.2}$$

where $y \neq 0$, ${}_1F_2$ is a generalized Hypergeometric series, $n_\ell(\xi)$ are the spherical Neumann functions [6]. Also, in [1], It is shown that:

$$g_\ell(\rho) = \frac{-4}{\pi}\rho^{2\ell+2}\left(\frac{d}{d\rho}\frac{1}{2\rho}\right)^\ell\frac{d}{d\rho}\left(\frac{d}{d\rho}\frac{1}{2\rho}\right)^\ell\sin(2\rho). \tag{2.3}$$

Moreover, it can be observed from eqn. (1.6) and eqn. (2.3) that;

$$g_\ell(\rho) = \frac{-4}{\pi}\left\{\rho^{2\ell+2}\left(\frac{d}{d\rho}\frac{1}{2\rho}\right)^\ell\frac{d}{d\rho}\left(\frac{d}{d\rho}\frac{1}{2\rho}\right)^\ell\left(\frac{1}{2\rho}\frac{d}{d\rho}\right)^\ell\frac{d}{d\rho}\left(\frac{1}{2\rho}\frac{d}{d\rho}\right)^\ell\rho^{2\ell+2}\right\}j_\ell^2(\rho) \tag{2.4}$$

This formal expression shows that the $g_\ell$'s are the results of a series of complicated operations on $j_\ell^2(\rho)$. For instance,

$$g_0(\rho) = \frac{-4}{\pi}\left\{\rho^2\frac{d}{d\rho}\cdot\frac{d}{d\rho}\rho^2\right\}j_0^2(\rho) = \frac{-4}{\pi}\rho^2\frac{d}{d\rho}\left[\frac{d}{d\rho}\rho^2 j_0^2(\rho)\right],$$

but

$$\frac{d}{d\rho}\left(\rho^2 j_0^2(\rho)\right) = \sin(2\rho).$$

Hence

$$g_0(\rho) = \frac{-4}{\pi}\rho^2 \frac{d}{d\rho}\left[\sin(2\rho)\right] = \frac{-8}{\pi}\,\rho^2\cos(2\rho).$$

An additional expression for the $g_\ell(\rho)$ functions, which was not known before is:

$$g_\ell(\rho) = -(-1)^\ell \frac{8}{\pi}\rho^2\cos(2\rho) + \frac{(-1)^\ell 16\ell(\ell+1)}{\pi}\sum_{k=0}^{\ell}\frac{(-1)^k 2^k(\ell+k)!}{(k+1)!(\ell-k)!}\,\rho^2\,\frac{j_k(2\rho)}{(2\rho)^k} \tag{2.5}$$

This expression was obtained in the course of this research after a careful comparison between $g_\ell(\rho)$ and $j_\ell(\rho)$ functions. For large $\rho$, the leading term in $g_\ell(\rho)$ is always $\left[\frac{8}{\pi}\rho^2\cos(2\rho)\right]$ with the corresponding $(-1)^{\ell+1}$ sign depending upon $\ell$; see table I.

Table 1

Spherical Bessel Functions $j_\ell(\rho)$, the $g_\ell(\rho)$ functions and the $\int \left[g_\ell(\rho)/\rho^2\right] d\rho$, for $\ell = 0, 1$ and 2

| $\ell =$ | 0 | 1 | 2 |
|---|---|---|---|
| $j_\ell(\rho) =$ | $\sin(\rho)/\rho$ | $\frac{\sin(\rho)}{\rho^2} - \frac{\cos(\rho)}{\rho}$ | $\left(\frac{3}{\rho^3} - \frac{1}{\rho}\right)\sin(\rho) - \frac{3}{\rho^2}\cos(\rho)$ |
| $g_\ell(\rho) =$ | $\frac{-8\rho^2}{\pi}\cos(2\rho)$ | $\frac{8}{\pi}\left\{(\rho^2 - 2)\cos(2\rho) + \left(\frac{1}{\rho} - 2\rho\right)\sin(2\rho)\right\}$ | $-\frac{8}{\pi}\left\{\left(\rho^2 - 18 + \frac{36}{\rho^2}\right)\cos(2\rho) + \left(-6\rho + \frac{33}{\rho} - \frac{18}{\rho^3}\right)\sin(2\rho)\right\}$ |
| $\int \frac{g_\ell(\rho)}{\rho^2} d\rho =$ | $\frac{-4}{\pi}\sin(2\rho)$ | $\frac{4}{\pi}\left\{\left(1 - \frac{1}{\rho^2}\right)\sin(2\rho) + \frac{2}{\rho}\cos(2\rho)\right\}$ | $\frac{-4}{\pi}\left\{\left(1 - \frac{15}{\rho^2} + \frac{9}{\rho^4}\right)\sin(2\rho) + \left(\frac{6}{\rho} - \frac{18}{\rho^3}\right)\cos(2\rho)\right\}$ |

The rest of the terms in $g_\ell(\rho)$ can be related to the $j_\ell(\rho)$'s by examining the terms involving $\sin(2\rho)$ and $\cos(2\rho)$ for a given $g_\ell(\rho)$ functions with the corresponding coefficient and sign of each term. For example, $g_0(\rho)$ has only the leading term $\left[\frac{-8}{\pi}\rho^2\cos(2\rho)\right]$ and $g_1(\rho)$ has the leading term and other terms involving $\sin(2\rho)$ and $\cos(2\rho)$. These two $g_\ell(\rho)$'s can be written as:

$$g_0(\rho) = \frac{-8}{\pi}\rho^2\cos(2\rho) + \frac{8}{\pi}\{ \quad 0 \quad 0 \quad 0 \quad \cdots\}$$

$$g_1(\rho) = \frac{8}{\pi}\rho^2\cos(2\rho) + \frac{8}{\pi}\left\{-2\rho\sin(2\rho) + \frac{\sin(2\rho)}{\rho} - 2\cos(2\rho)\right\}$$

Notice that $g_1(\rho)$ is just the leading term plus

$$\left\{\frac{(-32)\rho^2}{\pi}j_0(2\rho) + \frac{(64)}{\pi}\rho^2\frac{j_1(2\rho)}{(2\rho)}\right\}$$

or

$$\begin{aligned} g_1(\rho) &= \frac{8}{\pi}\rho^2\cos(2\rho) + \frac{(2)(16)}{\pi}\left\{-\rho^2 j_0(2\rho) + 2\rho^2\frac{j_1(2\rho)}{(2\rho)}\right\} \\ &= \frac{8}{\pi}\rho^2\cos(2\rho) + \underbrace{\ell(\ell+1)}_{\ell=1}\frac{16}{\pi}\left\{-\rho^2 j_0(2\rho) + 2\rho^2\frac{j_1(2\rho)}{(2\rho)}\right\} \end{aligned}$$

and

$$g_0(\rho) = \frac{-8}{\pi}\rho^2\cos(2\rho) + \underbrace{\ell(\ell+1)}_{\ell=0}\frac{16}{\pi}\left\{\rho^2 j_0(2\rho)\right\}$$

Hence, all the other terms other than the leading term can be related to the $j_\ell(2\rho)$'s by suitable summation that runs from $k = 0$ to $k = \ell$ with the corresponding coefficients and signs of each term. This can be accomplished by representing

different $\ell$-cases $(\ell = 0, 1, 2, 3, 4)$ in a way similar to the above examples. Then one can surmise the general behavior of the representation which leads to eqn. (2.5).

Furthermore, from [1], two integral properties of the $g_\ell(\rho)$'s are known:

$$\text{(i)} \qquad \int_0^\infty \frac{g_\ell(kr)}{k^2 r^2} dk = 4(-1)^{\ell+1}\delta(k), \tag{2.a}$$

$$\text{(ii)} \qquad \int_0^\infty \left[\frac{g_\ell(\rho)}{\rho^2} - \frac{8(-1)^{\ell+1}}{\pi}\cos(2\rho)\right] d\rho = 0. \tag{2.b}$$

Eqn. (2.b) implies that by integrating eqn. (2.5) one gets:

$$\int_0^\infty \left[\frac{g_\ell(\rho)}{\rho^2} - (-1)^{\ell+1}\frac{8}{\pi}\cos(2\rho)\right] d\rho = \frac{(-1)^\ell 16\ell(\ell+1)}{\pi} \int_0^\infty \sum_{k=0}^{\ell} \frac{(-1)^k 2^k (\ell+k)!}{(k+1)!(\ell-k)!} \times$$

$$\left\{\frac{j_k(2\rho)}{(2\rho)k}\right\} d\rho = 0.$$

Hence;

$$\frac{(-1)^\ell 16\ell(\ell+1)}{\pi} \sum_{k=0}^{\ell} \frac{(-1)^k 2^k (\ell+k)!}{(k+1)!(\ell-k)!} \int_0^\infty \frac{j_k(2\rho)}{(2\rho)^k} d\rho = 0.$$

But, from [7];

$$\int_0^\infty \frac{j_k(2\rho)}{(2\rho)^k} d\rho = \frac{\pi}{(k)!(2)^{k+2}} \Rightarrow \frac{(-1)^\ell 4\ell(\ell+1)}{\pi} \sum_{k=0}^{\ell} \frac{(-1)^k (\ell+k)!}{(k)!(k+1)!(\ell-k)!} = 0$$

or;

$$\ell(\ell+1) \sum_{k=0}^{\ell} \frac{(-1)^k (\ell+k)!}{(k)!(k+1)!(\ell-k)!} = 0$$

Equivalently;

$$\sum_{k=0}^{\ell} \frac{(-1)^\ell \ell!}{(k)!(\ell-k)!} \frac{(\ell+1)(\ell+k)!}{(k+1)!(\ell-1)!} = \sum_{k=0}^{\ell} (-1)^k (\ell+1) \binom{\ell}{k} \binom{\ell+k}{k+1} = 0.$$

Thus, the following relation involving the binomial coefficients holds

$$\sum_{k=0}^{\ell}(-1)^k \binom{\ell}{k} \binom{\ell+k}{k+1} = 0. \tag{2.6}$$

where $\binom{0}{1} = 0$ and $\ell = 0, 1, 2, 3, \ldots$.

To our knowledge, *this is a new relation.* A small Fortran-77 numerical program was used to test eqn. (2.6) for different $\ell$-values ($\ell = 0, 1, 2, \ldots$). It gives a very satisfactory result within the precision of the machine used. The program and typical output are shown in Appendix (D). Other relations involving binomial coefficients are given in the literature [8] and are given in Table II for reference and comparison.

## Table II

**Sums of the Binomial Coefficients:**

**($n$ is a natural number), [8]**

| | | |
|---|---|---|
| (I) | 1. | $\sum_{k=0}^{m} \binom{n+k}{n} = \binom{n+m+1}{n+1}$. |
| | 2. | $1 + \binom{n}{2} + \binom{n}{4} + \cdots = 2^{n-1}$ |
| | 3. | $\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots = 2^{n-1}$. |
| | 4. | $\sum_{k=0}^{m} (-1)^k \binom{n}{k} = (-1)^m \binom{n-1}{m}$ $[n \geq 1]$. |
| (II) | 1. | $\binom{n}{0} + \binom{n}{3} \binom{n}{6} + \cdots = \frac{1}{3}\left(2^n + 2\cos\frac{n\pi}{3}\right)$. |
| | 2. | $\binom{n}{1} + \binom{n}{4} + \binom{n}{7} + \cdots = \frac{1}{3}\left(2^n + 2\cos\frac{(n-2)\pi}{3}\right)$. |
| | 3. | $\binom{n}{2} + \binom{n}{5} + \binom{n}{8} + + \cdots = \frac{1}{3}\left(2^n + 2\cos\frac{(n-2)\pi}{3}\right)$. |
| (III) | 1. | $\binom{n}{0} + \binom{n}{4} + \binom{n}{8} + \cdots = \frac{1}{2}\left(2^{n-1} + 2^{n/2}\cos\frac{n\pi}{4}\right)$. |
| | 2. | $\binom{n}{1} + \binom{n}{5} + \binom{n}{9} + \cdots = \frac{1}{2}\left(2^{n-1} + 2^{n/2}\sin\frac{n\pi}{4}\right)$. |
| | 3. | $\binom{n}{2} + \binom{n}{6} + \binom{n}{10} + \cdots = \frac{1}{2}\left(2^{n-1} - 2^{n/2}\cos\frac{n\pi}{4}\right)$. |

(Continued)

(IV) 1. $\sum_{k=0}^{n}(k+1)\binom{n}{k} = 2^{n-1}(n+2) \quad [n \geq 0]$.

2. $\sum_{k=0}^{n}(-1)^{k+1}k\binom{n}{k} = 0 \quad [n \geq 2]$ .

3. $\sum_{k=0}^{n}(-1)^{k}\binom{N}{k}k^{n-1} = 0 \quad [N \geq n \geq 1;\ 0^0 \equiv 1]$.

4. $\sum_{k=0}^{n}(-1)^{k}\binom{n}{k}k^{n} = (-1)^n! \quad [N \geq 0;\ 0^0 \equiv 1]$.

5. $\sum_{k=0}^{n}(-1)^{k}\binom{n}{k}k^{n}(a+k)^n = (-1)^n n! \quad [n \geq 0;\ 0^0 \equiv 1]$.

(V) 1. $\sum_{k=1}^{n}\frac{(-1)^{k+1}}{k+1}\binom{n}{k} = \frac{n}{n+1}$.

2. $\sum_{k=0}^{n}\frac{1}{k+1}\binom{n}{k} = \frac{2^{n+1}-1}{n+1}$.

3. $\sum_{k=0}^{n}\frac{a^{k+1}}{k+1}\binom{n}{k} = \frac{(a+1)^{n+1}-1}{n+1}$.

4. $\sum_{k=1}^{n}\frac{(-1)^{k+1}}{k}\binom{n}{k} = \sum_{m=1}^{n}\frac{1}{m}$.

(Continued)

(VI) 1. $\sum_{k=0}^{p} \binom{n}{k} \binom{m}{p-k} = \binom{n+m}{p}$ [$m$ is a natural number ],

2. $\sum_{k=0}^{n-p} \binom{n}{k} \binom{n}{p+k} = \frac{(2n)!}{(n-p)!(n+p)!}$.

(VII) 1. $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$.

2. $\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^2 = (-1)^n \binom{2n}{n}$.

3. $\sum_{k=0}^{2n+1} (-1)^k \binom{2n+1}{k}^2 = 0$.

4. $\sum_{k=1}^{n} k \binom{n}{k}^2 = \frac{(2n-1)!}{[(n-1)!]^2}$.

It is of interest to study the behavior of the $g_\ell(\rho)$ functions in the large and small $\rho$ limits. First, from eqn. (2.3), it can be seen that for large $\rho$-values ($\rho \to \infty$ the *leading term* of $\underset{\rho\to\infty}{g_\ell(\rho)}$ is $\left((-1)^{\ell+1}\frac{8\rho^2}{\pi}\cos(2\rho)\right)$.

If instead one examines eqn. (2.5), it is obvious that for $\rho \to \infty$; $\lim\limits_{\rho\to\infty} j_\ell(2\rho) = \dfrac{\sin(2\rho - \ell\pi/2)}{(2\rho)}$ and thereby the leading term of $\left[\underset{\rho\to\infty}{g_\ell(\rho)}\right]$ is: $(-1)^{\ell+1}\frac{8\rho^2}{\pi}\cos(2\rho)$.

In agreement with the result which follows from eqn. (2.3).

Secondary, for small values of $\rho$, $(\rho \to 0)$;

$$j_k(2\rho) \underset{\rho\to 0}{\longrightarrow} \frac{(2\rho)^k}{(2k+1)!!}$$

where $(2k+1)!! = (2k+1)(2k-1)\cdots 5.3.1$ and hence, from eqn. (2.5);

$$\lim_{\rho\to 0} g_\ell(\rho) = (-1)^{\ell+1}\frac{8\rho^2}{\pi} + \frac{(-1)^\ell 16\ell(\ell+1)}{\pi}\sum_{k=0}^{\ell}\frac{(-1)^k(2)^k(\ell+k)!\rho^2}{(k+1)!(\ell-k)!(2k+1)!!} \tag{2.7}$$

However, from eqn. (2.2), it can be seen that for $(\rho \to 0)$

$$\lim_{\rho\to 0} g_\ell(\rho) = \frac{(-1)^\ell 16\rho^{3+\ell}\Gamma(1/2-\ell)}{\pi^{3/2}(\ell+1/2)B(3/2,\ell)}\int_0^1 y^{\ell+3}(1-y^2)^{\ell-1}\lim_{\rho\to 0}\left[n_\ell(2\rho y)\right]dy$$

where $y \neq 0$. But

$$\lim_{\rho\to 0}(n_\ell(2\rho y)) \to -\frac{(2\ell-1)!!}{(2\rho y)^{\ell+1}}.$$

Hence;

$$\lim_{\rho\to 0} g_\ell(\rho) = \frac{(-1)^{\ell+1}16\rho^{3+\ell}\Gamma(1/2-\ell)}{\pi^{3/2}(\ell+1/2)B(3/2,\ell)}\int_0^1 y^{\ell+3}\frac{(1-y^2)^{\ell-1}(2\ell-1)!!}{(2\rho)^{\ell+1}\,y^{\ell+1}}dy$$

$$= \frac{(-1)^{\ell+1}16}{\pi^{3/2}(2)^{\ell+1}} \frac{\rho^2\Gamma(1/2-\ell)(2\ell-1)!!}{(\ell+1/2)B(3/2,\ell)} \int_0^1 y^2(1-y^2)^{\ell-1}dy$$

$$= \frac{(-1)^{\ell+1}16}{\pi^{3/2}} \frac{\rho^2}{(2)^{\ell+2}} \frac{\Gamma(1/2-\ell)(2\ell-1)!!}{(\ell+1/2)B(3/2,\ell)} \int_0^1 (y^2)^{3/2-1}(1-y^2)^{\ell-1}d(y^2).$$

But

$$\int_0^1 (y^2)^{3/2-1}(1-y^2)^{\ell-1}d(y^2) = B(3/2,\ell)$$

and

$$\Gamma(1/2-\ell) = \frac{(-1)^\ell 2^\ell\sqrt{\pi}}{(2\ell-1)!!}.$$

Thus;

$$\lim_{\rho\to 0} g_\ell(\rho) = \frac{(-1)^{\ell+1}32\rho^2(-1)^\ell 2^\ell\sqrt{\pi}(2\ell-1)!!}{\pi^{3/2}(2)^{\ell+2}(2\ell-1)!!B(3/2,\ell)(2\ell+1)} B(3/2,\ell).$$

Therefore, for any $\ell$-value

$$\lim_{\rho\to 0} g_\ell(\rho) = \frac{-8\rho^2}{\pi(2\ell+1)}. \tag{2.8}$$

Inserting (2.8) in (2.7) one obtains

$$\frac{-8\rho^2}{\pi(2\ell+1)} = (-1)^{\ell+1}\frac{8}{\pi}\rho^2 + \frac{(-1)^\ell 16\ell(\ell+1)}{\pi}\sum_{k=0}^{\ell} \frac{(-1)^k(2)^k(\ell+k)!\rho^2}{(k+1)!(\ell-k)!(2k+1)!!}$$

or;

$$\frac{-8\rho^2-(2\ell+1)(-1)^{\ell+1}8\rho^2}{\pi(2\ell+1)} = \frac{(-1)^\ell 16\ell(\ell+1)}{\pi}\sum_{k=0}^{\ell} \frac{(-1)^k 2^k(\ell+k)!\rho^2}{(k+1)!(\ell-k)!(2k+1)!!}$$

Hence;

$$\left\{\frac{(2\ell+1)+(-1)^{\ell+1}}{2\ell(\ell+1)(2\ell+1)}\right\} = \sum_{k=0}^{\ell} \frac{(-1)^k(2)^k(\ell+k)!}{(k+1)!(\ell-k)!(2k+1)!!}$$

$$(\ell = 0,1,2,3\ldots.) \tag{2.9}$$

(i) <u>If $\ell =$ even</u>:

$$\frac{1}{(\ell+1)(2\ell+1)} = \sum_{k=0}^{\ell} \frac{(-1)^k (2)^k (\ell+k)!}{(k+1)!(\ell-k)!(2k+1)!!}$$

(ii) <u>If $\ell =$ odd</u>:

$$\frac{1}{\ell(2\ell+1)} = \sum_{k=0}^{\ell} \frac{(-1)^k (2)^k (\ell+k)!}{(k+1)!(\ell-k)!(2k+1)!!}$$

Now, since

$$(2k+1)!! = \frac{(2k+1)!}{(2k)!!} = \frac{(2k+1)!}{(2)^k (k)!}$$

Thus;

$$\sum_{k=0}^{\ell} \frac{(-1)^k (2)^k (\ell+k)!}{(k+1)!(\ell-k)!(2k+1)!!} = \sum_{k=0}^{\ell} \frac{(-1)^k (2)^{2k} (\ell+k)!(k)!}{(k+1)!(\ell-k)!(2k+1)!}$$

$$= \sum_{k=0}^{\ell} \frac{(-1)^k (4)^k}{(2k+1)(k+1)} \binom{\ell+k}{\ell-k}.$$

Eqn. (2.9) becomes;

$$\left\{ \frac{(2\ell+1)+(-1)^{\ell+1}}{2\ell(\ell+1)(2\ell+1)} \right\} = \sum_{k=0}^{\ell} \frac{(-1)^k (4)^k}{(2k+1)(k+1)} \binom{\ell+k}{\ell-k} \tag{2.9}$$

which, as far as we know, gives another new property of the binomial coefficients.

## <u>New Differential Properties Involving $j_\ell(\rho)$ & $n_\ell(\rho)$</u>:

Eqn. (1.6), a result obtained in [1] is

$$\left(\frac{d}{d\rho^2}\right)^{\ell} \frac{d}{d\rho} \left(\frac{d}{d\rho^2}\right)^{\ell} \left[\rho^{\ell+1} j_\ell(\rho)\right]^2 = \sin(2\rho).$$

In Appendix (A), it is proven that, similarly,

$$\left(\frac{d}{d\rho^2}\right)^{\ell} \frac{d}{d\rho} \left(\frac{d}{d\rho^2}\right)^{\ell} \left[\rho^{\ell+1} n_\ell(\rho)\right]^2 = -\sin(2\rho). \tag{2.10}$$

This eqn. can be easily verified for $\ell = 0, 1, 2$ and 3.

In Appendix (B), it is also proven that;

$$\underbrace{\left(\frac{d}{d\rho^2}\right)^{\ell} \frac{d}{d\rho} \left(\frac{d}{d\rho^2}\right)^{\ell} \left(\rho^{2\ell+2}\right)}_{\hat{\Theta}_\ell(\rho)} n_\ell(\rho) j_\ell(\rho) = -\cos(2\rho) \tag{2.11}$$

a result which can easily be verified for $\ell = 0, 1, 2$, and 3.

From (1.6), (2.10) and (2.11), it can be seen that

$$\hat{\Theta}_\ell(\rho) \left(j_\ell(\rho) + i n_\ell(\rho)\right)^2 = 2\sin(2\rho) - 2i\cos(2\rho)$$

or

$$\hat{\Theta}_\ell(\rho) \left[h_\ell^{(1)}(\rho)\right]^2 = -2ie^{+2i\rho} \tag{2.12}$$

where $h_\ell^{(1)}(\rho)$ are the spherical Hankel functions of the first kind [6] and,

$$\hat{\Theta}_\ell(\rho) \left(j_\ell(\rho) - i n_\ell(\rho)\right)^2 = 2\sin(2\rho) + 2i\cos(2\rho),$$

or

$$\hat{\Theta}_\ell(\rho) \left[h_\ell^{(2)}(\rho)\right]^2 = 2ie^{-2i\rho} \tag{2.13}$$

where $h_\ell^{(2)}(\rho)$ are the spherical Hankel functions of the second kind [6].

Furthermore;

$$\hat{\Theta}_\ell(\rho)\left(j_\ell^2(\rho)+n_\ell^2(\rho)\right)=0$$

Hence;

$$\hat{\Theta}_\ell(\rho)\left(h_\ell^{(1)}(\rho)h_\ell^{(2)}(\rho)\right)=0. \tag{2.14}$$

# CHAPTER 3

## Distorted Born Approximation and the Use of the Auxiliary Potential

It is shown in the literature [9] that the problem of solving the Schrödinger eqn. for two particles scattering from each other (masses $m_1$ and $m_2$) reduces to one of solving the relative motion wave eqn.

$$\left\{\frac{-\hbar^2}{2m}\frac{d^2}{dr^2}+\frac{\hbar^2\ell(\ell+1)}{2mr^2}+V(r)-E\right\}u(r)=0 \tag{3.1}$$

where $\frac{u(r)}{r} = R(r)$, $R(r)$ is the radial function of the scattered wave, $m$ is the reduced mass $\left(\frac{1}{m}=\frac{1}{m_1}+\frac{1}{m_2}\right)$, $E$ is the relative energy and $V(r)$ is the scattering potential in the relative angular momentum channel ($\ell$).

However, it is interesting to observe that $E$ is the kinetic energy of a particle whose mass is $m$ and whose speed is the relative speed $v$. Therefore, one can think of eqn. (3.1) as representing the scattering of a particle, of mass $m$, initial speed $v$ and kinetic energy $E=\frac{1}{2}mv^2$, from a fixed scattering potential $V(r)$ in the angular momentum channel ($\ell$). Here, $r$ is the radial distance of the particle $m$ from the scattering potential.

In the rest of this thesis, we will consider a particle of mass $m$ scattered off

potential $V_1(r)$, with the understanding that this discussion applies equally well to two-particle scattering via an interaction $V(\vec{r}_1 - \vec{r}_2)$ if one makes the *appropriate identifications.*

Now suppose one has two different potentials $V_1(r)$ and $V_2(r)$. Consider the problem of a particle of mass $m$ and energy $E$ scattered off each of these potentials in channel $(\ell)$. Eqn. (3.1) holds for each of these two potentials, namely,

$$\left\{\frac{-\hbar^2}{2m}\frac{d^2}{dr^2} + \frac{\hbar^2}{2m}\frac{\ell(\ell+1)}{r^2} + V_1(r) - E\right\} u_1(r) = 0$$

$$\left\{\frac{-\hbar^2}{2m}\frac{d^2}{dr^2} + \frac{\hbar^2}{2m}\frac{\ell(\ell+1)}{r^2} + V_2(r) - E\right\} u_2(r) = 0$$

where $\frac{u_1(r)}{r}$, $\frac{u_2(r)}{r}$ are the radial functions of the particle scattered from $V_1 r(r)$, $V_2(r)$ respectively.

Premultiplying the first by $u_2(r)$, the second by $u_1(r)$, then subtracting; one gets

$$\frac{-\hbar^2}{2m}\left\{u_2\frac{d^2}{dr^2}u_1 - u_1\frac{d^2}{dr^2}u_2\right\} + (V_1 - V_2)\, u_1(r)u_2(r) = 0$$

which is equivalent to

$$\frac{-\hbar^2}{2m}\left\{\frac{d}{dr}\left[u_2\frac{du_1}{dr} - u_1\frac{du_2}{dr}\right]\right\} + [V_1(r) - V_2(r)]\, u_1(r)u_2(r) = 0$$

or

$$\frac{d}{dr}\left[u_2\frac{du_1}{dr} - u_1\frac{du_2}{dr}\right] = \frac{2m}{\hbar^2}\,[V_1(r) - V_2(r)]\, u_1(r)u_2(r).$$

Integrating over $r$ from $(0, \infty)$; one gets

$$\left[u_2(r)\frac{du_1(r)}{dr} - u_1(r)\frac{du_2}{dr}\right]\Bigg|_0^\infty = \frac{2m}{\hbar^2}\int_0^\infty [V_1(r) - V_2(r)]u_1(r)u_2(r)dr \qquad (3.2)$$

Also, the radial wave function $R(r)$ is finite at the origin ($r = 0$), $\left(\frac{u(r)}{r}\Big|_{r=0} \rightarrow \text{ finite }\right)$.

Hence at $r = 0$, $u_1 = u_2 = 0$, while

$$\lim_{(r\to\infty)} R(r) = \lim_{(r\to\infty)} \frac{u(r)}{r} = \frac{A_\ell \sin\left(kr - \frac{\ell\pi}{2} + \delta_\ell(k)\right)}{kr}$$

where $k = \sqrt{\frac{2mE}{\hbar^2}}$, $\delta_\ell(k)$ is the phase shift of the wave function produced by the interaction (potential).

Therefore, for $u_1$ and $u_2$ as $r \rightarrow \infty$

$$\frac{u_1(r)}{r} \rightarrow \frac{A_1 \sin(kr - \ell\pi/2 + \delta_1)}{kr}$$

$$\frac{du_1(r)}{dr} \rightarrow A_1 \cos(kr - \ell\pi/2 + \delta_1)$$

$$\frac{u_2(r)}{r} \rightarrow \frac{A_2 \sin(kr - \ell\pi/2 + \delta_2)}{kr}$$

$$\frac{du_2}{dr} \rightarrow A_2 \cos(kr - \ell\pi/2 + \delta_2)$$

and,

$$\left[\left\{u_2\frac{du_1}{dr} - u_1\frac{du_2}{dr}\right\}\right]_{r\to\infty} \rightarrow \frac{A_1A_2 \sin(kr - \ell\pi/2 + \delta_2)}{k} \times$$
$$\cos(kr - \ell\pi/2 + \delta_1) - \frac{A_1A_2 \sin(kr - \ell\pi/2 + \delta_1)\cos(kr - \ell\pi/2 + \delta_2)}{k} = A_1A_2 \sin(\delta_2 - \delta_1)$$

Then, substituting this in eqn. (3.2), yields;

$$A_1A_2 \sin(\delta_2 - \delta_1) = \frac{-2mk}{\hbar^2}\int_0^\infty [V_2(r) - V_1(r)]\, u_1(r)u_2(r)dr. \qquad (3.3)$$

Eqn. (3.3) is the *Distorted Born Approximation* [10].

As a special case of this eqn., take

$$V_1 = 0 \Rightarrow \delta_1 = 0 \Rightarrow u_1(r) = rj_\ell(kr) \xrightarrow[r\to\infty]{} \frac{\sin(kr - \ell\pi/2)}{r}, \qquad \text{(i.e. } A_1 = 1)$$

then,

$$A_2 \sin(\delta_2) = \frac{-2mk}{\hbar^2} \int_0^\infty V_2(r) j_\ell(kr) u_2(r) r dr.$$

Now, approximating $u_2(r)$ by $(u_2(r) \simeq rj_\ell(kr))$, i.e. $A_2 = 1 \Rightarrow$

$$\sin(\delta_2) \simeq \frac{-2mk}{\hbar^2} \int_0^\infty V_2(r) j_\ell^2(kr) r^2 dr. \tag{3.4}$$

Eqn. (3.4) represents the *Simple Born Approximation* [11] Moreover, for small phase shifts ($\sin\delta_2 \approx \delta_2$), Eqn. (3.3) reduces to;

$$\frac{-\hbar^2}{2mk}\delta_\ell(k) \approx \int_0^\infty j_\ell^2(kr) V_e(r) r^2 dr$$

which is exactly eqn. (1.1) that was used before in [1]. It is thus a special case of the more general Distorted Born Approximation relation.

## Auxiliary Potential (Approach I):

As an aid to a better solution of the inverse scattering problem, eqn. (3.3) is considered with two similar potentials $V_1(r)$ and $V_2(r)$. In this case one can approximately take:

$$u_2(r) \approx u_1(r)$$

and

$$A_1(k) \approx A_2(k).$$

Assuming that $\underline{V_1(r)}$ is *known* and hence $\underline{u_1(r)}$, $\underline{A_1(k)}$ and $\underline{\delta_1(k)}$ are *known*. On the other hand, we assume $\underline{V_2(r)}$ is *not known*, but $\delta_2(k)$ is known. The main aim in inverse scattering is to estimate $V_2(r)$ from the knowledge of $\delta_2(k)$. This we do with the help of $V_1(r)$ and the corresponding $\delta_1(k)$.

Substituting $u_2(r) \approx u_1(r)$ and $A_1(k) \approx A_2(k)$ in eqn. (3.3)

$$A_1^2(k)\sin(\delta_2 - \delta_1) \approx \left(\frac{-2mk}{\hbar^2}\right)\int_0^\infty [V_2(r) - V_1(r)]\, u_1^2(r)dr. \qquad (3.a)$$

Taking $V_1(r)$ - the auxiliary potential - to be a finite range ($a_1$) square well (or barrier)

$$V_1(r) = \begin{cases} V_1 & r \leq a_1 \\ 0 & r > a_1 \end{cases}$$

where $V_1$ is a constant, either negative (attractive potential) or positive (repulsive potential). Then

$$u_1(r) = \begin{cases} C_\ell r j_\ell(Kr) & r \leq a_1 \\ A_\ell r\,[\cos(\delta_\ell) j_\ell(kr) - \sin(\delta_\ell) n_\ell(kr)] & r > a_1 \end{cases}$$

where;

$$K = \sqrt{\tfrac{2m}{\hbar^2}(E - V_1)}$$

$$k = \sqrt{\tfrac{2m}{\hbar^2}E}$$

$C_\ell$ and $A_\ell$ are constants.

Further, assume the unknown potential $V_2(r)$ is also of finite range $(a_2)$ such that $(a_2)$ is less than or equal to $(a_1)$ – the range of $V_1(r)$ – then, in eqn. (3.a) $u_1(r)$ can be replaced by $C_1 r j_\ell(Kr)$, i.e.

$$\frac{A_1^2(k)}{C_1^2(K)}\left(\frac{-\hbar^2}{2mk}\right)\sin(\delta_2-\delta_1) \approx \int_0^\infty [V_2(r)-V_1(r)]\, j_\ell^2(Kr) r^2 dr \qquad (3.b)$$

Notice that the upper limit of the integral in eqn. (3.b) is taken to be $(\infty)$ since the potentials $V_1(r)$, $V_2(r)$ are of finite ranges and are zero after those ranges $(a_1, a_2)$. Applying the differential operator;

$$\hat{\Theta}_\ell(K) \equiv (d/dK^2)^\ell d/dK (d/dK^2)^\ell K^{2\ell+2}$$

on the above relation (3.b) gives;

$$\int_0^\infty \hat{\Theta}_\ell(K) j^2(Kr)[V_2(r)-V_1(r)]\, r^2 dr \simeq \hat{\Theta}_\ell(K)$$

$$\left\{\frac{-A_1^2(k)}{C_1^2(K)}\left(\frac{\hbar^2}{2mk}\right)\sin(\delta_2-\delta_1)\right\}. \qquad (3.c)$$

Now, from eqn. (1.6), one gets

$$\hat{\Theta}_\ell(K) j_\ell^2(Kr) = r^{2\ell-1}\left(\frac{d}{d(Kr)^2}\right)^\ell \frac{d}{d(Kr)}\left(\frac{d}{d(Kr)^2}\right)^\ell \left\{(Kr)^{2\ell+2} j_\ell^2(Kr)\right\} = r^{2\ell-1}\sin(2Kr).$$

Thereby, (3.c) becomes;

$$\int_0^\infty \sin(2Kr)\,[V_2(r)-V_1(r)]\, r^{2\ell+1} dr \simeq \hat{\Theta}_\ell(K)\left\{\frac{-A_1^2}{C_1^2}\left(\frac{\hbar^2}{2mk}\right)\sin(\delta_2-\delta_1)\right\}.$$

Multiplying both sides by $\sin(2Kr')$ and integrating over $K$ from $(0 \to \infty)$, gives;

$$\int_0^\infty [V_2(r) - V_1(r)]\, r^{2\ell+1} dr \int_0^\infty \sin(2Kr)\sin(2Kr')dK \simeq$$

$$\int_0^\infty \sin(2Kr')dK\, \hat{\Theta}_\ell(K) \left[ -\frac{A_1^2}{C_1^2} \left( \frac{\hbar^2}{2mk} \right) \sin(\delta_2 - \delta_1) \right]. \tag{3.d}$$

But,

$$\int_0^\infty \sin(2Kr)\sin(2Kr')dK = \frac{\pi}{4} \{\delta(r - r') - \delta(r + r')\}.$$

Thus, the RHS of (3.d) becomes:

$$\frac{\pi}{4} \int_0^\infty [V_2(r) - V_1(r)]\, r^{2\ell+1} [\delta(r - r') - \delta(r + r')]\, dr$$

The second $\delta$-function, namely, $\delta(r + r')$ gives a zero contribution to the integral as explained before in Chapter I. Thus, eqn. (3.d) becomes

$$V_2(r') \simeq V_1(r') + \frac{4}{\pi r'^{2\ell+1}} \int_0^\infty \sin(2Kr')\, \hat{\Theta}_\ell(K) \left[ \frac{A_\ell^2}{C_\ell^2(K)} (-\hbar^2/2mk) \times \right.$$

$$\left. \sin(\delta_2(k) - \delta_1(k)) \right] dK. \tag{3.5}$$

One recalls that $A_\ell(k)$ and $C_\ell(K)$ arise from the wave function normalization:

$$\frac{u(r)}{r} = \begin{cases} C_\ell(K) j_\ell(Kr) & r \leq a_1 \\ A_\ell(k) \left[\cos(\delta) j_\ell(kr) - sin(\delta) n_\ell(kr)\right] & r > a_1 \end{cases}$$

Let us take two examples where one can easily find $(A_\ell/C_\ell)^2$ in eqn. (3.5).

<u>**Example (1)**</u>: Let $\ell = 0$, in this case:

$$\frac{u_1(r)}{r} = \begin{cases} C_0(K) j_0(Kr) = C_0(K)\frac{\sin(Kr)}{Kr} & r \le a_1 \\ A_0(k)\left\{\cos(\delta_0)\frac{\sin(kr)}{kr} + \sin(\delta_0)\frac{\cos(kr)}{kr}\right\} & r > a_1 \end{cases}$$

But $\frac{u_1(r)}{r}$ is continuous at $r = a_1$. Thus,

$$C_0(K)\frac{\sin(Ka_1)}{Ka_1} = A_0(k)\left\{\cos(\delta_0)\frac{\sin(ka_1)}{ka_1} + \sin(\delta_0)\frac{\cos(ka_1)}{ka_1}\right\},$$

$$\frac{A_0(k)}{C_0(K)} = \frac{\sin(Ka_1)}{K}\frac{k}{\sin(ka_1+\delta_0)} = \frac{k}{K}\frac{\sin(Ka_1)}{\sin(Ka_1+\delta_0)}.$$

Substituting this relation for $\left(\frac{A_0(k)}{C_0(K)}\right)$ in eqn. (3.5), one obtains, for $\ell = 0$;

$$V_2(r) = V_1(r) + \frac{4}{\pi r}\int_0^\infty \sin(2Kr)\left[\hat{\Theta}_{\ell=0}(K)\left\{\left(\frac{-\hbar^2}{2mk}\right)\frac{k^2}{K^2}\frac{\sin^2(Ka_1)}{\sin^2(ka_1+\delta_1)} \times \right.\right.$$

$$\left.\left.\sin(\delta_2-\delta_1)\right\}\right] dK. \tag{3.5.a}$$

where

$\delta_1$ is the phase shift corresponding to $V_1(r)$

$\delta_2$ is the phase shift corresponding to $V_2(r)$

Notice that these phase shifts are functions of $(k)$ or the corresponding energy. Moreover, they correspond to the $\ell = 0$ channel as indicated by the above example.

<u>**Example (2)**</u>: Let $\ell = 1$, in this case:

$$\frac{u_1(r)}{r} = \begin{cases} C(K)\left[\frac{\sin(Kr)}{(Kr)^2} - \frac{\cos(Kr)}{Kr)}\right] & r \le a_1 \\ A(k)\left[\cos(\delta_1)\left(\frac{\sin(kr)}{(kr)^2} - \frac{\cos(k_r)}{(kr)}\right) + \sin(\delta_1)\left(\frac{\cos(kr)}{(kr)^2} + \frac{\sin(kr)}{(kr)}\right)\right] & r > a_1 \end{cases}$$

Since $\frac{u_1(r)}{r}$ is continuous at $r = a_1$, thus

$$\left(\frac{A_1}{C_1}\right) = \frac{\left\{\frac{\sin(Ka_1)}{(Ka_1)^2} - \frac{\cos(Ka_1)}{(Ka_1)}\right\}}{\left\{\frac{\cos(\delta_1)\sin(ka_1)}{(ka_1)^2} - \frac{\cos(\delta_1)\cos(ka_1)}{(ka_1)} + \frac{\sin(\delta_1)\cos(ka_1)}{(ka_1)^2} + \frac{\sin(\delta_1)}{(ka_1)}\sin(ka_1)\right\}}$$

Simplifying this gives

$$\left(\frac{A_1}{C_1}\right) = \frac{k}{K}\left\{\frac{\frac{\sin(Ka_1)}{(Ka_1)} - \cos(Ka_1)}{\frac{\sin(ka_1+\delta_1)}{(ka_1)} - \cos(ka_1+\delta_1)}\right\}$$

Substituting this in eqn. (3.5), thus for $\ell = 1$, one obtains:

$$V_2(r) = V_1(r) + \frac{4}{\pi r^3}\int_0^\infty \hat{\Theta}_{\ell=1}(K)\left\{\left(\frac{-\hbar^2}{2m}\right)\frac{k^2}{K^2}\frac{\left[\frac{\sin(Ka_1)}{(Ka_1)} - \cos(Ka_1)\right]^2}{\left[\frac{\sin(ka_1+\delta_1)}{(ka_1)} - \cos(ka_1+\delta_1)\right]^2} \times \right.$$

$$\left. \sin(\delta_2 - \delta_1)\right\}\sin(2Kr)dK. \qquad (3.5.b)$$

Notice that $(k)$ is a function of $(K)$ since the actual variable is $E$ (energy) which is contained in both. Also $\delta_2$ and $\delta_1$ are functions of energy and hence functions of $(K)$. Here, (1) and (2) refer to potentials $V_1$ and $V_2$ respectively and the corresponding quantities (e.g., $\delta_1$ and $\delta_2$). The channel is $\ell = 1$ as indicated at the beginning of the example. In both eqns. (3.5.a), (3.5.b), one thus has a direct expression for $V_2(r)$ if $V_1(r)$, $\delta_1$ and $\delta_2$ are known.

Eqn. (3.5) contains, as a special case, eqn. (1.7) obtained in [1]. To see this, let $V_1(r)$ – the auxiliary potential – go to zero, hence $K \to k$, $\delta_1(k) \to 0$, $\frac{u_1(r)}{r} \to Bj(kr)$ for all $r$ and therefore $(A/C) \to 1$. Substituting these in eqn. (3.5) gives

$$V_2(r) \simeq \frac{4}{\pi r^{2\ell+1}}\int_0^\infty \left[\left(\frac{d}{dk^2}\right)^\ell \frac{d}{dk}\left(\frac{d}{dk^2}\right)^\ell k^{2\ell+1}\left\{\left(\frac{-\hbar^2}{2mk}\right) \times \right.\right.$$

$$\left.\left. \sin(\delta_2)\right\}\right]\sin(2kr)dk.$$

For small $\delta_2(k)$, $\sin(\delta_2) \sim \delta_2$ and

$$V_2(r) \approx \frac{4}{\pi r^{2\ell+1}} \int_0^\infty \left[\hat{\Theta}_\ell(k) \left\{\frac{-\hbar^2}{2mk}\, \delta_2(k)\right\}\right] \sin(2kr) dk$$

which is exactly eqn. (1.7). Consequently, eqn. (1.7) is a special case of eqn. (3.5) corresponding to $V_1 = 0$ and reasonably small phase shifts $\delta_2$.

## Auxiliary Potential (Approach II):

From eqn. (3.b),

$$\left(\frac{-\hbar^2}{2mk}\right) \frac{A_1^2}{C_1^2} \sin(\delta_2 - \delta_1) \simeq \int_0^\infty [V_2(r) - V_1(r)]\, j_\ell^2(Kr) r^2 dr. \tag{3.6}$$

Multiplying the above eqn. by $g_\ell(Kr')$ and integrating over $K$ from $(0 \to \infty)$;

$$\int_0^\infty \left[V_2(r) - V_1(r)\right] \int_0^\infty \left[j_\ell^2(Kr) g_\ell(Kr') dK\right] r^2 dr \simeq \int_0^\infty \left(\frac{-\hbar^2}{2mk}\right) \frac{A_1^2}{C_1^2} \sin(\delta_2 - \delta_1) \times$$

$$g_\ell(Kr') dK$$

But,

$$\int_0^\infty j_\ell^2(Kr) g_\ell(Kr') dK = \frac{r'^2}{r^2} \left\{\delta(r - r') + \delta(r + r') - (-1)^\ell 2\delta(r')\right\}$$

As explained before $\delta(r + r')$ does not contribute to the above integration. Hence:

$$r'^2 [V_2(r') - V_1(r')] - 2(-1)^\ell \delta(r') r'^2 \int_0^\infty [V_2(r) - V_1(r)]\, dr \simeq$$

$$\int_0^\infty \left(\frac{-\hbar^2}{2mk}\right) \frac{A_1^2}{C_1^2} \sin(\delta_2 - \delta_1) \cdot g_\ell(Kr') dK$$

or,

$$V_2(r') \simeq V_1(r') + \frac{1}{r'^2}\int_0^\infty \left(\frac{-\hbar^2}{2mk}\right)\frac{A_1^2}{C_1^2}\sin(\delta_2-\delta_1)g_\ell(Kr')dK$$

$$+2(-1)^\ell\delta(r')\int_0^\infty [V_2(r)-V_1(r)]\,dr. \tag{3.7}$$

However, from eqn. (3.6) as ($K \to \infty$) corresponding to ($E \to \infty$) one obtains;

$$\left(\frac{-\hbar^2}{2mk}\right)\frac{A_\ell^2}{C_\ell^2}\sin(\delta_2-\delta_1) \to \frac{1}{K^2}\int_0^\infty [V_2(r)-V_1(r)]\cdot\sin^2\left(Kr-\frac{\ell\pi}{2}\right)dr.$$

But,

$$\sin^2(\Theta) = \frac{1}{2}-\frac{1}{2}\cos(2\Theta).$$

Thus,

$$LHS \to \frac{1}{2K^2}\int_0^\infty [V_2(r)-V_1(r)]\,(1-\cos(2Kr-\ell r))dr.$$

But

$$\cos(\Theta-\ell\pi) = \cos(\Theta)\cos(\ell\pi)+\sin(\Theta)\sin(\ell\pi) = (-1)^\ell\cos(\Theta).$$

Hence:

$$LHS \to \frac{1}{2K^2}\int_0^\infty [V_2(r)-V_1(r)]\left(1+(-1)^{\ell+1}\cos(2Kr)\right)dr.$$

It was shown before that

$$\lim_{K\to\infty}\left\{\int_0^\infty [V_2(r)-V_1(r)]\cos(2Kr)dr\right\} = 0.$$

Therefore;

$$\lim_{K\to\infty}\left\{K^2\left(\frac{-\hbar^2}{2mk}\right)\frac{A_1^2}{C_1^2}\sin^2(\delta_2-\delta_1)\right\} = \frac{1}{2}\int_0^\infty [V_2(r)-V_1(r)]\,dr.$$

Substituting this in eqn. (3.7),

$$V_2(r) = V_1(r) + \frac{1}{r^2}\int_0^\infty -\left(\frac{-\hbar^2}{2mk}\right)\frac{A_\ell^2}{C_\ell^2}\sin(\delta_2-\delta_1)g_\ell(Kr)dK$$

$$+4(-1)^\ell\delta(r)\lim_{K'\to\infty}\left\{K'^2\left(\frac{-\hbar^2}{2mk'}\right)\frac{A_1'^2}{C_1'^2}\sin(\delta_2'-\delta_1')\right\}$$

where the limit term is differentiated from the other terms by the primes on the energy variables. From (2.a):

$$\int_0^\infty \frac{g_\ell(Kr)}{K^2r^2}dK = 4(-1)^{\ell+1}\delta(r).$$

Hence:

$$V_2(r) = V_1(r) + \frac{1}{r^2}\int_0^\infty\left(\frac{-\hbar^2}{2mk}\right)\frac{A_\ell^2}{C_\ell^2}\sin(\delta_2-\delta_1)g_\ell(Kr)dK$$

$$-\left(\int_0^\infty\frac{g_\ell(Kr)}{K^2r^2}dK\right) * \lim_{K'\to\infty}K'^2\cdot\left\{\left(\frac{-\hbar^2}{2mk'}\right)\frac{A_1'^2}{C_1'^2}\sin(\delta_2'-\delta_1')\right\}.$$

Thus, for any $\ell$:

$$V_2(r) = V_1(r) + \frac{1}{r^2}\int_0^\infty g_\ell(Kr)dK\left\{\left(\frac{-\hbar^2}{2mk}\right)\frac{A_\ell^2}{C_\ell^2}\sin(\delta_2-\delta_1) - \frac{1}{K^2}\lim_{K'\to\infty}K'^2\times\right.$$

$$\left.\left\{\left(\frac{-\hbar^2}{2mk'}\right)\frac{A_1'^2}{C_1'^2}\sin(\delta_2'-\delta_1')\right\}\right\}. \tag{3.8}$$

Moreover, as a special case of eqn. (3.8), take $V_1(r) = 0$ in which limit $K \to k$, $K' \to k'$, $\delta_1 = \delta_1' = 0$, $\frac{u_1(r)}{r} \to Bj_\ell(kr)$ for all $r$ and hence, $\frac{A_1}{C_1} = 1$. Substituting these in eqn. (3.8), one gets:

$$V_2(r) \simeq \frac{1}{r^2}\int_0^\infty g_\ell(kr)dk\left\{\left(\frac{-\hbar^2}{2mk}\right)\sin(\delta_2) - \frac{1}{k^2}\cdot\lim_{k'\to\infty}k'^2\times\right.$$

$$\left.\left\{\left(\frac{-\hbar^2}{2mk}\right)\sin(\delta_2')\right\}\right\}.$$

Further, if $\delta_2$ and $\delta_2'$ are reasonably small,

$$\sin(\delta_2) \sim \delta_2, \;\; \sin(\delta_2') \sim \delta_2'$$

Hence, in the limit $V_1 = 0$ and small $\delta_2$ one has:

$$V_2(r) \simeq \frac{1}{r^2}\int_0^\infty g_\ell(kr)dk\left\{\left(\frac{-\hbar^2}{2mk}\right)\delta_2(k) - \frac{1}{k^2}\lim_{k'\to\infty} k'^2 \times \right.$$

$$\left.\left\{\left(\frac{-\hbar^2}{2mk'}\right)\delta_2'(k)\right\}\right\}$$

which is exactly eqn. (1.5). Consequently, eqn. (1.5) is a special case of eqn. (3.8) for $V_1(r) = 0$ and small phase shifts $\delta_2$.

Let us consider two examples $\ell = 0$ and ($\ell = 1$) of eqn. (3.8):

<u>Example (3)</u>: In the case $\ell = 0$, $\frac{A}{C} = \frac{k}{K}\frac{\sin(Ka_1)}{\sin(ka_1+\delta_1)}$ a result obtained in example (1).

Substituting $(A/C)$ in eqn. (3.8):

$$V_2(r) \simeq V_1(r) + \frac{1}{r^2}\int_0^\infty g_0(Kr)dK\left\{\left(\frac{-\hbar^2}{2mk'}\right)\frac{k^2}{K^2}\frac{\sin^2(Ka_1)\cdot\sin(\delta_2-\delta_1)}{\sin^2(ka_1+\delta_1)} - \right.$$

$$\left.\frac{1}{K^2}\lim_{K'\to\infty} K'^2\left[\left(\frac{-\hbar^2}{2mk'}\right)\frac{\sin^2(K'a_1)}{\sin^2(ka_1+\delta_1')}\frac{k'^2}{K'^2}\sin(\delta_2'-\delta_1')\right]\right\}$$

Substituting the expression for $g_0(Kr)$ gives;

$$V_2(r) = V_1(r) + \frac{1}{r^2}\int_0^\infty \frac{-8}{\pi}K^2r^2\cos(2Kr)dK\left\{\left(\frac{-\hbar^2 k}{2m}\right)\frac{\sin^2(Ka_1)}{\sin^2(ka_1+\delta_1)} \times \right.$$

$$\left.\sin(\delta_2-\delta_1) - \frac{1}{K^2}\lim_{K'\to\infty} K'^2\left\{\left(\frac{-\hbar^2 k'}{2m}\right)\frac{\sin^2(K'a_1)}{\sin^2(k'a_1+\delta_1')}\frac{\sin(\delta_2'-\delta_1')}{K'^2}\right\}\right\}.$$

Thus

$$V_2(r) \simeq V_1(r) + \frac{8}{\pi}\int_0^\infty \cos(2Kr)dK \left\{\frac{\hbar^2 k}{2m}\frac{\sin^2(Ka_1)}{\sin^2(ka_1+\delta_1)}\sin(\delta_2-\delta_1)\right\} +$$

$$\lim_{K'\to\infty}\int_0^{K'} \frac{-8}{\pi}\cos(2Kr)dK \lim_{K'\to\infty}\left[\frac{\hbar^2 k'}{2m}\frac{\sin^2(K'a_1)}{\sin^2(k'a_1+\delta_1')}\sin^2(\delta_2'-\delta_1')\right].$$

But

$$\int_0^{K'} \frac{-8}{\pi}\cos(2Kr)dK = \frac{-4}{\pi r}\sin(2K'r),$$

thus,

$$V_2(r) \simeq V_1(r) + \frac{8}{\pi}\int_0^\infty \cos(2Kr)dK \left[\frac{\hbar^2 k}{2m}\frac{\sin^2(Ka_1)\sin(\delta_2-\delta_1)}{\sin^2(ka_1+\delta_1)}\right]$$

$$-\lim_{K'\to\infty}\left[\frac{4}{\pi r}\cdot\sin(2K'r)\cdot)\left(\frac{\hbar^2 k'}{2m}\right)\frac{\sin^2(K'a_1)\cdot\sin(\delta_2'-\delta_1')}{\sin^2(k'a_1+\delta_1')}\right] \quad (3.8.a)$$

where $\delta_1$, $\delta_1'$ and $\delta_2$, $\delta_2'$ are functions of energy and therefore functions of $(K)$ or $(K')$.

<u>Example (4)</u>: Let $(\ell = 1)$, in this case: $(A/C)^2 = \frac{k^2}{K^2}\left[\frac{\frac{\sin(Ka_1)}{(Ka_1)} - \cos(Ka_1)}{\frac{\sin(ka_1+\delta_1)}{ka_1} - \cos(ka_1+\delta_1)}\right]^2 \equiv \frac{k^2}{K^2}Y(Ka_1)$, which was obtained in example (2).

Substituting $(A/C)^2$ given above (for $\ell = 1$) in eqn. (3.8);

$$V_2(r) = V_1(r) + \frac{1}{r^2}\int_0^\infty \frac{g_1(Kr)}{K^2}\left(\frac{-\hbar^2 k)}{2m}\right)\sin(\delta_2-\delta_1)Y(Ka_1)dK$$

$$-\lim_{K'\to\infty}\left(\int_0^{K'}\frac{g_1(Kr)}{K^2r^2}dK\right)\cdot\lim_{K'\to\infty}\left[\frac{-\hbar^2 k'}{2m}\sin(\delta_2'-\delta_1')Y(K'a_1)\right]$$

From Table (I);

$$\int \frac{g_1(\rho)}{\rho^2} d\rho = \frac{4}{\pi}\left\{\left(1-\frac{1}{\rho^2}\right)\sin(2\rho)+\frac{2}{\rho}\cos(2\rho)\right\}. \tag{3.9}$$

Thus;

$$\begin{aligned} V_2(r) =& V_1(r) - \frac{1}{r^2}\int_0^\infty \left(\frac{\hbar^2 k}{2m}\right)\frac{\sin(\delta_2-\delta_1)g_1(Kr)}{K^2}Y(Ka_1)dK \\ &+ \lim_{K'\to\infty}\left\{\frac{4}{\pi r}\left[\left(1-\frac{1}{K'r)^2}\right)\sin(2K'r)+\frac{2}{K'r}\cos(2K'r)\right]\frac{\hbar^2 k'}{2m}\times \right. \\ &\left. \sin(\delta_2'-\delta_1')Y(K'a_1)\right\}. \end{aligned} \tag{3.8.b}$$

Notice that the RHS of eqn. (3.9) goes to zero at $\rho = 0$. Thus, eqns. (3.8.a) and (3.8.b) for the $\ell = 0$ and $\ell = 1$ channels respectively, represent two direct relations for $V_2$ if $V_1$, $\delta_1$ and $\delta_2$ are known.

In conclusion eqns. (3.5) and (3.8) represent the outcome of the two inversion techniques used in this research. They are direct relations for $V_2$ (for any $\ell$) if $V_1(r)$, and the corresponding phase shifts $\delta_1$ and $\delta_2$ are known. However, eqn. (3.4) involves a differential operator $\hat{\Theta}_\ell(K)$ which makes it inconvenient for numerical calculations. On the other hand, eqn. (3.8) does not involve any differential operations and will be used in the next chapter in some numerical calculations.

# CHAPTER 4

# Analytical and Numerical Examples

This chapter examines the accuracy of our technique for solving the inverse problem in quantum mechanical scattering theory. In obtaining eqns. (3.5) and (3.8) we made use of the Distorted Born approximation (eqn. 3.3)) and some mathematical techniques in order to obtain the unknown potential ($V_2(r)$) as a function of its phase shifts $\delta_2(k)$ and of some auxiliary potential $V_1(r)$ and the latter potential's phase shifts $\delta_1(k)$.

Before going to some numerical examples, let us consider the following *analytical examples*:

## Analytical Examples:

### Example (4.1):

Consider a square well (or barrier) potential $V_2(r)$ having a depth (or height) $V_2$ and range $a_2$, which produces phase shifts $\delta_2(k)$. The accuracy of our techniques will be reflected by how well one can reproduce this potential from eqn. (3.5) or eqn. (3.8).

Take an auxiliary square potential $V_1(r)$ of depth (or height) $V_1$ and range $a_1$ producing phase shifts $\delta_1(k)$. In this case, from eqn. (3.5):

$$V_2(r) \simeq V_1(r) + \frac{4}{\pi r^{2\ell+1}} \int_0^\infty \sin(2Kr)dK \cdot \hat{\Theta}_\ell(K) \left\{ \frac{A_\ell^2}{C_\ell^2} \left( \frac{-\hbar^2}{2mk} \right) \sin(\delta_2(k) - \delta_1(k)) \right\}. \tag{3.5}$$

But

$$\hat{\Theta}_\ell(K) = \left( \frac{d}{dK^2} \right)^\ell \frac{d}{dK} \left( \frac{d}{dK^2} \right)^\ell K^{2\ell+2}.$$

Moreover, from eqn. (3.a):

$$\frac{A_\ell^2}{C_\ell^2} \left( \frac{-\hbar^2}{2mk} \right) \sin(\delta_2(k) - \delta_1(k)) \simeq \int_0^\infty [V_2(r') - V_1(r')] j_\ell^2(Kr') r'^2 dr'. \tag{3.a}$$

Substituting these quantities in eqn. (3.5) shown above:

$$V_2(r) \simeq V_1(r) + \frac{4}{\pi r^{2\ell+1}} \int_0^\infty \sin(2Kr)dK \cdot \hat{\Theta}_\ell(K) \left\{ \int_0^\infty [V_2(r') - V_1(r')] j_\ell^2(Kr') r'^2 dr' \right\}.$$

Consider the simplest case $\ell = 0$:

$$V_2(r) \simeq V_1(r) + \frac{4}{\pi r} \int_0^\infty \sin(2Kr)dK \cdot \frac{d}{dK} \left\{ (K^2) \cdot \int_0^\infty [V_2(r') - V_1(r')] \frac{\sin^2(Kr')}{K^2 r'^2} r'^2 dr' \right\}$$

Integrating over $r'$:

$$V_2(r) \simeq V_1(r) + \frac{4}{\pi r} \int_0^\infty \sin(2Kr)dK \cdot \frac{d}{dK} \left\{ K^2 \left[ \frac{V_2 a_2 - V_1 a_1}{2K^2} + \frac{V_1 \sin(2Ka_1)}{4K^3} \right. \right.$$
$$\left. \left. \frac{-V_2 \sin(2Ka_2)}{4K^3} \right] \right\}$$

$$V_2(r) \simeq V_1(r) + \frac{4}{\pi r} \left[ \int_0^\infty \sin(2Kr) \{ V_1 a_1 \cos(2Ka_1) - V_2 a_2 \cos(2Ka_2) \} dK \right.$$
$$\left. + \int_0^\infty \sin(2Kr) \left( \frac{V_2 \sin(2Ka_2) - V_1 \sin(2Ka_1)}{4K^2} \right) dK \right]$$

$$V_2(r) \simeq V_1(r) + \frac{V_2}{\pi r}\left[\int_0^\infty \frac{\sin(2Kr)(\sin(2Ka_2)dK}{K^2} - a_2\int_0^\infty \frac{\sin(2K(r-a_2)) + \sin(2K(r+a_2))}{k}dK\right] +$$

$$\left(\frac{-V_1}{\pi r}\right)\left[\int_0^\infty \frac{\sin(2Kr)\sin(2Ka_1)}{K^2}dK - a_1\int_0^\infty \frac{\sin(2K(r-a_1)) + \sin(2K(r+a_1))}{K}dK\right] \tag{4.1}$$

But,

$$\int_0^\infty \frac{\sin(2Kr)\sin(2Ka)}{K^2}\,dK - a\int_0^\infty \frac{\sin(2K(r-a)) + \sin(2K(r+a))dK}{K}$$

$$= \begin{cases} \left(\left(\frac{\pi}{2}\right)2a - \pi a\right) = 0, & r > a \\ \left(\frac{\pi}{2}\right)2r = \pi r \quad , & r \le a \end{cases}$$

and

$$V_1(r) = \begin{cases} V_1 & r \le a_1 \\ 0 & r > a_1 \end{cases}$$

Therefore, eqn. (4.1) becomes:

<u>(i) $(a_1 \ge a_2)$</u>:

$$V_2(r) \simeq \begin{cases} V_1 + V_2 + (-V_1) = V_2 & r \le a_2 \le a_1 \\ V_1 + 0.0 + (-V_1) = 0.0 & a_2 < r \le a_1 \\ 0.0 + 0.0 + 0.0 = 0.0 & a_2 \le a_1 < r \end{cases}$$

<u>(ii) $(a_2 \ge a_1)$</u>:

$$V_2(r) \simeq \begin{cases} V_1 + V_2 + (-V_1) = V_2 & r \le a_1 \le a_2 \\ 0 + V_2 + 0 = V_2 & a_1 < r \le a_2 \\ 0 + 0 + 0 = 0 & a_1 \le a_2 < r \end{cases}$$

Cases (i) and (ii) give exactly the potential we started with at the beginning of this example.

**Example (4.2):**

Consider the same case discussed in example (4.1). Here, we make use of eqn. (3.8) instead of eqn. (3.5) namely:

$$V_2(r) \simeq \quad V_1(r) + \frac{1}{r^2}\int_0^\infty g_\ell(Kr)dK\left\{\left(\frac{-\hbar^2}{2mk}\right)\frac{A_\ell^2}{C_\ell^2}\cdot\sin(\delta_2(k)-\delta_1(k)) - \frac{1}{K^2}\lim_{K'\to\infty} \times\right.$$

$$\left. K'^2\left\{\left(\frac{-\hbar^2}{2mk'}\right)\frac{A_\ell'^1}{C_\ell'^2}\sin\left(\delta_2'(k')-\delta_1'(k')\right)\right\}\right. .$$

Making use of eqn. (3.a) in example (4.1) above

$$V_2(r) \simeq V_1(r) + \frac{1}{r^2}\int_0^\infty g_\ell(Kr)dK\left\{\left\{\int_0^\infty [V_2(r')-V_1(r')]j_\ell^2(Kr')r'^2dr'\right\} - \frac{1}{K^2}\lim_{K'\to\infty} \times\right.$$

$$\left. K'^2\left\{\int_0^\infty [V_2(r')-V_1(r')]j_\ell^2(K'r')r'^2dr'\right\}\right\}$$

Consider $\ell = 0$ (for simplicity):

$$V_2(r) \simeq V_1(r) + \frac{1}{r^2}\int_0^\infty \frac{-8K^2r^2}{\pi}\cos(2Kr)dK\left\{\int_0^\infty [V_2(r')-V_1(r')]\frac{\sin^2(Kr')}{K^2r'^2}r'^2dr'\right.$$

$$\left. -\frac{1}{K^2}\lim_{K'\to\infty} K'^2\left[\int_0^\infty [V_2(r')-V_1(r')]\frac{\sin^2(K'r'^2)}{K'^2r'^2}r'^2dr'\right]\right\}$$

Integrating over $r'$:

$$V_2(r) \simeq V_1(r) + \frac{1}{r^2}\int_0^\infty \frac{-8K^2r^2}{\pi}\cos(2Kr)dK\left\{\left[\frac{V_2a_2 - V_1a_1}{2K^2} + \frac{V_1\sin(2Ka_1) - V_2\sin(2Ka_2)}{4K^3}\right]\right.$$

$$\left. -\frac{1}{K^2}\lim_{K'\to\infty} K'^2\left[\frac{V_2a_2 - V_1a_1}{2K'^2} + \frac{V_1\sin(2K'a_1) - V_2\sin(2K'a_2)}{4K'^3}\right]\right\}$$

$$V_2(r) \simeq V_1(r) + \int_0^\infty\left(\frac{-4}{\pi}\right)V_2a_2\cos(2Kr)dK + \int_0^\infty\left(\frac{4V_1a_1}{\pi}\right)\cos(2Kr)dK$$

$$+\int_0^\infty\left(\frac{-2V_1}{\pi}\right)\frac{\sin(2Ka_1)\cos(2Kr)}{K}dK + \int_0^\infty\left(\frac{+2V_2}{\pi}\right)\frac{\sin(2Ka_2)\cos(2Kr)}{K}dK$$

$$+\int_0^\infty\left(\frac{4}{\pi}V_2a_2\right)\cos(2Kr)dK + \int_0^\infty\left(\frac{-4}{\pi}V_1a_1\right)\cos(2Kr)dK + 0.0.$$

Simplifying:

$$V_2(r) \simeq V_1(r) + \int_0^\infty \left(\frac{V_2}{\pi}\right)\left[\frac{\sin(2K(a_2-r)) + \sin(2K(a_2+r)}{K}\right] dK$$

$$- \int_0^\infty \left(\frac{V_1}{\pi}\right)\left[\frac{\sin(2K(a_1-r)) + \sin(2K(a_1+r)}{K}\right] dK$$

But

$$\int_0^\infty \frac{\sin(2K(a-r))}{K} + \frac{\sin(2K(a+r))}{K} dK =$$

$$\begin{cases} \pi & r \leq a \\ 0 & r > a \end{cases}$$

and

$$V_1(r) = \begin{cases} V_1 & r \leq a_1 \\ 0 & r > a_1 \end{cases}$$

<u>(i) $(a_1 \geq a_2)$</u>:

$$V_2(r) \simeq \begin{cases} V_1 + V_2 - V_1 = V_2 & r \leq a_2 \leq a_1 \\ V_1 + 0 - V_1 = 0 & a_2 < r \leq a_1 \\ 0 + 0 + 0 = 0 & a_2 \leq a_1 \leq r \end{cases}$$

<u>(ii) $(a_2 \geq a_1)$</u>:

$$V_2(r) \simeq \begin{cases} V_1 + V_2 - V_1 = V_2 & r \leq a_1 \leq a_2 \\ 0 + V_2 + 0 = V_2 & a_1 < r \leq a_2 \\ 0 + 0 + 0 = 0 & a_1 \leq a_2 < r \end{cases}$$

Cases (i) and (ii) give exactly the potential we started with in this example.

Therefore, as expected if the Distorted Born Approximation phase shifts (eqn. (3.a)) are substituted in eqn. (3.5) or eqn. (3.8), and the integration is taken over all $K(0 \rightarrow \infty)$ the potential $(V_2(r))$ is reproduced exactly. Notice that eqn. (3.a) gives expressions for the phase shifts for all $K$, which enables us to carry out the $K$ integration over all $K$ in obtaining the potential. However, if instead of using the Distorted Born Approximation phase shifts, we use exact or what we call "experimental" phase shifts (obtained by matching the exact wave functions and their derivatives at the edges ($r = a_1$ and $r = a_2$) of the square well (or barrier) potentials), up to what $K = K_{\max}$ can one integrate the expressions given by eqns. (3.5) and (3.8)? One notes that the Distorted Born Approximation eqn. (3.3) and therefore all the subsequent relations obtained from it (e.g. eqns.(3.5) and (3.8)) are based on non-relativistic quantum mechanical derivations as shown in chapter 3. Thus, our formalism is applicable *only* for non-relativistic energies and therefore $K = K_{\max}$ is fixed by some maximum non-relativistic kinetic energy

$$E_{\max} = \frac{1}{2} m v_{\max}^2 \quad \text{of the scattered particle} \left( \text{i.e.} K_{\max} = \sqrt{\frac{2m}{\hbar^2}(E_{\max} - V_1)} \right),$$

where $E_{\max}$ is smaller than $mc^2$.

## Numerical Examples:

Consider the scattering of a nucleon ($m \simeq 940\ MeV$) from a square well (or barrier) potential $(V_2(r))$ of depth (or height) $V_2$ and range $a_2$. In this case one

has to truncate the integrations given in eqns. (3.5) and (3.8) at some $K = K_{\max}$ where the non-relativistic formalism is applicable. Here, $m \simeq 940\ MeV$, thus up till $E = E_{\max} = 350\ \ MeV$ (a little more than a third of the rest mass $m$) the particle will behave non-relativistically and our formalism is applicable. Therefore, we choose $K_{\max} = \sqrt{\frac{2m}{\hbar^2}(350 - V_1)}$.

One thing still remains to be decided on. That is, what auxiliary potential $V_1(r)$ one should take to obtain the best results? One would expect that the best auxiliary potential is the one that produces the most similar phase shifts ($\delta_1(k)$) to the phase shifts of the actual potential ($\delta_2(k)$). To clarify this point, we will consider eqn. (3.8), because it is the easiest expression to use, in the following numerical examples.

**Example (4.3):**

Consider the scattering of a nucleon from a square barrier potential ($V_2(r)$)) in the channel $\ell = 1$, where $V_2(r)$ has height $V_2 = 10\ \ MeV$ and range $a_2 = 2\ fm$. Take for the auxiliary potential $V_1(r)$ a square barrier of height $V_1 = 6.5\ MeV$ and ranges 2.5, 2.2 and 2.0 $\ fm$, (see figures # 1, 2, and 3). In this case, the integral $\int_0^\infty (\cdots) dK$ becomes

$$\int_{K_{\min}}^{K_{\max}} (\cdots) dK$$

where, $K_{\min} = 0.0$, (since $V_1 \geq 0$ and $K = 0$ at $E = V_1 = 10\ \ MeV$) ,

$K_{\max} = \sqrt{\frac{2m}{\hbar^2}(350-10)} = \sqrt{\frac{2m}{\hbar^2}(340)}\ (1/fm).$

The corresponding "experimental" phase shifts and the results (reproduced potential $V_2(r)$) of the numerical calculations of eqn. (3.8.b) are shown in figures 1, 2 and 3.

Observe that in figure (1.b) the reproduced potential has a strange behavior at $r \approx 2.5$ fm.

In figure (1.c) the potential $V_2(r)$ is plotted together with the two contributing terms of eqn. (3.8.b):

$$V_2(r) = V_1(r) - \frac{1}{r^2}\int_0^\infty \left(\frac{\hbar^2 k}{2m}\right) \frac{\sin(\delta_2(k)-\delta_1(k))g_1(Kr)}{K^2} Y(Ka_1)dK$$

$$+ \lim_{K'\to\infty}\left\{\frac{4}{\pi r}\left[\left(1-\frac{1}{(K'r)^2}\right)\sin(2K'r) + \frac{2}{K'r}\cos(2K'r)\right] \times \right.$$

$$\left. \frac{\hbar k'}{2m}\sin(\delta_2(k)-\delta_1(k))Y(K'a_1)\right\}. \qquad (3.8.b)$$

or, $V_2(r) = V_1(r) -$ integral term

$$+ \text{ limit term} \qquad (3.8.b)$$

Since the range of $V_1(r)$ is taken to be 2.5 fm, while that of $V_2(r)$ is 2.0 fm, $V_2(r)$ has to go to zero after 2.0 $fm$ meaning that the integral term and the limit term should cancel the effect of $V_1(r)$ after 2.0 $fm$. From figure (1.c), we observe that the limit term starts to die off (oscillate around zero) after some distance $r$.

However, the integral term goes up to $\sim 6.5\ MeV$ between 2.0 and 2.5 $fm$ trying to cancel the effect of the auxiliary potential there (shown in figure (1.d)).

However, in figure (2.b) the reproduced potential is better than that in figure (1.b). In fact, the phase shifts in figure (2.a) are more similar than those in figure (1.a). Nevertheless, the behavior of the phase shifts in figure (2.a) as functions of energy or ($K$) is not completely similar (they have their turning points and inflection points at *different* values of $K$).

Finally, in figure (3.a), the phase shifts are so similar and behave perfectly the same as a function of $K$. Moreover, the reproduced potential is very good (if not excellent).

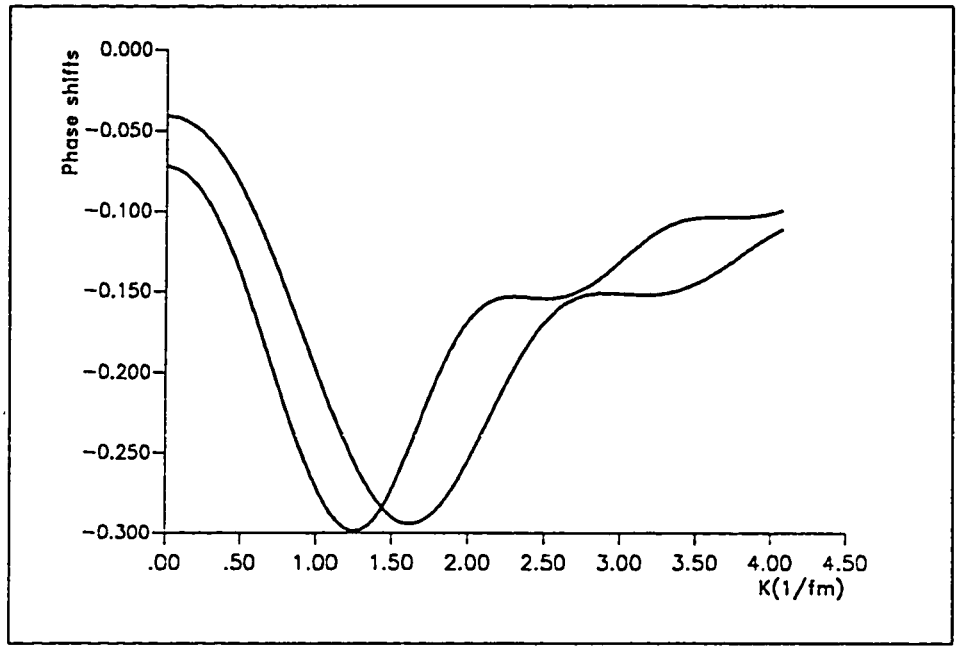


Figure (1.a): "Experimental" phase shifts produced by a nucleon scattering (in $\ell = 1$) from: (i) square barrier potential of height 10 MeV, width 2 fm (ii) square barrier potential of height 6.5 MeV, width 2.5 fm.

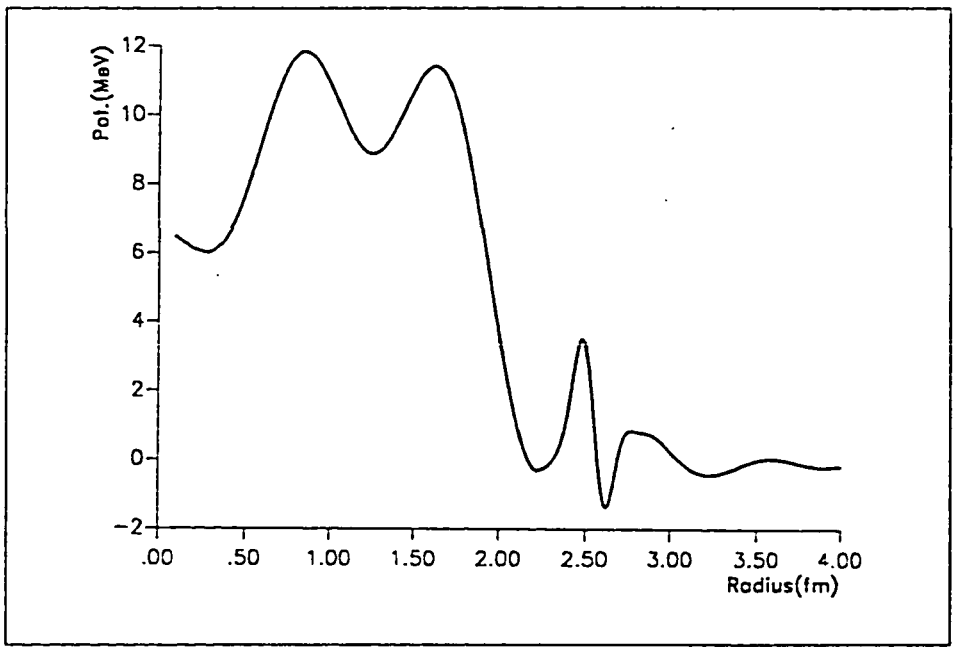


Figure (1.b): Potential calculated from eqn. (3.8.b) using the phase shifts shown above to reproduce the square barrier potential ($V_2(r)$) of height 10 MeV, width 2 fm when the auxiliary potential ($V_1(r)$) has height 6.5 MeV, width 2.5 fm

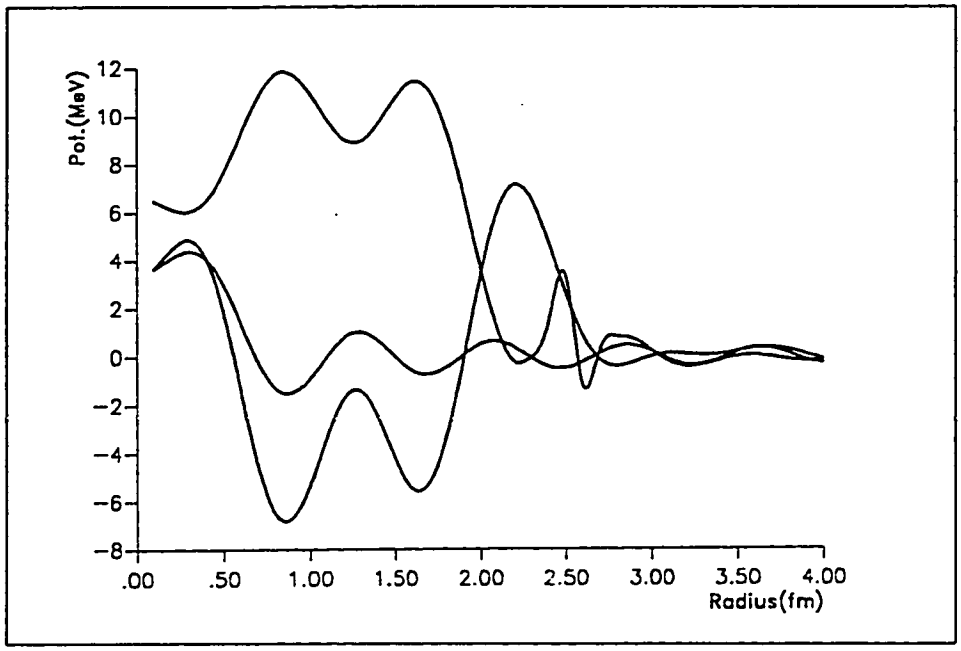


Figure (1.c): The potential shown in Figure (1.b) with the integral term and the limit term shown in eqn. (3.8.b).

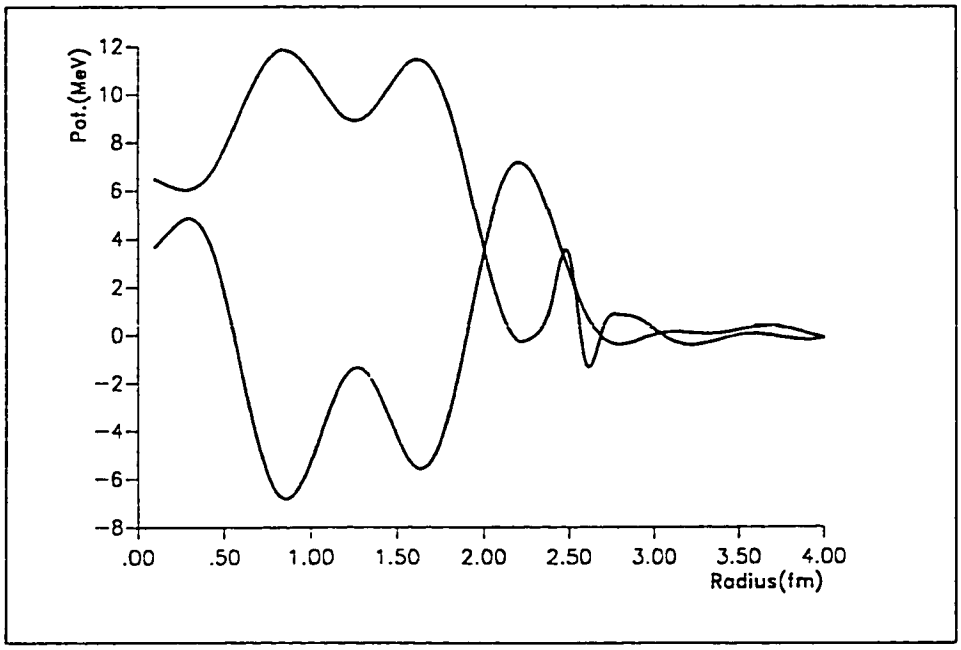


Figure (1.d): The potential shown in Figure (1.b) with the integral term only.

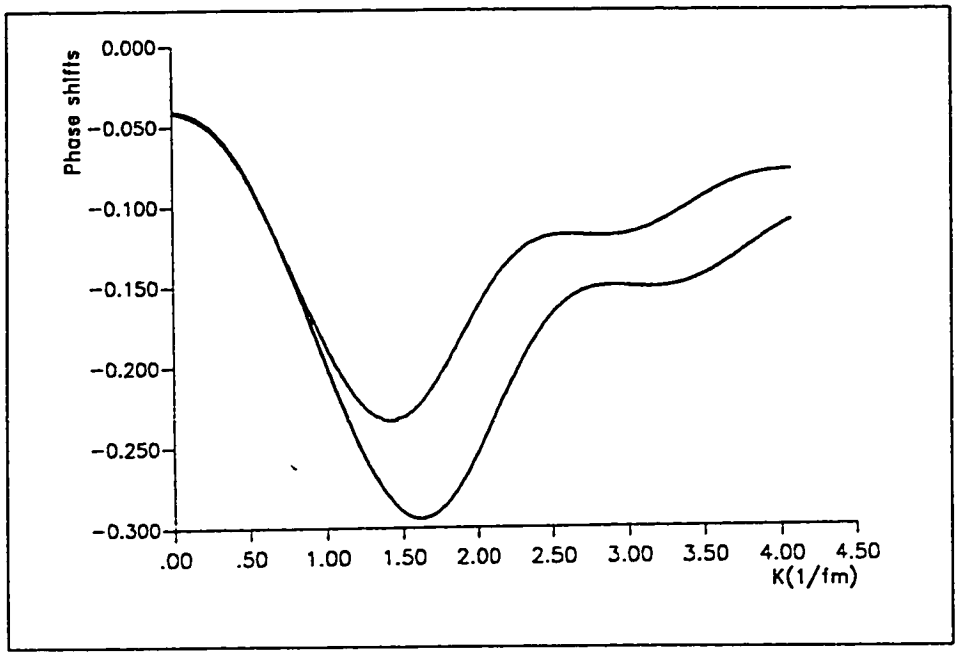


Figure (2.a): "Experimental" phase shifts produced by a nucleon scattering (in $\ell = 1$) from: (i) square barrier potential of height 10 MeV, width 2.0 fm (ii) square barrier potential of height 6.5 MeV, width 2.2 fm

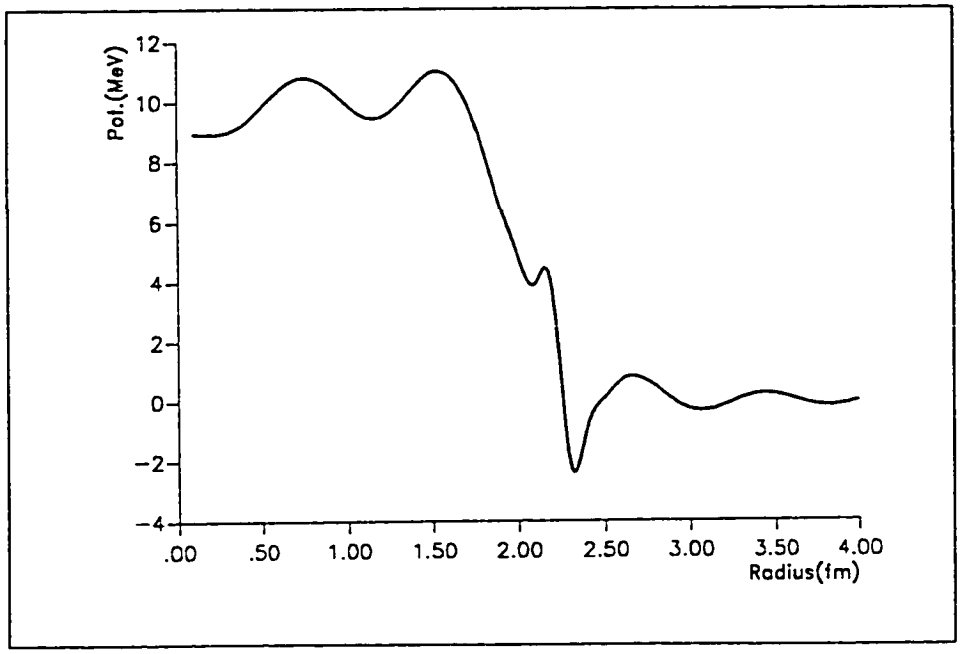


Figure (2.b): Potential calculated from eqn. (3.8.b) using the phase shifts shown above to reproduce the square barrier potential ($V_2(r)$) of height 10 MeV, width 2.0 fm when the auxiliary potential ($V_1(r)$) has height 6.5 MeV, width 2.2 fm

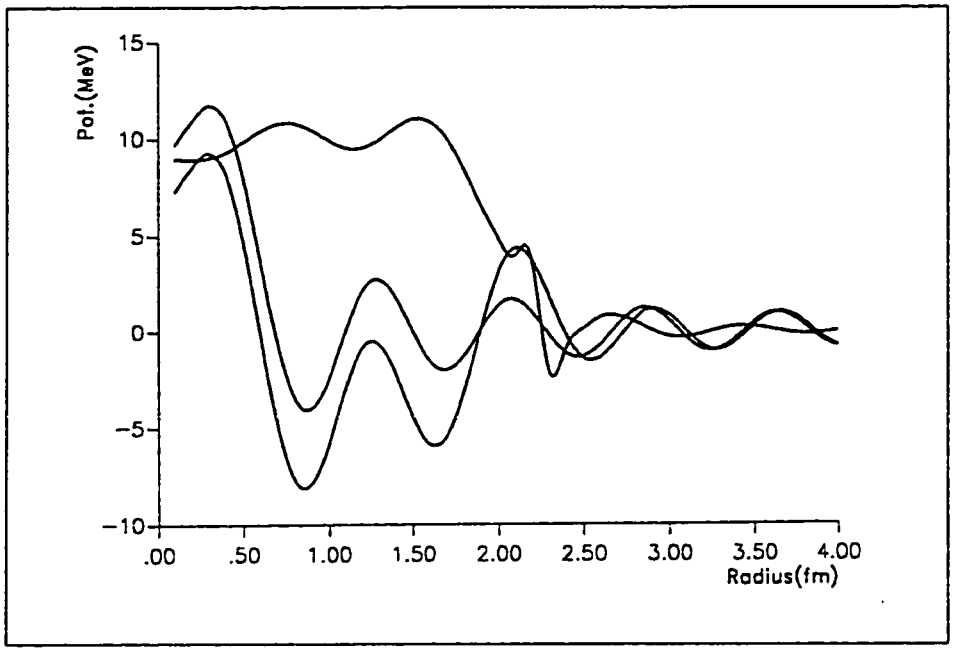


Figure (2.c): The potential shown in Figure (2.b) with the integral term and the limit term shown in eqn. (3.8.b).

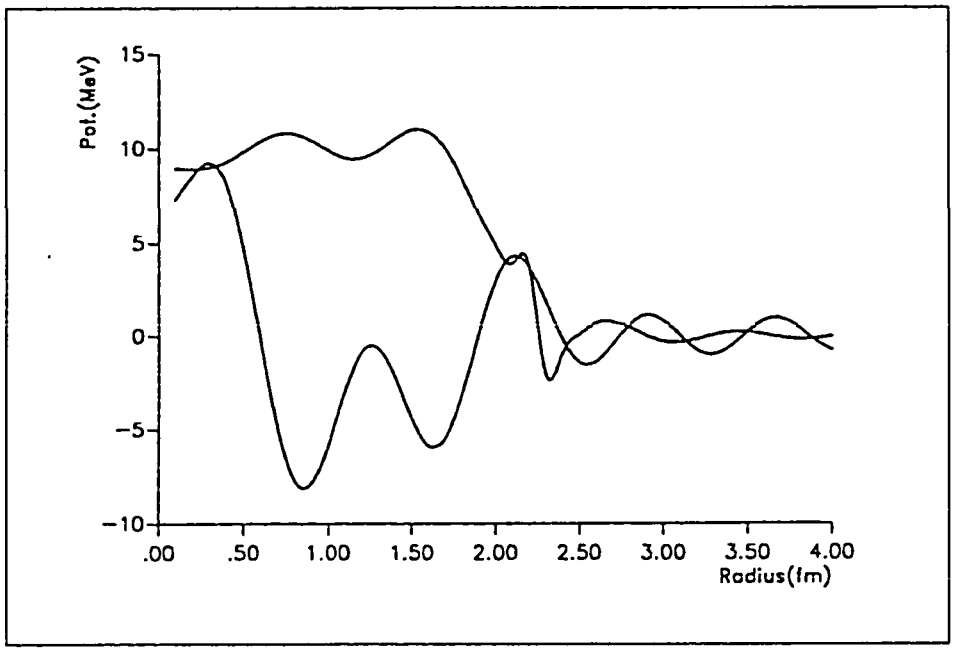


Figure (2.d): The potential shown in Figure (2.b) with the integral term only.

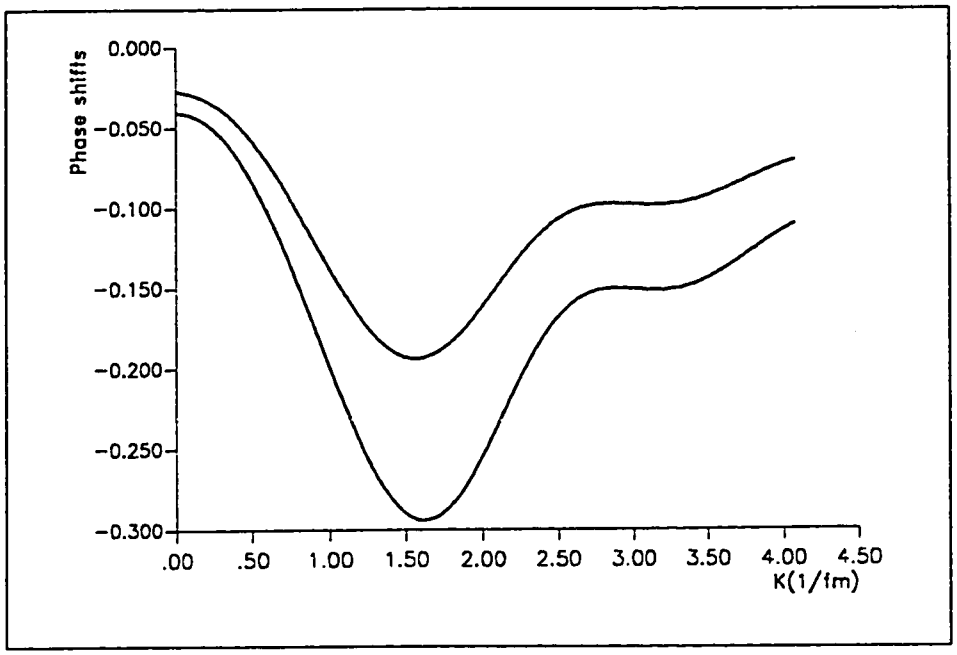


Figure (3.a): "Experimental" phase shifts produced by a nucleon scattering (in $\ell = 1$) from: (i) square barrier potential of height 10 MeV, width 2.0 fm (ii) square barrier potential of height 6.5 MeV, width 2.0 fm.

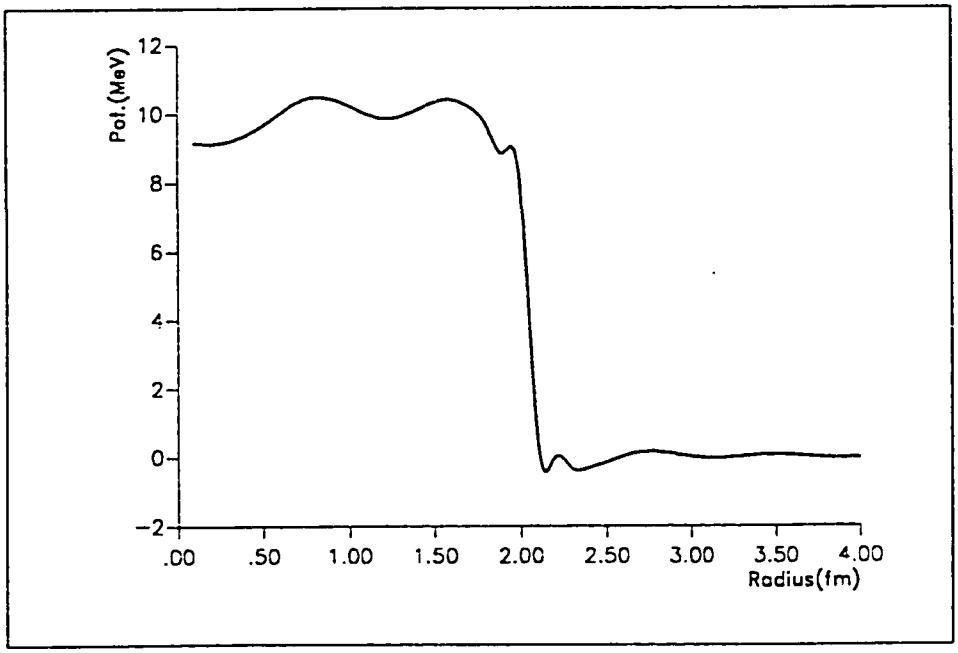


Figure (3.b): Potential calculated from eqn. (3.8.b) using the phase shifts shown above to reproduce the square barrier potential ($V_2(r)$) of height 10 MeV, width 2.0 fm when the auxiliary potential ($V_1(r)$) has height 6.5 MeV, width 2.0 fm

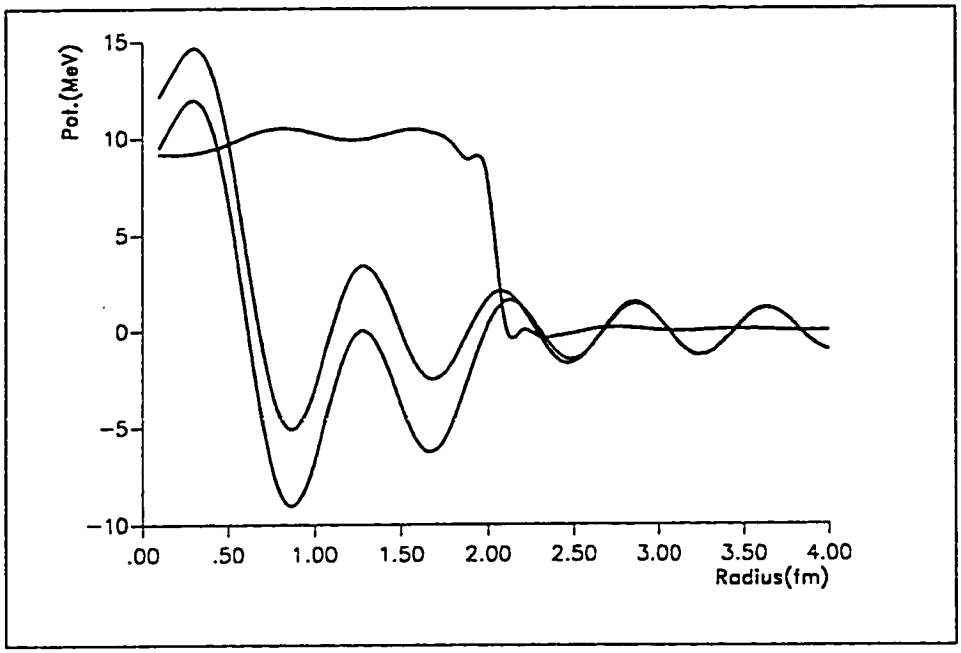


Figure (3.c): The potential shown in Figure (3.b) with the integral term and the limit term shown in eqn. (3.8.b).

In conclusion, the best auxiliary potential is the one that produces the most similar phase shifts ($\delta_1(k)$) to the phase shifts of the actual potential ($\delta_2(k)$). The most similar phase shifts ($\delta_1(k)$) are those that behave exactly (or very nearly) the same as ($\delta_2(k)$) of the unknown potential as a function of $K$ or energy (i.e. $\delta_1(k)$ achieves the turning points and inflection points at the same energy as $\delta_2(k)$ does). This occurs when $V_1(r)$ has the same range as $V_2(r)$. After fixing the range ($a_1$), one starts to adjust the strength of the auxiliary potential to make the difference between $\delta_2(k)$ and $\delta_1(k)$ as small as desired. This in turn will reflect positively on the calculation of the potential $V_2(r)$. In the following examples, we take an auxiliary potential of the same range as the actual one and choose different strength to see how good the calculations are for each case.

Example (4.4):

Consider the scattering of a nucleon from a square barrier potential ($V_2(r)$) in the channels $\ell = 0$ and $\ell = 1$.

Case (I): $V_2(r)$ has height $V_2 = 5\ MeV$ and range $a_2 = 2\ fm$. Use auxiliary potential $V_1(r)$ of width $a_1 = 2.0\ fm$ height $V_1 = 0.0$, 4.0 and 4.5 $MeV$. The phase shifts and the results of the numerical calculations of eqns. (3.8.a) and (3.8.b) are shown in figures 4, 5 and 6 for $\ell = 0$, and in figures 7, 8, 9 for $\ell = 1$. Notice that the closer the two sets of phase shifts are the better the reproduced potential $V_2(r)$,

which is as expected. Also, the higher the $\ell$ is the better the reproduced potential since then the phase shifts become smaller in magnitude, (because increasing $\ell$ also increases the centrifugal barrier: $\frac{\hbar^2 \ell(\ell+1)}{2mr^2}$). Furthermore, notice that in all the cases, the stronger potential produces larger phase shift in magnitude.

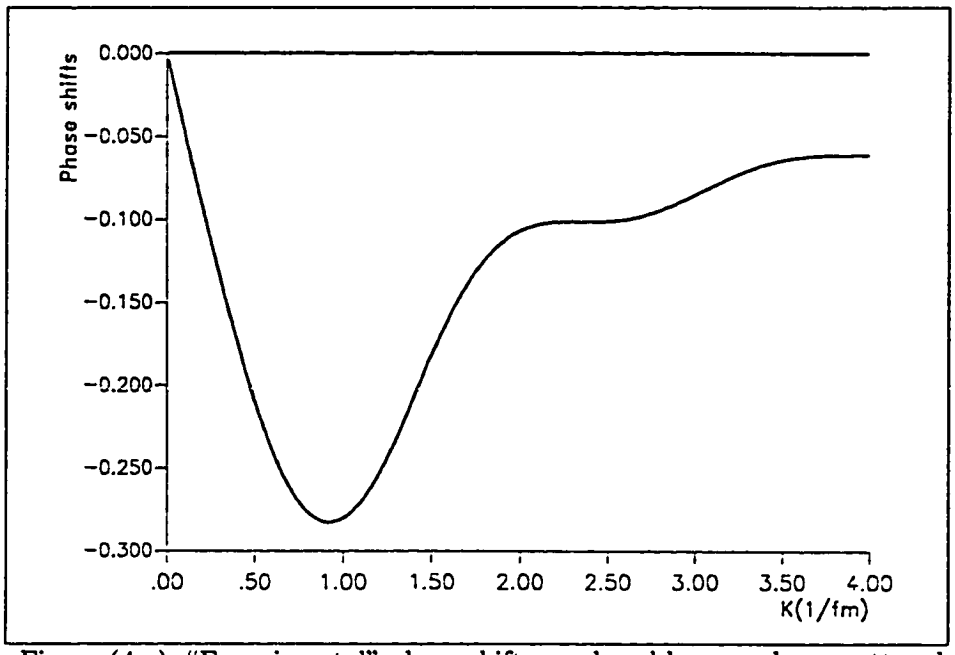


Figure (4.a): "Experimental" phase shifts produced by a nucleon scattered (in $\ell = 0$) from: (i) square barrier potential of height 5 MeV, width 2 fm ($V_2(r)$), and (ii) a square barrier potential of height 0.0 MeV, width 2 fm ($V_1(r)$).

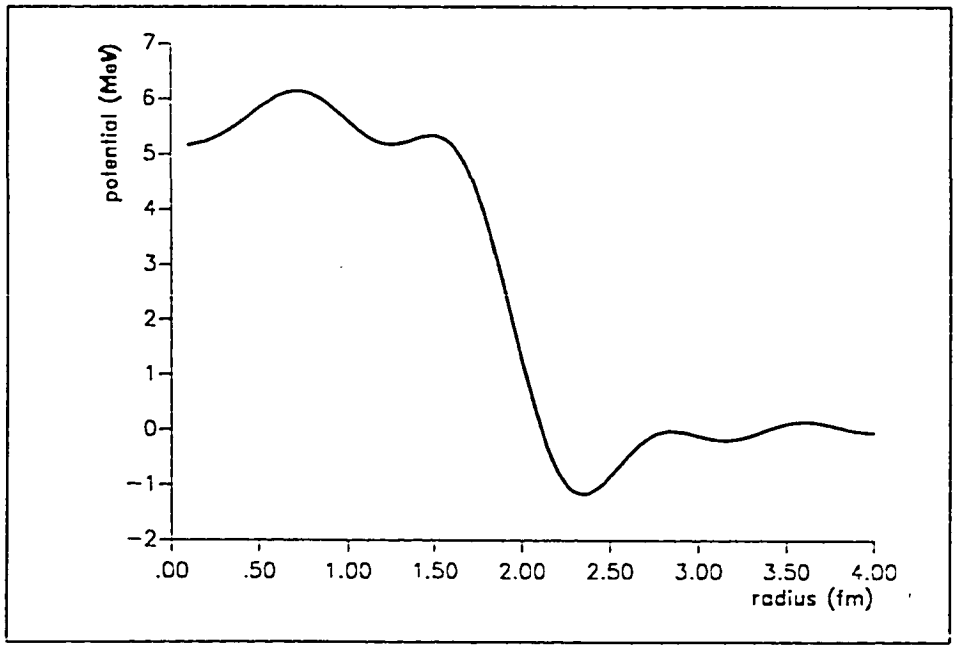


Figure (4.b): Potential calculated from eqn. (3.8.a) using the phase shifts shown above to reproduce the square barrier potential ($V_2(r)$) of height 5 MeV, width 2 fm when the auxiliary potential ($V_1(r)$) has height 0.0 MeV, width 2 fm

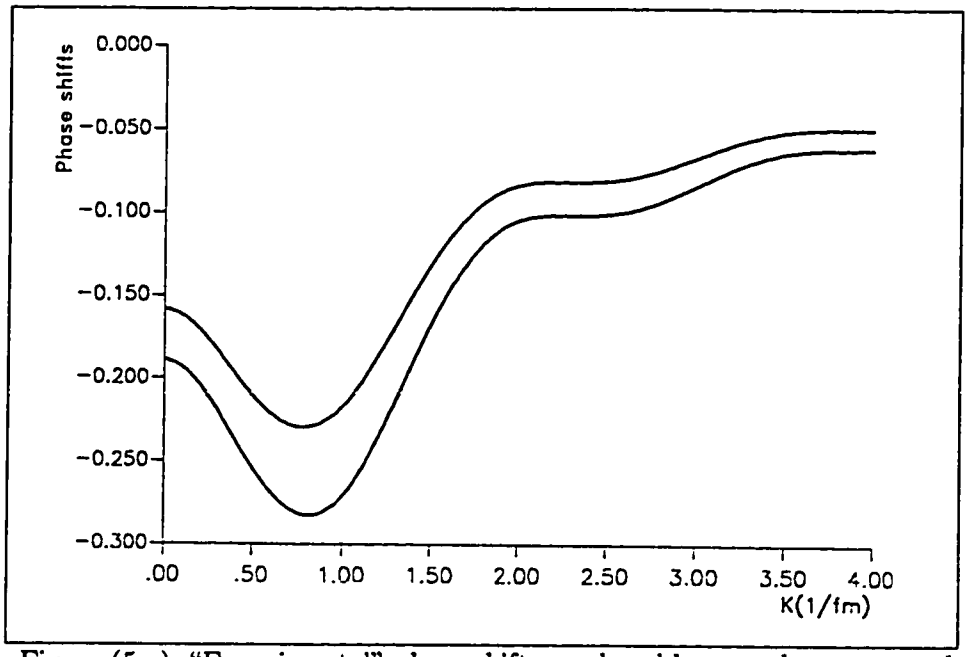


Figure (5.a): "Experimental" phase shifts produced by a nucleon scattered (in $\ell = 0$) from: (i) square barrier potential of height 5 MeV, width 2 fm ($V_2(r)$), and (ii) a square barrier potential of height 4 MeV, width 2 fm ($V_1(r)$).

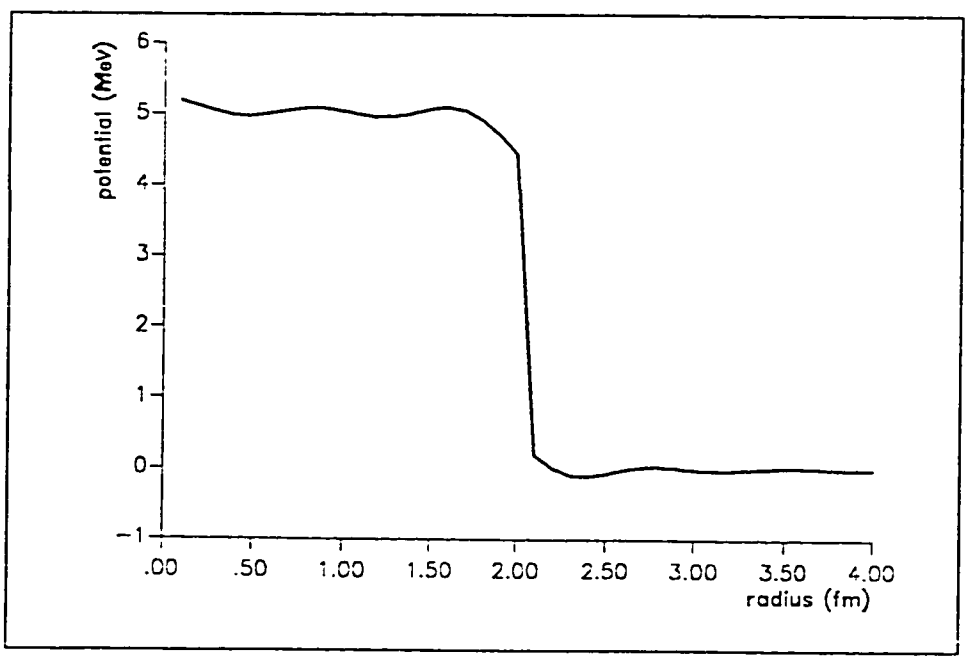


Figure (5.b): Potential calculated from eqn. (3.8.a) using the phase shifts shown above to reproduce the square barrier potential ($V_2(r)$) of height 5 MeV, width 2 fm when the auxiliary potential ($V_1(r)$) has height 4 MeV, width 2 fm

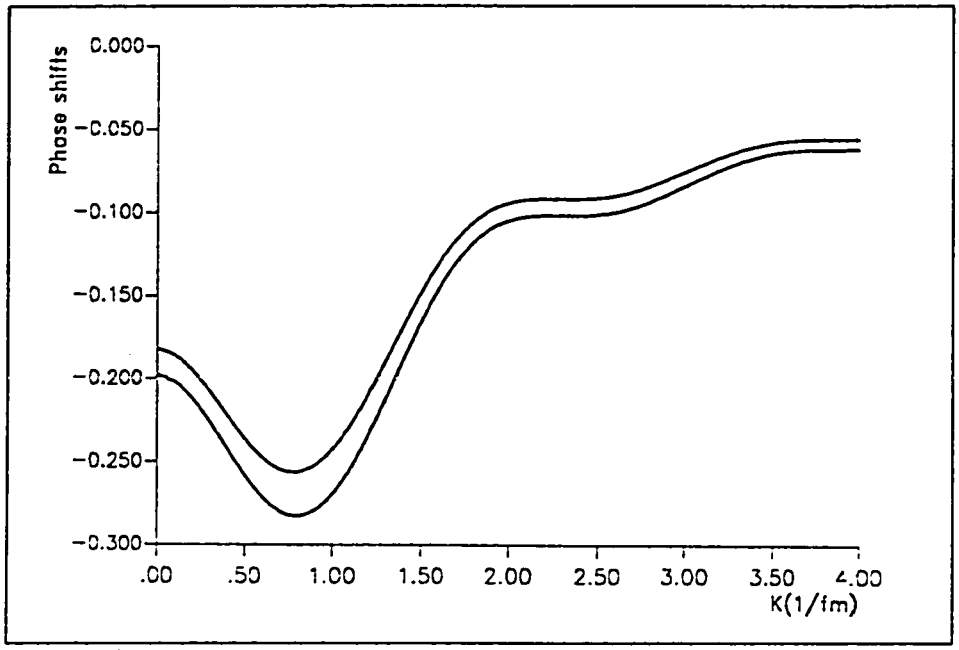


Figure (6.a): "Experimental" phase shifts produced by a nucleon scattered (in $\ell = 0$) from: (i) square barrier potential of height 5 MeV, width 2 fm ($V_2(r)$), and (ii) a square barrier potential of height 4.5 MeV, width 2 fm ($V_1(r)$).

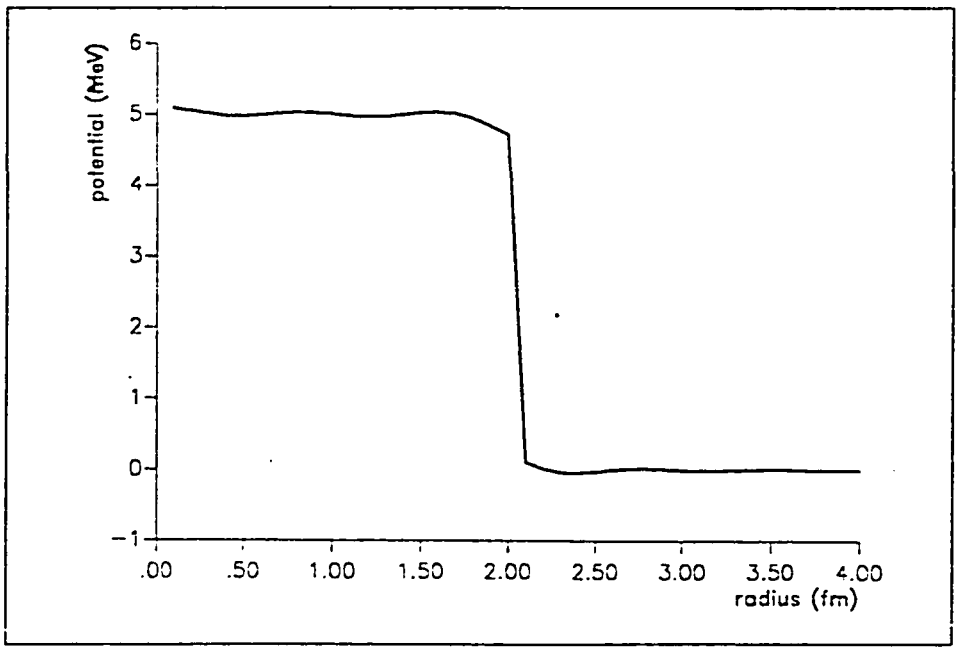


Figure (6.b): Potential calculated from eqn. (3.8.a) using the phase shifts shown above to reproduce the square barrier potential ($V_2(r)$) of height 5 MeV, width 2 fm when the auxiliary potential ($V_1(r)$) has height 4.5 MeV, width 2 fm

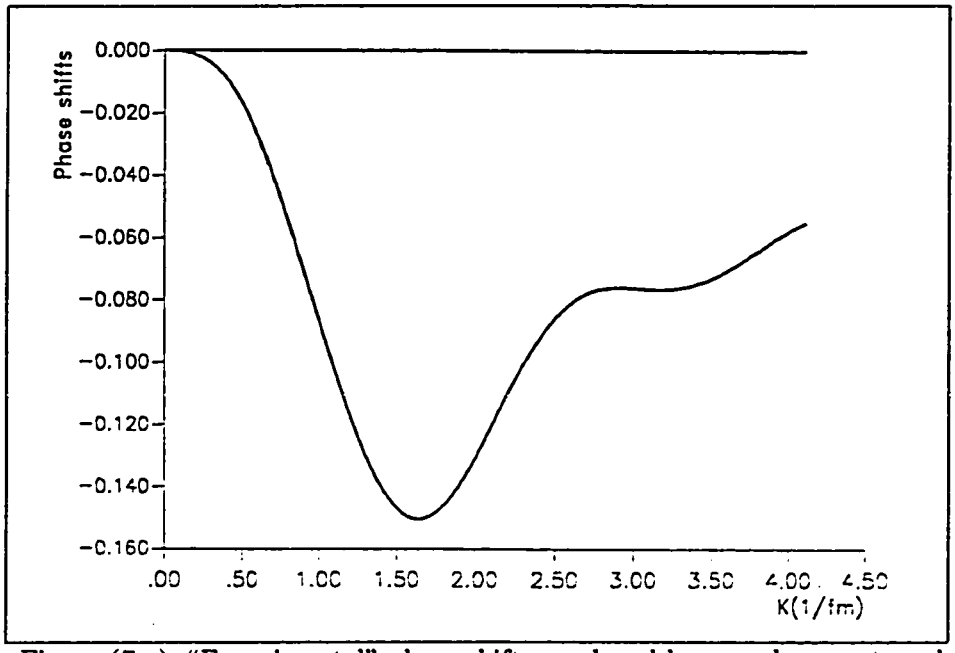


Figure (7.a): "Experimental" phase shifts produced by a nucleon scattered (in $\ell = 1$) from: (i) square barrier potential of height 5 MeV, width 2 fm ($V_2(r)$), and (ii) a square barrier potential of height 0.0 MeV, width 2 fm ($V_1(r)$).

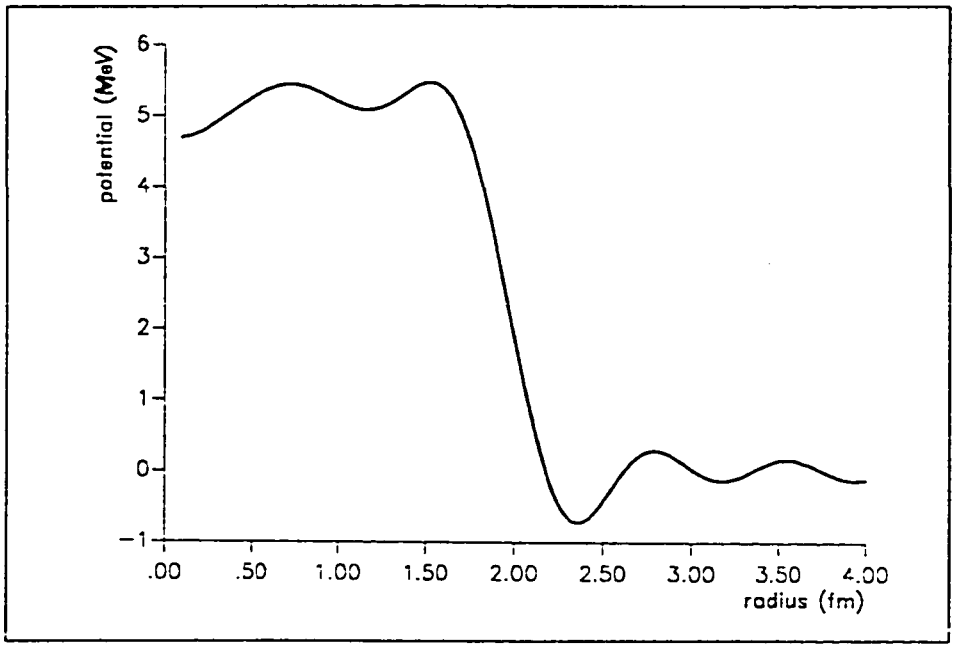


Figure (7.b): Potential calculated from eqn. (3.8.b) using the phase shifts shown above to reproduce the square barrier potential ($V_2(r)$) of height 5 MeV, width 2 fm when the auxiliary potential ($V_1(r)$) has height 0.0 MeV, width 2 fm

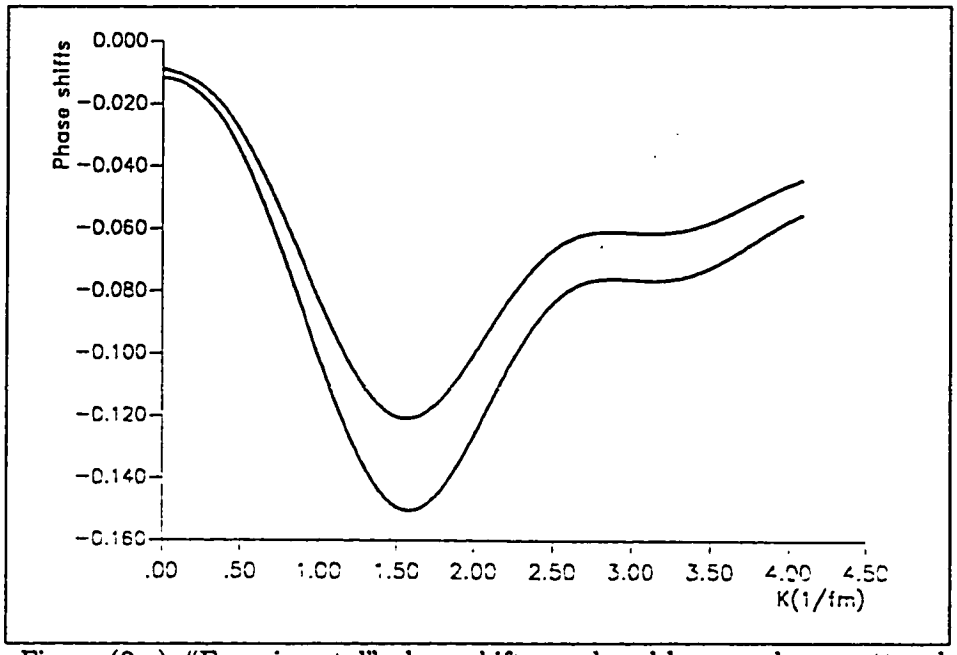


Figure (8.a): "Experimental" phase shifts produced by a nucleon scattered (in $\ell = 1$) from: (i) square barrier potential of height 5 MeV, width 2 fm ($V_2(r)$), and (ii) a square barrier potential of height 4 MeV, width 2 fm ($V_1(r)$).

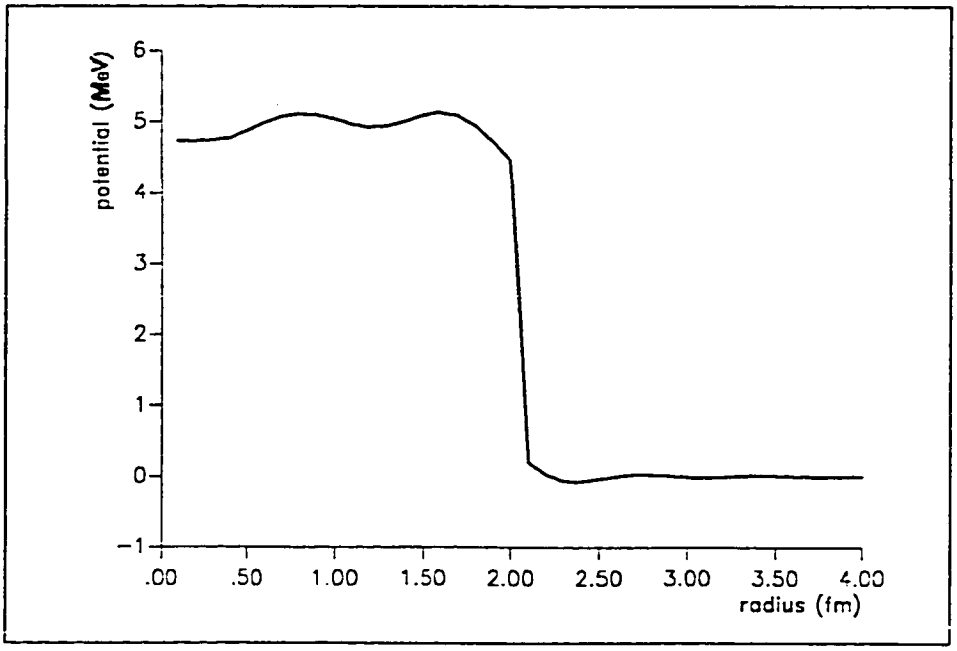


Figure (8.b): Potential calculated from eqn. (3.8.b) using the phase shifts shown above to reproduce the square barrier potential ($V_2(r)$) of height 5 MeV, width 2 fm when the auxiliary potential ($V_1(r)$) has height 4 MeV, width 2 fm

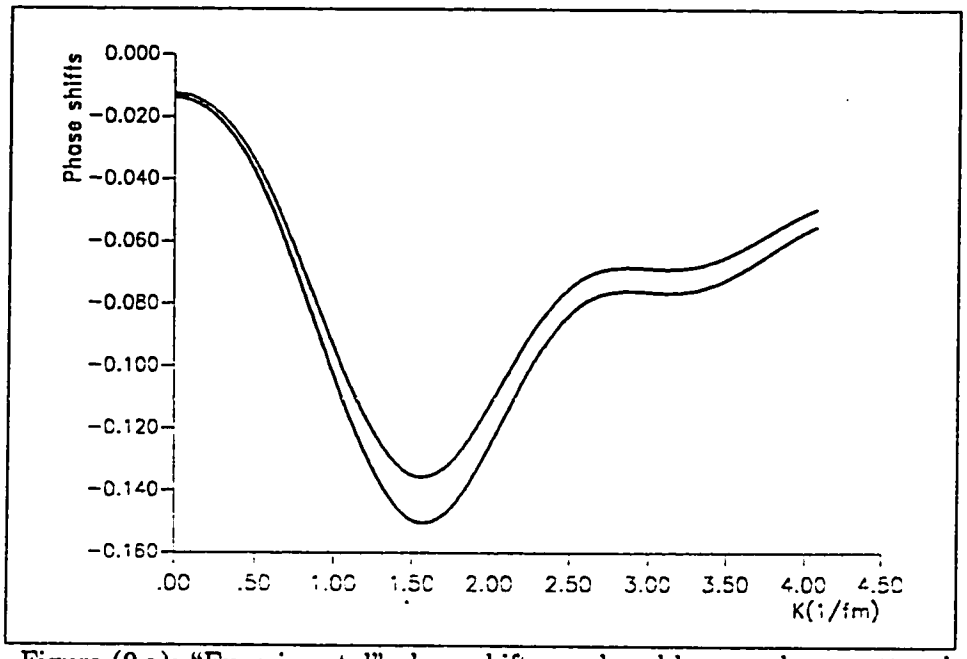


Figure (9.a): "Experimental" phase shifts produced by a nucleon scattered (in $\ell = 1$) from: (i) square barrier potential of height 5 MeV, width 2 fm ($V_2(r)$), and (ii) a square barrier potential of height 4.5 MeV, width 2 fm ($V_1(r)$).

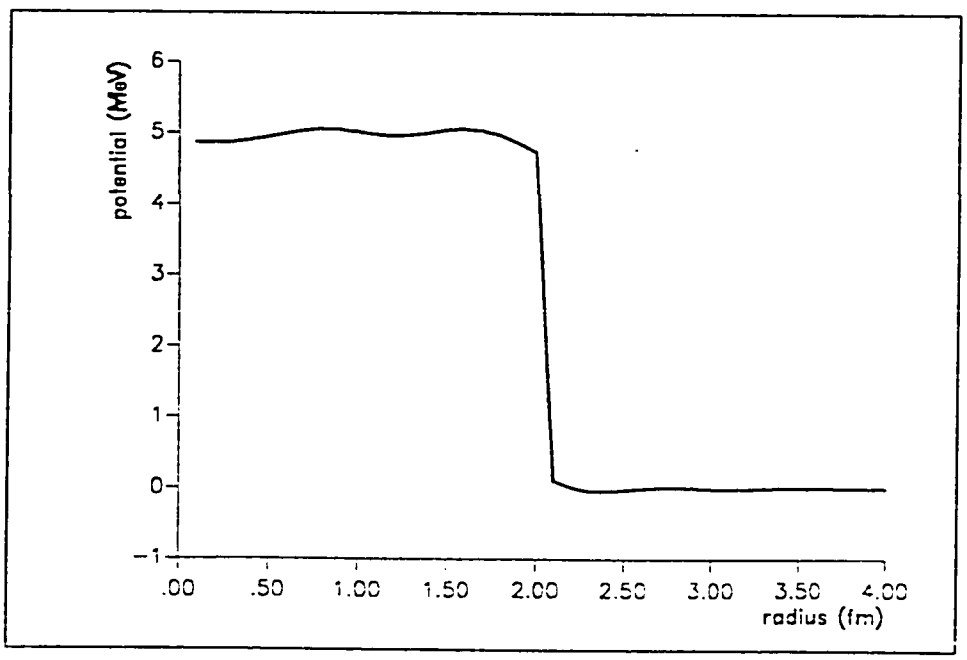


Figure (9.b): Potential calculated from eqn. (3.8.b) using the phase shifts shown above to reproduce the square barrier potential ($V_2(r)$) of height 5 MeV, width 2 fm when the auxiliary potential ($V_1(r)$) has height 4.5 MeV, width 2 fm

<u>Case (II)</u>: $V_2(r)$ has height $V_2 = 10\ MeV$ and range $a_2 = 2.0\ fm$. Use auxiliary potential $V_1(r)$ of width $a_1 = 2.\ fm$, height $V_1 = 0.0,\ \ 8.0$ and $9.0\ MeV$. The phase shifts and the results of the numerical calculations of eqns. (3.8.a) and (3.8.b) are shown in figures # 10, 11, and 12 for $\ell = 0$; and 13, 14, 15 for $\ell = 1$.

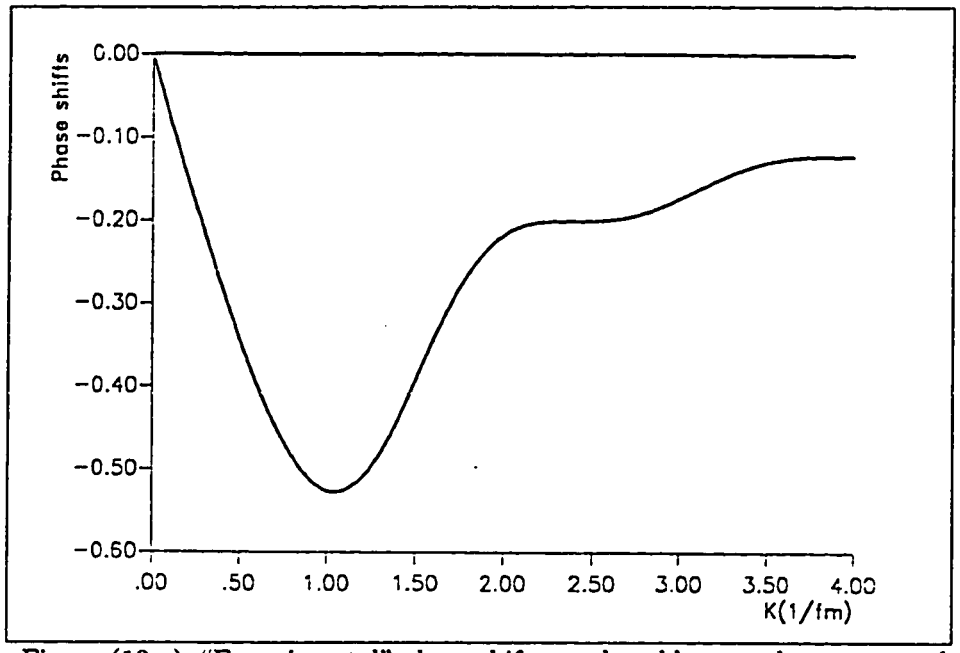


Figure (10.a): "Experimental" phase shifts produced by a nucleon scattered (in $\ell = 0$) from: (i) square barrier potential of height 10 MeV, width 2 fm ($V_2(r)$), and (ii) a square barrier potential of height 0.0 MeV, width 2 fm ($V_1(r)$).

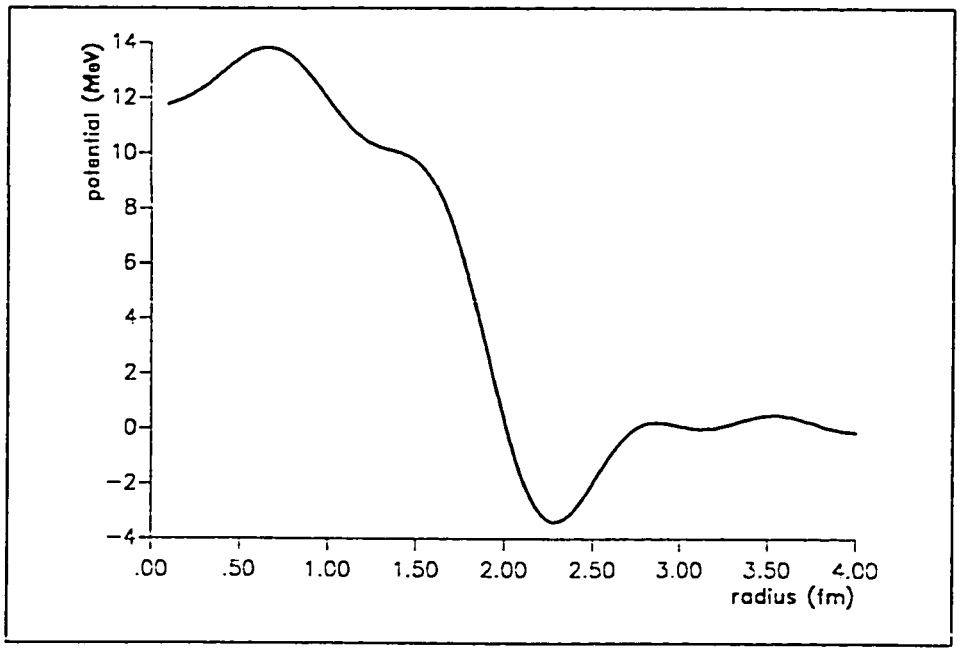


Figure (10.b): Potential calculated from eqn. (3.8.a) using the phase shifts shown above to reproduce the square barrier potential ($V_2(r)$) of height 10 MeV, width 2 fm when the auxiliary potential ($V_1(r)$) has height 0.0 MeV, width 2 fm

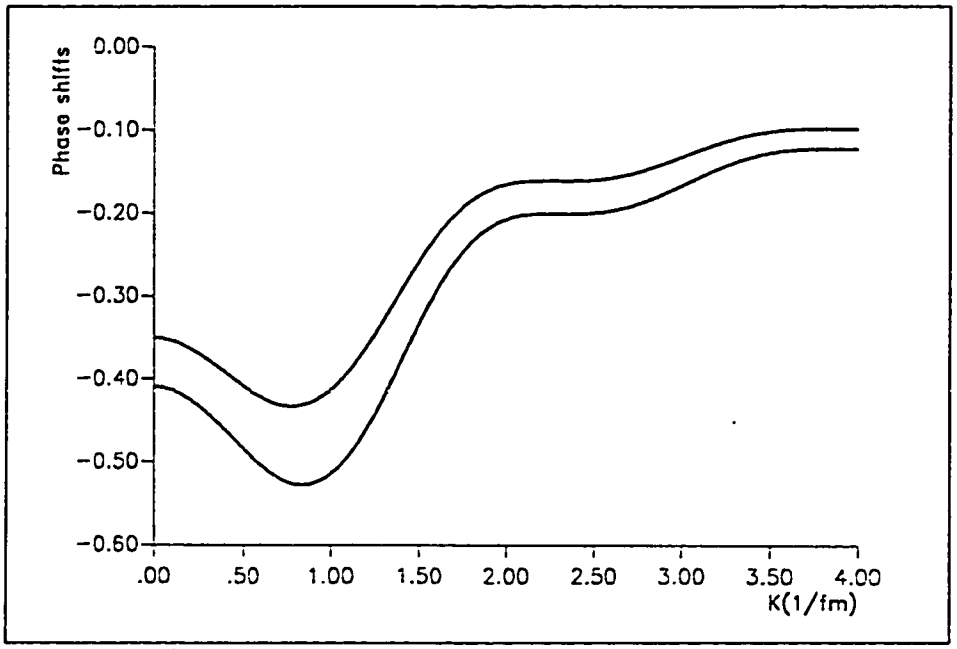


Figure (11.a): "Experimental" phase shifts produced by a nucleon scattered (in $\ell = 0$) from: (i) square barrier potential of height 10 MeV, width 2 fm ($V_2(r)$), and (ii) a square barrier potential of height 8 MeV, width 2 fm ($V_1(r)$).

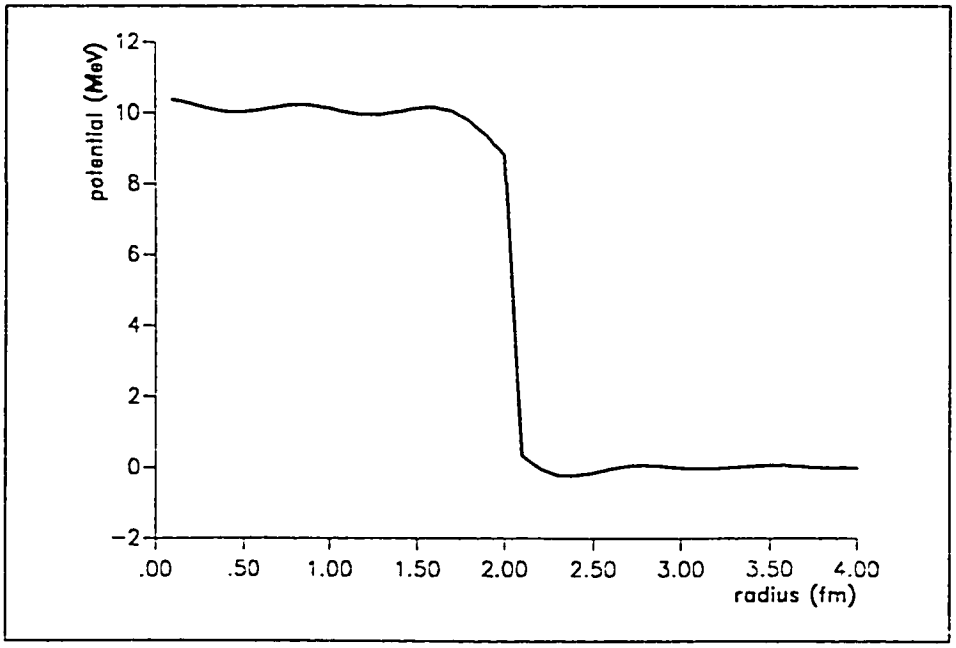


Figure (11.b): Potential calculated from eqn. (3.8.a) using the phase shifts shown above to reproduce the square barrier potential ($V_2(r)$) of height 10 MeV, width 2 fm when the auxiliary potential ($V_1(r)$) has height 8 MeV, width 2 fm

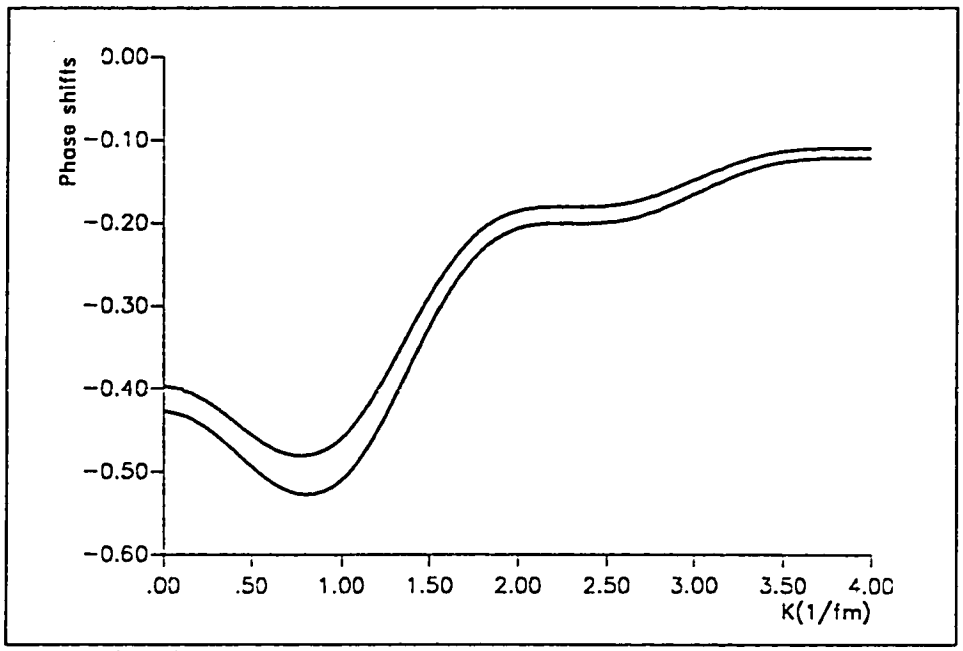


Figure (12.a): "Experimental" phase shifts produced by a nucleon scattered (in $\ell = 0$) from: (i) square barrier potential of height 10 MeV, width 2 fm ($V_2(r)$), and (ii) a square barrier potential of height 9 MeV, width 2 fm ($V_1(r)$).

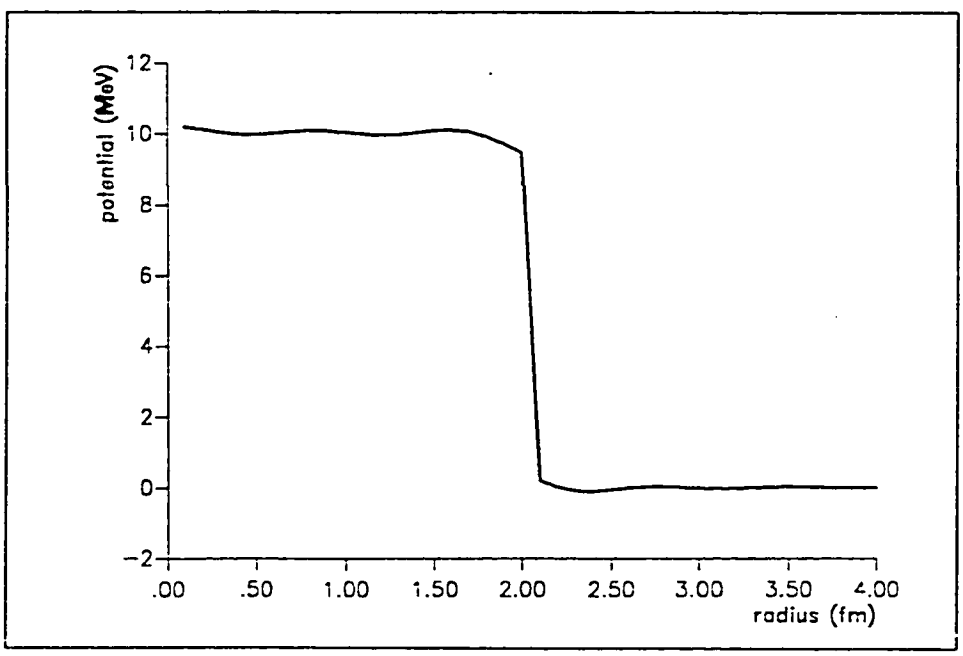


Figure (12.b): Potential calculated from eqn. (3.8.a.) using the phase shifts shown above to reproduce the square barrier potential ($V_2(r)$) of height 10 MeV, width 2 fm when the auxiliary potential ($V_1(r)$) has height 9 MeV, width 2 fm

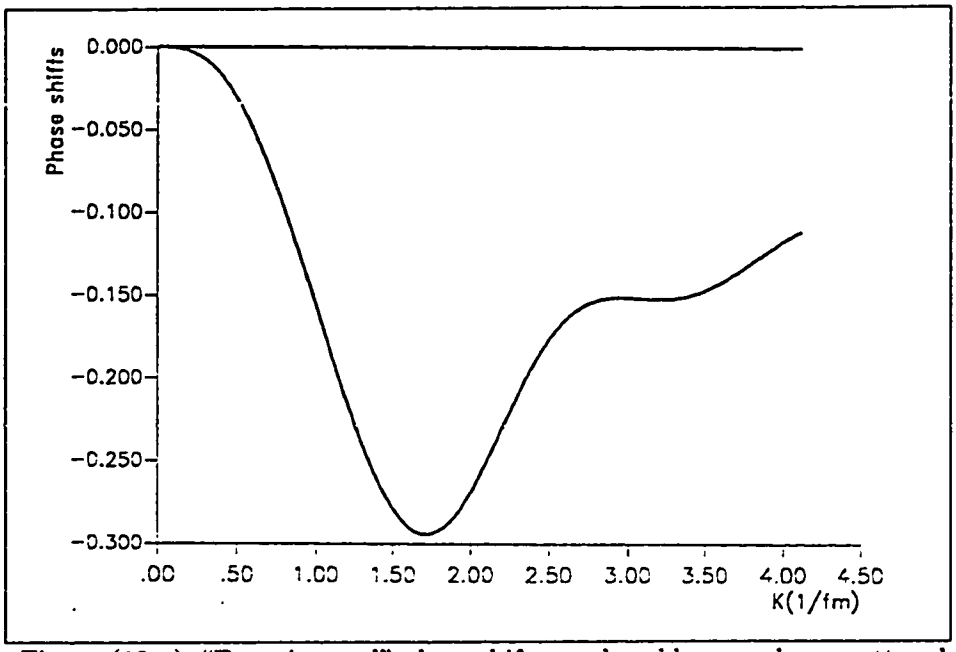


Figure (13.a): "Experimental" phase shifts produced by a nucleon scattered (in $\ell = 1$) from: (i) square barrier potential of height 10 MeV, width 2 fm ($V_2(r)$), and (ii) a square barrier potential of height 0.0 MeV, width 2 fm ($V_1(r)$).

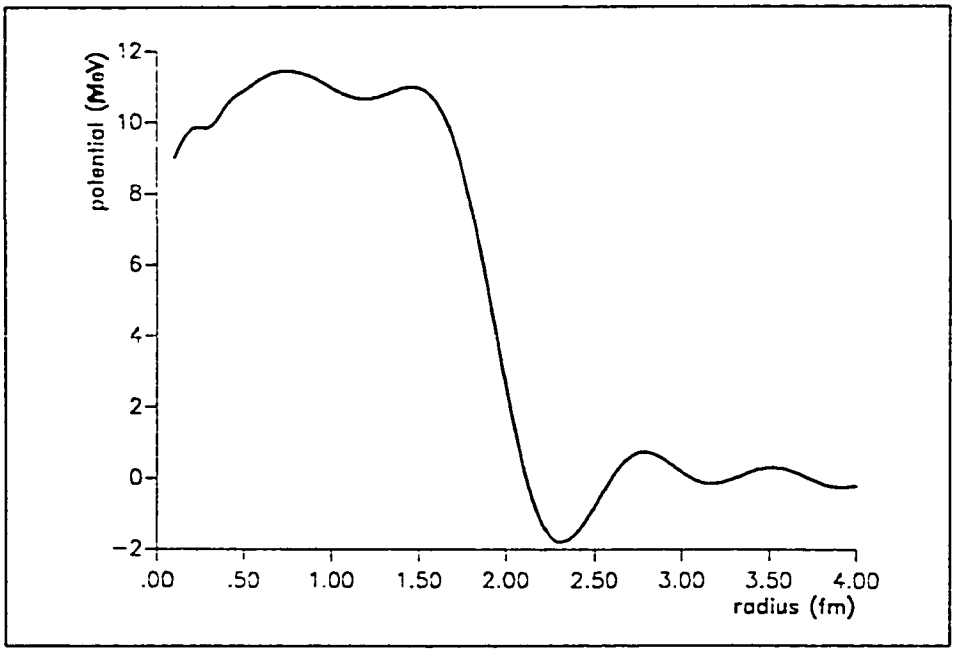


Figure (13.b): Potential calculated from eqn. (3.8.b) using the phase shifts shown above to reproduce the square barrier potential ($V_2(r)$) of height 10 MeV, width 2 fm when the auxiliary potential ($V_1(r)$) has height 0.0 MeV, width 2 fm

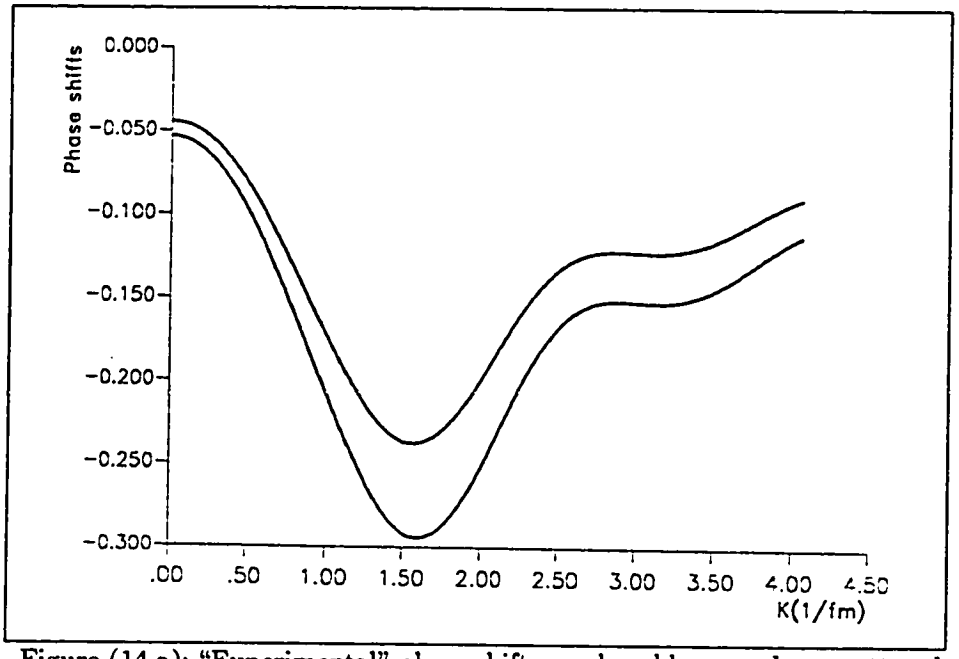


Figure (14.a): "Experimental" phase shifts produced by a nucleon scattered (in $\ell = 1$) from: (i) square barrier potential of height 10 MeV, width 2 fm ($V_2(r)$), and (ii) a square barrier potential of height 8 MeV, width 2 fm ($V_1(r)$).

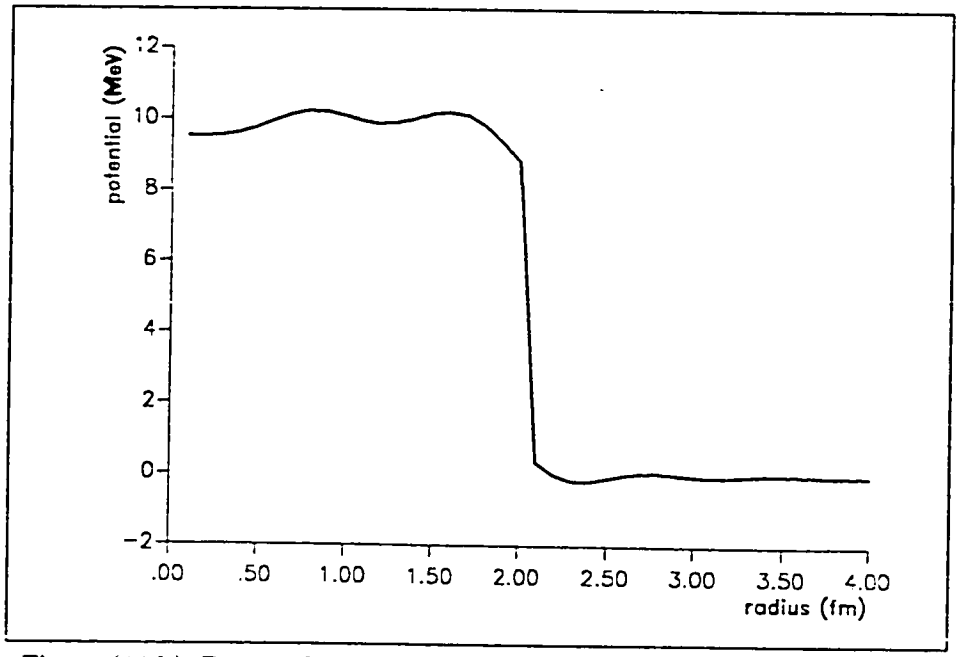


Figure (14.b): Potential calculated from eqn. (3.8.b) using the phase shifts shown above to reproduce the square barrier potential ($V_2(r)$) of height 10 MeV, width 2 fm when the auxiliary potential ($V_1(r)$) has height 8 MeV, width 2 fm

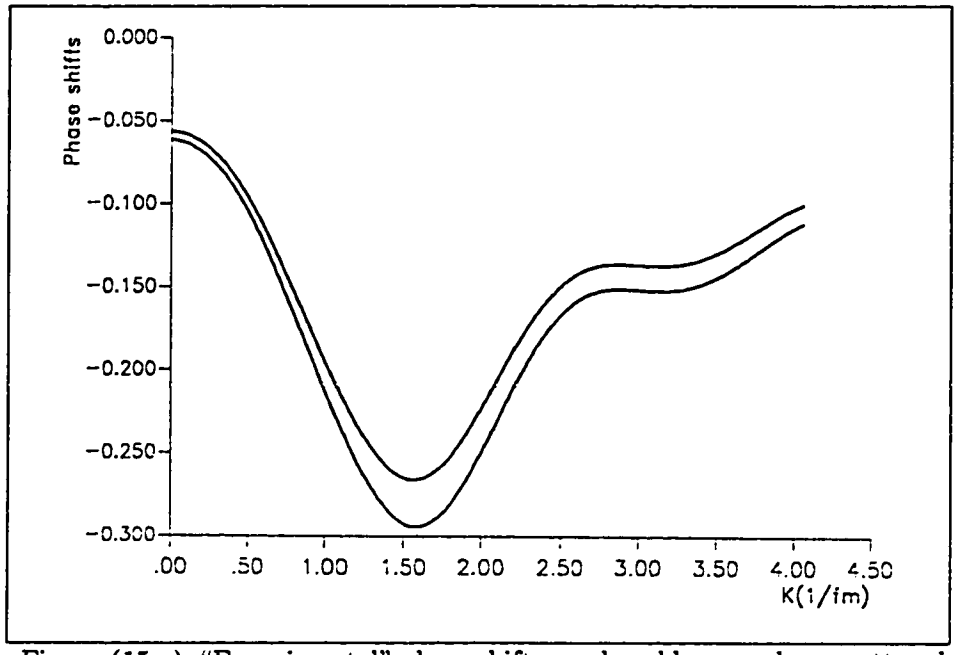


Figure (15.a): "Experimental" phase shifts produced by a nucleon scattered (in $\ell = 1$) from: (i) square barrier potential of height 10 MeV, width 2 fm ($V_2(r)$), and (ii) a square barrier potential of height 9 MeV, width 2 fm ($V_1(r)$).

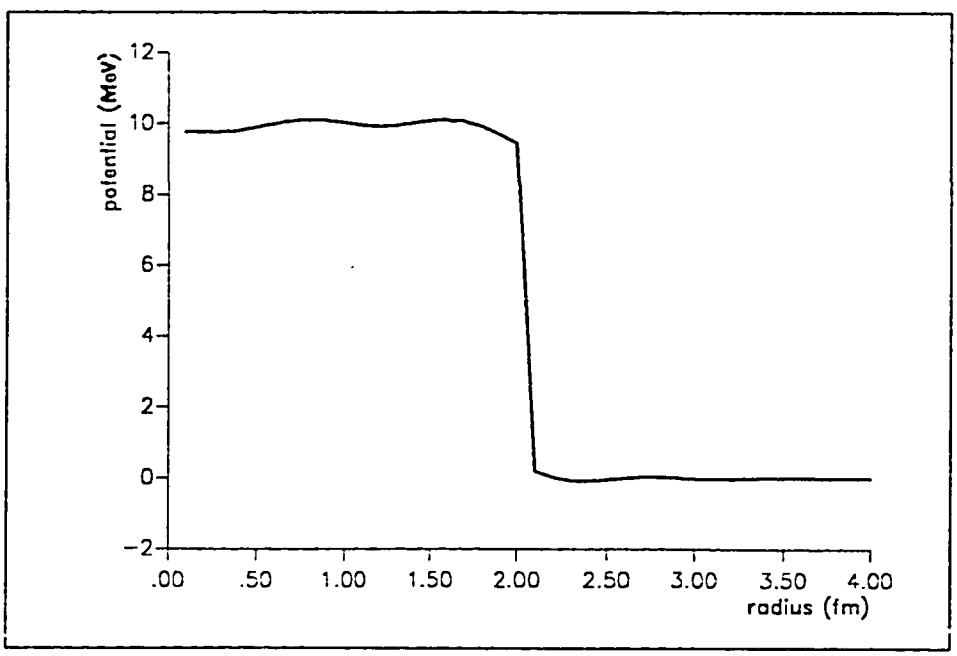


Figure (15.b): Potential calculated from eqn. (3.8.b) using the phase shifts shown above to reproduce the square barrier potential ($V_2(r)$) of height 10 MeV, width 2 fm when the auxiliary potential ($V_1(r)$) has height 9 MeV, width 2 fm

Case (III): $V_2(r)$ has height $V_2 = 40\ MeV$ and range $a_2 = 2.0\ fm$. Use auxiliary potential $V_1(r)$ of width $a_1 = 2.0\ fm$, height $V_1 = 0.0,\ 30$ and $36\ Mev$ for $\ell = 0$; and $V_1 = 0.0,\ 50$ and $45\ MeV$ for $\ell = 1$. See figures # 16, 17, and 18 for $\ell = 0$; and figures # 19, 20, and 31 for $\ell = 1$.

In this case we have taken the auxiliary potential to be weaker than the actual in one case ($\ell = 0$) and stronger than the actual in the other case ($\ell = 1$) to show that this does not have much effect on the reproduced potential. Here the small difference between $\ell = 0$ and $\ell = 1$ cases is due to the angular momentum effect and not due to the choice of stronger or weaker auxiliary potential.

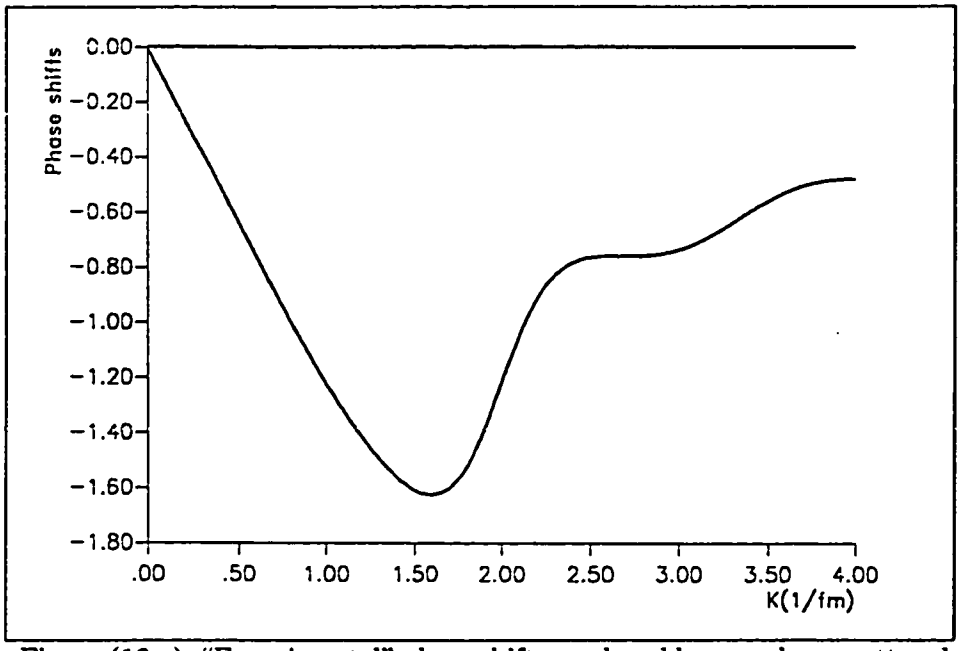


Figure (16.a): "Experimental" phase shifts produced by a nucleon scattered (in $\ell = 0$) from: (i) square barrier potential of height 40 MeV, width 2 fm ($V_2(r)$), and (ii) a square barrier potential of height 0.0 MeV, width 2 fm ($V_1(r)$).

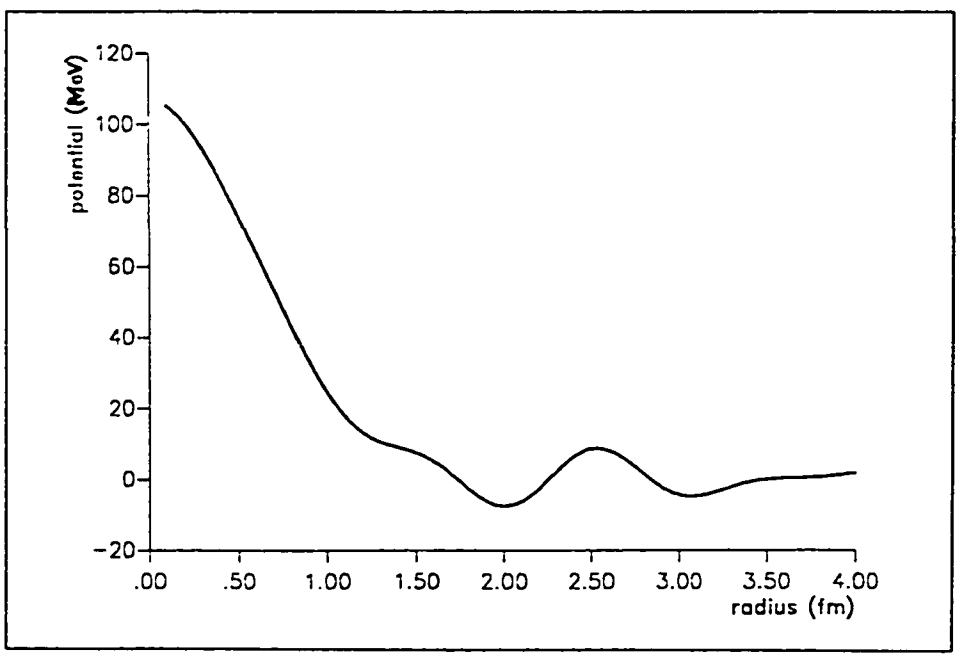


Figure (16.b): Potential calculated from eqn. (3.8.a) using the phase shifts shown above to reproduce the square barrier potential ($V_2(r)$) of height 40 MeV, width 2 fm when the auxiliary potential ($V_1(r)$) has height 0.0 MeV, width 2 fm

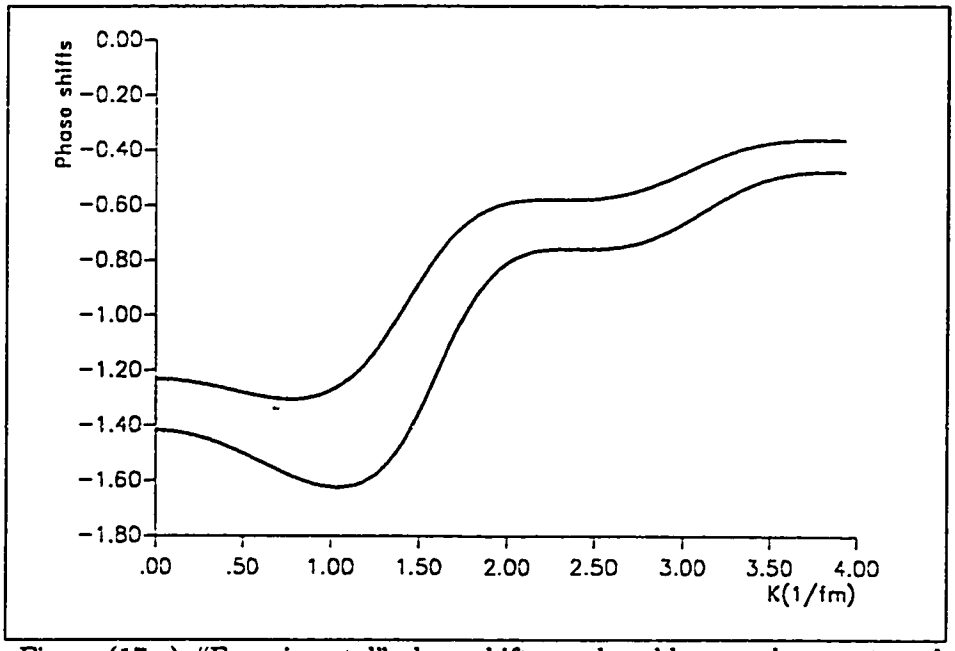


Figure (17.a): "Experimental" phase shifts produced by a nucleon scattered (in $\ell = 0$) from: (i) square barrier potential of height 40 MeV, width 2 fm ($V_2(r)$), and (ii) a square barrier potential of height 30 MeV, width 2 fm ($V_1(r)$).

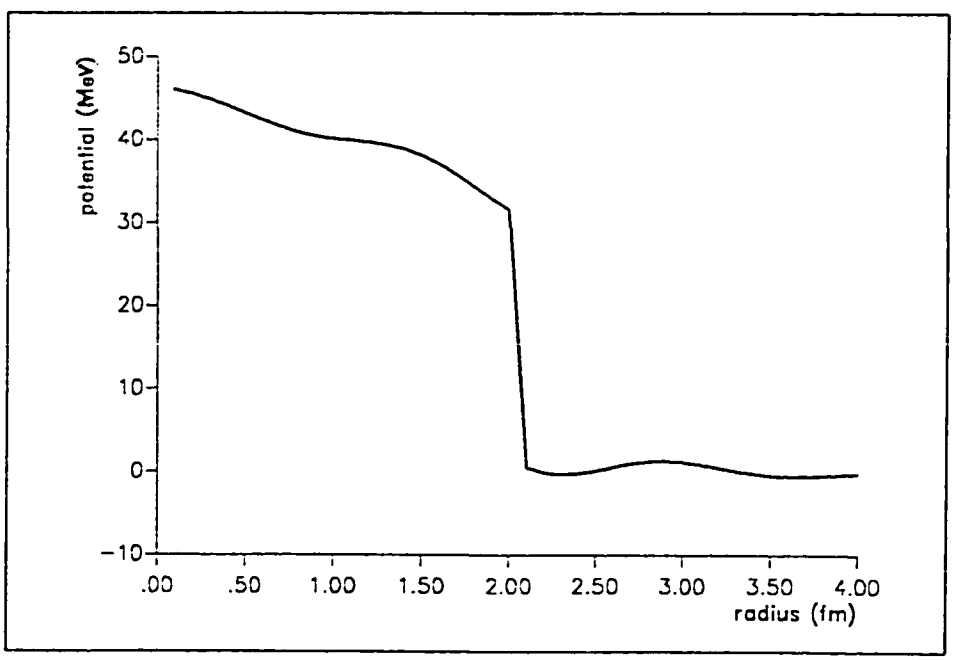


Figure (17.b): Potential calculated from eqn. (3.8.a) using the phase shifts shown above to reproduce the square barrier potential ($V_2(r)$) of height 40 MeV, width 2 fm when the auxiliary potential ($V_1(r)$) has height 30 MeV, width 2 fm

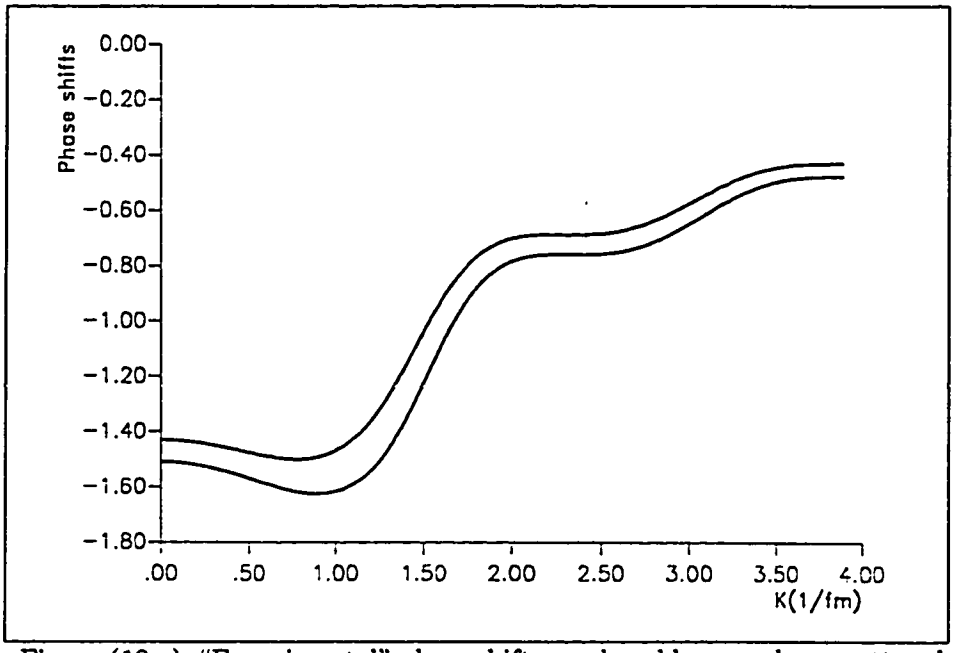


Figure (18.a): "Experimental" phase shifts produced by a nucleon scattered (in $\ell = 0$) from: (i) square barrier potential of height 40 MeV, width 2 fm ($V_2(r)$), and (ii) a square barrier potential of height 36 MeV, width 2 fm ($V_1(r)$).

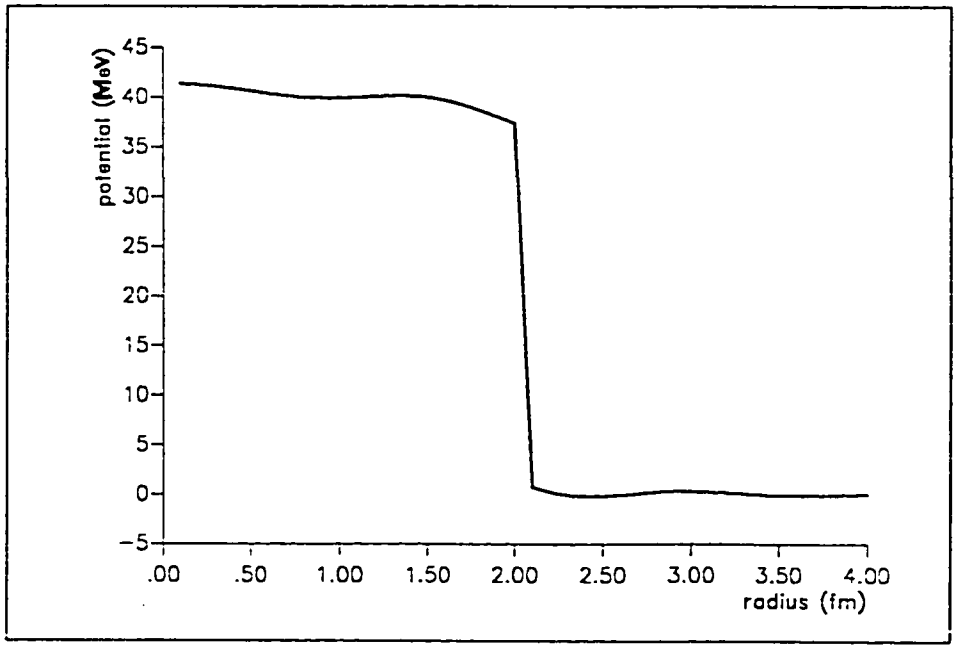


Figure (18.b): Potential calculated from eqn. (3.8.a) using the phase shifts shown above to reproduce the square barrier potential ($V_2(r)$) of height 40 MeV, width 2 fm when the auxiliary potential ($V_1(r)$) has height 36 MeV, width 2 fm

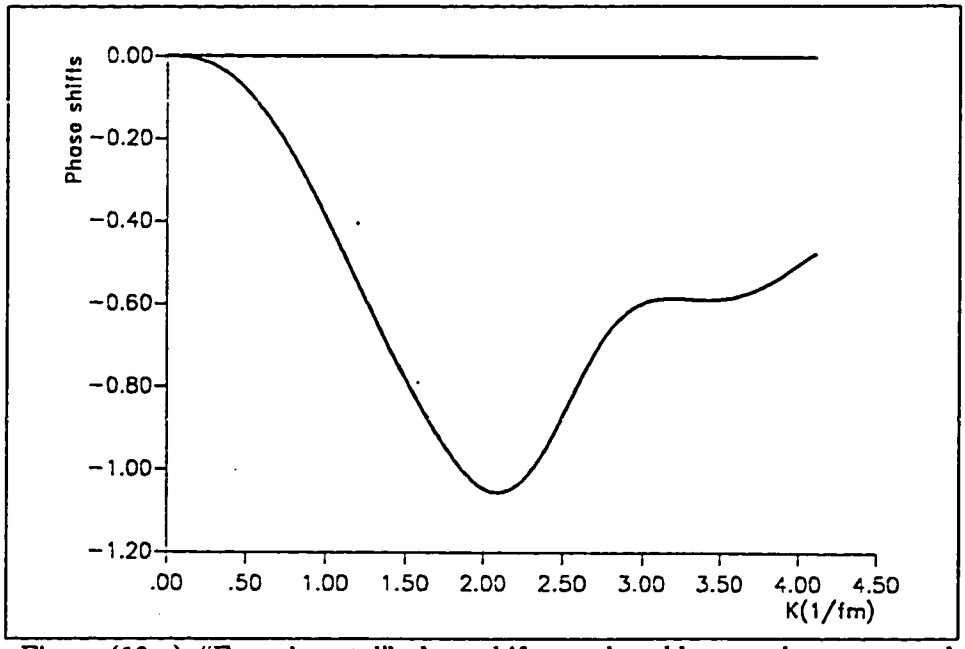


Figure (19.a): "Experimental" phase shifts produced by a nucleon scattered (in $\ell = 1$) from: (i) square barrier potential of height 40 MeV, width 2 fm ($V_2(r)$), and (ii) a square barrier potential of height 0.0 MeV, width 2 fm ($V_1(r)$).

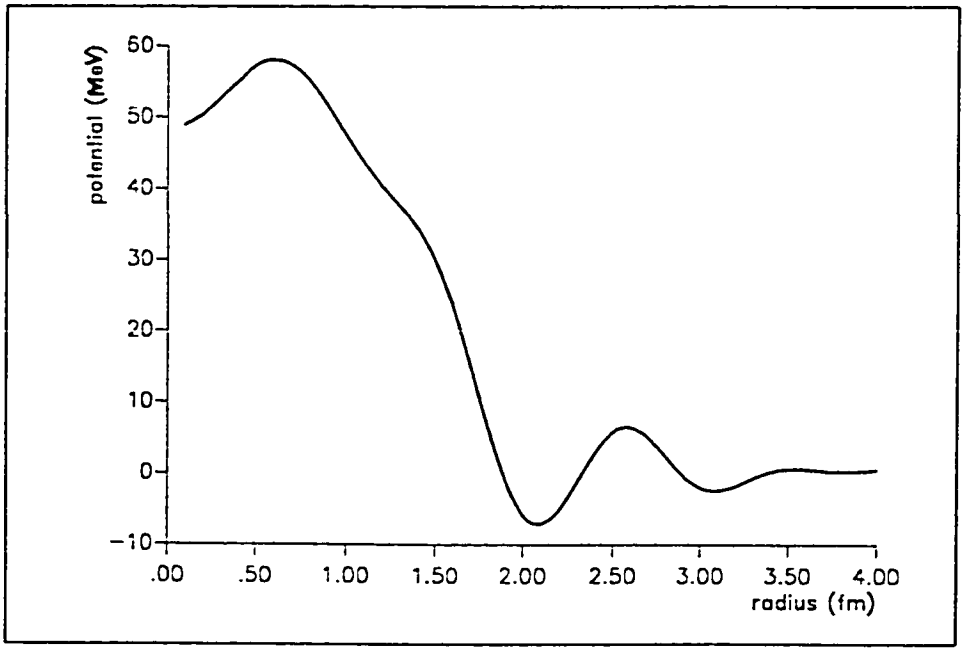


Figure (19.b): Potential calculated from eqn. (3.8.b) using the phase shifts shown above to reproduce the square barrier potential ($V_2(r)$) of height 40 MeV, width 2 fm when the auxiliary potential ($V_1(r)$) has height 0.0 MeV, width 2 fm

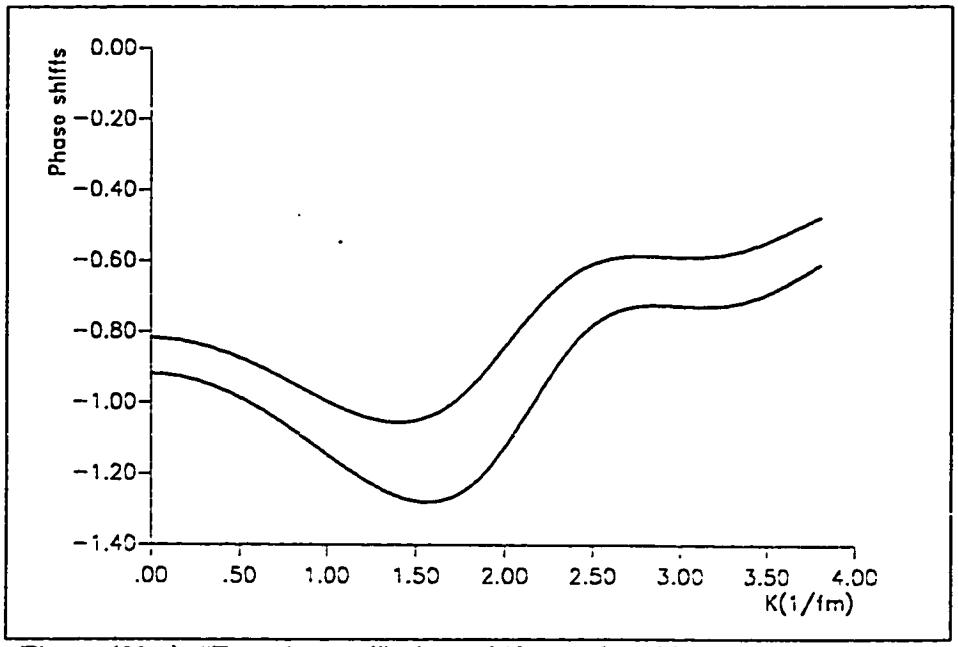


Figure (20.a): "Experimental" phase shifts produced by a nucleon scattered (in $\ell = 1$) from: (i) square barrier potential of height 40 MeV, width 2 fm ($V_2(r)$), and (ii) a square barrier potential of height 50 MeV, width 2 fm ($V_1(r)$).

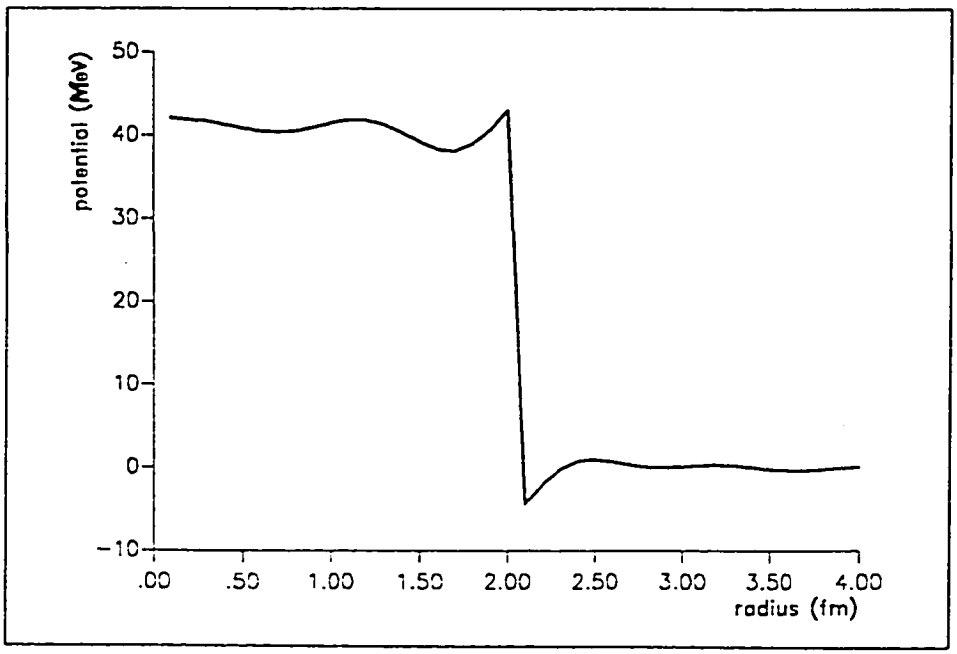


Figure (20.b): Potential calculated from eqn. (3.8.b) using the phase shifts shown above to reproduce the square barrier potential ($V_2(r)$) of height 40 MeV, width 2 fm when the auxiliary potential ($V_1(r)$) has height 50 MeV, width 2 fm

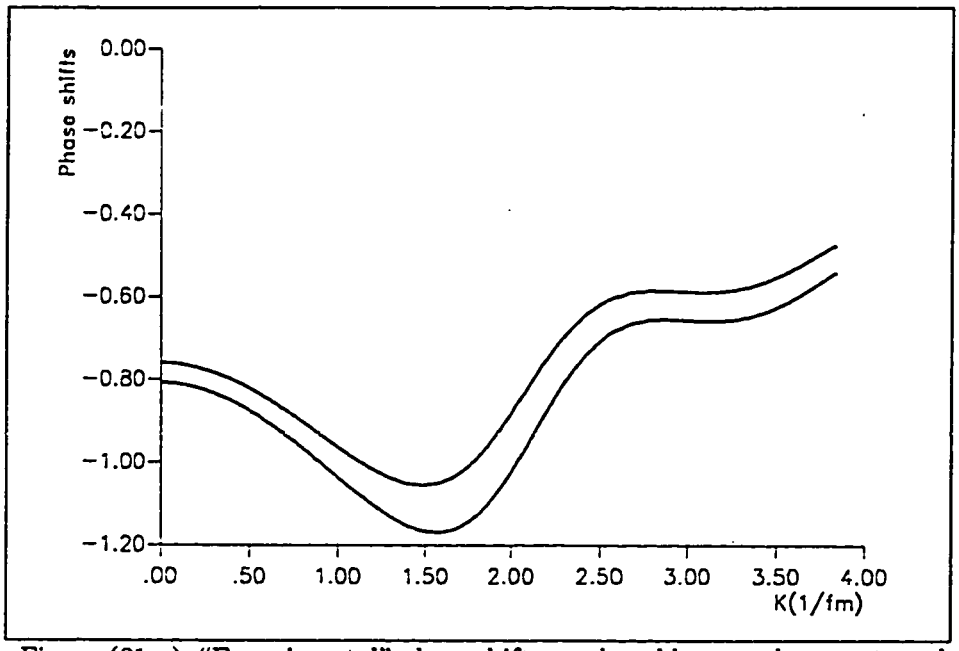


Figure (21.a): "Experimental" phase shifts produced by a nucleon scattered (in $\ell = 1$) from: (i) square barrier potential of height 40 MeV, width 2 fm ($V_2(r)$), and (ii) a square barrier potential of height 45 MeV, width 2 fm ($V_1(r)$).

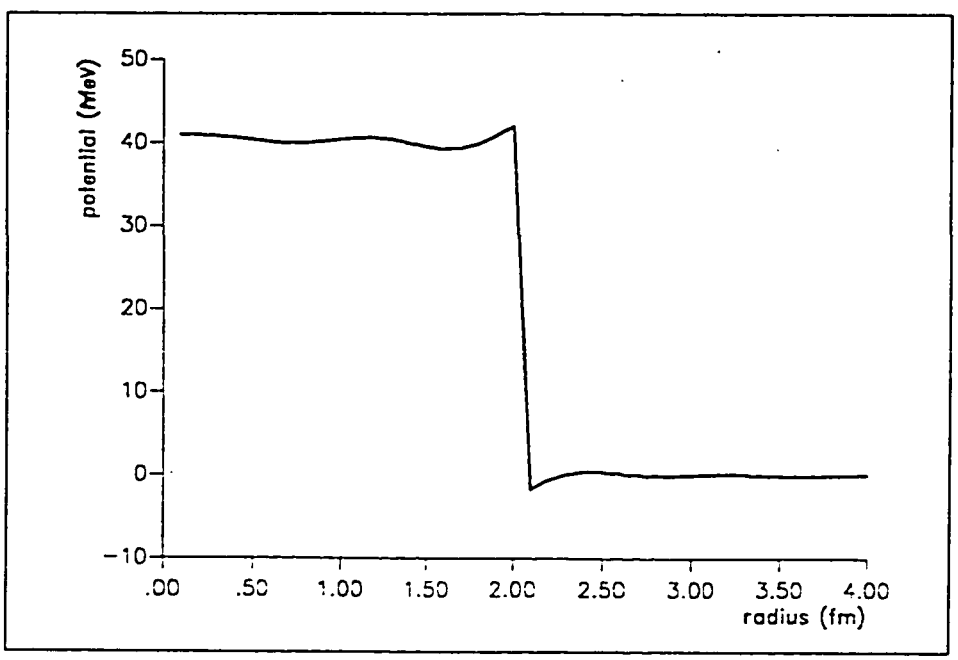


Figure (21.b): Potential calculated from eqn. (3.8.b) using the phase shifts shown above to reproduce the square barrier potential ($V_2(r)$) of height 40 MeV, width 2 fm when the auxiliary potential ($V_1(r)$) has height 45 MeV, width 2 fm

**Example (4.5):**

Consider the scattering of a nucleon from a square well potential ($V_2(r)$) in the channels $\ell = 0$ and $\ell = 1$. The previous discussion (for the examples on square barriers) about $K_{max}$ and the auxiliary potential applies equally well here. However, $K_{min}$ cannot be taken to be zero since $V_1 \leq 0.0$ and

$$K_{min} = \sqrt{\frac{2m}{\hbar^2}(E_{min} - V_1)},$$

$$\text{Since} \quad E_{min} = 0.0 \Rightarrow K_{min} = \sqrt{\frac{-2m}{\hbar^2} V_1} \geq 0.$$

This means that there is an additional error introduced in the calculations of the potentials in this case.

Case (I): $V_2(r)$ has depth $|V_2| = 5.0\ MeV$, and range $a_2 = 2.0\ fm$. Use auxiliary potential $V_1(r)$ of range $a_1 = 2.0\ fm$, depth: $|V_1| = 0.0$, 4.0 and 4.5 $MeV$. The phase shifts and the results of the calculations from eqns. (3.8.a) and (3.8.b) are shown in figures: 22, 23 and 24 for $\ell = 0$; and 25, 26, and 27 for $\ell = 1$.

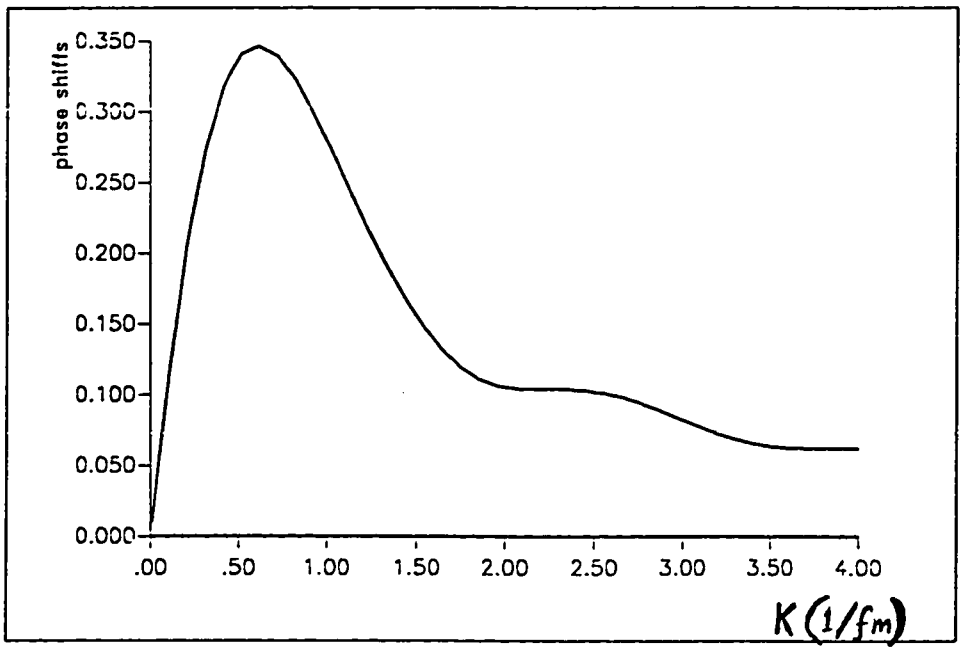


Figure (22.a): "Experimental" phase shifts produced by a nucleon scattered (in $\ell = 0$) from: (i) square well potential of depth 5 MeV, width 2 fm ($V_2(r)$), and (ii) a square well potential of depth 0.0 MeV, width 2 fm ($V_1(r)$).

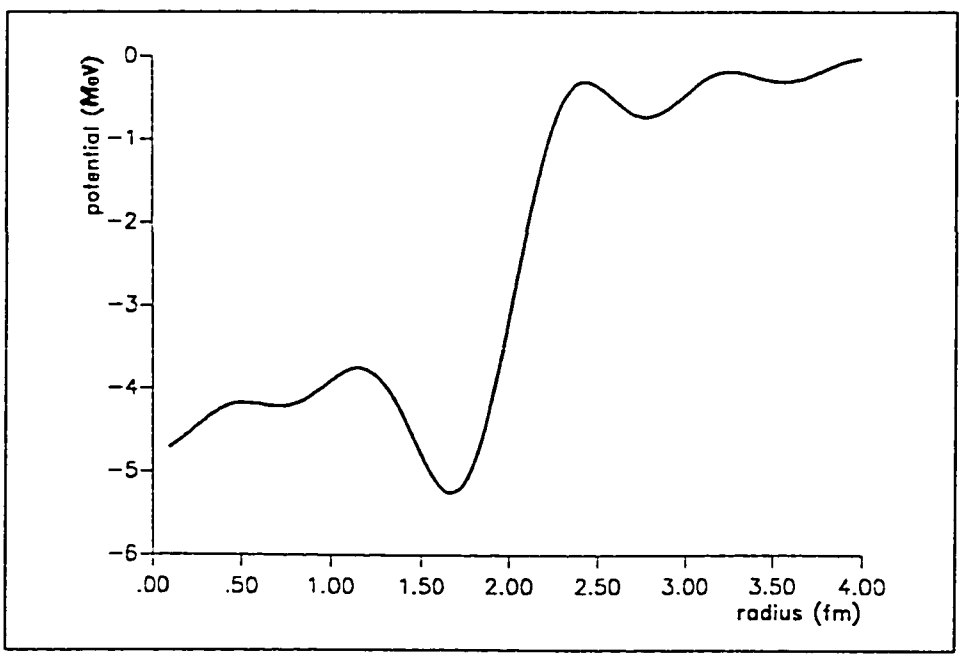


Figure (22.b): Potential calculated from eqn. (3.8.a) using the phase shifts shown above to reproduce the square well potential ($V_2(r)$) of depth 5 MeV, width 2 fm when the auxiliary potential ($V_1(r)$) has depth 0.0 MeV, width 2 fm

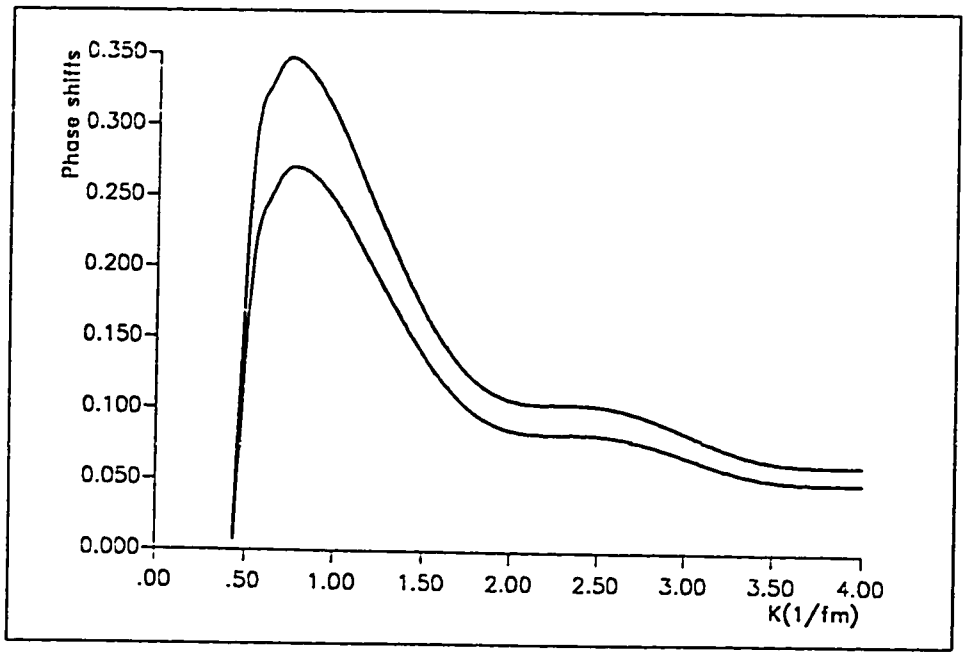


Figure (23.a): "Experimental" phase shifts produced by a nucleon scattered (in $\ell = 0$) from: (i) square well potential of depth 5 MeV, width 2 fm ($V_2(r)$); and (ii) a square well potential of depth 4 MeV, width 2 fm ($V_1(r)$).

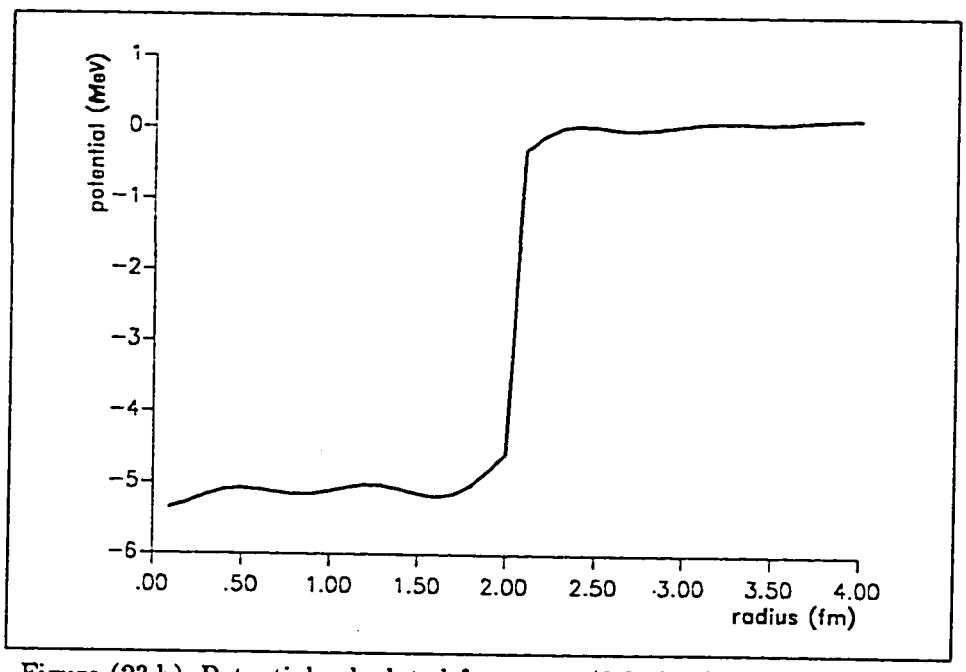


Figure (23.b): Potential calculated from eqn. (3.8.a) using the phase shifts shown above to reproduce the square well potential ($V_2(r)$) of depth 5 MeV, width 2 fm when the auxiliary potential ($V_1(r)$) has depth 4 MeV, width 2 fm

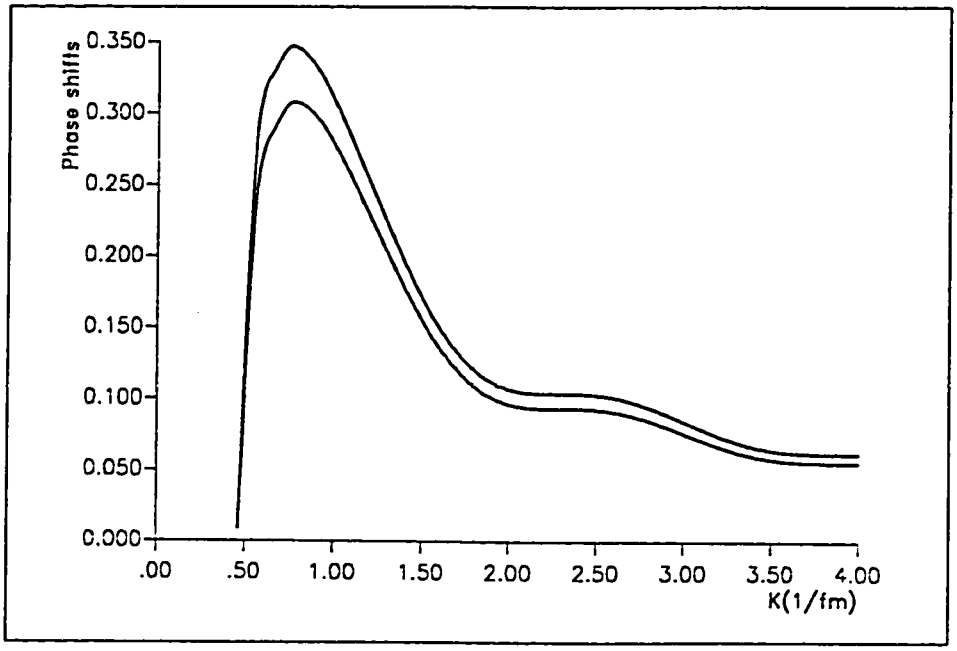


Figure (24.a): "Experimental" phase shifts produced by a nucleon scattered (in $\ell = 0$) from: (i) square well potential of depth 5 MeV, width 2 fm ($V_2(r)$), and (ii) a square well potential of depth 4.5 MeV, width 2 fm ($V_1(r)$).

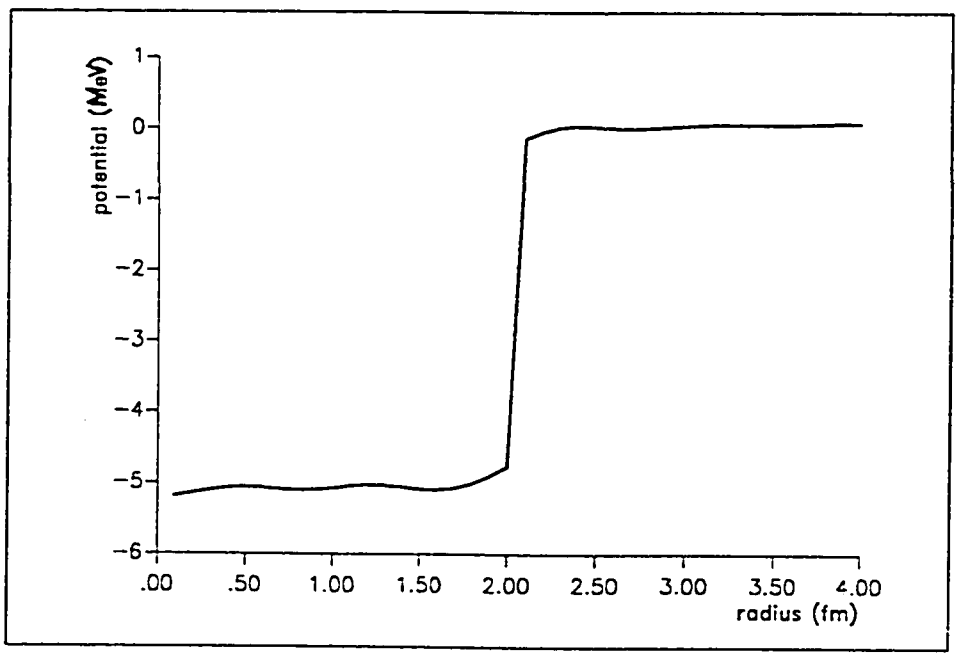


Figure (24.b): Potential calculated from eqn. (3.8.a) using the phase shifts shown above to reproduce the square well potential ($V_2(r)$) of depth 5 MeV, width 2 fm when the auxiliary potential ($V_1(r)$) has depth 4.5 MeV, width 2 fm

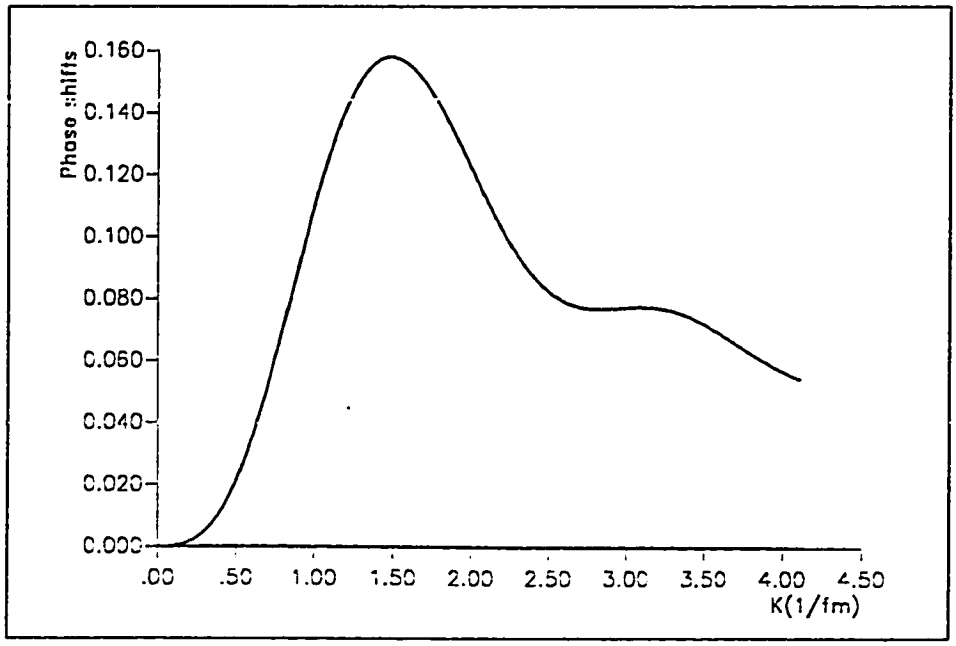


Figure (25.a): "Experimental" phase shifts produced by a nucleon scattered (in $\ell = 1$) from: (i) square well potential of depth 5 MeV, width 2 fm ($V_2(r)$), and (ii) a square well potential of depth 0.0 MeV, width 2 fm ($V_1(r)$).

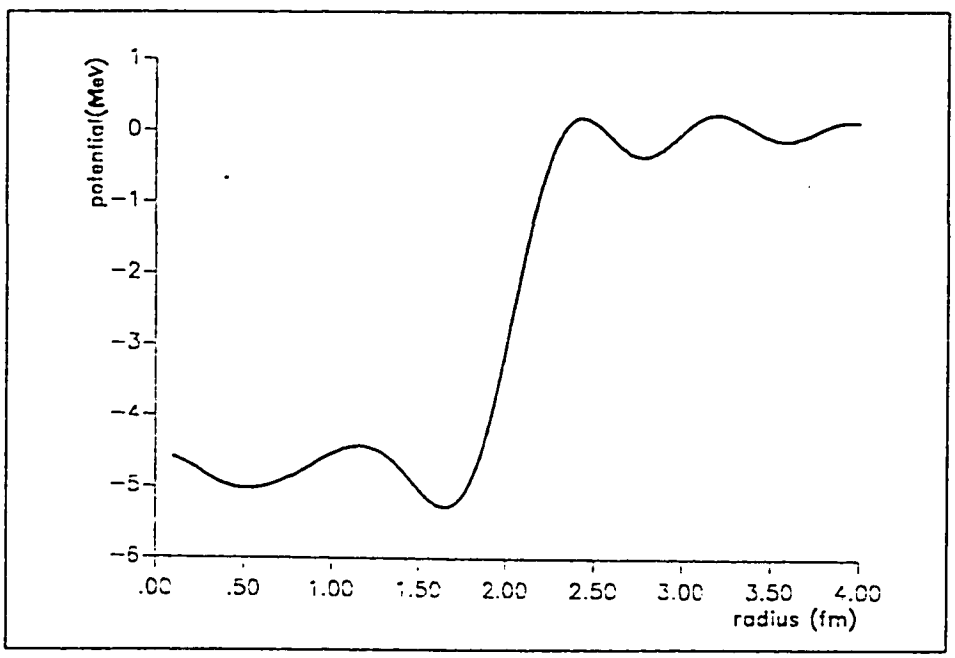


Figure (25.b): Potential calculated from eqn. (3.8.b) using the phase shifts shown above to reproduce the square well potential ($V_2(r)$) of depth 5 MeV, width 2 fm when the auxiliary potential ($V_1(r)$) has depth 0.0 MeV, width 2 fm

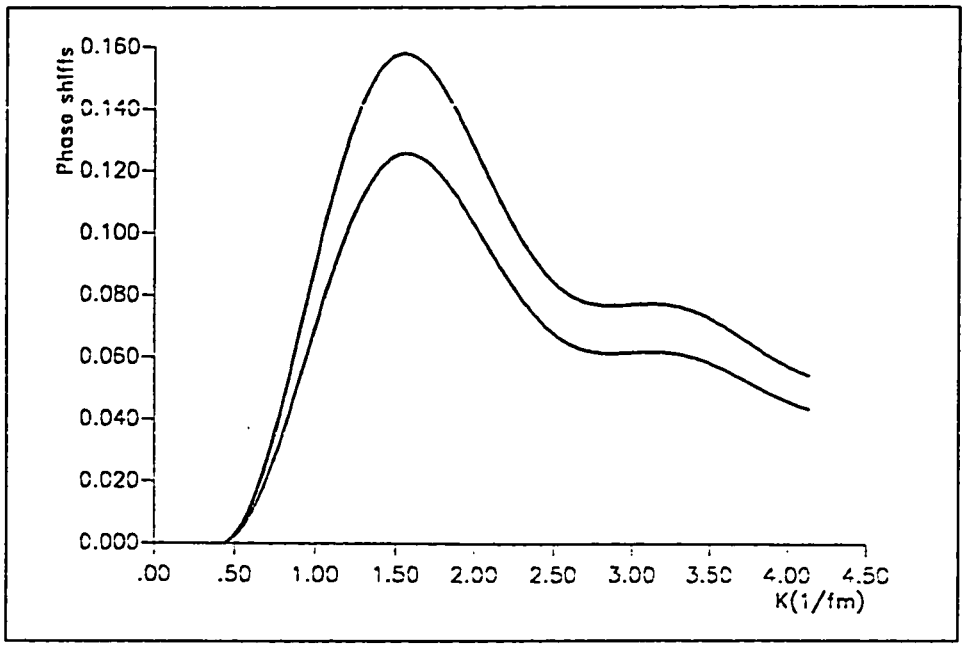


Figure (26.a): "Experimental" phase shifts produced by a nucleon scattered (in $\ell = 1$) from: (i) square well potential of depth 5 MeV, width 2 fm ($V_2(r)$), and (ii) a square well potential of depth 4 MeV, width 2 fm ($V_1(r)$).

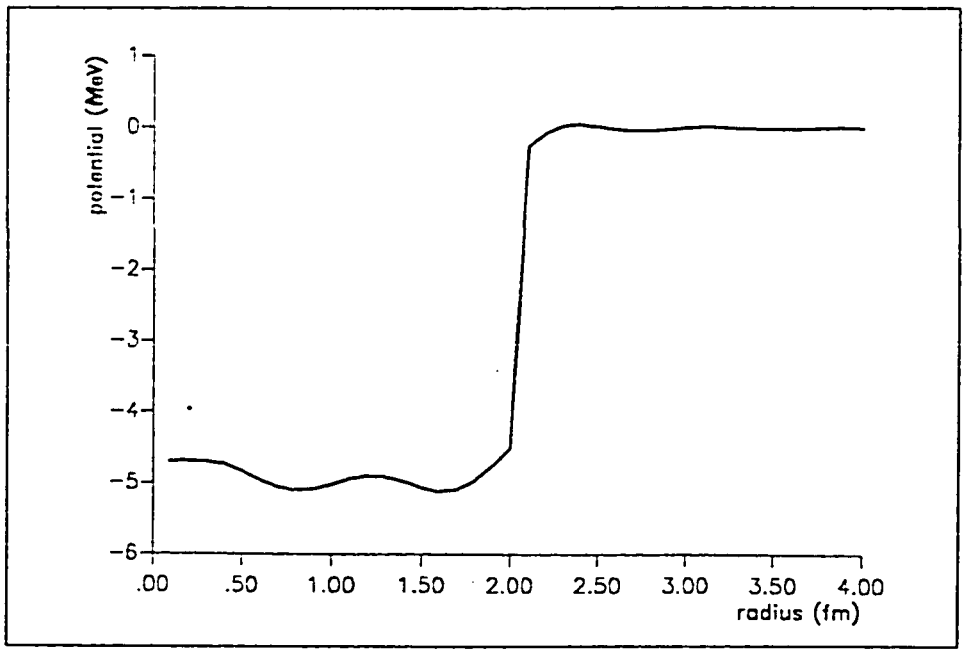


Figure (26.b): Potential calculated from eqn. (3.8.b) using the phase shifts shown above to reproduce the square well potential ($V_2(r)$) of depth 5 MeV, width 2 fm when the auxiliary potential ($V_1(r)$) has depth 4 MeV, width 2 fm

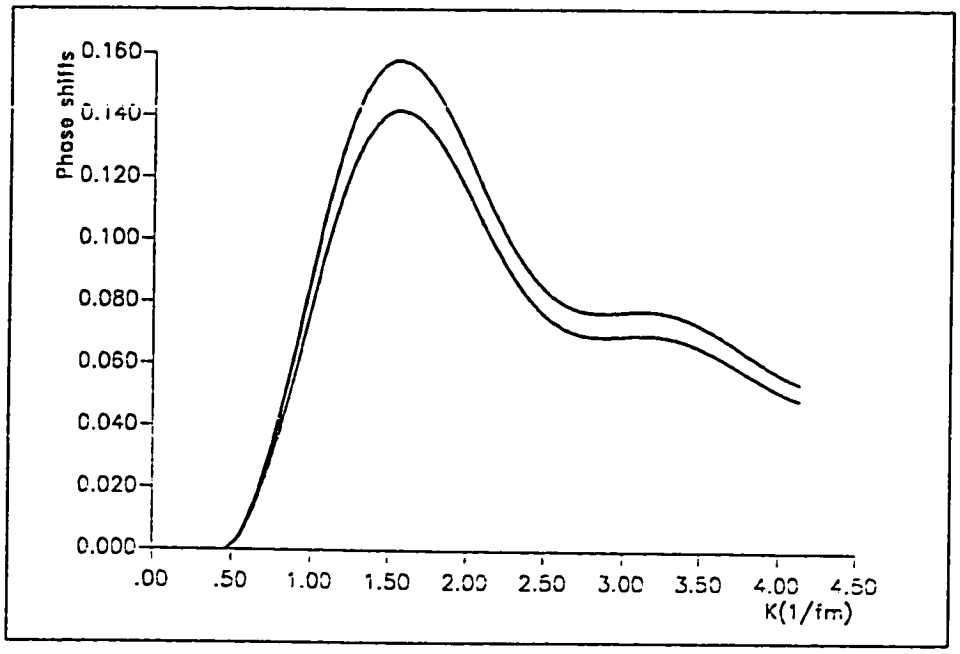


Figure (27.a): "Experimental" phase shifts produced by a nucleon scattered (in $\ell = 1$) from: (i) square well potential of depth 5 MeV, width 2 fm ($V_2(r)$), and (ii) a square well potential of depth 4.5 MeV, width 2 fm ($V_1(r)$).

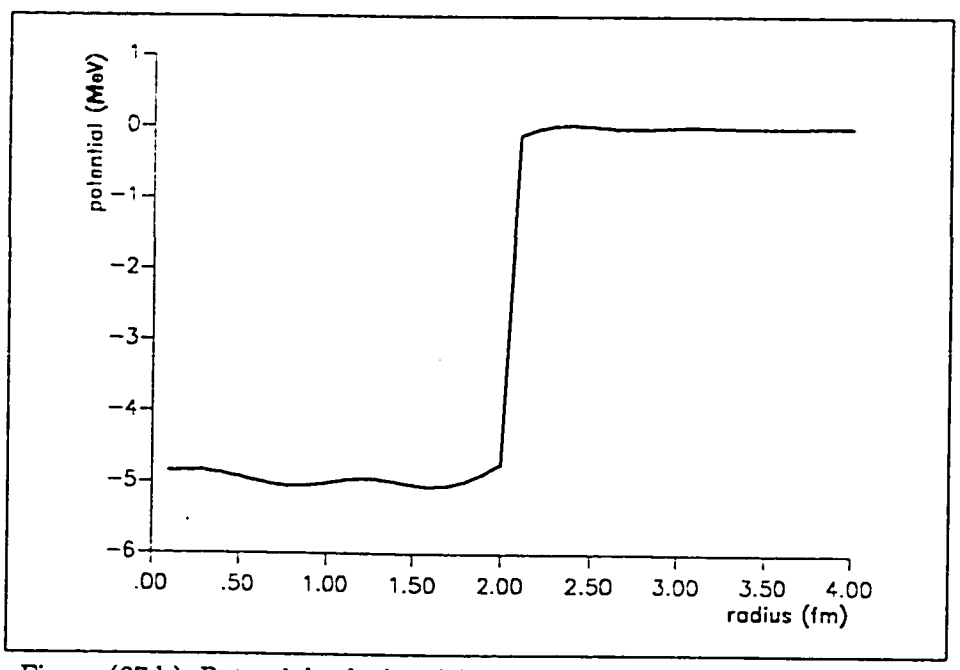


Figure (27.b): Potential calculated from eqn. (3.8.b) using the phase shifts shown above to reproduce the square well potential ($V_2(r)$) of depth 5 MeV, width 2 fm when the auxiliary potential ($V_1(r)$) has depth 4.5 MeV, width 2 fm

<u>Case (II)</u>: $V_2(r)$ has depth $|V_2| = 10\ MeV$, and range $a_2 = 2.0\ fm$. Use auxiliary potential $V_1(r)$ of range $a_1 = 2.0\ fm$, depth: $|V_1| = 0.0,\ 8.0$ and $9.0\ MeV$. See figures #: 28, 29 and 30 for $\ell = 0$; and 31, 32 and 33 for $\ell = 1$.

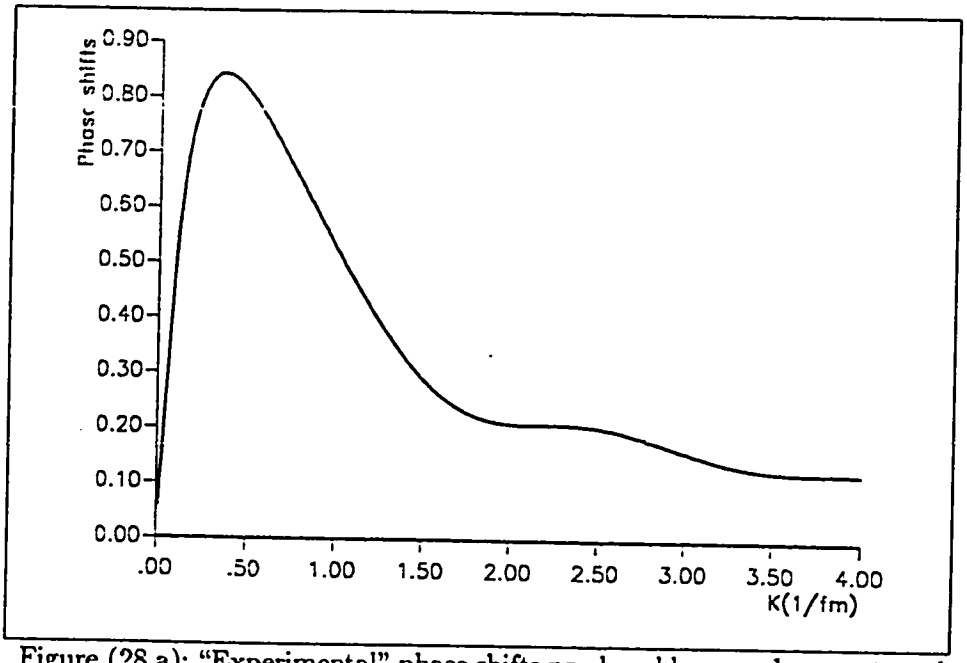


Figure (28.a): "Experimental" phase shifts produced by a nucleon scattered (in $\ell = 0$) from: (i) square well potential of depth 10 MeV, width 2 fm ($V_2(r)$), and (ii) a square well potential of depth 0.0 MeV, width 2 fm ($V_1(r)$).

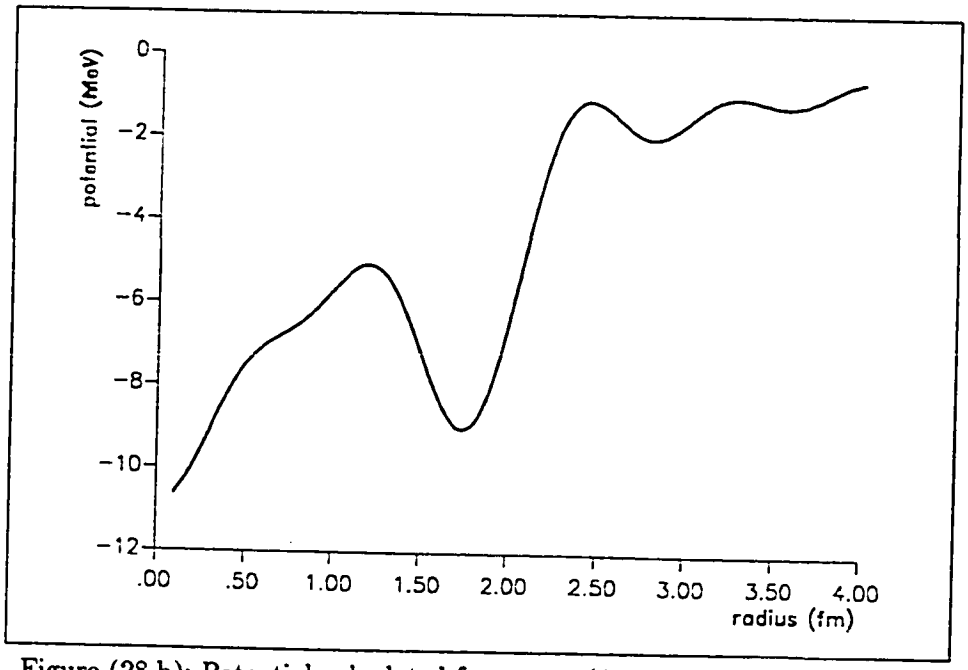


Figure (28.b): Potential calculated from eqn. (3.8.a) using the phase shifts shown above to reproduce the square well potential ($V_2(r)$) of depth 10 MeV, width 2 fm when the auxiliary potential ($V_1(r)$) has depth 0.0 MeV, width 2 fm

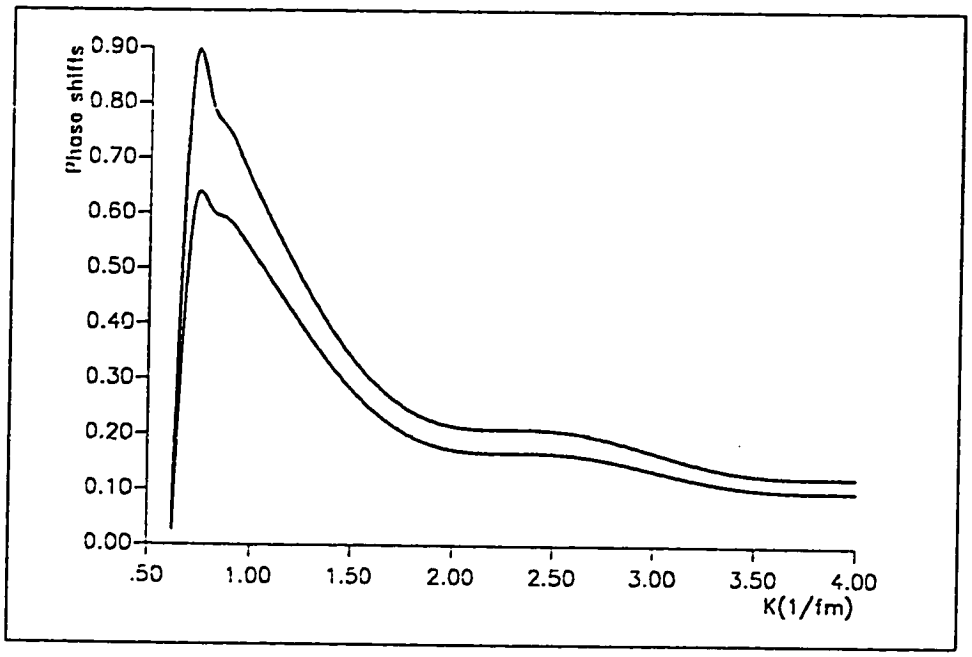


Figure (29.a): "Experimental" phase shifts produced by a nucleon scattered (in $\ell = 0$) from: (i) square well potential of depth 10 MeV, width 2 fm ($V_2(r)$), and (ii) a square well potential of depth 8 MeV, width 2 fm ($V_1(r)$).

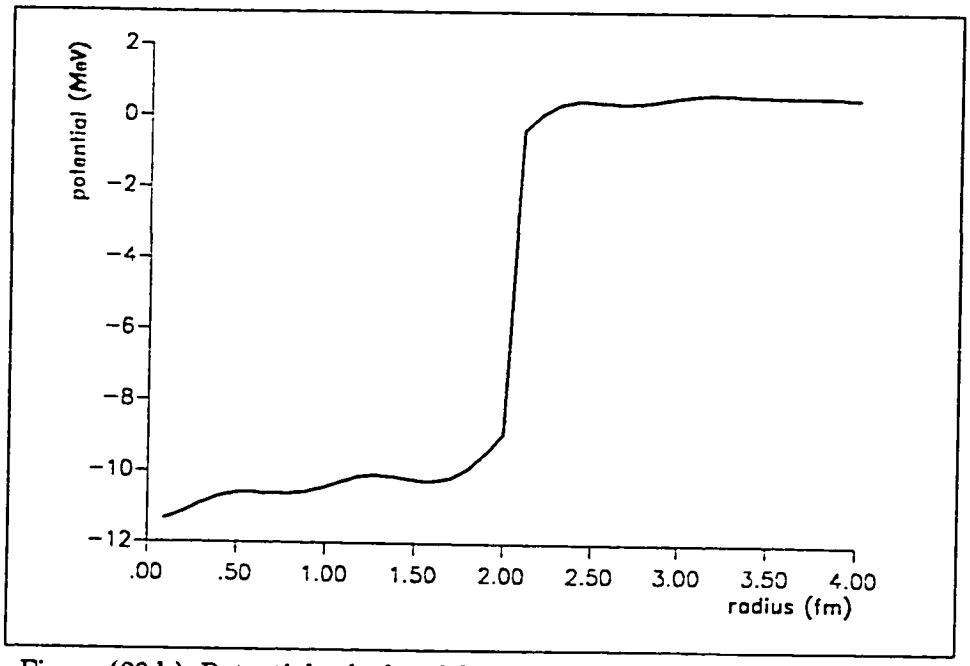


Figure (29.b): Potential calculated from eqn. (3.8.a.) using the phase shifts shown above to reproduce the square well potential ($V_2(r)$) of depth 10 MeV, width 2 fm when the auxiliary potential ($V_1(r)$) has depth 8 MeV, width 2 fm

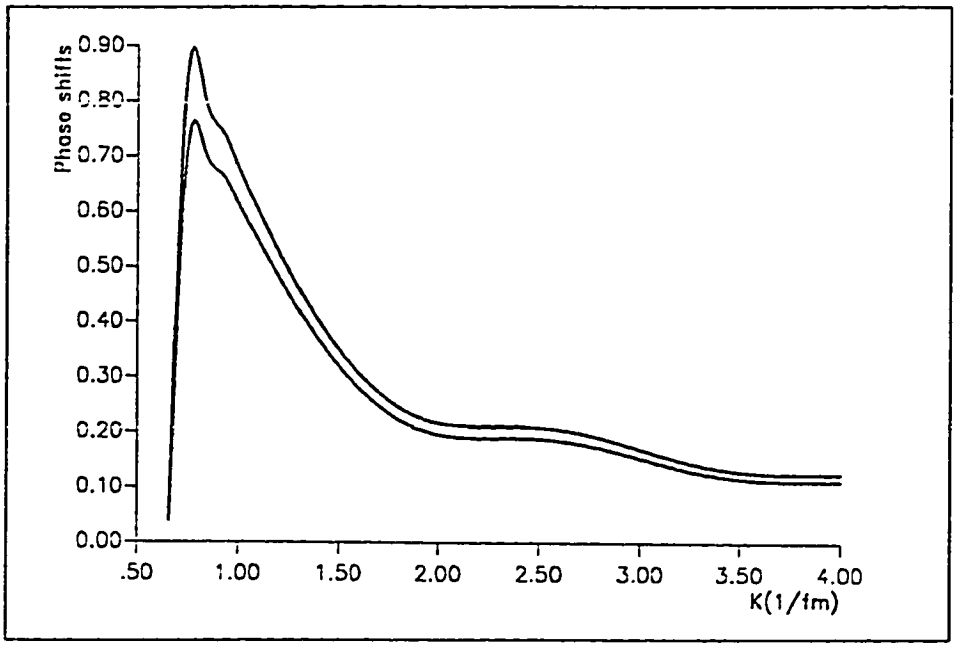


Figure (30.a): "Experimental" phase shifts produced by a nucleon scattered (in $\ell = 0$) from: (i) square well potential of depth 10 MeV, width 2 fm ($V_2(r)$), and (ii) a square well potential of depth 9 MeV, width 2 fm ($V_1(r)$).

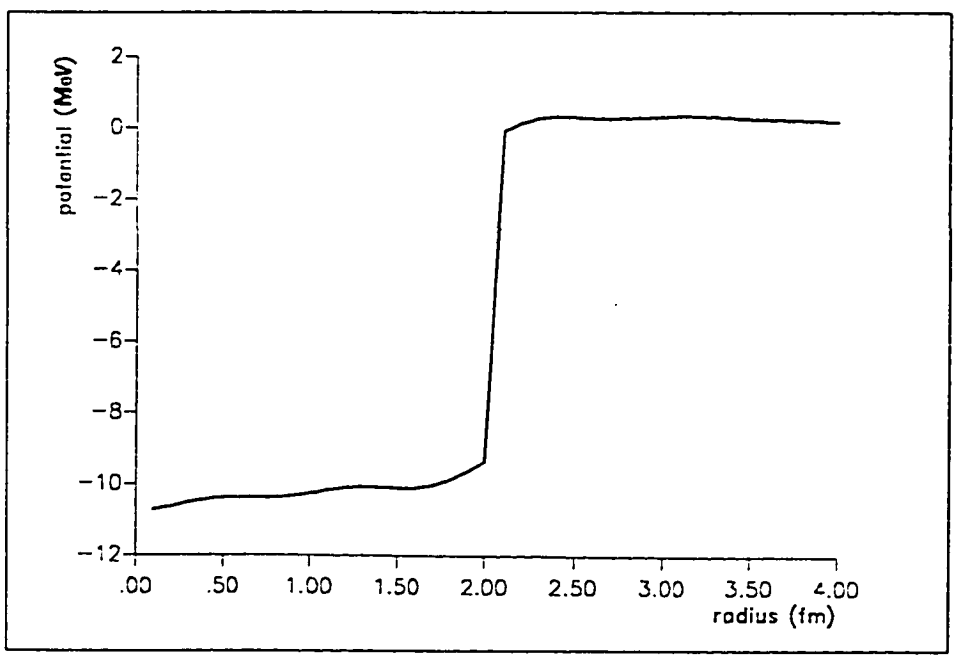


Figure (30.b): Potential calculated from eqn. (3.8.a) using the phase shifts shown above to reproduce the square well potential ($V_2(r)$) of depth 10 MeV, width 2 fm when the auxiliary potential ($V_1(r)$) has depth 9 MeV, width 2 fm

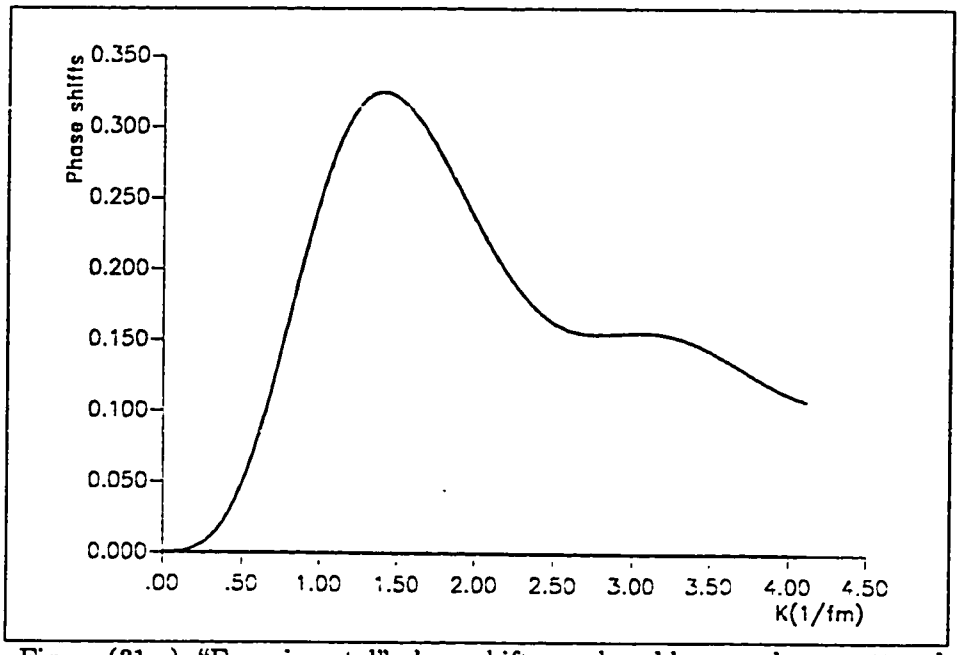


Figure (31.a): "Experimental" phase shifts produced by a nucleon scattered (in $\ell = 1$) from: (i) square well potential of depth 10 MeV, width 2 fm ($V_2(r)$), and (ii) a square well potential of depth 0.0 MeV, width 2 fm ($V_1(r)$).

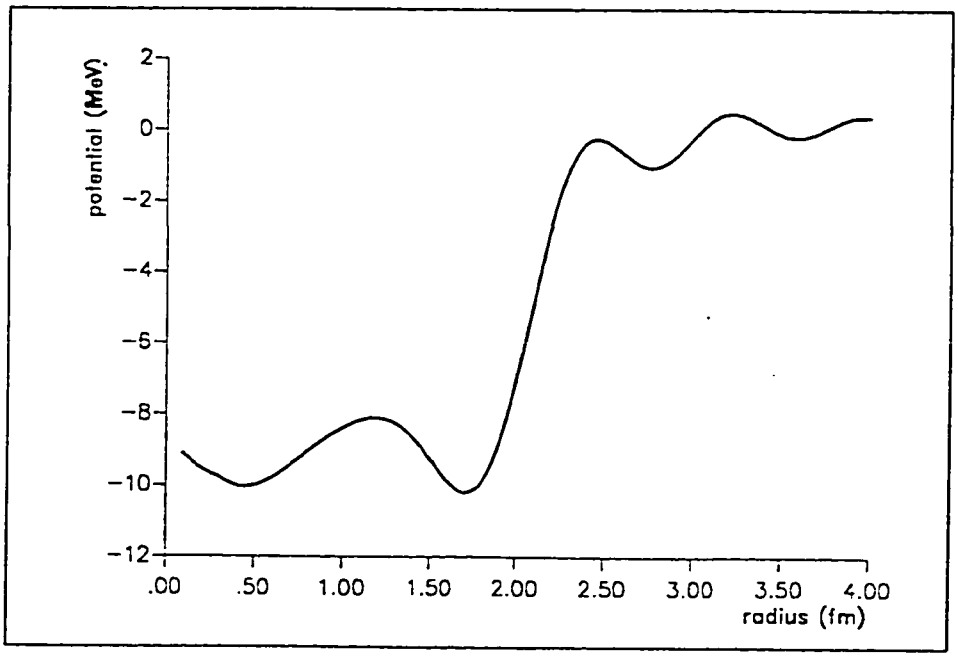


Figure (31.b): Potential calculated from eqn. (3.8.b) using the phase shifts shown above to reproduce the square well potential ($V_2(r)$) of depth 10 MeV, width 2 fm when the auxiliary potential ($V_1(r)$) has depth 0.0 MeV, width 2 fm

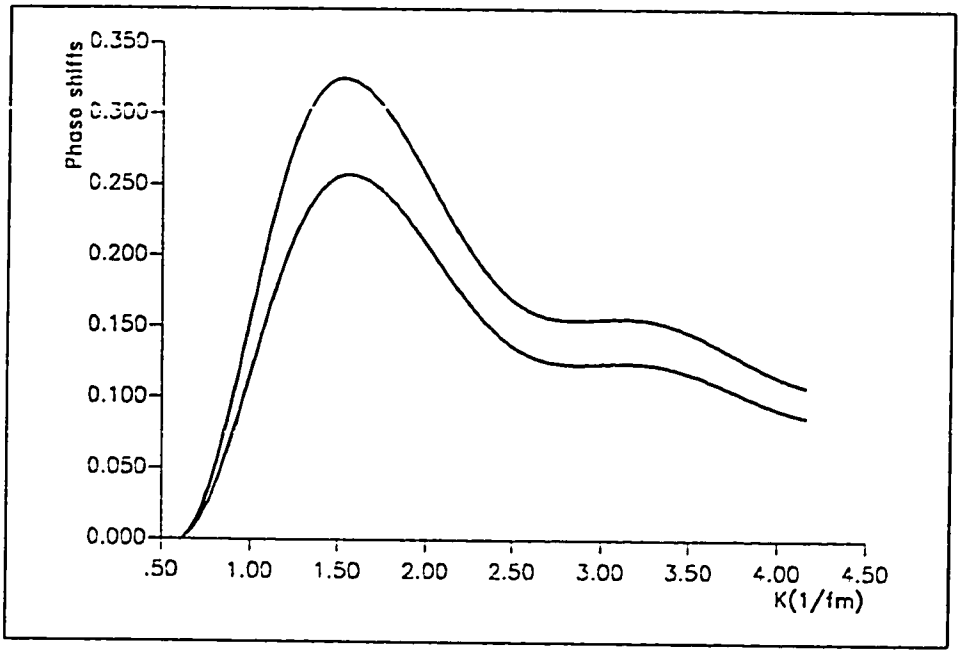


Figure (32.a): "Experimental" phase shifts produced by a nucleon scattered (in $\ell = 1$) from: (i) square well potential of depth 10 MeV, width 2 fm ($V_2(r)$), and (ii) a square well potential of depth 8 MeV, width 2 fm ($V_1(r)$).

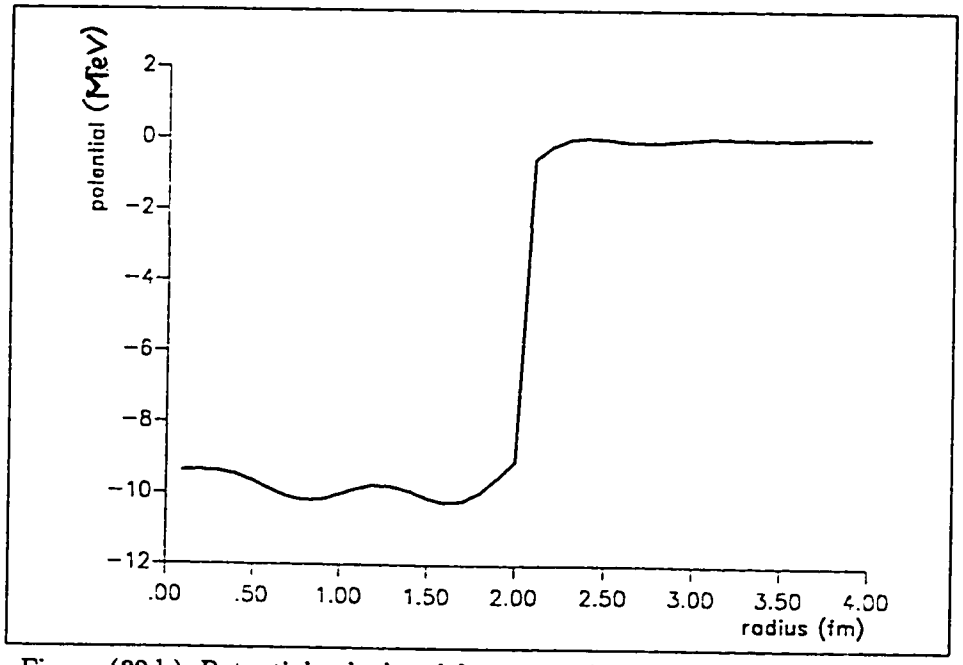


Figure (32.b): Potential calculated from eqn. (3.8.b) using the phase shifts shown above to reproduce the square well potential ($V_2(r)$) of depth 10 MeV, width 2 fm when the auxiliary potential ($V_1(r)$) has depth 8 MeV, width 2 fm

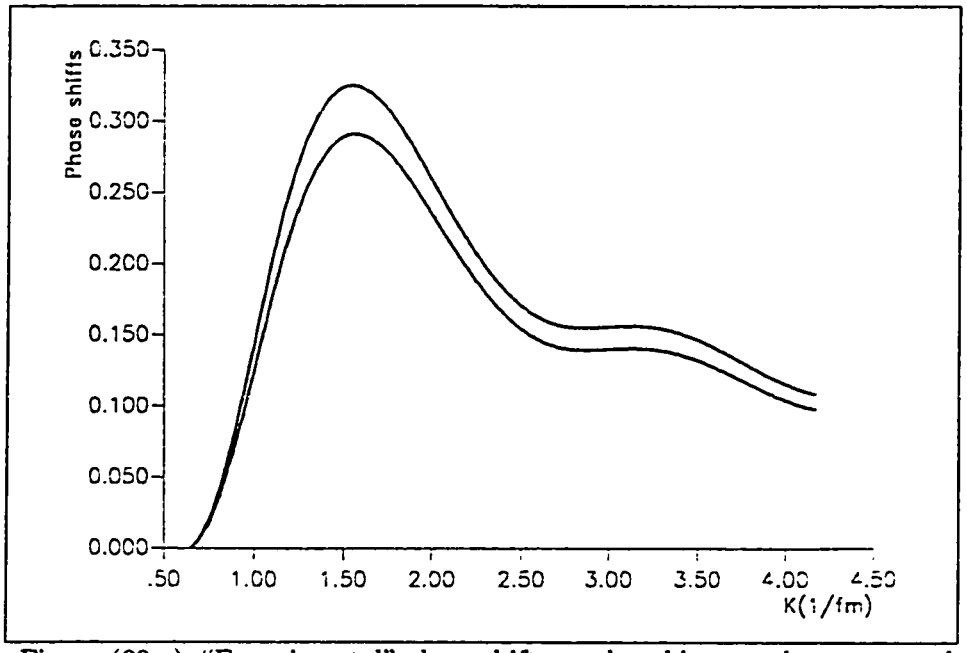


Figure (33.a): "Experimental" phase shifts produced by a nucleon scattered (in $\ell = 1$) from: (i) square well potential of depth 10 MeV, width 2 fm ($V_2(r)$), and (ii) a square well potential of depth 9.0 MeV, width 2 fm ($V_1(r)$).

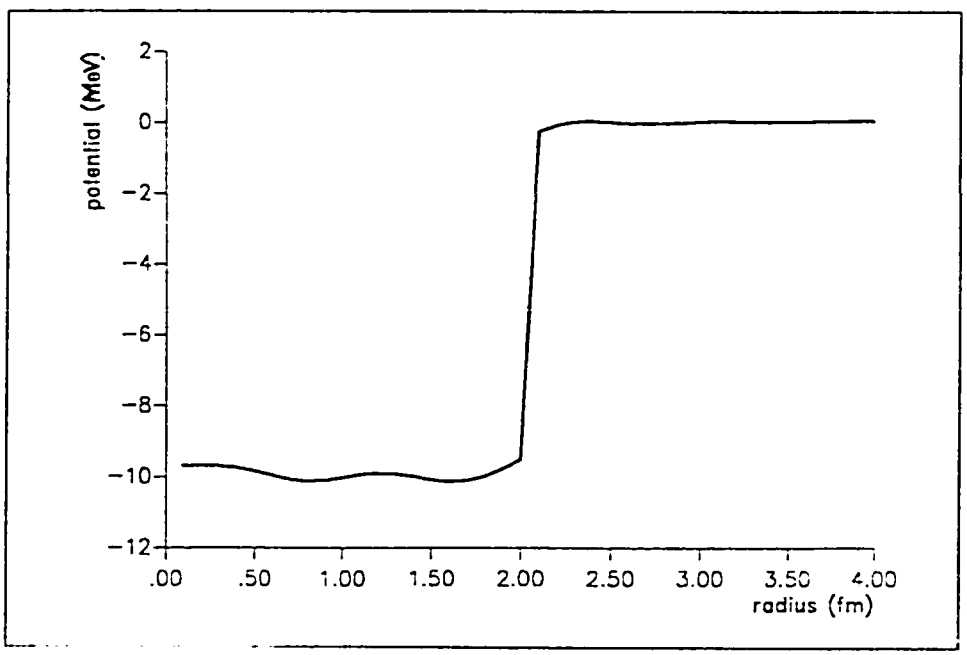


Figure (33.b): Potential calculated from eqn. (3.8.b) using the phase shifts shown above to reproduce the square well potential ($V_2(r)$) of depth 10 MeV, width 2 fm when the auxiliary potential ($V_1(r)$) has depth 9.0 MeV, width 2 fm

Case (III): $V_2(r)$ has depth $|V_2| = 40\ MeV$, and range $a_2 = 2.0\ fm$. Use an auxiliary potential of width $a_1 = 2.0\ fm$ and depth: $|V_1| = 0.0$, 43 and 42 $MeV$ for $\ell = 0$ and $|V_2| = 0.0$, 30 and 35 $MeV$ for $\ell = 1$. See figures #: 34, 35 and 36 for $\ell = 0$; and 37, 38 and 39 for $\ell = 1$.

In this case, for $\ell = 0$ the potential $V_2(r)$ has one bound state which means that the phase shifts should start at $\simeq \pi$ for $k = 0$ (see Appendix C for the "Experimental" phase shifts calculation). In this case, the *best* auxiliary potential should be stronger than $V_2(r)$ in order to make sure that its phase shifts also start at $\simeq \pi$. This is not true for the $\ell = 1$ case, since there is no bound $p$-state for $a_2 = 2.0\ fm$ and $|V_2| = 40\ MeV$.

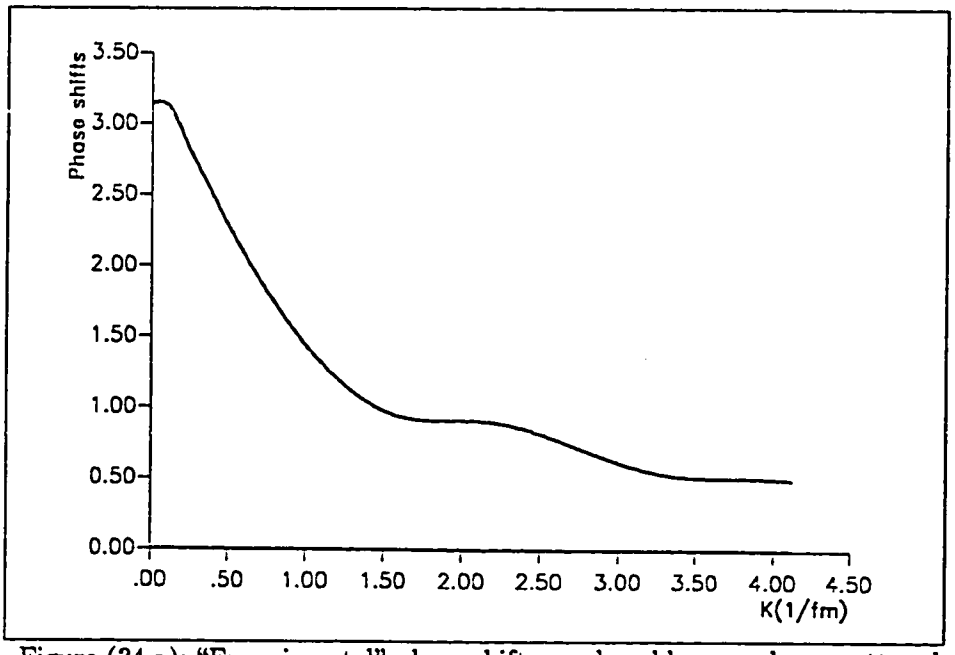


Figure (34.a): "Experimental" phase shifts produced by a nucleon scattered (in $\ell = 0$) from: (i) square well potential of depth 40 MeV, width 2 fm ($V_2(r)$), and (ii) a square well potential of depth 0.0 MeV, width 2 fm ($V_1(r)$).

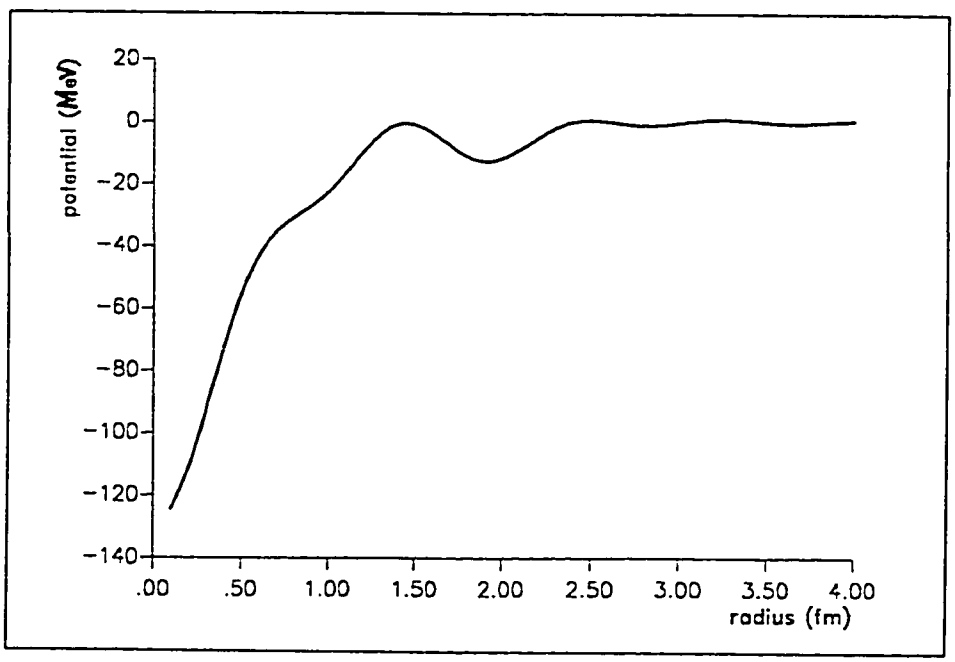


Figure (34.b): Potential calculated from eqn. (3.8.a.) using the phase shifts shown above to reproduce the square well potential ($V_2(r)$) of depth 40 MeV, width 2 fm when the auxiliary potential ($V_1(r)$) has depth 0.0 MeV, width 2 fm

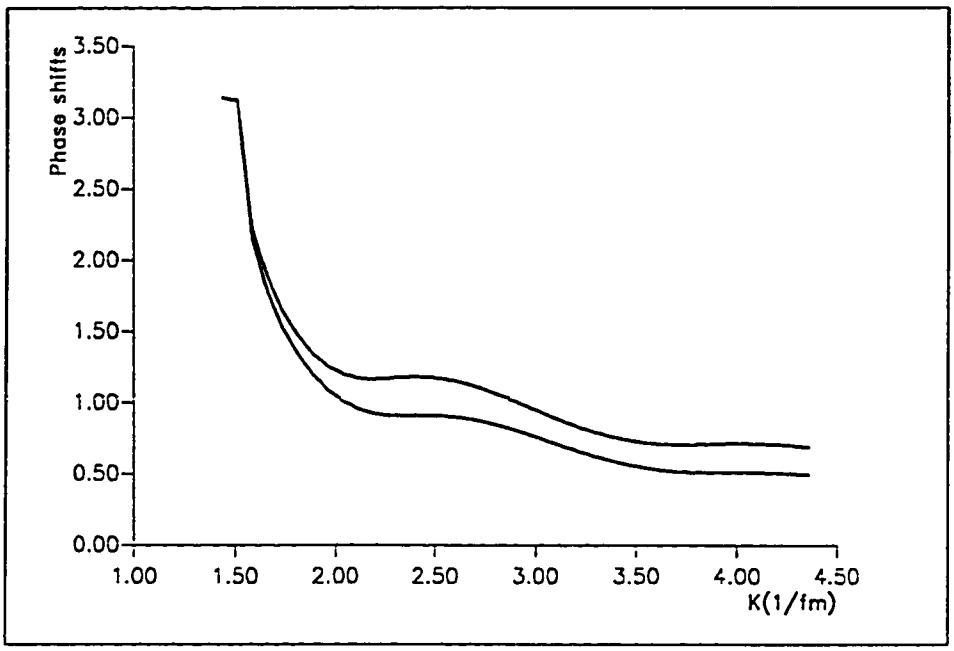


Figure (35.a): "Experimental" phase shifts produced by a nucleon scattered (in $\ell = 0$) from: (i) square well potential of depth 40 MeV, width 2 fm ($V_2(r)$), and (ii) a square well potential of depth 43 MeV, width 2 fm ($V_1(r)$).

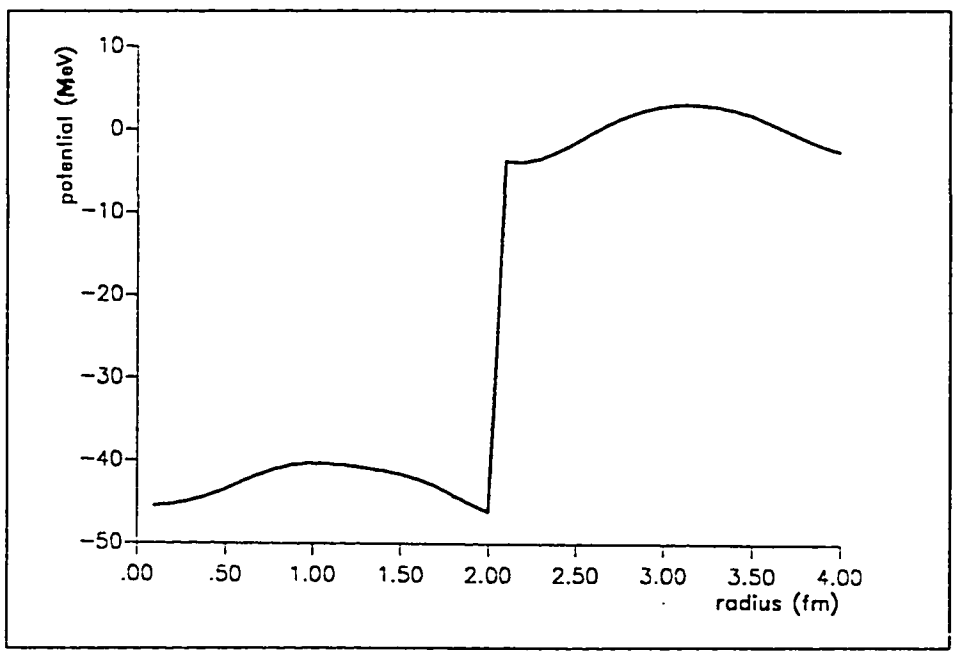


Figure (35.b): Potential calculated from eqn. (3.8.a.) using the phase shifts shown above to reproduce the square well potential ($V_2(r)$) of depth 40 MeV, width 2 fm when the auxiliary potential ($V_1(r)$) has depth 43 MeV, width 2 fm

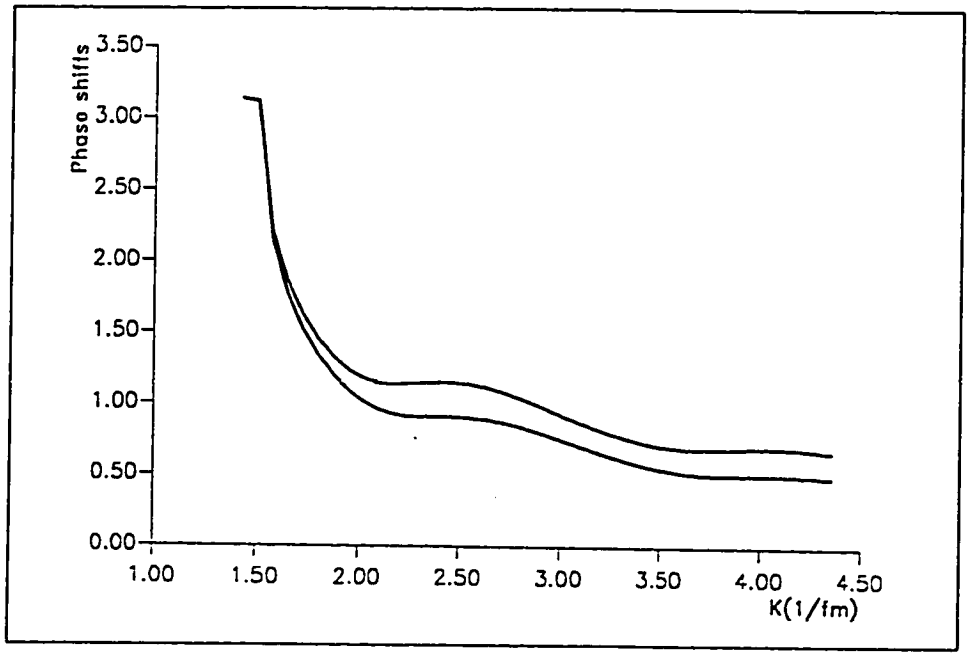


Figure (36.a): "Experimental" phase shifts produced by a nucleon scattered (in $\ell = 0$) from: (i) square well potential of depth 40 MeV, width 2 fm ($V_2(r)$), and (ii) a square well potential of depth 42 MeV, width 2 fm ($V_1(r)$).

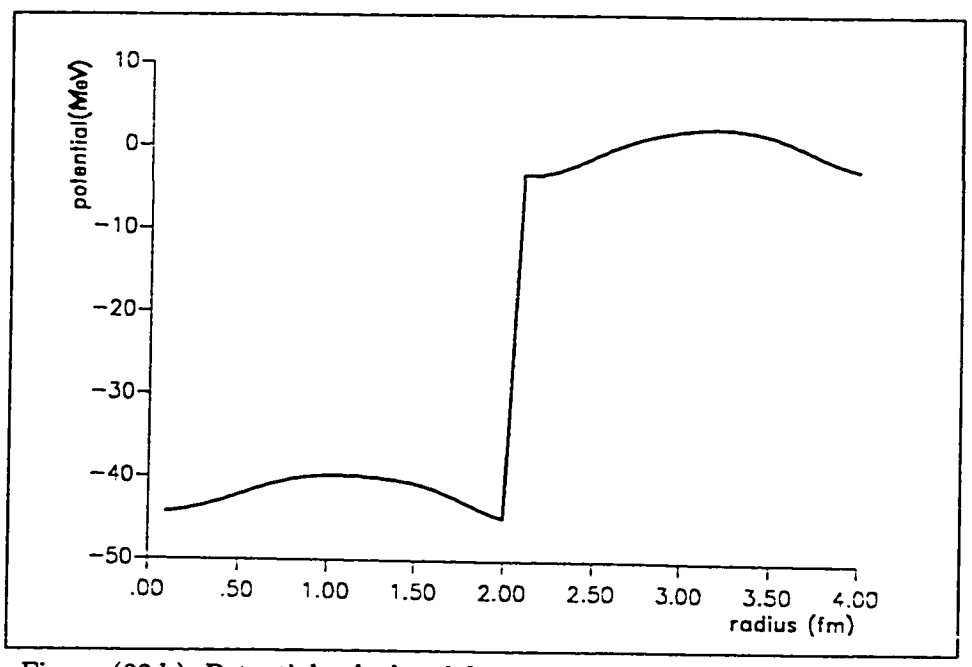


Figure (36.b): Potential calculated from eqn. (3.8.a) using the phase shifts shown above to reproduce the square well potential ($V_2(r)$) of depth 40 MeV, width 2 fm when the auxiliary potential ($V_1(r)$) has depth 42 MeV, width 2 fm

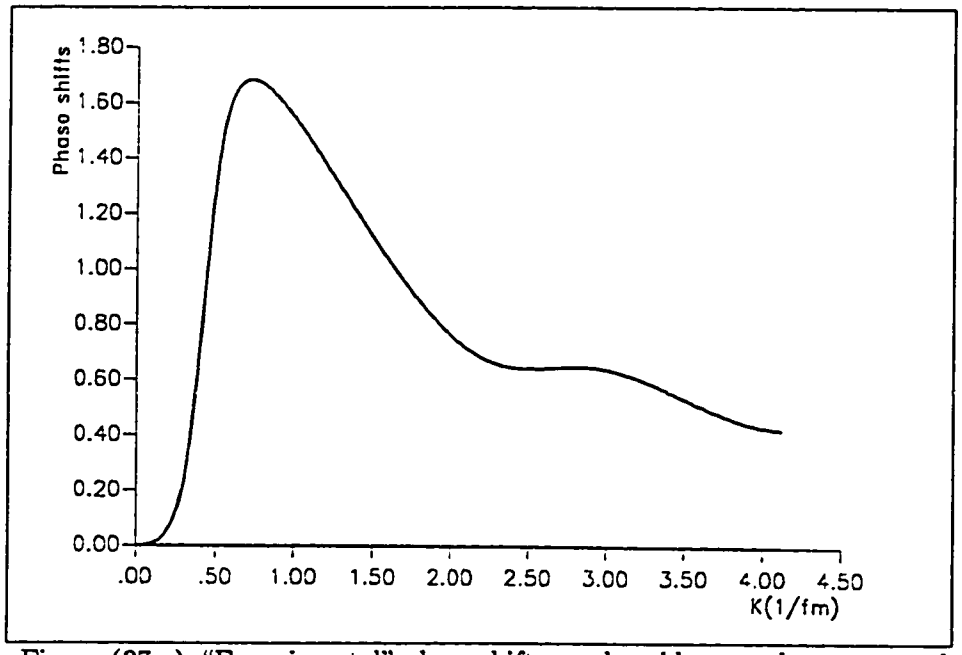


Figure (37.a): "Experimental" phase shifts produced by a nucleon scattered (in $\ell = 1$) from: (i) square well potential of depth 40 MeV, width 2 fm ($V_2(r)$), and (ii) a square well potential of depth 0.0 MeV, width 2 fm ($V_1(r)$).

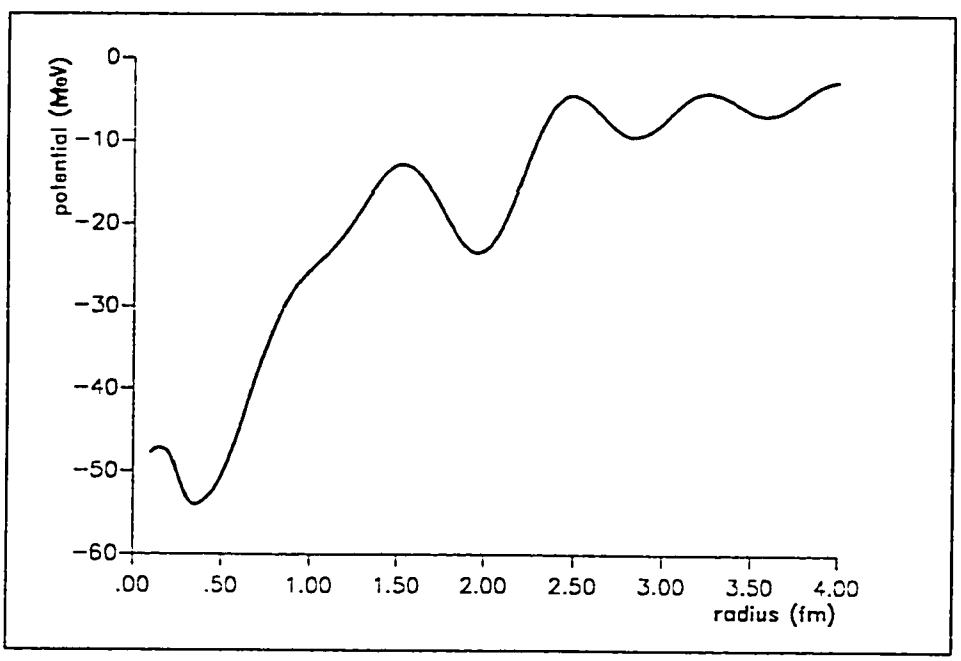


Figure (37.b): Potential calculated from eqn. (3.8.b) using the phase shifts shown above to reproduce the square well potential ($V_2(r)$) of depth 40 MeV, width 2 fm when the auxiliary potential ($V_1(r)$) has depth 0.0 MeV, width 2 fm

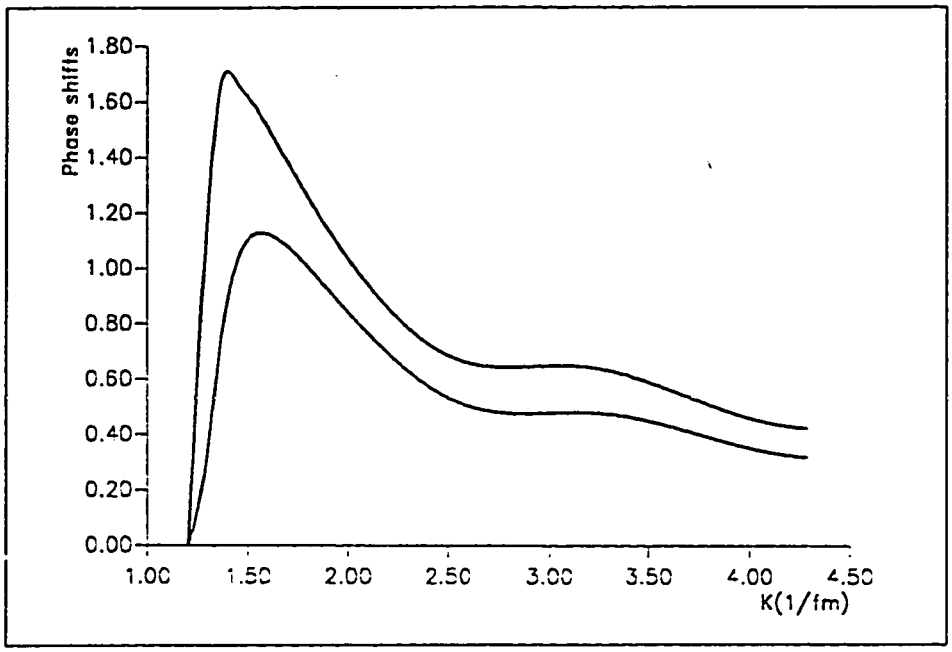


Figure (38.a): "Experimental" phase shifts produced by a nucleon scattered (in $\ell = 1$) from: (i) square well potential of depth 40 MeV, width 2 fm ($V_2(r)$), and (ii) a square well potential of depth 30 MeV, width 2 fm ($V_1(r)$).

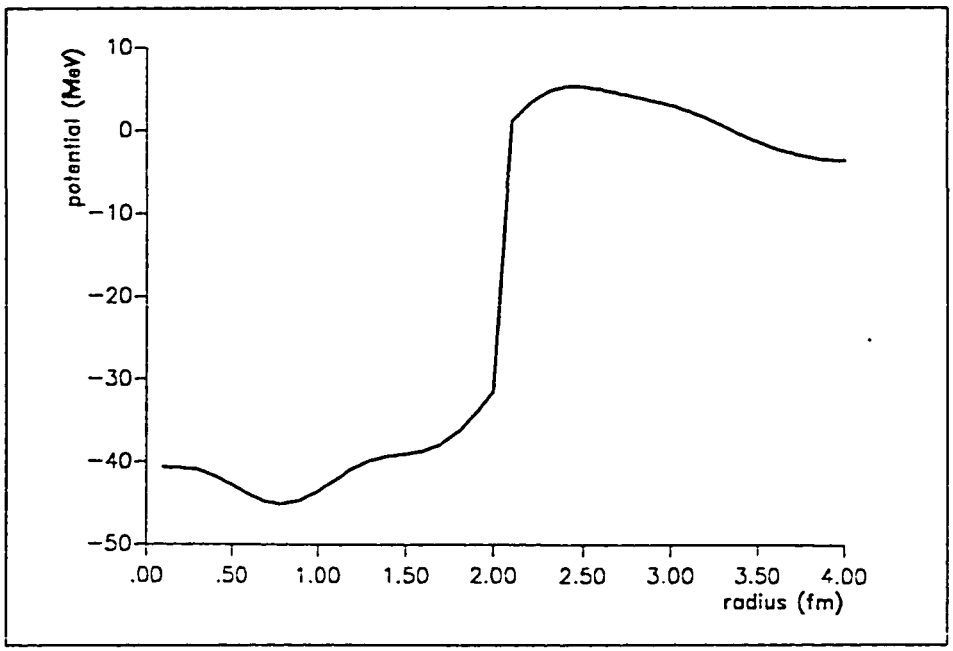


Figure (38.b): Potential calculated from eqn. (3.8.b) using the phase shifts shown above to reproduce the square well potential ($V_2(r)$) of depth 40 MeV, width 2 fm when the auxiliary potential ($V_1(r)$) has depth 30 MeV, width 2 fm

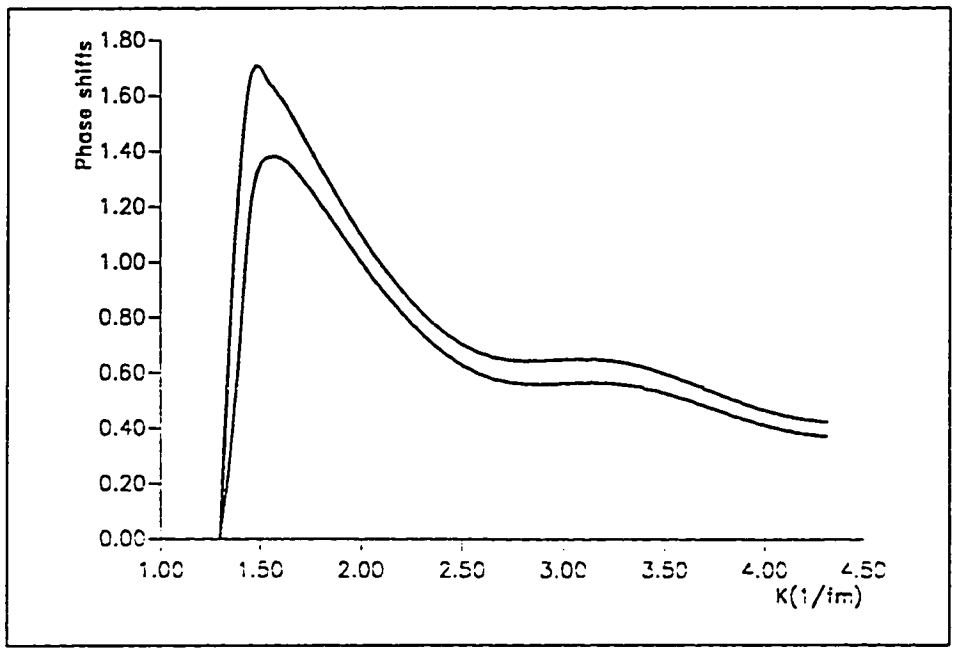


Figure (39.a): "Experimental" phase shifts produced by a nucleon scattered (in $\ell = 1$) from: (i) square well potential of depth 40 MeV, width 2 fm ($V_2(r)$), and (ii) a square well potential of depth 35 MeV, width 2 fm ($V_1(r)$).

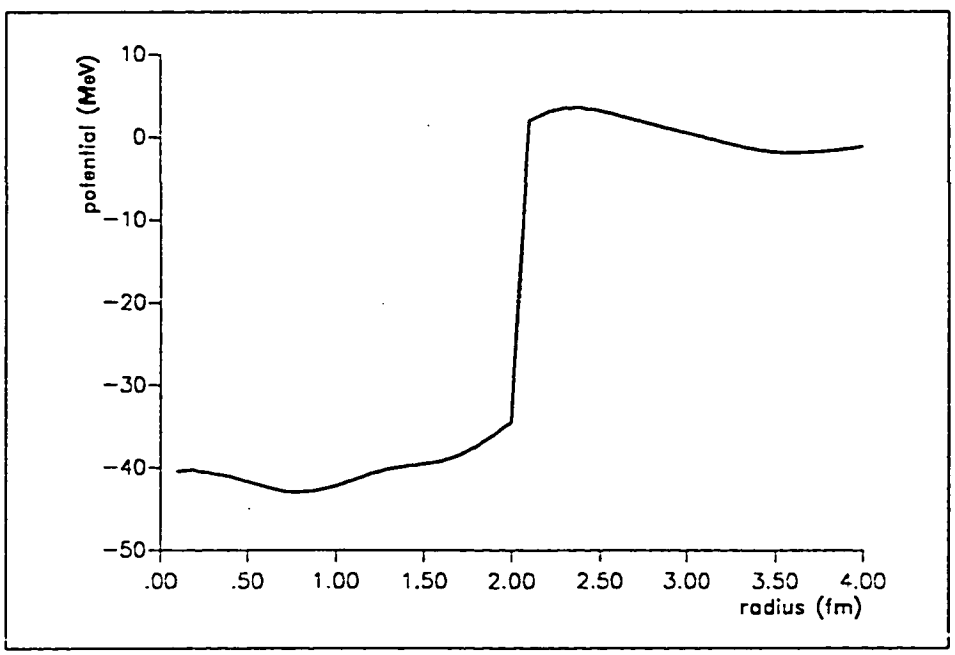


Figure (39.b): Potential calculated from eqn. (3.8.b) using the phase shifts shown above to reproduce the square well potential ($V_2(r)$) of depth 40 MeV, width 2 fm when the auxiliary potential ($V_1(r)$) has depth 35 MeV, width 2 fm

We conclude that our approach is very good and powerful for calculating the potential from its phase shifts, provided we use the *best* auxiliary potential. One recalls that our approaches require square well (or barrier) auxiliary potentials. Therefore, there are two parameters for fixing the best auxiliary potential. First, the range to which the phase shifts (and thus the calculated potential) are very sensitive. The best range to choose is the one at which the phase shifts ($\delta_2(k)$ and $\delta_1(k)$) become very similar in shape. Thus, one should change the range of the auxiliary potential with *some* fixed strength (height or depth) till he gets the most similar behavior for $\delta_1(k)$ as compared to $\delta_2(k)$. This corresponds to getting the best range. Afterwards, one can change the strength of the auxiliary potential with the range fixed at the best value found above. In this way one can obtain very similar and close phase shifts ($\delta_2(k)$ and $\delta_1(k)$) and thereby a potential which very closely approximates the *unknown* potential $V_2(r)$.

One further notes that our approach is accurate even for strong potentials (e.g., 40 $MeV$), whereas the approach used in [1] is not. The examples taken in this chapter are obtained by *assuming* some square potential $V_2(r)$ and then finding its phase shifts $\delta_2(k)$. Afterwards the best auxiliary potential is found as explained above. Finally, one calculates $V_2(r)$ as if it were unknown except for its phase shifts $\delta_2(k)$. The results came to be very satisfactory.

In general, one *only* has $\delta_2(k)$ of some *unknown* potential $V_2(r)$. However, the procedures used above apply equally well to *any* $V_2(r)$.

# APPENDICES

<u>Appendix A</u>: A proof of Eqn. (2.10):

The spherical Neuman functions are defined in terms of the cylindrical Bessel functions as:

$$n_\ell(\rho) = (-1)^{\ell+1}\left(\frac{\pi}{2\rho}\right)^{1/2} J_{\ell-\frac{1}{2}}(\rho)\rho \tag{A.1}$$

$$n_\ell^2(\rho) = \frac{\pi}{2\rho} J^2_{-\ell-\frac{1}{2}}(\rho) \tag{A.2}$$

where, $J_\ell(\rho)$ is the cylindrical Bessel function, $n_\ell(\rho)$ is the spherical Neumann function and $\ell$ is an integer $(0, 1, 2, 3, \ldots)$ [6]. From [12]:

$$n_\ell^2(\rho) = \frac{\pi}{2\rho}\sum_{k=0}^{\infty} \frac{(-1)^k(\rho)^{2k-2\ell-1}\Gamma(2k-2\ell)}{(2)^{2k-2\ell-1}\Gamma(k-2\ell)\Gamma(k-\ell+1/2)^2 k!}. \tag{A.3}$$

Using the duplication formula of Gamma function [13]; namely:

$$2^{2x-1}\Gamma(x)\Gamma(x+1/2) = \sqrt{\pi}\Gamma(2x), \tag{A.4}$$

A.3 becomes:

$$n_\ell^2(\rho) = \frac{\sqrt{\pi}}{2\rho}\sum_{k=0}^{\infty} \frac{(-1)^k(\rho)^{2k-2\ell-1}\Gamma(k-\ell)^2\Gamma(k-\ell+1/2)}{(2)^{\circ}\Gamma(k-2\ell)\Gamma(k-\ell)\Gamma(k-\ell+1/2)^2 k!}. \tag{A.5}$$

Applying (A.4) again, (A.5) will reduce to:

$$n_\ell^2(\rho) = \sum_{k=0}^{\infty} \frac{(-1)^k(\rho)^{2(k-\ell-1)}\Gamma(k-\ell)^2}{(2)^{2-2(k-\ell)}\Gamma(k-2\ell)\Gamma(2k-2\ell)k!}. \tag{A.5'}$$

But

$$\Gamma(n+1) = n\Gamma(n) = n! \quad n = \text{ integer } \quad (\geq 0)$$

Hence

$$n_\ell^2(\rho) = \frac{1}{(2)^{2\ell+2}} \sum_{k=0}^{\infty} \frac{(-1)^k (\rho)^{2(k-\ell-1)} (2)^{2k} ((k-\ell-1)!)^2}{(k-2\ell-1)!(2k-2\ell-1)!k!}. \tag{A.6}$$

Applying the differential operation $\frac{d}{d\rho^2}\ (\rho^{2\ell+2}) \equiv \frac{1}{2\rho}\frac{d}{d\rho}\ (\rho^{2\ell+2})$ on both sides;

$$\frac{1}{2\rho}\frac{d}{d\rho}\left(\rho^{2\ell+2} n_\ell^2(\rho)\right) = \frac{1}{(2)^{2\ell+3}} \sum_{k=0}^{\infty} \frac{(-1)^k (2)^{2k} \rho^{2k-2} (2k)((k-\ell-1)!)^2}{(k-2\ell-1)!(2k-2\ell-1)!k!}. \tag{A.7}$$

Let

$$\frac{1}{2\rho}\frac{d}{d\rho} \equiv \hat{P}(\rho),$$

and

$$\rho^{2\ell+2} n_\ell^2(\rho) \equiv \eta(\rho).$$

Hence applying $\frac{1}{2\rho}\frac{d}{d\rho}$ $\ell$ times on both sides,

$$(\hat{P}(\rho))^\ell (\eta(\rho)) = \left[2/(2)^{2\ell+3}\right] \sum_{k=0}^{\infty} \frac{(-1)^k (2)^{2k} \rho^{2k-2\ell} (k)(k-1)\cdots(k-\ell+1)((k-\ell-1)!)^2}{(k-2\ell-1)!(2k-2\ell-1)!k!}. \tag{A.8}$$

Then:

$$\frac{d}{d\rho}\left[\hat{P}(\rho)\right]^\ell (\eta(\rho)) = \left[2/2^{2\ell+3}\right] \sum_{k=0}^{\infty} \frac{(-1)^k (2)^{2k} (\rho)^{2k-2\ell-1} (2k-2\ell)(k)(k-1)\cdots}{(k-2\ell-1)!(2k-2\ell-1)!(k)!} \times$$

$$\frac{(k-\ell+1)((k-\ell-1)!)^2}{1} \tag{A.9}$$

$$LHS = (2/(2)^{2\ell+2}) \sum_{k=0}^{\infty} \frac{(-1)^k (2)^{2k} (\rho)^{2k-2\ell-1} (k)(k-1)\cdots(k-\ell+1)(k-\ell)(k-\ell-1)!(k-\ell-1)!}{(k-2\ell-1)(2k-2\ell-1)!(k)!}. \tag{A.10}$$

Simplifying;

$$LHS = (2/(2)^{2\ell+2}) \sum_{k=0}^{\infty} \frac{(-1)^k (2)^{2k} (\rho)^{2k-2\ell-1} (k-\ell-1)!}{(k-2\ell-1)!(2k-2\ell-1)!}. \tag{A.10'}$$

Applying $[\hat{P}(\rho)]^{\ell}$ again;

$$\left[\hat{P}(\rho)\right]^{\ell} \frac{d}{d\rho}\left[\hat{P}(\rho)\right]^{\ell}(\eta(\rho)) = (1/2^{3\ell+1})\sum_{k=0}^{\infty}\frac{(-1)^k(2)^{2k}(\rho)^{2k-4\ell-1}(2k-2\ell-1)(2k-2\ell-3)(2k-2\ell-5)\times}{(k-2\ell-1)!(2k-2\ell-1)!}$$

$$\frac{\cdots(2k-4\ell+1)(k-\ell-1)!}{1} \tag{A.11}$$

$$\Rightarrow LHS = (1/(2)^{3\ell+1})\sum_{k=0}^{\infty}\frac{(-1)^k(2)^{2k}(\rho)^{2k-4\ell-1}(2k-2\ell-1)(2k-2\ell-2)(2k-2\ell-3)\cdots\times}{(k-2\ell-1)!(2k-2\ell-1)!(2k-2\ell-2)(2k-2\ell-4)(2k-2\ell-6)\cdots}$$

$$\frac{\cdot(2k-4\ell+1)(2k-4\ell)\cdot(k-\ell-1)!}{\cdots(2k-4\ell)}$$

$$LHS = \left(1/(2)^{3\ell+1}\right)\sum_{k=0}^{\infty}\frac{(-1)^k(2)^{2k}(\rho)^{2k-4\ell-1}(2k-2\ell-1)(2k-2\ell-2)\cdots(2k-4\ell)\cdot(k-\ell-1)!}{(k-2\ell-1)!(2k-2\ell-1)!(2)^{\ell}(k-\ell-1)\cdots(k-2\ell)}$$

$$= \left(1/(2)^{4\ell+1}\right)\sum_{k=0}^{\infty}\frac{(-1)^k(2)^{2k}(\rho)^{2k-4\ell-1}(2k-2\ell-1)(2k-2\ell-2)\cdots(2k-4\ell)(2k-4\ell-1)!(k-\ell-1)!}{(k-2\ell-1)!(2k-2\ell-1)!(k-\ell-1)\cdots(k-2\ell)(2k-4\ell-1)!} \tag{A.11'}$$

Simplifying;

$$LHS = \sum_{k=0}^{\infty}\frac{(-1)^k(2)^{2k+3}(\rho)^{2k-4\ell-1}(k-\ell-1)!}{(k-2\ell-1)!(k-\ell-1)\cdots(k-2\ell)\cdot(2k-4\ell-1)!}\frac{1}{(2)^{4\ell+4}} \tag{A.12}$$

$$LHS = \sum_{k=0}^{\infty}\frac{(-1)^k(2)^{2k+3}(\rho)^{2k-4\ell-1}}{(2k-4\ell-1)!(2)^{4\ell+4}} \tag{A.12'}$$

But $(-n)! = \pm\infty$, where $n$ is +ve integer. Hence the only surviving terms in (A.12′) above are those for which $(2k-4\ell-1) \geq 0$ or,

$$2k \geq 4\ell + 1$$

or,

$$k \geq 2\ell + 1/2$$

However,

$$k = \text{ integer}$$

$$\ell = \text{ integer}$$

Hence,

$$k \geq 2\ell + 1$$

and

$$[\hat{p}(\rho)]^{\ell} \frac{d}{d\rho} [\hat{p}(\rho)]^{\ell} (\eta(\rho)) = \sum_{k=2\ell+1}^{\infty} \frac{(-1)^k (2)^{2k+3} (\rho)^{2k-4\ell-1}}{(2k-4\ell-1)!(2)^{4\ell+4}}. \tag{A.13}$$

Let

$$(k - 2\ell - 1) = m$$

Hence

$$LHS = \sum_{m=0}^{\infty} \frac{(-1)^{m+2\ell+1} (2)^{2m+4\ell+5} (\rho)^{2m+1}}{(2m+1)!(2)^{4\ell+4}} \tag{A.14}$$

$$LHS = \sum_{m=0}^{\infty} \frac{(-1)^{m+1} (2\rho)^{2m+1}}{(2m+1)!} = -\sin(2\rho) \tag{A.14'}$$

Hence

$$\boxed{\left(\tfrac{1}{2\rho}\tfrac{d}{d\rho}\right)^{\ell} \tfrac{d}{d\rho} \left(\tfrac{1}{d\rho}\tfrac{d}{2\rho}\right)^{\ell} (\rho^{2\ell+2} n_{\ell}^{2}(\rho)) = -\sin(2\rho).}$$

**Appendix B**: A proof of Eqn. (2.11):

Since,

$$j_\ell(\rho) = (\pi/2\rho)^{1/2} J_{\ell+1/2}(\rho) \tag{B.1}$$

$$n_\ell(\rho) = (-1)^{\ell+1} \left(\frac{\pi}{2\rho}\right)^{1/2} J_{\ell-\frac{1}{2}}(\rho) \tag{B.2}$$

where,

$j_\ell(\rho)$ is the spherical Bessel function

$n_\ell(\rho)$ is the spherical Neumann function

$J_{\ell+1/2}(\rho)$ and $J_{-\ell-1/2}(\rho)$ are the cylindrical Bessel functions

$\ell$ is an integer $(0, 1, 2, \ldots)$ [6].

Hence, from (B.1) and (B.2)

$$n_\ell(\rho) j_\ell(\rho) = (-1)^{\ell+1} (\frac{\pi}{2\rho}) J_{-\ell-1/2}(\rho) J_{\ell+1/2}(\rho) \tag{B.3}$$

Thus from [12], (B.3) bcomes:

$$n_\ell(\rho) j_\ell(\rho) = (-1)^{\ell+1} (\frac{\pi}{2\rho}) \sum_{k=0}^{\infty} \frac{(-1)^k (\rho)^{2k} \Gamma(2k+1)}{(2)^{2k} \Gamma(k+1) \Gamma(k-\ell+1/2) \Gamma(k+\ell+3/2) k!} \tag{B.4}$$

Hence

$$\rho^{2\ell+2} n_\ell(\rho) j_\ell(\rho) = (-1)^{\ell+1} \pi \sum_{k=0}^{\infty} \frac{(-1)^k (\rho)^{2k+2\ell+1} (2k)!}{(2)^{2k+1} (k-\ell-1/2)! (k+\ell+1/2)! [(k)!]^2} \tag{B.5}$$

Let

$$\rho^{2\ell+2} n_\ell(\rho) j_\ell(\rho) \equiv \xi(\rho)$$

$$\frac{1}{2\rho}\frac{d}{d\rho} \equiv \hat{p}(\rho)$$

Applying $\hat{p}(\rho)$ on both sides of (B.5):

$$\hat{p}(\rho)(\xi(\rho)) = \frac{(-1)^{\ell+1}\pi}{2}\sum_{k=0}^{\infty}\frac{(-1)^k(\rho)^{2k+2\ell-1}(2k+2\ell+1)(2k)!}{(2)^{2k+1}(k-\ell-1/2)!(k+\ell+1/2)!(k!)^2} \tag{B.6}$$

or,

$$[\hat{p}(\rho)]^{\ell}(\xi(\rho)) = \frac{(-1)^{\ell+1}\pi}{2^{\ell}}\sum_{k=0}^{\infty}\frac{(-1)^k(\rho)^{2k+1}(2k+2\ell+1)(2k+2\ell-1)\cdots(2k+3)(2k)!}{(2)^{2k+1}(k-\ell-1/2)!(k+\ell+1/2)!(k!)^2} \tag{B.7}$$

Hence,

$$LHS = \frac{(-1)^{\ell+1}\pi}{2^{\ell}}\sum_{k=0}^{\infty}\frac{(-1)^k(\rho)^{2k+1}(2k+2\ell+1)(2k+2\ell)(2k+2\ell-1)\cdots}{(2)^{2k+1}(k-\ell-1/2)!(k+\ell+1/2)!(2k+2\ell)(2k+2\ell-2)\cdots}\times$$

$$\frac{(2k+2\ell-2)\cdots(2k+3)(2k+2)(2k)!}{\cdots(2k+2)(k!)^2}$$

$$LHS = (-1)^{\ell+1}\pi\sum_{k=0}^{\infty}\frac{(-1)^k(\rho)^{2k+1}(2k+2\ell+1)!/(2k+1)}{(2)^{2k+\ell+1}(k-\ell-1/2)!(k+\ell+1/2)!(2k+2\ell)(2k+2\ell-2)\cdots(2k+2)(k!)^2} \tag{B.7'}$$

Applying $\frac{d}{d\rho}$ on both sides of (B.7′):

$$\frac{d}{d\rho}[\hat{p}(\rho)]^{\ell}(\xi(\rho)) = (-1)^{\ell+1}\pi\sum_{k=0}^{\infty}\frac{(-1)^k(\rho)^{2k+1}(2k+2\ell+1)!}{(2)^{2k+\ell+1+1}(2k+1)(k-\ell-1/2)!(k+\ell+1/2)!(k+\ell)(k+\ell-1)\ldots(k+1)(k!)(k)!}$$

$$LHS = (-1)^{\ell+1}\pi\sum_{k=0}^{\infty}\frac{(-1)^k(\rho)^{2k}(2k+2\ell+1)!}{(2)^{2k+2\ell+1}(k-\ell-1/2)!\Gamma(k+\ell+3/2)\Gamma(k+\ell+1)(k!)} \tag{B.8}$$

Using the duplication formula (A.4):

$$LHS = \frac{(-1)^{\ell+1}\pi}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{(-1)^k(\rho)^{2k}(2k+2\ell+1)!}{(2)^{2k+2\ell+1}(k-\ell-1/2)!(2)^{1-2k-2\ell}\Gamma(2k+2\ell+2)(k!)}. \tag{B.9}$$

Simplifying,

$$LHS = (-1)^{\ell+1}\sqrt{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k(\rho)^{2k}\Gamma(2k+2\ell+2)}{(k-\ell-1/2)!\Gamma(2k+2\ell+2)(k!)} \tag{B.9'}$$

Applying $[\hat{p}(\rho)]^\ell$ on both sides of (B.9′)

$$[\hat{p}(\rho)]^\ell \frac{d}{d\rho}[\hat{p}(\rho)]^\ell(\xi(\rho)) = (-1)^{\ell+1}\sqrt{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k(\rho)^{2k-2\ell}(2k)(2k-2)\cdots(2k-2\ell+2)}{(2)^\ell(k-\ell-1/2)!(k!)} \tag{B.10}$$

$$LHS = (-1)^{\ell+1}\sqrt{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k(\rho)^{2k-2\ell}(2)^\ell(k)(k-1)\cdots(k-\ell+1)}{(2)^\ell(k-\ell-1/2)!(k!)}$$

$$LHS = (-1)^{\ell+1}\sqrt{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k(\rho)^{2k-2\ell}(k)!}{(k-\ell)!(k-\ell-1/2)!(k)!}$$

$$LHS = (-1)^{\ell+1}\pi\sum_{k=0}^{\infty}\left\{(-1)^k(\rho)^{2k-2\ell}/(\Gamma(k-\ell+1)\Gamma(k-\ell+1/2))\right\} \tag{B.10′}$$

Using the duplication formula (A.4), eqn. (B.10') becomes

$$LHS = (-1)^{\ell+1}\sum_{k=0}^{\infty}\frac{(-1)^k(\rho)^{2k-2\ell}}{(2)^{1-2k+2\ell-1}\Gamma(2k-2\ell+1)} \tag{B.11}$$

$$LHS = \sum_{k=0}^{\infty}\left\{(-1)^{k+\ell+1}(\rho)^{2k-2\ell}/((2)^{2\ell-2k}\cdot(2k-2\ell)!)\right\} \tag{B.11′}$$

But

$$(-n)! = \pm\infty \qquad \text{where} \quad n \text{ is integer } (0,1,2,3,\ldots)$$

$\Rightarrow$ the only terms that are nonvanishing in (B.11') above are those for which

$$(2k-2\ell) \geq 0$$

$$k \geq \ell$$

$$\Rightarrow [\hat{p}(\rho)]^\ell \frac{d}{d\rho}[\hat{p}(\rho)]^\ell(\xi(\rho)) = 0 + \sum_{k=\ell}^{\infty}\frac{(-1)^{k+\ell+1}(2\rho)^{2k-2\ell}}{(2k-2\ell)!}. \tag{B.12}$$

Let $(k-\ell)=m$

$$\Rightarrow LHS = 0 + \sum_{m=0}^{\infty} \frac{(-1)^{m+2\ell+1}(2\rho)^{2m}}{(2m)!}$$

$$= -\sum_{m=0}^{\infty} \frac{(-1)^{m}(2\rho)^{2m}}{(2m)!} = -\cos(2\rho). \qquad \text{(B.12')}$$

Hence

$$\boxed{\left(\tfrac{d}{d\rho}\right)^{\ell} \tfrac{d}{d\rho} \left(\tfrac{d}{d\rho}\right)^{\ell} \left(\rho^{2\ell+2} n_\ell(\rho) j_\ell(\rho)\right) = -\cos(2\rho)}$$

**Appendix C:** The Derivation of The "Experimental" Phase Shifts $\delta_\ell(k)$ for $\ell = 0$ and $\ell = 1$.

The radial wave function for a particle of mass $m$, energy $E$ scattered from a 3-dimensional square well (or barrier) potential of depth (or height) $V$ and range $a$, in the channel $\ell$ is;

$$R(r) = \frac{u(r)}{r} = \begin{cases} C_\ell j_\ell(Kr) & r > a \\ A_\ell \left\{\cos(\delta_\ell) j_\ell(kr) - \sin(\delta_\ell) n_\ell(kr)\right\} & r \geq a \end{cases}$$

where,

$C_\ell$, $A_\ell$ are constants

$j_\ell$, $n_\ell$ are the spherical Bessel and Neumann functions respectively .

$K = \sqrt{\frac{2m}{\hbar^2}(E - V)}, \quad k = \sqrt{\frac{2m}{\hbar^2}E}$

$\delta_\ell$ is the phase shift

Then:

Case (I): $\ell = 0$

Since the wave function and its derivative are continuous at $r = a$

$$\Rightarrow C_\ell \frac{\sin(Ka)}{(Ka)} = A_\ell \left\{\frac{\sin(ka + \delta_0)}{ka}\right\} \tag{I}$$

$$C_\ell \left[\frac{K\cos(Ka)}{Ka} - \frac{\sin(Ka)}{Ka^2}\right] = A_\ell \left\{\frac{k\cos(ka + \delta)}{ka} - \frac{\sin(ka + \delta)}{ka^2}\right\} \tag{II}$$

Dividing (II) by (I)

$$K\frac{\cos(Ka)}{\sin(Ka)} - 1 = \frac{k\cos(ka+\delta)}{\sin(ka+\delta)} - 1$$

$$\Rightarrow \frac{\tan(Ka)}{K} = \frac{\tan(ka+\delta_0)}{k}$$

$$\Rightarrow \delta_0 = \tan^{-1}\left[\left(\frac{k}{K}\right)\tan(Ka)\right] - ka$$

<u>Case (II) $\ell = 1$</u>

Since the Wave function and its derivative are continuous at $r = a$,

$$C_1\left\{\frac{\sin(Ka)}{K^2a^2} - \frac{\cos(Ka)}{Ka}\right\} = A_1\left\{\cos(\delta_1)\left[\frac{\sin(ka)}{k^2a^2} - \frac{\cos(ka)}{ka}\right] + \sin(\delta_1)\ \times \right.$$

$$\left.\left[\frac{\cos(ka)}{k^2a^2} + \frac{\sin(ka)}{ka}\right]\right\} \qquad \text{(I)}$$

and

$$C_1\left\{\frac{\cos(Ka)}{Ka^2} - \frac{2\sin(Ka)}{K^2a^3} + \frac{\sin(Ka)}{a} + \frac{\cos(Ka)}{Ka^2}\right\} = A_1\left\{\cos(\delta_1)\left[\frac{2\cos(ka)}{ka^2}\right.\right.$$

$$\left.\left. - \frac{2\sin(ka)}{k^2a^3} + \frac{\sin(ka)}{a}\right] + \sin(\delta_1)\left[\frac{-\sin(ka)}{ka^2}\frac{-2\cos(ka)}{k^2a^3} + \frac{\cos(ka)}{a} - \frac{\sin(ka)}{ka^2}\right]\right\}$$

Using (I) to simplify (II),

$$C_1\sin(ka) - 2A_1\left\{\cos(\delta_1)\left[\frac{\sin ka}{k^2a^2} - \frac{\cos(ka)}{ka}\right] + \sin(\delta_1)\left[\frac{\cos(ka)}{k^2a^2} + \right.\right.$$

$$\left.\left.\frac{\sin(ka)}{(ka)}\right]\right. = A_1\left\{\cos(\delta_1)\sin(ka) + \sin(\delta_1)\cos(ka)\right\}$$

$$-2A_1\left\{\left[\frac{\sin(ka)}{k^2a^2} - \frac{\cos(ka)}{ka}\right]\cos(\delta_1) + \left[\frac{\cos(ka)}{k^2a^2} + \frac{\sin(ka)}{ka}\right]\sin(\delta_1)\right\} \qquad \text{(III)}$$

Then; (III) simplifies to;

$$C_1 \sin(Ka) = A \sin(ka + \delta_1) \tag{IV}$$

Dividing (II) by (IV);

$$\frac{\sin(Ka) - Ka\cos(Ka)}{(K^2a^2)\sin(Ka)} = \frac{\sin(ka+\delta_1) - ka\cos(ka+\delta_1)}{(k^2a^2)\sin(ka+\delta_1)}$$

or

$$\frac{k^2}{K^2}\{1 - (Ka)\cot(Ka)\} = \{1 - ka\cot(ka+\delta_1)\}$$

$$\Rightarrow (ka)\cot(ka+\delta_1) = 1 - \frac{k^2}{K^2} + \frac{k^2a}{K}\cot(Ka)$$

$$\Rightarrow \cot(ka+\delta_1) = \left\{\frac{1}{ka} - \frac{k}{K^2a} + \frac{k}{K}\cot(Ka)\right\}$$

$$\Rightarrow \delta_1 = \cot^{-1}\left\{\frac{1}{ka} - \frac{k}{K^2a} + \frac{k}{K}\cot(Ka)\right\} - ka$$

or

$$\delta_1(k) = \tan^{-1}\left\{\frac{1}{(\frac{1}{ka}) - (\frac{k}{K^2a}) + \frac{k}{K\tan(Ka)}}\right\} - ka$$

$$\delta_1(k) = \tan^{-1}\left\{\frac{akK^2\tan(Ka)}{\left(\frac{-2mV}{\hbar^2}\right)\tan(Ka) + aKk^2}\right\} - ka$$

where

$$\left(\frac{-2mV}{\hbar^2}\right) = (K^2 - k^2)$$

$$K = \sqrt{\frac{2m}{\hbar^2}(E - V)}$$

$$k = \sqrt{\frac{2m}{\hbar^2}(E)}.$$

## Appendix D

Fortran–77 Program which evaluates the summation relation given by eqn. (2.6), and its output.

```
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION FAC(50)
      OPEN(UNIT=22, FILE='B000', STATUS='NEW')
      FAC(1)=1.
      FAC(2)=1.
      DO 1 J=3,45
      FAC(J)=(J-1)*FAC(J-1)
 1    CONTINUE
 2    FORMAT(E10.9, I10)
      DO 10 L=1,12
      LL=L+ 1
      ANS=0
      DO 3 KK=1,LL
      K=KK-1
      ANSP=(-1)**K*FAC(K+L+1)/FAC(K+2)/FAC(L-K+1)/FAC(K+1)
      ANS=ANS+ANSP
 3    CONTINUE
      WRITE(22,*)L,       ANS
10    CONTINUE
      CALL EXIT
      END
```

```
 1  0.000000000000000000E+00
 2  0.000000000000000000E+00
 3  0.000000000000000000E+00
 4  0.000000000000000000E+00
 5  0.000000000000000000E+00
 6  0.000000000000000000E+00
 7  0.000000000000000000E+00
 8  0.000000000000000000E+00
 9  0.000000000000000000E+00
10  0.000000000000000000E+00
11  0.000000000000000000E+00
12  0.218278728425502777E-09
```

## Appendix E

Fortran–77 Program which evaluates the summation relation given by eqn. (2.9) and its output: (first column has numerical results, second one has analytical results).

```
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION FAC(50)
      OPEN(UNIT=23, FILE='BS45', STATUS='NEW')
      OPEN(UNIT=24, FILE='BS56', STATUS='NEW')
      FAC(1)=1.
      FAC(2)=1.
      DO 1 J=3,35
      FAC(J)=(J-1)*FAC(J-1)
 1    CONTINUE
 2    FORMAT(E10.5, I10)
      DO 10 I=1,13,2
      II=I+ 1
      A1=(I*(2*I+1))
      ODD=1/A1
      A2=((I+1)*(2*I+1))
      EVEN=1/A2
      ANS=0
      DO 3 KK=1,II
      K=KK-1
      ANSP=(-4)**K*FAC(K+I+1)/(FAC(2*K+1)*FAC(I-K+1)*(K+1)*(2*K+1))
      ANS=ANS+ANSP
 3    CONTINUE
      WRITE(23,*) I,      ANS,        ODD
 10   CONTINUE
      DO 20 L=2,12,2
      LL=L+ 1
      A1=(L*(2*L+1))
      ODD=1/A1
      A2=((L+1)*(2*L+1))
      EVEN=1/A2
      ANS=0
      DO 5 KK=1,LL
      K=KK-1
      ANSP=(-4)**K*FAC(K+L+1)/(FAC(2*K+1)*FAC(L-K+1)*(K+1)*(2*K+1))
      ANS=ANS+ANSP
 5    CONTINUE
      WRITE(24,*) L,      ANS,        EVEN
 20   CONTINUE
      CALL EXIT
      END
```

```
  1  0.333333333333333343        0.333333333333333329
  3  0.476190476190476719E-01    0.476190476190476190E-01
  5  0.181818181818163005E-01    0.181818181818181818E-01
  7  0.952380952375619927E-02    0.952380952380952380E-02
  9  0.584795321606179641E-02    0.584795321637426868E-02
 11  0.395256916272046510E-02    0.395256916996047404E-02
 13  0.284900210681371391E-02    0.284900284900284898E-02

  2  0.666666666666666519E-01    0.666666666666666657E-01
  4  0.222222222222223653E-01    0.222222222222222222E-01
  6  0.109890109890109500E-01    0.109890109890109890E-01
  8  0.653594771267762553E-02    0.653594771241829998E-02
 10  0.432900433770555537E-02    0.432900432900432900E-02
 12  0.307692306614626432E-02    0.307692307692307692E-02
```

## Appendix F

Fortran–77 Program which numerically calculates the potential from eqn. (3.8.a), ($\ell = 0$) for those cases where there are no bound states. Here the scattering is that of a nucleon ($m \simeq 940\ MeV$).

```
C       IMPLICIT REAL*8(A-H,O-Z)
         REAL  K,H,S1(4002),S2(4002),VV2(40)
         DATA  V1,V2,A1,A2,M,N/-4.5,-5.,2.0,2.0,4001,40/
          D1(K)=ATAN(P*TAN(K*A1)/K)-P*A1
          D2(K)=ATAN(P*TAN(Q*A2)/Q)-P*A2
          D22(K)=ATAN(P*TANH(O*A2)/O)-P*A2
      G(R)=26.284*PMAX*SIN(2*KMAX*R)*(SIN(KMAX*A1))**2*SIN(D2MAX-D1MAX)/
     C(R*(SIN(PMAX*A1+D1MAX))**2)
      F(K)=41.286*P*SIN(DT2-DT1)*COS(2*K*R)*(SIN(A1*K))**2/SIN(A1*P+DT1)
     C**2
         KMAX=SQRT(0.0484*(350-V1))
         IF (V1.GE.0.0) THEN
           V=0.010
         ELSE
           V=SQRT(0.0484*(0.002066-V1))
         ENDIF
          H=(KMAX-V)/(M-1)
C         OPEN(UNIT=51, FILE='PH35' ,STATUS='NEW')
C         OPEN(UNIT=52, FILE='PH45' ,STATUS='NEW')
          OPEN(UNIT=17, FILE='L117' ,STATUS='NEW')
C        WRITE(6,*)V1,V2, A1,A2,M
         DO 20 J=1,40
            R=J/10.0
            S=10.0/J
            SUM=0.0
C        WRITE(6,*)R ,J,SUM
          DO 30 I=1,M
            K=V+(I-1)*H
            X=(K**2)/(0.0484)+V1
            IF (X.LT.V2) THEN
              O=SQRT(.0484*(V2-X))
            ELSE
              Q=SQRT(.0484*(X-V2))
            ENDIF
            P=SQRT(.0484*X)
C        WRITE(6,*)X,K,Q,P
            S1(I)=D1(K)
          IF (ABS(S1(I)-S1(I-1)).GT.9.42) THEN
            S1(I)=S1(I)+9.4248
          ELSEIF (ABS(S1(I)-S1(I-1)).GT.6.28) THEN
           S1(I)=S1(I)+6.2832
          ELSEIF (ABS(S1(I)-S1(I-1)).GT.3.14) THEN
           S1(I)=S1(I)+3.1416
          ELSE
           S1(I)=S1(I)
          ENDIF
```

(Continued )

```
          IF (X.GT.V2) THEN
           S2(I)=D2(K)
          ELSE
           S2(I)=D22(K)
          ENDIF
         IF (ABS(S2(I)-S2(I-1)).GT.9.42) THEN
          S2(I)=S2(I)+9.4248
         ELSEIF (ABS(S2(I)-S2(I-1)).GT.6.28) THEN
          S2(I)=S2(I)+6.2832
         ELSEIF (ABS(S2(I)-S2(I-1)).GT.3.14) THEN
          S2(I)=S2(I)+3.1416
         ELSE
          S2(I)=S2(I)
         ENDIF
         DT1=S1(I)
         DT2=S2(I)
C        WRITE(51,*)K,S1(I),S2(I)
         IF (I.EQ.1.OR.I.EQ.M) THEN
          SUM=SUM+.5*F(K)
         ELSE
          SUM=SUM+F(K)
         ENDIF
  30     CONTINUE
           SUM=SUM*H*(1.273)
           EMAX=20.643*KMAX**2+V1
           QMAX=SQRT(.0484*(EMAX-V2))
           PMAX=SQRT(.0484*EMAX)
           D1MAX=S1(4001)
           D2MAX=S2(4001)
           IF (R.LE.A2) THEN
             U2=V1-G(R)+SUM
           ELSE
             U2=-G(R)+SUM
           ENDIF
             VV2(J)=U2
           WRITE(17,*)R,VV2(J),G(R),SUM
 20      CONTINUE
         STOP
         END
```

## Appendix G

Fortran–77 Program which numerically calculates the potential from eqn. (3.8.a), ($\ell = 0$) for those cases where there is a bound state. Here, the scattering is that of a nucleon ($m \simeq 940\ MeV$).

```
C       IMPLICIT REAL*8(A-H,O-Z)
        REAL  K,KMAX,H,S1(4001),S2(4001),VV2(40)
        DATA  V1,V2,A1,A2,M/-43.,-40.,2.00,2.1,4001/
         D1(K)=ATAN(P*TAN(K*A1)/K)-P*A1
         D2(K)=ATAN(P*TAN(Q*A2)/Q)-P*A2
         D22(K)=ATAN(P*TANH(O*A2)/O)-P*A2
      G(R)=26.284*PMAX*SIN(2*KMAX*R)*(SIN(KMAX*A1))**2*
     CSIN(D2MAX-D1MAX)/(R*(SIN(PMAX*A1+D1MAX))**2)
      F(K)=41.286*P*SIN(DT2-DT1)*COS(2*K*R)*(SIN(A1*K))**2/SIN(A1*P+DT1)
     C**2
        KMAX=SQRT(0.0484*(350-V1))
        IF (V1.GE.0.0) THEN
          V=0.010
        ELSE
          V=SQRT(0.0484*(0.002066-V1))
        ENDIF
         H=(KMAX-V)/(M-1)
C        OPEN(UNIT=51, FILE='PH35' ,STATUS='NEW')
C        OPEN(UNIT=52, FILE='PH45' ,STATUS='NEW')
         OPEN(UNIT=17, FILE='L010' ,STATUS='NEW')
C       WRITE(6,*)V1,V2, A1,A2,M,V,KMAX
        DO 20 J=1,40
           R=J/10.0
           SUM=0.0
C       WRITE(6,*)R ,J,SUM
         DO 30 I=1,M
           K=V+(I-1)*H
           X=(K**2)/(0.0484)+V1
           IF (X.LT.V2) THEN
             O=SQRT(.0484*(V2-X))
           ELSE
             Q=SQRT(.0484*(X-V2))
           ENDIF
           P=SQRT(.0484*X)
C       WRITE(6,*)X,K,Q,P
           S1(1)=3.141
           S1(I+1)=D1(K+H)
         IF (ABS(S1(I+1)-S1(I)).GT.9.) THEN
           S1(I+1)=S1(I+1)+9.4248
         ELSEIF (ABS(S1(I+1)-S1(I)).GT.6.) THEN
          S1(I+1)=S1(I+1)+6.2832
         ELSEIF (ABS(S1(I+1)-S1(I)).GT.3.14) THEN
          S1(I+1)=S1(I+1)+3.1416
         ELSE
          S1(I+1)=S1(I+1)
         ENDIF
```

(Continued)

```
          IF (X.GT.V2) THEN
           S2(1)=3.141
           S2(I+1)=D2(K+H)
          ELSE
           S2(1)=3.141
           S2(I+1)=D22(K+H)
          ENDIF
         IF (ABS(S2(I+1)-S2(I)).GT.9.) THEN
          S2(I+1)=S2(I+1)+9.4248
         ELSEIF (ABS(S2(I+1)-S2(I)).GT.6.) THEN
          S2(I+1)=S2(I+1)+6.2832
         ELSEIF (ABS(S2(I+1)-S2(I)).GT.3.14) THEN
          S2(I+1)=S2(I+1)+3.1416
         ELSE
          S2(I+1)=S2(I+1)
         ENDIF
         DT1=S1(I)
         DT2=S2(I)
C         WRITE(6,*)S1(I),S2(I),P
         IF (I.EQ.1.OR.I.EQ.M) THEN
          SUM=SUM+.5*F(K)
         ELSE
          SUM=SUM+F(K)
         ENDIF
  30     CONTINUE
           SUM=SUM*H*(1.273)
           EMAX=20.643*KMAX**2+V1
           QMAX=SQRT(.0484*(EMAX-V2))
           PMAX=SQRT(.0484*EMAX)
           D1MAX=S1(4001)
           D2MAX=S2(4001)
           IF (R.LE.A1) THEN
             U2=V1-G(R)+SUM
           ELSE
             U2=-G(R)+SUM
           ENDIF
             VV2(J)=U2
           WRITE(17,*)R,VV2(J)
 20      CONTINUE
         STOP
         END
```

## Appendix H

Fortran-77 Program which numerically calculates the potential from eqn. (3.8.b) ($\ell = 1$). Here, the scattering is that of a nucleon ($m \simeq 940\ MeV$).

```
C       IMPLICIT REAL*8(A-H,O-Z)
        REAL KMAX,K,K1,S,H,S1(4002),S2(4002),VV2(40)
        DATA V1,V2,A1,A2,M/6.5,10.,2.5,2.0,4001/
          E1(K)=A1*P*(K**2)*TAN(A1*K)/(-0.0484*V1*TAN(A1*K)+A1*K*(P**2))
          D1(K)=ATAN(E1(K))-P*A1
          E2(K)=A2*P*(Q**2)*TAN(A2*Q)/(-0.0484*V2*TAN(A2*Q)+A2*Q*(P**2))
          D2(K)=ATAN(E2(K))-P*A2
        E22(K)=A2*P*(O**2)*TANH(A2*O)/(0.0484*V2*TANH(A2*O)-A2*O*(P**2))
          D22(K)=ATAN(E22(K))-P*A2
        YMAX(K)=(((SIN(KMAX*A1)/(KMAX*A1))-COS(KMAX*A1))**2)/(((SIN(PMAX
      C*A1+D1MAX)/(PMAX*A1))-COS(PMAX*A1+D1MAX))**2)
       B1(R)=(1-(1/((KMAX*R)**2)))*SIN(2*KMAX*R)+2*COS(2*KMAX*R)/(KMAX*R)
        G(R)=26.284*PMAX*SIN(D2MAX-D1MAX)*B1(R)*YMAX(K)/R
        Y(K)=(((SIN(K*A1)/(K*A1))-COS(K*A1))**2)/(((SIN(P*A1+DT1)/(P*A1))
      C-COS(P*A1+DT1))**2)
       F1(K)=P*SIN(DT2-DT1)*Y(K)*COS(2*K*R)
       F2(K)=-2*P*SIN(DT2-DT1)*COS(2*K*R)*Y(K)*(S**2)/(K**2)
       F3(K)=P*SIN(DT2-DT1)*SIN(2*K*R)*Y(K)*(S**3)/(K**3)
       F4(K)=-2*P*SIN(DT2-DT1)*SIN(2*K*R)*Y(K)*S/(K)
           KMAX=SQRT(0.0484*(350.0-V1))
         IF (V1.GE.0.0) THEN
             K1=0.010
         ELSE
             K1=SQRT(0.0484*(0.002066-V1))
         ENDIF
          H=(KMAX-K1)/(M-1)
C         OPEN(UNIT=51, FILE='PH35' ,STATUS='NEW')
          OPEN(UNIT=17, FILE='L146' ,STATUS='NEW')
C       . WRITE(6,*)V1,V2,K1,A1,A2,KMAX
         DO 20 J=1,40
            R=J/10.0
            S=10./J
            SUM1=0.0
            SUM2=0.0
            SUM3=0.0
            SUM4=0.0
C        WRITE(6,*)R ,J,SUM
          DO 30 I=1,M
            K=K1+(I-1)*H
            X=(K**2)/0.04840000+V1
            IF (X.LT.V2) THEN
              O=SQRT(.0484*(V2-X))
            ELSE
              Q=SQRT(.0484*(X-V2))
            ENDIF
            P=SQRT(.0484*X)
C        WRITE(6,*)X,K,Q,P
            S1(I)=D1(K)
          IF (ABS(S1(I)-S1(I-1)).GT.9.) THEN
            S1(I)=S1(I)+9.4248
          ELSEIF (ABS(S1(I)-S1(I-1)).GT.6.) THEN
           S1(I)=S1(I)+6.2832
          ELSEIF (ABS(S1(I)-S1(I-1)).GT.3.) THEN
           S1(I)=S1(I)+3.1416
          ELSE
```

(Continued)

```
          S1(I)=S1(I)
         ENDIF
          IF (X.GT.V2) THEN
           S2(I)=D2(K)
          ELSE
           S2(I)=D22(K)
          ENDIF
         IF (ABS(S2(I)-S2(I-1)).GT.9.) THEN
          S2(I)=S2(I)+9.4248
         ELSEIF (ABS(S2(I)-S2(I-1)).GT.6.) THEN
          S2(I)=S2(I)+6.2832
         ELSEIF (ABS(S2(I)-S2(I-1)).GT.3.) THEN
          S2(I)=S2(I)+3.1416
         ELSE
          S2(I)=S2(I)
         ENDIF
         DT1=S1(I)
         DT2=S2(I)
C        WRITE(51,*)K,S1(I),S2(I)
C        WRITE(6,*)K,P,DT1,DT2,Y(K),B1(R),B2(R)
         IF (I.EQ.1.OR.I.EQ.M) THEN
          SUM1=SUM1+.5*F1(K)
          SUM2=SUM2+.5*F2(K)
          SUM3=SUM3+.5*F3(K)
          SUM4=SUM4+.5*F4(K)
         ELSE
          SUM1=SUM1+F1(K)
          SUM2=SUM2+F2(K)
          SUM3=SUM3+F3(K)
          SUM4=SUM4+F4(K)
         ENDIF
   30    CONTINUE
           SUM=(SUM1+SUM2+SUM3+SUM4)*H*(52.567)
           EMAX=20.66*(KMAX**2)+V1
           QMAX=SQRT(.0484*(EMAX-V2))
           PMAX=SQRT(.0484*EMAX)
           D1MAX=S1(4001)
           D2MAX=S2(4001)
C        WRITE(6,*)KMAX,PMAX,D2MAX,QMAX,YMAX(K)
           IF (R.LE.A1) THEN
             U2=V1+G(R)-SUM
           ELSE
             U2=+G(R)-SUM
           ENDIF
             VV2(J)=U2
           WRITE(17,*)R,VV2(J),G(R),SUM
 20      CONTINUE
         STOP
         END
```

# REFERENCES


1. H.A. Mavromatis and A.M. Jalal, J. Math. Phys. 31, 1991, p. 1181.
2. B.N. Zakhariev and A.A. Suzko, *Direct and Inverse Problems*, (Springer-Verlag 1990), and references therein.
3. H.A. Mavromatis and K. Schilcher, J. Math Phys. 9, 1968, p. 1627.
4. A. Erdel'yi et al., *Tables of Integral Transforms*, (Bateman Manuscript Project) (McGraw-Hill, New York, 1954), Vol. 1, p. 63.
5. A. Messiah, *Quantum Mechanics*, Vol. II. (North-Holland, Amesterdam, 1966). P. 813.
6. L.I. Schiff, *Quantum Mechanics*, 3rd ed., (McGraw-Hill, New York, 1968) p. 83.
7. I.S. Gradshteyn and I.M. Ryzhik, *Tables of Integrals, Series, and Products.* (Academic Press, New York, 1980), p. 672.
8. I.S. Gradshteyn and I.M. Ryzhik, Ref. 7, p. 3.
9. E.Merzbacher, *Quantum Mechanics* (Willey, New York, 1970), 2nd ed., P. 232.
10. Morse, P.M., and Feshbach, *Methods of Theoretical Physics*, 2 pts. (McGraw-Hill, New York, 1953).
11. E. Merzbacher, Ref. 9. p. 244.
12. I.S. Gradshteyn and I.M. Ryzhik, Ref. 7, p. 960.
13. Murray R. Spiegel, *Mathematical Handbook of Formulas and Tables*, (Mcgraw-Hill, 1968), p. 102.