MATLIS' SEMI-REGULARITY IN TRIVIAL RING EXTENSIONS AND RINGS WITH SEMI-REGULAR PROPER HOMOMORPHIC IMAGES

BY

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To whom dreamed of this end and waited for it minute by minute by praying and sometimes crying; my parents, my wife, my son Waleed, and to whom I consider him as member of my family my supervisor Prof. Salah; I dedicate this work ...

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THESIS ABSTRACT

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This Ph.D. thesis consists of two chapters which contribute to the study of homological aspects of commutative rings with zero-divisors. The first chapter is devoted to trivial ring extensions (also called Nagata idealizations). Namely, we investigate the transfer of the notion of (Matlis') semi-regular ring (also known as IF-ring) along with related concepts, such as (semi-)coherence and self fp-injectivity, in various contexts of these constructions. Section 1.2 investigates trivial extensions issued from (local) rings and Section 1.3 is devoted to trivial ring extensions issued from integral domains. In Section

1.4 (and also in Section 1.2), we put the new results in use to enrich the literature with new families of examples subject to the ring-theoretic notions involved in this study and also validate some questions left open in the literature.

In the second chapter, we prove an extension of Zaks' conjecture on integral domains with semi-regular proper homomorphic images (with respect to the finitely generated ideals) to the class of coherent rings (with zero-divisors). Section 2.2 features the main result of this chapter, which extends and recovers, in Section 2.3, Levy's related result on Noetherian rings [41, Theorem] and Matlis' related result on Prüfer domains [44, Theorem]. It also globalizes Couchot's related result on chained rings [15, Theorem 11]. In Section, 2.4, we use the main result in combination with our results in the first chapter to construct new examples of rings with semi-regular proper homomorphic images (with respect to the finitely generated ideals) via trivial ring extensions.

ملخص الرسالة

الاسم الكامل: خالد وليد عداربة.

عنوان الرسالة: حلقة ماتليز شبه المنتظمة في التوسعة الحلقية البدهية و حدس زاكس على الحلقات شبه المنتظمة المتبقية.

التخصص: الرياضيات

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هذه الرسالة تتكون من وحدتين و تهدف الى دراسة الطبيعة التناظرية للحلقات التبديلية. الوحدة الاولى تتناول التمديدات البديهية. سنتحقق من نقل مفهوم حلقة ماتليز شبه المنتظمة و بعض المفاهيم المرتبطه بها في حالات متعددة من هذا البناء. حيث أننا سنتحقق من التمديدات البديهية التي تنسج من الحلقة خاصة المحلية و أيضا التمديدات البديهية التي تنسج من المجال. سنستخدم النتائج الجديدة لاثراء المطبوعات بعائلات أمثلة على الحلقات المشمولة بالدراسة و في التحقق من صحة بعض الاسئلة التي تركت مفتوحة.

في الوحدة الثانية، سنثبت تعميم لحدس زاكس حول شبه المنتظم . النتيجة الاساسية تعمم و تسترد نتائج كل من ليفي على الحلقة النوثرية و نتائج ماتليز على المجال البروفري. أيضا هي تعولم نتائج كوشوت على الحلقة السلسلية. أيضا سنستخدم النتيجة الاساسية مع بعض النتائج من الوحدة الأولى لبناء مثال جديد على الحلقة شبه المنتظمة المتبقية باستخدام التمديدات البديهية.

CHAPTER 1

INTRODUCTION

Throughout, all rings considered are commutative with identity and all modules are unital. A ring R is coherent if every finitely generated ideal of R is finitely presented. The class of coherent rings includes strictly the classes of Noetherian rings, von Neumann regular rings (i.e., every module is flat), valuation rings, and semi-hereditary rings (i.e., every finitely generated ideal is projective). During the past three decades, the concept of coherence developed towards a full-fledged topic in commutative algebra under the influence of homology; and several notions grew out of coherence (e.g., finite conductor property, quasi-coherence, v-coherence, and n-coherence). For more details on coherence see please [25, 26] and for coherent-like properties see, for instance, [35, 36].

In 1982, Matlis proved that a ring R is coherent if and only if $\hom_R(M,N)$ is flat for any injective R-modules M and N [43, Theorem 1]. In 1985, he defined a ring R to be semi-coherent if $\hom_R(M,N)$ is a submodule of a flat R-module for any injective R-modules M and N. Then, inspired by this definition and von Neumann regularity, he defined a ring to be semi-regular if any module can be embedded in a flat module (or, equivalently; if every injective module is flat) [44]. He then proved that semi-regularity is a local property in the class of coherent rings [44, Proposition 2.3]. Moreover, he proved that in the class of reduced rings, von Neumann regularity collapses to semi-regularity [44, Proposition 2.7]; and under the Noetherian assumption, semi-regularity equals the self-injective property; i.e., R is quasi-Frobenius if and only if R is semi-regular and Noetherian [44, Proposition 3.4]. Beyond Noetherian settings, examples of semi-regular rings arise as factor rings of Prüfer domains over nonzero finitely generated ideals [44, Proposition 5.3]. It is worth noting, at this point, that the notion of a semi-regular ring was briefly mentioned by Sabbagh (1971) in [55, Section 2] and studied in non-commutative settings by Jain (1973) in [33], Colby (1975) in [14], and Facchini & Faith (1995) in [21], among others, where it was always termed as IF-ring. Also, it was extensively studied -under IF terminology- in (commutative) valuation settings by Couchot in [15, 16, 17].

For a ring A and an A-module E, the trivial ring extension of A by E is the ring $R := A \ltimes E$ where the underlying group is $A \times E$ and the multiplication is defined by $(a,e)(b,f) = (ab, af + be)$. The ring R is also called the (Nagata) idealization of E over A and is denoted by $A(+)E$. This construction was first introduced, in 1962, by Nagata [45] in order to facilitate interaction between rings and their modules and also to provide various families of examples of commutative rings containing zero-divisors. The literature abounds of papers on trivial extensions dealing with the transfer of ring-theoretic notions in various settings of these constructions (see, for instance, [1, 6, 19, 22, 27, 28, 29, 39, 40, 48, 49, 50, 51, 52, 53, 56]). For more details on commutative trivial extensions (or idealizations), we refer the reader to Glaz's and Huckaba's respective books [25, 32], and also D. D. Anderson & Winders relatively recent and comprehensive survey paper [5].

A domain R is Dedekind if every ideal of R is projective. In 1966, Levy proved a dual version for this result stating that, for a Noetherian ring R (possibly with zero-divisors), every proper homomorphic image of R is self-injective if and only if R is a Dedekind domain or a principal ideal ring with the descending chain condition or a local ring whose maximal ideal M has composition length 2 with $M^2 = 0$ [41, Theorem]. In 1985, Matlis proved that if R is a Prüfer domain, then R/I is semi-regular for every nonzero finitely generated ideal I of R [44, Proposition 5.3]. Then Abraham Zaks conjectured that the converse of this result should be true; i.e., an integral domain R is Prüfer if and only if R/I is semi-regular for every nonzero finitely generated ideal I of R . This was proved by Matlis in [44, Theorem, p. 371]; extending thus Levy's theorem in the case of integral domains. In this vein, recall Couchot's result that a chained ring is residually semi-regular [15, Theorem 11].

This Ph.D. thesis consists of two chapters which contribute to the study of homological aspects of commutative rings with zero-divisors. The first chapter is devoted to trivial ring extensions (also called Nagata idealizations). Namely, we investigate the transfer of the notion of (Matlis') semi-regular ring (also known as IF-ring) along with related concepts, such as (semi-)coherence and self fp-injectivity, in various contexts of these constructions. Section 1.2 investigates trivial extensions issued from (local) rings and Section 1.3 is devoted to trivial ring extensions issued from integral domains.

In Section 1.4 (and also in Section 1.2), we put the new results in use to enrich the literature with new families of examples subject to the ring-theoretic notions involved in this study and also validate some questions left open in the literature.

In the second chapter, we prove an extension of Zaks' aforementioned conjecture on integral domains with semi-regular proper homomorphic images (with respect to the finitely generated ideals) to the class of coherent rings (with zero-divisors). Section 2.2 features the main result of this chapter, which extends and recovers, in Section 2.3, Levy's and Matlis' aforementioned results on Noetherian rings and Prüfer domains, respectively. It also globalizes Couchot's related result on chained rings. In Section 2.4, we use the main result in combination with our results in the first chapter to construct new examples of rings with semi-regular proper homomorphic images (with respect to the finitely generated ideals) via trivial ring extensions.

For the reader's convenience, Figure 1.1 displays a diagram of implications summarizing the relations among the main notions involved in this work.

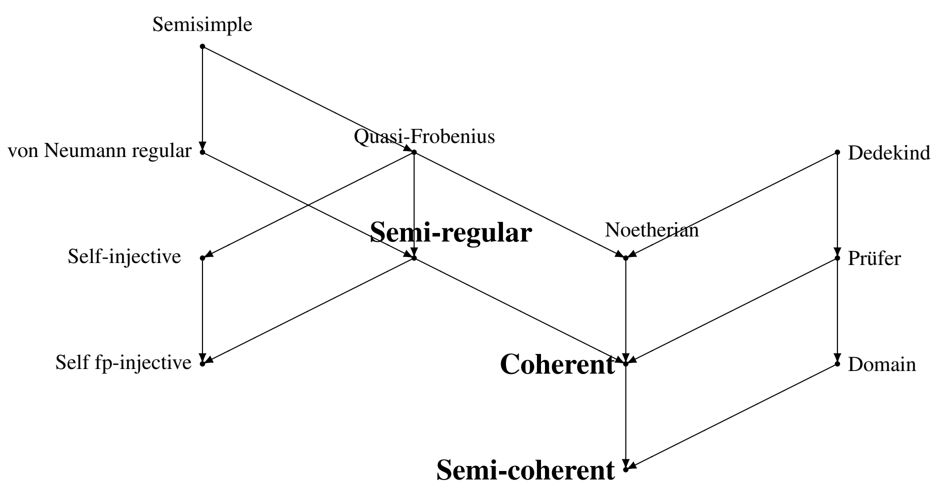

Figure 1.1: A ring-theoretic perspective for semi-regularity and (semi-)coherence

Throughout, for a ring A, let $Q(A)$ denote its total ring of quotients, $Z(A)$ denote the set of its zero-divisors, and $Max(A)$ denote the set of its maximal ideals. For an ideal I of A, $Ann(I)$ will denote the annihilator of I.

CHAPTER 2

MATLIS' SEMI-REGULARITY IN TRIVIAL RING EXTENSIONS

This chapter[1] is devoted to trivial ring extensions (also called Nagata idealizations). It investigates necessary and sufficient conditions for the transfer of Matlis' semi-regularity along with related concepts in various contexts of these constructions.

2.1 Introduction

Recall that a ring R is semi-coherent if $\hom_R(M,N)$ is a submodule of a flat R-module for any injective R-modules M and N; and R is a semi-regular ring (also known as IF-ring) if any module can be embedded in a flat module (or, equivalently; if every injective module is flat) [44]. An R-module E is fp-injective (also called absolutely pure) if $Ext_R^1(M,E)=0$ for every finitely presented R-module M [24, IX-3]; and R is self fp-injective if it is fp-injective as an R-module.

It is worthwhile recalling that a ring R is semi-regular if and only if R is self fp-injective and coherent [33, Theorem 3.10] or [14, Theorem 2] if and only if R is coherent and R_M is semi-regular for every maximal ideal M of R [44, Proposition 2.3] if and only if R is coherent and satisfies the double annihilator condition (i.e., $Ann_R(Ann_R(I)) = I$, for each finitely generated ideal I of A) [44, Proposition 4.1].

In this chapter, we investigate the transfer of the notion of a semi-regular ring along with related concepts, such as (semi-)coherence and self fp-injectivity, in various contexts of these constructions. Section 1.2 investigates trivial extensions issued from (local) rings and Section 1.3 is devoted to trivial ring extensions issued from integral domains. In Section 1.4 (and also Section 1.2), we put the new results in use to enrich the literature with new families of examples subject to the ring-theoretic notions involved in this study and also validate some questions left open in the literature.

2.2 Transfer of semi-regularity to trivial ring extensions issued from (local) rings

This section investigates the transfer of semi-coherence and semi-regularity in trivial ring extensions issued from (local) rings (i.e., possibly, with zero-divisors). Recall that a ring R is arithmetical if every finitely generated ideal of R is locally principal [23, 34]; and R is a chained ring if R is local and arithmetical (i.e., its ideals are linearly ordered with respect to inclusion) [10, 11, 32]. In [15, Theorem 10], Couchot established necessary and sufficient conditions for a chained ring (termed as valuation ring) to be semi-regular (termed as IF-ring). Recall also that a ring is quasi-Frobenius if it is Noetherian and self-injective.

Throughout, for a ring A and an A-module E, let E_t denote the set of torsion elements of E; namely,

$$E_t := \left\{ e \in E \mid ae = 0 \text{ for some } 0 \neq a \in A \right\}.$$

The first result of this section establishes conditions under which some trivial extensions of local rings inherit semi-regularity. It also establishes a correlation with the notions of quasi-Frobenius ring and chained ring. Recall, for convenience, that prime (resp., maximal) ideals of a trivial extension $A \ltimes E$ have the form $p \ltimes E$, where p is a prime (resp., maximal) ideal of A [32, Theorem 25.1(3)].

Theorem 2.2.1 *Let $(A, \mathfrak{m})$ be a local ring, E a nonzero A-module with $\mathfrak{m}E_t = 0$ (e.g., E torsion free or $\frac{A}{\mathfrak{m}}$-vector space), and $R := A \ltimes E$. The following statements are equivalent:*

(1) *R is semi-regular;*

(2) *R is quasi-Frobenius;*

(3) *A is a chained ring, $\mathfrak{m}^2 = 0$, and $E \cong A$.*

Moreover, if any one of the equivalent conditions holds, then R is principal if and only if A is a field.

Proof. A quasi-Frobenius ring is semi-regular [44, Proposition 3.4]. So, we will prove the implications (1) $\Rightarrow$ (3) $\Rightarrow$ (2).

(1) $\Rightarrow$ (3) Assume R is semi-regular and let us envisage two cases.

Case 1: Suppose $E_t = E$. In this case, observe that $\mathfrak{m}E = 0$. We first prove, by way of contradiction, that A is a field. Deny and let $0 \neq x \in \mathfrak{m}$. Then,

$$Ann_R(x,0) = Ann_A(x) \ltimes E.$$

Moreover, the facts $x \neq 0$ and $Ann_A(x) \ltimes E \neq 0$ yield, respectively,

$$Ann_A(x) \subseteq \mathfrak{m} \text{ and } Ann_R(Ann_A(x) \ltimes E) \subseteq \mathfrak{m} \ltimes E.$$

By semi-regularity of R, we obtain

$$\begin{aligned} Ax \ltimes 0 &= R(x,0) \\ &= Ann_R(Ann_R(x,0)) \\ &= Ann_R(Ann_A(x) \ltimes E) \\ &= \big(Ann_A(Ann_A(x)) \cap \mathfrak{m}\big) \ltimes E. \end{aligned}$$

It follows that $E = 0$, the desired contradiction. Therefore, A is a field. Next, let e be a nonzero vector in E. Clearly, semi-regularity of R combined with the fact that e is torsion free yields

$$(*) \qquad \begin{array}{rcl} 0 \ltimes Ae & = & R(0,e) \\ & = & Ann_R(Ann_R(0,e)) \\ & = & Ann_R(Ann_A(e) \ltimes E) \\ & = & Ann_R(0 \ltimes E) \\ & = & 0 \ltimes E \end{array}$$

so that $E = Ae \cong A$, as desired.

Case 2: Suppose $E_t \subsetneqq E$ and let $e \in E \setminus E_t$. The same arguments used in $(*)$ yield

$$E = Ae \cong A.$$

Therefore, we may assume that

$$R := A \ltimes A \text{ with } \mathfrak{m}Z(A) = 0$$

If $Z(A) = 0$, then, for any $0 \neq a \in A$, we have $Aa = A$ again by $(*)$, hence A is a field. If $Z(A) \neq 0$, then $\mathfrak{m} \subseteq Z(A)$, hence $\mathfrak{m} = Z(A)$, whence $\mathfrak{m}^2 = 0$. It remains to prove that A is a chained ring. Next, let t be a nonzero arbitrary element of $\mathfrak{m}$. Observe that, for $(x,y) \in Ann_R(\mathfrak{m} \ltimes A)$, $(x,y)(0,1) = 0$ yields $x = 0$, and $(0,y)(t,0) = 0$ yields $y \in \mathfrak{m}$.

So, we have

$$\begin{aligned} 0 \ltimes (t) &= Ann_R(Ann_R(0,t)) \\ &= Ann_R(\mathfrak{m} \ltimes A) \\ &= 0 \ltimes \mathfrak{m} \end{aligned}$$

so that $\mathfrak{m} = (t)$. Now, let I be a nonzero proper ideal of A (i.e., $0 \subsetneqq I \subseteq \mathfrak{m}$) and let $0 \neq a \in I$. Necessarily, $I = \mathfrak{m} = (a)$, proving that A is a chained ring (in fact, principal).

(3) $\Rightarrow$ (2) Assume A is a chained ring, $\mathfrak{m}^2 = 0$, and $E \cong A$. Observe first that the assumption $\mathfrak{m}^2 = 0$ forces $\mathfrak{m}Z(A) = 0$. So, we may assume that

$$R := A \ltimes A.$$

Also, notice that A is necessarily principal; for, let I be a nonzero proper ideal of A, $0 \neq a \in I$, and $x \in \mathfrak{m}$. Then either $x \in (a)$ or $a \in (x)$. The second case yields $a = ux$ for some unit $u \in A$, hence $I = \mathfrak{m} = (a)$, as desired. Hence A is Artinian and so is R by [5, Theorem 4.8]. Moreover, since A is a chained ring, every two isomorphic ideals of A are equal; that is, the socle of A is square free. Therefore, by Kourki's result [39, Theorem 3.6], R is quasi-Frobenius.

For the proof of the last statement of the theorem, recall first from [5, Theorem 4.10] that, given a ring A and a nonzero A-module E, the trivial extension $A \ltimes E$ is principal if and only if A is principal and E is cyclic with

$$Ann_A(E) = M_1 \cdots M_n$$

for some idempotent maximal ideals $M_1,\ldots,M_n$ of A. Now, assume $R := A \ltimes A$, where $(A,\mathfrak{m})$ is a chained ring with $\mathfrak{m}^2 = 0$. We proved above that A is principal. Then, the aforementioned result yields R is principal if and only if $0 = Ann_A(A) = \mathfrak{m}^2 = \mathfrak{m}$ if and only if A is a field, completing the proof of the theorem. □

For the special case of trivial extensions of local rings by vector spaces over their residue fields, we obtain the following result.

Corollary 2.2.2 *Let $(A,\mathfrak{m})$ be a local ring, E a nonzero $\frac{A}{\mathfrak{m}}$-vector space, and $R := A \ltimes E$. Then, the following statements are equivalent:*

(1) *R is semi-regular;*

(2) *R is quasi-Frobenius;*

(3) *R is a chained ring;*

(4) *A is a field and* $\dim_A E = 1$.

Proof. Combine Theorem 2.2.1 with [6, Theorem 3.1(3)] which handles the equivalence (3) ⇔ (4). □

A von Neumann regular ring is a reduced semi-regular ring [44, Proposition 2.7]. Matlis noticed that "(von Neumann) regular rings and quasi-Frobenius rings are seen to have a common denominator of definition–they are both extreme examples of semi-regular rings." One may easily appeal to trivial extensions (since these constructions are not reduced) to provide more examples discriminating between von Neumann regularity and semi-regularity, as shown below. Also, recall that the classic examples of quasi-Frobenius rings are semi-simple rings and quotient rings of principal domains modulo nonzero finitely generated ideals. Theorem 2.2.1 provides, readily, examples of original quasi-Frobenius rings, as shown below.

Example 2.2.3 $\frac{\mathbb{Z}}{4\mathbb{Z}} \ltimes \frac{\mathbb{Z}}{4\mathbb{Z}}$ is a quasi-Frobenius ring that is neither von Neumann regular nor principal.

Further, one may provide new examples of semi-regular rings. To proceed further, we need to recall the following fact: if S is a multiplicatively closed subset of the trivial extension $R := A \ltimes E$ and $S_o := S \cap A$, then the universal property of localization yields

$$S^{-1}R \cong S_o^{-1}A \ltimes S_o^{-1}E.$$

Example 2.2.4 Let A be any non-Noetherian von Neumann regular ring (e.g., infinite direct product of fields). Then $R := A \ltimes A$ is a semi-regular ring that is neither von Neumann regular nor quasi-Frobenius. Indeed, for every $\mathfrak{m} \in Max(A)$, $R_{\mathfrak{m} \ltimes A} = A_{\mathfrak{m}} \ltimes A_{\mathfrak{m}}$ is semi-regular by Corollary 2.2.2. Moreover, R is coherent by [25, Remark, p. 55]. By [44, Proposition 2.3], R is semi-regular. However, R is neither von Neumann regular (since not reduced) nor quasi-Frobenius (since not Noetherian).

Recall that semi-regularity is a local property in the class of coherent rings [44, Proposition 2.3]. Outside this class, the question was left open. The next example addresses this question. In this vein, recall that coherence is not a local property. Glaz provided an example of a locally Noetherian ring that is not coherent [25, Example, p. 51]. The next example features also a new locally Noetherian (in fact, locally principal) ring which is not coherent.

Example 2.2.5 Let k be a field and $F_i := k$, for every $i \in \mathbb{N}$. Let

$$\begin{aligned} A &:= \prod_{i\in\mathbb{N}} F_i \\ I &:= \bigoplus_{i\in\mathbb{N}} F_i \\ R &:= A \ltimes \tfrac{A}{I}. \end{aligned}$$

Then R is a locally principal quasi-Frobenius ring and, a fortiori, locally semi-regular; which is not coherent and, a fortiori, not semi-regular. Indeed, let P be a prime ideal of R; that is, $P := p \ltimes \frac{A}{I}$, for some prime ideal p of A. Then, we have

$$R_P \cong A_p \ltimes \frac{A_p}{I_p}$$

which is isomorphic to k if $I \nsubseteq p$ or to $k \ltimes k$ if $I \subseteq p$ and, in this case, R is a principal quasi-Frobenius ring by Corollary 2.2.2. Finally, observe that

$$Ann_R(0,\overline{1}) = I \ltimes \frac{A}{I}$$

is not finitely generated in R since I is not finitely generated in A. So, R is not coherent, as desired.

Next, we provide a global version for Theorem 2.2.1.

Corollary 2.2.6 *Let A be a ring, E a nonzero A-module, and $R := A \ltimes E$. Suppose* $\mathfrak{m}E_t = 0$, $\forall \mathfrak{m} \in Max(A)$. *Then, the following assertions are equivalent*

(1) *R is semi-regular;*

(2) *A is coherent arithmetical,* $E \cong A$*, and* $\mathfrak{m}^2 A_{\mathfrak{m}} = 0$, $\forall \mathfrak{m} \in Max(A)$.

Proof. One can check that $\mathfrak{m}E_t = 0$ forces $\mathfrak{m}(E_{\mathfrak{m}})_t = 0$ for every $\mathfrak{m} \in Max(A)$. Moreover, [44, Proposition 2.3] ensures that R is semi-regular if and only if R is coherent and $R_{\mathfrak{m}}$ is semi-regular for every $\mathfrak{m} \in Max(A)$. Also, if R is coherent, then so is its retract A [25, Theorem 4.1.5]. These facts combined with Theorem 2.2.1 lead to the conclusion, completing the proof of the theorem. □

The next result establishes the transfer of semi-coherence to trivial extensions over flat modules.

Proposition 2.2.7 *Let A be a ring, E a nonzero flat A-module, and $R := A \ltimes E$. Then, the following assertions are equivalent*

(1) *A is semi-coherent;*

(2) *R is semi-coherent.*

Proof. Notice first that A can be viewed as a subring of $R := A \ltimes E$ and hence R is a flat A-algebra (since E is by hypothesis flat). Assume A is semi-coherent and let M, N be two injective R-modules. Then, by [54, Theorem 3.44],

$$M\ (\cong Hom_R(R,M)) \text{ and } N\ (\cong Hom_R(R,N))$$

are injective A-modules. Hence, $Hom_A(M,N)$ is a submodule of a flat A-module F. So, we obtain

$$\begin{aligned} Hom_R(M,N) &\subseteq Hom_A(M,N) \\ &\subseteq F \\ &\subseteq F \otimes_A R \end{aligned}$$

where the first containment holds because $A \subseteq R$ and the third containment holds since F is A-flat. Moreover, $F \otimes_A R$ is R-flat. It follows that R is semi-coherent.

Conversely, assume R is semi-coherent and let M,N be two injective A-modules. By the adjoint isomorphism, $Hom_A(R,M)$ and $Hom_A(R,N)$ are injective R-modules. Next, consider the following mapping

$$\varphi : Hom_A(M,N) \longrightarrow Hom_R\big(Hom_A(R,M), Hom_A(R,N)\big)$$

defined by $\varphi(u)(f) = u \circ f$, for every $u \in Hom_A(M,N)$ and $f \in Hom_A(R,M)$. Clearly, φ is a linear map of A-modules. Moreover, we claim that φ is injective. Indeed, let $u \in Hom_A(M,N)$ with $\varphi(u) = 0$ and let $x \in M$. Consider the following A-map

$$f : R \longrightarrow M \ ; \ (a,e) \mapsto ax.$$

Then, we have

$$0 = \varphi(u)(f)(1,0) = u(f(1,0)) = u(x)$$

which yields $u = 0$, as desired. By hypothesis, we have

$$Hom_R\big(Hom_A(R,M), Hom_A(R,N)\big) \subseteq F$$

where F is a flat R-module, which is also a flat A-module since R is A-flat. Consequently, A is semi-coherent, completing the proof of the result. □

From [44], recall that a ring R is self semi-injective if every R-homomorphism from a finitely generated ideal of R to an R-module extends to R.

Corollary 2.2.8 *Let A be a semi-coherent ring (e.g., domain), E a nonzero flat A-module, and $R := A \ltimes E$. Then, the following assertions are equivalent*

(1) *R is semi-regular;*

(2) *R is self semi-injective.*

Proof. By Proposition 2.2.7, R is semi-coherent. Conclude via the facts that a ring R is semi-regular if and only if R is coherent and self semi-injective [44, Proposition 3.3] if and only if R is semi-coherent and self fp-injective [16, Proposition 1]. □

Next, we show how one can use the above results to provide new examples discriminating between the notions of semi-coherence, coherence, and semi-regularity. For this purpose, we first establish a lemma on coherence (which generalizes [36, Theorem 3.1(1)]).

Lemma 2.2.9 *Let A be a domain, E a torsion free A-module, and $R := A \ltimes E$. Then, the following assertions are equivalent*

(1) *R is coherent;*

(2) *A is coherent and E is finitely generated.*

Proof. Assume that R is coherent. Then A, being a retract of R, is coherent by [25, Theorem 4.1.5]. Moreover, let $0 \neq e \in E$. By [25, Theorem 2.3.2(7)], $Ann_R(0,e)$ is finitely generated. Since E is torsion free, we get

$$Ann_R(0,e) = 0 \ltimes E.$$

It follows that E is finitely generated. Conversely, assume A is a coherent domain and E is a finitely generated A-module. Then, E is a submodule of a finitely generated free A-module [54, Lemma 4.31], which is then coherent [25, Theorem 2.2.3]. Therefore, E is coherent. It follows that R is coherent by [25, Remark, p. 55]. □

Example 2.2.10 Let A be a domain which is not a field with quotient field K. Then:

(1) $K \ltimes K^2$ is a coherent ring which is not semi-regular by Lemma and Corollary.

(2) $A \ltimes K$ is a semi-coherent ring which is not coherent by Proposition and Lemma.

Recall that a ring R is mininjective (also called mini-injective) if every R-homomorphism from a simple ideal of R to an R-module extends to R. Harada proved that an Artinian mininjective ring is quasi-Frobenius [30, Theorem 13]. Next, we provide an example which shows that, unlike self semi-injectivity and self fp-injectivity, mininjectivity does not coincide with semi regularity in the class of coherent rings. Moreover, in [44, Proposition 2.2], Matlis proved that, for a ring R, if $Q(R)$ is semi-regular, then R is semi-coherent; and the converse was left open. The example shows that the converse does not hold, in general, even for R coherent.

Example 2.2.11 Let A be a coherent domain (e.g., Prüfer) and let $R := A \ltimes A^2$. Then:

(1) R is coherent by Lemma 2.2.9.

(2) R is mininjective by [39, Lemma 3.1 & Theorem 3.3].

(3) $Q(R)$ is not semi-regular. Indeed, one can easily check that $Z(R) = 0 \ltimes A^2$. Hence, for $S := R \setminus Z(R)$, we obtain $S_o := S \cap A = A \setminus \{0\}$ and thus

$$Q(R) \cong S_o^{-1}A \ltimes S_o^{-1}A^2 = K \ltimes K^2$$

where $K := Q(A)$. By Example 2.2.10, $Q(R)$ is not semi-regular, and hence neither is R by [44, Proposition 2.1].

We close this section by observing that the assumption "$\mathfrak{m}E_t = 0$" in Theorem 2.2.1 is (convenient but) not inevitable in order to construct quasi-Frobenius rings issued from trivial ring extensions, as shown by the next example.

Example 2.2.12 Let $(A, \mathfrak{m})$ be an Artinian local ring with residue field K and let E denote an injective envelope of K. Then, by Kourki's result [39, Theorem 3.6], $R := A \ltimes E$ is quasi-Frobenius. Indeed, it suffices to verify that the socle of $Ann_A(E) \times E$ is square free; that is, $Ann_{Ann_A(E) \times E}(\mathfrak{m})$ is either null or simple [39, Lemma 3.1]. In fact, we have

$$Ann_A(E) = 0 \text{ and } Ann_E(\mathfrak{m}) = K$$

which yield

$$
\begin{aligned}
Ann_{Ann_A(E)\times E}(\mathfrak{m}) &= Ann_{Ann_A(E)}(\mathfrak{m}) \times Ann_E(\mathfrak{m}) \\
&= Ann_{(0)}(\mathfrak{m}) \times K \\
&= 0 \times K, \text{ as desired.}
\end{aligned}
$$

2.3 Transfer of semi-regularity to trivial ring extensions issued from domains

This section investigates the transfer of semi-regularity to trivial ring extensions issued from domains. We first state some preliminary results which will make up the proof of the main result of this section (Theorem 2.3.8).

Recall that a module over a domain is divisible if each element of the module is divisible by every nonzero element of the domain [54]. The first lemma asserts that fp-injectivity and, a fortiori, divisibility of the module E are necessary conditions for the trivial extension $A \ltimes E$ to inherit semi-regularity.

Lemma 2.3.1 *Let A be a ring, E an A-module, and $R := A \ltimes E$. Then:*

(1) *If R is self fp-injective, then E is fp-injective.*

(2) *In particular, if A is a domain and R is semi-regular, then E is divisible.*

Proof. (1) Let

$$M := \sum_{1\leq i\leq n} Am_i$$

be a finitely generated submodule of A^n, for some positive integer n, and let

$$f : M \longrightarrow E$$

be an A-map. One can identify R^n with $A^n \ltimes E^n$ as R-modules under the natural scalar multiplication. Consider the finitely generated submodule of R^n given by

$$N := \sum_{1 \leq i \leq n} R(m_i, 0)$$

along with the R-maps

$$N \overset{p}{\twoheadrightarrow} M \overset{f}{\longrightarrow} E \overset{u}{\hookrightarrow} R$$

where p is defined by

$$p\big(\sum_{1 \leq i \leq n} (a_i, e_i)(m_i, 0)\big) = \sum_{1 \leq i \leq n} a_i m_i$$

and u is the canonical embedding. Then, $g := u \circ f \circ p$ extends to R^n via $\overline{g}$, since R is self fp-injective. It follows that f extends to A^n via the A-map

$$\overline{f} : A^n \overset{i}{\hookrightarrow} R^n \overset{\overline{g}}{\longrightarrow} R \overset{\pi}{\twoheadrightarrow} E$$

where i is the canonical embedding and π is the canonical surjection. Therefore, E is fp-injective [24, Theorem IX-3.1].

(2) Straightforward via (1) since a semi-regular ring is self fp-injective; and an fp-

injective module is divisible. □

Remark 2.3.2 The second statement of the lemma is still valid if A is an arbitrary ring (i.e., possibly with zero-divisors) and divisibility of E is taken over all non zero-divisors of A.

The next lemma shows that divisibility of the module E controls the finitely generated ideals of the trivial extension $A \ltimes E$.

Lemma 2.3.3 *Let A be a domain, E a divisible A-module, and $R := A \ltimes E$. Then, for any finitely generated ideal $\mathscr{I}$ of R, we have:*

- *Either $\mathscr{I} = I \ltimes E$ for some nonzero finitely generated ideal I of A,*
- *Or $\mathscr{I} = 0 \ltimes E'$ for some finitely generated submodule E' of E.*

Proof. First, note that if E' is a finitely generated submodule of E, then $0 \ltimes E'$ is a finitely generated ideal of R. Also, let

$$I := \sum_{1 \leq i \leq n} Aa_i$$

with $0 \neq a_i \in A$ for all i and let $e \in E$. Then, by divisibility, $e = a_1 e'$ for some $e' \in E$ and, hence, $(0,e) = (a_1,0)(0,e')$. It follows that

$$I \ltimes E = \sum_{1 \leq i \leq n} (a_i,0)R$$

is a finitely generated ideal of R.

Next, let

$$\mathscr{I} = \sum_{1\le i\le n} (x_i, e_i)R$$

with $x_i \in A$ and $e_i \in E$ for $i = 1, \ldots, n$. If $x_i = 0$ for all i, then

$$\begin{aligned}\mathscr{I} &= \textstyle\sum_{1\le i\le n} 0 \ltimes Ae_i \\ &= 0 \ltimes E'\end{aligned}$$

with $E' := \sum_{1\le i\le n} Ae_i$, as desired.

Next, assume the x_i's are not all null and, mutatis mutandis, let $r \in \{1, \ldots, n\}$ such that $x_i \neq 0$ for $i \le r$ and $x_i = 0$ for $i \ge r+1$. We claim that

$$\mathscr{I} = I \ltimes E \text{ with } I := \sum_{1\le i\le r} Ax_i.$$

Indeed, $\forall\, i \in \{1, \ldots, r\}$ and $\forall\, j \in \{r+1, \ldots, n\}$, we have

$$\begin{aligned}(x_i, e_i)R &\subseteq Ax_i \ltimes (Ex_i + Ae_i) \\ &\subseteq I \ltimes E\end{aligned}$$

and

$$\begin{aligned}(x_j, e_j)R &= 0 \ltimes Ae_j \\ &\subseteq I \ltimes E\end{aligned}$$

so that $\mathscr{I} \subseteq I \ltimes E$. For the reverse inclusion, let

$$z := (\sum_{1\le i\le r} a_i x_i, e) \in I \ltimes E.$$

We can write

$$z := (a_1x_1, e) + \sum_{2\leq i\leq r} (a_ix_i, 0).$$

So, it suffices to show that

$$(a_ix_i, e) \in (x_i, e_i)R$$

for any given $e \in E$ and $i \in \{1,\dots,r\}$. This holds if there is $e' \in E$ such that

$$e = x_ie' + a_ie_i.$$

Indeed, recall at this point that E is divisible and suppose $e = 0$. If $a_ie_i = 0$, take $e' := 0$; and if $a_ie_i \neq 0$, then $a_ie_i = x_ie'_i$ for some $e'_i \in E$ and hence take $e' := -e'_i$. Suppose $e \neq 0$ and let $e = x_ie''_i$ for some $e''_i \in E$. If $a_ie_i = 0$, take $e' := e''_i$; and if $a_ie_i \neq 0$, take $e' := e''_i - e'_i$, proving the claim. □

Remark 2.3.4 Notice that the converse of the above lemma is always true; namely, if all finitely generated ideals of R have the two aforementioned forms, then E is divisible. For, let x be a nonzero element of A. Then,

$$(x,0)R = xA \ltimes xE$$

is a finitely generated ideal of R with $xA \neq 0$, which forces $E = xE$.

Next, we examine the transfer of coherence in trivial extensions of domains by divisible modules. In this vein, we will use Fuchs-Salce's definition of a coherent module; that is, all its finitely generated submodules are finitely presented [24, Chapter IV] (i.e., the module itself doesn't have to be finitely generated). In Bourbaki, such a module is called "pseudo-coherent" [13] and Wisbauer called it "locally coherent" [57].

We first isolate the simple case when A is trivial. Namely, if $A := k$ is a field and E is a k-vector space, then a combination of Lemma 2.2.9 and [5, Theorem 4.8] yields: "*$k \ltimes E$ is coherent if and only if $k \ltimes E$ is Noetherian if and only if* $\dim_k E < \infty$." The next result handles the case when A is a non-trivial domain.

Proposition 2.3.5 *Let A be a domain which is not a field, E a divisible A-module, and $R := A \ltimes E$. Then, the following assertions are equivalent*

(1) *R is a coherent ring;*

(2) *A is a coherent domain, E is a torsion coherent module, and $Ann_E(x)$ is finitely generated for all $x \in A$.*

Proof. (1) $\Longrightarrow$ (2) Assume R is coherent. Then so are its retract A by [25, Theorem 4.1.5] and E by Glaz's remark following [25, Theorem 4.4.4] in page 146. Now, assume there is a torsion-free element $e \in E$ and let $0 \neq a \in A$. Then

$$\begin{aligned} Ann_R(0,e) &= Ann_A(e) \ltimes E \\ &= 0 \ltimes E \end{aligned}$$

is a finitely generated ideal of R. So E is a finitely generated A-module. Let $e_1, \ldots, e_n$

be a minimal generating set for E. By the divisibility assumption, we obtain

$$e_1 = a \sum_{1 \leq i \leq n} a_i e_i$$

for some $a_1, \ldots, a_n \in A$. If $1 - aa_1 \neq 0$, then

$$e_1 = (1 - aa_1) \sum_{1 \leq i \leq n} b_i e_i$$

for some $b_1, \ldots, b_n \in A$, forcing

$$e_1 \in \sum_{2 \leq i \leq n} Ae_i$$

which is absurd. So, necessarily, we have

$$1 - aa_1 = 0.$$

It follows that A is a field, the desired contradiction. Hence, E is a torsion module. Finally, let $0 \neq x \in A$. Then,

$$Ann_R(x, 0) = 0 \ltimes Ann_E(x)$$

is finitely generated in R. So $Ann_E(x)$ is a finitely generated submodule of E.

(2) $\Longrightarrow$ (1) We first show that the intersection of any two finitely generated ideals of R is finitely generated. Let I_1 and I_2 be two nonzero finitely generated ideals of A and

let E_1 and E_2 be two finitely generated submodules of E. Since A is a coherent domain, $I_1 \cap I_2$ is a nonzero finitely generated ideal of A. By Lemma 2.3.3,

$$(I_1 \ltimes E) \cap (I_2 \ltimes E) = (I_1 \cap I_2) \ltimes E$$

is a finitely generated ideal of R. Further, obviously,

$$(I_1 \ltimes E) \cap (0 \ltimes E_1) = 0 \ltimes E_1$$

is finitely generated. Moreover, since E is coherent, $E_1 \cap E_2$ is a finitely generated submodule of E [24, (D)–Page 128]. Hence,

$$(0 \ltimes E_1) \cap (0 \ltimes E_2) = 0 \ltimes (E_1 \cap E_2)$$

is a finitely generated ideal of R. In view of Lemma 2.3.3, we are done. By [25, Theorem 2.3.2(7)], it remains to show that $Ann_R(x,e)$ is finitely generated for any $(x,e) \in R$. Indeed, if $x \neq 0$, then

$$Ann_R(x,e) = 0 \ltimes Ann_E(x)$$

is finitely generated in R (since by hypothesis $Ann_E(x)$ is finitely generated). Next, assume $x = 0$. In view of the exact sequence

$$0 \to Ann_A(e) \to A \to Ae \to 0,$$

since E is torsion coherent, $Ann_A(e)$ is a nonzero finitely generated ideal of A. By Lemma 2.3.3,

$$Ann_R(0,e) = Ann_A(e) \ltimes E$$

is a finitely generated ideal of R, completing the proof of the proposition. □

In the above result, the assumption "$Ann_E(x)$ is finitely generated for all $x \in A$" is not superfluous in presence of the other assumptions, as shown by Example 2.4.4. In order to proceed further, we need to extend, to an A-module, Matlis' double annihilator condition in a ring A; that is

$$Ann_A(Ann_A(I)) = I$$

for each finitely generated ideal I of A [44, Section 4, Definition].

Definition 2.3.6 Let A be a ring. An A-module E is said to satisfy the double annihilator condition (in short, DAC) if the two following assertions hold:

(DAC1) $Ann_A(Ann_E(I)) = I$, for every finitely generated ideal I of A.

(DAC2) $Ann_E(Ann_A(E')) = E'$, for every finitely generated submodule E' of E.

Obviously, this definition coincides with Matlis' double annihilator condition when $E = A$. Moreover, all these conditions are unrelated in general, as shown by Example 2.4.5.

We need the next lemma which characterizes the double annihilator condition in a trivial ring extension via the (DAC) property of its divisible module.

Lemma 2.3.7 *Let A be a domain, E a divisible A-module, and $R := A \ltimes E$. Then, the following assertions are equivalent*

(1) *R satisfies Matlis' double annihilator condition;*

(2) *E satisfies (DAC).*

Proof. First, notice that

$$Ann_A(Ann_E(0)) = Ann_A(E) = 0$$

since $aE = E$, $\forall\ 0 \neq a \in A$. Now, by Lemma 2.3.3, the finitely generated ideals of R have the forms

$$I \ltimes E \text{ and } 0 \ltimes E'$$

where I is a nonzero finitely generated ideal of A and E' is a finitely generated submodule of E. Moreover, one can easily check that

$$Ann_R(I \ltimes E) = 0 \ltimes Ann_E(I)$$

and

$$Ann_R(0 \ltimes E') = Ann_A(E') \ltimes E.$$

It follows that

$$Ann_R(Ann_R(I \ltimes E)) = \big(Ann_A(Ann_E(I))\big) \ltimes E$$

and

$$Ann_R(Ann_R(0 \ltimes E')) = 0 \ltimes \big(Ann_E(Ann_A(E'))\big),$$

leading to the conclusion. □

Finally, we are ready to state the main theorem of this section on the transfer of semi-regularity to trivial ring extensions.

Theorem 2.3.8 *Let A be a domain, E an A-module, and $R := A \ltimes E$. Then, the following assertions are equivalent*

(1) *R is semi-regular;*

(2) *Either A is a field with $E \cong A$ or A is coherent, E is a divisible (resp., fp-injective) torsion coherent module which satisfies (DAC), and $\mathrm{Ann}_E(x)$ is finitely generated for all $x \in A$.*

Proof. Let us first isolate the simple case when A is trivial. Namely, if $A := k$ is a field and E is a nonzero k-vector space, then $k \ltimes E$ is semi-regular if and only if $k \ltimes E$ is quasi-Frobenius if and only if $\dim_k E = 1$. This is a particular case of Corollary 2.2.2. Now, assume that A is a domain which is not a field. Combine Lemma 2.3.1, Proposition 2.3.5, and Lemma 2.3.7 with Matlis' result that "*a ring is semi-regular if and only if it is coherent and satisfies the double annihilator condition (on finitely generated ideals)*" [44, Proposition 4.1]. □

2.4 Applications and examples

A nonzero fractional ideal I of a domain A with quotient field K is called divisorial provided

$$I = I_v := (I^{-1})^{-1}$$

where

$$I^{-1} := (R : I) = \{x \in K \mid xI \subseteq R\}.$$

A domain is called divisorial if all its nonzero (fractional) ideals are divisorial. Divisorial domains have been studied by, among others, Bass [8] and Matlis [42] for the Noetherian case, Heinzer [31] for the integrally closed case, Bastida-Gilmer [7] for the transfer to $D+M$ constructions, and Bazzoni [9] for more general settings. It is worthwhile recalling that a domain in which all finitely generated ideals are divisorial is not necessarily divisorial [9, Example 2.11].

Also, recall that a domain A is totally divisorial if every overring of A is a divisorial domain; and A is stable if every nonzero ideal of A is projective over its ring of endomorphisms [24, 47]. It is worthwhile knowing that a domain A is totally divisorial if and only if A is a stable divisorial domain [47, Theorem 3.12].

As an application of Theorem 2.3.8, the next corollary will allow us to enrich the literature with new families of examples subject to semi-regularity.

Corollary 2.4.1 *Let A be a coherent domain which is not a field. Then, the following assertions are equivalent*

(1) $A \ltimes \frac{Q(A)}{A}$ *is semi-regular;*

(2) *Each nonzero finitely generated ideal of A is divisorial.*

Proof. First, notice that $Q(A)$ is a coherent A-module since it is torsion-free [24, IV-2, Lemma 2.5]. Further, given any exact sequence of modules over a coherent ring

$$0 \to M' \to M \to M" \to 0,$$

if any two of the modules M', M, $M"$ are finitely presented, then so is the third [24, IV-2, Exercise 2.5]. It follows that $E := Q(A)/A$ is coherent. Moreover, E is clearly a divisible torsion module and

$$Ann_E(x) = \overline{(1/x)}A$$

for any nonzero $x \in A$. Therefore, by Theorem 2.3.8, $A \ltimes E$ is semi-regular if and only if E satisfies (DAC). So, we just need to prove the following claim:

E satisfies (DAC) $\Leftrightarrow$ Each nonzero finitely generated ideal of A is divisorial.

Indeed, let $\overline{W}$ be a nonzero finitely generated submodule of E. Note that W is necessarily a fractional ideal of A containing A. Therefore W^{-1} is an integral ideal of A and hence

$$\begin{aligned} Ann_A(\overline{W}) &= A \cap W^{-1} \\ &= W^{-1}. \end{aligned}$$

Moreover, let I be a nonzero finitely generated ideal of A. Then

$$Ann_E(I) = \overline{I^{-1}}.$$

So, we obtain

$$\begin{aligned} Ann_A(Ann_E(I)) &= (I^{-1})^{-1} \\ &= I_v \end{aligned}$$

and

$$\begin{aligned} Ann_E(Ann_A(\overline{W})) &= \overline{(W^{-1})^{-1}} \\ &= \overline{W_v} \end{aligned}$$

which lead easily to the proof of the claim. □

Next, we provide various examples of semi-regular trivial ring extensions which are neither von Neumann regular (since not reduced) nor quasi-Frobenius (since not Noetherian).

Example 2.4.2 Let A be a coherent domain which is not a field and let $R := A \ltimes \frac{Q(A)}{A}$. Note that R is not Noetherian since $\frac{Q(A)}{A}$ is not finitely generated.

(1) Assume A is integrally closed. Then, R is semi-regular if and only if A is Prüfer. Indeed, combine Corollary 2.4.1 with the fact that every invertible ideal is divisorial and Krull's result that "*an integrally closed domain in which all nonzero finitely generated ideals are divisorial is Prüfer*" (cf. [31, Proof of Theorem 5.1]). For an original example, take A to be any non-trivial Prüfer domain (e.g., $A := \mathbb{Z} + X\mathbb{Q}[X]$).

(2) If A is a divisorial domain, then R is semi-regular by Corollary 2.4.1. For an original example, take A to be any pseudo-valuation domain issued from a valuation domain (V, M) with M finitely generated and $[\frac{V}{M} : k] = 2$. Then, A is a (non-integrally closed) divisorial domain [7, Theorem 2.1 & Corollary 4.4], which is

coherent [20, Theorem 3] or [12, Theorem 3].

(3) Next, we provide a non-integrally closed non-divisorial domain A in which every finitely generated ideal is divisorial; and hence R is semi-regular by Corollary 2.4.1. Indeed, let D be a non-integrally closed pseudo-valuation domain which is divisorial and coherent (e.g., take D to be the domain A of (2) above) and let K be its quotient field. By [46, Theorem 2.6], D is not stable and hence not totally divisorial by [47, Theorem 3.12]. Let V be a valuation domain of the form $K+M$ and let $A := D+M$. Then, A is a non-integrally closed non-divisorial domain [7, Theorem 2.1 & Corollary 4.4] which is coherent [20, Theorem 3] or [12, Theorem 3]. Moreover, since D is divisorial, every finitely generated ideal of A is divisorial by [7, Theorem 2.1(k) & Theorem 4.3].

One may use Theorem 2.3.8 and Proposition 2.3.5 to enrich the literature with new examples of coherent rings which are not semi-regular, as shown below.

Example 2.4.3 Let A be a coherent domain which is not a field and let

$$\begin{aligned} E &:= \tfrac{Q(A)}{A} \oplus \tfrac{Q(A)}{A} \\ R &:= A \ltimes E. \end{aligned}$$

Then R is a coherent ring which is not semi-regular. Indeed, similar arguments used in the proof of Corollary 2.4.1 lead to the fact that E is a divisible torsion coherent module. Moreover, one can easily check that

$$Ann_E(x) = \overline{(1/x)}A \oplus \overline{(1/x)}A$$

for any nonzero $x \in A$. So, by Proposition 2.3.5, R is a coherent ring. However, E does not satisfy (DAC). Otherwise, deny and let x be any nonzero nonunit element of A. Then, we obtain

$$\begin{aligned} (\bar{0}) \oplus \overline{(1/x)}A &= Ann_E\big(Ann_A((\bar{0}) \oplus \overline{(1/x)}A)\big) \\ &= Ann_E(Ax) \\ &= \overline{(1/x)}A \oplus \overline{(1/x)}A \end{aligned}$$

yielding

$$\overline{(1/x)}A = (\bar{0})$$

the desired contradiction. Consequently, by Theorem 2.3.8, R is not semi-regular.

The next example shows that, in Proposition 2.3.5, the assumption "$Ann_E(x)$ is finitely generated for all $x \in A$" is not superfluous in presence of the other assumptions.

Example 2.4.4 Let A be a coherent domain which is not a field (e.g., any non-trivial Prüfer domain) and let

$$E := \bigoplus_{n \geq 0} E_n \text{ with } E_n := Q(A)/A$$

Then, E is a divisible coherent A-module [24, (C)–Page 37 & (B)–Page 128] and, clearly, E is torsion. However, the condition "$Ann_E(x)$ is finitely generated for all $x \in A$" does not hold. For, let x be any nonzero nonunit element of A. Then, one can easily check that

$$Ann_E(x) = \bigoplus_{n \geq 0} \overline{(1/x)}$$

which is not finitely generated.

The next example shows that the conditions of Definition 2.3.6 are unrelated in general.

Example 2.4.5 Let A be a ring and E a nonzero A-module.

(1) Assume $A := K$ is a field. Then, E satisfies (DAC1). Moreover, E satisfies (DAC2) if and only if $\dim_K(E) = 1$. Indeed, the first statement is straightforward. The second statement holds since $Ann_E(Ann_K(e)) = E$, for any nonzero $e \in E$.

(2) Assume $(A, \mathfrak{m})$ is local and $E := A/\mathfrak{m}$. Then, E satisfies (DAC2). Moreover, E satisfies (DAC1) if and only if $l(\mathfrak{m}) = 1$. Indeed, the first statement is straight since E has no nonzero proper submodules. The second statement holds since $Ann_A(Ann_E(x)) = \mathfrak{m}$, for any $x \in \mathfrak{m}$.

(3) Assume A satisfies Matlis' double annihilator condition (e.g., semi-regular) and E has a torsion-free element. Then, E satisfies (DAC) if and only if $E \cong A$. This is true since $Ann_E(Ann_A(e)) = E$, for any given torsion-free element $e \in E$.

CHAPTER 3

ZAKS' CONJECTURE ON RINGS WITH SEMI-REGULAR PROPER HOMOMORPHIC IMAGES

This chapter[1] proves an extension of Zaks' conjecture on integral domains with semi-regular proper homomorphic images (with respect to the finitely generated ideals) to the class of coherent rings (with zero-divisors) and provides new examples of rings with semi-regular proper homomorphic images via trivial ring extensions.

3.1 Introduction

A ring R is arithmetical if every finitely generated ideal of R is locally principal [23, 34, 38]; and R is a chained ring if R is local and arithmetical [10, 11, 32]. In the domain setting, these two notions coincide with Prüfer and valuation domains, respectively. In [15], Couchot investigated semi-regularity (termed as IF-ring) in the class of chained rings (termed as valuation rings). He proved that a chained ring is residually semi-regular [15, Theorem 11]. It is worthwhile recalling that, in the Noetherian setting, semi-regularity coincides with self-injectivity [44, Proposition 3.4]; and under coherence, it coincides with the double annihilator condition (i.e., $Ann(Ann(I)) = Ann(I)$, for every finitely generated ideal I) [44, Proposition 4.1].

For convenience, recall that a domain R is Dedekind if every ideal of R is projective. In 1966, Levy proved a dual version for this result stating that, for a Noetherian ring R (possibly with zero-divisors), every proper homomorphic image of R is self-injective if and only if R is a Dedekind domain or a principal ideal ring with descending chain condition or a local ring whose maximal ideal M has composition length 2 with $M^2 = 0$ [41, Theorem]. In 1985, Matlis proved that if R is a Prüfer domain, then R/I is semi-regular for every nonzero finitely generated ideal I of R [44, Proposition 5.3]. Then Abraham Zaks conjectured that the converse of this result should be true; i.e., an integral domain R is Prüfer if and only if R/I is semi-regular for every nonzero finitely generated ideal I of R. This was proved by Matlis in [44, Theorem, p. 371]; extending thus Levy's theorem in the case of integral domains.

In this chapter, we prove an extension of Zaks' aforementioned conjecture on integral domains with semi-regular proper homomorphic images (with respect to the finitely generated ideals) to the class of coherent rings (with zero-divisors). Section 2.2 features the main result of this chapter, which extends and recovers, in Section 2.3, Levy's and Matlis' aforementioned results on Noetherian rings and Prüfer domains, respectively. It also globalizes Couchot's related result on chained rings. In Section 2.4, we use the main result in combination with our results in the first chapter to construct new examples of rings with semi-regular proper homomorphic images (via trivial ring extensions).

3.2 Characterization of coherent residually semi-regular rings

We first provide a suitable terminology for rings with semi-regular homomorphic images (with respect to the finitely generated ideals); a ring R is called *residually semi-regular* if R/I is semi-regular, for each nonzero finitely generated ideal I of R.

Throughout, for an R-module M, $l(M)$ will denote the *composition length* of M ($= \infty$, if M has no composition series). Levy's (resp., Matlis') results asserts that a Noetherian domain (resp., a domain) R is residually semi-regular if and only if R is Dedekind (resp., Prüfer). In the non-domain setting, Levy's result ensures that a Noetherian ring with zero-divisors is residually semi-regular if and only if R is principal Artinian or (R,M) is local with $M^2 = 0$ and $l(M) = 2$. Recall that a semi-regular ring, being equal to its total ring of quotients, is always a Prüfer ring.

Next, we announce the main result of this chapter, which characterizes the notion of residually semi-regular ring in the class of coherent rings (possibly, with zero-divisors). Indeed, it extends Zaks's conjecture to the class of coherent rings, generalizing thus Levy's, Matlis', and Couchot's aforementioned results.

Theorem 3.2.1 *Let R be a coherent ring and consider the following conditions:*

($\mathscr{C}_1$) (R,M) *is local with* $M^2 = 0$ *and* $l(M) = 2$.

($\mathscr{C}_2$) *R is arithmetical and* R_M *is semi-regular for every* $M \in Max(R)$ *such that* $rR_M = 0$ *for some nonzero* $r \in R$.

Then, R is residually semi-regular if and only if R satisfies ($\mathscr{C}_1$) *or* ($\mathscr{C}_2$).

Notice, at this point, that a coherent arithmetical ring is not residually semi-regular, in general. This is evidenced by Example 3.4.1, which shows that the assumption "R_M is semi-regular for every $M \in Max(R)$ such that $rR_M = 0$ for some nonzero $r \in R$" is not redundant with the arithmetical property; and hence a global version for Couchot's result is not always true.

We break down the proof of the theorem into several lemmas.

Lemma 3.2.2 *Let R be a local residually semi-regular ring and let* I_1 *and* I_2 *be two finitely generated ideals of R with* $I_1 \cap I_2 \neq 0$. *Then:*

(i) $I_1 \cap I_2$ *is finitely generated.*

(ii) I_1 *and* I_2 *are comparable.*

Proof. (i) Let $0 \neq x \in I_1 \cap I_2$. Without loss of generality, we may assume that $Rx \subsetneqq I_1 \cap I_2$ and consider the semi-regular ring $\overline{R} := R/Rx$ which is coherent by [44, Proposition 3.3]. Then, $\overline{I_1} \cap \overline{I_2} = \overline{I_1 \cap I_2}$ is finitely generated in $\overline{R}$. Hence $I_1 \cap I_2$ is finitely generated in R.

(ii) First, note that if $0 \neq I, J$, and K are three finitely generated ideals of R with $I \subseteq J$ and $I \subseteq K$, then, by [44, Proposition 4.1], R/I satisfies the double annihilator condition on J/I and

$$Ann_{\frac{R}{I}}(\frac{J}{I}) + Ann_{\frac{R}{I}}(\frac{K}{I}) = Ann_{\frac{R}{I}}(\frac{J \cap K}{I})$$

that is,

$$(I : (I : J)) = J \tag{3.2.1}$$

and

$$(I : J) + (I : K) = (I : J \cap K) \tag{3.2.2}$$

where by $(I : J)$ we mean

$$(I :_R J) = \{x \in R \mid xJ \subseteq I\}.$$

Now, $0 \neq I_1 \cap I_2$ is finitely generated by (i). Hence, by (3.2.2), we obtain

$$\begin{aligned} (I_1 \cap I_2 : I_1) + (I_1 \cap I_2 : I_2) &= (I_1 \cap I_2 : I_1 \cap I_2) \\ &= R. \end{aligned}$$

Therefore, $1 = x + y$, for some $x \in (I_1 \cap I_2 : I_1)$ and $y \in (I_1 \cap I_2 : I_2)$. It follows that, for any $a_1 \in I_1$ and $a_2 \in I_2$, we have

$$(1 - y)a_1 = xa_1 \in I_2 \text{ and } ya_2 \in I_1$$

Since R is local, either y or $1-y$ is a unit, forcing I_1 and I_2 to be comparable. $\square$

Lemma 3.2.3 *Let (R,M) be a local coherent residually semi-regular ring and let $x,y \in R$.*

(i) $x^2 \neq 0$ *and* $y^2 \neq 0 \Rightarrow xy \neq 0 \Rightarrow (x)$ *and* (y) *are comparable.*

(ii) $x^2 = 0$ *and* $y^2 \neq 0 \Rightarrow (x) \subseteq (y)$.

Proof. (i) In view of Lemma 3.2.2, we only need to prove the first implication. Assume $x^2 \neq 0$ and $y^2 \neq 0$. Clearly, $x \neq 0$ and $y \neq 0$. Suppose, by way of contradiction, that $xy = 0$. Then, necessarily, (x) and (y) are incomparable. Next, let $I := (x,y)$. Since R is coherent, $Ann(y)$ is finitely generated and then

$$Ann(y) \subseteq I$$

by Lemma 3.2.2. Further, $y \notin I^2$; otherwise, $y = ax^2 + by^2$ for some $a,b \in R$ yields

$$y = ax^2(1-by)^{-1} \in (x)$$

which is absurd. So, $\bar{y} \neq 0$ in $\overline{R} := R/I^2$. We claim that

$$Ann(\bar{y}) = \bar{I} \text{ in } \overline{R}$$

Indeed, let $\bar{t} \in Ann(\bar{y})$. Then, there exist $a,b \in R$ such that

$$y(t-by) = ax^2 \in (x) \cap (y).$$

By Lemma 3.2.2, $y(t-by)=0$. Hence

$$t-by \in Ann(y) \subseteq I.$$

Whence $\bar{t} \in \bar{I}$. The reverse inclusion is obvious, proving the claim. Now, the fact that $\overline{R}$ is semi-regular yields

$$\begin{array}{rcl} \bar{I} & \subseteq & Ann(\bar{I}) \\ & = & Ann(Ann(\bar{y})) \\ & = & (\bar{y}) \\ & \subseteq & \bar{I}. \end{array} \tag{3.2.3}$$

It follows that $(\bar{y})=\bar{I}$ and therefore

$$\begin{array}{rcl} I & = & (y)+I^2 \\ & = & (y)+MI. \end{array} \tag{3.2.4}$$

By Nakayama's lemma, we get $I=(y)$, the desired contradiction.

(ii) Assume $x^2=0$ and $y^2 \neq 0$. Clearly, $y \neq 0$. Without loss of generality, we may assume $x \neq 0$ and y is not a unit. If $xy \neq 0$, then (x) and (y) are comparable and necessarily $(x) \subseteq (y)$. Next, suppose that $xy=0$ and let $I:=(x,y)$. Similarly to (i), coherence implies $Ann(y) \subseteq I$, and $\bar{y} \neq 0$ in $\overline{R}:=R/I^2$; otherwise, $y=ay^2$ for some $a \in R$ yields $y(1-ay)=0$, absurd (since $1-ay$ is a unit). Also, $ty=ay^2$ for some $a \in R$ yields $t-ay \in Ann(y) \subseteq I$ and so $t \in I$. That is,

$$Ann(\bar{y})=\bar{I} \text{ in } \overline{R}.$$

Similar arguments as in (3.2.3) and (3.2.4) lead to $I=(y)$, as desired. □

Lemma 3.2.4 *Let R be a local coherent residually semi-regular ring and I a finitely generated ideal of R. Then:*

- *Either I is principal,*
- *Or I is generated by two elements with $I^2=0$.*

Proof. Notice first that, for any $0\neq x,y,z\in R$, (x,y) and (x,z) are comparable by Lemma 3.2.2. It follows that any finitely generated ideal is generated by at most two elements. So, $I=(x,y)$ for some $x,y\in R$. If $xy\neq 0$ or $x^2\neq 0$ or $y^2\neq 0$, then I is principal by Lemma 3.2.2 and Lemma 3.2.3, completing the proof of the lemma. □

Lemma 3.2.5 *Let (R,M) be a local coherent residually semi-regular ring. Then, R is Gaussian. Moreover, if $(Z(R))^2=0$, then $Z(R)=M$.*

Proof. By [11, Theorem 2.2], R is Gaussian if and only if $\forall\, a,b\in R$:

- $(a,b)^2=(a^2)$ or (b^2)
- and if $(a,b)^2=(a^2)$ and $ab=0$, then $b^2=0$.

Next, let $a,b\in R$. The case $a^2\neq 0$ and $b^2\neq 0$ is handled by Lemma 3.2.3(i) and the case $a^2\neq 0$ and $b^2=0$ is handled by Lemma 3.2.3(ii). If $a^2=b^2=0$, then $ab=0$ by Lemma 3.2.2, whence $(a,b)^2=0$, completing the proof of the first statement.

The assumption $(Z(R))^2=0$ forces $Z(R)=Ann(a)$, for every $0\neq a\in Z(R)$. So, coherence implies that $Z(R)$ is finitely generated ideal of R. Next, let $x\in R\setminus Z(R)$.

By Lemma 3.2.2, $Z(R) \subseteq Rx$. Further, $Z(R)$ is a prime ideal since R is local Gaussian. Hence, one can check that

$$(Z(R) : Rx) = Z(R).$$

Therefore, by (3.2.1), we obtain

$$\begin{aligned} Rx &= (Z(R) : (Z(R) : Rx)) \\ &= (Z(R) : Z(R)) \\ &= R. \end{aligned}$$

Therefore, x is a unit and hence $Z(R) = M$, completing the proof of the lemma. □

Recall that an ideal is quasi-projective if it is projective modulo its annihilator; and R is an fqp-ring if every finitely generated ideal of R is quasi-projective [1, 18]. We always have:

Arithmetical

$\Downarrow$

fqp

$\Downarrow$

Gaussian

and the fqp notion is a local property in the class of coherent rings [18, Proposition 4.4] or [1, Corollary 3.15].

Lemma 3.2.6 *Let (R,M) be a local coherent residually semi-regular ring. Then, $Ann(x) = Ann(y)$, for any nonzero $x, y \in R$ such that (x) and (y) are incomparable.*

Proof. If $M^2 = 0$, then $M = Ann(x)$ for every $x \in M$ and the result trivially holds. Next, assume $M^2 \neq 0$ and let x, y be two nonzero elements of R such that (x) and (y) are incomparable. By Lemmas 3.2.2 and 3.2.3, we get

$$(x) \cap (y) = 0 \text{ and } x^2 = y^2 = xy = 0.$$

Hence

$$x, y \in Ann(x) \cap Ann(y).$$

So, coherence and Lemma 3.2.2 imply that $Ann(x)$ and $Ann(y)$ are comparable; say, $Ann(x) \subseteq Ann(y)$. Next, we prove the reverse inclusion. Let $t \in Ann(y)$ and assume, by way of contradiction, that $tx \neq 0$. First, notice that, via (3.2.1), we have

$$\begin{aligned} (tx, y) &\subseteq \Big((tx) : \big((tx) : (y)\big)\Big) \\ &\subseteq \Big((tx) : \big((tx) : (tx, y)\big)\Big) \\ &= (tx, y). \end{aligned}$$

Moreover, $\big((tx) : (x)\big)$ and $\big((tx) : (y)\big)$ are finitely generated by coherence and

$$0 \neq x \in \big((tx) : (x)\big) \cap \big((tx) : (y)\big).$$

So, by Lemma 3.2.2, $\big((tx) : (x)\big)$ and $\big((tx) : (y)\big)$ are comparable. If $\big((tx) : (x)\big) \subseteq \big((tx) :$

$(y)\big)$, then we obtain via (3.2.1)

$$\begin{aligned} (tx,y) &= \Big((tx) : \big((tx) : (y)\big)\Big) \\ &\subseteq \Big((tx) : \big((tx) : (x)\big)\Big) \\ &= (x) \end{aligned}$$

yielding $(y) \subseteq (x)$, absurd. So, suppose $\big((tx) : (y)\big) \subseteq \big((tx) : (x)\big)$. Then same argument as above yields $(x) \subseteq (tx,y)$. That is,

$$x - atx \in (x) \cap (y) = 0$$

for some $a \in R$. Hence, $x(1-at) = 0$, whence $1 - at \in Ann(x) \subseteq Ann(y)$. It follows that $y = yat = 0$, absurd. □

Lemma 3.2.7 *A local coherent residually semi-regular ring is an fqp-ring.*

Proof. Let I be a finitely generated ideal of R. We shall prove that I is quasi projective. By [1, Theorem 2.3], we only need to prove that

$$I \cong \left(\frac{R}{J}\right)^n$$

for some ideal J of R and integer $n \geq 0$. By Lemma 3.2.4, either I is principal or I is generated by two elements with $I^2 = 0$. If $I = Rx$, then

$$I \cong \frac{R}{Ann(x)}$$

as desired. Next, suppose that $I = (x, y)$ is not principal. We claim that

$$I \cong \left(\frac{R}{Ann(x)} \right)^2.$$

To this purpose, consider the surjective R-map

$$\varphi : R^2 \to I$$

defined by $\varphi(a,b) = ax + by$. Now, $\varphi(a,b) = 0$ yields

$$ax = -by \in (x) \cap (y) = 0$$

by Lemma 3.2.2 since (x) and (y) are incomparable. Therefore, $a \in Ann(x)$ and $b \in Ann(y) = Ann(x)$ by Lemma 3.2.6. It follows that

$$Ker(\varphi) = Ann(x) \times Ann(x)$$

and thus

$$I \cong \frac{R^2}{Ann(x) \times Ann(x)} \cong \left(\frac{R}{Ann(x)} \right)^2$$

completing the proof of the lemma. □

Lemma 3.2.8 *Let (R, M) be a local ring with $M^2 = 0$. Then, the following assertions are equivalent*

(1) *R is semi-regular;*

(2) *R is coherent with* $l(M) \leq 1$.

Proof. Assume R is semi-regular. Hence R is coherent [44, Proposition 2.3]. We may assume that R is not a field and let $0 \neq x \in M$. Then, we have

$$\begin{aligned} xR &= Ann_R(Ann_R(xR)) \\ &= Ann_R(M) \\ &= M. \end{aligned}$$

Consequently, $l(M) = 1$. Conversely, assume that R is coherent with $l(M) = 1$ and let $0 \neq x \in M$. Then, we have

$$\begin{aligned} xR &= M \\ &= Ann_R(M) \\ &= Ann_R(Ann_R(xR)). \end{aligned}$$

It follows that R satisfies the double annihilator condition on finitely generated ideals. Therefore, R is semi-regular by [44, Proposition 4.1]. □

Proof of Theorem 3.2.1 We first prove sufficiency. Let I be a nonzero finitely generated proper ideal of R. Then, R/I is coherent by [25, Theorem 2.4.1]. Assume that $(\mathscr{C}_1)$ holds. Therefore, $l(M/I) \leq 1$ and hence, by Lemma 3.2.8, R/I is a semi-regular ring, as desired. Next, assume that $(\mathscr{C}_2)$ holds. Let $M \in Max(R)$ with $I \subseteq M$ and let $IR_M = rR_M$, for some $0 \neq r \in R$. If $rR_M \neq 0$, then

$$(R/I)_{M/I} \cong R_M/rR_M$$

is semi-regular by [15, Theorem 11(1)]. If $rR_M = 0$, then

$$(R/I)_{M/I} \cong R_M$$

is semi-regular by hypothesis. Therefore, by [44, Proposition 2.3], R is a residually semi-regular ring.

Conversely, assume R is residually semi-regular and let us envisage two cases.

Case 1: Assume there is $M \in Max(R)$ such that $M^2 = 0$. Necessarily, (R,M) is local with M being the only prime ideal of R. We will show that either R is a chained ring or $l(M) = 2$. Without loss of generality, we may assume that R is not a field (i.e., $M \neq 0$). If $(a) \cap (b) \neq 0$ for every nonzero $a,b \in M$, then, by Lemma 3.2.2, R is a chained ring. Further, let I be a nonzero proper ideal of R, $0 \neq a \in I$, and $x \in M$. Then either $x \in (a)$ or $a \in (x)$. The second case yields $a = ux$ for some unit $u \in R$, hence $I = M = (a)$; i.e., $l(M) = 1$. By Lemma 3.2.8, R is semi-regular so that $(\mathscr{C}_2)$ is satisfied. Next, assume that there exist nonzero $a_o, b_o \in M$ such that

$$(a_o) \cap (b_o) = 0.$$

Then, (a_o) and (b_o) are incomparable and, moreover, the assumption $M^2 = 0$ yields the following property for any $0 \neq a,b \in M$:

$$(b) \nsubseteq (a) \Rightarrow M = (a,b). \tag{3.2.5}$$

Indeed, we obviously have

$$M \subseteq (a:b) \text{ and } M \subseteq (a:M).$$

Hence $(a:b) = M$ since $(a:b) \neq R$ and whence

$$(a:(a:b)) = (a:M) = M$$

since $(a:M) \neq R$. So, we obtain

$$\begin{aligned} M &= (a:(a:b)) \\ &\subseteq (a:(a:(a,b))) \\ &= (a,b) \\ &\subseteq M \end{aligned}$$

where the second equality is ensured by (3.2.1), yielding $M = (a,b)$, as claimed. It follows that $M = (a_o, b_o)$ and thus R is Artinian. Hence

$$2 \leq l(M) < \infty.$$

Next, let I be an ideal of R with $0 \subsetneqq I \subsetneqq M$ and let $0 \neq a \in I$. Therefore, for any $b \in R$, if $b \notin (a)$, then $M = (a,b)$ by (3.2.5). It follows that $I = (a)$ and no ideal can be inserted between I and M. Consequently, $l(M) = 2$ so that $(\mathscr{C}_1)$ is satisfied.

Case 2: Assume that $M^2 \neq 0$, for every $M \in Max(R)$ (and observe that $M^2 R_M$ might be null). Let $M \in Max(R)$ and, without loss of generality, assume that R_M is not a field. Note first that if $rR_M = 0$ for some nonzero $r \in R$, then

$$R_M \cong R_M/rR_M \cong (R/rR)_{M/rR}$$

is semi-regular, as desired. It remains to show that R_M is a chained ring. To this purpose, let us envisage two subcases.

SUBCASE 2.1: Suppose that $M^2R_M = 0$. Necessarily, $R_M \cong (R/M^2)_{M/M^2}$ is semi-regular. Hence, by Lemma 3.2.8, $l(MR_M) = 1$; whence $xR_M = MR_M$, for any $0 \neq x \in MR_M$. In particular, R_M is a chained ring.

SUBCASE 2.2: Suppose that $M^2R_M \neq 0$. By Lemma 3.2.7, R_M is an fqp-ring. Assume, by way of contradiction, that R_M is not a chained ring. Then, by [1, Lemmas 3.12 & 4.5], we have

$$(Nil(R_M))^2 = 0 \text{ and } Z(R_M) = Nil(R_M).$$

That is, $(Z(R_M))^2 = 0$. But, by Lemma 3.2.5, $Z(R_M) = MR_M$, the desired contradiction. So, in both cases, R_M is a chaired ring and, hence, R is an arithmetical ring. □

3.3 Applications: recovering Levy's and Matlis' classical results

As a first application, we recover Matlis' result which solves Zak's conjecture on residually semi-regular integral domains.

Corollary 3.3.1 ([44, Theorem, p. 371]) *An integral domain R is residually semi-regular if and only if R is Prüfer.*

Proof. Sufficiency is straightforward by Theorem 3.2.1 (since a Prüfer domain is coherent), and necessity is straightforward by Lemma 3.2.2(ii) (since the residually semi-regular property is stable under localization). □

Next, we recover Levy's result on Noetherian rings with self-injective proper homomorphic images. In this vein, recall for convenience that, under Noetherian assumption, semi-regularity coincides with self-injectivity.

Corollary 3.3.2 ([41, Theorem]) *Let R be a Noetherian ring and consider the following conditions:*

$(\mathscr{C}_1)$ *R is a Dedekind domain.*

$(\mathscr{C}_2)$ *R is a principal Artinian ring.*

$(\mathscr{C}_3)$ *(R,M) is local with $M^2 = 0$ and $l(M) = 2$.*

Then, R is residually semi-regular if and only if R satisfies $(\mathscr{C}_1)$ or $(\mathscr{C}_2)$ or $(\mathscr{C}_3)$.

Proof. In view of Corollary 3.3.1, we may assume that R is not a domain. For sufficiency, it suffices to consider the case where R is principal Artinian. Then, obviously, R is arithmetical. Moreover, let $M \in Max(R)$. Then, $MR_M = (t)$ for some $0 \neq t \in R_M$ with $t^n = 0$ for some minimal integer $n \geq 2$. So, the only nonzero ideals of R_M are (t^k) where $k = 1, \ldots, n-1$, and one can easily check that

$$Ann_{R_M}(Ann_{R_M}(t^k)) = Ann_{R_M}(t^{n-k}) = (t^k).$$

Therefore, R_M is semi-regular and thus Theorem 3.2.1 leads to the conclusion. For necessity, in view of Theorem 3.2.1, we only need to consider the case when R is an arithmetical residually semi-regular ring and check that R is principal Artinian. Indeed, let $M \in Max(R)$. So, R_M is a chained Noetherian ring. If R_M is a domain, then it is semi-regular (since R is not a domain) and a fortiori a field. If R_M is not a domain, assume P is a non-maximal prime ideal of R_M. Then,

$$0 \subsetneq P \subseteq \cap_{n \geq 1} M^n R_M = 0$$

which is absurd. So, in both cases, we have $\dim(R_M) = 0$. Consequently, $\dim(R) = 0$ and thus R is Artinian. It follows that R is principal by the structure theorem for Artinian rings (since the arithmetical property is stable under factor rings), completing the proof of the corollary. □

Another application of Theorem 3.2.1 shows that, in the class of semi-regular rings, the arithmetical property coincides with the notion of residually semi-regular ring.

Corollary 3.3.3 *Let R be a semi-regular ring. Then, the following assertions are equivalent*

(1) *R is arithmetical;*

(2) *R is residually semi-regular.*

Proof. Combine Theorem 3.2.1 with Lemma 3.2.8 for sufficiency and [44, Proposition 2.1] for necessity. □

We will appeal to this corollary, in the next section, to provide new examples of residually semi-regular rings, arising as arithmetical semi-regular rings.

3.4 Examples: trivial ring extensions subject to residually semi-regularity

We first provide an example of a coherent arithmetical ring which is not residually semi-regular. This shows that the assumption "R_M is semi-regular for every $M \in Max(R)$ such that $rR_M = 0$ for some nonzero $r \in R$" within Condition $(\mathscr{C}_2)$ of Theorem 3.2.1 is not redundant with the arithmetical property; and then Couchot's result [15, Theorem 11] that "*a chained ring is residually semi-regular*" does not carry up to arithmetical rings.

Recall that a ring is semi-hereditary if all its finitely generated ideals are projective. We have the following (irreversible) implications [10, 25, 26]:

Prüfer domain $\qquad\qquad\qquad\qquad$ arithmetical ring

$\searrow \qquad\qquad \nearrow$

semi-hereditary ring

$\nearrow \qquad\qquad \searrow$

von Neumann regular ring $\qquad\qquad\qquad\qquad$ coherent ring

Example 3.4.1 Let R be a semi-hereditary ring which is neither a (Prüfer) domain nor a von Neumann regular ring. First, note that, for every $M \in Max(R)$, R_M is a (valuation) domain and hence there is $0 \neq r \in R$ such that $\frac{r}{1} = 0$ in R_M, since R has zero-divisors. Moreover, a semi-regular domain is necessarily a field. So, there is $M \in Max(R)$ such that R_M is not semi-regular, since R is not von Neumann regular. Consequently, R is a coherent arithmetical ring which does not satisfy $(\mathscr{C}_2)$ of Theorem 3.2.1; that is, R is not residually semi-regular, as desired.

Next, we use Theorem 3.2.1 to construct original examples of non-local coherent residually semi-regular rings beyond Matlis', Levy's, and Couchot's contexts. For this purpose, we investigate the transfer of this notion to trivial extensions. Recall that the trivial extension of a ring A by an A-module E is the ring $R := A \ltimes E$, where the underlying group is $A \times E$ and the multiplication is given by $(a,e)(b,f) = (ab, af + be)$.

The next result investigates the transfer of the notion of residually semi-regular ring to trivial ring extensions issued from local rings.

Proposition 3.4.2 *Let (A,M) be a local ring, E a nonzero A-module, and $R := A \ltimes E$. Consider the following conditions:*

$(\mathscr{C}_1)$ *A is a field and* $\dim_A(E) \leq 2$.

$(\mathscr{C}_2)$ *$M^2 = 0$ with $l(M) = 1$ and $E \cong A/M$.*

$(\mathscr{C}_3)$ *A is a non-trivial valuation domain, E is a uniserial divisible torsion coherent module, and $Ann_E(x)$ is finitely generated for all $x \in A$.*

Then, R is a coherent residually semi-regular ring if and only if any one of the above three conditions holds.

Proof. Assume that R is residually semi-regular. By Theorem 3.2.1, $(M \ltimes E)^2 = 0$ with $l(M \ltimes E) = 2$ or R is a chained ring. The first case yields $M^2 = 0$ and $ME = 0$ (i.e., E is an A/M-vector space) with $l(M) + l(E) = 2$. It follows that either A is a field with $l(E) = 2$ (i.e., $\dim_A(E) = 2$) or $l(M) = 1$ and $l(E) = 1$ (i.e., $E \cong A/M$). Next, assume that R is a chained ring. If A is a field, then $\dim_A(E) = 1$ by [6, Theorem 3.1]. If A is not a field, then a combination of [18, Proposition 1.1] and Proposition 1.3.5 leads to the conclusion. Conversely, suppose that $(\mathscr{C}_1)$ or $(\mathscr{C}_2)$ holds. Then, R is coherent by [36, Theorem 2.6], and $(M \ltimes E)^2 = 0$ with $l(M \ltimes E) = 2$. By Theorem 3.2.1, R is residually semi-regular. Next, suppose that $(\mathscr{C}_3)$ holds. By Proposition 1.3.5, R is coherent and, by [18, Proposition 1.1], R is a chained ring and hence residually semi-regular by Theorem 3.2.1. □

Notice that coherent residually semi-regular rings issued via $(\mathscr{C}_1)$ or $(\mathscr{C}_2)$ of Proposition 3.4.2 are necessarily Noetherian. However, one may use $(\mathscr{C}_3)$ to provide examples of non-local non-Noetherian coherent residually semi-regular rings with zero-divisors (i.e., beyond Matlis', Levy's, and Couchot's contexts), as shown below.

Example 3.4.3 Let A be a non-local non-Noetherian Prüfer domain, $E := \frac{Q(A)}{A}$, and $R := A \ltimes E$. Then R is a non-local non-reduced non-Noetherian coherent residually semi-regular ring. Indeed, R is not reduced (as it is the case of any trivial extension) and it is neither local nor Noetherian since A is not. Moreover, R is a semi-regular (and, a fortiori, coherent) ring by Example 1.3.12. Next, let $M \in Max(R)$. Then, $M = \mathfrak{m} \ltimes E$, for some maximal ideal $\mathfrak{m}$ of A and hence

$$R_M = A_{\mathfrak{m}} \ltimes E_{\mathfrak{m}} = A_{\mathfrak{m}} \ltimes \frac{Q(A_{\mathfrak{m}})}{A_{\mathfrak{m}}}$$

with $A_{\mathfrak{m}}$ being a valuation domain. Now, $Q(A_{\mathfrak{m}})$ is a coherent $A_{\mathfrak{m}}$-module (since it is torsion-free) and so is $E_{\mathfrak{m}}$. Moreover, $E_{\mathfrak{m}}$ is clearly a divisible torsion module and

$$Ann_{E_{\mathfrak{m}}}(x) = \overline{(1/x)}A_{\mathfrak{m}}$$

for any nonzero $x \in A_{\mathfrak{m}}$. It follows that R_M is residually semi-regular by Proposition 3.4.2. Consequently, R is locally residually semi-regular and hence residually semi-regular by Corollary 3.3.3, since semi-regularity is stable under localization and the arithmetical notion is a local property.

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