# FIXED POINT THEORY OF MONOTONE MAPPINGS WITH A GRAPH APPROACH

BY


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*To my parents,*
*my sibling and my wife*

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Finally, with great gratitude, I want to thank my family for believing in me. I highly appreciate their tutelage and support.

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# LIST OF ABBREVIATIONS

| | |
|---|---|
| $X$ | An abstract set |
| $T$ | A map |
| $Fix(T)$ | Set of all fixed points of the map $T$ |
| $d(x, y)$ | Distance (space distance) from $x$ to $y$ |
| $(X, d)$ | Metric space |
| $\preceq$ | Partial order |
| $(X, \preceq)$ | Partially ordered set |
| $(X, d, \preceq)$ | Partially ordered metric space |
| $\mathcal{M}(n)$ | The set of all $n \times n$ matrices |
| $\mathcal{H}(n)$ | The set of all $n \times n$ Hermitian matrices |
| $\mathcal{P}(n)$ | The set of all $n \times n$ positive-definite matrices |
| $\\|\cdot\\|$ | The spectral norm |
| $\\|\cdot\\|_1$ | The trace norm |
| $G$ | Directed graph |
| $V(G)$ | The set of vertices |
| $E(G)$ | The set of directed edges |
| $G^{-1}$ | The Conversion of the digraph $G$ |
| $\tilde{G}$ | The undirected graph obtained from $G$ |
| $(X, d, G)$ | A metric space endowed with the digraph $G$ |
| $\bar{d}$ | Extended distance |
| $(X, \bar{d})$ | An extended metric space |

# THESIS ABSTRACT

**NAME:** Mohammed Hamoud Aljohani

**TITLE OF STUDY:** Fixed Point Theory of Monotone Mappings with a Graph Approach

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*In this thesis, we work on some extensions of the Banach Contraction Principle in metric spaces endowed with a partial order or more generally a directed graph. These extensions were initiated by Ran and Reurings in the partially ordered case and by Jachymski in the case of a directed graph. In particular, we give a unified approach that does not involve any binary relation by introducing a new concept called extended distances. This concept leads to a new structure called extended metric spaces. This novel approach is original and may help shed a deeper light into some extensions of the Banach Contraction Principle*

# ملخص الرسالة

**الاسم الكامل**: محمد بن حمود بن سلامه الجهني

**عنوان الرسالة: نظرية النقطة الثابتة للرواسم المطردة من خلال نظرية الرسومات**

**التخصص**: رياضيات

**تاريخ الدرجة العلمية**: أبريل ٢٠١٦

في هذه الرسالة العلمية، تم دراسة نظرية النقطة الثابتة لباناخ وكذلك بعض تمديداتها في الفضاءات المترية المرتبطة بعلاقة الترتيب الجزئية أو بصورة عامة المخططات الموجهة. بدأت هذه التمديدات بدراسة النظرية في الفضاءات المترية المرتبة جزئياً بواسطة ران و رورينق. بعد ذلك قام جشمسكي بتوسعة النظرية من خلال دراسة خواصها في الفضاءات المترية المرتبطة بالمخططات الموجهة. في هذا العمل قمنا بطرح طريقة جديدة لدراسة نظرية باناخ ، تعتمد هذه الطريقة على مفهوم جديد قمنا بتسميته المتريات الممتدة، باستخدام هذا المفهوم استطعنا تعريف بنية رياضية اسميناها الفضاءات المترية الممتدة. كذلك قمنا بإثبات نظرية باناخ للفضاءات المترية الممتدة. هذه المفاهيم جديدة وقد سمحت لنا بالنظر إلى جميع التمديدات لنظرية باناخ كحالة خاصة من نظرية باناخ في الفضاءات المترية الممتدة.

# CHAPTER 1

# INTRODUCTION

Let X be a non-empty set and $T$ be a map from $X$ into itself. A point $x \in X$ is said to be a fixed point of $T$ if $T(x) = x$. The set of fixed points of $T$ will be denoted by $Fix(T)$. Fixed point theory studies the conditions on both $T$ and $X$ that guarantee the existence or both the existence and the uniqueness of a fixed point. Fixed point theorems are considered to be among the most important and powerful tools in mathematics and specially in nonlinear analysis, due to its use across large number of disciplines of mathematics, like analysis, topology, geometry, game theory and set theory, and its applications in other branches of science such as economics and population biology.

Historically, fixed point theory finds its root in 1886 in the work of Poincaré [17] . It can be divided into three main subfields namely topological, metric, and order-theoretic fixed point theories according to the discovery of Brouwer's, Banach's, Tarski's fixed point theorems respectively.

In 1912, Brouwer [12], proved that *any continuous function from the closed unit* *ball in the finite dimensional space $\mathbb{R}^n$ into itself has a fixed point.* Although it is proved in an equivalent form in the work of Poincaré, the proof of the current form was given first in 1909, by Brouwer for n= 3. Then, it is proved for n arbitrary independently by Hadamard in 1910, and Brouwer in 1912. It is considered as one of the major fixed point theorems which is used considerably in solving differential and integral equations. One can see that convexity and compactness of the domain and continuity of the mapping are crucial in Brouwer's fixed point theorem. The drawback with Brouwer's fixed point theorem is the lack of information to how to approximate or approach the fixed point.

In 1922, Banach [8] proved in his thesis the famous and interesting Banach fixed point theorem, also known as the Banach Contraction Principle. It gives conditions which guarantee that, if they are satisfied, the process of iterating a mapping produce a fixed point constructively. The conditions in Banach Contraction Principle are the contraction-Lipschitz condition of the map and the completeness of the metric space. By contrast with Brouwer's fixed point theorem, Banach Contraction Principle provides a constructive method to find a fixed point.

The development of fixed point theory in partially ordered sets was initiated by knaster in 1927 [18]. He proved his fixed point theorem for monotone increasing mappings. In 1939, this result was extended to monotone increasing mappings on complete lattice by Tarski as follow: *any monotone increasing mapping i.e., for all $x, y \in X$ we have $x \preceq y$ implies $T(x) \preceq T(y)$, on a complete lattice i.e., every subset $Y \subseteq X$ has a least upper bound (supremum) and a greatest lower* *bound (infimum) in $X$, has a fixed point. Moreover, the set of all fixed points of $T$ is a complete sublattice of $X$.*

Beside the extension of Knaster's theorem, Tarski's fixed point theorem has some important applications in many areas of mathematics such as set theory and topology as well as in artificial intelligence. It is worth mentioning that Tarski's fixed point theorem was unpublished until 1955 [18].

Recently a new direction has been attracting some attention dealing with the extension of the Banach Contraction Principle to metric spaces endowed with a partial order or a directed graph. This excitement followed the publication of Ran and Reurings paper [14]. The ideas behind the main fixed point theorem of [14] are found in the original paper [9]. In particular, the authors of [9, 14] showed how this extension is useful when solving some special matrix equations for which the technique involved in solving them is similar to the one used in the Banach Contraction Principle. Following the publication of [14], Nieto and Rodríguez-López [13] extended the conclusion of [14] by dropping the continuity assumption of the map. Then used such arguments in solving some differential equations. Jachymski [10] was the first one to give an extension of the Banach Contraction Principle in metric spaces endowed with a directed graph. Jachymski's work obviously extends the ideas developed by Ran and Reurings as well as Nieto and Rodríguez-López. Since then many mathematicians used Jachymski's framework to come up with more fixed point results [2, 3, 4, 5, 6, 11]. Before we close these historical facts, let us point out that the first attempt to generalize the Banach Contraction Principle to partially ordered metric spaces was carried by Turinici in [15, 16].

In this thesis, we unify these extensions by considering a new concept of metric spaces endowed with a general distance allowed to take the value infinity.

# CHAPTER 2

# PRELIMINARIES

The purpose of this chapter is to provide some basic concepts and results of metric spaces and metric spaces endowed with a partial order or a digraph which will be used throughout this thesis.

## 2.1 Metric Spaces

### 2.1.1 Definitions and Basic Concepts

Since we are going to study the metric fixed point theory, particularly Banach Contraction Principle and its extensions, we will start by giving the definition of the underlying structure.

**Definition 2.1** *Let $X$ be a non-empty set. A mapping $d : X \times X \to [0, \infty)$ is called a* ***metric or a distance*** *on $X$ if the following conditions are satisfied:*

*(i)* $d(x, y) = 0 \Leftrightarrow x = y$;

*(ii)* $d(y,x) = d(x,y)$*, for all* $x, y \in X$*; (symmetry)*

*(iii)* $d(x,z) \leq d(x,y) + d(y,z)$*, for all* $x, y, z \in X$ *(the triangle inequality).*

*A set* $X$ *endowed with a metric* $d$ *is called* ***metric space*** *and it is denoted by* $(X, d)$*.*

**Definition 2.2** *Let* $(X, d)$ *be a metric space.*

*(i) A sequence* $\{x_n\}$ *in* $X$ *is said to be* ***convergent*** *if there exists* $x \in X$ *such that* $\lim\limits_{n \to +\infty} d(x_n, x) = 0$*.*

*(ii) A sequence* $\{x_n\}$ *in* $X$ *is said to be* ***Cauchy*** *if for all* $\varepsilon > 0$*, there exists* $N \in \mathbb{N}$ *such that* $d(x_n, x_m) < \varepsilon$*, for all* $m, n \geq N$*.*

*(iii)* $(X, d)$ *is said to be* ***complete*** *whenever any Cauchy sequence in* $X$ *is convergent in* $X$*.*

**Lemma 2.1** *Let* $(X, d)$ *be a metric space. If the sequence* $\{x_n\}$ *in* $X$ *is convergent, it's limit is unique.*

### 2.1.2 Banach Contraction Principle

The classical Banach Contraction Principle appeared first in Banach's thesis and used to prove the existence of a solution to an integral equation. It is applicable to vast branches of mathematics. This is because the simplicity of its assumptions which are the completeness of the metric space and the Lipschitz condition of the mapping.

**Definition 2.3** *Let $(X, d)$ be a metric space. The map $T : X \to X$ is called* ***Lipschitzian*** *if there exists a constant $k > 0$ (called* ***Lipschitz constant****) such that*

$$d(T(x), T(y)) \leq k\ d(x, y),$$

*for all $x, y \in X$. When $k < 1$, we will say $T$ is a* ***contraction****.*

Now we are ready to state the well-known Banach Contraction Principle.

**Theorem 2.1** *[8] Let $(X, d)$ be a complete metric space and let $T : X \to X$ be a contraction mapping. Then $T$ has a unique fixed point $x_0$, and for each $x \in X$, we have*

$$\lim_{n\to\infty} T^n(x) = x_0.$$

*Moreover, for each $x \in X$, we have*

$$d(T^n(x), x_0) \leq \frac{k^n}{1-k} d(T(x), x).$$

**Proof.** Since $T$ is a contraction, there exists $k \in [0, 1)$ such that

$$d(T(x), T(y)) \leq k\ d(x, y),$$

for all $x, y \in X$. Hence

$$d(T^n(x), T^n(y)) \leq k^n\ d(x, y),$$

for all $x, y \in X$ and $n \in \mathbb{N}$. In particular, we have

$$d(T^{n+1}(x), T^n(x)) \leq k^n \; d(T(x), x),$$

for all $x \in X$ and $n \in \mathbb{N}$. Using the triangle inequality of the distance $d$, we get

$$\begin{aligned} d(T^n(x), T^{n+h+1}(x)) &\leq \sum_{i=n}^{n+h} d(T^i(x), T^{i+1}(x)). \\ &\leq \sum_{i=n}^{n+h} k^i d(x, T(x)). \\ &= k^n \frac{1-k^h}{1-k} d(x, T(x)), \end{aligned}$$

for any $x \in X$ and $n, h \in \mathbb{N}$. Since $k < 1$, we get $1 - k^h < 1$, for any $h \in \mathbb{N}$. Hence

$$d(T^n(x), T^{n+h}(x)) \leq \frac{k^n}{1-k} d(x, T(x)), \tag{1}$$

for any $x \in X$ and $n, h \in \mathbb{N}$. Since $k < 1$, we have $\lim\limits_{n \to +\infty} k^n = 0$. In combination with (1), we obtain that $\{T^n(x)\}$ is a Cauchy sequence. Since $X$ is complete, there exists $x_0 \in X$ such that $\lim\limits_{n \to \infty} T^n(x) = x_0$. Next, we show that $x_0$ is fixed point of $T$. We have

$$d(T^{n+1}(x), T(x_0)) \leq k \; d(T^n(x), x_0),$$

for any $n \in \mathbb{N}$. This inequality will imply that $\{T^{n+1}(x)\}$ converges to $T(x_0)$. Since it also converges to $x_0$ being a subsequence of $\{T^n(x)\}$, the uniqueness of the limit will force $T(x_0) = x_0$, i.e., $x_0$ is a fixed point of $T$ as claimed. Next, we prove the uniqueness of the fixed point of $T$. Let $z \in X$ be another fixed point of $T$. Then we have

$$d(x_0, z) = d(T(x_0), T(z)) \leq kd(x_0, z)$$

Since $k < 1$, we obtain $d(x_0, z) = 0$, i.e., $x_0 = z$. Therefore, the fixed point $x_0$ is independent of the point $x \in X$. In other words, we have

$$\lim_{n \to \infty} T^n(x) = x_0,$$

for any $x \in X$. Moreover, if we let $h \to +\infty$ in the inequality (1), we obtain

$$d(T^n(x), x_0) \leq \frac{k^n}{1-k} d(T(x), x),$$

for all $x \in X$ and $n \in \mathbb{N}$. ∎

## 2.2 Partially Ordered Metric Spaces

In some applications, the contraction condition satisfied by the mapping in the Banach Contraction Principle may not be satisfied on the entire space but only on some subset. The existence as well as the uniqueness of a fixed point is worthy of investigation. This is the case with Ran and Reurings extension of the Banach Contraction Principle in metric spaces endowed with a partial order.

### 2.2.1 Basic Definitions and Notations

**Definition 2.4** *Let $X$ be a set. An* ***order*** *(or* ***partial order****) on $X$ is a binary relation $\preceq$ on $X$ such that, for all $x, y, z \in X$, we have*

*(i)* $x \preceq x$ *(reflexivity),*

*(ii)* $x \preceq y$ *and* $y \preceq x$ *implies* $x = y$ *(antisymmetry),*

*(iii)* $x \preceq y$ *and* $y \preceq z$ *implies* $x \preceq z$ *(transitivity).*

*A set $X$ equipped with an ordered relation $\preceq$ is said to be a* ***partially ordered set*** *and it is denoted by $(X, \preceq)$. $(X, \preceq)$ is called*

*(i) A* ***lattice*** *if each two-element subset $\{x, y\} \subseteq X$ has a least upper bound (supremum) and a greatest lower bound (infimum) in X.*

*(ii) A* ***complete lattice*** *if every subset $Y \subseteq X$ has a supremum and an infinimum in X.*

A metric space $(X, d)$ endowed with a partial order $\preceq$ will be known as a **partially ordered metric space** and will be denoted by $(X, d, \preceq)$.

**Definition 2.5** *Let $(X, \preceq)$ be an ordered set. A mapping $T : X \to X$ is said to be* ***monotone increasing*** *(resp.* ***decreasing****) if for all $x, y \in X$, we have $x \preceq y$ implies $T(x) \preceq T(y)$ (resp. $T(y) \preceq T(x)$). $T$ is called monotone if it is monotone increasing or decreasing.*

Recall the Knaster-Tarski fixed point theorem:

**Theorem 2.2** *[18] Let $(X, \preceq)$ be a complete lattice. Suppose that $T : X \to X$ is monotone increasing. Then $Fix(T)$, the set of fixed points of $T$, is a nonempty complete sublattice of $X$.*

### 2.2.2 Banach Contraction Principle in Partially Ordered Metric Spaces

In this subsection, we discuss the extensions of the Banach Contraction Principle in partially ordered metric spaces based on Ran and Reurings [14] and Nieto and Rodríguez-Lopez[13] articles.

**Definition 2.6** *Let $(X, d, \preceq)$ be a partially ordered metric space. A mapping $T : X \to X$ is said to be* ***monotone Lipschitzian*** *if $T$ is monotone and there exists $k \geq 0$ such that*

$$d(T(x), T(y)) \leq k\ d(x, y),$$

*for any $x, y \in X$ such that $x \preceq y$. If $k < 1$, then $T$ is called a* ***monotone contraction****.*

As we said earlier, the contraction condition is not satisfied on the entire space but only on comparable elements. In particular, monotone Lipschitzian mappings may not be continuous while Lipschitzian mappings are not only continuous, but they are uniformly continuous.

Let $(X, d, \preceq)$ be a partially ordered metric space and $T : X \to X$ be a monotone contraction. Clearly if $x_0$ is a fixed point of $T$, then we have $x_0 \preceq T(x_0)$ and

$T(x_0) \preceq x_0$. Next, we build a simple example of a monotone contraction $T$ such that $T(x)$ and $x$ are not comparable for any $x \in X$. In this case $T$ has no fixed point.

**Example 2.2.1** *Consider the Euclidean plane* $(\mathbb{R}^2, d)$. *Define* $L_i = \{(i, y);\ y \in \mathbb{R}\}$, *for* $i = 1, 2$. *It is clear that* $(L_1 \cup L_2, d)$ *is a complete metric space. Define the partial order on* $L_1 \cup L_2$ *by*

$$(x_1, y_1) \preceq (x_2, y_2)\ if\ and\ only\ if x_1 = x_2\ and\ y_1 \leq y_2.$$

*Now we want to define a monotone function which is contraction only on comparable elements. Consider the mapping* $T : L_1 \cup L_2 \to L_1 \cup L_2$ *defined by*

$$\begin{cases} T(1, y) = (2, \frac{1}{2}y); \\ T(2, y) = (1, \frac{1}{2}y). \end{cases}$$

*Since any point in* $L_1$ *and any point in* $L_2$ *are not comparable, there is no element* $(x, y) \in L_1 \cup L_2$ *such that* $(x, y)$ *and* $T(x, y)$ *are comparable. Therefore,* $T$ *has no fixed point. Moreover,* $T$ *is a monotone contraction. Indeed, let* $(x_1, y_1) \preceq (x_2, y_2)$. *Then we have* $x_1 = x_2$ *and* $y_1 \leq y_2$. *Set* $x = x_1 = x_2$. *Hence*

$$d(T(x, y_1), T(x, y_2)) = \sqrt{(x - x)^2 + \left(\frac{1}{2}y_1 - \frac{1}{2}y_2\right)^2} = \frac{1}{2}\sqrt{(y_1 - y_2)^2} = \frac{1}{2}d((x, y_1), (x, y_2)),$$

*i.e.,* $T$ *is a contraction on comparable elements with* $k = \frac{1}{2}$. *It is clear that* $T$ *is not a contraction on the whole* $L_1 \cup L_2$.

Therefore, to hope for the existence of a fixed point of $T$, we must assume that there exists $x_0 \in X$ such that $x_0$ and $T(x_0)$ are comparable. Hence it is natural to ask what assumptions are needed to make sure that $T$ has a fixed point. In order to understand what was done for monotone contraction mappings, we will need the following technical lemma.

**Lemma 2.2** *Let $(X, d, \preceq)$ be a partially ordered metric space and $T : X \to X$ be a monotone contraction. If $x$ and $y$ are comparable, then we have*

$$d(T^n(x), T^n(y)) \leq k^n \; d(x, y),$$

*for any $n \in \mathbb{N}$. In particular, if there exists $x_0 \in X$ such that $x_0$ and $T(x_0)$ are comparable, then the orbit $\{T^n(x_0)\}$ is a Cauchy sequence.*

**Proof.** Since $x$ and $y$ are comparable and $T$ is monotone, we deduce that $T^n(x)$ and $T^n(y)$ are comparable. Since $T$ is a monotone contraction, there exists $k \in [0, 1)$ such that

$$d(T(u), T(v)) \leq k \; d(u, v),$$

for any comparable elements $u, v \in X$. Hence

$$d(T^n(x), T^n(y)) \leq k \; d(T^{n-1}(x), T^{n-1}(y)) \leq k^n \; d(x, y),$$

for any $n \in \mathbb{N}$. Next, let $x_0 \in X$ such that $x_0$ and $T(x_0)$ are comparable. Hence

$$d(T^n(x_0), T^{n+1}(x_0)) \leq k^n \; d(x_0, T(x_0)),$$

for any $n \in \mathbb{N}$. As we have done in the proof of the Banach Contraction Principle, we obtain

$$d(T^n(x_0), T^{n+h}(x_0)) \leq \frac{k^n}{1-k}\ d(x_0, T(x_0)),$$

for any $n, h \in \mathbb{N}$, which implies that $\{T^n(x_0)\}$ is a Cauchy sequence. ∎

Using Lemma 2.2, we obtain the following result.

**Theorem 2.3** *Let $(X, d, \preceq)$ be a complete partially ordered metric space. Let $T : X \to X$ be a monotone contraction. Assume there exists $x_0 \in X$ such that $x_0$ and $T(x_0)$ are comparable. Then $\{T^n(x_0)\}$ converges to $\omega \in X$. Moreover, for any $x \in X$ comparable with $x_0$, we have $\lim_{n\to+\infty} T^n(x) = \omega$.*

**Proof.** Lemma 2.2 implies that $\{T^n(x_0)\}$ is a Cauchy sequence. Since $X$ is complete, then $\{T^n(x_0)\}$ is convergent. Set $\omega = \lim_{n\to+\infty} T^n(x_0)$. Let $x \in X$ be comparable to $x_0$. Again Lemma 2.2 implies

$$d(T^n(x_0), T^n(x)) \leq k^n\ d(x_0, x),$$

for any $n \in \mathbb{N}$. In particular, we have $\lim_{n\to+\infty} d(T^n(x_0), T^n(x)) = 0$ since $k < 1$, which implies $\lim_{n\to+\infty} T^n(x) = \omega$. ∎

Following the conclusion of Theorem 2.3, it is natural to wonder when the limit of an orbit is a fixed point. The first such result is given by Ran and Reurings.

**Theorem 2.4** *Let $(X, d, \preceq)$ be a complete partially ordered metric space. Let $T : X \to X$ be a continuous monotone contraction. Assume there exists $x_0 \in X$*

*such that $x_0$ and $T(x_0)$ are comparable. Then $\{T^n(x_0)\}$ converges to a fixed point $\omega$ of $T$.*

**Proof.** Theorem 2.3 implies that $\{T^n(x_0)\}$ converges to $\omega \in X$. Since $T$ is continuous, we get

$$\omega = \lim_{n \to +\infty} T^{n+1}(x_0) = \lim_{n \to +\infty} T(T^n(x_0)) = T(\omega),$$

i.e., $\omega$ is a fixed point of $T$. ∎

In order to fully have an extension of the Banach Contraction Principle, we need to discuss the uniqueness of the fixed point of $T$. Note that under the assumptions of Theorem 2.4, we have

$$\lim_{n \to +\infty} T^n(x) = \omega,$$

for any $x \in X$ comparable with $x_0$. Therefore, if $X$ is lattice, then for any $x \in X$ not necessarily comparable to $x_0$, there exists $z \in X$ such that $x \preceq z$ and $x_0 \preceq z$ (here we only used the existence of a supremum of two elements of $X$). Then we know that $\lim_{n \to +\infty} T^n(z) = \omega$ and using Lemma 2.2, we also have $\lim_{n \to +\infty} d(T^n(z), T^n(x)) = 0$ which implies $\lim_{n \to +\infty} T^n(x) = \omega$. In other words, we have $\lim_{n \to +\infty} T^n(x) = \omega$, for any $x \in X$. This is exactly the main conclusion of the Banach Contraction Principle.

Now we are ready to state Ran and Reurings' extension of the Banach Contraction Principle in partially ordered metric spaces.

**Theorem 2.5** *[14] Let $(X, d, \preceq)$ be a complete partially ordered metric space. Assume that $(X, \preceq)$ is a lattice. Let $T : X \to X$ be a continuous monotone contraction. Assume there exists $x_0 \in X$ such that $x_0$ and $T(x_0)$ are comparable. Then $\{T^n(x_0)\}$ converges to the unique fixed point $\omega$ of $T$. Moreover, we have*

$$\lim_{n \to +\infty} T^n(x) = \omega,$$

*for any $x \in X$.*

The motivation behind Theorem 2.5 is an example that deals with some matrix equations. This example is given next. First, we will need some notations:

- $\mathcal{M}(n)$ : the set of all $n \times n$ matrices.

- $\mathcal{H}(n) \subseteq \mathcal{M}(n)$ : the set of all $n \times n$ Hermitian matrices.

- $\mathcal{P}(n) \subseteq \mathcal{H}(n)$ : the set of all $n \times n$ positive definite matrices. Also, for $X \in \mathcal{P}(n)$ we will denote it by $X > 0$.

- $X \geq 0$ : means that $X$ is positive semi-definite.

- $X \leq Y$ if and only if $0 \leq Y - X$.

- $\| \cdot \|$ is the spectral norm defined by $\|A\| = \sqrt{\lambda^+(A^*A)}$, where $\lambda^+(A^*A)$ is the largest eigenvalue of $A^*A$.

- $\| \cdot \|_1$ is the trace norm defined by $\|A\|_1 = tr(A) = \sum_{j=1}^{n} s_j(A)$, where $s_j(A), j = 1, 2, 3, ..., n$ are the singular values of $A$, i.e., the square roots of the eigenvalues of $A^*A$.

- Fix $Q \in \mathcal{P}(n)$, the norm $\|A\|_{1,Q}$ is defined to be equal to $\|Q^{\frac{1}{2}}AQ^{\frac{1}{2}}\|_1$.

**Example 2.2.2** *[14] Does the following matrix equation have a unique Hermitian solution?*

$$X = Q \pm \sum_{j=1}^{m} A_j^* \mathcal{F}(X) A_j, \tag{2}$$

*where $Q$ is a positive definite matrix and $A_1, ..., A_m$ are arbitrary $n \times n$ matrices, and where $\mathcal{F}$ is a continuous and monotone map such that $\mathcal{F}(\mathcal{P}(n)) \subseteq \mathcal{P}(n)$.*

*Such equation usually arises in variety of applications for example dynamic programming and stochastic filtering. Note that, if we define $G_{\pm} : \mathcal{H}(n) \longrightarrow \mathcal{H}(n)$, where $\mathcal{H}(n)$ is the set of all $n \times n$ Hermitian matrices, by*

$$G_{\pm}(X) = Q \pm \sum_{j=1}^{m} A_j^* \mathcal{F}(X) A_j.$$

*Then equation (2) is a fixed point problem since $X$ is solution to (2) if and only if $G_{\pm}(X) = X$.*

*Since $\mathcal{H}(n)$ is a finite dimensional vector space, the distance $d$ induced by the norm $\|A\|_{1,Q}$ is complete. Also it is easy to check that $(\mathcal{H}(n), \leq)$ is a lattice. For the sake of simplicity, we consider the case when $\mathcal{F}(X) = X$ in (2). In order to prove that $G_{\pm}$ is a monotone contraction, we will need the following lemma:*

***Lemma 2.3*** *[14] Let $A, B$ be two $n \times n$ matrices such that $A \geq 0$ and $B \geq 0$. Then $0 \leq tr(AB) \leq \|A\| \cdot tr(B)$.*

*We will prove the contraction property for $G_+$. The same argument can be applied for $G_-$. Let $\tilde{Q} \in \mathcal{P}(n)$ and $X, Y \in \mathcal{H}(n)$ such that $X \leq Y$. Then $G_+(X) \leq G_+(Y)$ and*

$$\begin{aligned}
\|G_+(Y) - G_+(X)\|_{1,\tilde{Q}} &= \|\tilde{Q}^{\frac{1}{2}}(G_+(Y) - G_+(X))\tilde{Q}^{\frac{1}{2}}\|_1 \\
&= tr[\tilde{Q}^{\frac{1}{2}}(Q + \sum_{j=1}^{m} A_j^* Y A_j - Q + \sum_{j=1}^{m} A_j^* X A_j)\tilde{Q}^{\frac{1}{2}}] \\
&= tr[\sum_{j=1}^{m} \tilde{Q}^{\frac{1}{2}} A_j^*(Y - X)A_j\tilde{Q}^{\frac{1}{2}}] \\
&= \sum_{j=1}^{m}[tr(\tilde{Q}^{\frac{1}{2}} A_j^*(Y - X)A_j\tilde{Q}^{\frac{1}{2}})] \\
&= \sum_{j=1}^{m}[tr(A_j^*\tilde{Q}A_j(Y - X))] \\
&= \sum_{j=1}^{m}[tr(A_j^*\tilde{Q}A_j\tilde{Q}^{\frac{-1}{2}}\tilde{Q}^{\frac{1}{2}}(Y - X)\tilde{Q}^{\frac{1}{2}}\tilde{Q}^{\frac{-1}{2}})] \\
&= \sum_{j=1}^{m}[tr(\tilde{Q}^{\frac{-1}{2}} A_j^*\tilde{Q}A_j\tilde{Q}^{\frac{-1}{2}}(\tilde{Q}^{\frac{1}{2}}(Y - X)\tilde{Q}^{\frac{1}{2}}))] \\
&= tr[\sum_{j=1}^{m}(\tilde{Q}^{\frac{-1}{2}} A_j^*\tilde{Q}A_j\tilde{Q}^{\frac{-1}{2}}(\tilde{Q}^{\frac{1}{2}}(Y - X)\tilde{Q}^{\frac{1}{2}}))].
\end{aligned}$$

*Using Lemma 2.3, we get*

$$\|G_+(Y) - G_+(X)\|_{1,\tilde{Q}} \leq \|\sum_{j=1}^{m} \tilde{Q}^{\frac{-1}{2}} A_j^*\tilde{Q}A_j\tilde{Q}^{\frac{-1}{2}}\| \cdot \|Y - X\|_{1,\tilde{Q}}.$$

*If we assume $\sum_{j=1}^{m} A_j^*\tilde{Q}A_j < \tilde{Q}$, then $k = \|\sum_{j=1}^{m} \tilde{Q}^{\frac{-1}{2}} A_j^*\tilde{Q}A_j\tilde{Q}^{\frac{-1}{2}}\| < 1$, i.e., $G_+$ is a contraction. Since $G_+(0) = Q > 0$, by Theorem 2.5, we conclude that the matrix equation (2) has a unique solution $\tilde{X}$ and for any matrix $X \in \mathcal{H}(n)$, we have $\lim_{n\to+\infty} G_+^n(X) = \tilde{X}$. This result is useful when dealing with approximation of the solution of the equation (2).*

Nieto and Rodríguez-Lopez [13] weakened the continuity assumption in Theorem 2.5 by assuming that the partial order and the distance enjoy a nice property for monotone sequences. Indeed, the continuity of the mapping was used to prove that the limit of the orbit is a fixed point while the uniqueness was obtained under the assumption that the space is a lattice though the proof suggests that we only need to assume that any two elements have either an upper bound or a lower bound. Looking at the specific example of continuous functions, which Nieto and Rodríguez-Lopez used as their underlying metric space in their example, it is easy to see that any convergent monotone increasing sequence is lower than their limit and any convergent monotone decreasing sequence is greater than their limit. The set of continuous functions is very commonly used when dealing with differential or integral equations. It is this property that Nieto and Rodríguez-Lopez assumed to obtain their fixed point result.

*A partially ordered metric space $(X, d, \preceq)$ is said to satisfy $(P_1)$ if and only if any increasing (resp. decreasing) sequence $\{x_n\}$, if $x_n \to x$, then $x_n \preceq x$ (resp. $x \preceq x_n$) for all $n \in \mathbb{N}$.*

Next we give Nieto and Rodríguez-Lopez's fixed point result.

**Theorem 2.6** *[13] Let $(X, d, \preceq)$ be a complete partially ordered metric space. Assume that $(X, d, \preceq)$ satisfies the property $(P_1)$. Let $T : X \to X$ be a monotone increasing contraction. If there exist $x_0 \in X$ such that $x_0$ and $T(x_0)$ are com-*

*parable, then $T$ has a fixed point. Moreover, if $(X, \preceq)$ is such that every pair of elements of $X$ has an upper or a lower bound, then $f$ is a Picard Operator, i.e., $T$ has a unique fixed point $\omega$ and $\lim_{n \to +\infty} T^n(x) = \omega$, for any $x \in X$.*

**Proof.** Using Theorem 2.3, we know that the orbit $\{T^n(x_0)\}$ is convergent to $\omega$. Without loss of generality, we assume that $x_0 \preceq T(x_0)$. In this case, the sequence $\{T^n(x_0)\}$ is monotone increasing. Using the property $(P_1)$, we conclude that $T^n(x_0) \preceq \omega$, for any $n \in \mathbb{N}$. Since $T$ is a monotone contraction, there exists $k \in [0, 1)$ such that

$$d(T(x), T(y)) \leq k\ d(x, y),$$

for any $x, y \in X$ such that $x \preceq y$. In particular, we have

$$d(T^{n+1}(x_0), T(\omega)) \leq k\ d(T^n(x_0), \omega),$$

for any $n \in \mathbb{N}$. This inequality implies that $\lim_{n \to +\infty} T^{n+1}(x_0) = T(\omega)$. Since $\{T^{n+1}(x_0)\}$ is a subsequence of $\{T^n(x_0)\}$, we conclude that $\omega = T(\omega)$, i.e., $\omega$ is a fixed point of $T$. The proof shows that $\lim_{n \to +\infty} T^n(x) = \omega$, for any $x \in X$, follows the same ideas developed before. ∎

**Remark 2.2.1** *A careful look at the proof of Theorem 2.6 suggests that we can weaken the property $(P_1)$ and still get the same conclusion. The weaker property is:*

*A partially ordered metric space $(X, d, \preceq)$ is said to satisfy $(P_1')$ if and only if any increasing (resp. decreasing) sequence $\{x_n\}$, if $x_n \to x$, then $x_{\varphi(n)} \preceq x$ (resp.*

$x \preceq x_{\varphi(n)}$) *for all* $n \in \mathbb{N}$, *for some subsequence* $\{x_{\varphi(n)}\}$ *of* $\{x_n\}$.

We finish this subsection by discussing the example given by Nieto and Rodríguez-Lopez.

**Example 2.2.3** *[13] Consider the following differential equation*

$$\begin{cases} u'(t) = f(t, u(t)), \qquad t \in I = [0, T]; \\ u(0) = u(T), \end{cases} \tag{3}$$

*where* $T > 0$ *and the continuous function* $f : I \times \mathbb{R} \to \mathbb{R}$ *are given. The task in this example is to prove the existence and the uniqueness of a solution for the first-order periodic problem (3) by converting the differential equation to an integral equation which is a fixed point problem. The following technical assumption is needed:*

> ***Assumption:*** *Assume that there exist* $\lambda > 0$ *and* $\mu > 0$, *with* $\lambda > \mu$ *such that*
>
> $$0 \leq f(t, u) + \lambda u - [f(t, v) + \lambda v] \leq \mu(u - v). \tag{4}$$
>
> *for all* $u, v \in \mathbb{R}$ *such that* $u \geq v$ *and all* $t \in I$.

*We can rewrite (3) to get*

$$\begin{cases} u'(t) + \lambda u(t) = f(t, u(t)) + \lambda u(t), \qquad t \in I = [0, T]; \\ u(0) = u(T). \end{cases} \tag{5}$$

*Note that a solution of (3) must be in $C^1(I,\mathbb{R})$, the set of all continuously differentiable functions on $I=[0,T]$. Equation (5) is equivalent to the integral equation*

$$u(t)=\int_0^T G(t,s)[f(t,u(s))+\lambda u(s)]ds,$$

*where $G(t,s)$ is the Green function*

$$G(t,s)=\begin{cases}\dfrac{e^{\lambda(T+s-t)}}{e^{\lambda T}-1}, & 0\le s<t\le T;\\ \\ \dfrac{e^{\lambda(s-t)}}{e^{\lambda T}-1}, & 0\le t<s\le T.\end{cases}$$

*Now define $\mathcal{A}:C(I,\mathbb{R})\to C(I,\mathbb{R})$ by*

$$[\mathcal{A}u](t)=\int_0^T G(t,s)[f(t,u(s))+\lambda u(s)]ds. \tag{6}$$

*It is clear that $u\in C(I,\mathbb{R})$, the set of all continuous functions on $I$, is a solution of (6) if and only if $\mathcal{A}u=u$ if and only if $u\in C^1(I,\mathbb{R})$ is a solution to (5) and (4). Let us investigate the properties of the mapping $\mathcal{A}$. First, it is clear that the underlying set $X=C(I,\mathbb{R})$ is a complete metric space with the distance defined by*

$$d(u,v)=\sup_{t\in I}|u(t)-v(t)|,$$

*for any $u,v\in C(I,\mathbb{R})$. Note that $C(I,\mathbb{R})$ is also partially ordered by*

$$u\preceq v \quad if\ and\ only\ if \quad u(t)\le v(t),\ for\ all\ t\in I.$$

*Therefore,* $(C(I,\mathbb{R}),d)$ *is a complete partially ordered metric space. Since the infimum and supremum of two functions in* $C(I,\mathbb{R})$ *are in* $C(I,\mathbb{R})$*, we conclude that* $C(I,\mathbb{R})$ *is a lattice. It is also straightforward to show that* $C((I,\mathbb{R}),d,\preceq)$ *satisfies the property (*$P_1$*). Next, we show that the map* $\mathcal{A}$ *is a monotone contraction.*

Monotonicity: *let* $u,v\in C(I,\mathbb{R})$ *be such that* $u\preceq v$*, then*

$$f(t,u(t))+\lambda u(t)\leq f(t,v(t))+\lambda v(t),$$

*for all* $t\in I$*. Since* $G(t,s)>0$*, we get*

$$\begin{aligned}[\mathcal{A}u](t) &= \textstyle\int_0^T G(t,s)[f(t,u(s))+\lambda u(t)]ds.\\ &\leq \textstyle\int_0^T G(t,s)[f(t,v(s))+\lambda v(t)]ds\\ &= [\mathcal{A}v](t),\end{aligned}$$

*for any* $t\in I$*. So* $\mathcal{A}$ *is monotone increasing.*

Contraction: *let* $u, v \in C(I, \mathbb{R})$ *be such that* $u \preceq v$*, then*

$$\begin{aligned}
d([\mathcal{A}u],[\mathcal{A}v]) &= \sup_{t\in I} \mid [\mathcal{A}u](t) - [\mathcal{A}v](t) \mid \\
&= \sup_{t\in I} \left| \int_0^T G(t,s)[f(t,u(s)) + \lambda u(s) - f(t,v(s)) - \lambda v(s)]ds \right| \\
&\leq \sup_{t\in I} \int_0^T G(t,s) \mid [f(t,u(s)) + \lambda u(s) - f(t,v(s)) - \lambda v(s)] \mid ds \\
&\overset{by(4)}{\leq} \sup_{t\in I} \int_0^T G(t,s) \mid \mu(u(s) - v(s)) \mid ds \\
&\leq \mu \sup_{t\in I} |u(t) - v(t)| \sup_{t\in I} \int_0^T G(t,s)ds \\
&= \mu\ d(u,v) \sup_{t\in I} \left( \int_0^t \frac{e^{\lambda(T+s-t)}}{e^{\lambda T} - 1} ds + \int_t^T \frac{e^{\lambda(s-t)}}{e^{\lambda T} - 1} ds \right) \\
&= \mu\ d(u,v) \sup_{t\in I} \frac{1}{e^{\lambda T} - 1} \left( \frac{1}{\lambda} e^{\lambda(T+s-t)} \mid_0^t + \frac{1}{\lambda} e^{\lambda(s-t)} \mid_t^T \right) \\
&= \mu\ d(u,v)\ \frac{1}{\lambda}.
\end{aligned}$$

*In other words, we have*

$$d([\mathcal{A}u],[\mathcal{A}v]) \leq \frac{\mu}{\lambda} d(u,v),$$

*for any* $u, v \in C(I, \mathbb{R})$ *be such that* $u \preceq v$*. Since* $k = \dfrac{\mu}{\lambda} < 1$*, we conclude* *that $\mathcal{A}$ is a monotone contraction.*

*In order to prove the existence of a fixed point of $\mathcal{A}$, we will need to assume that a lower solution exists. Recall that a function $u \in C^1(I, \mathbb{R})$ is said to be a* ***lower solution*** *of (3) [13] if*

$$\begin{cases} u'(t) \leq f(t, u(t)), \qquad t \in I = [0, T]; \\ u(0) \leq u(T). \end{cases}$$

*Let $x_0 \in C(I, \mathbb{R})$ be a lower solution. Then, $x_0 \preceq \mathcal{A}(x_0)$. Using Theorem 2.6, we conclude that the differential equation (3) has a unique solution $u^*$ and for any function $u \in C(I, \mathbb{R})$, we have $\lim_{n \to +\infty} \mathcal{A}^n(u) = u^*$. This result is useful when dealing with approximation of the solution of the equation (3).*

## 2.3 Metric Spaces Endowed with Digraph

Once the extensions of the Banach Contraction Principle in metric spaces endowed with a partial order were published, the attention of some mathematicians turned to the case of structures more general than the partial order for which the conclusion of these extensions is still valid. This is the approach taken by Jachymski [10] who used directed graphs. Indeed, it is easy to naturally connect a directed graph to any partial order. Let us initiate this section with the necessary vocabulary and definitions of metric spaces endowed with a directed graph which will be used in our study.

### 2.3.1 Basic Definitions and Concepts

**Definition 2.7** *A* ***directed graph*** *or* ***digraph*** *$G$ consists of a set of objects called vertices, denoted by $V(G)$, together with a (possibly empty) set $E(G)$ of ordered pairs of vertices, called arcs or directed edges. $E(G)$ represents a binary relation on the set of vertices $V(G)$, i.e., $E(G) \subset V(G) \times V(G)$.*

We are concerned here with directed graphs that have a loop at every vertex, i.e., $(v, v) \in E(G)$ for all $v \in V(G)$. Such digraphs are called reflexive. Moreover, we assume that there exists a distance function $d$ defined on the set of vertices $V(G)$. In this case, we may treat $G$ as a weighted digraph by assigning to each edge the distance between its vertices. By $G^{-1}$ we denote the conversion of a digraph $G$, i.e., the digraph obtained from $G$ by reversing the direction of its edges. Thus we have

$$E(G^{-1}) = \{(y, x) | (x, y) \in E(G)\}.$$

A digraph $G$ is called an oriented digraph if whenever $(x, y) \in E(G)$, then $(y, x) \notin E(G)$. The letter $\widetilde{G}$ denotes the undirected graph (or simply graph) obtained from $G$ by disregarding the direction of edges. Actually, it will be more convenient to treat $\widetilde{G}$ as a directed graph for which the set of its edges is symmetric. In this case, we have

$$E(\widetilde{G}) = E(G) \cup E(G^{-1}).$$

Given a digraph $G = (V(G), E(G))$, a (di)path of G is a sequence of distinct vertices $x_0, x_2, ..., x_n, ...$ with $(x_i, x_{i+1}) \in E(G)$ for each $i \in \mathbb{N}$. A finite path

$(x_0, x_1, ..., x_n)$ is said to have length $n+1$, for $n \in \mathbb{N}$. A closed directed path of length $n > 1$ from $x$ to $y$, i.e., $x = y$, is called a directed cycle. An acyclic digraph is a digraph that has no directed cycle. A digraph is connected if there is a finite (di)path joining any two of its vertices and it is weakly connected if $\widetilde{G}$ is connected. Given an acyclic digraph $G$, we can always define a partial order $\preceq_G$ on the set of vertices of $G$ by $x \preceq_G y$ whenever there is a directed path from $x$ to $y$.

We call $(V^*, E^*)$ a subgraph of $G$ if $V^* \subseteq V(G)$, $E^* \subseteq E(G)$ and for any edge $(x, y) \in E^*$, we have $x, y \in V^*$.

If $G$ is such that $E(G)$ is symmetric and $x$ is a vertex in $G$, then the subgraph $G_x$ consisting of all edges and vertices which are contained in some path beginning at $x$ is called the component of $G$ containing $x$. In this case $V(G_x) = [x]_G$, where $[x]_G$ is the equivalence class of the succeeding relation $\mathcal{R}$ defined on $V(G)$ by the rule:

$$y \ \mathcal{R} \ z \text{ if there is a (directed) path in } G \text{ from } y \text{ to } z.$$

Clearly $G_x$ is connected, for any $x \in V(G)$.

**Example 2.3.1** *Let $(X, \preceq)$ be a partially ordered set. We define the oriented graph $G_{\preceq}$ on $X$ by $V(G_{\preceq}) = X$ and for any $x, y \in X$, we have $(x, y) \in E(G_{\preceq})$ if and only if $x \preceq y$. It is easy to check that $G_{\preceq}$ has no parallel arcs as $x \preceq y$ and $y \preceq x$ imply $x = y$.*

**Definition 2.8** *Let $G = (V(G), E(G))$ be a digraph.*

1. *$G$ is **transitive** if*

$$(x, y) \in E(G) \text{ and } (y, z) \in E(G) \Rightarrow (x, z) \in E(G),$$

*for all $x, y, z \in V(G)$.*

2. *A $G$-**interval** is any subset of the form*

   *(i)* $[a, \rightarrow) = \{x \in V(G) : (a, x) \in E(G)\}$,

   *(ii)* $(\leftarrow, a] = \{x \in V(G) : (x, a) \in E(G)\}$,

   *for any $a \in V(G)$.*

In the next example, we discuss a transitive cyclic digraph which can not be generated by a partial order.

**Example 2.3.2** *Consider the Hilbert space $l_2$ defined by*

$$\ell_2 = \{(x_n) \in \mathbb{R}^{\mathbb{N}} : \sum_{n\in\mathbb{N}} | x_n |^2 < +\infty\}$$

*Define the digraph $G$ on $\ell_2$ by:*

$$(x, y) \in E(G) \; if \text{ and only } if \; x_n \leq y_n, \; n \geq 2,$$

*where $x = (x_n)$ and $y = (y_n)$ are in $\ell_2$. Then $G$ is reflexive, transitive for which $G$-intervals are convex and closed. Note that $G$ contains cycles. Indeed, we have*

*$(x, y) \in E(G)$ and $(y, x) \in E(G)$ where*

$$x = (1, 0, 0, ...) \text{ and } y = (2, 0, 0, 0, ...).$$

*Therefore the graph $G$ will not be generated by a partial order.*

Throughout this section, we will denote a metric space $(X, d)$ endowed with a digraph $G$ by $(X, d, G)$, where $V(G) = X$.

**Definition 2.9** *Let $(X, d, G)$ be a metric space endowed with a digraph $G$, and $C$ a nonempty subset of $X$.*

*(i) We say that a mapping $T : C \to C$ is $G$-**edge preserving or $G$-monotone** if for any $x, y \in C$, we have*

$$(x, y) \in E(G) \Rightarrow (T(x), T(y)) \in E(G).$$

*(ii) We say that a mapping $T : C \to C$ is a $G$-**contraction** if $T$ is $G$-edge preserving and there exists $k \in [0, 1)$ such that*

$$(x, y) \in E(G) \Rightarrow d(T(x), T(y)) \leq k \ d(x, y),$$

*for any $x, y \in C$.*

**Definition 2.10** *[10] Let $(X, d)$ be a metric space. We say that sequences $\{x_n\}$ and $\{y_n\}$ are **Cauchy equivalent** if both sequences are Cauchy and*

$\lim_{n \to +\infty} d(x_n, y_n) = 0.$

We have the following technical lemma.

**Lemma 2.4** *[10] Let $(X, d, G)$ be a metric space endowed with a digraph, and $T : X \longrightarrow X$ be a $G$-contraction. Then, given $x \in X$ and $y \in [x]_{\tilde{G}}$, there is $r(x, y) \geq 0$ such that*

$$d(T^n(x), T^n(y)) \leq k^n \; r(x, y)$$

*for all $n \in \mathbb{N}$.*

**Proof.** Let $y \in [x]_{\tilde{G}}$, then there is a finite path $(x_0, x_1, ..., x_m)$ in $\tilde{G}$ from $x = x_0$ to $x_m = y$ such that $(x_i, x_{i+1}) \in E(\tilde{G})$ where $i = 0, 1, ..., m$. Since $T$ is a $G$-contraction, $T$ is $\tilde{G}$-contraction. By induction we have $(T^n(x_i), T^n(x_{i+1})) \in E(\tilde{G})$ and

$$d(T^n(x_i), T^n(x_{i+1})) \leq k^n d(x_i, x_{i+1}),$$

for all $n \in \mathbb{N}$ and $i = 0, 1, ..., m$. Hence, by the triangle inequality we get

$$d(T^n(x), T^n(y)) \leq \sum_{i=0}^{m} d(T^n(x_i), T^n(x_{i+1})) \leq \sum_{i=0}^{m} k^n d(x_i, x_{i+1})$$

for all $n \in \mathbb{N}$ and $i = 0, 1, ..., m$. If we let $r(x, y) = \sum_{i=0}^{m} d(x_i, x_{i+1})$, we get

$$d(T^n(x), T^n(y)) \leq k^n \; r(x, y).$$

∎

**Proposition 2.1** *[10] Let $(X, d, G)$ be a metric space endowed with a digraph, and $T : X \longrightarrow X$ be a $G$-contraction. Assume there exists $x_0 \in X$ such that $T(x_0) \in [x_0]_{\tilde{G}}$. Let $\tilde{G}_{x_0}$ be the component of $\tilde{G}$ containing $x_0$. Then $[x_0]_{\tilde{G}}$ is $T$-invariant and $T\mid_{[x_0]_{\tilde{G}}}$ is $\tilde{G}_{x_0}$-contraction. Moreover, if $x, y \in [x_0]_{\tilde{G}}$ , then $\{T^n(x)\}$ and $\{T^n(y)\}$ are Cauchy equivalent.*

**Proof.** To show that $[x_0]_{\tilde{G}}$ is $T$-invariant, let $x \in [x_0]_{\tilde{G}}$. Then there is a finite path $(x_0, x_1, ..., x_n)$ in $\tilde{G}$ from $x_0$ to $x_n = x$ such that $(x_i, x_{i+1}) \in E(\tilde{G})$ where $i = 0, 1, ..., n$. Note that since $T$ is a $G$-contraction, then $T$ is a $\tilde{G}$ contraction. Hence $(T(x_i), T(x_{i+1})) \in E(\tilde{G})$, for $i = 0, 1, ..., n$, i.e., there is a finite path $(T(x_0), T(x_1), ..., T(x_n))$ in $\tilde{G}$ from $T(x_0)$ to $T(x_n) = T(x)$. Thus $T(x) \in [T(x_0)]_{\tilde{G}}$. Since $T(x_0) \in [x_0]_{\tilde{G}}$, we have $[T(x_0)]_{\tilde{G}} = [x_0]_{\tilde{G}}$ which implies $T(x) \in [x_0]_{\tilde{G}}$, i.e., $[x_0]_{\tilde{G}}$ is $T$-invariant. To show that $T\mid_{[x_0]_{\tilde{G}}}$ is a $\tilde{G}_{x_0}$-contraction, we first prove that $T\mid_{[x_0]_{\tilde{G}}}$ is monotone. Let $(x, y) \in \tilde{G_{x_0}}$, then there is a finite path $(x_0, x_1, ..., x_n)$ in $\tilde{G}_{x_0}$ from $x_0$ to $x_n = y$ where $x_{n-1} = x$ such that $(x_i, x_{i+1}) \in E(\tilde{G}_{x_0})$ where $i = 0, 1, ..., n$. Let $(y_0, y_1, ..., y_m)$ in $\tilde{G}$ from $y_0 = x_0$ to $y_m = T(x_0)$. Therefore, $(y_0, y_1, ..., y_m, T(x_1), T(x_2), ..., T(x_n))$ is a finite path in $\tilde{G}$ from $y_0 = x_0$ to $T(x_n) = T(y)$. In particular, $(T(x_{n-1}), T(x_n)) \in E(\tilde{G}_{x_0})$ i.e., $(T(x), T(y)) \in E(\tilde{G}_{x_0})$. Since $E(\tilde{G}_{x_0}) \subseteq E(\tilde{G})$, we deduce that $T\mid_{[x_0]_{\tilde{G}}}$ is a $\tilde{G}_{x_0}$-contraction. The last statement will be proved in Theorem 2.7. ∎

**Theorem 2.7** *[10] Let $(X, d, G)$ be a metric space endowed with a digraph, then the following statements are equivalent:*

*(i) $G$ is weakly connected;*

*(ii) for any $G$-contraction $T : X \longrightarrow X$ and $x, y \in X$ , the sequences $\{T^n(x)\}$ and $\{T^n(y)\}$ are Cauchy equivalent;*

*(iii) for any $G$-contraction $T : X \longrightarrow X$, $T$ has at most one fixed point.*

**Proof.** First we prove $(i) \Rightarrow (ii)$. Assume that $G$ is weakly connected. Then for any $x \in X$, we have $X = [x]_{\tilde{G}}$ and $T(x) \in [x]_{\tilde{G}}$. Assume that $T$ is a $G$-contraction. Then Lemma 2.4 implies

$$d(T^n(x), T^{n+1}(x)) \leq k^n \ r(x, T(x)),$$

for some $r(x, T(x)) \geq 0$ and for all $n \in \mathbb{N}$. Since $k < 1$, the series $\sum d(T^n(x), T^{n+1}(x))$ is convergent which implies that $\{T^n(x)\}$ is Cauchy. Similarly, we have $\{T^n(y)\}$ is a Cauchy sequence for any $y \in X$. Using again Lemma 2.4, we get

$$d(T^n(x), T^n(y)) \leq k^n \ r(x, y),$$

for some $r(x, y) \geq 0$ and for all $n \in \mathbb{N}$. Since $k < 1$, we deduce that $\lim\limits_{n \to +\infty} d(T^n(x), T^n(y)) = 0$, i.e., the sequences $\{T^n(x)\}$ and $\{T^n(y)\}$ are Cauchy equivalent, which completes the proof of (ii).

Next we prove that $(ii) \Rightarrow (iii)$. Let $T : X \longrightarrow X$ be a $G$-contraction. Let $x, y \in X$ be two fixed points of $T$. The property (ii) implies that the sequences $\{T^n(x)\}$ and $\{T^n(y)\}$ are Cauchy equivalent. In particular, we have $\lim\limits_{n \to +\infty} d(T^n(x), T^n(y)) = 0$ and since $T^n(x) = x$ and $T^n(y) = y$, for any $n \in \mathbb{N}$, we conclude that $x = y$, i.e., $T$ has at most one fixed point.

Finally, we prove that $(iii) \Rightarrow (i)$. Assume to the contrary that there exists a digraph $G$ not weakly connected for which any $G$-contraction $T : X \longrightarrow X$ has at most one fixed point. Let $x_0 \in X$, then $[x_0]_{\tilde{G}}$ and $X \setminus [x_0]_{\tilde{G}}$ are not empty. Let $y_0 \in X \setminus [x_0]_{\tilde{G}}$ and define $T : X \longrightarrow X$ by

$$T(x) = \begin{cases} x_0, & \text{if } x \in [x_0]_{\tilde{G}}; \\ y_0, & \text{if } x \in X \setminus [x_0]_{\tilde{G}}. \end{cases}$$

It is clear that $x_0$ and $y_0$ are fixed points of $T$. Next, we show that $T$ is $G$-contraction. Indeed, let $x, y \in X$ such that $(x, y) \in E(G)$. Then we have $y \in [x]_{\tilde{G}}$ *and* $x \in [y]_{\tilde{G}}$. Therefore, either we have $x, y \in [x_0]_{\tilde{G}}$ or $x, y \in X \setminus [x_0]_{\tilde{G}}$. In both cases, we have $T(x) = T(y)$. Since $G$ contains all loops, $(T(x), T(y)) \in E(G)$, i.e., $T$ is $G$-edge preserving and

$$d(T(x), T(y)) = 0 \leq \frac{1}{2} d(x, y),$$

which implies that $T$ is a $G$-contraction contradicting (iii). ∎

As a direct consequence to Theorem 2.7, we get the following result.

**Corollary 2.3.1** *[10] Let $(X, d, G)$ be a complete metric space endowed with a digraph, Then the following statements are equivalent:*

*(i) $G$ is weakly connected;*

*(ii) for any $G$-contraction $T : X \longrightarrow X$, there is $x^*$ such that $\lim_{n \to +\infty} T^n(x) = x^*$ for all $x \in X$.*

According to Corollary 2.3.1, an extension of the Banach Contraction Principle to metric spaces endowed with a digraph is obtainable provided the limit of the orbits is a fixed point. This will happen if we assume that the map is continuous. This is the version of Ran and Reurings fixed point theorem in metric spaces endowed with graphs.

**Theorem 2.8** *Let $(X, d, G)$ be a complete metric space endowed with a digraph. Assume that $G$ is weakly connected. Then any continuous and $G$-contraction mapping $T : X \to X$ has a unique fixed point $x^* \in X$. Moreover, the orbit $\{T^n(x)\}$ converges to $x^*$, for all $x \in X$.*

**Proof.** Corollary 2.3.1 the existence of $x^* \in X$ such $\lim_{n\to+\infty} T^n(x) = x^*$, for any $x \in X$. Since $T$ is continuous, we have $x^* = \lim_{n\to+\infty} T^{n+1}(x) = T(x^*)$, i.e., $x^*$ is a fixed point of $T$. The uniqueness of the fixed point may be obtained in different ways. For example, we may use (iii) of Theorem 2.7. ∎

The next question is how to relax the continuity assumption in Theorem 2.8. Jachymski [10] followed the same ideas developed by Nieto and Rodríguez-Lopez in the case of partially ordered metric spaces. In particular, Jachymski introduced the following property:

> *A metric space endowed with a digraph $(X, d, G)$ is said to satisfy (P2) if and only if for any $\{x_n\}$ in $X$, if $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$, for $n \in \mathbb{N}$, then there is a subsequence $\{x_{k_n}\}$, with $(x_{k_n}, x) \in E(G)$, for $n \in \mathbb{N}$.*

The graph version of Nieto and Rodríguez-Lopez's fixed point theorem obtained by Jachymski is the following theorem:

**Theorem 2.9** *[10] Let $(X, d, G)$ be a complete metric space endowed with a digraph. Assume that $(X, d, G)$ satisfies the property (P2). Let $T : X \to X$ be $G$-contraction and $X_T := \{x \in X : (x, T(x)) \in E(G)\}$. Then the following statements hold:*

1. *For any $x \in X_T$ , $T|_{[x]_{\widetilde{G}}}$ is a PO, that is $T$ has a unique fixed point $x^* \in [x]_{\widetilde{G}}$ and for each $x \in [x]_{\widetilde{G}}$, $\lim_{n\to\infty} T^n(x) = x^*$.*
2. *If $X_T \neq \emptyset$ and $G$ is weakly connected, then $T$ is a PO, that is $T$ has a unique fixed point $x^* \in X$ and for each $x \in X$, $\lim_{n\to\infty} T^n(x) = x^*$.*
3. *$cardFix\ T$=card$\{[x]_{\widetilde{G}} : x \in X_T\}$.*
4. *$T$ has a unique fixed point if and only if there exists an $x_0 \in X_T$ such that $X_T \subseteq [x_0]_{\widetilde{G}}$.*
5. *$Fix\ T \neq \emptyset$ if and only if $X_T \neq \emptyset$.*

**Proof.**

1. Let $x \in X_T$, then $T(x) \in [x]_{\widetilde{G}}$. If $y \in [x]_{\widetilde{G}}$, and since $T$ is a $G$-contraction, $\{T^n(x)\}$ and $\{T^n(y)\}$ are Cauchy equivalent. Since $X$ is complete $\{T^n(x)\}$ and $\{T^n(y)\}$ converge to some $x^* \in X$. Since $(x, T(x)) \in E(G)$ and $T$ is $G$-contraction, we have

$$(T^n(x), T^{n+1}(x)) \in E(G), \quad for\ any\ n \in \mathbb{N}. \tag{*}$$

Using property (P2), there exists a subsequence $\{T^{k_n}(x)\}$ such that

$(T^{k_n}(x), x^*) \in E(G)$, for all $n \in \mathbb{N}$. By $(*)$ there is a path $(x, T^1(x), T^2(x), ..., T^{k_1}(x), x^*)$ in $G$ (and $\tilde{G}$) from $x$ to $x^*$, then $x^* \in [x]_{\tilde{G}}$. Since $T$ is a $G$-contraction, the triangle inequality implies

$$\begin{aligned} d(T(x^*), x^*) &\leq d(T(x^*), T^{k_{n+1}}(x)) + d(T^{k_{n+1}}(x), x^*) \\ &\leq k\ d(x^*, T^{k_n}(x)) + d(T^{k_{n+1}}(x), x^*), \end{aligned}$$

for any $n \in \mathbb{N}$. If we let $n \to \infty$, we get $d(T(x^*), x^*) = 0$, i.e., $T(x^*) = x^*$ which implies that $T|_{[x]_{\tilde{G}}}$ is a PO.

2. Assume that $G$ is weakly connected. Then we have $[x]_{\tilde{G}} = X$, for any $x \in X$. Using (1), we conclude that $T$ is a PO on $X$.

3. Define the following map

$$f(x) = [x]_{\tilde{G}}\ for\ all\ x \in Fix(T).$$

If we show that $f$ is a bijection from $Fix(T)$ onto $\{[x]_{\tilde{G}};\ x \in X_T\}$, the conclusion of (3) will follow. Let $x_1, x_2 \in Fix(T)$ such that $f(x_1) = f(x_2)$, i.e., $[x_1]_{\tilde{G}} = [x_2]_{\tilde{G}}$. Then $x_2 \in [x_1]_{\tilde{G}}$. Using (1), we have

$$\lim_{n \to +\infty} T^n(x_2) \in [x_1]_{\tilde{G}} \cap Fix(T) = \{x_1\},$$

i.e., $x_1 = x_2$, which implies that $f$ is injective or 1-to-1. Since all loops contained in $E(G)$, $Fix(T) \subseteq X_T$ which implies $f(FixT) \subseteq \{[x]_{\tilde{G}};\ x \in$

$X_T\}$. On the other hand, if $x \in X_T$, then by (1) we have $\lim\limits_{n\to+\infty} T^n(x) \in [x]_{\tilde{G}} \cap Fix(T)$ which yield $f(\lim\limits_{n\to+\infty} T^n(x)) = [x]_{\tilde{G}}$. Therefore, $f$ is surjective or onto. Hence $f$ is a bijection.

4. (4) and (5) are an easy consequences of (3).

∎

### 2.3.2 Conclusion

So far, we have seen that an extension of the Banach Contraction Principle exists where the contraction condition is satisfied by special elements of the metric space and may not be satisfied by the entire space. Wonderful examples of application are given by Ran and Reurings and Nieto and Rodríguez-Lopez in metric spaces endowed with a partial order. Therefore, this extension is worthy of investigation which explains the excitement generated over the last decade. In the next chapter, we will propose to unify all these extensions done in partially ordered metric spaces or in metric spaces endowed with a digraph.

# CHAPTER 3

# EXTENDED METRIC SPACES

In this Chapter, we will revisit most of the results discussed in the previous sections. We will offer a new structure that unifies partially ordered metric spaces and metric spaces endowed with a directed graph. Then we give a more general fixed point theorem that reduces to Ran and Reurings and Nieto and Rodríguez-Lopez's fixed point theorems in partially ordered metric spaces and Jachymski's fixed point theorem in metric spaces endowed with a directed graph.

## 3.1 Basic results

As we said before, most of these new extensions of the Banach Contraction Principle rely on the contraction-Lipschitz condition satisfied by special elements of the space not necessarily the entire space. This is how we got the idea to focus on these special elements and eliminate the others. This idea lead to the concept of extended metric spaces.

**Definition 3.1** *Let $X$ be an abstract set. A function $\bar{d} : X \times X \to [0, +\infty]$ is called an* ***extended distance*** *if the following conditions are satisfied:*

*(i)* $\bar{d}(x, y) = 0$ *if and only if* $x = y$;

*(ii)* $\bar{d}(x, y) = \bar{d}(y, x)$, *for all* $x, y \in X$

*(iii)* $\bar{d}(x, y) \leq \bar{d}(x, z) + \bar{d}(z, y)$, *for all* $x, y, z \in X$ *such that* $\bar{d}(x, y) < \infty$, $\bar{d}(x, z) < \infty$ *and* $\bar{d}(z, y) < \infty$.

*In this case, the pair $(X, \bar{d})$ is called* ***an extended metric space****.*

Next, we discuss the concept of convergence in extended metric spaces.

**Definition 3.2** *Let $(X, \bar{d})$ be an extended metric space.*

*(i) A sequence $\{x_n\}$ in $X$ is said to be* ***convergent*** *if there exists $x \in X$ such that* $\lim_{n \to +\infty} \bar{d}(x_n, x) = 0$.

*(ii) A nonempty subset $C$ of $X$ is said to be* ***closed*** *if for any sequence $\{x_n\}$ in $C$ which converges to $x \in X$, we have $x \in C$.*

*(iii) A sequence $\{x_n\}$ in $X$ is said to be* ***strongly Cauchy*** *if the series* $\sum_{n=0}^{\infty} \bar{d}(x_n, x_{n+1})$ *converges.*

*(iv) $(X, \bar{d})$ is said to be* ***strongly complete*** *whenever any strongly Cauchy sequence in $X$ is convergent in $X$.*

**Remark 3.1.1** *Note that in extended metric spaces the uniqueness of the limit may not happen. In fact, if a sequence $\{x_n\}$ in $X$ converges to $x$ and $y$, then $x = y$ provided $\bar{d}(x, y) < \infty$.*

Note that in a metric space $(X, d)$ every strongly Cauchy sequence is Cauchy, but the converse may not be true. But it is well known that a Cauchy sequence always contains a strongly Cauchy subsequence [7]. Clearly, this allows us to see that a metric space $(X, d)$ is complete if and only if it is strongly complete.

## 3.2 Banach Contraction Principle in Extended Metric Spaces

In order to discuss the Banach contraction principle in extended metric spaces, we will need to introduce the concept of Lipschitzian mappings in these spaces.

**Definition 3.3** *Let $(X, \bar{d})$ be an extended metric space. A mapping $T : X \to X$ is said to be $\bar{d}$-Lipschitzian if there exists $k > 0$ such that*

$$\bar{d}(T(x), T(y)) \leq k\ \bar{d}(x, y),$$

*for all $x, y \in X$. If $k < 1$, then $T$ is said to be a $\bar{d}$-contraction mapping.*

**Remark 3.2.1** *Let $(X, \bar{d})$ be an extended metric space. Let $T : X \to X$ be a $\bar{d}$-contraction mapping. Then for any $x, y \in Fix(T)$, we have $x = y$ whenever $\bar{d}(x, y) < \infty$.*

The following technical lemma will be useful when studying the Banach Contraction Principle in extended metric spaces.

**Lemma 3.1** *Let $(X, \bar{d})$ be an extended metric space and $T : X \to X$ be a $\bar{d}$-contraction. Set $X_T = \{x \in X;\ \bar{d}(x, T(x)) < \infty\}$. For any $x_0 \in X_T$, the orbit $\{T^n(x_0)\}$ is strongly Cauchy. Moreover if $\{T^n(x_0)\}$ converges to $x \in X$, then $T(x) = x$ or $\bar{d}(x, T(x)) = \infty$.*

**Proof.** Since T is a $\bar{d}$-contraction mapping, there exists $k \in (0, 1)$ such that

$$\bar{d}(T(x), T(y)) \leq k\ \bar{d}(x, y),$$

which implies

$$\bar{d}(T^n(x), T^n(y)) \leq k^n\ \bar{d}(x, y),$$

for all $x, y \in X$ and $n \in \mathbb{N}$. Let $x_0 \in X_T$, i.e., $\bar{d}(x_0, T(x_0)) < \infty$. Then

$$\bar{d}(T^n(x_0), T^{n+1}(x_0)) \leq k^n\ \bar{d}(x_0, T(x_0)),$$

for any $n \in \mathbb{N}$. Hence $\sum\limits_{n=0}^{\infty} \bar{d}(T^n(x_0), T^{n+1}(x_0))$ is convergent, i.e., $\{T^n(x_0)\}$ is strongly Cauchy, since $k < 1$ and $\bar{d}(x_0, T(x_0) < \infty$. Assume that $\{T^n(x_0)\}$ converges to $x \in X$. Assume that $x \in X_T$, i.e., $\bar{d}(x, T(x)) \neq \infty$. Since $\{T^n(x_0)\}$ converges to $x$, there exists $N \in \mathbb{N}$ such that $\bar{d}(T^n(x_0), x) < \infty$, for any $n \geq N$. Hence

$$\bar{d}(T^{n+1}(x_0), T(x)) \leq k\ \bar{d}(T^n(x_0), x) < \infty,$$

for any $n \geq N$. Using the property (iii) satisfied by $\bar{d}$, we get

$$\begin{aligned} \bar{d}(x, T(x)) &\leq \bar{d}(x, T^{n+1}(x_0)) + \bar{d}(T^{n+1}(x_0), T(x)) \\ &\leq \bar{d}(x, T^{n+1}(x_0)) + k\ \bar{d}(x, T^n(x_0)), \end{aligned}$$

for any $n \geq N$. Since $\{T^n(x_0)\}$ converges to $x$, we get $\bar{d}(x, T(x)) = 0$, i.e., $T(x) = x$. ∎

In order to insure that the map $T$ in Lemma 3.1 has a fixed point, we need the following property:

> *The extended metric space $(X, \bar{d})$ is said to satisfy the property (P3) if $\bar{d}(x, y) < \infty$, for any $\{x_n\}$ in $X$ which converges to $x$, and $y \in X$ such that $\bar{d}(x_n, y) < \infty$, for any $n \in \mathbb{N}$.*

Next, we give the analogue to the Banach Contraction Principle in extended metric spaces.

**Theorem 3.1** *Let $(X, \bar{d})$ be a strongly complete extended metric space and $T : X \to X$ be a $\bar{d}$-contraction. Assume that $X$ satisfies the property (P3). If $X_T = \{x \in X;\ \bar{d}(x, T(x)) < \infty\} \neq \emptyset$, then $T$ has a fixed point.*

**Proof.** Let $x_0 \in X_T$. As we did in the proof of Lemma 3.1, we know that $\{T^n(x_0)\}$ is strongly Cauchy. Since $(X, \bar{d})$ is strongly complete, $\{T^n(x_0)\}$ converges to some $x \in X$. Since $\{T^n(x_0)\}$ converges to $x$, there exists $N \in \mathbb{N}$ such that $\bar{d}(T^n(x_0), x) < \infty$, for any $n \geq N$. Hence

$$\bar{d}(T^{n+1}(x_0), T(x)) \leq k\ \bar{d}(T^n(x_0), x) < \infty,$$

for any $n \geq N$. Clearly $\{T^{n+1}(x_0)\}_{n \geq N}$ is a subsequence of $\{T^n(x_0)\}$ which converges to $x$, the property (P3) implies that $\bar{d}(x, T(x)) < \infty$. Using Lemma 3.1, we conclude that $T(x) = x$, i.e., $x$ is a fixed point of $T$. ∎

In the following example, we show how Theorem 3.1 unifies the fixed point theorems of Ran and Reurings [14], Nieto Rodríguez-Lopez [13] and Jachymski [10].

**Example 3.2.1** *Let $(X, d, G)$ be a metric space endowed with a digraph $G$. Define $\bar{d} : X \times X \to [0, +\infty]$ by*

$$\bar{d}(x,y) = \begin{cases} d(x,y), & \text{if } x \text{ and } y \text{ are adjacent;} \\ \infty, & \text{if } x \text{ and } y \text{ are not adjacent.} \end{cases}$$

*Recall that $x, y \in X$ are adjacent if $(x, y) \in E(G)$ or $(y, x) \in E(G)$. It is easy to check that $(X, \bar{d})$ is an extended metric space. Recall that graph intervals are any subset*

$$(\leftarrow, a] = \{x \in X; (x,a) \in E(G)\} \text{ or } [a, \rightarrow) = \{x \in X; (a,x) \in E(G)\},$$

*for any $a \in X$. Assume that graph intervals are closed. Then $(X, \bar{d})$ enjoys the property (P3). Indeed, let $\{x_n\}$ be in $X$ which converges to $x$, i.e., $\lim_{n\to\infty} \bar{d}(x_n, x) = 0$ which implies $\lim_{n\to\infty} d(x_n, x) = 0$. Let $y \in X$ be such that $\bar{d}(x_n, y) < \infty$, for any $n \in \mathbb{N}$. Using the definition of $\bar{d}$, we know that $x_n$ and $y$ are adjacent, for any $n \in \mathbb{N}$. In particular, either $\{m \in \mathbb{N}; (y, x_m) \in E(G)\}$ or $\{m \in \mathbb{N}; (x_m, y) \in E(G)\}$ is infinite. This will imply that there exists a subsequence $\{x_{\varphi(n)}\}$ of $\{x_n\}$ such that*

*$(x_{\varphi(n)}, y) \in E(G)$ or $(y, x_{\varphi(n)}) \in E(G)$, for any $n \in \mathbb{N}$. Since graph intervals are closed and $\lim_{n\to\infty} d(x_{\varphi(n)}, x) = 0$, we conclude that $(x, y) \in E(G)$ or $(y, x) \in E(G)$, i.e., $x$ and $y$ are adjacent which implies $\bar{d}(x, y) < \infty$. Moreover, if we assume that $(X, d)$ is complete, then $(X, \bar{d})$ is strongly complete. Indeed, let $\{x_n\}$ be strongly Cauchy, i.e., $\sum_{n\in\mathbb{N}} \bar{d}(x_n, x_{n+1})$ is convergent. Hence $\bar{d}(x_n, x_{n+1}) < \infty$, for any $n \in \mathbb{N}$, which implies that $\sum_{n\in\mathbb{N}} d(x_n, x_{n+1})$ is convergent. Hence $\{x_n\}$ is a Cauchy sequence in $(X, d)$. Since $(X, d)$ is complete, there exists $x \in X$ such that $\lim_{n\to\infty} d(x_n, x) = 0$. Our assumption on $\{x_n\}$ and by definition of $\bar{d}$, we have $x_n$ and $x_{n+1}$ are adjacent, for any $n \in \mathbb{N}$. Using a similar argument as before, we show that $x_n$ and $x$ are adjacent, for any $n \in \mathbb{N}$. Hence $\bar{d}(x_n, x) = d(x_n, x)$, for any $n \in \mathbb{N}$. Clearly this will imply that $\lim_{n\to\infty} \bar{d}(x_n, x) = 0$, i.e., $\{x_n\}$ converges to $x$ in $(X, \bar{d})$. Next, we discuss the class of $\bar{d}$-contractions. Let $T : X \to X$ be a map. We will say that $T$ is a $G$-contraction if*

*(i)* *$(x, y) \in E(G)$ implies $(T(x), T(y)) \in E(G)$, for any $x, y \in X$,*

*(ii)* *there exists $k \in [0, 1)$ such that*

$$d(T(x), T(y)) \leq k\ d(x, y),$$

*for any $x, y \in X$ such that $x$ and $y$ are adjacent.*

*$G$-contractions are the maps for which Jachymski proved his fixed point theorem in metric space endowed with a digraph. Let $T : X \to X$ be a $G$-contraction. Let $x, y \in X$ be such that $\bar{d}(x, y) < \infty$. Then by definition of $\bar{d}$, $x$ and $y$ are adjacent.*

*Since $T$ is monotone, then $T(x)$ and $T(y)$ are adjacent. Hence $\bar{d}(T(x), T(y)) = d(T(x), T(y))$ and $\bar{d}(x, y) = d(x, y)$, which implies*

$$\bar{d}(T(x), T(y)) \leq k \; \bar{d}(x, y),$$

*where $k \in [0, 1)$ is given by (ii). Therefore, any $G$-contraction in the sense of Jachymski is also a $\bar{d}$-contraction.*

## 3.3 Future Problems

The concept of extended metric spaces allowed us to look at many extensions of the Banach Contraction Principles as one. Recently, the authors of [1] gave an extension of the Caristi's fixed point theorem in metric spaces endowed with a digraph. It is our hope that we can have a more general Caristi's fixed point theorem in extended metric spaces. Then prove that the extension of this theorem to metric spaces endowed with a digraph is a particular case of the new Caristi's fixed point theorem in extended metric spaces.

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