TEMPERATURE DISTRIBUTION IN A CIRCULAR CYLINDER WITH MIXED BOUNDARY CONDITIONS

BY

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To My Family

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All praises is for Allah (S.W.T.), the most high, most gracious and most merciful. May His peace, blessings and mercy be upon His noble messenger and prophet Muhammad (S.A.W), his family, his companions, and those who follow their footsteps till the last day.

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THESIS ABSTRACT

Full NAME: Rahmatullah Ibrahim Nuruddeen

TITLE OF STUDY: Temperature Distribution in a Circular Cylinder with Mixed Boundary Conditions

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The heat conduction in solids is one of the important areas in engineering problems. More often, the boundary of the solids are kept at a prescribed temperature or insulated. However, in many situations the surface of the solid is part heated/cooled and part insulated. In this thesis we discuss the heat conduction in circular cylinders subjected to mixed boundary conditions using the Wiener-Hopf technique. The temperature distribution and heat flux in the form of closed integrals are obtained. These integrals have been evaluated in some cases of interest.

ملخص الرسالة

الاسم الكامل: رحمة الله إبراهيم نور الدين

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يُعد انتقال الحرارة في المواد الصلبة أحد المواضيع ذات الأهمية في المسائل الهندسية. غالباً ما يتم عزل أطراف السطوح الصلبة حرارياً أو وضعها تحت درجة حرارة معينة، وفي كثير من الحالات يكون جزء من سطح المواد الصلبة معزولاً ويتم تسخين أو تبريد الجزء الآخر.

نناقش في هذه الرسالة انتقال الحرارة في اسطوانة دائرية تحت شروط حدية مختلطة باستخدام طريقة (فينر وهوبف)، حيث تم الحصول على الحل لانتقال الحرارة وللتدفق الحراري في صيغة تكاملات تم إيجاد قيمها في بعض الحالات الهامة.

CHAPTER ONE

INTRODUCTION AND LITERATURE REVIEW

1.0 Introduction

Heat conduction problems are encountered in many engineering applications. However, the most important common term that cuts across all sorts of heat conduction problems irrespective of their application and nature is the term “Temperature”. In this regard, we note that even if the process or method of heating is invisible, still the temperature is observed. Furthermore, one can clearly note that when a metal bar is heated, its temperature at the other end will eventually begin to rise. This transfer of energy or heat is due to molecular activity. That is, molecules at the hot region exchange their energies with neighboring layers through random collisions between the molecules. Heat conduction process is one of the three ways of heat transfer in addition to radiation and convection. The heat conduction or heat flow under a variety of boundary conditions of different conducting bodies has been of great importance in many engineering problems. Further, one can find an intensive study regarding the heat conduction problem occurring in rods, cylinders, spheres and plates among others ranging from one-dimensional, two-dimensional and three-dimensional coordinate systems in literature (see for example [4], [9] and [10]). Heat conduction problems are broadly categorized as steady-state (time-independent) and transient (time-dependent). The methods commonly used are separation of variables, Green’s function, integral transforms and numerical schemes. However, in case of mixed boundary conditions in which case the boundary of the solid is subjected to

different boundary conditions in different parts of the an interface. In such problems, the Wiener-Hopf technique has been extensively used in the literature. This technique depends on the utilization of the known integral transforms, mainly the Laplace transform and Fourier transform as used in [2], [3], [5], [6], [25] and [28] among others.

The technique “Wiener-Hopf technique”, came into existence in 1931 after the efforts made by Norbert Wiener (1894-1964) and Eberhard Hopf (1902-1983) in trying to solve some integral equations of certain form. Furthermore, in 1952, Douglas Samuel Jones (1922-2013) modified the Wiener-Hopf technique to solve mixed boundary value problem directly without having to formulate it as an integral equation. [17].

In this work, the modified Wiener-Hopf technique due to Jones would be used due to its direct application in comparison with the afore mentioned methods which are found to be inadequate. In the modified approach, Jones gave a method of obtaining the functional Wiener-Hopf equation and subsequently solving the boundary value problem. We shall use the method to determine the temperature distribution in hollow or solid infinite homogeneous circular cylinders with mixed boundary conditions. In the physical sense, part of the external surface of the cylinder will be subjected to a constant (or given) temperature while the other part has a known temperature flux or is insulated.

1.1 Literature Review

The problem of heat conduction in homogeneous and isotropic bodies has been extensively reported in literature. Carslaw and Jaeger [4] have formulated such problems in space, half-space, cylinders and beams, and other composite media. They have given temperature distribution and heat flux at the surface of such materials. The steady-state as well as transient problem have been considered in this classical reference. The heat conduction problem with mixed boundary conditions on the interface or surface, or having different conducting bodies is our primary interest in this work. Often such body is subjected to more than one condition on the boundary. For instance, one part maybe assumed to be insulated while the other part to be kept at a constant temperature; or one part assumed to be immersed in a fluid and the other left outside with just surface temperature.

Chakrabarti [6] gave the explicit solution of the sputtering temperature of a cooling cylindrical rod with an insulated core when allowed to enter into a cold fluid of large extent with a uniform speed v in the positive semi-infinite range while the negative semi-infinite range is kept outside, and a simple integral expression is derived for the value of the sputtering temperature of the rod at the points of entry (see also Chakrabarti [7]). Georgiadis et al [11] considered infinite dissimilar materials which are joined and brought in contact over half of their common boundary and the other half insulated all along the common boundary (interface). The solution is then obtained for the heat conduction problem after assuming the two conducting bodies to be kept initially at

different uniform temperatures, which are brought into contact over part of their surfaces at time t = 0. Chakrabarti and Bera [5] studied a mixed boundary-valued problem associated with the diffusion equation which involves the physical problem of cooling of an infinite slab in a two-fluid medium. An analytical solution is derived for the temperature distribution at the quench fronts being created by two different layers of cold fluids having different cooling abilities moving on the upper surface of the slab at a constant speed. Similarly, Zaman [24] studied a heat conduction problem across a semi-infinite interface in layered plates. The two plates are kept in contact, in which the contact between the layers takes place in one part of the interface while the outer part is perfectly insulated. In Bera and Chakrabarti [3], the explicit solutions are obtained for the temperature distributions on the surface of a cylindrical rod without an insulated core as well as that inside a cylindrical rod with an insulated inner core when the rod, in either of the two cases, is allowed to enter, with a uniform speed, into two different layers of fluid with different cooling abilities. The modification of Wiener-Hopf technique is used. Zaman and Al-Khairy [26] considered a steady state temperature distribution in a homogeneous rectangular infinite plate. They assumed that the lower part to be cooled by a fluid flowing at a constant velocity while the upper part satisfies the general mixed boundary conditions. In addition, Zaman and Al-khairy [27] again discussed the cooling problem of a composite layered plate comprising of a dissimilar layers of uniform thickness having mixed interface thereby finding the closed forms of both the temperature distribution and the heat flux of the plate using the modified Wiener-Hopf technique.

Satapathy [20] considered a two-dimensional quasi-steady conduction equation governing conduction controlling rewetting of an infinite cylinder with heat generation. The analytical solution obtained by Wiener–Hopf technique yields the quench front temperature as a function of various model parameters used. It is good to note that the process of rewetting or quenching is to re-establishment of liquid contact with a solid surface whose initial temperature exceeds the rewetting temperature thereby creating a mixed condition on the surface where Wiener-Hopf technique can be applied. Shafei and Nekoo [21] solved the heat conduction problem of a finite hollow cylinder using generalized finite Hankel method which is based on the use of the integral transformations method. The cylinder is assumed to be of finite length, and the finite element method is used verify the closeness of the solution obtained by the method. To sum up, Kedar and Deshmukh [16], considered the inverse heat conduction problem in a semi-infinite hollow circular cylinder using integral transform method and the result is given in series form in terms of Bessel functions. Further, the hollow circular cylinder is subjected to a known temperature under transient condition. Initially the cylinder is assumed to be at zero temperature and temperature at the lower surface is also assumed to have zero heat flux.

Finally**,** in this thesis, we intend to consider two problems: the first is the determination of the analytical solution of the transient heat conduction in a homogenous hollow infinite circular cylinder that is subjected to different boundary conditions on the outer surface while the inner surface is kept at zero temperature throughout.

In the second, we consider an infinite solid circular cylinder in which part of the boundary is being heated while the other part has a prescribed flux. The resulting mixed boundary value problem from both problems is solved using the Jones' modification method of the Wiener-Hopf technique. The temperature distribution and the heat flux are determined in both.

1.2 Basic Definitions of Some Terms

1.2.1 Bessel Differential Equation

$$r^2 T_{rr} + rT_r + (n^2 - r^2)T = 0.$$

1.2.2 Solution of the Bessel Differential Equation

$$T(r) = AJ_n(r) + BY_n(r).$$

Where $J_n(r)$ and $Y_n(r)$ are the Bessel functions of first and second kinds respectively.

1.2.3 Modified Bessel Differential Equation

$$r^2 T_{rr} + rT_r - (n^2 + r^2)T = 0.$$

1.2.4 Solution of the Modified Bessel Differential Equation

$$T(r) = AI_n(r) + BK_n(r).$$

Where $I_n(r)$ and $K_n(r)$ are the modified Bessel functions of first and second kinds respectively.

Note:

We take note of the following relations and definitions all related to Bessel differential equation

1. $J_n(r) = \sum_{k=0}^{\infty} \frac{(-1)^k (r/2)^{n+2k}}{k!(n+k)!}$
2. $Y_n(r) = \frac{J_n(r)\cos(n\pi) - J_{-n}(r)}{\sin(n\pi)}$
3. $I_n(r) = \sum_{k=0}^{\infty} \frac{(r/2)^{n+2k}}{k!(n+k)!}$
4. $K_n(r) = \frac{\pi}{2\sin(n\pi)}\{I_{-n}(r) - I_n(r)\}$
5. $J_n'(r) = \frac{n}{r}J_n(r) - J_{n+1}(r)$
6. $Y_n'(r) = \frac{n}{r}Y_n(r) - Y_{n+1}(r)$
7. $I_n'(r) = \frac{n}{r}I_n(r) + I_{n+1}(r)$
8. $K_n'(r) = \frac{n}{r}K_n(r) - K_{n+1}(r)$
9. $I_n(r) = i^{-n}J_n(ir)$

1.3 Heat Conduction Problems

The specification of temperature and heat flux in the region of a solid or metal where conduction is taking place brings about temperature distribution and heat flows. To do that, the description of the point or region needs to be known. That is, the condition and the boundary condition of the specify area. However, in order to describe the temperature distribution, the special coordinates must be known. In general, heat conduction is described as one-dimensional, two-dimensional, and three-dimensional depending upon the variables describing the temperature distribution in the given region.

1.3.1 One-Dimensional Heat Conduction Problem

The one-dimensional heat conduction problem is the simplest form of heat conduction equation in which the temperature T depends only on one space variable x and on the time variable t. The equation is given as:

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{k}\frac{\partial T}{\partial t}.$$

Where, x belongs to some finite or infinite interval, $t > 0$ and k is the thermal diffusivity of the material.

1.3.2 Three-Dimensional Heat Conduction Problem

In three-dimensional heat conduction problem in rectangular coordinate system; the temperature distribution T depends on three space variables x, y and z and the on time variable t. The equation is given below

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{k}\frac{\partial T}{\partial t}.$$

Transient (unsteady-state) means that the temperature at any location or region changes with time; it usually happens due to the sudden change of conditions.

1.3.3 Three-Dimensional Heat Conduction Problem in Cylindrical Coordinates

The three-dimensional transient heat conduction problem in cylindrical coordinate system is defined in such a way that the temperature distribution T depends on three space variables r, θ and z and the on time variable t.

The equation is given below

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{k}\frac{\partial T}{\partial t}.$$

1.3.4 Three-Dimensional (Transient) Heat Conduction Problem with Axial Symmetry

The three-dimensional transient heat conduction problem in cylindrical coordinate system, axial symmetry is defined in such a way that the temperature distribution T depends on only the two space variables r and z and the on time variable t {the space variables are independent of angle}. The equation is given as:

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{k}\frac{\partial T}{\partial t}.$$

1.4 Initial and Boundary Conditions

In order for us to obtain a temperature distribution it is necessary to solve the governing heat conduction equation subject to some boundary and initial conditions. Boundary conditions are mathematical equations describing what takes place physically at/on the boundary, while an initial condition describes the temperature distribution at time $t = 0$ or at a fixed time t_0. (See, Jiji (2009) [13]).

For instance, figure **A** below shows four typical boundary conditions for two-dimensional heat conduction in a rectangular plate, while figure **B** shows an interface of two materials. Two boundary conditions are associated with this case.

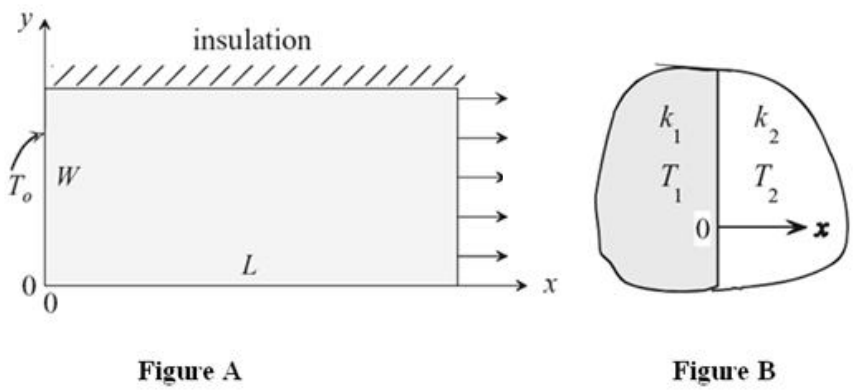

Figure 1.1: Boundary and Interfacial Conditions

1.4.1 Specified Temperature

The specified temperature along the boundary is $(0, y)$ in figure **A** is T_o. This temperature can be uniform or can vary along y as well as with time. Mathematically this condition is expressed as

$$T(0, y, t) = T_o.$$

1.4.2 Specified Heat Flux

The specified heat flux along boundary (L, y) in figure **A** is $q(L, y, t)$. According to Fourier's law this condition is expressed as

$$q(L, y, t) = -k\frac{\partial T(L, y, t)}{\partial x}.$$

In particular, the boundary at (x, W) is thermally insulated in figure **A**. Thus, the specified heat flux would now be

$$\frac{\partial T(x, W, t)}{\partial x} = 0.$$

1.4.3 Interface Condition

Figure **B** shows a composite wall of two materials with thermal diffusivities k_1 and k_2. For a perfect interface contact, the two temperatures must be the same at the interface. Thus,

$$T_1\ (0, y, t) = T_1\ (0, y, t).$$

Conservation of energy at the interface requires that the two fluxes be identical. Application of Fourier's law gives

$$k_1 \frac{\partial T_1(0, y, t)}{\partial x} = k_2 \frac{\partial T_2(0, y, t)}{\partial x}.$$

1.5 Solution of Heat Conduction Problem

To solve the heat conduction problem, that is, the given heat conduction partial differential equation irrespective of the coordinate system and the boundary conditions (or initial condition) means that finding the temperature distribution or temperature field function that depends on various space parameters (such as x, y, z or r, θ, z) and on the time variable that is consistent with the conditions defined on the boundary. In this regard, several techniques are available with separation of variable method as the most widely used method and then the integral transform methods. However, the boundary conditions also play a very important role in choosing which method to be used. For instance, while using the Fourier transform method, we put our concern on the nature of the boundary conditions; on doing that, we determine whether to use Fourier Sine or Fourier cosine method as sub-classes of Fourier transform method.

CHAPTER TWO

INTEGRAL TRANSFORMS AND WIENER-HOPF TECHNIQUE

2.0 Introduction

In this chapter, we present the methodology to be followed in order to solve our intended problems in chapters three and four. To solve our mixed boundary value problem, we use the technique that is based upon the integral transforms. The transforms to be used are the Laplace transform in the time variable and the Fourier transform in the space variables.

However, many problems of practical interest with our problem inclusive give rise to singular integral equations defined in $(0, \infty)$ range which the above mentioned transforms and others like Mellin transform do not allow the use of the convolution theorem thereby rendering these transforms inapplicable. Thus, we present a method due to Wiener and Hopf called Wiener-Hopf technique or method to solve such integral equations. Besides, we also present the Modified Wiener-Hopf technique by Jones which simplifies the difficulties faced in dealing with the integral equations by simply applying the technique to the governing partial differential equation and its boundary conditions. Some theorems are also presented.

2.1 Laplace Transform

The Laplace transform in the time variable t is defied {whenever it exists} by

$$\mathcal{L}\{T(t)\} = \int_0^\infty T(t)e^{-st}dt = \bar{T}(s), \qquad \text{and} \tag{2.1}$$

Laplace inverse transform in the Laplace parameter s is defined {whenever it exists} by

$$\mathcal{L}^{-1}\{\bar{T}(s)\} = \frac{1}{2\pi i}\int_{-i\infty+h}^{i\infty+h} T(s)e^{st}ds = T(t). \tag{2.2}$$

2.2 Fourier Transform

The Fourier transform is taken in one of the space variables. The transform and its corresponding inverse are defined by

$$\mathcal{F}\{T(x)\} = \int_{-\infty}^{\infty} T(x)e^{i\alpha x}dx = T^*(\alpha), \quad \text{and} \tag{2.3}$$

$$\mathcal{F}^{-1}\{T^*(\alpha)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} T^*(\alpha)e^{-i\alpha x}d\alpha = T(x). \tag{2.4}$$

If these integrals exist.

2.3 Fourier Transform of One-Sided Functions

We first introduce the one-sided functions due to their usefulness in the context of mixed boundary value problems. We define $\{T_+(x)\}$ an upper (right) and $\{T_-(x)\}$ a lower (left) half functions called one-sided functions as

$$T_+(x) = \begin{cases} T(x), & x > 0 \\ 0, & x < 0 \end{cases} \quad \text{and} \quad T_-(x) = \begin{cases} 0, & x > 0 \\ T(x), & x < 0 \end{cases}.$$

Having defined the Fourier transform which is defined over $(-\infty, \infty)$ range; we also need to define the Fourier transform of the so-called one sided functions. That is, a function defined only on a half-range.

So, the Fourier transform of $T_+(x)$ would be:

$$\mathcal{F}\{T_+(x)\} = \int_{-\infty}^{\infty} T_+(x) e^{i\alpha x} dx = \int_0^{\infty} T_+(x) e^{i\alpha x} dx = T_+^*(\alpha). \tag{2.5}$$

where, $\alpha = \sigma + i\tau$, and $T_+(x) = O(e^{-ax})$ as $x \to \infty$, i.e.

$$|T_+(x)| \leq C_1 |e^{-ax}| \; as \;\; x \to \infty.$$

Then,

$$\left| \int_0^{\infty} T_+(x) e^{i(\sigma + i\tau)x} dx \right| \leq C_1 \int_0^{\infty} e^{-(a+\tau)x} dx.$$

Thus, $\mathcal{F}\{T_+(x)\}$ is defined as analytic function if $\tau > -a$.

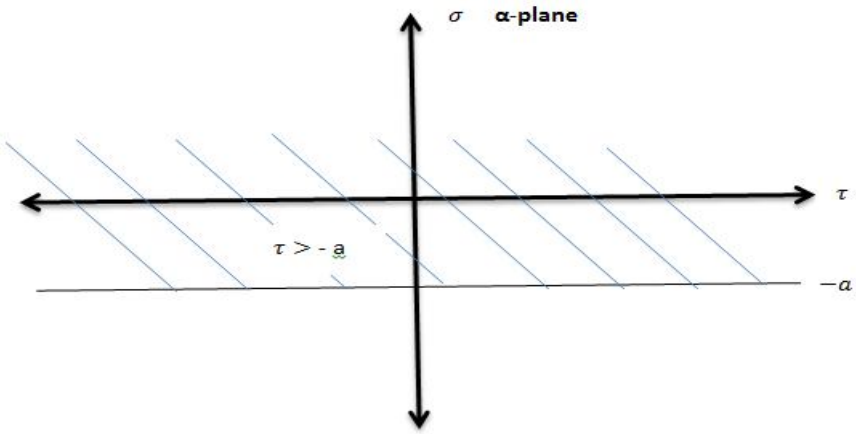

Figure 2.1: Upper Half-Plane

In the same way, we define $T_{-}^{*}(\alpha)$ as:

$$\mathcal{F}\{T_{-}(x)\} = \int_{-\infty}^{\infty} T_{-}(x)e^{i\alpha x}dx = \int_{-\infty}^{0} T_{-}(x)e^{i\alpha x}dx = T_{-}^{*}(\alpha). \tag{2.6}$$

where $\alpha = \sigma + i\tau$, and $T_{-}(x) = O(e^{ax})$ as $x \to -\infty$. i.e.

$$|T_{-}(x)| \leq C_2|e^{ax}| \; as \;\; x \to -\infty.$$

Then,

$$\left|\int_{-\infty}^{0} T_{-}(x)e^{i(\sigma+i\tau)x}dx\right| \leq C_2 \int_{-\infty}^{0} e^{(a-\tau)x}dx$$

Thus, $\mathcal{F}\{T_{-}(x)\}$ is defined as analytic function if $\tau < a$.

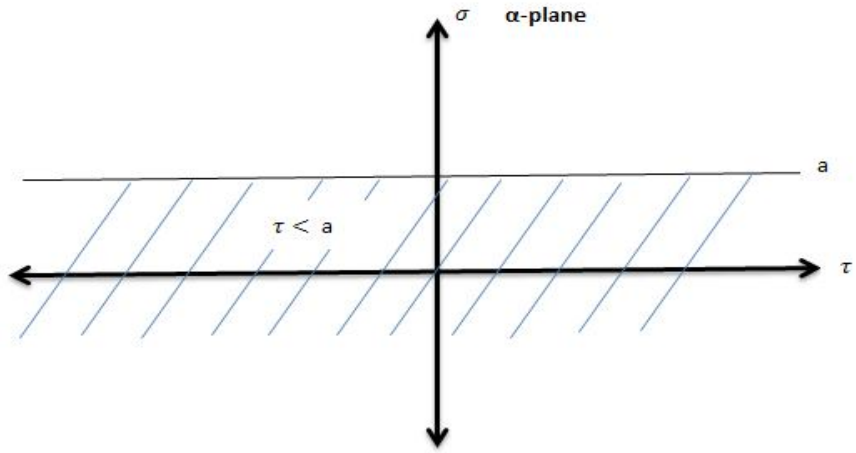

Figure 2.2: Lower Half-Plane

Hence,

$$T^*(\alpha) = \int_{-\infty}^{\infty} T(x)e^{i\alpha x}dx = \int_{-\infty}^{0} T_-(x)e^{i\alpha x}dx + \int_{0}^{\infty} T_+(x)e^{i\alpha x}dx.$$

So that ,
$$T^*(\alpha) = T_-^*(\alpha) + T_+^*(\alpha). \tag{2.7}$$

Where, $T^*(\alpha)$ is an analytic function of α if $-a < \tau < a$ for $T(x) = O\left(e^{-a|x|}\right)$ as $|x| \to \infty$. The region given by $-a < \tau < a$ is called the **strip of analyticity,** of $T^*(\alpha)$.

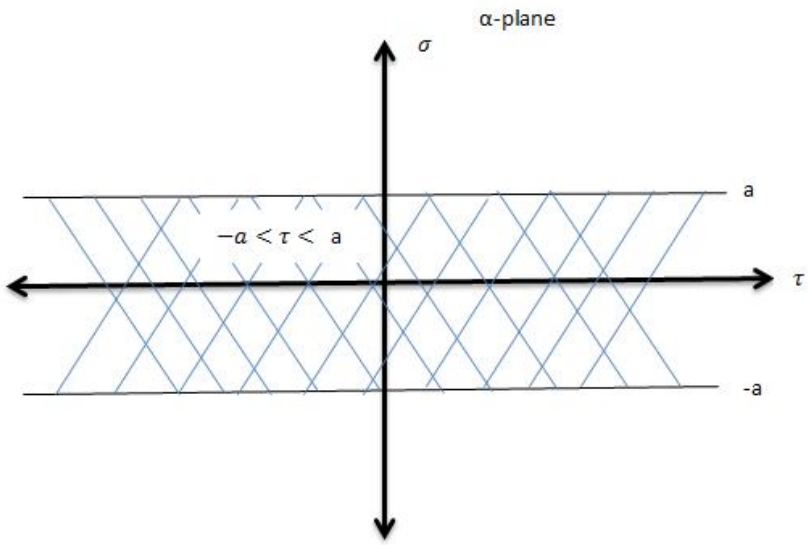

Figure 2.3: Strip of Analyticity

2.4 Theorems

The theorems to be used in this work are listed and stated as follows

2.4.1 Additive Decomposition Theorem

Let $f(\alpha)$ be an analytic function of $\alpha = \sigma + i\tau$, regular in the strip $\tau_- < \tau < \tau_+$, such that $|f(\sigma + i\tau)| < C|\sigma|^{-p}, p > 0$, for $|\sigma| \to \infty$, the inequality holding uniformly for all τ in the strip $\tau_- + \varepsilon \leq \tau \leq \tau_+ - \varepsilon, \varepsilon > 0$. Then, for $\tau_- < c < \tau < d < \tau_+$;

$$f(\alpha) = f_-(\alpha) + f_+(\alpha)$$

with

$$f_+(\alpha) = \frac{1}{2\pi i}\int_{-\infty+ic}^{\infty+ic} \frac{f(\zeta)}{\zeta - \alpha} d\zeta \text{ and } \quad f_-(\alpha) = -\frac{1}{2\pi i}\int_{-\infty+id}^{\infty+id} \frac{f(\zeta)}{\zeta - \alpha} d\zeta,$$

where $f_+(\alpha)$ is regular for all $\tau > \tau_-$ and $f_-(\alpha)$ is regular for all $\tau < \tau_+$ respectively. [19]

2.4.2 Multiplicative Factorization Theorem

If $K(\alpha)$ satisfies the conditions of theorem above, which implies in particular that $K(\alpha)$ is regular and non-zero in a strip $\tau_- < \tau < \tau_+$, $-\infty < \sigma < \infty$ and $K(\alpha) \to +1$ as $\sigma \to \pm\infty$ in the strip, then we can write

$$K(\alpha) = K_-(\alpha)\, K_+(\alpha)$$

with

$$K_+(\alpha) = exp\left[\frac{1}{2\pi i}\int_{-\infty+ic}^{\infty+ic} \frac{In\, K\,(\zeta)}{\zeta-\alpha} d\zeta\right] \text{ and } K_-(\alpha) = exp\left[-\frac{1}{2\pi i}\int_{-\infty+id}^{\infty+id} \frac{In\, K(\zeta)}{\zeta-\alpha} d\zeta\right],$$

where $K_-(\alpha)$, and $K_+(\alpha)$ are regular, bounded, and non-zero in $\tau > \tau_-$ and $\tau < \tau_+$, respectively. [19]

2.4.3 Extended form of Liouville's Theorem

If $f(z)$ is an entire function such that $|f(z)| \leq M|z|^p$ as $|z| \to \infty$ where M, p are constants, then $f(z)$ is a polynomial pf degree less than or equal to $[p]$ where $[p]$ is the integral part of p. [19]

2.4.4 Residue Theorem

Suppose $f(z)$ is analytic inside and on a simple closed contour C except for isolated singularities at $z_1, z_2, \dots, z_n$ inside C. Then

$$\oint f(z) = 2\pi i \sum_{i=1}^{n} Res[f(z); z_i],$$

[29].

2.4.5 Infinite Product Theorem

Consider an entire function $K(\alpha)$ which

a). is an even function of α, that is $K(\alpha) = K(-\alpha)$,

b). has simple zeros at $\alpha = \pm i\alpha_n, n = 1, 2, \dots$, and $\alpha_n \to an + b$, $n \to \infty$.

Then $K(\alpha)$ can be represented by either of the two forms

$$K(\alpha) = \begin{cases} K(0) \prod_{n=1}^{\infty} (1 - \frac{\alpha}{i\alpha_n})(1 + \frac{\alpha}{i\alpha_n}) \\ K(0) \prod_{n=1}^{\infty} (1 - \frac{\alpha}{i\alpha_n}) e^{-\frac{i\alpha}{an}} \prod_{n=1}^{\infty} (1 + \frac{\alpha}{i\alpha_n}) e^{+\frac{i\alpha}{an}} \end{cases}$$

[23].

2.4.6 Jordan's Lemma

Suppose we have a circular arc C_R with center 0; If $m > 0$ and $f(z) = \frac{p(z)}{q(z)}$ such that degree of $p \geq 1 + degree\ of\ q$, then

$$\lim_{R\to\infty} \oint f(z)e^{\pm imz}dz = 0$$

With if $m > 0$: C_R is closed in the upper half plane

And if $m < 0$: C_R is closed in the lower half plane. [23]

2.5 The Wiener-Hopf Equation

We describe the method by considering a singular integral equation that arises in integral equation formulation of the mixed boundary value problems.

Consider the integral equation

$$\int_0^\infty k(x-\xi)f(\xi)d\xi = \mu f(x) + g(x), \quad 0 < x < \infty \tag{2.8}$$

where μ, $g(x)$ and $f(x)$ are given and wish to find $f(x), 0 < x < \infty$.

The first step is to extend the range of integration to infinite interval $-\infty < x < \infty$ as follows

$$\int_{-\infty}^\infty k(x-\xi)f(\xi)d\xi = \begin{cases} \mu f(x) + g(x), & 0 < x < \infty \\ h(x), & -\infty < x < 0 \end{cases} \tag{2.9}$$

with

$$h(x) = \int_{-\infty}^{0} k(x-\xi) f(\xi) d\xi$$

where

$$\begin{cases} f(x) = 0; \ \ 0 < x \\ g(x) = 0; \ 0 < x \\ h(x) = 0; \ \ x > 0 \end{cases}.$$

So, we call the above functions $f_+(x)$, $g_+(x)$ and $h_-(x)$ respectively. Thus, functions $f_+(x)$ and $g_+(x)$ which varnish for negative x are said to be right-sided functions, whereas, $h_-(x)$ is a left-sided function. Therefore equation (2.9) can be rewritten as

$$\int_{-\infty}^{\infty} k(x-\xi) f_+(\xi) d\xi = \mu f_+(x) + g_+(x) + h_-(x), \qquad -\infty < x < \infty. \quad (2.10)$$

We give some list the assumptions under which we will attempt to solve equation (2.11):

I. $k(x) = 0(e^{-c|x|})$ as $|x| \to \infty, c > 0$.

II. $g(x) = 0\left(e^{d'x}\right)$ as $x \to +\infty, d' < c$.

The integral on the left-side exists if $f(x)$ is of exponential order at infinity with exponential smaller than c. Thus we shall look for a solution $f(x)$ satisfying

III. $f(x) = 0\left(e^{d''x}\right)$ as $x \to +\infty, d'' < c$.

From I and III, it then follows that

IV. $h(x) = 0(e^{-c|x|})$ as $x \to -\infty$.

And on taking the Fourier transform of equation (2.10) we get

$$k^*(\alpha)f_+^*(\alpha) = \mu f_+^*(\alpha) + g_+^*(\alpha) + h_-^*(\alpha). \tag{2.11}$$

Hence, equation (2.11) is called the *Wiener-Hopf equation*, where the transforms are defined and analytic in the following regions:

$k^*(\alpha)$ in the strip $-c < \mathrm{Im}(\alpha) < c$;

$f_+^*(\alpha)$ in the upper half-plane $\mathrm{Im}(\alpha) > d''$;

$g_+^*(\alpha)$ in the upper half-plane $\mathrm{Im}(\alpha) > d'$;

$h_-^*(\alpha)$ in the lower half-plane $\mathrm{Im}(\alpha) < c$.

If we let $d = \max(d', d'')$, then all the functions are valid in the strip $d < \mathrm{Im}(\alpha) < c$ of analyticity. (see Stakgold [22] for more details).

2.6 Solution of the Wiener-Hopf Equation

We can rewrite equation (2.11) as

$$f_+^*(\alpha)\ \{k^*(\alpha) - \mu\} - g_+^*(\alpha) = h_-^*(\alpha). \tag{2.12}$$

From equation (2.12), the function $k^*(\alpha) - \mu$ is neither positive nor negative function, therefore we need to factorize it into two non-zero functions by the use of factorization theorem 2.4.2. That is, we factorize it as:

$$k^*(\alpha) - \mu = k_-(\alpha)k_+(\alpha)$$

Therefore, equation (2.12) becomes

$$f_+^*(\alpha)k_+(\alpha) - \frac{g_+^*(\alpha)}{k_-(\alpha)} = \frac{h_-^*(\alpha)}{k_-(\alpha)}. \tag{2.13}$$

In the same way, we notice from equation (2.13) that $\frac{g_+^*(\alpha)}{k_-(\alpha)}$ is a mixed function; therefore it needs to be decomposed by the use of additive decomposition theorem 2.4.1, thus,

$$\frac{g_+^*(\alpha)}{k_-(\alpha)} = m_+(\alpha) + m_-(\alpha).$$

Finally, equation (2.13) becomes

$$f_+^*(\alpha)k_+(\alpha) - m_+^*(\alpha) = \frac{h_-^*(\alpha)}{k_-(\alpha)} + m_-^*(\alpha) = J(\alpha). \tag{2.14}$$

As the left hand side of the equation (2.14) is analytic in the upper half-plane while the right hand side is analytic in the lower half-pane, both sides are equal in the common strip; we conclude that these define an entire function $J(\alpha)$ by analytic continuation.

By an extended form of Liouville's theorem 2.4.3, this analytic function is constant. In most practical cases this constant can be evaluated to be zero by the asymptotic behavior of the left or right hand side.

Thus, the unknown function in equation (2.14) is

$$f_+^*(\alpha) = \frac{m_+^*(\alpha)}{k_+(\alpha)}. \tag{2.15}$$

Finally, on taking the Fourier inverse transform of equation (2.15), we simply get our function $f(x)$ as

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{m_+^*(\alpha)}{k_+(\alpha)} e^{-i\alpha z}\, d\alpha. \tag{2.16}$$

2.7 Modified Jones Method

In the modification due to D. S. Jones [15], the Wiener-Hopf functional equation is derived directly from the boundary value problem rather than having to first reduce it to the integral equation. To illustrate the main steps, we demonstrate the method by considering an incoming acoustic wave incident on a rigid half-plane. We consider the wave to be time harmonic but that is not a limitation as for the transient waves, we can first apply the Laplace transform in time and then proceed. We thus consider

$$u_{xx} + u_{yy} + k^2 u = 0, \tag{2.17}$$

where u is the diffracted wave given by $u = u_t - u_i$, where u_t is the velocity potential and u_i is the incident wave, and k is the wave number having a positive imaginary part. The following conditions apply:

(i) $\frac{\partial u_t}{\partial y} = 0$ on $y = 0$, $-\infty < x \leq 0$, so that

$$\frac{\partial u}{\partial y} = ik \sin\theta \; e^{-ik\,x\cos\theta}, y = 0, \; -\infty < x \leq 0. \tag{2.18}$$

(ii) $\frac{\partial u_t}{\partial y}$ and therefore $\frac{\partial u}{\partial y}$ are continuous on $y = 0$, $-\infty < x < \infty$.

(iii) u_t and therefore u are continuous on $y = 0$, $0 < x < \infty$.

(iv) For any fixed y, $y \geq 0$ or $y \leq 0$

(a) $|u| < C_3 e^{(k_2 \cos\theta - k_2|y|\sin\theta)}$, for $-\infty < x < -|y|\cot\theta$.

(b) $|u| < C_4 e^{-\{k_2(x^2+y^2)^{\frac{1}{2}}\}}$, for $-|y|\cot\theta < x < \infty$.

Finally near the edge of the half-plane at the origin we assume

(v) $\left(\frac{\partial u_t}{\partial y}\right) \to C_5 x^{-\frac{1}{2}}$ as $x \to +0$ on $y = 0$,

$u_t \to C_6$ as $x \to +0$ on $y = 0$,

$u_t \to C_7$ as $x \to +0$ on $y = +0$,

$u_t \to C_8$ as $x \to +0$ on $y = -0$.

We define from the sided-functions the half-range Fourier transform as

$$\mathrm{U}(\alpha, y) = \mathrm{U}_-(\alpha, y) + \mathrm{U}_+(\alpha, y) = \int_{-\infty}^{0} u(x,y)\, e^{i\alpha x} dx + \int_0^{\infty} u(x,y)\, e^{i\alpha x} dx. \tag{2.19}$$

Now, from condition (iv), for a given y, $|u| < D_1 e^{-k_2 x}$ as $x \to +\infty$ and

$|u| < D_2 e^{k_2 x \cos\theta}$ as $x \to -\infty$ where D_1, D_2 are constants.

Thus, $\mathsf{U}_+$is analytic for $\tau > -k_2$, U_- is analytic for $\tau < k_2 \cos\theta$, and U is analytic in the strip $-k_2 < \tau < k_2 \cos\theta$.

Also,

$$|\mathsf{U}| \leq |\mathsf{U}_-| + |\mathsf{U}_+|.$$

If we now take the Fourier transform in x to equation (2.17), we get

$$\frac{dU(\alpha, y)}{dy} - \gamma^2 U(\alpha, y) = 0, \quad \gamma = (\alpha^2 - k^2)^{\frac{1}{2}}. \tag{2.20}$$

More details on $\gamma = (\alpha^2 - k^2)^{\frac{1}{2}}$ are explained in Nobel [19]. Thus, the solution of equation (2.20) takes the form:

$$U(\alpha, y) = \begin{cases} A_1(\alpha) e^{-\gamma x} + B_1(\alpha) e^{\gamma x}, & y \geq 0 \\ A_2(\alpha) e^{-\gamma x} + B_2(\alpha) e^{\gamma x}, & y \leq 0 \end{cases}, \tag{2.21}$$

where $A_1, B_1 A_2,$ and $B_2$are functions of α and U is discontinuous at $y = 0$. Further, from equation (2.20), the real part of γ is always positive in $-k_2 < \tau < k_2$ and therefore in equation (2.21) we must take $B_1 = A_2 = 0$. From condition (ii), $\frac{\partial u}{\partial y}$ is continuous across$y = 0$. Hence $\frac{\partial U}{\partial y}$ is continuous across $y = 0$ and we can set

$$A_1(\alpha) = -B_2(\alpha) = A(\alpha), \qquad \text{say}.$$

Hence

$$U(\alpha, y) = \begin{cases} A(\alpha)e^{-\gamma x}, & y \geq 0 \\ A(\alpha)e^{\gamma x}, & y \leq 0. \end{cases}$$

(2.22)

Now, when a transform is discontinuous across $y = 0$, we extend the notation by writing for instance $U(\alpha, +0)$ or $U(\alpha, -0)$ to mean that the limit as y tends to zero approached from positive value of y or approached from the left respectively. Thus, from condition (ii), $U'_+(\alpha, +0) = U'_+(\alpha, -0) = U'_+(\alpha, 0)$, and similarly for $U'_-(\alpha, 0)$. Likewise from condition (iii), $U_+(\alpha, +0) = U_+(\alpha, -0) = U(\alpha, 0)$.

Thus, applying the definition above to equation (2.22), we get the following

$$U_+(\alpha, 0) + U_-(\alpha, +0) = A(\alpha), \tag{2.23a}$$

$$U_+(\alpha, 0) + U_-(\alpha, -0) = -A(\alpha), \tag{2.23b}$$

$$U'_+(\alpha, 0) + U'_-(\alpha, 0) = -\gamma A(\alpha). \tag{2.23c}$$

Eliminating $A(\alpha)$ in equation (2.33); adding equations (2.33a) and (2.33b) we get

$$2U_+(\alpha, 0) = -U_-(\alpha, +0) - U_-(\alpha, -0). \tag{2.24}$$

On subtracting equation (2.33b) from (2.33a) and eliminating $A(\alpha)$ between the resulting equation and equation (2.33a), we obtain

$$U'_+(\alpha, 0) + U'_-(\alpha, 0) = -\frac{1}{2}\gamma\{U_-(\alpha, +0) - U_-(\alpha, -0)\}. \tag{2.25}$$

Thus, from equation (2.25), $U'_-(\alpha, 0)$ is known from condition (i) after evaluating it as

$$U'_-(\alpha, 0) = \int_{-\infty}^{0} ik \sin\theta \; e^{-ik\,x\cos\theta}\, e^{i\alpha x} dx = \frac{k\sin\theta}{\alpha - k\cos\theta}. \tag{2.26}$$

For simplicity, we let

$$U_-(\alpha, +0) - U_-(\alpha, -0) = 2D_-(\alpha) \text{ and } U_-(\alpha, +0) + U_-(\alpha, -0) = 2S_-(\alpha), \tag{2.27}$$

where $D_-(\alpha)$ and $S_-(\alpha)$ represent the difference and sum of two functions respectively that are both analytic for $\tau < k_2 \cos\theta$.

Equations (2.24) and (2.25) become:

$$U_+(\alpha, 0) = -S_-(\alpha), \tag{2.28}$$

$$U'_+(\alpha, 0) + \frac{k\sin\theta}{\alpha - k\cos\theta} = -\gamma D_-(\alpha). \tag{2.29}$$

Thus, equations (2.28) and (2.29) give the Wiener-Hopf functional equation with four unknowns $U_+(\alpha, 0), U'_+(\alpha, 0), S_-(\alpha)$ and $D_-(\alpha)$ that hold in the common strip of analyticity $-k_2 < \tau < k_2 \cos\theta$.

Finally, to solve equations (2.28) and (2.29), we apply the same procedure discussed above in section 2.6.

CHAPTER THREE

HEAT CONDUCTION IN A CIRCULAR HOLLOW CYLINDER AMIDST MIXED BOUNDARY CONDITIONS

3.0 Introduction

The heat conduction in circular cylinders is of interest in many engineering applications. In case of a reactor, the circular cylinders containing radioactive source are cooled by water. In such cases, part of the cylinder may be immersed in water giving rise to mixed boundary conditions on the outer surface of the cylinder. In this chapter, we determine the analytical solution of the transient heat conduction in a homogenous isotropic hollow infinite cylinder that is subjected to different boundary conditions on the outer surface while the inner surface temperature is kept at zero temperature throughout. Thus, the dimensions of the hollow cylinder are $-\infty < z < \infty$ in the axial direction and the cylinder occupies $a < r < b$ where a and b are positive and $a \neq b$. We further assume axial symmetry so that all field variables are independent of angle. The Jones' modification method of the Wiener-Hopf technique is utilized due to the mixed nature of the boundary conditions on the outer surface of the hollow cylinder.

3.1 Formulation of the Problem

We consider the classical three dimensional transient heat conduction equation in cylindrical coordinate system that is axially symmetric. The temperature distribution of an arbitrary point r, z, t on the hollow cylinder is given by $T(r, z, t)$. The infinite hollow cylinder under consideration is assumed to be kept at a constant zero temperature from within the cylinder at $r = a$, that is, the inner surface temperature from $-\infty < z < \infty$. Furthermore, on the outer surface, the respective temperature and heat flux are assumed on the different semi-infinite parts respectively as shown in Figure 3.1.

We write the heat conduction equation as follow

$$T_{rr} + \frac{1}{r}T_r + T_{zz} = \frac{1}{k}T_t \qquad a < r < b \qquad (3.1)$$

where, $T(r, z, t)$ is the temperature, z the horizontal length and r is the radius of the circular hollow cylinder. Moreover, k is the thermal diffusivity constant.

The boundary and initial conditions are as follows

i) The initial condition

$$T(r, z, 0) = 0 \quad \text{for } a < r < b \text{ and } -\infty < z < \infty. \qquad (3.2)$$

ii) The inner surface temperature on $r = a$ satisfies

$$T(a, z, t) = 0 \quad \text{for } -\infty < z < \infty, t > 0. \qquad (3.3)$$

iii) The outer surface temperature on $r = b$ satisfies

$T(b,z,t) = T_b e^{-\lambda z}$, for $0 < z < \infty, t > 0$; T_b and $\lambda > 0$ are constants. (3.4)

iv) The heat flux on $r = b$ is given by

$T_r(b,z,t) = 0$ for $-\infty < z < 0, t > 0$. (3.5)

In solving the above system, we put $T(r,z,t) = e^{-k\mu^2 t}u(r,z)$, { k is the thermal diffusivity, and μ is constant} in equation (3.1) to get

$$u_{rr} + \frac{1}{r}u_r + u_{zz} + \mu^2 u = 0. \quad (3.6)$$

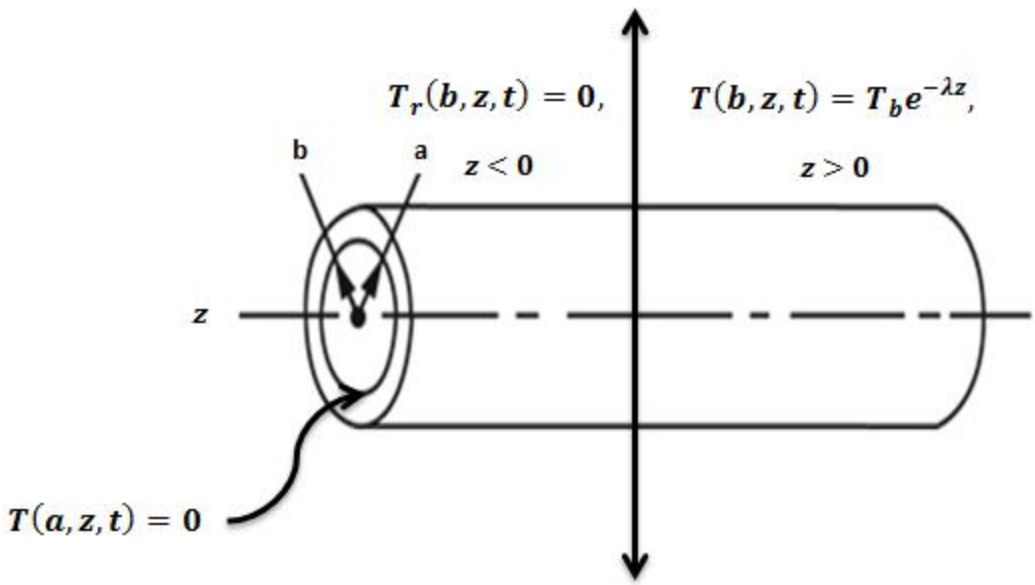

Figure 3.1: Geometry of the problem

3.2 Wiener-Hopf Equation

We define the Fourier transform in z and its corresponding inverse transform in α {if exist} as:

$$\mathcal{F}\{u(z)\} = \int_{-\infty}^{\infty} u(z)e^{i\alpha z}dz = u^*(\alpha), \tag{3.7}$$

$$\mathcal{F}^{-1}\{u^*(\alpha)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{u}(\alpha)e^{-i\alpha z}d\alpha = u(z). \tag{3.8}$$

We also give the half range Fourier transforms as

$$\int_{0}^{\infty} u(z)e^{i\alpha z}dx = u^*{}_+(\alpha), \tag{3.9}$$

$$\int_{-\infty}^{0} u(z)e^{i\alpha z}dx = u^*{}_-(\alpha). \tag{3.10}$$

We use the Fourier transform and half-range functions of Fourier transform described in chapter 2, and write

$$u^*(\alpha) = u^*{}_+(\alpha) + u^*{}_-(\alpha), \tag{3.11}$$

$u(\alpha) = 0(e^{\tau_- z})$ as $z \to \infty$ and $u(\alpha) = 0(e^{\tau_+ z})$ as $z \to -\infty$. Thus $u^*{}_+(\alpha)$ is an analytic function of α in the upper half-plane $\tau > \tau_-$, while $u^*{}_-(\alpha)$ is an analytic function of α in the lower half-plane $\tau < \tau_+$ respectively. Thus, $u^*(\alpha)$ defines an analytic function in the common strip $\tau_- < \tau < \tau_+$ with $\tau = Im(\alpha)$ where $\alpha = \sigma + i\tau$.

Now, taking the Fourier transform in z of equation (3.6), we obtain

$$u^*{}_{rr} + \frac{1}{r}u^*{}_r + (\mu^2 - \alpha^2)u^* = 0. \tag{3.12}$$

Similarly taking the Fourier transform of the boundary conditions, we obtain

i) $u^*(a, \alpha) = 0,$ (3.13)

ii) $u^*{}_+(b, \alpha) = i\frac{T_b}{\alpha + i\lambda},$ (3.14)

iii) $u^{*\prime}{}_-(b, \alpha) = 0.$ (3.15)

Solution of equation (14) is given by

$$u^*(r, \alpha) = A\,J_0(wr) + B\,Y_0(wr), \tag{3.16}$$

where $J_0(wr)$ and $Y_0(wr)$ are Bessel functions of first and second kinds respectively. Further, $w(\alpha)$ denotes the square-root function

$$w(\alpha) = \sqrt{\mu^2 - \alpha^2}, \tag{3.17}$$

which is defined in the complex α −plane, with cuts along $\alpha = \mu$ to $\alpha = \mu + i\infty$ and $\alpha = -\mu$ to $\alpha = -\mu - i\infty$ such as $w(0) = \mu$ (see [12]).

Thus, from the boundary condition given in equation (3.13) we obtain

$$A\,J_0(wa) + B\,Y_0(wa) = 0. \tag{3.18}$$

Similarly from boundary conditions in equations (3.14) and (3.14) we get the following respective equations

$$u^*{}_-(b, \alpha) + i\frac{T_b}{\alpha + i\lambda} = A\,J_0(wb) + B\,Y_0(wb), \tag{3.19}$$

$$u^{*\prime}{}_+(b, \alpha) = -w\{A\,J_1(wb) + B\,Y_1(wb)\}. \tag{3.20}$$

Therefore, solving for B from equation (3.18), as

$$B = -A\frac{J_0(wa)}{Y_0(wa)},$$

and substituting it into equations (3.19) and (3.20) we get the following equations

$$u^*{}_-(b,\alpha) + i\frac{T_b}{\alpha+i\lambda} = \frac{A}{Y_0(wa)}\{Y_0(wa)J_0(wb) - J_0(wa)Y_0(wb)\}, \tag{3.21}$$

$$u^{*\prime}{}_+(b,\alpha) = -w\frac{A}{Y_0(wa)}\{Y_0(wa)\,J_1(wb) - J_0(wa)Y_1(wb)\}. \tag{3.22}$$

Hence, from equations (3.21) and (3.22), we get the *Winer-Hopf equation* given in

$$u^*{}_-(b,\alpha) + i\frac{T_b}{\alpha+i\lambda} = \frac{T_1(a,b,\alpha)}{T_2(a,b,\alpha)}\,\hat{u}'_+(b,\alpha), \tag{3.23}$$

where,

$$T_1(a,b,\alpha) = Y_0(wa)J_0(wb) - J_0(wa)Y_0(wb), \tag{3.24a}$$

$$T_2(a,b,\alpha) = w\{J_0(wa)Y_1(wb) - Y_0(wa)\,J_1(wb)\}. \tag{3.24b}$$

For brevity sake, we write

$$M(\alpha) = \frac{T_1(a,b,\alpha)}{T_2(a,b,\alpha)},$$

that is,

$$M(\alpha) = \frac{T_1(a,b,\alpha)}{T_2(a,b,\alpha)} = \frac{Y_0(wa)J_0(wb) - J_0(wa)Y_0(wb)}{w\{J_0(wa)Y_1(wb) - Y_0(wa)\,J_1(wb)\}}. \tag{3.25}$$

3.3 Solution of the Wiener-Hopf Equation

To solve the Wiener-Hopf equation (3.23), we use the factorization theorem in chapter 2 over $M(\alpha)$ in equation (3.25) and factorize it into the product of $M_+(\alpha)$ and $M_-(\alpha)$ in

such a way that $M_+(\alpha)$ is analytic in the upper half-plane $\tau > \tau_-$ and $M_-(\alpha)$ analytic in the lower half-plane $\tau < \tau_+$ respectively given theoretically as

$$M_+(\alpha) = \exp\left\{\frac{1}{2\pi i}\int_{-\infty+ic}^{\infty+ic} \frac{In\, M(\zeta)}{\zeta-\alpha} d\zeta\right\}, \tag{3.26}$$

$$M_-(\alpha) = \exp\left\{-\frac{1}{2\pi i}\int_{-\infty+id}^{\infty+id} \frac{In\, M(\zeta)}{\zeta-\alpha} d\zeta\right\}. \tag{3.27}$$

From equations (3.26) and (3.27), c and d are chosen within the analytic region of $In\, M(\zeta)$. That is, $\tau_- < c < \tau < d < \tau_+$.

Thus, $M(\alpha)$ can be expressed using the infinite product factorization theorem [19] as

$$M(\alpha) = F^{\frac{1}{2}} \prod_{n=1}^{\infty} \left(\frac{\alpha^2 + \alpha_n^2}{\alpha^2 + \beta_n^2}\right); \;\; \alpha_n, \beta_n > 0,$$

where,

$$F = \prod_{n=1}^{\infty} \left\{\frac{\beta_n^2}{\alpha_n^2}\right\} \frac{\{Y_0(\mu a)J_0(\mu b) - J_0(\mu a)Y_0(\mu b)\}}{\mu\{J_0(\mu a)Y_1(\mu b) - Y_0(\mu a)\, J_1(\mu b)\}}.$$

That is,

$$M(\alpha) = F^{\frac{1}{2}} \prod_{n=1}^{\infty} \left(\frac{\alpha^2+\alpha_n^2}{\alpha^2+\beta_n^2}\right) = M_+(\alpha)M_-(\alpha), \tag{3.28}$$

where, $\pm i\alpha_n$ and $\pm\, i\beta_n$ are the simple zeros of $T_1(a, b, \alpha)$ and $T_2(a, b, \alpha)$ respectively for $n = 1,2, \ldots$ Furthermore, the explicit functions of $M_+(\alpha)$ and $M_-(\alpha)$ are and given in the Appendix I.

Finally, equation (3.23) becomes

$$\frac{u^*{}_-(b,\alpha)}{M_-(\alpha)} + i\frac{T_b}{(\alpha+i\lambda)M_-(\alpha)} = M_+(\alpha)u^{*\prime}{}_+(b,\alpha). \tag{3.29}$$

In the same way, the mixed term $i\frac{T_b}{(\alpha+i\lambda)M_-(\alpha)}$ in equation (3.29) needs to be decomposed either by the observation or through the use of additive decomposition theorem given in chapter 2, given by

$$i\frac{T_b}{(\alpha + i\lambda)M_-(\alpha)} = i\frac{T_b}{M_-(\alpha)} - i\frac{T_b}{M_-(-i\lambda)},$$

where, $i\frac{T_b}{M_-(\alpha)}$ is an analytic function in the lower half-plane, $\tau < \tau_+$, while $i\frac{T_b}{M_-(-i\lambda)}$ is an analytic function in the upper half-plane, $\tau > \tau_-$, respectively. Thus, equation (3.29) becomes

$$\frac{iT_b}{(\alpha+i\lambda)M_-(\alpha)} = i\frac{T_b}{(\alpha+i\lambda)}\left\{\frac{1}{M_-(\alpha)} - \frac{1}{M_-(-i\lambda)}\right\} + i\frac{T_b}{(\alpha+i\lambda)M_-(-i\lambda)}. \tag{3.30}$$

Thus, from equation (29), we obtain

$$\frac{\hat{u}_-(b,\alpha)}{M_-(\alpha)} + i\frac{T_b}{(\alpha+i\lambda)}\left\{\frac{1}{M_-(\alpha)} - \frac{1}{M_-(-i\lambda)}\right\} = -i\frac{T_b}{(\alpha+i\lambda)M_-(-i\lambda)} + M_+(\alpha)\hat{u}'_+(b,\alpha). \tag{3.31}$$

As the left hand side of the equations (3.31) is analytic in the lower half-plane$\tau < \tau_+$, while the right hand side is analytic in the upper half-pane, $\tau > \tau_-$, both sides are equal in the common strip $\tau_- < \tau < \tau_+$; we conclude that these define an entire function by analytic continuation. Thus, by an extended form of Liouville's theorem 2.4.3, this

analytic function is constant taken to be zero by the asymptotic behavior of the left or right hand side. Thus, we obtain our unknown functions as:

$$u^*{}_-(b,\alpha) = -i\frac{T_b M_-(\alpha)}{(\alpha+i\lambda)}\left\{\frac{1}{M_-(\alpha)} - \frac{1}{M_-(-i\lambda)}\right\}, \tag{3.32}$$

and

$$u^{*\prime}{}_+(b,\alpha) = i\frac{T_b}{(\alpha+i\lambda)M_-(-i\lambda)M_+(\alpha)}. \tag{3.33}$$

Equations (3.32) and (3.33) give the explicit expressions of the temperature distribution and heat flux of the hollow cylinder under consideration respectively in the transformed domain attached with $M_-(\alpha)$ and $M_+(\alpha)$; half-range analytic functions that are given in the Appendix I.

Having determined the unknown functions of $u^*{}_-(b,\alpha)$ and $u^{*\prime}{}_+(b,\alpha)$, we then proceed to find the unknown constants of A and B in equation (3.16) as:

$$A = \frac{Y_0(wa)u^{*\prime}{}_+(b,\alpha)}{T_2(a,b,\alpha)} \text{ and } B = -\frac{J_0(wa)u^{*\prime}{}_+(b,\alpha)}{T_2(a,b,\alpha)}$$

where,

$$T_2(a,b,\alpha) = w\{J_0(wa)Y_1(wb) - Y_0(wa)\,J_1(wb)\}.$$

We get the overall temperature distribution in the body from equations (3.16), (3.24) and (3.33) as follows:

$$u^*(r,\alpha) = \frac{u^{*\prime}{}_+(b,\alpha)}{T_2(a,b,\alpha)}\{Y_0(wa)J_0(wr) - J_0(wa)Y_0(wr)\}.$$

Where, $u^{*\prime}_{+}(b,\alpha)$ is already determined in equation (3.33). The Fourier inverse transform would now be taken to obtain the temperature $u(r,z)$ and the flux $u_r(r,z,)$ in the whole body in a space variable respectively. Hence, on taking the Fourier inverse transform, we get the overall temperature distribution in the body from equations follows:

$$u(r,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u^{*}(r,\alpha)e^{-i\alpha z}\, d\alpha.$$

That is,

$$u(r,z) = \frac{iT_b}{2\pi\, M_{-}(-i\lambda)}\int_{-\infty}^{\infty}\frac{Y_0(wa)J_0(wr)-J_0(wa)Y_0(wr)}{(\alpha+i\lambda)M_{+}(\alpha)T_2(a,b,\alpha)}e^{-i\alpha z}\, d\alpha. \qquad (3.34)$$

Substituting the values of $M_{+}(\alpha)$ and $T_2(a,b,\alpha)$ in equation (3.34) as expressed in the Appendix I; we get

$$u(r,z) = \frac{iT_b}{2\pi F^{\frac{1}{2}}\, T_2(a,b,0)M_{-}(-i\lambda)}\int_{-\infty}^{\infty}\frac{Y_0(wa)J_0(wr)-J_0(wa)Y_0(wr)}{(\alpha+i\lambda)\prod_{n=1}^{\infty}\{\alpha+i\alpha_n\}\prod_{n=1}^{\infty}\{\alpha-i\beta_n\}}e^{-i\alpha z}\, d\alpha. \qquad (3.35)$$

Evaluating the integral in equation (3.35) using the residue calculus together with *Jordan's lemma*; in which the integrand is having simple poles at $\alpha = -i\lambda, -i\alpha_n$ and $i\beta_n$ for $n = 1,2,3,\ldots$, we thus obtain;

$$u(r,z) = \frac{T_b}{F^{\frac{1}{2}}\, T_2(a,b,0) M_-(-i\lambda)} \left\{ \frac{Y_0(w_1 a) J_0(w_1 r) - J_0(w_1 a) Y_0(w_1 r)}{\prod_{n=1}^{\infty}\{-i\lambda + i\alpha_n\} \prod_{n=1}^{\infty}\{-i\lambda - i\beta_n\}} e^{-\lambda z} \right.$$

$$+ \sum_{j=1}^{\infty} \frac{Y_0(w_2 a) J_0(w_2 r) - J_0(w_2 a) Y_0(w_2 r)}{(-i\alpha_j + i\lambda) \prod_{n=1, n\neq j}^{\infty}\{-i\alpha_j + i\alpha_n\} \prod_{n=1}^{\infty}\{-i\alpha_j - i\beta_n\}} e^{-\alpha_j z}$$

$$\left. - \sum_{j=1}^{\infty} \frac{Y_0(w_3 a) J_0(w_3 r) - J_0(w_3 a) Y_0(w_3 r)}{(i\beta_j + i\lambda) \prod_{n=1}^{\infty}\{i\beta_j + i\alpha_n\} \prod_{n=1, n\neq j}^{\infty}\{i\beta_j - i\beta_n\}} e^{\beta_j z} \right\},$$

(3.36)

where from equation(3.36), $w_1 = \sqrt{\mu^2 + \lambda^2}$, $w_2 = \sqrt{\mu^2 + \alpha_j^2}$ and $w_3 = \sqrt{\mu^2 + \beta_j^2}$ for $j = 1,2,3,\ldots$.

Thus, the overall temperature distribution of the hollow cylinder under the assumption made earlier that $T(r,z,t) = e^{-k\mu^2 t} u(r,z)$ is

$$T(r,z,t) = \frac{T_b}{F^{\frac{1}{2}}\, T_2(a,b,0) M_-(-i\lambda)} \left\{ \frac{Y_0(w_1 a) J_0(w_1 r) - J_0(w_1 a) Y_0(w_1 r)}{\prod_{n=1}^{\infty}\{-i\lambda + i\alpha_n\} \prod_{n=1}^{\infty}\{-i\lambda - i\beta_n\}} e^{-\lambda z} \right.$$

$$+ \sum_{j=1}^{\infty} \frac{Y_0(w_2 a) J_0(w_2 r) - J_0(w_2 a) Y_0(w_2 r)}{(-i\alpha_j + i\lambda) \prod_{n=1, n\neq j}^{\infty}\{-i\alpha_j + i\alpha_n\} \prod_{n=1}^{\infty}\{-i\alpha_j - i\beta_n\}} e^{-\alpha_j z}$$

$$\left. - \sum_{j=1}^{\infty} \frac{Y_0(w_3 a) J_0(w_3 r) - J_0(w_3 a) Y_0(w_3 r)}{(i\beta_j + i\lambda) \prod_{n=1}^{\infty}\{i\beta_j + i\alpha_n\} \prod_{n=1, n\neq j}^{\infty}\{i\beta_j - i\beta_n\}} e^{\beta_j z} \right\} e^{-k\mu^2 t}.$$

(3.37)

3.4 Heat Flux on the Surface

In practical problems, we are more concerned about the heat flux rather than the temperature distribution. Now, we define the heat flux by

$$q_r(r,z,t) = -ke^{-k\mu^2 t}\frac{\partial u(r,z)}{\partial r}, \tag{3.38}$$

given that $T(r,z,t) = e^{-k\mu^2 t}u(r,z)$.

Now, on taking the Fourier transform in z of equation (3.38), we get

$$q_r^*(r,\alpha,t) = -ke^{-k\mu^2 t}\frac{\partial u^*(r,\alpha)}{\partial r}. \tag{3.39}$$

Thus, substituting the heat flux obtained in equation (3.33) at $r = b$ into equation (3.39), we get

$$q_r^*(b,\alpha,t) = -ke^{-k\mu^2 t}i\frac{T_b}{(\alpha + i\lambda)M_-(-i\lambda)M_+(\alpha)}. \tag{3.40}$$

Now, taking the Fourier inverse transform 'α', we get

$$q_r(b,z,t) = -\frac{ke^{-k\mu^2 t}iT_b}{2\pi M_-(-i\lambda)}\int_{-\infty}^{\infty}\frac{e^{-i\alpha z}}{(\alpha + i\lambda)M_+(\alpha)}\,d\alpha. \tag{3.41}$$

Again, since $M_+(\alpha)$ is given in the Appendix I as: $M_+(\alpha) = F^{\frac{1}{2}}\prod_{n=1}^{\infty}\left\{\frac{\alpha+i\alpha_n}{\alpha+i\beta_n}\right\}$, where $\pm i\alpha_n$ and $\pm i\beta_n$ are simple zeros of $T_1(a,b,\alpha)$ and $T_2(a,b,\alpha)$ respectively.

Thus, equation (3.41) becomes

$$q_r(b,z,t) = -\frac{ke^{-k\mu^2 t} iT_b}{2\pi F^{\frac{1}{2}} M_-(-i\lambda)} \int_{-\infty}^{\infty} \frac{\prod_{n=1}^{\infty}(\alpha + i\beta_n)\, e^{-i\alpha z}}{\prod_{n=1}^{\infty}(\alpha + i\alpha_n)\,(\alpha + i\lambda)}\, d\alpha. \tag{3.42}$$

Finally, applying the residue theorem by enclosing the contour in the lower half-plane, the contributions of the poles as $\alpha = -i\lambda$ and $-i\alpha_n$ for $n = 1,2,3 \ldots$ give the overall heat flux on the surface of the cylinder as

$$q_r(b,z,t) = \frac{kT_b}{F^{\frac{1}{2}} M_-(-i\lambda)} \left[\sum_{j=1}^{\infty} \frac{\prod_{n=1}^{\infty}(i\beta_n - i\alpha_j) e^{-\alpha_j z}}{(i\alpha_j - i\lambda) \prod_{n=1, n\neq j}^{\infty}(i\alpha_n - i\alpha_j)} - \frac{e^{-\lambda z}}{M_+(-i\lambda)} \right] e^{-k\mu^2 t}. \tag{3.43}$$

CHAPTER FOUR

HEAT CONDUCTION IN A CIRCULAR CYLINDER WITH GENERAL BOUNDARY CONDITIONS

4.0 Introduction

In chapter two, we introduced in details the Jones' modification of the Wiener-Hopf technique and there after used to solve a wave diffraction problem in two-dimension subject to known mixed boundary conditions. There again, we determined the solution of

$$u_{xx} + u_{yy} + k^2 u = 0 \text{ in } y \geq 0, -\infty < x < \infty,$$

such that u represents the outgoing waves at infinity; also $y = 0$, the mixed conditions were given on $u = 0, x > 0$ and $u_y = ik \sin\theta \; e^{-ikx\cos\theta}$. Furthermore, Nobel [19] stated that the same problem can have generalized mixed boundary conditions, that is, on $y = 0$; $u = f(x)$ for $x > 0$ and $u_y = g(x)$ for $x < 0$ respectively, where $|f(x)| < C_1 e^{\tau - x}$ as $x \to \infty$ and $|g(x)| < C_2 e^{\tau + x}$ as $x \to -\infty$ with C_1 and C_2 are constants.

However, in this chapter, we intend to study and solve the heat conduction problem in an infinite circular solid cylinder with finite radius when subjected to general mixed boundary conditions on its outer surface. The generalized mixed boundary conditions present on both the half-plane surfaces of the cylinder allow the use of the Jones' modification of the Wiener-Hopf technique based on the application of Fourier

transforms. In this regard, the solution of the boundary value problem is aimed at determining the temperature distribution and that of the heat flux of the problem under consideration. Lastly, we determine an explicit analytical solution of the problem for some special boundary conditions of interest.

4.1 Formulation of the Problem

We consider an infinite cylindrical rod of uniform cross section and finite radius r. The cylinder is assumed to have the general mixed boundary conditions on the boundary; that is, the temperature is given to by $f(r, z, t)$ in $-\infty < z < 0$ while $g(r, z, t)$ represents the heat flux in $0 < z < \infty$ as shown in Figure 4.1. The temperature T in the cylinder satisfies the governing equation

$$T_{rr} + \frac{1}{r}T_r + \frac{1}{r^2}T_{\theta\theta} + T_{zz} = \frac{1}{k}T_t \tag{4.1}$$

and further assume that the heat conduction is axially symmetry, so that equation (4.1) becomes

$$T_{rr} + \frac{1}{r}T_r + T_{zz} = \frac{1}{k}T_t \tag{4.2}$$

where k is the thermal diffusivity, expressed as $k = \frac{\upsilon}{\rho C}$ with υ thermal conductivity, ρ density and c specific heat constant all depending on the material. The initial and boundary conditions are as follows

(i) The initial condition

$$T(r,z,t) = 0; \;\; at\ t = 0. \tag{4.3}$$

(ii) The temperature on $z < 0$,

$$T(r,z,t) = f(r,z,t); \quad -\infty < z < 0, \quad 0 \leq r \leq a. \tag{4.4}$$

(iii) The heat flux on $z > 0$,

$$T_z(r,z,t) = g(r,z,t); \quad 0 < z < \infty, \quad 0 \leq r \leq a. \tag{4.5}$$

(iv) The continuity condition at $z = 0$,

$$T^{+}(r,z,t) = T^{-}(r,z,t); \; at\ z = 0. \tag{4.6}$$

In addition, the functions $f(r,z,t)$ and $g(r,z,t)$ are assumed to be of exponential order as $|z|$ turns ∞.

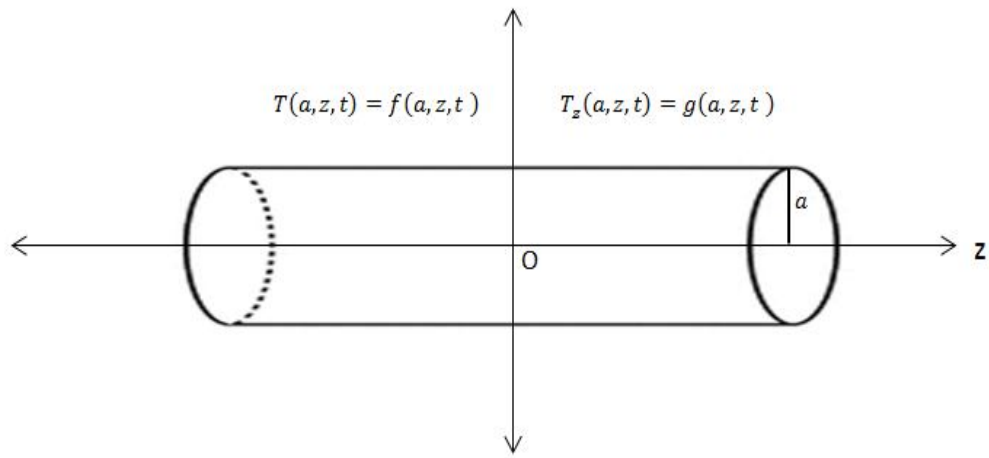

Figure 4.1: Geometry of the problem

4.2 Wiener-Hopf Equation

The Laplace transform in the time variable t and its inverse transform in s are defined {if exist} by:

$$\mathcal{L}\{T(t)\} = \int_0^\infty T(t)e^{-st}dt = \bar{T}(s) \tag{4.7}$$

and

$$\mathcal{L}^{-1}\{\bar{T}(s)\} = \frac{1}{2\pi i}\int_{-i\infty+h}^{i\infty+h} \bar{T}(s)e^{st}ds = T(t) \tag{4.8}$$

In the same way, we define the Fourier transform in z and its corresponding inverse Fourier transform in α {if exist} by:

$$\mathcal{F}\{T(z)\} = \int_{-\infty}^{\infty} T(z)e^{i\alpha z}dz = T^*(\alpha) \tag{4.9}$$

and

$$\mathcal{F}^{-1}\{T^*(\alpha)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} T^*(\alpha)e^{-i\alpha z}d\alpha = T(z) \tag{4.10}$$

with $\alpha = \sigma + i\tau$.

Moreover, we also introduce the half range Fourier transforms as

$$\int_{0}^{\infty} T(z)e^{i\alpha z}dz = T_+^*(\alpha) \tag{4.11}$$

and

$$\int_{-\infty}^{0} T(z)e^{i\alpha z}dz = T_-^*(\alpha) \tag{4.12}$$

So that,

$$T^*(\alpha) = T_+^*(\alpha) + T_-^*(\alpha), \tag{4.13}$$

$T(z) = 0(e^{\tau_- z})$ as $z \to \infty$ and $T(z) = 0(e^{\tau_+ z})$ as $z \to -\infty$. Thus $T_+^*(\alpha)$ is an analytic function of α in the upper half-plane $\tau > \tau_-$, while $T_-^*(\alpha)$ is an analytic function of α in the lower half-plane $\tau < \tau_+$ respectively. Thus, $T^*(\alpha)$ defined an analytic function in the common strip $\tau_- < \tau < \tau_+$ with $\tau = Im(\alpha)$.

We now take the Laplace transform in t and Fourier transform in z of equation (4.2) to get

$$\bar{T}_{rr}^* + \frac{1}{r}\bar{T}_r^* - \left(\alpha^2 + \frac{s}{k}\right)\bar{T}^* = 0. \tag{4.14}$$

The transformed boundary conditions are given by

(i) $\bar{T}^*(r, \alpha, 0) = 0,$ (4.15)

(ii) $\bar{T}_-^*(a, \alpha, s) = \bar{f}_-^*(a, \alpha, s),$ (4.16)

(iii) $\bar{T}_+^{*\prime}(a, \alpha, s) = \bar{g}_+^*(a, \alpha, s),$ (4.17)

(iv) $\bar{T}_-^*(r, 0, s) = \bar{T}_+^*(r, 0, s).$ (4.18)

Where T' is the differentiation of T with respect to z variable and later transformed to α after taking the Fourier transform of T with respect to z. Hence, the solution of equation (4.14) is given by

$$\bar{T}^*(r,\alpha,s) = A(\alpha)I_0(qr) + B(\alpha)K_0(qr). \tag{4.19}$$

Where $I_0(qr)$ and $K_0(qr)$ are modified Bessel functions of first and second kinds respectively, and $q(\alpha) = \sqrt{\alpha^2 + \frac{s}{k}}$. Furthermore, for the boundedness of our solution, we obtain our solution from (4.19) as

$$\bar{T}^*(r,\alpha,s) = A(\alpha)I_0(qr) \text{ for } 0 \le r \le a. \tag{4.20}$$

Thus,

$$\bar{T}^*(r,\alpha,s) = \begin{cases} A_1(\alpha)I_0(qr), & \text{for } -\infty < z < 0 \\ A_2(\alpha)I_0(qr), & \text{for } \quad 0 < z\ \infty \end{cases}. \tag{4.21}$$

From boundary conditions (3.16) and (3.17) we get

$$\bar{T}_+^*(a,\alpha,s) + \bar{f}_-^*(a,\alpha,s) = A_1(\alpha)I_0(qa), \tag{4.22}$$

and

$$\bar{T}_-^{*\prime}(a,\alpha,s) + \bar{g}_+^*(a,\alpha,s) = A_2(\alpha)\ a\ \alpha\frac{I_1(qa)}{q}. \tag{4.23}$$

Also, from continuity boundary condition in equation (4.18) at $z = 0$ we get

$$A_1(\alpha)\ I_0\left(\sqrt{\frac{s}{k}}a\right) = A_2(\alpha)\ I_0\left(\sqrt{\frac{s}{k}}a\right)\ \forall\, r \in [0,a].$$

Thus,

$$A_1(\alpha) = A_2(\alpha) = A(\alpha). \tag{4.24}$$

Thus, equations (4.22) and (4.23) become

$$\bar{T}_{+}^{*}(a,\alpha,s) + \bar{f}_{-}^{*}(a,\alpha,s) = A(\alpha)I_0(qa), \tag{4.25}$$

and

$$\bar{T}_{-}^{*\prime}(a,\alpha,s) + \bar{g}_{+}^{*}(a,\alpha,s) = A(\alpha)a\,\alpha\frac{I_1(qa)}{q}. \tag{4.26}$$

From equations (4.24) and (4.26), we get the Wiener-Hopf equation given by

$$\bar{T}_{-}^{*\prime}(a,\alpha,s) + \bar{g}_{+}^{*}(a,\alpha,s) = \frac{a\alpha\, I_1(qa)}{qI_0(qa)}\left\{\bar{T}_{+}^{*}(a,\alpha,s) + \bar{f}_{-}^{*}(a,\alpha,s)\right\}. \tag{4.27}$$

4.3 Solution of the Wiener-Hopf Equation

In equation (4.27), the mixed term

$$\frac{a\alpha\, I_1(qa)}{qI_0(qa)} = \frac{a}{i}\alpha\frac{J_1(iqa)}{qJ_0(iqa)},$$

is denoted by $K(\alpha)$. {Note that $I_n(z) = i^{-n}J_n(iz)$ as stated in chapter 1}.

We then factorize $K(\alpha)$ being a product of two meromorphic functions using factorization theorem *2.4.2* as

$$K(\alpha) = \frac{a\,\alpha\, I_1(qa)}{i\,q\,I_0(qa)} = K_{+}(\alpha)K_{-}(\alpha) \tag{4.28}$$

(see Mittra and Lee [18] and Nobel [19]). Where $K_{+}(\alpha)$ and $K_{-}(\alpha)$ are analytic in the upper and lower half planes respectively. The expression for $K_{+}(\alpha)$ and $K_{-}(\alpha)$ are given in Appendix II in equations D_1 and D_2 respectively. Thus, equation (4.27) becomes

$$\frac{\bar{T}_{-}^{*\prime}(a,\alpha,s)}{K_{-}(\alpha)} + \frac{\bar{g}_{+}^{*}(a,\alpha,s)}{K_{-}(\alpha)} = K_{+}(\alpha)\bar{T}_{+}^{*}(a,\alpha,s) + K_{+}(\alpha)\bar{f}_{-}^{*}(a,\alpha,s). \tag{4.29}$$

Again, decomposing the mixed terms in equation (4.29) using additive decomposition theorem *2.4.1* we get as follows

$$M(\alpha) = \frac{\bar{g}_+^*(a,\alpha,s)}{K_-(\alpha)} - K_+(\alpha)\bar{f}_-^*(a,\alpha,s) = M_+(\alpha) + M_-(\alpha), \tag{4.30}$$

where $M_+(\alpha)$ and $M_-(\alpha)$ are given to be

$$M_+(\alpha) = \frac{1}{2\pi i}\int_{-\infty+ic}^{\infty+ic}\left\{\frac{\bar{g}_+^*(a,\zeta,s)}{K_-(\zeta)} - K_+(\zeta)\bar{f}_-^*(a,\zeta,s)\right\}\frac{1}{\zeta-\alpha}d\zeta, \tag{4.31}$$

and

$$M_-(\alpha) = -\frac{1}{2\pi i}\int_{-\infty+id}^{\infty+id}\left\{\frac{\bar{g}_+^*(a,\zeta,s)}{K_-(\zeta)} - K_+(\zeta)\bar{f}_-^*(a,\zeta,s)\right\}\frac{1}{\zeta-\alpha}d\zeta, \tag{4.32}$$

respectively, where c and d are in the analytic region.

That is, $\tau_- < c < Im(\alpha) < d < \tau_+$. Finally, equation (4.30) becomes

$$\frac{\bar{T}_-^{*\prime}(a,\alpha,s)}{K_-(\alpha)} + M_-(\alpha) = K_+(\alpha)\bar{T}_+^*(a,\alpha,s) - M_+(\alpha). \tag{4.33}$$

Equation (4.33) defines an entire function in the whole $\alpha - plane$ by analytic continuation. In which the left hand side is analytic in the lower half-plane $\tau < \tau_+$, while the right hand side is analytic in the upper half plane $\tau > \tau_-$ respectively. In addition, both sides can be shown to be zero by the extended form of Liouville's theorem 2.4.3 as $\alpha \to \pm\infty$.

Thus, our unknown functions are found to be:

$$\bar{T}_+^*(a,\alpha,s) = \frac{M_+(\alpha)}{K_+(\alpha)}, \tag{4.34}$$

and

$$\bar{T}_-^{*\prime}(a,\alpha,s) = -M_-(\alpha)K_-(\alpha). \tag{4.35}$$

We note that $K_-(\alpha)$ and $K_+(\alpha)$ are given by

$$K_+(\alpha) = \frac{\sqrt{a}}{\sqrt{i}}\frac{L_+(\alpha)}{P_+(\alpha)},$$

$$K_-(\alpha) = \frac{\sqrt{a}}{\sqrt{i}}\alpha\frac{L_-(\alpha)}{P_-(\alpha)},$$

and their explicit details are given in Appendix II in equations D_1 and D_2, while $M_-(\alpha)$ and $M_+(\alpha)$ are given by equations (4.31) and (4.32) in terms of known functions

respectively. Equations (4.34) and (4.35) can be then used together with (4.25) and (4.26) to determine the overall temperature distribution in the transformed domain as

$$\bar{T}^*(r,\alpha,s) = \frac{\bar{T}_+^*(a,\alpha,s) + \bar{f}_-^*(a,\alpha,s)}{I_0(qa)} I_0(qr).$$

The inverse Laplace transform and the inversion Fourier transform can then be taken to obtain the temperature distribution $T(r,z,t)$ and the heat flux $T_z(r,z,t)$ respectively. Hence, on taking these inversions we got the overall temperature distribution of the body under consideration as follows:

$$T(r,z,t) = \frac{1}{4\pi^2 i} \int_{-\infty}^{\infty} \int_{-i\infty+h}^{i\infty+h} \left\{ \frac{\bar{T}_+^*(a,\alpha,s) + \bar{f}_-^*(a,\alpha,s)}{I_0(qa)} I_0(qr) \right\} e^{st} e^{-i\alpha z}\, ds\, d\alpha. \qquad (4.36)$$

4.4 Evaluation of the Temperature Distribution & Heat Flux in Some Special Cases

Here, we intend to consider some boundary conditions specified on the outer surface of the cylinder under consideration. That is, we consider a special case of transient heat conduction, that is, the steady periodic heat conduction of the type

$$T(r,z,t) = T(r,z)\, e^{-i\omega t},$$

where ω is the angular frequency. However, to determine the explicit analytical solutions of the temperature distribution and the heat flux, we use the method of transforming a transient heat conduction problem to a steady state problem suggested by Nobel [19]. That is, we transform the Laplace transform parameter s to $i\omega$ in the above equations (4.31), (4.32), (4.34) and (4.35) respectively and also in $K_-(\alpha)$ and $K_+(\alpha)$ functions given in the Appendix II respectively.

4.4.1 Case I

In case I, we assume the boundary conditions of the type

$$T(r,z,t) = T_o, \quad z < 0 \quad and \quad T_z(r,z,t) = 0, \; z > 0 \text{ at } r = a \tag{4.37}$$

Transforming the boundary conditions after taking Laplace transform in t and Fourier transform in z we get,

$$\bar{T}^*_-(a,\alpha,s) = \frac{T_o}{i\alpha s} \text{ and } \bar{T}^{*\prime}_+(a,\alpha,s) = 0, \tag{4.38}$$

where T_- and $T_+{}'$ are the temperature distribution for $z < 0$ and is the heat flux for $z > 0$ at $r = a$ respectively, and the prime stands for the derivative with respect to z.

Here, we use the method by Nobel [19] of transforming s to $i\omega$ in the equation (4.38), that is, $s \leftrightarrow i\omega$, we get

$$T^*_-(a,\alpha) = -\frac{T_o}{\alpha\omega} \quad \text{and} \quad T^{*\prime}_+(a,\alpha) = 0 \tag{4.39}$$

So, on substituting the above transformed boundary conditions into equations (4.31) and (4.32), we get

$$M_+(\alpha) = \frac{T_o}{2\pi i}\frac{\sqrt{a}}{\omega\sqrt{i}}\int_{-\infty+ic}^{\infty+ic}\frac{L_+(\zeta)}{\zeta(\zeta-\alpha)P_+(\zeta)}\,d\,\zeta, \tag{4.40}$$

and

$$M_-(\alpha) = -\frac{T_o}{2\pi i}\frac{\sqrt{a}}{\omega\sqrt{i}}\int_{-\infty+id}^{\infty+id}\frac{L_+(\zeta)}{\zeta(\zeta-\alpha)P_+(\zeta)}\,d\,\zeta, \tag{4.41}$$

where, $P_+(\zeta) = \sqrt{J_0(i\sqrt{\frac{i\omega}{k}}a)}\prod_{m=1}^{\infty}(1+\frac{\zeta}{i\alpha_m})$ and $L_+(\zeta) = \sqrt{J_1(i\sqrt{\frac{i\omega}{k}}a)}\prod_{n=1}^{\infty}(1+\frac{\zeta}{i\alpha_n})$.

Thus, to evaluate the integrals in equations (4.40) and (4.41); we examine the simple poles there present and make use of the complex residue calculus considering the contours of integration closed in the upper half-plane by a semi-circle for both $M_{-}(\alpha)$ and $M_{+}(\alpha)$ as evaluated below:

For $M_{+}(\alpha)$; we have simple poles at $\zeta = 0$, $\zeta = \alpha$ and $\zeta = -i\alpha_m = \beta_m$ for $P_{+}(\zeta)$, where $\alpha_m = i\sqrt{\left(\frac{j_{0,m}}{a}\right)^2 + \frac{i\omega}{k}}$ for $m = 1, 2, \ldots$ and $j_{0,m}$ are the zeros of Bessel function of first kind of order zero (see Appendix III).

Hence, equation (4.40) is evaluated below after closing the contour in the upper half-plane such that all the poles lie inside, $\{\tau_{-} < c < d\}$ as:

$$M_{+}(\alpha) = \frac{T_o\sqrt{a}}{\omega\sqrt{i}}\left[\sum_{m=1}^{\infty}\frac{A_m}{\alpha - \beta_m} + \frac{L_{+}(\alpha)}{\alpha P_{+}(\alpha)} - \frac{L_{+}(0)}{\alpha P_{+}(0)}\right], \tag{4.42}$$

where, $A_m = \frac{-L_{+}(\beta_m)}{\beta_m P_{+}'(\beta_m)}$.

Similarly, for $M_{-}(\alpha)$, we have simple poles at $\beta_k, \beta_{k+1}, \ldots,$ $k \ll m$. Where $\beta_k{}'s$ are above the line $-\infty + id$ to $\infty + id$ as such those $c < d < \tau_{+}$. Thus, equation (4.41) is evaluated as:

$$M_{-}(\alpha) = \frac{T_o\sqrt{a}}{\omega\sqrt{i}}\sum_{k=1}^{\infty}\frac{\boldsymbol{B_k}}{\alpha - \beta_k}, \tag{4.43}$$

where $\boldsymbol{B_k} = \frac{L_{+}(\beta_k)}{\beta_k P_{+}'(\beta_k)}$.

4.4.1.1 Evaluation of the Temperature Distribution

To determine the temperature distribution on the surface of the cylinder at $r = a$ for $z > 0$ having determined the explicit function of $M_+(\alpha)$ in equation (4.42); we get the following formula from equation (4.34):

$$T_+^*(\alpha) = \frac{M_+(\alpha)}{K_+(\alpha)},$$

where

$$K_+(\alpha) = \frac{\sqrt{a}}{\sqrt{i}} \frac{L_+(\alpha)}{P_+(\alpha)}$$

Thus, our transformed temperature for $z > 0$ from equations (4.34) and (4.42) and after using the expression of $K_+(\alpha)$ we obtain:

$$T_+^*(a, \alpha) = \frac{T_o}{\omega}\left[\sum_{m=1}^{\infty} A_m \frac{P_+(\alpha)}{(\alpha - \beta_m)L_+(\alpha)} + \frac{1}{\alpha} - \frac{L_+(0)P_+(\alpha)}{\alpha P_+(0)L_+(\alpha)}\right]. \tag{4.44}$$

Now, to evaluate the temperature distribution for $z > 0$, in z-variable; we take the Fourier inverse transform of equation (4.44) with respect to z as

$$T_+(a, z) = \frac{T_o}{\omega}\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\sum_{m=1}^{\infty} A_m \frac{P_+(\alpha)}{(\alpha - \beta_m)L_+(\alpha)} + \frac{1}{\alpha} - \frac{L_+(0)P_+(\alpha)}{\alpha P_+(0)L_+(\alpha)}\right] e^{-i\alpha z} d\alpha. \tag{4.45}$$

Here, evaluation of equation (4.45) follows the same procedure applied to equations (4.40) and (4.41), that is, via residue calculus.

Equation (4.45) has simple poles at $\alpha = 0, \alpha = \beta_m$ for $m = 1,2, \ldots$ and

$\alpha = -i\alpha_n = \beta_n$ for $L_+(\alpha)$, where $\alpha_n = i\sqrt{\left(\frac{j_{1,n}}{a}\right)^2 + \frac{i\omega}{k}}$ for $n = 1, 2, \ldots$ and

$\beta_m = \sqrt{\left(\frac{j_{0,m}}{a}\right)^2 + \frac{i\omega}{k}}$ for $m = 1,2, \ldots$ with $j_{0,m}$ and $j_{1,n}$ are the zeros of Bessel function of first kind of order zero and order one respectively.

So, from equation (4.45), we apply *Jordan's lemma 2.4.6* as follows:

For $z > 0$; we close the contour in the lower half-plane and since there is no pole in the closed contour, then

$$T_+(a, z) = 0 \text{ for } z > 0. \tag{4.46}$$

For $z < 0$; we close the contour in the upper half-plane, but in such a way that $\alpha = 0$ would lie in the upper half-plane (i.e. would be inside the semi-circle). Thus, we get

$$T_+(a, z) = \frac{T_o}{\omega} i \left\{\sum_{j=1}^{\infty}\left[A_j \frac{P_+(\beta_j)}{L_+(\beta_j)} e^{-i\beta_j z} + \sum_{n=1, n\neq j}^{\infty} A_j \frac{P_+(\beta_n)}{(\beta_n - \beta_j) L'_+(\beta_n)} e^{-i\beta_n z}\right] - \sum_{n=1}^{\infty} \frac{L_+(0) P_+(\beta_n)}{\beta_n P_+(0) L'_+(\beta_n)} e^{-i\beta_n z}\right\}, \quad \text{for, } z < 0 \tag{4.47}$$

where, $A_j = \frac{-L_+(\beta_j)}{\beta_j P'_+(\beta_j)}$.

Thus the overall temperature distribution is obtained by adding equations (4.46) and (4.47) as

$$T_+(a, z) = \frac{T_o}{\omega} i \left\{\sum_{j=1}^{\infty}\left[A_j \frac{P_+(\beta_j)}{L_+(\beta_j)} e^{-i\beta_j z} + \sum_{n=1, n\neq j}^{\infty} A_j \frac{P_+(\beta_n)}{(\beta_n - \beta_j) L'_+(\beta_n)} e^{-i\beta_n z}\right] - \sum_{n=1}^{\infty} \frac{L_+(0) P_+(\beta_n)}{\beta_n P_+(0) L'_+(\beta_n)} e^{-i\beta_n z}\right\} \tag{4.48}$$

Hence, to regain the time parameter in the temperature distribution, we get from our earlier assumption that

$$T_+(a,z,t) = \frac{T_o}{\omega} i \left\{ \sum_{j=1}^{\infty} \left[A_j \frac{P_+(\beta_j)}{L_+(\beta_j)} e^{-i\beta_j z} + \sum_{n=1, n\neq j}^{\infty} A_j \frac{P_+(\beta_n)}{(\beta_n - \beta_j) L'_+(\beta_n)} e^{-i\beta_n z} \right] - \sum_{n=1}^{\infty} \frac{L_+(0) P_+(\beta_n)}{\beta_n P_+(0) L'_+(\beta_n)} e^{-i\beta_n z} \right\} e^{-i\omega t}. \tag{4.49}$$

4.4.1.2 Evaluation of the Heat Flux

The heat flux of the cylinder on $r = a$ for $z < 0$ is to be determined from equation (4.35) given that the heat flux on $z > 0$ is zero from equation (4.37); as

$$T^{*\prime}_-(a,\alpha) = -M_-(\alpha) K_-(\alpha),$$

where

$$K_-(\alpha) = \frac{\sqrt{a}}{\sqrt{i}} \alpha \frac{L_-(\alpha)}{P_-(\alpha)}$$

Thus, we get the heat flux function in the transformed domain for $z < 0$ from equations (4.37) and (4. 43) and also by the use of $K_-(\alpha)$ function as:

$$T^{*\prime}_-(a,\alpha) = -\frac{T_o a}{\omega i} \sum_{k=1}^{\infty} \frac{B_k}{\alpha - \beta_k} \alpha \frac{L_{(\alpha)}}{P_{(\alpha)}}. \tag{4.50}$$

And on taking the Fourier inversion transform in z, we obtain:

$$T^{*\prime}_-(a,z) = -\frac{T_o a}{\omega i} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \sum_{k=1}^{\infty} B_k \frac{\alpha L_{(\alpha)}}{(\alpha - \beta_k) P_{(\alpha)}} e^{-i\alpha z} d\alpha. \tag{4.51}$$

So, we apply *Jordan's lemma 2.4.6* as follow to evaluate equation (4.51) as follows: For $z > 0$, we close the contour in the lower half-plane with simple poles at $\alpha = -\beta_m$, $m = 1,2,\ldots$ we obtain

$$T^{*\prime}_{-}(a,z) = \frac{T_o a}{\omega}\sum_{j=1}^{\infty}\left\{\sum_{m=1}^{\infty}\boldsymbol{B_j}\frac{\beta_m L_{-}(-\beta_m)}{(\beta_m+\beta_j)P'_{-}(-\beta_m)}e^{i\beta_m z}\right\}, z > 0 \tag{4.52}$$

Similarly, for $z < 0$, we close the contour in the upper half-plane with the simple poles at $\alpha = \beta_k$, $k = 1,2,\ldots$ we obtain

$$T^{*\prime}_{-}(a,z) = -\frac{T_o a}{\omega}\sum_{j=1}^{\infty}\boldsymbol{B_j}\frac{\beta_j L_{-}(\beta_j)}{P'_{-}(\beta_j)}e^{-i\beta_j z}, z < 0 \tag{4.53}$$

Thus, the overall heat flux for $z < 0$ is the summation of equations (4.52) and (4.53) as

$$T^{*\prime}_{-}(a,z) = -\frac{T_o a}{\omega}\sum_{j=1}^{\infty}\left\{\boldsymbol{B_j}\frac{\beta_j L_{-}(\beta_j)}{P'_{-}(\beta_j)}e^{-i\beta_j z} - \sum_{m=1}^{\infty}\boldsymbol{B_j}\frac{\beta_m L_{-}(-\beta_m)}{(\beta_m+\beta_j)P'_{-}(-\beta_m)}e^{i\beta_m z}\right\} \tag{4.54}$$

where $\boldsymbol{B}_j = \frac{L_{+}(\beta_j)}{\beta_j P'_{+}(\beta_j)}$.

Also to regain the time variable from our assumption, equation (4.54) for heat flux is:

$$T^{*\prime}_{-}(a,z,t) = -\frac{T_o a}{\omega}\sum_{j=1}^{\infty}\left\{\boldsymbol{B_j}\frac{\beta_j L_{-}(\beta_j)}{P'_{-}(\beta_j)}e^{-i\beta_j z} - \sum_{m=1}^{\infty}\boldsymbol{B_j}\frac{\beta_m L_{-}(-\beta_m)}{(\beta_m+\beta_j)P'_{-}(-\beta_m)}e^{i\beta_m z}\right\}e^{-i\omega t}. \tag{4.55}$$

4.4.2 Case II

We consider the boundary conditions of the type:

$T(r,z,t) = T_o e^{\lambda z}, \lambda > 0,\ z < 0$ and $T_z(r,z,t) = 0, z > 0$ at $r = a$. (4.56)

So, transforming the boundary conditions after taking Laplace transform in t and Fourier transform in z we get,

$\bar{T}^*_-(a,\alpha,s) = \frac{T_o}{is(\alpha - i\lambda)}$ and $\bar{T}^{*\prime}_+(a,\alpha,s) = 0,$ (4.57)

where T_- and $T_+{}'$ are the temperature distribution for $z < 0$ and the heat flux for $z > 0$ at $r = a$ respectively, whereas the prime $\{'\}$ stands for the derivative of T with respect to z.

Here, we use the method by Nobel [19] of transforming s to $i\omega$ in the equation (4.56), that is, $s \leftrightarrow i\omega$, we get

$T^*_-(a,\alpha) = -\frac{T_o}{\omega(\alpha - i\lambda)}$ and $T^{*\prime}_+(a,\alpha) = 0.$ (4.58)

So, on substituting the above transformed boundary conditions into equations (4.31) and (4.32), we get

$$M_+(\alpha) = \frac{T_o}{2\pi i}\frac{\sqrt{a}}{\omega\sqrt{i}}\int_{-\infty+ic}^{\infty+ic}\frac{L_+(\zeta)}{(\zeta - i\lambda)(\zeta - \alpha)P_+(\zeta)}\,d\zeta \tag{4.59}$$

and

$$M_-(\alpha) = -\frac{T_o}{2\pi i}\frac{\sqrt{a}}{\omega\sqrt{i}}\int_{-\infty+id}^{\infty+id}\frac{L_+(\zeta)}{(\zeta - i\lambda)(\zeta - \alpha)P_+(\zeta)}\,d\zeta. \tag{4.60}$$

Where $P_+(\zeta) = \sqrt{J_0(i\sqrt{\frac{i\omega}{k}}a)\prod_{m=1}^{\infty}(1+\frac{\zeta}{i\alpha_m})}$ and $L_+(\zeta) = \sqrt{J_1(i\sqrt{\frac{i\omega}{k}}a)\prod_{n=1}^{\infty}(1+\frac{\zeta}{i\alpha_n})}$.

Thus, to evaluate the integrals in equations (4.59) and (4.60); we examine the simple poles there present and make use of the complex residue calculus considering the contours of integration.

For $M_+(\alpha)$; we have simple poles at $\zeta = i\lambda$, $\zeta = \alpha$ and $\zeta = -i\alpha_m = \beta_m$ for $P_+(\zeta)$, where $\alpha_m = i\sqrt{\left(\frac{j_{0,m}}{a}\right)^2 + \frac{i\omega}{k}}$ for $m = 1, 2, \ldots$ and $j_{0,m}$ are the zeros of Bessel function of first kind of order zero.

Hence, equation (4.59) is evaluated using the method as in above as

$$M_+(\alpha) = \frac{T_o\sqrt{a}}{\omega\sqrt{i}}\left[\sum_{m=1}^{\infty}\frac{A_m}{\alpha-\beta_m} + \frac{L_+(\alpha)}{(\alpha-i\lambda)P_+(\alpha)} - \frac{L_+(i\lambda)}{(\alpha-i\lambda)P_+(i\lambda)}\right], \tag{4.61}$$

where $A_m = \frac{-L_+(\beta_m)}{(\beta_m - i\lambda)P_+'(\beta_m)}$.

Similarly for $M_-(\alpha)$; we have simple poles at $\beta_k, \beta_{k+1}, \ldots,$ $k \ll m$. Where $\beta_k's$ are above the line $-\infty + id$ to $\infty + id$ as given in the above. Thus,

$$M_-(\alpha) = \frac{T_o\sqrt{a}}{\omega\sqrt{i}}\sum_{k=1}^{\infty}\frac{B_k}{\alpha-\beta_k}, \tag{4.62}$$

where $B_k = \frac{L_+(\beta_k)}{(\beta_k - i\lambda)P_+'(\beta_k)}$.

4.4.2.1 Evaluation of the Temperature Distribution

To determine the temperature distribution on the surface of the cylinder at $r = a$ for $z > 0$ having determined the explicit function of $M_+(\alpha)$ in equation (4.61); we got the following formula from equation (4.34):

$$T_+^*(\alpha) = \frac{M_+(\alpha)}{K_+(\alpha)},$$

where

$$K_+(\alpha) = \frac{\sqrt{a}}{\sqrt{i}} \frac{L_+(\alpha)}{P_+(\alpha)}$$

Thus, our temperature equation in the transformed domain for $z > 0$ is derived from equations (4.34) and (4.61) and indeed after using the expression of $K_+(\alpha)$ as follows

$$T_+^*(a,\alpha) = \frac{T_o}{\omega}\left[\sum_{m=1}^{\infty} A_m \frac{P_+(\alpha)}{(\alpha-\beta_m)L_+(\alpha)} + \frac{1}{\alpha - i\lambda} - \frac{L_+(i\lambda)P_+(\alpha)}{(\alpha - i\lambda)P_+(i\lambda)L_+(\alpha)}\right]. \tag{4.63}$$

Now, to evaluate the temperature distribution for $z > 0$, in z-variable; we take the Fourier inverse transform of equation (4.63) with respect to z as

$$T_+(a,z) = \frac{T_o}{\omega}\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\sum_{m=1}^{\infty} A_m \frac{P_+(\alpha)}{(\alpha-\beta_m)L_+(\alpha)} + \frac{1}{\alpha-i\lambda} - \frac{L_+(i\lambda)P_+(\alpha)}{(\alpha-i\lambda)P_+(i\lambda)L_+(\alpha)}\right] e^{-i\alpha z} d\alpha. \tag{4.64}$$

Here, evaluation of equation (4.64) follows the same procedure applied to equations (4.59) and (4.60), that is, via residue calculus.

Equation (4.64) has simple poles at $\alpha = i\lambda$, $\alpha = \beta_m$ for $m = 1,2,\ldots$ and

$\alpha = -i\alpha_n = \beta_n$ for $L_+(\alpha)$, where $\alpha_n = i\sqrt{\left(\frac{j_{1,n}}{a}\right)^2 + \frac{i\omega}{k}}$ for $n = 1, 2, \ldots$ and

$\beta_m = \sqrt{\left(\frac{j_{0,m}}{a}\right)^2 + \frac{i\omega}{k}}$ for $m = 1,2, \ldots$ with $j_{0,m}$ and $j_{1,n}$ the zeros of Bessel function of first kind of order zero and order one respectively.

So, from equation (4.64), we apply *Jordan's lemma* as follow: for $z > 0$; we close the contour in the lower half-plane and since there is no pole in the closed contour, then

$$T_+(a, z) = 0 \text{ for } z > 0. \tag{4.65}$$

For $z < 0$; we close the contour in the upper half-plane, but in such a way $\alpha = i\lambda$ would lie in the upper half-plane (i.e. would be inside the semi-circle). Thus, we get

$$T_+(a, z) = \frac{T_o}{\omega} i \left\{ \sum_{j=1}^{\infty} \left[A_j \frac{P_+(\beta_j)}{L_+(\beta_j)} e^{-i\beta_j z} + \sum_{n=1, n\neq j}^{\infty} A_j \frac{P_+(\beta_n)}{(\beta_n - \beta_j) L'_+(\beta_n)} e^{-i\beta_n z} \right] - \sum_{n=1}^{\infty} \frac{L_+(i\lambda) P_+(\beta_n)}{(\beta_n - i\lambda) P_+(i\lambda) L'_+(\beta_n)} e^{-i\beta_n z} \right\}$$

$$\text{for, } z < 0 \tag{4.66}$$

where $A_j = \frac{-L_+(\beta_j)}{(\beta_j - i\lambda) P'_+(\beta_j)}$.

Thus, the overall temperature distribution is obtained by adding equations (4.65) and (4. 66) as

$$T_+(a, z) = \frac{T_o}{\omega} i \left\{ \sum_{j=1}^{\infty} \left[A_j \frac{P_+(\beta_j)}{L_+(\beta_j)} e^{-i\beta_j z} + \sum_{n=1, n\neq j}^{\infty} A_j \frac{P_+(\beta_n)}{(\beta_n - \beta_j) L'_+(\beta_n)} e^{-i\beta_n z} \right] - \sum_{n=1}^{\infty} \frac{L_+(i\lambda) P_+(\beta_n)}{(\beta_n - i\lambda) P_+(i\lambda) L'_+(\beta_n)} e^{-i\beta_n z} \right\} \tag{4.67}$$

Hence, to regain the time parameter in the temperature distribution, we get from our earlier assumption that

$$T_+(a,z,t) = \frac{T_o}{\omega} i \left\{ \sum_{j=1}^{\infty} \left[A_j \frac{P_+\left(\beta_j\right)}{L_+\left(\beta_j\right)} e^{-i\beta_j z} + \sum_{n=1, n\neq j}^{\infty} A_j \frac{P_+\left(\beta_n\right)}{(\beta_n - \beta_j) L_+'\left(\beta_n\right)} e^{-i\beta_n z} \right] \right.$$

$$\left. - \sum_{n=1}^{\infty} \frac{L_+(i\lambda) P_+\left(\beta_n\right)}{(\beta_n - i\lambda) P_+(i\lambda) L_+'\left(\beta_n\right)} e^{-i\beta_n z} \right\} e^{-i\omega t}$$

(4.68)

4.4.2.2 Evaluation of the Heat Flux

The heat flux for $z < 0$ at $r = a$ as it is given to be zero on the other side by equation (4.56) is to be determined from equation (4.35) given by

$$T^{*\prime}_-(a,\alpha) = -M_-(\alpha) K_-(\alpha),$$

where,

$$K_-(\alpha) = \frac{\sqrt{a}}{\sqrt{i}} \alpha \frac{L_-(\alpha)}{P_-(\alpha)}$$

Thus, we get the heat flux function in the transformed domain for $z < 0$ from equations (4.57) and (4.62) and also by the use of $K_-(\alpha)$ function as:

$$T^{*\prime}_-(a,\alpha) = -\frac{T_o a}{\omega i} \sum_{k=1}^{\infty} \frac{B_k}{\alpha - \beta_k} \alpha \frac{L_-(\alpha)}{P_-(\alpha)}. \tag{4.69}$$

And on taking the Fourier inversion transform in z, we obtain:

$$T^{*\prime}_-(a,z) = -\frac{T_o a}{\omega i} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \sum_{k=1}^{\infty} B_k \frac{\alpha L_-(\alpha)}{(\alpha - \beta_k) P_-(\alpha)} e^{-i\alpha z} d\alpha \tag{4.70}$$

So, we apply *Jordan's lemma* to evaluate equation (4.70) as follows: for $z > 0$, we close the contour in the lower half-plane with simple poles at $\alpha = -\beta_m, m = 1,2, \ldots$ we obtain

$$T^{*\prime}_{-}(a,z) = \frac{T_o a}{\omega}\sum_{j=1}^{\infty}\left\{\sum_{m=1}^{\infty}\boldsymbol{B_j}\frac{\beta_m L_{-}(-\beta_m)}{(\beta_m+\beta_j)P'_{-}(-\beta_m)}e^{i\beta_m z}\right\}, z > 0 \tag{4.71}$$

Similarly, for $z < 0$, we close the contour in the upper half-plane with the simple poles at $\alpha = \beta_k,\ k = 1,2,\ldots$ we obtain

$$T^{*\prime}_{-}(a,z) = -\frac{T_o a}{\omega}\sum_{j=1}^{\infty}\boldsymbol{B_j}\frac{\beta_j L_{-}(\beta_j)}{P'_{-}(\beta_j)}e^{-i\beta_j z}, z < 0 \tag{4.72}$$

Thus, the overall heat flux for $z < 0$ is the summation of equations (4.71) and (4.72) as

$$T^{*\prime}_{-}(a,z) = -\frac{T_o a}{\omega}\sum_{j=1}^{\infty}\left\{\boldsymbol{B}_j\frac{\beta_j L_{-}(\beta_j)}{P'_{-}(\beta_j)}e^{-i\beta_j z} - \sum_{m=1}^{\infty}\boldsymbol{B}_j\frac{\beta_m L_{-}(-\beta_m)}{(\beta_m+\beta_j)P'_{-}(-\beta_m)}e^{i\beta_m z}\right\} \tag{4.73}$$

where $\boldsymbol{B_j} = \frac{L_{+}(\beta_j)}{(\beta_j - i\lambda)P'_{+}(\beta_j)}$.

Thus, to regain the time variable from our assumption, equation (4.73) for heat flux is:

$$T^{*\prime}_{-}(a,z,t) = -\frac{T_o a}{\omega}\sum_{j=1}^{\infty}\left\{\boldsymbol{B}_j\frac{\beta_j L_{-}(\beta_j)}{P'_{-}(\beta_j)}e^{-i\beta_j z} - \sum_{m=1}^{\infty}\boldsymbol{B}_j\frac{\beta_m L_{-}(-\beta_m)}{(\beta_m+\beta_j)P'_{-}(-\beta_m)}e^{i\beta_m z}\right\}e^{-i\omega t}. \tag{4.74}$$

4.4.3 Case III

In case III, we consider the boundary conditions of the form:

$$T(r,z,t) = T_o e^{-\mu t}, \mu > 0,\ z < 0 \text{ and } T_z(r,z,t) = 0,\ z > 0 \text{ at } r = a \tag{4.75}$$

So, transforming the boundary conditions after taking Laplace transform in t and Fourier transform in z we get,

$\bar{T}_{-}^{*}(a,\alpha,s) = \frac{T_o}{i\alpha(s+\mu)}$ and $\bar{T}^{*\prime}_{+}(a,\alpha,s) = 0,$ (4.76)

where T_{-} and $T_{+}{}'$ are the temperature distribution for $z < 0$ and is the heat flux for $z > 0$ at $r = a$ respectively, and the prime stands for the derivative with respect to z.

Here, we s to $i\omega$ in the equation (4.76), so that:

$T_{-}^{*}(a,\alpha) = -\frac{T_o}{\alpha(\omega-i\mu)}$ and $T_{+}^{*\prime}(a,\alpha) = 0.$ (4.77)

So, on substituting the above transformed boundary conditions into equations (4.31) and (4.32), we get

$$M_{+}(\alpha) = \frac{T_o}{2\pi i}\frac{\sqrt{a}}{(\omega - i\mu)\sqrt{i}}\int_{-\infty+ic}^{\infty+ic}\frac{L_{+}(\zeta)}{\zeta(\zeta-\alpha)P_{+}(\zeta)}\,d\,\zeta \tag{4.78}$$

and

$$M_{-}(\alpha) = -\frac{T_o}{2\pi i}\frac{\sqrt{a}}{(\omega - i\mu)\sqrt{i}}\int_{-\infty+id}^{\infty+id}\frac{L_{+}(\zeta)}{\zeta(\zeta-\alpha)P_{+}(\zeta)}\,d\,\zeta, \tag{4.79}$$

where $P_{+}(\zeta) = \sqrt{J_0(i\sqrt{\frac{i\omega}{k}}a)}\prod_{m=1}^{\infty}(1+\frac{\zeta}{i\alpha_m})$ and $L_{+}(\zeta) = \sqrt{J_1(i\sqrt{\frac{i\omega}{k}}a)}\prod_{n=1}^{\infty}(1+\frac{\zeta}{i\alpha_n})$.

Thus, to evaluate the integrals in equations (4.78) and (4.79); we examine the simple poles as in above to evaluate $M_{-}(\alpha)$ and $M_{+}(\alpha)$ as follows:

For $M_{+}(\alpha)$; we have simple poles at $\zeta = 0$, $\zeta = \alpha$ and $\zeta = -i\alpha_m = \beta_m$ for $P_{+}(\zeta)$, where $\alpha_m = i\sqrt{\left(\frac{j_{0,m}}{a}\right)^2 + \frac{i\omega}{k}}$ for $m = 1, 2, \ldots$ and $j_{0,m}$ are the zeros of Bessel function of first kind of order zero.

Hence, equation (4.78) is evaluated as:

$$M_+(\alpha) = \frac{T_o\sqrt{a}}{(\omega - i\mu)\sqrt{i}}\left[\sum_{m=1}^{\infty}\frac{A_m}{\alpha - \beta_m} + \frac{L_+(\alpha)}{\alpha P_+(\alpha)} - \frac{L_+(0)}{\alpha P_+(0)}\right] \tag{4.80}$$

where, $A_m = \frac{-L_+(\beta_m)}{\beta_m P'_+(\beta_m)}$.

Similarly for $M_-(\alpha)$; we have simple poles at $\beta_k, \beta_{k+1}, \ldots,$ $k \ll m$. Where $\beta_k's$ are above the line $-\infty + id$ to $\infty + id$ as described above. Thus,

$$M_-(\alpha) = \frac{T_o\sqrt{a}}{(\omega - i\mu)\sqrt{i}}\sum_{k=1}^{\infty}\frac{B_k}{\alpha - \beta_k}, \tag{4.81}$$

where $B_k = \frac{L_+(\beta_k)}{\beta_k P'_+(\beta_k)}$.

4.4.3.1 Evaluation of the Temperature Distribution

To determine the temperature distribution on the surface of the cylinder at $r = a$ for $z > 0$ having determined the explicit function of $M_+(\alpha)$ in equation (4.80); we got the following formula from equation (4.34):

$$T_+^*(\alpha) = \frac{M_+(\alpha)}{K_+(\alpha)},$$

where

$$K_+(\alpha) = \frac{\sqrt{a}}{\sqrt{i}}\frac{L_+(\alpha)}{P_+(\alpha)}$$

Thus, our temperature equation the transformed domain for $z > 0$ is derived from equations (4.34) and (4.80) and indeed after using the expression of $K_+(\alpha)$ as follows

$$T_+^*(a,\alpha) = \frac{T_o}{(\omega - i\mu)}\left[\sum_{m=1}^{\infty}A_m\frac{P_+(\alpha)}{(\alpha - \beta_m)L_+(\alpha)} + \frac{1}{\alpha} - \frac{L_+(0)P_+(\alpha)}{\alpha P_+(0)L_+(\alpha)}\right]. \tag{4.82}$$

Now, to evaluate the temperature distribution for $z > 0$, in z-variable; we take the Fourier inverse transform of equation (4.82) with respect to z as

$$T_+(a,z) = \frac{T_o}{(\omega - i\mu)} \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\sum_{m=1}^{\infty} A_m \frac{P_+(\alpha)}{(\alpha - \beta_m) L_+(\alpha)} + \frac{1}{\alpha} - \frac{L_+(0) P_+(\alpha)}{\alpha P_+(0) L_+(\alpha)} \right] e^{-i\alpha z} d\alpha. \qquad (4.83)$$

Here, evaluation of equation (4.83) follows the same procedure applied to equations (4.78) and (4.79), that is, via residue calculus.

Equation (4.83) has simple poles at $\alpha = 0$, $\alpha = \beta_m$ for $m = 1,2,\ldots$ and

$\alpha = -i\alpha_n = \beta_n$ for $L_+(\alpha)$, where $\alpha_n = i\sqrt{\left(\frac{j_{1,n}}{a}\right)^2 + \frac{i\omega}{k}}$ for $n = 1, 2, \ldots$ and

$\beta_m = \sqrt{\left(\frac{j_{0,m}}{a}\right)^2 + \frac{i\omega}{k}}$ for $m = 1,2,\ldots$ with $j_{0,m}$ and $j_{1,n}$ the zeros of Bessel function of first kind of order zero and order one respectively.

So, from equation (4.83), we apply *Jordan's lemma* as follow: for $z > 0$; we close the contour in the lower half-plane and since there is no pole in the closed contour, then

$$T_+(a,z) = 0 \text{ for } z > 0. \qquad (4.84)$$

For $z < 0$; we close the contour in the upper half-plane, but in such a way $\alpha = 0$ would lie in the upper half-plane (i.e. would be inside the semi-circle). Thus, we get

$$T_+(a,z) =$$

$$\frac{T_o}{(\omega - i\mu)} i \left\{ \sum_{j=1}^{\infty} \left[A_j \frac{P_+(\beta_j)}{L_+(\beta_j)} e^{-i\beta_j z} + \sum_{n=1, n\neq j}^{\infty} A_j \frac{P_+(\beta_n)}{(\beta_n - \beta_j) L'_+(\beta_n)} e^{-i\beta_n z} \right] - \sum_{n=1}^{\infty} \frac{L_+(0) P_+(\beta_n)}{\beta_n P_+(0) L'_+(\beta_n)} e^{-i\beta_n z} \right\},$$

$$z < 0, \qquad (4.85)$$

where, $A_j = \frac{-L_+(\beta_j)}{\beta_j P'_+(\beta_j)}$.

Thus the overall temperature distribution is obtained by adding equations (4.84) and (4.85) as

$T_+(a,z) =$

$$\frac{T_o}{(\omega-i\mu)} i \left\{ \sum_{j=1}^{\infty} \left[A_j \frac{P_+(\beta_j)}{L_+(\beta_j)} e^{-i\beta_j z} + \sum_{n=1,n\neq j}^{\infty} A_j \frac{P_+(\beta_n)}{(\beta_n-\beta_j)L'_+(\beta_n)} e^{-i\beta_n z} \right] - \sum_{n=1}^{\infty} \frac{L_+(0)P_+(\beta_n)}{\beta_n P_+(0)L'_+(\beta_n)} e^{-i\beta_n z} \right\}. \tag{4.86}$$

Hence, to regain the time parameter in the temperature distribution, we get from our earlier assumption that

$T_+(a,z,t) =$

$$\frac{T_o}{(\omega-i\mu)} i \left\{ \sum_{j=1}^{\infty} \left[A_j \frac{P_+(\beta_j)}{L_+(\beta_j)} e^{-i\beta_j z} + \sum_{n=1,n\neq j}^{\infty} A_j \frac{P_+(\beta_n)}{(\beta_n-\beta_j)L'_+(\beta_n)} e^{-i\beta_n z} \right] - \sum_{n=1}^{\infty} \frac{L_+(0)P_+(\beta_n)}{\beta_n P_+(0)L'_+(\beta_n)} e^{-i\beta_n z} \right\} e^{-i\omega t} \tag{4.87}$$

4.4.3.2 Evaluation of the Heat Flux

To determine the heat flux for $z < 0$ and $r = a$, as it is given to be zero on $z > 0$ by equation (4.75); is to be determined from equation (4.35) given by

$$T^{*\prime}_{-}(a,\alpha) = -M_-(\alpha)K_-(\alpha),$$

where

$$K_-(\alpha) = \frac{\sqrt{a}}{\sqrt{i}} \alpha \frac{L_-(\alpha)}{P_-(\alpha)}$$

Thus, we get the heat flux equation in the transformed domain for $z < 0$ from equations (4.77) and (4.81) and also by the use of $K_-(\alpha)$ function as:

$$T^{*\prime}_{-}(a,\alpha) = -\frac{T_o a}{(\omega - i\mu)i}\sum_{k=1}^{\infty}\frac{B_k}{\alpha - \beta_k}\,\alpha\frac{L_{-}(\alpha)}{P_{-}(\alpha)} \tag{4.88}$$

And on taking the Fourier inversion transform in z, we obtain:

$$T^{*\prime}_{-}(a,z) = -\frac{T_o a}{(\omega - i\mu)i}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\sum_{k=1}^{\infty}B_k\frac{\alpha L_{(\alpha)}}{(\alpha - \beta_k)P_{(\alpha)}}\,e^{-i\alpha z}d\alpha. \tag{4.89}$$

So, we apply *Jordan's lemma* as follow to evaluate equation (4.89) as follows: for $z > 0$, we close the contour in the lower half-plane with simple poles at $\alpha = -\beta_m, m = 1,2,\ldots$ we obtain

$$T^{*\prime}_{-}(a,z) = \frac{T_o a}{(\omega - i\mu)}\sum_{j=1}^{\infty}\left\{\sum_{m=1}^{\infty}\boldsymbol{B_j}\frac{\beta_m L_{-}(-\beta_m)}{(\beta_m + \beta_j)P'_{-}(-\beta_m)}e^{i\beta_m z}\right\}, z > 0 \tag{4.90}$$

Similarly, for $z < 0$, we close the contour in the upper half-plane with the simple poles at $\alpha = \beta_k,\ k = 1,2,\ldots$ we obtain

$$T^{*\prime}_{-}(a,z) = -\frac{T_o a}{(\omega - i\mu)}\sum_{j=1}^{\infty}\boldsymbol{B_j}\frac{\beta_j L_{-}(\beta_j)}{P'_{-}(\beta_j)}e^{-i\beta_j z}, z < 0 \tag{4.91}$$

Thus, the overall heat flux for $z < 0$ is the summation of equations (4.90) and (4.91) as

$$T^{*\prime}_{-}(a,z) = -\frac{T_o a}{(\omega - i\mu)}\sum_{j=1}^{\infty}\left\{\boldsymbol{B_j}\frac{\beta_j L_{-}(\beta_j)}{P'_{-}(\beta_j)}e^{-i\beta_j z} - \sum_{m=1}^{\infty}\boldsymbol{B_j}\frac{\beta_m\, L_{-}(-\beta_m)}{(\beta_m + \beta_j)P'_{-}(-\beta_m)}e^{i\beta_m z}\right\} \tag{4.92}$$

where, $\boldsymbol{B}_j = \frac{L_{+}(\beta_j)}{\beta_j P'_{+}(\beta_j)}$.

Thus, to regain the time variable from our assumption, equation (4.92) for heat flux is:

$$T^{*\prime}_{-}(a,z,t) = -\frac{T_o a}{(\omega - i\mu)}\sum_{j=1}^{\infty}\left\{\boldsymbol{B_j}\frac{\beta_j L_{-}(\beta_j)}{P'_{-}(\beta_j)}e^{-i\beta_j z} - \sum_{m=1}^{\infty}\boldsymbol{B_j}\frac{\beta_m\, L_{-}(-\beta_m)}{(\beta_m + \beta_j)P'_{-}(-\beta_m)}e^{i\beta_m z}\right\}e^{-i\omega t}$$

(4.93)

CHAPTER FIVE

CONCLUSION AND RECOMMENDATIONS

5.1 CONCLUSION

In conclusion, we have first considered a mixed boundary value problem arising from heat conduction of an infinite hollow circular cylinder and solved it using the Jone's modification of the Wiener-Hopf technique. The closed form solution of the temperature distribution and that of heat flux on the surface of the cylinder are later used to evaluate the overall temperature distribution of the body.

Further, in the later part, a mixed boundary value problem arising from heat conduction in an infinite homogeneous solid cylinder has been considered and solved using the same method. The boundary of the cylinder has been subjected to two different boundary conditions (general mixed boundary conditions), that is, one part of the boundary is held at a prescribed temperature while the heat flux is prescribed on the other part. The solutions of the respective temperature distribution and that of heat flux in the half-ranges of the cylinder have finally been obtained under some special cases of interest.

5.2 RECOMMENDATIONS

This work can be extended by:

- Including the angle dependence in the problem thereby dropping the assumption made for axial symmetry.
- Considering the heat conduction in concentric/coaxial cylinders consisting of two different materials. The interface conditions can be those of perfect contact with mixed boundary conditions on the surface or imperfect contact resulting in mixed conditions at the interface.

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APPENDICES

APPENDIX I

$$M(\alpha) = \frac{T_1(a,b,\alpha)}{T_2(a,b,\alpha)} = \frac{Y_0(wa)J_0(wb) - J_0(wa)Y_0(wb)}{w\{J_0(wa)Y_1(wb) - Y_0(wa)\,J_1(wb)\}}.$$

Where

$$T_1(a,b,\alpha) = Y_0(wa)J_0(wb) - J_0(wa)Y_0(wb), \qquad (A_1)$$

$$T_2(a,b,\alpha) = w\{J_0(wa)Y_1(wb) - Y_0(wa)\,J_1(wb)\}. \qquad (A_2)$$

The infinite product representation [] of the above expressions are given below:

$$T_1(a,b,\alpha) = T_1(a,b,0)\,\frac{\prod_{n=1}^{\infty}\{\alpha + i\alpha_n\}\prod_{n=1}^{\infty}\{\alpha - i\alpha_n\}}{\prod_{n=1}^{\infty}\alpha_n},$$

$$T_2(a,b,\alpha) = T_2(a,b,0)\,\frac{\prod_{n=1}^{\infty}\{\alpha + i\beta_n\}\prod_{n=1}^{\infty}\{\alpha - i\beta_n\}}{\prod_{n=1}^{\infty}\beta_n}.$$

Where $\pm i\alpha_n$ and $\pm i\beta_n$ are simple zeros of $T_1(a,b,\alpha)$ and $T_2(a,b,\alpha)$ respectively.

Thus,

$$M(\alpha) = \frac{T_1(a,b,\alpha)}{T_2(a,b,\alpha)} = F\prod_{n=1}^{\infty}\left\{\frac{\alpha + i\alpha_n}{\alpha + i\beta_n}\right\}\prod_{n=1}^{\infty}\left\{\frac{\alpha - i\alpha_n}{\alpha - i\beta_n}\right\},$$

with

$$M_{+}(\alpha) = F^{\frac{1}{2}} \prod_{n=1}^{\infty} \left\{ \frac{\alpha + i\alpha_n}{\alpha + i\beta_n} \right\} e^{i\frac{\alpha}{n\pi}\{b-a\}+\chi(\alpha)}, \qquad (B_1)$$

and

$$M_{-}(\alpha) = F^{\frac{1}{2}} \prod_{n=1}^{\infty} \left\{ \frac{\alpha - i\alpha_n}{\alpha - i\beta_n} \right\} e^{-i\frac{\alpha}{n\pi}\{b-a\}-\chi(\alpha)}, \qquad (B_2)$$

where,

$$F = \prod_{n=1}^{\infty} \left\{ \frac{\beta_n^2}{\alpha_n^2} \right\} \frac{\{Y_0(\mu a)J_0(\mu b) - J_0(\mu a)Y_0(\mu b)\}}{\mu\{J_0(\mu a)Y_1(\mu b) - Y_0(\mu a)\, J_1(\mu b)\}}$$

and

$$\chi(\alpha) = i\frac{\alpha}{\pi}\left[1 - C + \ln\left(\frac{2\pi}{b-a}\right) + i\frac{\pi}{2}\right].$$

where C is the Euler's Constant given by $0.57721\ldots$

APPENDIX II

Recall that from equation (4.28),

$$K(\alpha) = \frac{a\alpha\, J_1(iqa)}{i\, qJ_0(iqa)} = \frac{a}{i}\alpha\left\{\frac{J_1(iqa)}{q}\right\}\left\{\frac{1}{J_0(iqa)}\right\} = \frac{a}{i}\alpha\frac{L(\alpha)}{P(\alpha)} = \frac{a}{i}\alpha\frac{L_+(\alpha)}{P_+(\alpha)}\,\frac{L_-(\alpha)}{P_-(\alpha)},$$

thus,

$$K(\alpha) = K_+(\alpha)\, K_-(\alpha).$$

with

$$L(\alpha) = \frac{J_1(iqa)}{q} \text{ and } P(\alpha) = J_0(iqa).$$

For $L(\alpha) = \frac{J_1(iqa)}{q}$:

$\frac{J_1(iqa)}{q}$ is an even function of α that has no branch points or poles, but does have zeros when $J_1(iqa) = 0, q \neq 0$. The small zeros $j_{1,n}$ of $J_1(z)$ are tabulated in Abramowitz and Stegun [1]. Now,

$$L(\alpha) = L_+(\alpha)L_-(\alpha), \qquad (A_1)$$

where $L(\alpha)$ is given below using infinite product theorem [];

$$L(\alpha) = \frac{J_1(iqa)}{q} = \frac{J_0\left(i\sqrt{\frac{s}{k}}a\right)}{\sqrt{\frac{s}{k}}}\prod_{n=1}^{\infty}\{1 - \frac{\alpha}{i\alpha_n}\}\, e^{-\frac{i\alpha a}{n\pi}}\prod_{n=1}^{\infty}\{1 + \frac{\alpha}{i\alpha_n}\}\, e^{\frac{i\alpha a}{n\pi}}. \qquad (A_2)$$

Where and $\alpha_n = \pm i\sqrt{\left(\frac{j_{1,n}}{a}\right)^2 + i\frac{\omega}{k}}$, for $n = 1,2,3,\ldots$are the simple zeros of $\frac{J_1(iqa)}{q}$ given that $q = \sqrt{\alpha^2 + \frac{s}{k}}$.

Thus,

$$L_+(\alpha) = \sqrt{\frac{J_0\left(i\sqrt{\frac{s}{k}}a\right)}{\sqrt{\frac{s}{k}}}} \prod_{n=1}^{\infty}\{1 + \frac{\alpha}{i\alpha_n}\}\, e^{\frac{i\alpha a}{n\pi}+\chi(\alpha)} \qquad (A_3)$$

$$L_-(\alpha) = \sqrt{\frac{J_0\left(i\sqrt{\frac{s}{k}}a\right)}{\sqrt{\frac{s}{k}}}} \prod_{n=1}^{\infty}\{1 - \frac{\alpha}{i\alpha_n}\}\, e^{-\frac{i\alpha a}{n\pi}-\chi(\alpha)} \qquad (A_4)$$

where

$$\chi(\alpha) = i\frac{\alpha a}{\pi}\left[1 - C + \ln\left(\frac{2\pi}{ka}\right) + i\frac{\pi}{2}\right],$$

with C is the Euler's Constant given by $0.57721\ldots$

For $P(\alpha) = J_0(iqa)$:

$J_0(iqa)$ is an even function of α that has no branch points or poles, but does have zeros when $J_0(iqa) = 0$. The small zeros $j_{0,n}$ of $J_0(z)$ are tabulated in Abramowitz and Stegun [1] too.

$$P(\alpha) = P_+(\alpha)P_-(\alpha), \qquad (B_1)$$

$$P(\alpha) = J_0(iqa) = J_0\left(i\sqrt{\frac{s}{k}}a\right) \prod_{m=1}^{\infty}\{1 - \frac{\alpha}{i\alpha_m}\}\, e^{-\frac{i\alpha a}{m\pi}} \prod_{m=1}^{\infty}\{1 + \frac{\alpha}{i\alpha_m}\}\, e^{\frac{i\alpha a}{m\pi}} \qquad (B_2)$$

where $\alpha_m = \pm i\sqrt{\left(\frac{j_{0,n}}{a}\right)^2 + i\frac{\omega}{k}}$, for $m = 1,2,3,\ldots$are the simple zeros of $J_0(iqa)$.

Thus,

$$P_{+}(\alpha) = \sqrt{J_0\left(i\sqrt{\frac{s}{k}}a\right)}\prod_{m=1}^{\infty}\{1 + \frac{\alpha}{i\alpha_m}\}\, e^{\frac{i\alpha a}{m\pi}}, \qquad (B_3)$$

$$P_{(\alpha)} = \sqrt{J_0\left(i\sqrt{\frac{s}{k}}a\right)}\prod_{m=1}^{\infty}\left\{1 - \frac{\alpha}{i\alpha_m}\right\} e^{-\frac{i\alpha a}{m\pi}}. \qquad (B_4)$$

Thus, from appendices (A) and (B) above; we obtain

$$K(\alpha) = \frac{a\alpha\, J_1(iqa)}{i\, qJ_0(iqa)} = \frac{a}{i}\,\alpha\,\frac{L_{+}(\alpha)}{P_{+}(\alpha)}\,\frac{L_{-}(\alpha)}{P_{-}(\alpha)} = K_{+}(\alpha)\, K_{-}(\alpha), \qquad (C_1)$$

where,

$$K_{+}(\alpha) = \frac{\sqrt{a\, J_1\left(i\sqrt{\frac{s}{k}}a\right)}}{\sqrt{i\sqrt{\frac{s}{k}}\, J_0\left(i\sqrt{\frac{s}{k}}a\right)}}\,\frac{\prod_{n=1}^{\infty}\{1+\frac{\alpha}{i\alpha_n}\}}{\prod_{m=1}^{\infty}\{1+\frac{\alpha}{i\alpha_m}\}}\, e^{\frac{i\alpha a}{\pi}\left(\frac{1}{n}-\frac{1}{m}\right)+\chi(\alpha)}, \qquad (D_1)$$

$$K_{-}(\alpha) = \frac{\sqrt{a\, J_1\left(i\sqrt{\frac{s}{k}}a\right)}}{\sqrt{i\sqrt{\frac{s}{k}}\, J_0\left(i\sqrt{\frac{s}{k}}a\right)}}\,\frac{\alpha\prod_{n=1}^{\infty}\{1-\frac{\alpha}{i\alpha_n}\}}{\prod_{m=1}^{\infty}\{1-\frac{\alpha}{i\alpha_m}\}}\, e^{-\frac{i\alpha a}{\pi}\left(\frac{1}{n}-\frac{1}{m}\right)-\chi(\alpha)}. \qquad (D_2)$$

APPENDIX III

Roots of Bessel functions

The n-th roots of $J_0(z) = 0$ **The n-th roots of $J_1(z)$**

n	Root $J_0(z)$	n	Root $J_1(z)$
1	2.40482555769577	1	3.8317059702075
2	5.52007811028631	2	7.0155866698156
3	8.65372791291101	3	10.1734681350627
4	11.7915344390142	4	13.3236919363142
5 . . .	14.9309177084877	5 . . .	16.4706300508776

The n-th roots of $Y_0(z) = 0$ **The n-th roots of $Y_1(z)$**

n	Root $Y_0(z)$	n	Root $Y_1(z)$
1	0.8935769662791	1	2.1971413260310
2	3.9576784193148	2	5.4296810407941
3	7.0860510603017	3	8.5960058683311
4	10.2223450434964	4	11.7491548308398
5 . . .	13.3610974738727	5 . . .	14.8974421283367

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