SYMMETRY ANALYSIS OF THE WAVE EQUATION ON TWO SPHERICALLY SYMMETRIC SPACETIMES

BY

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To the memory of my parents

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Thesis Abstract

Name: Usamah Sadeg Al-Ali

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Lie symmetry method is the most general technique that can be used to investigate questions related to differential equations. One of the significant applications of Lie symmetry theory is to find exact solutions of differential equations with some natural symmetries inherited from the physical phenomena governing the differential equations.

This research is concerned with carrying out a symmetry analysis of wave equations on two spherically symmetric spacetimes, precisely

- the non-static spacetime with the metric

$$g = dt^2 - dr^2 - e^{t-r}\left(dy^2 + \sin^2(y)dz^2\right)$$

- and the Schwarzschild spacetime with the metric

$$g = \left(1 - \frac{2M}{r}\right)dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2\left(dy^2 + \sin^2(y)dz^2\right).$$

If $\Box$ denotes the Laplacian on the spacetime then the wave equations under study are equations of the form

$$\Box u = 0 \qquad (*)$$

and the aim is, for each case of the metric,

- To determine symmetries of the wave equation (*)
- To carry out different reductions of the wave equation (*) to a PDE in two variables utilizing a combination of symmetries
- To obtain exact solutions of wave equation (*) by further analyzing the reduced PDEs.

ملخص الرسالة

الاسم : أسامة صادق محمد العلي

عنوان الرسالة : التحليل التناظري للمعادلات التفاضلية في زمكانين متناظرين كرويا.

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طريقة تناظر " لي " هي الأكثر عمومية من بين الطرق التي تعنى ببحث المسائل المتعلقة بالمعادلات التفاضلية. أحد أهم التطبيقات لهذه الطريقة هو إيجاد الحلول الدقيقة للمعادلات التفاضلية ذات التناظرات الطبيعية الناشئة من الظواهر الفيزيائية التي تحكم هذه المعادلات.

هذا البحث يعنى بدراسة التحليل التناظري للمعادلات الموجية في زمكانين متناظرين كرويا. هذان الزمكانان هما :

- الزمكان غير الثابت المعرف بالدالة المترية

$$g = dt^2 - dr^2 - e^{t-r}\left(dy^2 + \sin^2(y)dz^2\right)$$

- الزمكان الشوارزشيلدي المعرف بالدالة المترية

$$g = \left(1 - \frac{2M}{r}\right)dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2\left(dy^2 + \sin^2(y)dz^2\right).$$

إذا كان $\Box$ يرمز لمؤثر لابلاس في الزمكان رباعي الابعاد فان المعادلات الموجية قيد الدراسة تكون على الصورة

$$\Box u = 0 \qquad (*)$$

و أهداف البحث بالنسبة للزمكانين قيد الدراسة هي :

- إيجاد التناظرات للمعادلة الموجية (*)
- تقليص المعادلة الموجية الى معادلة تفاضلية من الدرجة الثانية في متغيرين.
- إيجاد حلول دقيقة للمعادلة الموجية (*) وذلك بتحليل المعادلات المختزلة.

Introduction

The theory of General Relativity is currently the most widely accepted theory of gravitation [1]. It is at present the most accurately tested physical theory [2] for example on account of the observations of the Taylor-Hulse binary pulsar, for which Taylor and Hulse were awarded the Nobel Prize in Physics[3].

Einstein's theory regards the motion of objects not as being due to some driving force (as in Newton's theory) but as due to the curvature of spacetime, where the usual 3-dimensional space and 1-dimensional time are inextricably tied together as an indivisible whole spacetime [4]. A class of physically significant spacetimes are spherically symmetric which have been completely classified according to their isometries in [5], [6].

In this work, two types of spherically symmetric spacetimes will be considered. These consist of a non-static spacetime and the Schwarzschild spacetime. In both cases, the wave equation on the metrics associated to these spacetimes will be studied. The tool playing the central role in this study is going to be the method of Lie symmetries. Similar work can also be found in [7].

The method of studying differential equations using their symmetries was introduced by the Norwegian mathematician Sophus Lie more than a century ago. In this research, the method of Lie symmetries will be briefly reviewed and then will be applied to investigate wave equation on the two spacetimes mentioned above. Chapters 1 and 2 provide an introduction to symmetries of ODEs whereas chapter 3 presents the Lie symmetry method for studying PDEs.

A non-static spherically symmetric spacetime with the metric

$$g = dt^2 - dr^2 - e^{t-r}\left(dy^2 + \sin^2(y)dz^2\right)$$

is considered in chapter 4, and a symmetry analysis of the corresponding wave equation

$$u_t + u_{tt} + u_r - u_{rr} - e^{r-t}\cot(y)u_y - e^{r-t}u_{yy} - e^{r-t}\csc^2(y)u_{zz} = 0 \qquad (*)$$

is carried out. Precisely, symmetries of the wave equation (*) obtained and are further applied to find the following four new solutions of (*).

The first solution is

$$u_1(t,r,y,z) = \frac{1}{2}e^r\left[h(t+r)\ln\left|\sin(y)\cos(z)-1\right| - h(t+r)\ln\left|\sin(y)\cos(z)+1\right| + 2g(t+r)\right].$$

The second solution is

$$u_2(t,r,y,z) = ke^{\frac{1}{2}(a-1)t+\frac{1}{2}(a+1)r}\left[\ln\left|1+\sin(y)\cos(z)\right| + \ln\left|1-\sin(y)\cos(z)\right| + c_1\right].$$

The third solution is

$$u_3(t,r,y,z) = ke^{\frac{1}{2}(a-1)t+\frac{1}{2}(a+1)r}\left[\ln\left|1+\sin(y)\sin(z)\right| + \ln\left|1-\sin(y)\sin(z)\right| + c_1\right].$$

The fourth solution is

$$u_4(t,r,y,z) = \cos\left(\frac{b}{a-c}\ln\left|\csc(y)-\cot(y)\right| + c_2\right)e^{\frac{1}{2}\left[1+\frac{b}{c-a}\right](r-t)+c_1+\frac{b}{c-a}(t-z)}.$$

The final chapter is focused towards studying the wave equation on Schwarzschild spacetime given by

$$u_{tt} - \frac{2}{r}u_r + \frac{6M}{r^2}u_r - u_{rr} + \frac{4M}{r}u_{rr} - \frac{4M^2}{r^3}u_r - \frac{4M^2}{r^2}u_{rr} - \frac{\cot(y)}{r^2}u_y - \frac{1}{r^2}u_{yy}$$

$$+\frac{2M\cot(y)}{r^3}u_y + \frac{2M}{r^3}u_{yy} - \frac{1}{r^2\sin^2(y)}u_{zz} + \frac{2M}{r^3\sin^2(y)}u_{zz} = 0 \qquad (**)$$

where the Schwarzschild spacetime is described by the metric

$$g=\left(1-\frac{2M}{r}\right)dt^2-\left(1-\frac{2M}{r}\right)^{-1}dr^2-r^2\left(dy^2+\sin^2(y)dz^2\right).$$

The symmetry algebra is determined and is employed to find two new solutions of equation (**).

The first solution is

$$u_1(t,r,y,z)=k_1(2M-r)^{2M}e^{r+t-z}\cos\left[\ln\left|\csc(y)-\cot(y)\right|\right]HeunC\left(-4M,4M,0,8M^2,-8M^2,1-\frac{r}{2M}\right)$$

The second solution is

$$u_2(t,r,y,z)=k_2(2M-r)^{-2M}e^{r+t-z}\cos\left[\ln\left|\csc(y)-\cot(y)\right|\right]HeunC\left(-4M,-4M,0,8M^2,-8M^2,1-\frac{r}{2M}\right)$$

CHAPTER 1

INTRODUCTION TO SYMMETRIES OF DIFFERENTIAL EQUATIONS

This chapter provides an introduction to the notion of symmetries of differential equations. It seems that most people have heard the word ' symmetry ' for the first time in elementary schools. In particular, geometry classes in primary schools made our first exposure to this term. This is the reason why most people perceive the term 'symmetry' in a geometrical sense. However, nowadays we know that the concept of symmetry is not restricted to geometry. It is becoming a fundamental concept in many areas of science.

In mathematical sciences for instance, this concept appears in many branches like mathematical logic, algebra and differential equations. The concept has been also implemented in other areas of science like biology, chemistry and physics.

In fact, in some cases symmetry plays a very central role in the development of science. In the case of modern physics, symmetries underlie some of the most profound results found in this field (for example general relativity and quatntum mechanics) [8]. Its role has been extremely important to the extent that inspired the noble Laureate PW Anderson to write in one of his articles the following statement:

' It is only slightly overstating the case to say that physics is the study of symmetry' [9].

Although the meanings of symmetry are distinguishable in different contexts, there is something in common between all the contexts. It is harmony and similarity. In our context in this thesis we are going to deal with a specific type of symmetry. It is the symmetry of differential equations.

Actually, studying symmetries of differential equations has proved to be a very efficient tool in simplifying and finding exact solutions to these equations. As we are going to see in the next chapters, the symmetry based approach can solve wide range of differential equations of familiar and unfamiliar types. Nowadays, the symmetry analysis of differential equations is a significant area of research in applied mathematics. This fact is reasonable since most of the differential equations that we face in science and engineering are hard to solve.

1.1 The Basic Definition and Examples

A symmetry in general is a mapping of one mathematical object into itself or into another mathematical object that preserves some properties of the object. In the sense of geometry, a symmetry of a geometrical object is a transformation whose action leaves the object apparently unchanged. For instance, if we rotate an equilateral triangle anticlockwise about its center by an angle of $2\pi/3$ (or $4\pi/3$ or 2π) then the triangle will look the same as it did before the rotation. In this case we say that this transformation is a symmetry for the equilateral triangle. In the case of the sphere it is evident that rotation by any angle will leave the sphere unchanged. This means that the sphere is invariant under a continuous group of rotational symmetries. Hence, we call the symmetries of the sphere continuous symmetries while that of the equilateral triangle are called discrete symmetries.

In fact, the concept of invariance of geometrical objects under a symmetry transformation can be generalized to differential equations. Thus, any transformation mapping a differential equation into an equivalent equation of the same form is said to be a symmetry of the equation. Such transformation maps the solution space of the differential equation into itself. This argument leads us into the following definition.

Definition

A symmetry of a differential equitation is a transformation that keeps its form

invariant.

In order to understand this definition in more depth we consider the following examples.

Example 1.1.1

Decide whether or not the transformation

$$\bar{t} = t + a \ , \ \bar{x} = x \ , \ \bar{u} = u$$

is a symmetry of the heat equation

$$u_t - u_{xx} = 0$$

Solution

Obviously,

$$\frac{d\bar{u}}{d\bar{t}} = \frac{du}{d(t+a)} = \frac{du}{dt}$$

Also,

$$\frac{d^2\bar{u}}{d\bar{x}^2} = \frac{d^2u}{dx^2}$$

Thus,

$$u_t - u_{xx} = 0 \Rightarrow \bar{u}_{\bar{t}} - \bar{u}_{\bar{x}\bar{x}} = 0.$$

Hence, the heat equation is invariant under the given transformation.

Example 1.1.2

Show that the Ri+ccati equation

$$\frac{dy}{dx}=\frac{y+1}{x}+\frac{y^{2}}{x^{3}}$$

has the symmetry

$$\bar{x}=\frac{x}{1-\varepsilon x}\ ,\ \bar{y}=\frac{y}{1-\varepsilon x}$$

Solution

$$d\bar{y}=\frac{(1-\varepsilon x)dy+\varepsilon ydx}{(1-\varepsilon x)^{2}}$$

$$d\bar{x}=\frac{(1-\varepsilon x)dx+\varepsilon xdx}{(1-\varepsilon x)^{2}}$$

$$\frac{d\bar{y}}{d\bar{x}}=\left(\frac{(1-\varepsilon x)dy+\varepsilon ydx}{(1-\varepsilon x)^{2}}\right)\left(\frac{(1-\varepsilon x)dx+\varepsilon xdx}{(1-\varepsilon x)^{2}}\right)^{-1}$$

$$=\left(\frac{(1-\varepsilon x)dy}{(1-\varepsilon x)dx}+\frac{\varepsilon ydx}{(1-\varepsilon x)dx}\right)\left(\frac{(1-\varepsilon x)dx}{(1-\varepsilon x)dx}+\frac{\varepsilon xdx}{(1-\varepsilon x)dx}\right)^{-1}$$

$$=\left(\frac{dy}{dx}+\frac{\varepsilon y}{1-\varepsilon x}\right)\left(1+\frac{\varepsilon x}{1-\varepsilon x}\right)^{-1}=\left(\frac{y+1}{x}+\frac{y^{2}}{x^{3}}+\frac{\varepsilon y}{1-\varepsilon x}\right)\left(\frac{1-\varepsilon x+\varepsilon x}{1-\varepsilon x}\right)^{-1}$$

$$=\frac{(1-\varepsilon x)(y+1)}{x}+\frac{y^{2}(1-\varepsilon x)}{x^{3}}+\varepsilon y$$

$$=\frac{y+1-\varepsilon x}{x}+\frac{y^{2}(1-\varepsilon x)}{x^{3}}$$

$$=\frac{y}{x}+\frac{1-\varepsilon x}{x}+\frac{y^{2}}{x^{2}}\left(\frac{1-\varepsilon x}{x}\right)$$

$$=\frac{\bar{y}}{\bar{x}}+\frac{1}{\bar{x}}+\frac{\bar{y}^{2}}{\bar{x}^{2}}\left(\frac{1}{\bar{x}}\right)$$

$$=\frac{\bar{y}+1}{\bar{x}}+\frac{\bar{y}^{2}}{\bar{x}^{3}}$$

Therefore,

$$\frac{dy}{dx}=\frac{y+1}{x}+\frac{y^{2}}{x^{3}}\Rightarrow\frac{d\bar{y}}{d\bar{x}}=\frac{\bar{y}+1}{\bar{x}}+\frac{\bar{y}^{2}}{\bar{x}^{3}}$$

Thus, the given transformation is a symmetry of the Riccati equation.

1.2 Symmetries of First Order ODEs

In order to understand the concept of symmetry of differential equations let us consider the simplest case which is the first order ODE. In particular, consider the equation

$$\frac{dy}{dx}=f(x). \tag{1.2.1}$$

We know from elementary course in differential equations that equation (1.2.1) is easily solved by putting

$$dy=f(x)dx.$$

Then we integrate both sides of the above equation to obtain the required solution. Obviously, equation (1.2.1) is simple to solve because it is separable.

However, there is a deeper property that enabled us to solve this equation. In order to understand this property notice that the slopes of the solution curves dy/dx are independent of y. This means that we can shift any solution curve in the y-direction into any of the other solution curves by means of the transformation $(x,y)\rightarrow(x,y+c)$ as shown in figure 1.2.1.

Similarly, if we consider the ODE given by

$$\frac{dy}{dx}=g(y). \tag{1.2.2}$$

Then, it is evident that this equation is separable as well since we can put

$$\frac{dy}{g(y)}=dx.$$

Again here the slopes of the solution curves dy/dx are independent of x which means that we can shift any solution curve in the x-direction into any other by means Of the transformation $(x, y) \to (x+c, y)$ as shown in figure 1.2.2 below.

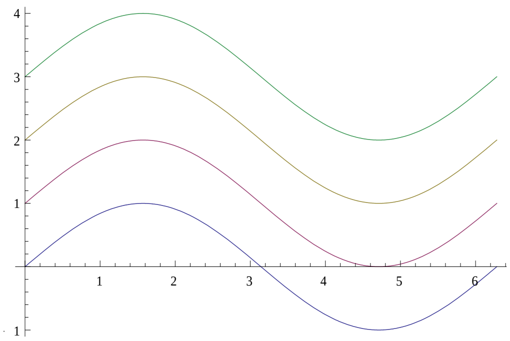

Figure 1.2.1 : Solution curves shifted in the y-direction.

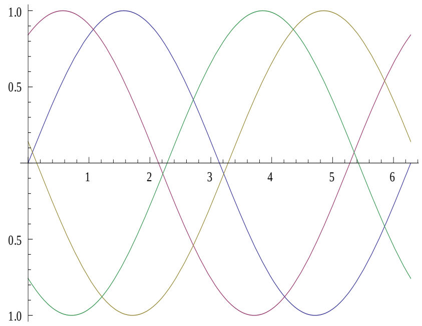

Figure 1.2.2 : Solution curves shifted in the x-direction.

Actually, we can easily check that equation (1.2.1) is invariant under the transformation $(x, y) \to (x, y+c)$ while equation (1.2.2) is invariant under the transformation $(x, y) \to (x+c, y)$. In other words, the first transformation is a symmetry of equation (1.2.1) and the second transformation is a symmetry of equation (1.2.2). Now we go back to our main inquiry : why equations (1.2.1) and (1.2.2) were easy to solve? What was the deeper property that made them solvable?

The answer is getting clear for us now. It is the presence of a continuous transformation that takes each solution curve into another. In other words, the presence of a Lie symmetry is the deeper property that enabled us to solve equations (1.2.1) and (1.2.2).

For more details on this subject see [10].

1.3 Separability of First Order ODEs

In the previous section we have seen that a first order ODE can be transformed into a separable equation if its set of solution curves is invariant under transformation in some coordinate system. Actually, knowing the proper coordinate system to be utilized makes it easy to transform the considered equation into a separable one. Consider the following example for better clarification of this idea.

pExample 1.3.1

Transform the differential equation

$$\frac{dy}{dx} = \frac{x^2 + y^2}{xy} \tag{1.3.1}$$

into a separable equation using the substitution

$$u(x, y) = \frac{y}{x}$$

then find its solution.

Solution

Obviously,

$$u(x, y) = \frac{y}{x}$$

implies that

$$du = \frac{xdy - ydx}{x^2} \Rightarrow x^2 du = xdy - ydx .$$

Hence,

$$x^2 \frac{du}{dx} = x \frac{dy}{dx} - y .$$

Dividing by x gives

$$x \frac{du}{dx} = \frac{dy}{dx} - \frac{y}{x}$$

or equivalently,

$$\frac{dy}{dx} = x \frac{du}{dx} + u .$$

Substituting the above equation into equation (1.3.1) we get

$$x\frac{du}{dx}+u=\frac{x^2+y^2}{xy}\Rightarrow x\frac{du}{dx}+u=\frac{\frac{x^2}{y^2}+1}{\frac{x}{y}}$$

Thus,

$$x\frac{du}{dx}+u=\frac{u^2+1}{u}$$

$$x\frac{du}{dx}+u=u+\frac{1}{u}\Rightarrow x\frac{du}{dx}=\frac{1}{u}$$

Therefore, the above expression leads us to the separable equation

$$udu=\frac{dx}{x}$$

We can solve the upper separable equation by integrating both sides to get

$$\frac{u^2}{2}=\ln|x|+c$$

or equivalently,

$$\frac{1}{2}u^2-\ln|x|=c.$$

Therefore,

$$\frac{1}{2}\left(\frac{y}{x}\right)^2-\ln|x|=c.$$

Hence, the final answer can be expressed as

$$y=x\sqrt{2\ln|x|+c}.$$

Remark

Consider the following transformation

$$\bar{x} = e^{\varepsilon} x$$
$$\bar{y} = e^{\varepsilon} y \,.$$

Then, it is obvious that equation (1.3.1) is invariant under the given transformation. In other words, equation (1.3.1) implies that

$$\frac{d\bar{y}}{d\bar{x}} = \frac{\bar{x}^2 + \bar{y}^2}{\overline{xy}}$$

Moreover, we can easily check that the transformation satisfies the following properties.

i) $\varepsilon = 0$ gives the identity transformation

$$\bar{x} = x \,, \bar{y} = y \,.$$

ii) $-\varepsilon$ characterizes the inverse transformation

$$\bar{x} = e^{-\varepsilon} x \,, \bar{y} = e^{-\varepsilon} y \,.$$

iii) The product of transformations is also a member of the set of transformations.

A transformation satisfying these three properties is said to be a one parameter group of transformation. This group will be studied in details in the next chapter.

CHAPTER 2

SYMMETRIES OF ODEs

In the previous chapter we have seen that a symmetry of a first order ODE is a transformation that leave the differential equation invariant. As a matter of fact, the concept of symmetry can be generalized to any differential equation including higher order ODEs and PDEs.

Indeed, the method of symmetry can be thought of as a general method that determines the coordinate system which simplifies the equation so that it can be easier to solve. The question that need to be raised is how to obtain this coordinate system?. In order to answer this question we need first to know how to obtain the symmetries of a given differential equation.

In fact, the definition of symmetry that we gave in the previous chapter is good enough to determining if a given transformation is a symmetry of a given differential equation. However, this definition does not give us information about how to obtain the symmetries of the equation. Therefore, we need a more technical definition that we can use as a criterion to obtain the symmetries. In this chapter our main goal will be to derive the criterion that will enable us to obtain symmetries of ODEs.

2.1 One Parameter Group of Transformations in Plane

Definition (point transformations)

A transformation in the xy-plane $T: R^2 \to R^2$ which transform a point $p=(x,y)$ into another point $T(p)=\bar{p}=(\bar{x},\bar{y})$ through the functions

$$\bar{x}=\phi(x,y)$$

$$\bar{y}=\psi(x,y)$$

is called a point transformation.

The table below gives some examples of point transformations.

The transformation	Action
$\bar{x}=x+c$ $\bar{y}=y$	Translate the point (x,y) horizontally by c units.
$\bar{x}=x$ $\bar{y}=y+c$	Translate the point (x,y) vertically by c units.
$\bar{x}=x\cos(\theta)-y\sin(\theta)$ $\bar{y}=x\sin(\theta)+y\cos(\theta)$	Rotate the point (x,y) counterclockwise through an angle of θ.
$\bar{x}=e^{k}x$ $\bar{y}=e^{k}y$	Rescale the point (x,y) uniformly.
$\bar{x}=e^{k_1}x$ $\bar{y}=e^{k_2}y$	Rescale the point (x,y) non- uniformly.

Table 2.1.1 : Some famous examples of point transformations.

Definition (one parameter group of point transformations in plane)

The set of transformations T_ε with $\varepsilon \in I \subset R$ is called a one parameter group of transformations if the following conditions are satisfied.

i) The existence of the identity transformation T_0 so that

$$T_0(x,y)=(x,y).$$

ii) For every transformation T_ε there exists the inverse transformation $T_\varepsilon^{-1} = T_{-\varepsilon}$ such that

$$T_\varepsilon T_\varepsilon^{-1} = T_\varepsilon^{-1} T_\varepsilon = T_0 .$$

iii) The composition of two transformations in the group is a member of the set of transformation in the group (closure property).

Remark

The one parameter group of transformations satisfies the associativity of group multiplication. That is,

$$T_a (T_b T_c) = (T_a T_b) T_c .$$

Example 2.1.1

Show that the set of transformations $\{T_\varepsilon\}$ given by

$$T_\varepsilon (x, y) = (x + \varepsilon, y + m\varepsilon)$$

is a one parameter group of transformation.

Solution

i) It is clear that T_0 is the identity transformation since

$$T_0 (x, y) = (x + 0, y + 0) = (x, y).$$

ii) $T_{-\varepsilon}$ is the inverse transformation because

$$T_\varepsilon T_{-\varepsilon} (x, y) = T_\varepsilon (x - \varepsilon, y - m\varepsilon) = (x - \varepsilon + \varepsilon, y - m\varepsilon + m\varepsilon) = (x, y)$$

$$T_{-\varepsilon} T_\varepsilon (x, y) = T_{-\varepsilon} (x + \varepsilon, y + m\varepsilon) = (x + \varepsilon - \varepsilon, y + m\varepsilon - m\varepsilon) = (x, y).$$

iii) The closure property is satisfied since

$$T_{\varepsilon_1} T_{\varepsilon_2} (x, y) = T_{\varepsilon_1} (x + \varepsilon_2, y + m\varepsilon_2) = (x + \varepsilon_2 + \varepsilon_1, y + m\varepsilon_2 + m\varepsilon_1)$$

$$= \big(x + (\varepsilon_1 + \varepsilon_2), y + m(\varepsilon_1 + \varepsilon_2)\big) = T_{\varepsilon_1 + \varepsilon_2} .$$

Therefore, the given transformation is a one parameter group of transformation.

2.2 Infinitesimal Generator of One Parameter Group of Transformations

Definition

Given a one parameter group of transformation

$$\bar{x} = x(\varepsilon) = \phi(x, y, \varepsilon)$$

$$\bar{y} = y(\varepsilon) = \psi(x, y, \varepsilon)$$

then the vector field

$$X = \zeta(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y}$$

is called an infinitesimal generator of the group if

$$\zeta(x, y) = \left.\frac{\partial \phi}{\partial \varepsilon}\right|_{\varepsilon=0} \quad \text{and} \quad \eta(x, y) = \left.\frac{\partial \psi}{\partial \varepsilon}\right|_{\varepsilon=0}$$

so that the vector (ζ, η) gives the tangent to the curve

$$\alpha(\varepsilon) = (\bar{x}, \bar{y}) = \big(\phi(x, y, \varepsilon), \psi(x, y, \varepsilon)\big)$$

at the point $p(x, y)$.

Example 2.2.1

Find the infinitesimal generator of the following group

$$\bar{x} = \frac{x}{1-\varepsilon y}\ ,\ \bar{y} = \frac{y}{1-\varepsilon y}$$

Solution

$$\phi(x,y,\varepsilon)=\frac{x}{1-\varepsilon y}\Rightarrow\frac{d\phi}{d\varepsilon}=\frac{xy}{(1-\varepsilon y)^2}$$

$$\zeta(x,y)=\left.\frac{d\phi}{d\varepsilon}\right|_{\varepsilon=0}=xy$$

$$\psi(x,y,\varepsilon)=\frac{y}{1-\varepsilon y}$$

$$\frac{d\psi}{d\varepsilon}=\frac{y^2}{\left(1-\varepsilon y\right)^2}$$

$$\eta(x,y)=\left.\frac{d\psi}{d\varepsilon}\right|_{\varepsilon=0}=y^2.$$

Hence, the infinitesimal generator of the given group is

$$X=xy\frac{\partial}{\partial x}+y^2\frac{\partial}{\partial y}.$$

Group Corresponding to Generator

We have seen so far that if we have a one parameter group of transformations then we can find its infinitesimal generator. The opposite of this argument also holds. That is, if we know the infinitesimal generator then we can generate the one parameter group associated with it. This is explained in more details below.

Result

Given the infinitesimal generator

$$X=\zeta(x,y)\frac{\partial}{\partial x}+\eta(x,y)\frac{\partial}{\partial y}$$

The corresponding one parameter group of transformations

$$\bar{x}=\phi(x,y,\varepsilon)$$

$$\bar{y}=\psi(x,y,\varepsilon)$$

with

$$\phi(x, y, 0) = x \ , \ \psi(x, y, 0) = y$$

can be obtained by solving the ODE

$$\frac{dx}{d\varepsilon} = \zeta(x, y)$$

with initial condition

$$x(0) = x \ ,$$

and the ODE

$$\frac{dy}{d\varepsilon} = \eta(x, y)$$

with initial condition

$$y(0) = y \, .$$

Remark

The existence and uniqueness of solutions to the above ODEs is derived from fundamental existence and uniqueness theorems of systems of first order ODEs.

Example 2.2.2

Find the group generated by the infinitesimal generator

$$X = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}$$

Solution

We need to solve two systems of ODEs. The first one is

$$\frac{dx}{d\varepsilon} = x^2 \quad , \quad x(0) = x$$

The second one is

$$\frac{dy}{d\varepsilon} = xy \quad , \quad y(0) = y$$

The first equation can be solved as follows

$$\frac{dx}{x^2} = d\varepsilon$$

$$-\frac{1}{x} = \varepsilon + c$$

$$x(\varepsilon) = -\frac{1}{\varepsilon + c}$$

The initial condition $x(0) = x$ implies that

$$x = -\frac{1}{c}$$

$$c = -\frac{1}{x}$$

Hence,

$$x(\varepsilon) = -\frac{1}{\varepsilon - \frac{1}{x}}$$

Or equivalently,

$$x(\varepsilon) = \frac{x}{1 - x\varepsilon}$$

We then solve the second equation by putting

$$\frac{dy}{xy} = d\varepsilon$$

This leads into

$$\frac{1}{x}\ln|y| = \varepsilon + c$$

Multiplying both sides of the above equation by x gives

$$\ln|y| = x\,\varepsilon + cx$$

$$y(\varepsilon) = e^{x\,\varepsilon + cx}$$

The initial condition $y(0) = y$ implies that

$$y = e^{cx}$$

$$cx = \ln|y|$$

Or equivalently,

$$c = \frac{1}{x}\ln|y|$$

Therefore,

$$y(\varepsilon) = e^{x\,\varepsilon} e^{\frac{1}{x}(\ln|y|x}$$

or equivalently,

$$y(\varepsilon) = y e^{x\,\varepsilon}.$$

Hence, the group generated by the given generator is

$$\bar{x} = \frac{x}{1 - x\,\varepsilon} \quad , \quad \bar{y} = y e^{x\,\varepsilon}.$$

2.3 Prolongation of Vector Fields

A vector field

$$X = \zeta(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y} \tag{2.3.1}$$

in a plane transforms a point (x, y) into another point $(\bar{x}, \bar{y})$ at a small scale. Therefore, the infinitesimal transformation of the corresponding group can be expressed using the Taylor series as

$$\bar{x} = x + \varepsilon\zeta(x, y) + O(\varepsilon^2)$$

$$\bar{y} = y + \varepsilon\eta(x, y) + O(\varepsilon^2).$$

For the sake of symmetry approach method we will need to know how to transform the derivatives infinitesimally via the given vector field (or the transformation group). The required information can be obtained by the prolongation of the vector field. In this context, a first prolongation of the vector field will provide the infinitesimal transformations for each of x, y, y'. If we need to transform the second derivative y'' also then we have to find the second prolongation. In general the kth prolongation $X^{[k]}$ of the vector field (2.3.1) gives the infinitesimal transformations for each $x, y, y', y'', \ldots\ldots, y^{(k)}$ where $y^{(k)}$ denotes the kth derivative of y.

First Prolongation

The first prolongation extends the vector field (2.3.1) into the vector field

$$X^{[1]} = \zeta(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y} + \eta^{[1]}(x, y, y')\frac{\partial}{\partial y'} \qquad (2.3.2)$$

so that $y' = \dfrac{dy}{dx}$ is transformed to $\bar{y}' = \dfrac{d\bar{y}}{d\bar{x}}$

Thus, we should be able to express $\bar{y}'$ as

$$\bar{y}' = y' + \varepsilon\eta^{[1]}(x, y, y') + O(\varepsilon^2).$$

Now, our main aim is to obtain a formula for $\eta^{[1]}(x, y, y')$.

Notice that,

$$\bar{x} = x + \varepsilon\zeta(x, y) \Rightarrow d\bar{x} = dx + \varepsilon d\zeta$$

$$\bar{y} = y + \varepsilon\eta(x, y) \Rightarrow d\bar{y} = dy + \varepsilon d\eta$$

Hence,

$$\bar{y} = \frac{d\bar{y}}{d\bar{x}} = \frac{dy + \varepsilon d\eta}{dx + \varepsilon d\zeta}$$

Dividing the numerator and the denominator of the above expression by dx we obtain

$$\bar{y} = \frac{\frac{dy}{dx} + \varepsilon \frac{d\eta}{dx}}{1 + \varepsilon \frac{d\zeta}{dx}} = \left(y' + \varepsilon \frac{d\eta}{dx} \right) \left(1 + \varepsilon \frac{d\zeta}{dx} \right)^{-1}$$

Using the binomial theorem the above expression becomes

$$\bar{y} = \left(y' + \varepsilon \frac{d\eta}{dx} \right) \left(1 - \varepsilon \frac{d\zeta}{dx} + O(\varepsilon^2) \right) = y' + \varepsilon \left(\frac{d\eta}{dx} - y' \frac{d\zeta}{dx} \right) + O(\varepsilon^2)$$

Clearly, from the above result we conclude that

$$\eta^{[1]} = \frac{d\eta}{dx} - y' \frac{d\zeta}{dx}$$

If we use the definition of total differentiation of a function as

$$\frac{d\zeta(x, y)}{dx} = \frac{\partial \zeta}{\partial x} + \frac{\partial \zeta}{\partial y} \frac{dy}{dx}$$

$$\frac{d\eta(x, y)}{dx} = \frac{\partial \eta}{\partial x} + \frac{\partial \eta}{\partial y} \frac{dy}{dx}$$

Then the expression for $\eta^{[1]}$ can be written as

$$\eta^{[1]}(x, y, y') = \eta_x + y'(\eta_y - \zeta_x) - (y')^2 \zeta_y .$$

Definition

The first prolongation of the vector field (2.3.1) is given by

$$X^{[1]} = \zeta(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y} + \left(\eta_x + y'(\eta_y - \zeta_x) - (y')^2 \zeta_y \right) \frac{\partial}{\partial y'} .$$

The next example illustrates how to obtain the first prolongation for a given vector field.

Example 2.3.1

Find the first prolongation of the vector field

$$X = x^2 \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y}$$

Solution

$$\zeta(x, y) = x^2 \Rightarrow \zeta_x = 2x \text{ and } \zeta_y = 0$$

Similarly,

$$\eta(x, y) = y^2 \Rightarrow \eta_x = 0 \text{ and } \eta_y = 2y$$

$$\eta^{[1]}(x, y, y') = \eta_x + y'(\eta_y - \zeta_x) - (y')^2 \zeta_y = 2yy' - 2xy'$$

Therefore, the first prolongation is given by

$$X^{[1]} = x^2 \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y} + (2yy' - 2xy') \frac{\partial}{\partial y'}.$$

Second Prolongation

Now, we are seeking to find a transformation of the second derivative y''. In other words, our goal is to extend the vector field (2.3.1) to the vector field

$$X^{[2]} = \zeta(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y} + \eta^{[1]}(x, y, y') \frac{\partial}{\partial y'} + \eta^{[2]}(x, y, y', y'') \frac{\partial}{\partial y''}$$

so that $y'' = \dfrac{dy'}{\partial x}$ is transformed to $\bar{y}'' = \dfrac{d\bar{y}'}{\partial \bar{x}}$ such that

$$\bar{y}'' = y'' + \varepsilon \eta^{[2]}(x, y, y', y'') + O(\varepsilon^2).$$

Hence, we need to find a formula for $\eta^{[2]}(x, y, y', y'')$. In order to derive the desired formula we can proceed as follows.

$$\bar{y}' = y' + \varepsilon\eta^{[1]}(x, y, y') \Rightarrow d\bar{y}' = dy' + \varepsilon d\eta^{[1]}$$

$$\bar{y}'' = \frac{d\bar{y}'}{d\bar{x}} = \frac{dy' + \varepsilon d\eta^{[1]}}{dx + \varepsilon d\zeta}$$

Dividing the numerator and denominator in the above expression by dx yields

$$\bar{y}'' = \left(\frac{dy'}{dx} + \varepsilon\frac{d\eta^{[1]}}{dx}\right)\left(1 + \varepsilon\frac{d\zeta}{dx}\right)^{-1} = \left(y'' + \varepsilon\frac{d\eta^{[1]}}{dx}\right)\left(1 + \varepsilon\frac{d\zeta}{dx}\right)^{-1}$$

Using the binomial expansion the above expression becomes

$$\bar{y}'' = \left(y'' + \varepsilon\frac{d\eta^{[1]}}{dx}\right)\left(1 + \varepsilon\frac{d\zeta}{dx}\right)^{-1} = \left(y'' + \varepsilon\frac{d\eta^{[1]}}{dx}\right)\left(1 - \varepsilon\frac{d\zeta}{dx} + O(\varepsilon^2)\right)$$

$$= y'' + \varepsilon\left(\frac{d\eta^{[1]}}{dx} - y''\frac{d\zeta}{dx}\right) + O(\varepsilon^2)$$

From the above result we get

$$\eta^{[2]}(x, y, y', y'') = \frac{d\eta^{[1]}}{dx} - y''\frac{d\zeta}{dx}$$

Notice that the above expression can be computed using the total differentiation and the chain rule to get

$$\eta^{[2]}(x, y, y', y'') = \eta_{xx} + \left(2\eta_{xy} - \zeta_{xx}\right)y' + \left(\eta_{yy} - 2\zeta_{xy}\right)\left(y'\right)^2 - \zeta_{yy}\left(y'\right)^3 + \left(\eta_y - 2\zeta_x - 3y'\zeta_y\right)y''.$$

Definition

The second prolongation of the vector field (2.3.1) is given by

$$X^{[2]} = \zeta(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y} + \left(\eta_x + y'(\eta_y - \zeta_x) - \left(y'\right)^2\zeta_y\right)\frac{\partial}{\partial y'}$$

$$+\left(\eta_{xx} + \left(2\eta_{xy} - \zeta_{xx}\right)y' + \left(\eta_{yy} - 2\zeta_{xy}\right)\left(y'\right)^2 - \zeta_{yy}\left(y'\right)^3 + \left(\eta_y - 2\zeta_x - 3y'\zeta_y\right)y''\right)\frac{\partial}{\partial y''}$$

Example 2.3.2

Find the second prolongation of the vector field

$$X = x^2 \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y}$$

Solution

The first prolongation for this vector field has been computed in example 2.3.1 and it was found to be

$$X^{[1]} = x^2 \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y} + (2yy' - 2xy')\frac{\partial}{\partial y'} .$$

Therefore, all we need to compute now is $\eta^{[2]}(x, y, y', y'')$. Thus, using the derived formula given in the previous definition we have,

$$\eta^{[2]}(x, y, y', y'') = \eta_{xx} + \left(2\eta_{xy} - \zeta_{xx}\right)y' + \left(\eta_{yy} - 2\zeta_{xy}\right)\left(y'\right)^2 - \zeta_{yy}\left(y'\right)^3 + \left(\eta_y - 2\zeta_x - 3y'\zeta_y\right)y''$$

$$= 2\left(y'\right)^2 + 2yy'' - 2y' - 4xy''$$

Therefore, the second prolongation of the given vector field is given by

$$X^{[2]} = x^2 \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y} + (2yy' - 2xy')\frac{\partial}{\partial y'} + \left(2\left(y'\right)^2 + 2yy'' - 2y' - 4xy''\right)\frac{\partial}{\partial y''}$$

Higher Order Prolongation

We can show by induction that if we extend the vector field (2.3.1) then the coefficient of $\partial / \partial y^{(n)}$ in the extended vector is given by

$$\eta^{[i]} = \frac{d\eta^{(i-1)}}{dx} - y^{(i)} \frac{d\zeta}{dx}$$

where $i = 1, 2, 3, ..., n$ and $\eta^{[0]} = \eta$.

Therefore, we can now generalize the definition of the prolongation of the vector field (2.3.1).

Definition

The nth prolongation of the vector field (2.3.1) is given by

$$X^{[n]} = \zeta(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} + \eta^{[1]}(x,y,y')\frac{\partial}{\partial y'} + \eta^{[2]}(x,y,y',y'')\frac{\partial}{\partial y''}$$

$$+.........+\eta^{[n]}(x,y,y',y'',...,y^{(n)})\frac{\partial}{\partial y^{(n)}}$$

where

$$\eta^{[i]} = \frac{d\eta^{(i-1)}}{dx} - y^{(i)}\frac{d\zeta}{dx}.$$

2.4 Invariance Criterion for Finding Symmetries of ODEs

We are now very close to achieving our main goal of this chapter which is to find a criterion for obtaining symmetries of ODEs. However, before we proceed to this point it is worthwhile to introduce the concept of differential invariants of a vector field.

Definition (differential invariants of a vector field)

A function $f(x,y)$ is a differential invariant of a vector field if the form of $f(x,y)$ does not change under the infinitesimal transformation

$$\bar{x} = x + \varepsilon\zeta(x,y)$$

$$\bar{y} = y + \varepsilon\eta(x,y)$$

That is, $f(\bar{x},\bar{y}) = f(x,y)$

Or equivalently, the function $f(x,y)$ is invariant of the vector field X if

$$X(f) = 0.$$

Example 2.4.1

Find the general differential invariant of the vector field

$$X = y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}. \qquad (2.4.1)$$

Solution

The differential invariant $f(x, y)$ has to satisfy the equation

$$X(f) = 0 .$$

This leads to

$$y \frac{\partial f}{\partial x} + x \frac{\partial f}{\partial y} = 0$$

The characteristic system of the above PDE is

$$\frac{dx}{y} = \frac{dy}{x} \Rightarrow xdx = ydy$$

This implies that,

$$\frac{x^2}{2} = \frac{y^2}{2} + c$$

Or equivalently,

$$x^2 - y^2 = c .$$

Therefore, the most general differential invariant of the given vector field is given by

$$f(x, y) = \phi(x^2 - y^2) .$$

Definition (Invariance Criterion)

The infinitesimal transformation

$$\bar{x} = x + \varepsilon\zeta(x, y)$$

$$\bar{y} = y + \varepsilon\eta(x, y)$$

or equivalently, the generator

$$X = \zeta(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y}$$

is a symmetry of the ODE

$$F(x, y, y', ..., y^{(n)}) = 0$$

if

$$X^{[n]}F\Big|_{F=0} = 0$$

where $X^{[n]}$ is the nth prolongation of the generator X.

In fact, the invariance criterion is useful in two aspects. It can be used to check if a given generator is a symmetry of a given ODE. Moreover, it can be applied to obtain the symmetries of an ODE. The following example will clarify the first point.

Example 2.4.2

Show that the generator

$$X = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}$$

is a symmetry of the differential equation

$$xy' - y = e^{\frac{x}{y}}.$$

Solution

Since the given ODE is first order then we only need to find the first prolongation of X. Notice that,

$$\zeta(x, y) = x^2 \Rightarrow \frac{d\zeta}{dx} = 2x$$

$$\eta(x, y) = xy \Rightarrow \frac{d\eta}{dx} = y + xy'$$

$$\eta^{[1]}(x, y, y') = \frac{d\eta}{dx} - y'\frac{d\zeta}{dx} = y + xy' - 2xy' = y - xy'$$

Thus, the first prolongation of X is given by

$$X^{[1]} = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y} + (y - xy') \frac{\partial}{\partial y'}$$

Let

$$F = xy' - y - e^{\frac{x}{y}}.$$

Then

$$X^{[1]}F = x^2 \frac{\partial}{\partial x}\left(xy' - y - e^{\frac{x}{y}} \right) + xy \frac{\partial}{\partial y}\left(xy' - y - e^{\frac{x}{y}} \right) + (y - xy') \frac{\partial}{\partial y'}\left(xy' - y - e^{\frac{x}{y}} \right)$$

$$= x^2 \left(y' - \frac{1}{y} e^{\frac{x}{y}} \right) + xy \left(-1 + \frac{x}{y^2} e^{\frac{x}{y}} \right) + (y - xy')(x)$$

$$= x^2 y' - \frac{x^2}{y} e^{\frac{x}{y}} - xy + \frac{x^2}{y} e^{\frac{x}{y}} + xy - x^2 y' = 0$$

Hence, $X^{[1]}F\big|_{F=0} = 0$ which implies that the given generator is a symmetry of the given equation.

2.5 Finding the Symmetries of ODEs

We have mentioned in the previous section that the invariance criterion $X^{[n]}F\big|_{F=0} = 0$ plays the key role in finding the symmetries of ODEs. The idea is to apply this criterion to the given ODE which lead to a system of PDEs in ζ, η. This system is called the system of determining equations (which in general will be overdetermined). Solving the determining equations will give us all the functions $\zeta(x, y)$, $\eta(x, y)$ that satisfy the invariance condition.

Procedure for Finding Symmetries of ODEs

We can summarize the procedure for finding symmetries of ODEs as follows.

Step # 1

Apply the invariance criterion

$$X^{[n]}F\Big|_{F=0}=0$$

where $F(x,y,y',y'',......,y^{(n)})=0$ is the given differential equation.

$X^{[n]}$ is the nth prolongation of the symmetry generator X.

Step # 2

Compare the coefficients of the derivatives of y in step # 1 to obtain the determining equations.

Step # 3

Simplify and solve the determining equations.

The procedure of obtaining symmetries of ODEs is described in details in many textbooks, see for example [11], [12], [13], [14], [15], [16], [17], [18].

Example 2.5.1

Find the symmetries of the differential equation

$$y''+y^{2}=0. \tag{2.5.1}$$

Solution

Notice that the symmetry generator of the given differential equation is of the form

$$X=\zeta(x,y)\frac{\partial}{\partial x}+\eta(x,y)\frac{\partial}{\partial y} \tag{2.5.2}$$

Step # 1

We apply the invariance criterion which is in this case

$$X^{[2]}F\Big|_{F=0}=0$$

This leads to

$$X^{[2]}F = \zeta(x,y)\frac{\partial}{\partial x}\left(y''+y^2\right)+\eta(x,y)\frac{\partial}{\partial y}\left(y''+y^2\right)+\eta^{[1]}\frac{\partial}{\partial y'}\left(y''+y^2\right)+\eta^{[2]}\frac{\partial}{\partial y''}\left(y''+y^2\right)=0$$

Or equivalently,

$$2y\,\eta(x,y)+\eta^{[2]}=0$$

This implies that,

$$2y\,\eta(x,y)+\eta_{xx}+\left(2\eta_{xy}-\zeta_{xx}\right)y'+\left(\eta_{yy}-2\zeta_{xy}\right)(y')^2-\zeta_{yy}\,(y')^3+\left(\eta_y-2\zeta_x-3y'\zeta_y\right)y''=0$$

Using $y''=-y^2$ then substituting in the above equation we get

$$2y\,\eta(x,y)+\eta_{xx}+\left(2\eta_{xy}-\zeta_{xx}\right)y'+\left(\eta_{yy}-2\zeta_{xy}\right)(y')^2-\zeta_{yy}\,(y')^3-y^2\left(\eta_y-2\zeta_x-3y'\zeta_y\right)=0$$

Step # 2

We derive now the determining equations.

This is done by equating the coefficients of different powers of derivatives of y to zero.

From step # 1 above we have the following coefficients

Coefficient of $(y')^0$ is : $2y\eta+2y^2\zeta_x-y^2\eta_y+\eta_{xx}$

Coefficient of y' is : $3y^2\zeta_y+2\eta_{xy}-\zeta_{xx}$

Coefficient of $(y')^2$ is : $\eta_{yy}-2\zeta_{xy}$

Coefficient of $(y')^3$ is : $-\zeta_{yy}$

Therefore, the determining equations can be written as

e1 : $\zeta_{yy}=0$

e2 : $\eta_{yy}-2\zeta_{xy}=0$

e3 : $3y^2\zeta_y+2\eta_{xy}-\zeta_{xx}=0$

e4 : $2y\eta + 2y^2\zeta_x - y^2\eta_y + \eta_{xx} = 0$

Step # 3

We solve the above determining system as follows.

Differentiating e2 with respect to y gives after simplification the following

$$\eta_{yyy} = 0 . \tag{2.5.3}$$

Differentiating e3 with respect to y then simplify to get

$$6y\zeta_y + 2\eta_{xyy} - \zeta_{xxy} = 0 . \tag{2.5.4}$$

Differentiating (2.5.4) with respect to y then simplify to get

$$\zeta_y = 0 . \tag{2.5.5}$$

Putting (2.5.5) in e2 we obtain

$$\eta_{yy} = 0 . \tag{2.5.6}$$

Also, putting (2.5.5) into e3 we obtain

$$2\eta_{xy} = \zeta_{xx} . \tag{2.5.7}$$

Now, (2.5.5) implies that

$$\zeta = A(x) . \tag{2.5.8}$$

Moreover, (2.5.6) implies that

$$\eta(x, y) = B(x)y + C(x) . \tag{2.5.9}$$

Thus, putting (2.5.8) and (2.5.9) into (2.5.7) we obtain

$$2B'(x) = A''(x) . \tag{2.5.10}$$

Thus, e4 becomes

$$2B(x)y^2 + 2C(x)y + 2y^2 A'(x) - B(x)y^2 + B''(x)y + C''(x) = 0 .$$

Therefore, the above equation implies that

$$B(x)+2A'(x)=0.$$

Differentiating the above equation with respect to x gives

$$B'(x)+2A''(x)=0. \tag{2.5.11}$$

Equations (2.5.10) and (2.5.11) imply that

$$A''(x)=0$$

$$A(x)=c_1x+c_2 \tag{2.5.12}$$

Also, (2.5.8) and (2.5.12) imply that

$$\zeta=c_1x+c_2. \tag{2.5.13}$$

Furthermore, putting (2.5.12) into (2.5.11) we get

$$B'(x)=0$$

$$B(x)=c_3$$

Hence, (2.5.9) becomes after substitution

$$\eta(x,y)=c_3y+C(x). \tag{2.5.14}$$

Now, substituting (2.5.13) and (2.5.14) into e4 gives

$$c_3y^2+2C(x)y+2c_1y^2+C''(x)=0.$$

The above equation leads to

$$c_3=-2c_1 \quad \text{and} \quad C(x)=0.$$

Therefore, the final solution of the determining system can be rewritten as

$$\zeta=c_1x+c_2$$

$$\eta=-2c_1y$$

Now, to find the symmetries we proceed as follows.

Set $c_1 = 1$, $c_2 = 0$ then substitute in the symmetry generator (2.5.2) to get the first symmetry which is given by

$$X_1 = x \frac{\partial}{\partial x} - 2y \frac{\partial}{\partial y} .$$

Similarly, to get the second symmetry we set $c_2 = 1$, $c_1 = 0$ then we substitute in (2.5.2) to get

$$X_2 = \frac{\partial}{\partial x} .$$

It is a standard result that the set of all symmetries of an ODE forms a Lie algebra with commutator operation

$$[X_1, X_2](f) = X_1[X_2(f)] - X_2[X_1(f)].$$

Notice that the Lie algebra generated by the symmetries X_1, X_2 of equation (2.5.1) is given in table (2.5.1).

	X_1	X_2
X_1	0	$-X_2$
X_2	X_2	0

Table 2.5.1 : The commutator table generated by symmetries Of equation (2.5.1).

2.6 Reduction of ODEs Using Symmetries

ODEs can be reduced into simpler forms if we know the proper coordinate system that can be utilized to simplify them. The symmetries of an ODE provide us with the desired coordinate system. This task can be achieved using invariant functions expressed as new independent variables. These functions are called the similarity variables. They are obtained by solving the characteristic system produced by the equation

$$X^{[n]}I = 0 .$$

The symmetry transformation converts a first order ODE into a separable equation and reduces the order of a higher order ODE by one. This means that a second order ODE can be reduced into a first order. To illustrate this idea consider the Riccati equation given in the following example.

Example 2.6.1

Reduce the Riccati equation given by

$$\frac{dy}{dx} + y^2 - \frac{2}{x^2} = 0$$

using the symmetry

$$X = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$$

into a separable equation (algebraic equation) then find its exact solutions.

Solution

Setting

$$X(I) = 0$$

we get the characteristic system

$$\frac{dx}{x} = \frac{dy}{-y} .$$

Integrating both sides of the above equation yields

$$\ln|x| = -\ln|y| + c$$

where c denotes constant.

Equivalently, this leads into

$$\ln|x| + \ln|y| = c$$

Hence,

$$\ln|xy| = c$$

$$xy = k$$

where k denotes constant.

Therefore, the general form of invariant solution is

$$xy = k$$

or equivalently,

$$y = \frac{k}{x} \tag{2.6.1}$$

Thus,

$$\frac{dy}{dx} = -\frac{k}{x^2}$$

Now, substitute for dy/dx in the given Riccati equation we obtain an algebraic equation given by

$$k^2 - k - 2 = 0.$$

In order to obtain exact solution for the Riccati equation we solve this algebraic equation to obtain two solutions

$$k_1 = -1 \ , \ k_2 = 2 .$$

Substituting these two solutions in equation (2.6.1) yields

$$y_1 = \frac{-1}{x} \quad \text{and} \quad y_2 = \frac{2}{x} .$$

Table (2.6.1) provides some examples of symmetries of some nonlinear first order differential equations.

No.	Equation	Symmetry
1	$y' = F(y)$	$X = \frac{\partial}{\partial x}$
2	$y' = F(ax + by)$	$X = b\frac{\partial}{\partial x} - a\frac{\partial}{\partial y}$
3	$y' = F(\frac{y}{x})$	$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$
4	$y' = x^{k-1}F\left(\frac{y}{x^k}\right)$	$X = x\frac{\partial}{\partial x} + ky\frac{\partial}{\partial y}$
5	$xy' = F(xe^{-y})$	$X = x\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$
6	$y' = yF(ye^{-y})$	$X = \frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$
7	$y' = \frac{y}{x} + xF\left(\frac{y}{x}\right)$	$X = \frac{\partial}{\partial x} + \frac{y}{x}\frac{\partial}{\partial y}$
8	$y' = \frac{y}{x + F(y)}$	$X = y\frac{\partial}{\partial x}$
9	$xy' = y + F(x)$	$X = x\frac{\partial}{\partial y}$
10	$xy' = y\left[\ln y + F(x)\right]$	$X = xy\frac{\partial}{\partial y}$

Table 2.6.1 : A known symmetry of some 1st order nonlinear ODEs [15]

CHAPTER 3

SYMMETRIES OF PDEs

This chapter is focused towards generalizing the notion of symmetries of ODEs to PDEs. The underlying concept in the case of PDEs is the same. Since the symmetry of an ODE is a transformation that preserves the form of the ODE or a transformation that keeps the ODE invariant, then the symmetry of a PDE is a transformation that keeps it unchanged. Without loss of generality we will restrict our attention in this chapter to second order PDEs that involve one dependent variable and two independent variables. However, the ideas and concepts can be extended to PDEs of any type. In the next two chapters we will see that the symmetry approach can be applied successfully to more complicated PDEs.

3.1 Definition of Symmetries of PDEs

A one parameter group of transformations

$$\bar{x} = f(x, y, u, \varepsilon)$$

$$\bar{y} = g(x, y, u, \varepsilon)$$

$$\bar{u} = h(x, y, u, \varepsilon)$$

is called a symmetry of the PDE

$$F(x,y,u,u_x,u_y,u_{xx},u_{xy},u_{yy},....,\partial^n u)=0$$

if the form of PDE remains unchanged under the transformations. In other words,

$$T(x,y,u)\to(\bar{x},\bar{y},\bar{u})\Rightarrow F(\bar{x},\bar{y},\bar{u},\partial\bar{u},....,\partial^n\bar{u})=0$$

where $\partial^n u$ denotes the nth partial derivative of u .

3.2 Prolongation of Infinitesimal Generators of Symmetries of PDEs

Consider the second order PDE in two independent variables of the form

$$F(x,y,u,u_x,u_y,u_{xx},u_{xy},u_{yy})=0.$$

The symmetry operator of this PDE is going to be of the form

$$\zeta(x,y,u)\frac{\partial}{\partial x}+\eta(x,y,u)\frac{\partial}{\partial y}+\phi(x,y,u)\frac{\partial}{\partial u} \tag{3.2.1}$$

Thus, the infinitesimal group is

$$\bar{x}=x+\varepsilon\zeta(x,y,u)+O(\varepsilon^2)$$

$$\bar{y}=y+\varepsilon\eta(x,y,u)+O(\varepsilon^2)$$

$$\bar{u}=u+\varepsilon\phi(x,y,u)+O(\varepsilon^2)$$

To find the transformation of the partial derivatives of first order u_x,u_y we need to derive a formula for the first prolongation. The prolongation formulas will be derived using the total differentiation operator defined below

Definition (Total Differentiation Operator)

Given a function $F\left(x,y,u(x,y),f^1(x,y),f^2(x,y),.....,f^n(x,y)\right)$.

The total differentiation operators with respect to x and y are respectively defined as

$$D_x = \frac{\partial}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial f^1}{\partial x}\frac{\partial}{\partial f^1} + \frac{\partial f^2}{\partial x}\frac{\partial}{\partial f^2} + \dots\dots\dots + \frac{\partial f^n}{\partial x}\frac{\partial}{\partial f^n}$$

$$D_y = \frac{\partial}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial}{\partial u} + \frac{\partial f^1}{\partial y}\frac{\partial}{\partial f^1} + \frac{\partial f^2}{\partial y}\frac{\partial}{\partial f^2} + \dots\dots\dots + \frac{\partial f^n}{\partial y}\frac{\partial}{\partial f^n}$$

First Prolongation

We are looking for the functions $\eta^{[x]}(x, y, u, u_x, u_y)$ and $\eta^{[y]}(x, y, u, u_x, u_y)$ such that

$$\bar{u}_{\bar{x}} = u_x + \varepsilon\eta^{[x]}(x, y, u, u_x, u_y) + O(\varepsilon^2)$$

$$\bar{u}_{\bar{y}} = u_y + \varepsilon\eta^{[y]}(x, y, u, u_x, u_y) + O(\varepsilon^2)$$

Notice that

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

Hence,

$$d\bar{x} = dx + \varepsilon d\zeta + O(\varepsilon^2)$$

$$= dx + \varepsilon\left(\zeta_x dx + \zeta_y dy + \zeta_u du\right) + O(\varepsilon^2)$$

$$= dx + \varepsilon\left(\zeta_x dx + \zeta_y dy + \zeta_u\left(\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\right)\right) + O(\varepsilon^2)$$

$$= \left[1 + \varepsilon\left(\zeta_x + \frac{\partial u}{\partial x}\zeta_u\right)\right]dx + \varepsilon\left(\zeta_y + \frac{\partial u}{\partial y}\zeta_u\right)dy + O(\varepsilon^2)$$

$$d\bar{x} = \left[1 + \varepsilon D_x\zeta\right]dx + \varepsilon\left[D_y\zeta\right]dy + O(\varepsilon^2) \qquad (3.2.2)$$

Similarly, we have

$$d\bar{y} = \varepsilon\left(D_x\eta\right)dx + \varepsilon\left(1 + D_y\eta\right)dy + O(\varepsilon^2) \qquad (3.2.3)$$

Furthermore, we have

$$d\bar{u} = du + \varepsilon d\phi + O(\varepsilon^2) = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \varepsilon\left[\phi_x dx + \phi_y dy + \phi_u\left(\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\right)\right] + O(\varepsilon^2)$$

$$d\bar{u} = \left[\frac{\partial u}{\partial x} + \varepsilon D_x\phi\right]dx + \left[\frac{\partial u}{\partial y} + \varepsilon D_y\phi\right]dy + O(\varepsilon^2) \tag{3.2.4}$$

Notice that $\bar{u} = \bar{u}(\bar{x}, \bar{y})$ implies that

$$d\bar{u} = \frac{\partial \bar{u}}{\partial \bar{x}}d\bar{x} + \frac{\partial \bar{u}}{\partial \bar{y}}d\bar{y}. \tag{3.2.5}$$

Putting equations (3.2.2), (3.2.3) and (3.2.4) in equation (3.2.5) we get

$$\left[\frac{\partial u}{\partial x} + \varepsilon D_x\phi\right]dx + \left[\frac{\partial u}{\partial y} + \varepsilon D_y\phi\right]dy + O(\varepsilon^2)$$

$$= \frac{\partial \bar{u}}{\partial \bar{x}}\left[\left(1+\varepsilon D_x\zeta\right)dx + \varepsilon D_y\zeta dy + O(\varepsilon^2)\right] + \frac{\partial \bar{u}}{\partial \bar{y}}\left[\varepsilon D_x\eta dx + \left(1+\varepsilon D_y\eta\right)dy + O(\varepsilon^2)\right]$$

$$= \left(\frac{\partial \bar{u}}{\partial \bar{x}}\left[1+\varepsilon D_x\zeta\right] + \varepsilon\left[D_x\eta\right]\frac{\partial \bar{u}}{\partial \bar{y}}\right)dx + \left(\frac{\partial \bar{u}}{\partial \bar{x}}\varepsilon\left[D_y\zeta\right] + \frac{\partial \bar{u}}{\partial \bar{y}}\left[1+\varepsilon D_y\eta\right]\right)dy + O(\varepsilon^2)$$

But since dx and dy are linearly independent, the previous relation implies

$$\frac{\partial u}{\partial x} + \varepsilon D_x\phi = \frac{\partial \bar{u}}{\partial \bar{x}}\left[1+\varepsilon D_x\zeta\right] + \varepsilon\left(D_x\eta\right)\frac{\partial \bar{u}}{\partial \bar{x}}$$

$$\frac{\partial u}{\partial y} + \varepsilon D_y\phi = \frac{\partial \bar{u}}{\partial \bar{x}}\varepsilon\left(D_y\zeta\right) + \frac{\partial \bar{u}}{\partial \bar{y}}\left[1+\varepsilon D_y\eta\right]$$

The above two relations can be put in a matrix form as follows

$$\begin{pmatrix} \dfrac{\partial u}{\partial x} + \varepsilon D_x\phi \\ \dfrac{\partial u}{\partial y} + \varepsilon D_y\phi \end{pmatrix} = \begin{pmatrix} 1+\varepsilon D_x\zeta & \varepsilon D_x\eta \\ \varepsilon D_y\zeta & 1+\varepsilon D_y\eta \end{pmatrix}\begin{pmatrix} \dfrac{\partial \bar{u}}{\partial \bar{x}} \\ \dfrac{\partial \bar{u}}{\partial \bar{y}} \end{pmatrix}$$

Set $A = \begin{pmatrix} 1+\varepsilon D_x\zeta & \varepsilon D_x\eta \\ \varepsilon D_y\zeta & 1+\varepsilon D_y\eta \end{pmatrix}$

Also, let $B=\begin{pmatrix} D_x\zeta & D_x\eta \\ D_y\zeta & D_y\eta \end{pmatrix}$

Then, we have

$$A = I + \varepsilon B$$

which implies that

$$A^{-1} = \left(I + \varepsilon B\right)^{-1} = I - \varepsilon B + O(\varepsilon^2).$$

From equation (3.2.4) we get

$$\begin{pmatrix} \dfrac{\partial \bar{u}}{\partial \bar{x}} \\ \dfrac{\partial \bar{u}}{\partial \bar{y}} \end{pmatrix} = A^{-1} \begin{pmatrix} \dfrac{\partial u}{\partial x} + \varepsilon D_x \phi \\ \dfrac{\partial u}{\partial y} + \varepsilon D_y \phi \end{pmatrix} + O(\varepsilon^2)$$

Or equivalently,

$$\begin{pmatrix} \dfrac{\partial \bar{u}}{\partial \bar{x}} \\ \dfrac{\partial \bar{u}}{\partial \bar{y}} \end{pmatrix} = \left(I - \varepsilon B\right) \begin{pmatrix} \dfrac{\partial u}{\partial x} + \varepsilon D_x \phi \\ \dfrac{\partial u}{\partial y} + \varepsilon D_y \phi \end{pmatrix} + O(\varepsilon^2) .$$

Simplifying we obtain

$$\begin{pmatrix} \dfrac{\partial u}{\partial x} + \varepsilon \eta^{[x]}(x, y, u, u_x, u_y) + O(\varepsilon^2) \\ \dfrac{\partial u}{\partial y} + \varepsilon \eta^{[y]}(x, y, u, u_x, u_y) + O(\varepsilon^2) \end{pmatrix} = \begin{pmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{pmatrix} + \varepsilon \begin{pmatrix} D_x \phi \\ D_y \phi \end{pmatrix} - \varepsilon B \begin{pmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{pmatrix} + O(\varepsilon^2)$$

which implies that

$$\begin{pmatrix} \eta^{[x]} \\ \eta^{[y]} \end{pmatrix} = \begin{pmatrix} D_x \phi \\ D_y \phi \end{pmatrix} - \begin{pmatrix} D_x\zeta & D_x\eta \\ D_y\zeta & D_y\eta \end{pmatrix} \begin{pmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{pmatrix}$$

Writing the above equation in terms of ζ, η and ϕ gives

$$\begin{pmatrix} \eta^{[x]} \\ \eta^{[y]} \end{pmatrix} = \begin{pmatrix} \phi_x + \left(\phi_u - \zeta_x\right)u_x - \eta_x u_y - \zeta_u u_x^2 - \eta_u u_x u_y \\ \phi_y + \left(\phi_u - \eta_y\right)u_y - \zeta_y u_x - \zeta_u u_x u_y - \eta_u u_y^2 \end{pmatrix}$$

Therefore,

$$\eta^{[x]} = \phi_x + \left(\phi_u - \zeta_x\right)u_x - \eta_x u_y - \zeta_u u_x^2 - \eta_u u_x u_y \tag{3.2.6}$$

$$\eta^{[y]} = \phi_y + \left(\phi_u - \eta_y\right)u_y - \zeta_y u_x - \zeta_u u_x u_y - \eta_u u_y^2 \tag{3.2.7}$$

Definition

The first prolongation of the vector field (3.2.1) is given by

$$X^{[1]} = \zeta(x,y,u)\frac{\partial}{\partial x} + \eta(x,y,u)\frac{\partial}{\partial y} + \phi(x,y,u)\frac{\partial}{\partial u} + \left(\phi_x + \left(\phi_u - \zeta_x\right)u_x - \eta_x u_y - \zeta_u u_x^2 - \eta_u u_x u_y\right)\frac{\partial}{\partial u_x}$$

$$+\left(\phi_y + \left(\phi_u - \eta_y\right)u_y - \zeta_y u_x - \zeta_u u_x u_y - \eta_u u_y^2\right)\frac{\partial}{\partial u_y}$$

Second Prolongation

We want to prolong the vector field X given in (3.2.1) to the generator

$$X^{[2]} = X^{[1]} + \eta^{[xx]}\frac{\partial}{\partial u_{xx}} + \eta^{[xy]}\frac{\partial}{\partial u_{xy}} + \eta^{[yy]}\frac{\partial}{\partial u_{yy}}$$

so that the functions $\eta^{[xx]}, \eta^{[xy]}$ and $\eta^{[yy]}$ provide transformations for the 2nd derivatives. That is,

$$\bar{u}_{\bar{x}\bar{x}} = u_{xx} + \varepsilon\eta^{[xx]}(x,y,u,u_x,u_y,u_{xx},u_{xy},u_{yy}) + O(\varepsilon^2)$$

$$\bar{u}_{\bar{x}\bar{y}} = u_{xy} + \varepsilon\eta^{[xy]}(x,y,u,u_x,u_y,u_{xx},u_{xy},u_{yy}) + O(\varepsilon^2)$$

$$\bar{u}_{\bar{y}\bar{y}} = u_{yy} + \varepsilon\eta^{[yy]}(x,y,u,u_x,u_y,u_{xx},u_{xy},u_{yy}) + O(\varepsilon^2)$$

It can be shown by doing similar computation as in the first prolongation that the functions $\eta^{[xx]}, \eta^{[xy]}$ and $\eta^{[yy]}$ are given by

$$\eta^{[xx]} = \phi_{xx} + \left(2\phi_{ux} - \zeta_{xx}\right)u_x - \eta_{xx}u_y + \left(\phi_{uu} - 2\zeta_{xu}\right)u_x^2 - 2\eta_{ux}u_x u_y - \zeta_{uu}u_x^3$$

$$-\eta_{uu}u_x^2 u_y + \left(\phi_u - 2\zeta_x\right)u_{xx} - 3\zeta_u u_{xx}u_x - \eta_u u_{xx}u_y - 2\eta_x u_{yx} - 2\eta_u u_{xy}u_x .$$

$$\eta^{[xy]} = \phi_{xy} + \left(\phi_{uy} - \zeta_{xy}\right)u_x - \left(\phi_{xu} - \eta_{xy}\right)u_y - \zeta_{uy}u_x^2 + \left(\phi_{uu} - \zeta_{xu} - \eta_{uy}\right)u_x u_y$$

$$-\eta_{xu}u_y^2 - \zeta_{uu}u_x^2 u_y - \eta_{uu}u_x u_y^2 + \left(\phi_u - \zeta_x - \eta_y\right)u_{xy} - 2\zeta_u u_x u_{xy} - 2\eta_u u_y u_{xy} - \eta_x u_{yy}$$

$$-\eta_u u_x u_{yy} - \zeta_u u_{xx} - \zeta_u u_y u_{xx} .$$

$$\eta^{[yy]} = \phi_{yy} + \left(2\phi_{uy} - \eta_{yy}\right)u_y - \zeta_{yy}u_x - 2\zeta_{uy}u_y u_x + \left(\phi_{uu} - 2\eta_{yu}\right)u_y^2 - \zeta_{uu}u_y^2 u_x - \eta_{uu}u_y^3$$

$$-2\zeta_y u_{xy} - 2\zeta_u u_y u_{xy} + \left(\phi_u - 2\eta_y\right)u_{yy} - \zeta_u u_x u_{yy} - 3\eta_u u_y u_{yy} .$$

Example 3.2.1

Find the second prolongation of the vector field

$$X = x^2 \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y} + u \frac{\partial}{\partial u} .$$

Solution

$$\zeta(x, y, u) = x^2$$

$$\eta(x, y, u) = y^2$$

$$\phi(x, y, u) = u$$

Substituting in the formulas of $\eta^{[x]}, \eta^{[y]}, \eta^{[xx]}, \eta^{[xy]}, \eta^{[yy]}$ we obtain the following.

$$\eta^{[x]} = \left(1 - 2x\right)u_x$$

$$\eta^{[y]} = \left(1 - 2y\right)u_y$$

$$\eta^{[xx]} = -2u_x + \left(1 - 4x\right)u_{xx}$$

$$\eta^{[xy]} = (1-2x-2y)u_{xy}$$

$$\eta^{[yy]} = -2u_y + (1-4y)u_{yy}$$

Substituting these values in the formula of $X^{[2]}$ we obtain the required second prolongation of X to be

$$X^{[2]} = x^2\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y} + u\frac{\partial}{\partial u} + (1-2x)u_x\frac{\partial}{\partial u_x} + (1-2y)u_y\frac{\partial}{\partial u_y} + (-2u_x + u_{xx} - 4xu_{xx})\frac{\partial}{\partial u_{xx}}$$

$$+(1-2x-2y)u_{xy}\frac{\partial}{\partial u_{xy}} + (-2u_y + u_{yy} - 4yu_{yy})\frac{\partial}{\partial u_{yy}}.$$

3.3 Invariance Criterion for Finding Symmetries of PDEs

An alternative definition of symmetries of PDEs can be given now. This definition will be based on prolongation. As in the case of ODEs in the previous chapter, this definition will play the key role in obtaining symmetries of PDEs.

Definition (Invariance Criterion for Symmetries of PDEs)

The infinitesimal transformation

$$\bar{x} = x + \varepsilon\zeta(x,y,u) + O(\varepsilon^2)$$

$$\bar{y} = y + \varepsilon\eta(x,y,u) + O(\varepsilon^2)$$

$$\bar{u} = u + \varepsilon\phi(x,y,u) + O(\varepsilon^2)$$

or equivalently, the generator

$$X = \zeta(x,y,u)\frac{\partial}{\partial x} + \eta(x,y,u)\frac{\partial}{\partial y} + \phi(x,y,u)\frac{\partial}{\partial u}$$

is a symmetry of the PDE

$$F\left(x,y,u,u_x,u_y,u_{xx},u_{xy},u_{yy},.....,\partial^n u\right) = 0 \qquad (3.3.1)$$

if $X^{[n]}F\big|_{F=0} = 0$

where $X^{[n]}$ is the nth prolongation of X .

Example 3.3.1

Show that the generator

$$X = y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$

is a symmetry of the wave equation

$$u_{xx} - u_{yy} = 0.$$

Solution

Since the wave equation is a second order then the invariance criterion will be

$$X^{[2]}F\big|_{F=0} = 0$$

where

$$F = u_{xx} - u_{yy}$$

Thus,

$$X^{[2]}F\big|_{F=0} = \eta^{[xx]} - \eta^{[yy]} = 0 .$$

Notice that,

$$X^{[2]}F = \phi_{xx} + \left(2\phi_{ux} - \zeta_{xx}\right)u_x - \eta_{xx}u_y + \left(\phi_{uu} - 2\zeta_{xu}\right)u_x^2 - 2\eta_{ux}u_xu_y - \zeta_{uu}u_x^3 - \eta_{uu}u_x^2u_y + \left(\phi_u - 2\zeta_x\right)u_{xx}$$

$$-3\zeta_u u_{xx}u_x - \eta_u u_{xx}u_y - 2\eta_x u_{yx} - 2\eta_u u_{xy}u_x - \phi_{yy} - \left(2\phi_{uy} - \eta_{yy}\right)u_y + \zeta_{yy}u_x + 2\zeta_{uy}u_yu_x - \left(\phi_{uu} - 2\eta_{yu}\right)u_y^2$$

$$+\zeta_{uu}u_y^2u_x + \eta_{uu}u_y^3 + 2\zeta_y u_{xy} + 2\zeta_u u_y u_{xy} - \left(\phi_u - 2\eta_y\right)u_{yy} + \zeta_u u_x u_{yy} + 3\eta_u u_y u_{yy}$$

Also, for the given generator we have

$$\zeta(x,y,u)=y$$

$$\eta(x,y,u)=x$$

$$\phi(x,y,u)=0$$

Substituting the values of $\zeta(x,y,u),\eta(x,y,u)$ and $\phi(x,y,u)$ in the above equation we obtain

$$X^{[2]}F=-2u_{yx}+2u_{xy}=0$$

Therefore, the given generator is a symmetry of the given wave equation.

3.4 Finding the Symmetries of PDEs

In order to obtain the symmetries of the PDE (3.3.1) we follow the procedure that is similar to the one mentioned in chapter 2. In particular the steps are as follows.

Step # 1

Apply the invariance criterion

$$X^{[n]}F\Big|_{F=0}=0$$

Step # 2

Compare the coefficients of different powers of derivatives of u in step 2 to obtain the determining equations.

Step # 3

Simplify and solve the determining equations.

Furthermore, for more details in the procedure of obtaining symmetries of PDEs see [11], [12], [13], [14], [15], [17], [18].

Example 3.4.1

Find the symmetry algebra of the differential equation

$$u_{xx} - u_{yy} = u. \qquad (3.4.1)$$

Solution

The symmetry generator is given by

$$X = \zeta(x,y,u)\frac{\partial}{\partial x} + \eta(x,y,u)\frac{\partial}{\partial y} + \phi(x,y,u)\frac{\partial}{\partial u} \qquad (3.4.2)$$

Step # 1

The invariance criterion is

$$X^{[2]}F\Big|_{F=0} = 0$$

where $F = u_{xx} - u_{yy} - u$.

Thus, the above condition leads to

$$X^{[2]}F = \zeta(x,y,u)\frac{\partial F}{\partial x} + \eta(x,y,u)\frac{\partial F}{\partial y} + \phi(x,y,u)\frac{\partial F}{\partial u} + \eta^{[x]}\frac{\partial F}{\partial u_x} + \eta^{[y]}\frac{\partial F}{\partial u_y} + \eta^{[xx]}\frac{\partial F}{\partial u_{xx}}$$

$$+\eta^{[xy]}\frac{\partial F}{\partial u_{xy}} + \eta^{[yy]}\frac{\partial F}{\partial u_{yy}} = 0$$

This leads to

$$-\phi(x,y,u) + \eta^{[xx]} - \eta^{[yy]} = 0.$$

Substituting for $\eta^{[xx]}$ and $\eta^{[yy]}$ in the above equation yields

$$-\phi(x,y,u) + \phi_{xx} + (2\phi_{ux} - \zeta_{xx})u_x - \eta_{xx}u_y + (\phi_{uu} - 2\zeta_{xu})u_x^2 - 2\eta_{ux}u_xu_y - \zeta_{uu}u_x^3 - \eta_{uu}u_x^2u_y$$

$$+(\phi_u - 2\zeta_x)u_{xx} - 3\zeta_u u_{xx}u_x - \eta_u u_{xx}u_y - 2\eta_x u_{yx} - 2\eta_u u_{yx}u_x - \phi_{yy} - (2\phi_{uy} - \eta_{yy})u_y + \zeta_{yy}u_x$$

$$+2\zeta_{uy}u_xu_y - (\phi_{uu} - 2\eta_{uy})u_y^2 + \zeta_{uu}u_y^2u_x + \eta_{uu}u_y^3 + 2\zeta_y u_{xy} + 2\zeta_u u_y u_{xy} - (\phi_u - 2\eta_y)u_{yy}$$

$$+\zeta_u u_x u_{yy} + 3\eta_u u_{yy}u_y = 0$$

Step # 2

The coefficients of powers of the derivatives of u are given as follows

Constant terms : $\phi_{xx} - \phi_{yy} - \phi$

Coefficient of u_x is : $2\phi_{ux} - \zeta_{xx} + \zeta_{yy}$

Coefficient of u_y is : $-\eta_{xx} - 2\phi_{uy} + \eta_{yy}$

Coefficient of u_x^2 is : $\phi_{uu} - 2\zeta_{xu}$

Coefficient of $u_x u_y$ is : $-2\eta_{ux} + 2\zeta_{uy}$

Coefficient of u_x^3 is : $-\zeta_{uu}$

Coefficient of $u_x^2 u_y$ is : $-\eta_{uu}$

Coefficient of u_{xx} is : $\phi_u - 2\zeta_x$

Coefficient of $u_{xx} u_x$ is : $-3\zeta_u$

Coefficient of u_{xy} is : $2\zeta_y - 2\eta_x$

Coefficient of u_y^2 is : $-\phi_{uu} + 2\eta_{uy}$

Coefficient of $u_y^2 u_x$ is : ζ_{uu}

Coefficient of u_y^3 is : η_{uu}

Coefficient of $u_y u_{xy}$ is : $2\zeta_u$

Coefficient of u_{yy} is : $2\eta_y - \phi_u$

Coefficient of $u_x u_{yy}$ is : ζ_u

Coefficient of $u_y u_{yy}$ is : $3\eta_u$

Coefficient of $u_{xy} u_x$ is : $-2\eta_u$

Coefficient of $u_{xx}u_y$ is : $-\eta_u$

Equating these coefficients to zero we obtain after simplifying and removing the redundant equations the following system of PDEs

e1 : $\zeta_u = 0$

e2 : $\eta_u = 0$

e3 : $\eta_{ux} - \zeta_{uy} = 0$

e4 : $\zeta_y - \eta_x = 0$

e5 : $2\phi_{ux} - \zeta_{xx} + \zeta_{yy} = 0$

e6 : $2\phi_{uy} + \eta_{xx} - \eta_{yy} = 0$

e7 : $\phi_{uu} - 2\zeta_{xu} = 0$

e8 : $\phi_u - 2\zeta_x = 0$

e9 : $\phi_u - 2\eta_y = 0$

e10 : $\phi_{xx} - \phi_{yy} - \phi = 0$

Step # 3

we solve the system as follows.

Notice that e1 implies that

$$\zeta = \zeta(x, y).$$

Also, e2 implies that

$$\eta = \eta(x, y).$$

Moreover, e8 and e9 imply that

$$\zeta_x = \eta_y. \qquad (3.4.3)$$

Differentiating (3.4.3) with respect to x gives

(3.4.4)

$$\zeta_{xx} = \eta_{yx}.$$

Differentiating e4 with respect to y implies that

$$\zeta_{yy} = \eta_{xy}. \tag{3.4.5}$$

Notice that (3.4.4) and (3.4.5) lead to

$$\zeta_{xx} = \zeta_{yy}. \tag{3.4.6}$$

Putting (3.4.6) into e5 we obtain

$$\phi_{ux} = 0. \tag{3.4.7}$$

Differentiating e4 with respect to x yields

$$\eta_{xx} = \zeta_{xy}. \tag{3.4.8}$$

Differentiating (3.4.3) with respect to y yields

$$\eta_{yy} = \zeta_{xy}. \tag{3.4.9}$$

Equations (3.4.8) and (3.4.9) imply that

$$\eta_{xx} = \eta_{yy}. \tag{3.4.10}$$

Putting (3.4.10) in e6 we get

$$\phi_{uy} = 0. \tag{3.4.11}$$

Also, putting e1 into e7 we obtain

$$\phi_{uu} = 0. \tag{3.4.12}$$

Therefore, equations (3.4.7), (3.4.11) and (3.4.12) lead to

$$\phi = c_1 u + B(x, y). \tag{3.4.13}$$

Putting (3.4.13) in e8 we obtain

$$c_1 = 2\zeta_x \Rightarrow \zeta_x = \frac{1}{2}c_1.$$

Integrating the above equation with respect to x gives

$$\zeta = \frac{1}{2}c_1 x + f(y). \tag{3.4.14}$$

Substituting (3.4.14) into (3.4.3) yields

$$\eta_y = \frac{1}{2}c_1$$

Integrating the above equation with respect to y yields

$$\eta = \frac{1}{2}c_1 y + g(x) \tag{3.4.15}$$

Substituting (3.4.14) and (3.4.15) into e4 yields

$$f'(y) = g'(x).$$

This implies that

$$f'(y) = g'(x) = c_2.$$

Therefore,

$$f(y) = c_2 y + c_3$$

$$g(x) = c_2 x + c_4$$

Substituting for $f(y)$ into (3.4.14) we obtain

$$\zeta = \frac{1}{2}c_1 x + c_2 y + c_3. \tag{3.4.16}$$

Furthermore, Substituting for $g(x)$ into (3.4.15) gives

$$\eta = \frac{1}{2}c_1 y + c_2 x + c_4. \tag{3.4.17}$$

Putting (3.4.13) into e10 we get

$$B_{xx} - B_{yy} - c_1 u - B = 0.$$

The above equation implies that $c_1 = 0$ and

$$B_{xx} - B_{yy} - B = 0.$$

Putting the value of c_1 into (3.4.13), (3.4.16) and (3.4.17) we obtain the final solution to the determining system which is given by

$$\zeta = c_2 y + c_3$$

$$\eta = c_2 x + c_4$$

$$\phi = B(x, y)$$

where $B(x, y)$ is any function satisfying equation (3.4.1).

Now, to obtain the symmetries we first set

$c_2 = 1$ and $c_3 = c_4 = 0$ then we substitute in the symmetry generator (3.4.2) to get

$$X_1 = y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}.$$

Similarly, set $c_3 = 1$ and $c_2 = c_4 = 0$ then we substitute in (3.4.2) to obtain

$$X_2 = \frac{\partial}{\partial x}.$$

Finally, set $c_4 = 1$ and $c_2 = c_3 = 0$ then substituting into (3.4.2) we get

$$X_3 = \frac{\partial}{\partial y}.$$

Furthermore, we have also the symmetry

$$X_\infty = B(x, y) \frac{\partial}{\partial u}$$

where $B(x, y)$ is any function satisfying equation (3.4.1).

The commutator table of the Lie algebra formed by X_1, X_2, X_3 is given below.

	X_1	X_2	X_3
X_1	0	$-X_3$	$-X_2$
X_2	X_3	0	0
X_3	X_2	0	0

Table 3.4.1 : Commutator table generated by symmetries of equation (3.4.1)

3.5 Reduction of PDEs Using Symmetries

The PDEs can be reduced into simpler forms using symmetries. As in the case of ODEs the task will be achieved using invariant functions. These functions will be obtained by solving the characteristic system produced by the equation

$$X(I) = 0.$$

However, a symmetry of a PDE will reduce the number of variables in the equation by one. In particular, this means that a second order PDE with two independent variables can be converted into an ODE by a symmetry transformation. The method of reduction is illustrated in the following example.

Example 3.5.1

Reduce the equation

$$u_{xx} - u_{yy} = u$$

using the symmetry

$$X_1 = y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

Solution

We solve the equation

$$X(I) = 0.$$

The characteristic system produced by the above equation is

$$\frac{dx}{y}=\frac{dy}{x}=\frac{du}{0}$$

$$xdx = ydy$$

Hence,

$$x^2 - y^2 = c_1 .$$

Moreover,

$$du = 0 \Rightarrow u = c_2 .$$

Therefore, the new similarity variables are

$$\zeta(x, y, u) = x^2 - y^2$$

$$V(\zeta) = u$$

Now, using the chain rule we differentiate u as follows

$$u_x = 2xV_\zeta$$

$$u_{xx} = 2V_\zeta + 4x^2V_{\zeta\zeta}$$

$$u_y = -2yV_\zeta$$

$$u_{yy} = -2V_\zeta + 4y^2V_{\zeta\zeta}$$

Substituting the partial derivatives of u in the differential equation we obtain

$$2V_\zeta + 4x^2V_{\zeta\zeta} + 2V_\zeta - 4y^2V_{\zeta\zeta} - V = 0$$

$$4V_\zeta + \left(4x^2 - 4y^2\right)V_{\zeta\zeta} - V = 0$$

$$4V_\zeta + 4\left(x^2 - y^2\right)V_{\zeta\zeta} - V = 0$$

Therefore, the final reduced equation is given by

$$4V_{\zeta} + 4\zeta V_{\zeta\zeta} - V = 0\,.$$

which is an ODE obtained by reducing the given PDE by one variable.

CHAPTER 4

SYMMETRY ANALYSIS OF THE WAVE EQUATION ON A NON-STATIC SPACETIME

A spacetime is said to be a spherically symmetric if its metric is invariant under rotation. In more technical terms, a spherically symmetric spacetime is a spacetime whose isometry group contains a subgroup which is isomorphic to the rotation group $SO(3)$ and the orbits of this group are 2-dimensional spheres. These isometries are interpreted as rotations [4].

Recall that an isometry of a spacetime is a transformation that preserves its metric and the Killing vector fields represent the infinitesimal form of isometries. The reader is referred to [19] for detailed introduction to isometry and Killing vector fields.

In fact, spherical symmetry is a characteristic feature of many solutions to the Einstein field equations. In this chapter, we are going to consider a spherically symmetric spacetime where the wave equation on the metric associated with it is going to be studied. The Lie symmetries method will be applied since it is an appropriate method for dealing with a natural equation on a four dimensional spacetime where four independent variables appear in the equation under study.

4.1 The Non-static Spacetime

The spacetime being considered is the spherically symmetric non-static spacetime described by the following metric

$$g = dt^2 - dr^2 - e^{t-r}\left(dy^2 + \sin^2(y)dz^2\right) \tag{4.1.1}$$

This spacetime does not admit $\partial/\partial t$ as a Killing vector field and hence is non-static [19]. Notice that the spacetime described by the metric (4.1.1) is like a twisted cross product between a plane and a sphere. This metric is written in a matrix form as

$$g_{ij} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -e^{t-r} & 0 \\ 0 & 0 & 0 & -e^{t-r}\sin^2(y) \end{bmatrix}$$

The inverse of the above matrix is given by

$$g^{ij} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -e^{r-t} & 0 \\ 0 & 0 & 0 & -e^{r-t}\csc^2(y) \end{bmatrix}$$

The wave equation on the metric (4.1.1) is obtained using the following formula

$$\Box u = \frac{1}{\sqrt{\left|\det g_{ij}\right|}} \sum_{i=1}^{4} \frac{\partial}{\partial x_i}\left(\sqrt{\left|\det g_{ij}\right|}\, g^{ij} \frac{du}{dx_j}\right) \tag{4.1.2}$$

Applying the above formula to the metric (4.1.1) we obtain the wave equation of this metric which is given by

$$u_t + u_{tt} + u_r - u_{rr} - e^{r-t}\cot(y)u_y - e^{r-t}u_{yy} - e^{r-t}\csc^2(y)u_{zz} = 0\,. \tag{4.1.3}$$

Obviously, equation (4.1.3) is a 2nd order PDE in four independent variables and is not easy to solve using standard methods. Therefore, Lie symmetry methods will be used to find exact solutions of this equation.

4.2 Symmetry Algebra of the Wave Equation on the Non-static Spacetime

Our main objective in this section is to apply the symmetry analysis to obtain all the symmetries of equation (4.1.3).

Since equation (4.1.3) is a 2nd order PDE in four independent variables and one dependent variable then its symmetry generator will take the following form

$$X = \zeta^{[t]}(t,r,y,z,u)\frac{\partial}{\partial t} + \zeta^{[r]}(t,r,y,z,u)\frac{\partial}{\partial r} + \zeta^{[y]}(t,r,y,z,u)\frac{\partial}{\partial y} + \zeta^{[z]}(t,r,y,z,u)\frac{\partial}{\partial z}$$
$$+\phi(t,r,y,z,u)\frac{\partial}{\partial u}. \tag{4.2.1}$$

In order to obtain all the symmetries of equation (4.1.3) we have to apply the three steps mentioned in the previous chapter.

Step # 1

The invariance criterion is

$$X^{[2]}F\Big|_{F=0} = 0$$

where F is the left hand side of equation (4.1.3).

This criterion leads to

$$X^{2}F\Big|_{F=0} = \zeta^{[t]}(t,r,y,z,u)\frac{\partial F}{\partial t} + \zeta^{[r]}(t,r,y,z,u)\frac{\partial F}{\partial r} + \zeta^{[y]}(t,r,y,z,u)\frac{\partial F}{\partial y} +$$

$$\zeta^{[z]}(t,r,y,z,u)\frac{\partial F}{\partial z} + \phi(t,r,y,z,u)\frac{\partial F}{\partial u} + \eta^{[t]}\frac{\partial F}{\partial u_t} + \eta^{[r]}\frac{\partial F}{\partial u_r} + \eta^{[y]}\frac{\partial F}{\partial u_y} + \eta^{[z]}\frac{\partial F}{\partial u_z}$$

$$+\eta^{[tt]}\frac{\partial F}{\partial u_{tt}} + \eta^{[tr]}\frac{\partial F}{\partial u_{tr}} + \eta^{[ty]}\frac{\partial F}{\partial u_{ty}} + \eta^{[tz]}\frac{\partial F}{\partial u_{tz}} + \eta^{[rr]}\frac{\partial F}{\partial u_{rr}} + \eta^{[ry]}\frac{\partial F}{\partial u_{ry}} + \eta^{[rz]}\frac{\partial F}{\partial u_{rz}} +$$

$$\eta^{[yy]}\frac{\partial F}{\partial u_{yy}} + \eta^{[yz]}\frac{\partial F}{\partial u_{yz}} + \eta^{[zz]}\frac{\partial F}{\partial u_{zz}} = 0$$

Step # 2

The lengthy calculation of step 1 can be computed using maple to obtain a system of 26 PDEs given below

e 1 : $\zeta^t_z = 0$

e 2 : $\zeta^t_u = 0$

e 3 : $\zeta^t_y = 0$

e 4 : $\zeta^r_u = 0$

e 5 : $\zeta^r_y = 0$

e 6 : $\zeta^r_z = 0$

e 7 : $\zeta^y_y = 0$

e 8 : $\zeta^y_u = 0$

e 9 : $\zeta^y_t = 0$

e 10 : $\zeta^y_r = 0$

e11 : $\zeta^z_r = 0$

e 12 : $\zeta^z_u = 0$

e 13 : $\zeta^z_t = 0$

e 14 : $\zeta^y_{zz} = -\zeta^y$

e 15 : $\zeta^z_y = -\csc^2(y)\zeta^y_z$

e 16 : $\zeta^z_z = -\cot(y)\zeta^y$

e 17 : $\zeta^t_t = \frac{1}{2}\zeta^t - \frac{1}{2}\zeta^r$

e18 : $\zeta_r^r = \frac{1}{2}\zeta^t - \frac{1}{2}\zeta^r$

e19 : $\zeta_{rr}^t = \frac{1}{4}\zeta^t - \frac{1}{4}\zeta^r - \frac{1}{2}\zeta_r^t$

e 20 : $\zeta_t^r = \zeta_r^t$

e 21 : $\eta_{ur} = \frac{1}{4}\zeta^t - \frac{1}{4}\zeta^r - \frac{1}{2}\zeta_r^t$

e 22 : $\eta_{uy} = 0$

e 23 : $\eta_{uz} = 0$

e24 : $\eta_{uu} = 0$

e25 : $\eta_{ut} = -\frac{1}{4}\zeta^t + \frac{1}{4}\zeta^r + \frac{1}{2}\zeta_r^t$

e26 : $\eta_{tt} = (e^r\eta_{zz} + e^r\sin^2(y)\eta_{yy} - e^t\sin^2(y)\eta_r + e^t\sin^2(y)\eta_{rr}$
$-e^t\sin^2(y)\eta_t + e^r\cos(y)\sin(y)\eta_y)/e^t\sin^2(y)$

Step # 3

We solve the determining system as follows.

Notice that e7 , e8 , e9 and e10 implies

$$\zeta_t^y = \zeta_r^y = \zeta_y^y = \zeta_u^y = 0$$

which means that

$$\zeta^y = \zeta^y(z).$$

Also , e14 implies that

$$\zeta_{zz}^y + \zeta^y = 0.$$

This leads to

$$\zeta^y = c_1\cos(z) + c_2\sin(z). \qquad (4.2.2)$$

Furthermore , e11 , e12 and e13 imply

$$\zeta_r^z = \zeta_u^z = \zeta_t^z = 0$$

which means that

$$\zeta^z = \zeta^z(y, z)$$

Differentiating (4.2.2) with respect to z we get

$$\zeta_z^y = -c_1 \sin(z) + c_2 \cos(z). \tag{4.2.3}$$

Furthermore, e15 and (4.2.3) lead to

$$\zeta_y^z = -\csc^2(y)\zeta_z^y = -\csc^2(y)[-c_1 \sin(z) + c_2 \cos(z)]$$

$$\zeta_y^z = c_1 \sin(z)\csc^2(y) - c_2 \cos(z)\csc^2(y)$$

Integrating the above equation with respect to y leads to

$$\zeta^z = -c_1 \sin(z)\cot(y) + c_2 \cos(z)\cot(y) + g(z). \tag{4.2.4}$$

Differentiating (4.2.4) with respect to z we obtain

$$\zeta_z^z = -c_1 \cos(z)\cot(y) - c_2 \sin(z)\cot(y) + g'(z) \tag{4.2.5}$$

Putting (4.2.2) and (4.2.5) in e16 produces

$$-\left(c_1 \cos(z) + c_2 \sin(z)\right)\cot(y) = -c_1 \cos(z)\cot(y) - c_2 \sin(z)\cot(y) + g'(z).$$

This implies that

$$g'(z) = 0 \Rightarrow g(z) = c_3.$$

Hence , (4.2.4) becomes

$$\zeta^z = -c_1 \sin(z)\cot(y) + c_2 \cos(z)\cot(y) + c_3. \tag{4.2.6}$$

Moreover , e24 implies

$$\eta = A(t, r, y, z)u + B(t, r, y, z). \tag{4.2.7}$$

Hence, e22 and e23 with (4.2.7) lead to

$$\frac{dA(t,r,y,z)}{dz}=\frac{dA(t,r,y,z)}{dy}=0$$

$$A=A(t,r)$$

Thus , (4.2.7) becomes

$$\eta=A(t,r)u+B(t,r,y,z)\,. \tag{4.2.8}$$

Also, e 21 and e 25 imply

$$\eta_{ut}=-\eta_{ur}\,. \tag{4.2.9}$$

Now , (4.2.8) and (4.2.9) lead to

$$\frac{dA(t,r)}{dt}=-\frac{dA(t,r)}{dr} \tag{4.2.10}$$

Differentiating e21 with respect to r yields

$$\eta_{urr}=\frac{1}{4}\zeta_r^t-\frac{1}{4}\zeta_r^r-\frac{1}{2}\zeta_{rr}^t\,. \tag{4.2.11}$$

Substituting e18 and e19 into (4.2.11) gives

$$\eta_{urr}=\frac{1}{4}\zeta_r^t-\frac{1}{4}\left[\frac{1}{2}\zeta^t-\frac{1}{2}\zeta^r\right]-\frac{1}{2}\left[\frac{1}{4}\zeta^t-\frac{1}{4}\zeta^r-\frac{1}{2}\zeta_r^t\right]$$

$$\eta_{urr}=\frac{1}{2}\zeta_r^t+\frac{1}{4}\zeta^r-\frac{1}{4}\zeta^t\,. \tag{4.2.12}$$

Comparing (4.2.12) with e21 yield

$$\eta_{urr}=-\eta_{ur}\,. \tag{4.2.13}$$

Equations (4.2.13) and (4.2.8) imply

$$\frac{dA^2(t,r)}{dr^2}=-\frac{dA(t,r)}{dr}\Rightarrow\frac{dA^2(t,r)}{dr^2}+\frac{dA(t,r)}{dr}=0\,.$$

Integrating the above equation with respect to r yields

$$A(t,r)+\frac{dA(t,r)}{dr}=g_1(t)$$

$$A(t,r) = g_2(t)e^{-r} + g_1(t) \tag{4.2.14}$$

Differentiating (4.2.14) with respect to r yields

$$\frac{dA(t,r)}{dr} = -g_2(t)e^{-r}\,. \tag{4.2.15}$$

Also , differentiating (4.2.14) with respect to t implies

$$\frac{dA(t,r)}{dt} = g_2'(t)e^{-r} + g_1'(t)\,. \tag{4.2.16}$$

Substituting (4.2.15) and (4.2.16) into (4.2.10) we get

$$g_2'(t)e^{-r} + g_1'(t) = g_2(t)e^{-r} \Rightarrow g_1'(t) = 0\,.$$

Hence ,

$$g_1(t) = c_4$$

Also ,

$$g_2'(t) = g_2(t) \Rightarrow g_2(t) = c_5 e^{t}\,.$$

Substituting $g_1(t)$ and $g_2(t)$ into (4.2.14) gives

$$A(t,r) = c_5 e^{t-r} + c_4\,. \tag{4.2.17}$$

Now , substituting (4.2.17) into (4.2.8) we obtain

$$\eta = (c_5 e^{t-r} + c_4)u + B(t,r,y,z)\,. \tag{4.2.18}$$

Notice that e26 implies that $B(t,r,y,z)$ is any function satisfying equation (4.1.3).

Furthermore , e1 ,e2 and e3 imply

$$\zeta_z^t = \zeta_u^t = \zeta_y^t = 0 \Rightarrow \zeta^t = \zeta^t(r,t)\,.$$

Similarly , e4 , e5 and e6 imply

$$\zeta_u^r = \zeta_y^r = \zeta_z^r = 0 \Rightarrow \zeta^r = \zeta^r(r,t).$$

Notice that, (4.2.18) implies that

$$\eta_{ut} = c_5 e^{t-r}\,. \tag{4.2.19}$$

Substituting (4.2.19) into e25 gives

$$c_5 e^{t-r} = -\frac{1}{4}\zeta^t + \frac{1}{4}\zeta^r + \frac{1}{2}\zeta^t_r .$$

Multiplying the above equation by -2 will give

$$\frac{1}{2}\zeta^t - \frac{1}{2}\zeta^r - \zeta^t_r = -2c_5 e^{t-r} .$$

Putting e18 in the above equation we obtain

$$\zeta^r_r - \zeta^t_r = -2c_5 e^{t-r} .$$

Integrating the above equation with respect to r gives

$$\zeta^r - \zeta^t = 2c_5 e^{t-r} + f_1(t) .$$

Multiplying the above equation by $-\frac{1}{2}$ leads to

$$\frac{1}{2}\zeta^t - \frac{1}{2}\zeta^r = -c_5 e^{t-r} - \frac{1}{2} f_1(t) . \tag{4.2.20}$$

Putting e18 again in (4.2.20) gives

$$\zeta^r_r = -c_5 e^{t-r} - \frac{1}{2} f_1(t) .$$

Integrating the above equation with respect to r yields

$$\zeta^r = c_5 e^{t-r} - \frac{1}{2} r f_1(t) + f_2(t) . \tag{4.2.21}$$

Differentiating (4.2.21) with respect to t we get

$$\zeta^r_t = c_5 e^{t-r} - \frac{1}{2} r f_1'(t) + f_2'(t) . \tag{4.2.22}$$

Thus, e20 and (4.2.22) imply

$$\zeta^t_r = c_5 e^{t-r} - \frac{1}{2} r f_1'(t) + f_2'(t) .$$

Integrating the above equation with respect to r yields

$$\zeta^t = -c_5 e^{t-r} - \frac{1}{4} r^2 f_1'(t) + r f_2'(t) + f_3(t) . \tag{4.2.23}$$

Equations e17 and e18 imply

$$\zeta^t_t = \zeta^r_r \,. \tag{4.2.24}$$

Differentiating (4.2.23) with respect to t gives

$$\zeta^t_t = -c_5 e^{t-r} - \frac{1}{4} r^2 f_1''(t) + r f_2''(t) + f_3'(t). \tag{4.2.25}$$

Putting (4.2.21) and (4.2.25) into (4.2.24) yields

$$-c_5 e^{t-r} - \frac{1}{4} r^2 f_1''(t) + r f_2''(t) + f_3'(t) = -c_5 e^{t-r} - \frac{1}{2} f_1(t)\,.$$

Therefore ,

$$\begin{aligned} & f_1''(t) = 0 \\ & f_1(t) = c_6 t + c_7 \end{aligned}\,.$$

Similarly ,

$$\begin{aligned} & f_2''(t) = 0 \\ & f_2(t) = c_8 t + c_9 \end{aligned}\,.$$

Also ,

$$f_3'(t) = -\frac{1}{2} f_1(t)$$

$$f_3'(t) = -\frac{1}{2} c_6 t - \frac{1}{2} c_7$$

Thus ,

$$f_3(t) = -\frac{1}{4} c_6 t^2 - \frac{1}{2} c_7 t + c_{10}\,.$$

Substituting for $f_1(t)$ and $f_2(t)$ into (4.2.21) yields

$$\zeta^r = c_5 e^{t-r} - \frac{1}{2} c_6 t r - \frac{1}{2} c_7 r + c_8 t + c_9\,. \tag{4.2.26}$$

Moreover , substituting for $f_1(t)$, $f_2(t)$ and $f_3(t)$ into (4.2.23) gives

$$\zeta^t = -c_5 e^{t-r} - \frac{1}{4}c_6 r^2 + c_8 r - \frac{1}{4}c_6 t^2 - \frac{1}{2}c_7 t + c_{10}. \tag{4.2.27}$$

Differentiating (4.2.26) with respect to r yields

$$\zeta^r_r = -c_5 e^{t-r} - \frac{1}{2}c_6 t - \frac{1}{2}c_7. \tag{4.2.28}$$

Differentiating (4.2.27) with respect to t yields

$$\zeta^t_t = -c_5 e^{t-r} - \frac{1}{2}c_6 t - \frac{1}{2}c_7. \tag{4.2.29}$$

Putting (4.2.26), (4.2.27) and (4.2.29) into e17 leads to

$$-c_5 e^{t-r} - \frac{1}{2}c_6 t - \frac{1}{2}c_7 = -c_5 e^{t-r} - \frac{1}{8}c_6 r^2 + \frac{1}{2}c_8 r - \frac{1}{8}c_6 t^2 - \frac{1}{4}c_7 t + \frac{1}{2}c_{10} + \frac{1}{4}c_6 tr + \frac{1}{4}c_7 r - \frac{1}{2}c_8 t - \frac{1}{2}c_9$$

The above equation leads to

$$c_6 = 0$$

Furthermore,

$$c_{10} - c_9 = 2c_8 = -c_7. \tag{4.2.30}$$

Therefore , equations (4.2.26) and (4.2.27) become

$$\zeta^r = c_5 e^{t-r} + c_8 r + c_8 t + c_9 \tag{4.2.31}$$

$$\zeta^t = -c_5 e^{t-r} + c_8 r + c_8 t + c_{10} \tag{4.2.32}$$

Notice that ζ^r can be rewritten as

$$\zeta^r = c_5 e^{t-r} + c_8 r + c_8 t + c_{10} - 2c_8 \tag{4.2.33}$$

Now, we can obtain the symmetries from (4.2.2), (4.2.6), (4.2.18), (4.2.32) and (4.2.33) as follows.

Set $c_1 = 1$ and $c_i = 0$ for $i \neq 1$ then substitute in the symmetry generator (4.2.1) to obtain the first symmetry which is given by

$$X_1 = -\sin(z)\cot(y)\frac{\partial}{\partial z} + \cos(z)\frac{\partial}{\partial y}.$$

Set $c_2 = 1$ and $c_i = 0$ for $i \neq 2$ then substitute in (4.2.1) to get

$$X_2 = \cos(z)\cot(y)\frac{\partial}{\partial z} + \sin(z)\frac{\partial}{\partial y}.$$

Set $c_3 = 1$ and $c_i = 0$ for $i \neq 3$ then substitute in (4.2.1) to get

$$X_3 = \frac{\partial}{\partial z}.$$

Set $c_4 = 1$ and $c_i = 0$ for $i \neq 4$ then substitute in (4.2.1) to get

$$X_4 = u\frac{\partial}{\partial u}.$$

Set $c_5 = 1$ and $c_i = 0$ for $i \neq 5$ then substitute in (4.2.1) to get

$$X_5 = e^{t-r}u\frac{\partial}{\partial u} - e^{t-r}\frac{\partial}{\partial t} + e^{t-r}\frac{\partial}{\partial r}.$$

Set $c_8 = 1$ and $c_i = 0$ for $i \neq 8$ then substitute in (4.2.1) to get

$$X_6 = (r + t - 2)\frac{\partial}{\partial r} + (r + t)\frac{\partial}{\partial t}.$$

Set $c_{10} = 1$ and $c_i = 0$ for $i \neq 10$ then substitute in (4.2.1) to get

$$X_7 = \frac{\partial}{\partial r} + \frac{\partial}{\partial t}.$$

Moreover, we have

$$X_\infty = B(t, r, y, z)\frac{\partial}{\partial u}$$

where $B(t, r, y, z)$ is any function satisfying equation (4.1.3).

Finally, the commutation relation for Lie algebra generated by $X_1, X_2, X_3, X_4, X_5, X_6$ and X_7 is given in the table 4.2.1. This table will help us to choose the proper subalgebra that will be utilized in the reduction of the wave equation (4.1.3). In the next four sections we will reduce equation (4.1.3) to four equivalent equations using combinations of the symmetries that we have obtained.

	X_1	X_2	X_3	X_4	X_5	X_6	X_7
X_1	0	$-X_3$	X_2	0	0	0	0
X_2	X_3	0	$-X_1$	0	0	0	0
X_3	$-X_2$	X_1	0	0	0	0	0
X_4	0	0	0	0	0	0	0
X_5	0	0	0	0	0	$-2X_5$	0
X_6	0	0	0	0	$2X_5$	0	$-2X_7$
X_7	0	0	0	0	0	$2X_7$	0

Table 4.2.1 : Commutator table generated by symmetries of equation (4.1.3).

4.3 Reduction and Exact Solution of the Wave Equation Using the Subalgebra $\langle X_2, X_5 \rangle$

It is obvious from the commutator table in section 4.2 thati

$$[X_2, X_5] = 0.$$

This means that X_2 and X_5 commute. Hence, we can use the abelian subalgebra $\langle X_2, X_5 \rangle$ to reduce equation (4.1.3) by two variables. This reduction can be done as follows.

Level 1 Reduction

This reduction will be done using the symmetry

$$X = X_2 = \cos(z)\cot(y)\frac{\partial}{\partial z} + \sin(z)\frac{\partial}{\partial y}.$$

Solving the equation $X(I) = 0$ produces the following characteristic system

$$\frac{dt}{0}=\frac{dr}{0}=\frac{dy}{\sin(z)}=\frac{dz}{\cos(z)\cot(y)}=\frac{du}{0}.$$

Solving the above system gives

$$t=c_1$$

$$r=c_2$$

$$\sin(y)\cos(z)=c_3$$

$$u=c_4$$

The new similarity variables are

$$\zeta_1(t,r,y,z)=t$$

$$\zeta_2(t,r,y,z)=r$$

$$\zeta_3(t,r,y,z)=\sin(y)\cos(z)$$

$$V(\zeta_1,\zeta_2,\zeta_3)=u \qquad (4.3.1)$$

We use the chain rule to differentiate u as follows

$$u_t=V_{\zeta_1}$$

$$u_{tt}=V_{\zeta_1\zeta_1}$$

$$u_r=V_{\zeta_2}$$

$$u_{rr}=V_{\zeta_2\zeta_2}$$

$$u_y=\cos(y)\cos(z)V_{\zeta_3}$$

$$u_{yy}=-\sin(y)\cos(z)V_{\zeta_3}+\cos^2(y)\cos^2(z)V_{\zeta_3\zeta_3}$$

$$u_z=-\sin(y)\sin(z)V_{\zeta_3}$$

$$u_{zz} = -\sin(y)\cos(z)V_{\zeta_3} + \sin^2(y)\sin^2(z)V_{\zeta_3\zeta_3}$$

Substituting the new variables into equation (4.1.3) yields

$$V_{\zeta_1} + V_{\zeta_1\zeta_1} + V_{\zeta_2} - V_{\zeta_2\zeta_2} - e^{r-t}\cot(y)\cos(y)\cos(z)V_{\zeta_3} + e^{r-t}\sin(y)\cos(z)V_{\zeta_3} - e^{r-t}\cos^2(y)\cos^2(z)V_{\zeta_3\zeta_3}$$

$$+e^{r-t}\csc^2(y)\sin(y)\cos(z)V_{\zeta_3} - e^{r-t}\csc^2(y)\sin^2(y)\sin^2(z)V_{\zeta_3\zeta_3} = 0$$

Simplifying the above equation gives

$$V_{\zeta_1} + V_{\zeta_1\zeta_1} + V_{\zeta_2} - V_{\zeta_2\zeta_2} + 2\zeta_3 e^{\zeta_2-\zeta_1}V_{\zeta_3} + \left(\zeta_3^2 - 1\right)e^{\zeta_2-\zeta_1}V_{\zeta_3\zeta_3} = 0. \tag{4.3.2}$$

Level 2 Reduction

Next we perform further reduction using the symmetry

$$X = X_5 = e^{t-r}\frac{\partial}{\partial r} - e^{t-r}\frac{\partial}{\partial t} + e^{t-r}u\frac{\partial}{\partial u}$$

Notice that

$$X(\zeta_1) = X_5(t) = -e^{t-r}$$

$$X(\zeta_2) = X_5(r) = e^{t-r}$$

$$X(\zeta_3) = X_5\left(\sin(y)\cos(z)\right) = 0$$

$$X(V) = X_5(u) = e^{t-r}V$$

so that the symmetry

$$X = X(\zeta_1)\frac{\partial}{\partial \zeta_1} + X(\zeta_2)\frac{\partial}{\partial \zeta_2} + X(\zeta_3)\frac{\partial}{\partial \zeta_3} + X(V)\frac{\partial}{\partial V}$$

inherited by equation (4.3.2) is given by

$$X = -e^{t-r}\frac{\partial}{\partial \zeta_1} + e^{t-r}\frac{\partial}{\partial \zeta_2} + 0\frac{\partial}{\partial \zeta_3} + e^{t-r}V\frac{\partial}{\partial V}.$$

Now, equation $X(I) = 0$ produces the following characteristic system

$$\frac{d\zeta_1}{-e^{t-r}}=\frac{d\zeta_2}{e^{t-r}}=\frac{d\zeta_3}{0}=\frac{dV}{e^{t-r}V}$$

Solving the system we obtain

$$\zeta_3=c_1$$

$$\zeta_1+\zeta_2=c_2$$

$$\ln|V|-\zeta_2=c_3$$

The new similarity variables are

$$x(\zeta_1,\zeta_2,\zeta_3)=\zeta_3$$

$$s(\zeta_1,\zeta_2,\zeta_3)=\zeta_1+\zeta_2$$

$$W(x,s)=\ln|V|-\zeta_2$$

Therefore,

$$V=e^{W(x,s)+\zeta_2} \tag{4.3.3}$$

Using the chain rule to differentiate V we get

$$V_{\zeta_1}=W_s e^{W+\zeta_2}$$

$$V_{\zeta_1\zeta_1}=W_{ss}e^{W+\zeta_2}+W_s^{\,2}e^{W+\zeta_2}$$

$$V_{\zeta_2}=(W_s+1)e^{W+\zeta_2}$$

$$V_{\zeta_2\zeta_2}=W_{ss}e^{W+\zeta_2}+(W_s+1)^2e^{W+\zeta_2}$$

$$V_{\zeta_3}=W_x e^{W+\zeta_2}$$

$$V_{\zeta_3\zeta_3}=W_{xx}e^{W+\zeta_2}+W_x^{\,2}e^{W+\zeta_2}$$

Substituting the above variables in equation (4.3.2) we obtain

$$W_s e^{W+\zeta_2} + W_{ss} e^{W+\zeta_2} + W_s^{\,2} e^{W+\zeta_2} + (W_s + 1) e^{W+\zeta_2} - W_{ss} e^{W+\zeta_2} - (W_s + 1)^2 e^{W+\zeta_2} + 2x W_x e^{W+\zeta_2} e^{\zeta_2-\zeta_1}$$

$$+(x^2-1)^2 W_{xx} e^{W+\zeta_2} e^{\zeta_2-\zeta_1} + (x^2-1)^2 W_x^{\,2} e^{W+\zeta_2} e^{\zeta_2-\zeta_1} = 0$$

Simplifying the above equation we obtain our final reduced equation which is given by

$$2xW_x + x^2 W_{xx} - W_{xx} + x^2 W_x^{\,2} - W_x^{\,2} = 0$$

or equivalently,

$$\left(x^2-1\right)W_{xx} + \left(x^2-1\right)W_x^{\,2} + 2xW_x = 0 \qquad (4.3.4)$$

The above equation is a 2nd order in two independent variables since W is a function of x and s.

Exact Solution

We can obtain exact solution to the wave equation (4.1.3) by solving the reduced equation (4.3.4) above.

In order to solve (4.3.4) we will divide both sides of the equation by $\left(x^2-1\right)W_x$ to get

$$\frac{W_{xx}}{W_x} + W_x + \frac{2x}{x^2-1} = 0\,.$$

Integrating the above equation gives

$$\ln\left|W_x\right| + W + \ln\left|x^2-1\right| = A(s)\,. \qquad (4.3.5)$$

Assume

$$W = \ln\left|f(x,s)\right|\,. \qquad (4.3.6)$$

Putting (4.3.6) into (4.3.5) we get

$$\ln\left|f(x,s)\left(x^2-1\right)W_x\right| = A(s)$$

$$f(x,s)\left(x^2-1\right)W_x = e^{A(s)} = h(s)$$

$$W_x = \frac{h(s)}{(x^2-1)^2 f(x,s)} \tag{4.3.7}$$

Also from equation (4.3.6) we have

$$W_x = \frac{\frac{df(x,s)}{dx}}{f(x,s)} \tag{4.3.8}$$

Equating (4.3.7) and (4.3.8) yields

$$\frac{h(s)}{(x^2-1)^2 f(x,s)} = \frac{\frac{df(x,s)}{dx}}{f(x,s)}.$$

This leads to

$$\frac{df(x,s)}{dx} = \frac{h(s)}{x^2-1}.$$

Integrating the above equation with respect to x gives

$$f(x,s) = \frac{h(s)}{2}\ln|x-1| - \frac{h(s)}{2}\ln|x+1| + g(s).$$

Therefore, substituting for $f(x,s)$ into equation (4.3.6) gives

$$W = \ln\left|\frac{h(s)}{2}\ln|x-1| - \frac{h(s)}{2}\ln|x+1| + g(s)\right|. \tag{4.3.9}$$

Substituting (4.3.9) into equation (4.3.3) gives

$$V = e^{\ln\left|\frac{h(s)}{2}\ln|x-1| - \frac{h(s)}{2}\ln|x+1| + g(s)\right| + \zeta_2}$$

$$V = \frac{1}{2}e^{r}\left[h(s)\ln|\zeta_3 - 1| - h(s)\ln|\zeta_3 + 1| + 2g(\zeta_1 + \zeta_2)\right] \tag{4.3.10}$$

Finally, we can find our final solution by substituting (4.3.10) into equation (4.3.1) to obtain

$$u(t,r,y,z) = \frac{1}{2}e^{r}\left[h(t+r)\ln|\sin(y)\cos(z) - 1| - h(t+r)\ln|\sin(y)\cos(z) + 1| + 2g(t+r)\right]$$

4.4 Reduction and Exact Solution of the Wave Equation Using the Subalgebra $\langle X_7 + aX_4, X_2 \rangle$

Obviously, from the commutator table we notice that $\langle X_7 + aX_4, X_2 \rangle$ forms an abelian subalgebra since

$$[X_7 + aX_4, X_2] = 0$$

Thus, we can use the two-dimensional subalgebra $\langle X_7 + aX_4, X_2 \rangle$ to reduce the wave equation (4.1.3) by two variables.

Level 1 Reduction

Using the symmetry

$$X = X_7 + aX_4 = \frac{\partial}{\partial r} + \frac{\partial}{\partial t} + au\frac{\partial}{\partial u},$$

the equation $X(I) = 0$ gives the following system

$$\frac{dt}{1} = \frac{dr}{1} = \frac{dy}{0} = \frac{dz}{0} = \frac{du}{au}.$$

Solving the system gives

$$y = c_1$$

$$z = c_2$$

$$r - t = c_3$$

$$\ln|u| - at = c_4$$

The new similarity variables are

$$\zeta_1(t, r, y, z) = y$$

$$\zeta_2(t, r, y, z) = z$$

$$\zeta_3(t, r, y, z) = r - t$$

$$V(\zeta_1,\zeta_2,\zeta_3)=\ln|u|-at$$

Therefore,

$$u=e^{at+V}\ . \tag{4.4.1}$$

The derivatives of *u* are found using the chain rule as follows

$$u_t=\left[a-V_{\zeta_3}\right]e^{at+V}$$

$$u_{tt}=V_{\zeta_3\zeta_3}e^{at+V}+\left(a-V_{\zeta_3}\right)^2e^{at+V}$$

$$u_r=V_{\zeta_3}e^{at+V}$$

$$u_{rr}=V_{\zeta_3\zeta_3}e^{at+V}+V_{\zeta_3}^2e^{at+V}$$

$$u_y=V_{\zeta_1}e^{at+V}$$

$$u_{yy}=V_{\zeta_1\zeta_1}e^{at+V}+V_{\zeta_1}^2e^{at+V}$$

$$u_z=V_{\zeta_2}e^{at+V}$$

$$u_{zz}=V_{\zeta_2\zeta_2}e^{at+V}+V_{\zeta_2}^2e^{at+V}$$

Substituting the new variables in equation (4.1.3) gives

$$\left(a-V_{\zeta_3}\right)e^{at+V}+V_{\zeta_3\zeta_3}e^{at+V}+\left(a-V_{\zeta_3}\right)^2e^{at+V}+V_{\zeta_3}e^{at+V}-V_{\zeta_3\zeta_3}e^{at+V}-V_{\zeta_3}^2e^{at+V}-\cot(\zeta_1)V_{\zeta_1}e^{r-t}e^{at+V}$$

$$-V_{\zeta_1\zeta_1}e^{r-t}e^{at+V}-V_{\zeta_1}^2e^{r-t}e^{at+V}-\csc^2(\zeta_1)V_{\zeta_2\zeta_2}e^{r-t}e^{at+V}-\csc^2(\zeta_1)V_{\zeta_2}^2e^{r-t}e^{at+V}=0$$

Simplifying the above equation we obtain

$$\begin{aligned}&a+a^2-2aV_{\zeta_3}-e^{\zeta_3}\cot(\zeta_1)V_{\zeta_1}-e^{\zeta_3}V_{\zeta_1\zeta_1}-e^{\zeta_3}V_{\zeta_1}^2-e^{\zeta_3}\csc^2(\zeta_1)V_{\zeta_2\zeta_2}\\&-e^{\zeta_3}\csc^2(\zeta_1)V_{\zeta_2}^2=0\end{aligned} \tag{4.4.2}$$

Level 2 Reduction

Since the symmetry

$$X = X_2 = \sin(z)\frac{\partial}{\partial y} + \cot(y)\cos(z)\frac{\partial}{\partial z}$$

commute with $X_7 + aX_4$ so the symmetry given by

$$X = X(\zeta_1)\frac{\partial}{\partial \zeta_1} + X(\zeta_2)\frac{\partial}{\partial \zeta_2} + X(\zeta_3)\frac{\partial}{\partial \zeta_3} + X(V)\frac{\partial}{\partial V}$$

$$= X(y)\frac{\partial}{\partial \zeta_1} + X(z)\frac{\partial}{\partial \zeta_2} + X(r-t)\frac{\partial}{\partial \zeta_3} + X\left(\ln|u| - at\right)\frac{\partial}{\partial V}$$

is inherited by equation (4.4.2). In other words, the generator

$$X = \sin(\zeta_2)\frac{\partial}{\partial \zeta_1} + \cot(\zeta_1)\cos(\zeta_2)\frac{\partial}{\partial \zeta_2} + 0\frac{\partial}{\partial \zeta_3} + 0\frac{\partial}{\partial V}.$$

is a symmetry of equation (4.4.2).

The equation $X\left(I\right) = 0$ produces the system

$$\frac{d\zeta_1}{\sin(\zeta_2)} = \frac{d\zeta_2}{\cot(\zeta_1)\cos(\zeta_2)} = \frac{d\zeta_3}{0} = \frac{dV}{0}.$$

The solution of the system is

$$\zeta_3 = c_1$$

$$\sin(\zeta_1)\cos(\zeta_2) = c_2$$

$$V = c_3$$

The new similarity variables are

$$x(\zeta_1, \zeta_2, \zeta_3) = \zeta_3$$

$$s(\zeta_1, \zeta_2, \zeta_3) = \sin(\zeta_1)\cos(\zeta_2)$$

$$W(x, s) = V \tag{4.4.3}$$

The partial derivatives of V are given below

$$V_{\zeta_1} = \cos(\zeta_1)\cos(\zeta_2)W_s$$

$$V_{\zeta_1\zeta_1} = -\sin(\zeta_1)\cos(\zeta_2)W_s + \cos^2(\zeta_1)\cos^2(\zeta_2)W_{ss}$$

$$V_{\zeta_2} = -\sin(\zeta_1)\sin(\zeta_2)W_s$$

$$V_{\zeta_2\zeta_2} = -\sin(\zeta_1)\cos(\zeta_2)W_S + \sin^2(\zeta_1)\sin^2(\zeta_2)W_{ss}$$

$$V_{\zeta_3} = W_x$$

$$V_{\zeta_3\zeta_3} = W_{xx}$$

Substituting these variables in equation (4.4.2) we obtain

$$a + a^2 - 2aW_x - e^x \cot(\zeta_1)\cos(\zeta_1)\cos(\zeta_2)W_s + e^x \sin(\zeta_1)\cos(\zeta_2)\cos(\zeta_2)W_s - e^x \cos^2(\zeta_1)\cos^2(\zeta_2)W_{ss}$$

$$-e^x \cos^2(\zeta_1)\cos^2(\zeta_2)W_s^{\ 2} + e^x \csc^2(\zeta_1)\sin(\zeta_1)\cos(\zeta_2)W_s - e^x \csc^2(\zeta_1)\sin^2(\zeta_2)W_{ss}$$

$$-e^x \csc^2(\zeta_1)\sin^2(\zeta_1)\sin^2(\zeta_2)W_s^{\ 2} = 0$$

The above equation can be simplified to get our final reduced equation which is

$$\left(a + a^2 - 2aW_x\right)e^{-x} + 2sW_s - \left(1 - s^2\right)W_{ss} - \left(1 - s^2\right)W_s^{\ 2} = 0\,. \qquad (4.4.4)$$

Exact Solution

Now we are going to solve equation (4.4.4) to get exact solution to the wave equation (4.1.3).

In order to solve equation (4.4.4) we will assume first that

$$W = f(x) + g(s) \qquad (4.4.5)$$

This implies that

$$W_x = f'(x) \Rightarrow w_{xx} = f''(x)$$

$$W_s = g'(s) \Rightarrow w_{ss} = g''(s)$$

Substituting into equation (4.4.4) yields

$$\left(a + a^2 - 2af'(x)\right)e^{-x} + 2sg'(s) - \left(1 - s^2\right)g''(s) - \left(1 - s^2\right)g'^2(s) = 0$$

$$\left(a+a^2-2af'(x)\right)e^{-x} = -2sg'(s)+\left(1-s^2\right)g''(s)+\left(1-s^2\right)g'^2(s)=c$$

where c denotes constant.

Taking the constant to be zero for simplicity then, the above equation leads into two different equations. These equations are

$$\left(a+a^2-2af'(x)\right)e^{-x} = 0 \tag{4.4.6}$$

$$-2sg'(s)+\left(1-s^2\right)g''(s)+\left(1-s^2\right)g'^2(s)=0 \tag{4.4.7}$$

Equation (4.4.6) is easily solvable by putting

$$a+a^2-2af'(x)=0$$

which leads to

$$f'(x)=\frac{1}{2}+\frac{1}{2}a$$

Integrating the above equation we obtain the solution of equation (4.4.6) which is given by

$$f(x)=\frac{1}{2}x+\frac{1}{2}ax+c\,. \tag{4.4.8}$$

Now, we are to solve equation (4.4.7).

let's first divide the equation by $g'(s)\left(1-s^2\right)$ to get

$$g'(s)+\frac{g''(s)}{g'(s)}-\frac{2s}{1-s^2}=0\,.$$

Integrating the above equation we get

$$g(s)+\ln\left|g'(s)\right|+\ln\left|1-s^2\right|=0 \tag{4.4.9}$$

Assume that

$$g(s)=\ln\left|h(s)\right| \tag{4.4.10}$$

Putting (4.4.10) into (4.4.9) yields

$$\ln\left|\left(1-s^2\right)h'(s)\right| = 0$$

or equivalently,

$$\left(1-s^2\right)h'(s) = 1$$

$$h'(s) = \frac{1}{1-s^2}$$

$$h'(s) = \frac{1}{2}\left[\frac{1}{1+s} + \frac{1}{1-s}\right]$$

Integrating we obtain

$$h(s) = \frac{1}{2}\ln|1+s| - \frac{1}{2}\ln|1-s| + c \qquad (4.4.11)$$

Substituting (4.4.11) into (4.4.10) produces

$$g(s) = \ln\left|\frac{1}{2}\ln|1+s| - \frac{1}{2}\ln|1-s| + c\right|. \qquad (4.4.12)$$

Substituting (4.4.8) and (4.4.12) into equation (4.4.5) we obtain

$$W = \ln\left|\frac{1}{2}\ln|1+s| - \frac{1}{2}\ln|1-s| + c_1\right| + \frac{1}{2}ax + \frac{1}{2}x + c_2. \qquad (4.4.13)$$

Substituting (4.4.13) into (4.4.3) we get after making the necessary transformations

$$V = \ln\left|\frac{1}{2}\ln|1+\sin(\zeta_1)\cos(\zeta_2)| - \frac{1}{2}\ln|1-\sin(\zeta_1)\cos(\zeta_2)| + c_1\right| + \frac{1}{2}a\zeta_3 + \frac{1}{2}\zeta_3 + c_2 \qquad (4.4.14)$$

Finally, substitute equation (4.4.14) into equation (4.4.1) and make the necessary transformations to get our final solution which is

$$u(t,r,y,z) = ke^{\frac{1}{2}(a-1)t+\frac{1}{2}(a+1)r}\left[\ln|1+\sin(y)\cos(z)| - \ln|1-\sin(y)\cos(z)| + c_1\right].$$

4.5 Reduction and Exact Solution of the Wave Equation Using the Subalgebra $\langle X_7 + aX_4, X_1 \rangle$

Since we have

$$[X_7 + aX_4, X_1] = 0$$

we can use the abelian subalgebra $\langle X_7 + aX_4, X_1 \rangle$ to reduce the wave equation (4.1.3) by two variables as we did in the previous section.

Level 1 Reduction

First we use the symmetry

$$X = X_7 + aX_4 = \frac{\partial}{\partial r} + \frac{\partial}{\partial t} + au\frac{\partial}{\partial u}.$$

We have seen in section 4.4 that this symmetry reduced the wave equation (4.1.3) into equation (4.4.2). Therefore, we will jump immediately into the second level of reduction.

Level 2 Reduction

Using the symmetry

$$X_1 = -\sin(z)\cot(y)\frac{\partial}{\partial z} + \cos(z)\frac{\partial}{\partial y},$$

the symmetry inherited by equation (4.4.2) is given by

$$X = X(\zeta_1)\frac{\partial}{\partial \zeta_1} + X(\zeta_2)\frac{\partial}{\partial \zeta_2} + X(\zeta_3)\frac{\partial}{\partial \zeta_3} + X(V)\frac{\partial}{\partial V}$$

where the similarity variables $\zeta_1, \zeta_2, \zeta_3$ and V are the same variables that we have obtained in section 4.4.

Thus, the above expression can be written as

$$X = X(y)\frac{\partial}{\partial \zeta_1} + X(z)\frac{\partial}{\partial \zeta_2} + X(r-t)\frac{\partial}{\partial \zeta_3} + X(\ln u - at)\frac{\partial}{\partial V}.$$

Or equivalently,

$$X = -\cos(\zeta_2)\frac{\partial}{\partial\zeta_1} + \cot(\zeta_1)\sin(\zeta_2)\frac{\partial}{\partial\zeta_2} + 0\frac{\partial}{\partial\zeta_3} + 0\frac{\partial}{\partial V}.$$

The characteristic system produced by the equation $X(I) = 0$ is given by

$$\frac{d\zeta_1}{-\cos(\zeta_2)} = \frac{d\zeta_2}{\cot(\zeta_1)\sin(\zeta_2)} = \frac{d\zeta_3}{0} = \frac{dV}{0}.$$

The solution of the system is given by

$$\zeta_3 = c_1$$

$$\sin(\zeta_1)\sin(\zeta_2) = c_2$$

$$V = c_3$$

The new similarity variables are

$$x(\zeta_1, \zeta_2, \zeta_3) = \zeta_3$$

$$s(\zeta_1, \zeta_2, \zeta_3) = \sin(\zeta_1)\sin(\zeta_2)$$

$$W(x, s) = V \tag{4.5.1}$$

The partial derivatives of V are

$$V_{\zeta_1} = \cos(\zeta_1)\sin(\zeta_2)W_s$$

$$V_{\zeta_1\zeta_1} = -\sin(\zeta_1)\sin(\zeta_2)W_s + \cos^2(\zeta_1)\sin^2(\zeta_2)W_{ss}$$

$$V_{\zeta_2} = \sin(\zeta_1)\cos(\zeta_2)W_s$$

$$V_{\zeta_2\zeta_2} = -\sin(\zeta_1)\sin(\zeta_2)W_s + \sin^2(\zeta_1)\cos^2(\zeta_2)W_{ss}$$

$$V_{\zeta_3} = W_x$$

$$V_{\zeta_3\zeta_3} = W_{xx}$$

Substituting these variables in equation (4.4.2) we obtain our reduced equation which is given by

$$a+a^2-2aW_x - e^x\cot(\zeta_1)\cos(\zeta_1)\sin(\zeta_2)W_s + e^x\sin(\zeta_1)\sin(\zeta_2)W_s - e^x\cos^2(\zeta_1)\sin^2(\zeta_2)W_{ss}$$

$$-e^x\cos^2(\zeta_1)sin^2(\zeta_2)W_s^{\,2} + e^x\csc^2(\zeta_1)\sin(\zeta_1)\sin(\zeta_2)W_s - e^x\csc^2(\zeta_1)\sin^2(\zeta_1)\cos^2(\zeta_2)W_{ss}$$

$$-e^x\csc^2(\zeta_1)\sin^2(\zeta_1)\cos^2(\zeta_2)W_s^{\,2}=0$$

The above equation can be simplified as

$$\left(a+a^2-2aW_x\right)e^{-x}+2sW_S-\left(1-s^2\right)W_{ss}-\left(1-s^2\right)W_s^{\,2}=0\,.$$

This reduced equation is exactly the same as equation (4.4.4) in the previous section. Therefore, we can pick its solution immediately from section 4.4.Thus, this solution is

$$W=\ln\left|\frac{1}{2}\ln|1+s|-\frac{1}{2}\ln|1-s|+c_1\right|+\frac{1}{2}ax+\frac{1}{2}x+c_2\,.$$

Substituting this solution in equation (4.5.1) and transforming the variables we get

$$V=\ln\left|\frac{1}{2}\ln|1+\sin(\zeta_1)\sin(\zeta_2)|-\frac{1}{2}\ln|1-\sin(\zeta_1)\sin(\zeta_2)|+c_1\right|+\frac{1}{2}a\zeta_3$$

$$+\frac{1}{2}\zeta_3+c_2 \tag{4.5.2}$$

Finally, putting (4.5.2) into (4.4.1) and making the transformations between variables we obtain the final solution which is given by

$$u(t,r,y,z)=ke^{\frac{1}{2}(a-1)t+\frac{1}{2}(a+1)r}\left[\ln|1+\sin(y)\sin(z)|-\ln|1-\sin(y)\sin(z)|+c_1\right]$$

4.6 Reduction and Exact Solution of the Wave Equation Using the Subalgebra $\langle X_3+X_7, aX_3+bX_4+cX_7\rangle$

Notice that X_3, X_4 and X_7 commute with one another since,

$$\left[X_3, X_4\right] = \left[X_3, X_7\right] = \left[X_4, X_7\right] = 0.$$

Therefore, the two dimensional abelian subalgebra $\left\langle X_3 + X_7, aX_3 + bX_4 + cX_7 \right\rangle$ can reduce the equation (4.1.3) by two variables.

Level 1 Reduction

Using the symmetry

$$X = X_3 + X_7 = \frac{\partial}{\partial z} + \frac{\partial}{\partial t} + \frac{\partial}{\partial r},$$

the invariants of $X(I) = 0$ are obtained by solving the characteristic system

$$\frac{dz}{1} = \frac{dr}{1} = \frac{dt}{1} = \frac{dy}{0} = \frac{du}{0}.$$

The solution of the system is

$$y = c_1$$

$$r - t = c_2$$

$$t - z = c_3$$

$$u = c_4$$

The similarity variables are

$$\zeta_1(t, r, y, z) = y$$

$$\zeta_2(t, r, y, z) = r - t$$

$$\zeta_3(t, r, y, z) = t - z$$

$$V(\zeta_1, \zeta_2, \zeta_3) = u \qquad (4.6.1)$$

The partial derivatives of u are found as follows

$$u_t = -V_{\zeta_2} + V_{\zeta_3}$$

$$u_{tt} = V_{\zeta_2\zeta_2} - 2V_{\zeta_2\zeta_3} + V_{\zeta_3\zeta_3}$$

$$u_r = V_{\zeta_2}$$

$$u_{rr} = V_{\zeta_2\zeta_2}$$

$$u_y = V_{\zeta_1}$$

$$u_{yy} = V_{\zeta_1\zeta_1}$$

$$u_z = -V_{\zeta_3}$$

$$u_{zz} = V_{\zeta_3\zeta_3}$$

Substituting the partial derivatives in equation (4.1.3) gives

$$V_{\zeta_3} - 2v_{\zeta_2\zeta_3} + v_{\zeta_3\zeta_3} - e^{\zeta_2}\cot(\zeta_1)V_{\zeta_1} - e^{\zeta_2}V_{\zeta_1\zeta_1} - e^{\zeta_2}\csc^2(\zeta_1)V_{\zeta_3\zeta_3} = 0\,. \qquad (4.6.2)$$

Level 2 Reduction

Next reduction is obtained using the symmetry

$$X = aX_3 + bX_4 + cX_7 = a\frac{\partial}{\partial z} + bu\frac{\partial}{\partial u} + c\frac{\partial}{\partial r} + c\frac{\partial}{\partial t}$$

The symmetry

$$X = X(\zeta_1)\frac{\partial}{\partial \zeta_1} + X(\zeta_2)\frac{\partial}{\partial \zeta_2} + X(\zeta_3)\frac{\partial}{\partial \zeta_3} + X(V)\frac{\partial}{\partial V}$$

$$= X(y)\frac{\partial}{\partial \zeta_1} + X(r-t)\frac{\partial}{\partial \zeta_2} + X(t-z)\frac{\partial}{\partial \zeta_3} + X(u)\frac{\partial}{\partial V}$$

is inherited by equation (4.6.2). Therefore, the generator

$$X = 0\frac{\partial}{\partial \zeta_1} + 0\frac{\partial}{\partial \zeta_2} + (c-a)\frac{\partial}{\partial \zeta_3} + bV\frac{\partial}{\partial V}$$

is a symmetry of (4.6.2).

The invariants of $X(I) = 0$ are found by solving the system

$$\frac{d\zeta_1}{0}=\frac{d\zeta_2}{0}=\frac{d\zeta_3}{c-a}=\frac{dV}{bV}\,.$$

Solving the system gives

$$\zeta_1=c_1$$

$$\zeta_2=c_2$$

$$\ln|V|-\frac{b}{c-a}\zeta_3=c_4$$

The similarity variables are

$$x(\zeta_1,\zeta_2,\zeta_3)=\zeta_1$$

$$s(\zeta_1,\zeta_2,\zeta_3)=\zeta_2$$

$$W(x,s)=\ln|V|-\frac{b}{c-a}\zeta_3$$

The last equation leads to

$$V=e^{w+\frac{b}{c-a}\zeta_3}\,. \qquad (4.6.3)$$

Differentiating V using the chain rule produces

$$V_{\zeta_1}=w_x e^{w+\frac{b}{c-a}\zeta_3}$$

$$v_{\zeta_1\zeta_1}=w_{xx}e^{w+\frac{b}{c-a}\zeta_3}+w_x^2 e^{w+\frac{b}{c-a}\zeta_3}$$

$$v_{\zeta_2}=w_s e^{w+\frac{b}{c-a}\zeta_3}$$

$$v_{\zeta_2\zeta_2}=w_{ss}e^{w+\frac{b}{c-a}\zeta_3}+w_s^2 e^{w+\frac{b}{c-a}\zeta_3}$$

$$v_{\zeta_3}=\frac{b}{c-a}e^{w+\frac{b}{c-a}\zeta_3}$$

$$v_{\zeta_3\zeta_3}=\left(\frac{b}{c-a}\right)^2 e^{w+\frac{b}{c-a}\zeta_3}$$

$$v_{\zeta_2\zeta_3}=\frac{b}{c-a}w_s e^{w+\frac{b}{c-a}\zeta_3}$$

Substituting the new variables into (4.6.2) yields

$$\frac{b}{c-a}e^{W+\frac{b}{c-a}\zeta_3}-\frac{2b}{c-a}W_s e^{w+\frac{b}{c-a}\zeta_3}+\left(\frac{b}{c-a}\right)^2 e^{W+\frac{b}{c-a}\zeta_3}-e^S\cot(x)W_x e^{W+\frac{b}{c-a}\zeta_3}-W_{xx}e^s e^{w+\frac{b}{c-a}\zeta_3}$$

$$-W_x^2 e^s e^{w+\frac{b}{c-a}\zeta_3}-\csc^2(x)\left(\frac{b}{c-a}\right)^2 e^s e^{W+\frac{b}{c-a}\zeta_3}=0$$

Simplifying the above equation gives our reduced equation which is

$$\frac{b}{c-a}-\frac{2b}{c-a}W_s+\left(\frac{b}{c-a}\right)^2-e^s\cot(x)W_x-W_{xx}e^s-W_x^2e^s-\csc^2(x)\left(\frac{b}{c-a}\right)^2e^s=0\,.$$

Multiplying the above equation by e^{-s} yields

$$\frac{b}{c-a}e^{-s}-\frac{2b}{c-a}e^{-s}W_s+\left(\frac{b}{c-a}\right)^2e^{-s}-\cot(x)W_x-W_{xx}-W_x^2$$

$$-\csc^2(x)\left(\frac{b}{c-a}\right)^2=0 \qquad (4.6.4)$$

Exact Solution

We are now to solve equation (4.6.4).

Assume that

$$W(x,s)=f(x)+g(s)\,. \qquad (4.6.5)$$

Putting equation (4.6.5) into (4.6.4) gives

$$\frac{b}{c-a}e^{-s}-\frac{2b}{c-a}e^{-s}g'(s)+\left(\frac{b}{c-a}\right)^2e^{-s}-\cot(x)f'(x)-f''(x)-f'^2(x)-\csc^2(x)\left(\frac{b}{c-a}\right)^2=0\,.$$

The above equation leads to

$$\frac{b}{c-a}e^{-s}-\frac{2b}{c-a}e^{-s}g'(s)+\left(\frac{b}{c-a}\right)^2 e^{-s}=\cot(x)f'(x)+f''(x)+f'^2(x)+\csc^2(x)\left(\frac{b}{c-a}\right)^2=c$$

where c denotes constant.

Taking the constant to be zero then the above expression leads into two different equations. These equations are

$$\frac{b}{c-a}e^{-s}-\frac{2b}{c-a}e^{-s}g'(s)+\left(\frac{b}{c-a}\right)^2 e^{-s}=0 \qquad (4.6.6)$$

$$\cot(x)f'(x)+f''(x)+f'^2(x)+\csc^2(x)\left(\frac{b}{c-a}\right)^2=0 \qquad (4.6.7)$$

First of all, we are going to solve equation (4.6.6), so multiplying (4.6.6) by e^s then it will become

$$\frac{b}{c-a}-\frac{2b}{c-a}g'(s)+\left(\frac{b}{c-a}\right)^2=0.$$

Or equivalently,

$$g'(s)=\frac{b}{2(c-a)}+\frac{1}{2}.$$

Integrating we get

$$g(s)=\frac{1}{2}\left[1+\frac{b}{c-a}\right]s+c_1. \qquad (4.6.8)$$

Now we solve equation (4.6.7).

Assume that

$$f'(x)=\csc(x)h(x). \qquad (4.6.9)$$

Differentiating we obtain

$$f''(x)=-\csc(x)\cot(x)h(x)+\csc(x)h'(x). \qquad (4.6.10)$$

Substituting (4.6.9) and (4.6.10) into equation (4.6.7) we obtain

$$\cot(x)\csc(x)h(x)-\csc(x)\cot(x)h(x)+\csc(x)h'(x)+\csc^2(x)h^2(x)+\csc^2(x)\left(\frac{b}{c-a}\right)^2=0$$

Simplifying we get

$$\csc(x)h'(x)+\csc^2(x)h^2(x)+\csc^2(x)\left(\frac{b}{c-a}\right)^2=0\,.$$

Multiplying the above equation by $\sin^2(x)$ we obtain

$$h^2(x)+\sin(x)h'(x)+\left(\frac{b}{c-a}\right)^2=0\,. \tag{4.6.11}$$

To solve (4.6.11) assume that

$$h(x)=\frac{b}{c-a}\tan[A(x)]\,. \tag{4.6.12}$$

Differentiating we obtain

$$h'(x)=\frac{b}{c-a}A'(x)\sec^2[A(x)]\,. \tag{4.6.13}$$

Substituting (4.6.12) and (4.6.13) into (4.6.11) gives

$$\left[\frac{b}{c-a}\right]^2\tan^2[A(x)]+\frac{b}{c-a}\sin(x)A'(x)\sec^2[A(x)]+\left(\frac{b}{c-a}\right)^2=0\,.$$

Therefore, we have

$$\left[\frac{b}{c-a}\right]^2\left(\tan^2[A(x)]+1\right)+\frac{b}{c-a}\sin(x)A'(x)\sec^2[A(x)]=0$$

$$\left[\frac{b}{c-a}\right]^2\sec^2[A(x)]+\frac{b}{c-a}\sin(x)A'(x)\sec^2[A(x)]=0$$

This implies that

$$\sec^2[A(x)]\left(\left[\frac{b}{c-a}\right]^2+\frac{b}{c-a}\sin(x)A'(x)\right)=0$$

This leads into

$$\left[\frac{b}{c-a}\right]^2+\frac{b}{c-a}\sin(x)A'(x)=0$$

.

$$A'(x)=\frac{b}{a-c}\csc(x)$$

Integrating the above equation gives

$$A(x)=\frac{b}{a-c}\ln\left|\csc(x)-\cot(x)\right|+c_2. \qquad (4.6.14)$$

Substituting (4.6.14) into (4.6.12) we obtain

$$h(x)=\frac{b}{c-a}\tan\left[\frac{b}{a-c}\ln\left|\csc(x)-\cot(x)\right|+c_2\right]. \qquad (4.6.15)$$

Putting (4.6.15) into (4.6.9) yields

$$f'(x)=\frac{b}{c-a}\csc(x)\tan\left[\frac{b}{a-c}\ln\left|\csc(x)-\cot(x)\right|+c_2\right].$$

Integrating the above equation we obtain

$$f(x)=\ln\left|\cos\left[\frac{b}{a-c}\ln\left|\csc(x)-\cot(x)\right|+c_2\right]\right|+c_3. \qquad (4.6.16)$$

Substituting (4.6.8) and (4.6.16) into (4.6.5) we get

$$W=\ln\left|\cos\left[\frac{b}{a-c}\ln\left|\csc(x)-\cot(x)\right|+c_2\right]\right|+\frac{1}{2}\left[1+\frac{b}{c-a}\right]s+c_1. \qquad (4.6.17)$$

Now substituting (4.6.17) into (4.6.3) (taking into account that $x=\zeta_1, s=\zeta_2$) we get

$$V=\cos\left(\frac{b}{a-c}\ln\left|\csc(\zeta_1)-\cot(\zeta_1)\right|+c_2\right)e^{\frac{1}{2}\left[1+\frac{b}{c-a}\right]\zeta_2+c_1+\frac{b}{c-a}\zeta_3}. \qquad (4.6.18)$$

Finally, to obtain the final solution of the wave equation (4.1.3) we substitute equation (4.6.18) into (4.6.1) (taking into account that $\zeta_1=y, \zeta_2=r-t, \zeta_3=t-z$) to obtain the final solution which is given by

$$u(t,r,y,z)=\cos\left(\frac{b}{a-c}\ln\left|\csc(y)-\cot(y)\right|+c_2\right)e^{\frac{1}{2}\left[1+\frac{b}{c-a}\right](r-t)+c_1+\frac{b}{c-a}(t-z)}$$

4.7 Summary of the results

We can summarize the results of chapter 4 in the tables below. The first table contains the symmetries of the wave equation on the non-static spacetime. The other tables contain the solutions to the equation that we have obtained.

The Metric	$g=dt^2-dr^2-e^{t-r}\left(dy^2+\sin^2(y)dz^2\right)$
Wave Equation	$u_t+u_{tt}+u_r-u_{rr}-e^{r-t}\cot(y)u_y-e^{r-t}u_{yy}-e^{r-t}\csc^2(y)u_{zz}=0$
Symmetries	$X_1=-\sin(z)\cot(y)\frac{\partial}{\partial z}+\cos(z)\frac{\partial}{\partial y}$ $X_2=\cos(z)\cot(y)\frac{\partial}{\partial z}+\sin(z)\frac{\partial}{\partial y}$ $X_3=\frac{\partial}{\partial z}$ $X_4=u\frac{\partial}{\partial u}$ $X_5=e^{t-r}u\frac{\partial}{\partial u}-e^{t-r}\frac{\partial}{\partial t}+e^{t-r}\frac{\partial}{\partial r}$ $X_6=(r+t-2)\frac{\partial}{\partial r}+(r+t)\frac{\partial}{\partial t}$ $X_7=\frac{\partial}{\partial r}+\frac{\partial}{\partial t}$

Table 4.7.1 : The symmetries of the wave equation (4.1.3).

The exact solutions of the wave equation (4.1.3) are given in the following tables

First Exact Solution

Subalgebra	$S_1=\langle X_2,X_5\rangle$
Generator	$\left\langle\cos(z)\cot(y)\frac{\partial}{\partial z}+\sin(z)\frac{\partial}{\partial y},e^{t-r}u\frac{\partial}{\partial u}-e^{t-r}\frac{\partial}{\partial t}+e^{t-r}\frac{\partial}{\partial r}\right\rangle$
Reduced equation	$\left(x^2-1\right)W_{xx}+\left(x^2-1\right)W_x^2+2XW_x=0$
Exact solution	$u(t,r,y,z)=\frac{1}{2}e^r\left[h(t+r)\ln\left\|\sin(y)\cos(z)-1\right\|-h(t+r)\ln\left\|\sin(y)\cos(z)+1\right\|+2g(t+r)\right]$

Table 4.7.2 : First exact solution of the wave equation (4.1.3).

Second Exact Solution

Subalgebra	$S_2 = \langle X_7 + aX_4, X_2 \rangle$
Generator	$\left\langle \frac{\partial}{\partial r} + \frac{\partial}{\partial t} + au\frac{\partial}{\partial u}, \cos(z)\cot(y)\frac{\partial}{\partial z} + \sin(z)\frac{\partial}{\partial y} \right\rangle$
Reduced equation	$\left(a + a^2 - 2aW_x\right)e^{-x} + 2sW_s - \left(1 - s^2\right)W_{ss} - \left(1 - s^2\right)W_s^{\,2} = 0$
Exact solution	$u(t,r,y,z) = ke^{\frac{1}{2}(a-1)t + \frac{1}{2}(a+1)r}\left[\ln\lvert 1 + \sin(y)\cos(z)\rvert - \ln\lvert 1 - \sin(y)\cos(z)\rvert + c_1\right]$

Table 4.7.3 : Second exact solution of the wave equation (4.1.3).

Third Exact Solution

Subalgebra	$S_3 = \langle X_7 + aX_4, X_1 \rangle$
Generator	$\left\langle \frac{\partial}{\partial r} + \frac{\partial}{\partial t} + au\frac{\partial}{\partial u}, -\sin(z)\cot(y)\frac{\partial}{\partial z} + \cos(z)\frac{\partial}{\partial y} \right\rangle$
Reduced equation	$\left(a + a^2 - 2aW_x\right)e^{-x} + 2sW_s - \left(1 - s^2\right)W_{ss} - \left(1 - s^2\right)W_s^{\,2} = 0$
Exact solution	$u(t,r,y,z) = ke^{\frac{1}{2}(a-1)t + \frac{1}{2}(a+1)r}\left[\ln\lvert 1 + \sin(y)\sin(z)\rvert - \ln\lvert 1 - \sin(y)\sin(z)\rvert + c_1\right]$

Table 4.7.4 : Third exact solution of the wave equation (4.1.3).

Fourth Exact Solution

Subalgebra	$S_4 = \langle X_3 + X_7, aX_3 + bX_4 + cX_7 \rangle$
Generator	$\left\langle \frac{\partial}{\partial r} + \frac{\partial}{\partial t} + \frac{\partial}{\partial z}, a\frac{\partial}{\partial z} + bu\frac{\partial}{\partial u} + c\frac{\partial}{\partial r} + c\frac{\partial}{\partial t} \right\rangle$
Reduced equation	$\frac{b}{c-a}e^{-s} - \frac{2b}{c-a}e^{-s}W_s + \left(\frac{b}{c-a}\right)^2 e^{-s} - \cot(x)W_x - W_{xx} - W_x^{\,2}$ $-\csc^2(x)\left(\frac{b}{c-a}\right)^2 = 0$
Exact solution	$u(t,r,y,z) = \cos\left(\frac{b}{a-c}\ln\lvert\csc(y) - \cot(y)\rvert + c_2\right)e^{\frac{1}{2}\left[1+\frac{b}{c-a}\right](r-t) + c_1 + \frac{b}{c-a}(t-z)}$

Table 4.7.5 : Fourth exact solution of the wave equation (4.1.3)

CHAPTER 5

SYMMETRY ANALYSIS OF THE WAVE EQUATION ON SCHWARZSCHILD SPACETIME

5.1 Historical Background

Schwarzschild is the name of a mathematician who was the first one to find an exact solution to Einstein field equations . This solution, which is one of the simplest and most useful solutions, appeared just few weeks after Einstein had published his famous work on General Relativity in 1915/1916 [4].

The Einstein field equations are given by a system of partial differential equations that describes the relationships between two types of physical quantities. The left hand side of these equations involves an expression of mass, energy and stress of a given physical system while the right hand side involves an expression of the curvature of spacetime in the same system [4].

Actually, solving the Einstein field equations is a very difficult task as they form a coupled system of ten nonlinear PDEs in four dimensional spacetime in general. Therefore, the Schwarzschild approach in solving the system was to simplify it by considering the curvature of spacetime produced by a massive nonrotating spherical object. This means that Schwarzschild spacetime is consisting of one spherical object that is isolated from other masses as it lies in an empty space.

Unfortunately, Schwarzschild had little time to think about his solution as he died shortly after his paper was published. He was serving in the German army during World War I when he suffered from a disease that led ultimately into his death.

However, this solution became later the most famous solution to the field equations as it attracted the attention of physicists and mathematicians. Studying this spacetime had proved to be a very powerful tool to increasing our perception of gravity in the basis of general relativity.

5.2 Significance of the Schwarzschild Spacetime

The Schwarzschild solution is the most interesting spherically symmetric, static vacuum solution of the Einstein field equations. The importance of this solution arises from the fact that it provides the first verification of the Einstein theory of gravity. Indeed, studies with reference to this solution led to a few physical predictions concerning our solar system. These predictions were important in increasing our understanding of the solar system in particular and the whole universe in general as they were more accurate than the predictions made by the classical theory of Newtonian gravity. The reader is refereed to [20] for more details.

In summary, we can list three important predictions made by studying the Schwarzschild spacetime.

Firstly, it provides an explanation for the perihelion shift of Mercury that were not explained initially by Newtonian gravity.

Secondly, it predicted successfully the bending of starlight passing near the sun that Newton's Theory was unable to completely account for.

Finally, studying the Schwarzschild spacetime paved the way for the discovery and study of the black holes.

Moreover, in classical limits the relativity theory approximate to the Newton theory.

5.3 The Schwarzschild Metric

Schwarzschild metric describes the gravitational field exterior to a static spherical, uncharged mass (usually a star, a planet or a black hole) without angular momentum and isolated from all other masses [4]. This metric is given by

$$g = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2 \left(dy^2 + \sin^2(y) dz^2\right) \tag{5.3.1}$$

where t is the time coordinate, r, y and z are radial, zenith and azimuthal coordinates respectively, and M is the mass of the star (or planet).

This metric applies to the region $r > 2M$. The region for which $r < 2M$ is called a black hole because the gravitational field at this region is strong to the extent that nothing can escape, not even light.

It is obvious that some components of the metric blow up at $r = 0$ and $r = 2M$. This means that the metric has singularities at these radii. However, the singularity at $r = 2M$ is not a real singularity. It appears because of bad choice of coordinate. However, the singularity at $r = 0$ is a genuine singularity as the curvature of spacetime at this point is infinite. The reader is referred to [21] and [22] for more details on the Schwarzschild solution and the black holes.

Wave Equation on Schwarzschild Metric

The Schwarzschild metric can be written in a matrix form as

$$g_{ij} = \begin{bmatrix} 1 - \frac{2M}{r} & 0 & 0 & 0 \\ 0 & -\left(1 - \frac{2M}{r}\right)^{-1} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^2 y \end{bmatrix}$$

The inverse of the above matrix is given by

$$g^{ij}=\begin{bmatrix}\left(1-\frac{2M}{r}\right)^{-1} & 0 & 0 & 0\\ 0 & -\left(1-\frac{2M}{r}\right) & 0 & 0\\ 0 & 0 & \frac{-1}{r^2} & 0\\ 0 & 0 & 0 & \frac{-1}{r^2\sin^2 y}\end{bmatrix}$$

We can derive the wave equation on Schwarzschild spacetime by applying the formula (4.1.2) of the previous chapter. The formula applied to the metric (5.3.1) leads to the following wave equation

$$u_{tt}-\frac{2}{r}u_r+\frac{6M}{r^2}u_r-u_{rr}+\frac{4M}{r}u_{rr}-\frac{4M^2}{r^3}u_r-\frac{4M^2}{r^2}u_{rr}-\frac{\cot(y)}{r^2}u_y-\frac{1}{r^2}u_{yy}$$

$$+\frac{2M\cot(y)}{r^3}u_y+\frac{2M}{r^3}u_{yy}-\frac{1}{r^2\sin^2(y)}u_{zz}+\frac{2M}{r^3\sin^2(y)}u_{zz}=0 \qquad (5.3.2)$$

Evidently, the above equation is a very complicated one as it involves 13 terms and depends on four independent variables, t,r,y and z. Therefore, the classical techniques are not very helpful to obtain exact solutions to this equation. Thus, our task now is to implement symmetry analysis to reduce this equation into 2nd order PDE in two independent variables. This reduction will make the equation easier to handle.

5.4 The Symmetry Algebra of the Wave Equation on Schwarzschild Spacetime

In order to find symmetry algebra of equation (5.3.2) we first take the infinitesimal generator of symmetry algebra of the form

$$X = \zeta^t(t,r,y,z,u)\frac{\partial}{\partial t} + \zeta^r(t,r,y,z,u)\frac{\partial}{\partial r} + \zeta^y(t,r,y,z,u)\frac{\partial}{\partial y} + \zeta^z(t,r,y,z,u)\frac{\partial}{\partial z}$$
$$+\phi(t,r,y,z,u)\frac{\partial}{\partial u} \tag{5.4.1}$$

Then, we apply the three steps mentioned in section 3.4.

We start by applying the invariance condition (step#1) which is

$$X^{[2]}F\Big|_{F=0} = 0$$

where

$$F = u_{tt} - \frac{2}{r}u_r + \frac{6M}{r^2}u_r - u_{rr} + \frac{4M}{r}u_{rr} - \frac{4M^2}{r^3}u_r - \frac{4M^2}{r^2}u_{rr} - \frac{\cot(y)}{r^2}u_y - \frac{1}{r^2}u_{yy}$$
$$+\frac{2M\cot(y)}{r^3}u_y + \frac{2M}{r^3}u_{yy} - \frac{1}{r^2\sin^2(y)}u_{zz} + \frac{2M}{r^3\sin^2(y)}u_{zz}$$

The above criterion leads to a system of 22 linear PDEs (step#2) given below

e1 : $\zeta^y_{zz} = -\zeta^y$

e2 : $\zeta^y_u = 0$

e3 : $\zeta^y_t = 0$

e4 : $\zeta^y_r = 0$

e5 : $\zeta^y_y = 0$

e6 : $\zeta^t_t = 0$

e7 : $\zeta^t_r = 0$

e8 : $\zeta^t_y = 0$

e9 : $\zeta^t_z = 0$

$$\text{e10}: \overset{t}{\zeta}_u = 0$$

$$\text{e11}: \overset{z}{\zeta}_t = 0$$

$$\text{e12}: \overset{z}{\zeta}_r = 0$$

$$\text{e13}: \overset{z}{\zeta}_y = -\csc^2(y)\overset{y}{\zeta}_z$$

$$\text{e14}: \overset{z}{\zeta}_z = -\overset{y}{\zeta}\cot(y)$$

$$\text{e15}: \overset{z}{\zeta}_u = 0$$

$$\text{e16}: \overset{r}{\zeta} = 0$$

$$\text{e17}: \phi_{uy} = 0$$

$$\text{e18}: \phi_{ru} = 0$$

$$\text{e19}: \phi_{zu} = 0$$

$$\text{e20}: \phi_{tu} = 0$$

$$\text{e21}: \phi_{uu} = 0$$

$$\text{e22}: \quad \phi_{tt} = \frac{1}{r^3\sin^2(y)}[r\phi_{zz} - 2\phi_y m\cos(y)\sin(y) - 2\phi_{yy} m\sin^2(y) + 4\phi_r m^2\sin^2(y)$$

$$-2\phi_{zz} m - 6\phi_r mr\sin^2(y) + 4\phi_{rr} m^2 r\sin^2(y) - 4\phi_{rr} mr^2\sin^2(y) + \phi_{yy} r\sin^2(y)$$

$$+\cos(y)\phi_y r\sin(y) + \phi_{rr} r^3\sin^2(y) + 2\phi_r r^2\sin^2(y)] = 0$$

We now solve the system (step 3) as follows.

Notice that equations e6, e7, e8, e9 and e10 imply

$$\overset{t}{\zeta}(t,r,y,z,u) = c_1. \tag{5.4.2}$$

Also, e2, e3, e4 and e5 implies

$$\overset{y}{\zeta} = \overset{y}{\zeta}(z).$$

Thus, e1 leads to

$$\overset{y}{\zeta} = c_2 \cos(z) + c_3 \sin(z). \tag{5.4.3}$$

Moreover, e11, e12 and e15 imply

$$\overset{z}{\zeta} = \overset{z}{\zeta}(y, z).$$

Putting (5.4.3) into e14 gives

$$\zeta^z_z = -c_2 \cos(z)\cot(y) - c_3 \sin(z)\cot(y).$$

Integrating the above equation with respect to z yields

$$\zeta^z = -c_2 \sin(z)\cot(y) + c_3 \cos(z)\cot(y) + f(y). \tag{5.4.4}$$

Differentiating the above equation with respect to y gives

$$\zeta^z_y = c_2 \sin(z)\csc^2(y) - c_3 \cos(z)\csc^2(y) + f'(y). \tag{5.4.5}$$

Putting equation (5.4.3) into e13 yields

$$\zeta^z_y = c_2 \sin(z)\csc^2(y) - c_3 \cos(z)\csc^2(y). \tag{5.4.6}$$

Comparing (5.4.5) with (5.4.6) gives

$$f'(y) = 0 \Rightarrow f(y) = c_4.$$

Substituting the value of $f(y)$ into (5.4.4) gives

$$\zeta^z = -c_2 \sin(z)\cot(y) + c_3 \cos(z)\cot(y) + c_4. \tag{5.4.7}$$

Furthermore, equations e17, e18, e19, e20 and e21 imply

$$\phi = c_5 u + B(t, r, y, z). \tag{5.4.8}$$

Finally, e22 implies that $B(t,r,y,z)$ is any function satisfying the wave equation (5.3.2).

Now, we can obtain the required symmetries from equations (5.4.2), (5.4.3), (5.4.7) and (5.4.8) as follows.

In order to obtain the first symmetry we set

$c_1 = 1$ and $c_i = 0$ for $i \neq 1$

then we substitute in the symmetry generator (5.4.1) to get

$$X_1 = \frac{\partial}{\partial t}.$$

Similarly, to obtain the second symmetry we set

$c_2 = 1$ and $c_i = 0$ for $i \neq 2$

then we substitute again in (5.4.1) to get

$$X_2 = \cos(z)\frac{\partial}{\partial y} - \cot(y)\sin(z)\frac{\partial}{\partial z}.$$

We proceed in a similar manner to obtain the remaining symmetries as

$$X_3 = \sin(z)\frac{\partial}{\partial y} + \cot(y)\cos(z)\frac{\partial}{\partial z}$$

$$X_4 = \frac{\partial}{\partial z}$$

$$X_5 = u\frac{\partial}{\partial u}$$

$$X_\infty = B(t,r,y,z)\frac{\partial}{\partial u}$$

where $B(t,r,y,z)$ is any function satisfying (5.3.2).

The commutator table for the Lie algebra generated by X_1, X_2, X_3, X_4 and X_5 is given below.

	X_1	X_2	X_3	X_4	X_5
X_1	0	0	0	0	0
X_2	0	0	X_4	X_3	0
X_3	0	$-X_4$	0	$-X_2$	0
X_4	0	$-X_3$	X_2	0	0
X_5	0	0	0	0	0

Table 5.4.1: **The commutator table generated by symmetries of equation (5.3.2)**.

5.5 Reduction of the Wave Equation on Schwarzschild Spacetime Using Symmetries

As we have seen in the previous chapter, reducing the wave equation (5.3.2) by two variables is achieved using two-dimensional subalgebra. It is clear from the commutator table 5.4.1 that X_1, X_4 and X_5 commute with one another since

$$[X_1, X_4] = [X_1, X_5] = [X_4, X_5] = 0$$

Therefore, an appropriate subalgebra that will provide us with the desired reduction is the abelian algebra given by $\langle aX_1 + bX_4 + cX_5, dX_1 + eX_4 + fX_5 \rangle$.

The reduction can be done as follows.

Level 1 Reduction

We will first use the symmetry given by

$$X = aX_1 + bX_4 + cX_5$$

That is,

$$X = a\frac{\partial}{\partial t} + b\frac{\partial}{\partial z} + cu\frac{\partial}{\partial u}$$

Setting $X(I) = 0$, then we have

$$X(I) = a\frac{\partial I}{\partial t} + b\frac{\partial I}{\partial z} + cu\frac{\partial I}{\partial u} = 0$$

The derived characteristic system is

$$\frac{dt}{a} = \frac{dz}{b} = \frac{dr}{0} = \frac{dy}{0} = \frac{du}{cu}$$

The solution of the above characteristic system is given by

$$r = c_1$$

$$y = c_2$$

$$bt - az = c_3$$

$$ct - a\ln|u| = c_4$$

Therefore, the similarity variables are

$$\zeta_1(t, r, y, z) = r$$

$$\zeta_2(t, r, y, z) = y$$

$$\zeta_3(t, r, y, z) = bt - az$$

$$V(\zeta_1, \zeta_2, \zeta_3) = ct - a\ln|u|$$

Hence, the solution of equation (5.3.2) can be expressed as follows

$$u = e^{(c/a)t - (1la)V(\zeta_1, \zeta_2, \zeta_3)} \tag{5.5.1}$$

Now we use the chain rule to differentiate u as follows.

$$u_t = (\frac{c}{a} - \frac{b}{a}V_{\zeta_3})e^{(c/a)t - (1la)V}$$

$$u_{tt} = \frac{-b^2}{a}V_{\zeta_3\zeta_3}e^{(c/a)t - (1la)V} + (\frac{c}{a} - \frac{b}{a}V_{\zeta_3})^2 e^{(c/a)t - (1la)V}$$

$$u_r = -\frac{1}{a}V_{\zeta_1}e^{(c/a)t-(1la)V}$$

$$u_{rr} = -\frac{1}{a}V_{\zeta_1\zeta_1}e^{(c/a)t-(1la)V} + \frac{1}{a^2}V_{\zeta_1}^2 e^{(c/a)t-(1la)V}$$

$$u_y = -\frac{1}{a}V_{\zeta_2}e^{(c/a)t-(1la)V}$$

$$u_{yy} = -\frac{1}{a}V_{\zeta_2\zeta_2}e^{(c/a)t-(1la)V} + \frac{1}{a^2}V_{\zeta_2}^2 e^{(c/a)t-(1la)V}$$

$$u_z = V_{\zeta_3}e^{(c/a)t-(1la)V}$$

$$u_{zz} = -aV_{\zeta_3\zeta_3}e^{(c/a)t-(1la)V} + V_{\zeta_3}{}^2 e^{(c/a)t-(1la)V}$$

Hence, Equation (5.3.2) can be expressed using the new variables as

$$-\frac{b^2}{a}V_{\zeta_3\zeta_3} + (\frac{c}{a} - \frac{b}{a}V_{\zeta_3})^2 + \frac{2}{a\zeta_1}V_{\zeta_1} - \frac{6M}{a\zeta_1^2}V_{\zeta_1} + \frac{1}{a}V_{\zeta_1\zeta_1} - \frac{1}{a^2}V_{\zeta_1}{}^2 - \frac{4M}{a\zeta_1}V_{\zeta_1\zeta_1} + \frac{4M}{a^2\zeta_1}V_{\zeta_1}{}^2$$

$$+\frac{4M^2}{a\zeta_1^3}V_{\zeta_1} + \frac{4M^2}{a\zeta_1^2}V_{\zeta_1\zeta_1} - \frac{4M^2}{a^2\zeta_1^2}V_{\zeta_1}{}^2 + \frac{\cot(\zeta_2)}{a\zeta_1^2}V_{\zeta_2} + \frac{1}{a\zeta_1^2}V_{\zeta_2\zeta_2} - \frac{1}{a^2\zeta_1^2}V_{\zeta_2}{}^2 - \frac{2M\ \cot(\zeta_2)}{a\zeta_1^3}V_{\zeta_2}$$

$$-\frac{2M}{a\zeta_1^3}V_{\zeta_2\zeta_2} + \frac{2M}{a^2\zeta_1^3}V_{\zeta_2}{}^2 + \frac{a}{\zeta_1^2\sin^2(\zeta_2)}V_{\zeta 3\zeta 3} - \frac{1}{\zeta_1^2\sin^2(\zeta_2)}V_{\zeta_3}{}^2 - \frac{2aM}{\zeta_1^3\sin^2(\zeta_2)}V_{\zeta_3\zeta_3}$$

$$+\frac{2M}{\zeta_1^3\sin^2(\zeta_2)}V_{\zeta_3}{}^2 = 0 \qquad (5.5.2)$$

Level 2 Reduction

Next we will use the symmetry $X = dX_1 + eX_4 + fX_5$.

Hence,

$$X = d\frac{\partial}{\partial t} + e\frac{\partial}{\partial z} + fu\frac{\partial}{\partial u}.$$

Notice that

$$X = X(\zeta_1)\frac{\partial}{\partial\zeta_1} + X(\zeta_2)\frac{\partial}{\partial\zeta_2} + X(\zeta_3)\frac{\partial}{\partial\zeta_3} + X(V)\frac{\partial}{\partial V}$$

is the inherited symmetry of equation (5.5.2). This symmetry can be written as

$$X = X(r)\frac{\partial}{\partial\zeta_1} + X(y)\frac{\partial}{\partial\zeta_2} + X(bt - az)\frac{\partial}{\partial\zeta_3} + X(ct - a\ln|u|)\frac{\partial}{\partial V}$$

or equivalently,

$$X = 0\frac{\partial}{\partial\zeta_1} + 0\frac{\partial}{\partial\zeta_2} + \left(bd - ae\right)\frac{\partial}{\partial\zeta_3} + \left(cd - af\right)\frac{\partial}{\partial V}.$$

$X(I) = 0$ implies that

$$0\frac{dI}{d\zeta_1} + 0\frac{dI}{d\zeta_2} + (bd - ae)\frac{dI}{d\zeta_3} + (cd - af)\frac{dI}{dV} = 0.$$

The above equation leads to the characteristic system given by

$$\frac{d\zeta_1}{0} = \frac{d\zeta_2}{0} = \frac{d\zeta_3}{bd - ae} = \frac{dV}{cd - af}.$$

Solving the characteristic system yields

$$\zeta_1 = c_1$$

$$\zeta_2 = c_2$$

$$(cd - af)\zeta_3 + (ae - bd)V = c_3$$

Therefore , the new similarity variables are

$$x(\zeta_1, \zeta_2, \zeta_3) = \zeta_1$$

$$s(\zeta_1, \zeta_2, \zeta_3) = \zeta_2$$

$$W(x, s) = (cd - af)\zeta_3 + (ae - bd)V$$

Hence, $V(\zeta_1, \zeta_2, \zeta_3)$ can be expressed as

$$V = \frac{1}{ae-bd}W(x,s) - \frac{cd-af}{ae-bd}\zeta_3 \tag{5.5.3}$$

Using the chain rule we differentiate $V(\zeta_1,\zeta_2,\zeta_3)$ as follows

$$V_{\zeta_1} = \frac{1}{ae-bd}W_x$$

$$V_{\zeta_1\zeta_1} = \frac{1}{ae-bd}W_{xx}$$

$$V_{\zeta_2} = \frac{1}{ae-bd}W_s$$

$$V_{\zeta_2\zeta_2} = \frac{1}{ae-bd}W_{ss}$$

$$V_{\zeta_3} = -\frac{cd-af}{ae-bd}$$

$$V_{\zeta_3\zeta_3} = 0$$

Substituting the new variables into equation (5.5.2) yields

$$c^2x^2(ae-bd)^2 + 2cbx^2(ae-bd)(cd-af) + b^2x^2(cd-af)^2 + 2ax(ae-bd)W_x - 6aM(ae-bd)W_x$$

$$+ax^2(ae-bd)W_{xx} - x^2W_x^2 - 4aMx(ae-bd)W_{xx} + 4MxW_x^2 + \frac{4aM^2}{x}(ae-bd)W_x + 4aM^2(ae-bd)W_{xx}$$

$$-4M^2W_x^2 + a\cot(s)(ae-bd)W_s + a(ae-bd)W_{ss} - W_s^2 - \frac{2aM\cot(s)}{x}(ae-bd)W_s - \frac{2aM}{x}(ae-bd)W_{ss}$$

$$+\frac{2M}{x}W_s^2 - \frac{a^2}{\sin^2(s)}(cd-af)^2 + \frac{2a^2M}{x\sin^2(s)}(cd-af)^2 = 0 \tag{5.5.4}$$

Obviously, we can fix the values of the constants to simplify equation (5.5.4), so let

$a=1$, $b=1$, $c=0$, $d=1$, $e=2$, $f=-1$.

Then we substitute these values in equation (5.5.4) and simplify to obtain our reduced form which is

$$\frac{x^3}{x-2M}+(2x-2M)W_x+(x^2-2Mx)W_{xx}-(x^2-2Mx)W_x^{\,2}+\cot(s)W_s+W_{ss}-W_s^{\,2}$$

$$-\csc^2(s)=0 \tag{5.5.5}$$

5.6 Solutions of the Wave Equation on Schwarzschild Spacetime

Clearly, equation (5.5.5) is a 2nd order PDE in 2 independent variables. However, this equation is not a standard one. It cannot be solved by the classical techniques. Therefore, we will try to implement special tricks to find exact solutions.

Firstly, we will assume that the solution $W(x,s)$ will take the following form

$$W(x,s)=f(x)+g(s) \tag{5.6.1}$$

This means that

$$W_x=f'(x)$$

$$W_s=g'(s)$$

Thus, equation (5.5.5) becomes

$$\frac{x^3}{x-2M}+(2x-2M)f'(x)+(x^2-2Mx)f''(x)-(x^2-2Mx)f'^2(x)+\cot(s)g'(s)$$

$$+g''(s)-g'^2(s)-\csc^2(s)=0$$

This leads to

$$\frac{x^3}{x-2M}+(2x-2M)f'(x)+(x^2-2Mx)f''(x)-(x^2-2Mx)f'^2(x)$$

$$=-\cot(s)g'(s)-g''(s)+g'^2(s)+\csc^2(s)=c$$

Taking the constant c to be zero we obtain two equations. These are

$$-\cot(s)g'(s) - g''(s) + g'^2(s) + \csc^2(s) = 0 \tag{5.6.2}$$

$$\frac{x^3}{x-2M} + (2x-2M)f'(x) + (x^2-2Mx)f''(x) - (x^2-2Mx)f'^2(x) = 0 \tag{5.6.3}$$

Evidently, equation (5.6.2) is similar to equation (4.6.7) but with different coefficients. Hence, (5.6.2) can be solved using the same trick that has been implemented in solving (4.6.7).

Thus, we will start by assuming that

$$g'(s) = \csc(s)h(s)\,. \tag{5.6.4}$$

Differentiating (5.6.4) gives

$$g''(s) = -\csc(s)\cot(s).h(s) + \csc(s)h'(s)\,. \tag{5.6.5}$$

Substituting (5.6.4) and (5.6.5) into (5.6.2) yields

$$\csc(s)h'(s) - \csc^2(s)h^2(s) - \csc^2(s) = 0$$

Or equivalently,

$$\csc(s)\left(h'(s) - \csc(s)h^2(s) - \csc(s)\right) = 0$$

Hence,

$$h'(s) - \csc(s)h^2(s) - \csc(s) = 0\,. \tag{5.6.6}$$

Now, to solve (5.6.6) we will assume that

$$h(s) = \tan[A(s)]\,. \tag{5.6.7}$$

Putting (5.6.7) into (5.6.6) gives

$$A'(s)\sec^2[A(s)] - \csc(s)\tan^2[A(s)] - \csc(s) = 0$$

$$A'(s)\sec^2[A(s)] - \csc(s)\left[\tan^2[A(s)] + 1\right] = 0$$

$$A'(s)\sec^2[A(s)] - \csc(s)\sec^2[A(s)] = 0$$

Therefore,

$$\sec^2[A(s)]\big(A'(s)-\csc(s)\big)=0\,.$$

Thus,

$$A'(s)=\csc(s)\,.$$

Integrating the above equation we obtain

$$A(s)=\ln\left|\csc(s)-\cot(s)\right|+c_1\,. \tag{5.6.8}$$

Putting (5.6.8) into (5.6.7) we get

$$h(s)=\tan\Big[\ln\left|\csc(s)-\cot(s)\right|+c_1\Big]. \tag{5.6.9}$$

Putting (5.6.9) into (5.6.4) yields

$$g'(s)=\csc(s)\tan\Big[\ln\left|\csc(s)-\cot(s)\right|+c_1\Big]. \tag{5.6.10}$$

Integrating (5.6.10) we get the solution of (5.6.2) which is

$$g(s)=-\ln\left|\cos\Big(\ln\left|\csc(s)-\cot(s)\right|+c_1\Big)\right|+c_2$$

Or equivalently,

$$g(s)=\ln\left|\sec\Big[\ln\left|\csc(s)-\cot(s)\right|+c_1\Big]\right|+c_2\,. \tag{5.6.11}$$

Now , we are to solve equation (5.6.3)

Let $f'(x)=\psi(x)$ then (5.6.3) can be written as

$$\frac{d\psi}{dx}=\psi^2(x)-\frac{2x-2M}{x^2-2Mx}\psi(x)-\frac{x^4}{(x^2-2Mx)^2}\,. \tag{5.6.12}$$

Equation (5.6.12) is a Riccati equation in a standard form. Using maple to solve (5.6.12) we obtain two solutions $\psi_1(x)$ and $\psi_2(x)$. Then , integrating these solutions will produce two functions $f_1(x)$ and $f_2(x)$ .These functions are solutions of equation (5.6.3). The first function $f_1(x)$ is given by

$$f_1(x) = -\ln\left| c_1(-x+2M)^{2M} e^{x} HeunC\left(-4M, 4M, 0, 8M^2, -8M^2, \frac{-\frac{1}{2}x+M}{M}\right)\right|$$

where

$HeunC\left(-4M, 4M, 0, 8M^2, -8M^2, \frac{-\frac{1}{2}x+M}{M}\right)$ is a special function called the Heun function.

The second function $f_2(x)$ is given by

$$f_2(x) = -\ln\left| c_2(-x+2M)^{-2M} e^{x} HeunC\left(-4M, -4M, 0, 8M^2, -8M^2, \frac{-\frac{1}{2}x+M}{M}\right)\right|$$

where

$HeunC\left(-4M, -4M, 0, 8M^2, -8M^2, \frac{-\frac{1}{2}x+M}{M}\right)$ is also a Heun function.

First of all, we will consider the first solution $f_1(x)$.

Substituting (5.6.11) and $f_1(x)$ into (5.6.1) we obtain

$$W(x,s) = \ln\left| \frac{\sec\left[\ln\left|\csc(s)-\cot(s)\right|\right]\left(-x+2M\right)^{-2M} e^{-x}}{c_1 HeunC(-4M, 4M, 0, 8M^2, -8M^2, \frac{-\frac{1}{2}x+M}{M})}\right| + c_3 \quad (5.6.13)$$

Substituting equation (5.6.13) into equation (5.5.3) (taking into account that $x=\zeta_1$, $s=\zeta_2$ and the constants have been chosen to be $a=b=d=1, c=0, e=2, f=-1$) we obtain

$$V(\zeta_1,\zeta_2,\zeta_3)=\ln\left|\frac{\sec\left[\ln\left|\csc(\zeta_2)-\cot(\zeta_2)\right|\right](-\zeta_1+2M)^{-2M}e^{-\zeta_1}}{c_1 HeunC\left(-4M,4M,0,8M^2,-8M^2,\frac{-\frac{1}{2}\zeta_1+M}{M}\right)}\right|-\zeta_3+c_3 \qquad (5.6.14)$$

Finally, we substitute (5.6.14) into (5.5.1) (taking into account that $\zeta_1=r,\zeta_2=y$ and $\zeta_3=t-z$) we get the first exact solution to (5.3.2) which is given by

$$u_1(t,r,y,z)=k_1(2M-r)^{2M}e^{r+t-z}\cos\left[\ln\left|\csc(y)-\cot(y)\right|\right]HeunC\left(-4M,4M,0,8M^2,-8M^2,1-\frac{r}{2M}\right)$$

where k_1 is a constant.

In a similar manner, if we consider the second solution $f_2(x)$ then we make the necessary substitutions and transformations we reach the second solution $u_2(t,r,y,z)$ which is given by

$$u_2(t,r,y,z)=k_2(2M-r)^{-2M}e^{r+t-z}\cos\left[\ln\left|\csc(y)-\cot(y)\right|\right]HeunC\left(-4M,-4M,0,8M^2,-8M^2,1-\frac{r}{2M}\right)$$

where k_2 is a constant.

It is worth noting that the two solutions are dependent on the four variables, t,r,y and z, which make them very interesting solutions from physical point of view.

Finally, we can combine the two solutions together to obtain a solution which is more general than the two obtained ones. This combined solution is given by

$$u(t,r,y,z)=k_1(2M-r)^{2M}e^{r+t-z}\cos\left[\ln\left|\csc(y)-\cot(y)\right|\right]HeunC\left(-4M,4M,0,8M^2,-8M^2,1-\frac{r}{2M}\right)$$
$$+k_2(2M-r)^{-2M}e^{r+t-z}\cos\left[\ln\left|\csc(y)-\cot(y)\right|\right]HeunC\left(-4M,-4M,0,8M^2,-8M^2,1-\frac{r}{2M}\right)$$

5.7 Summary of the Results

We can summarize the results of chapter 5 in the two tables below. The first table contains the symmetries of the wave equation on Schwarzschild spacetime while the second one presents the obtained solutions of this equation.

The Metric	$g=\left(1-\frac{2M}{r}\right)dt^2-\left(1-\frac{2M}{r}\right)^{-1}dr^2-r^2\left(dy^2+\sin^2(y)dz^2\right)$
Wave Equation	$u_{tt}-\frac{2}{r}u_r+\frac{6M}{r^2}u_r-u_{rr}+\frac{4M}{r}u_{rr}-\frac{4M^2}{r^3}u_r-\frac{4M^2}{r^2}u_{rr}-\frac{\cot(y)}{r^2}u_y-\frac{1}{r^2}u_{yy}$ $+\frac{2M\cot(y)}{r^3}u_y+\frac{2M}{r^3}u_{yy}-\frac{1}{r^2\sin^2(y)}u_{zz}+\frac{2M}{r^3\sin^2(y)}u_{zz}=0$
Symmetries	$X_1=\frac{\partial}{\partial t}$ $X_2=\cos(z)\frac{\partial}{\partial y}-\cot(y)\sin(z)\frac{\partial}{\partial z}$ $X_3=\sin(z)\frac{\partial}{\partial y}+\cot(y)\cos(z)\frac{\partial}{\partial z}$ $X_4=\frac{\partial}{\partial z}$ $X_5=u\frac{\partial}{\partial u}$

Table 5.7.1 : The symmetries of the wave equation (5.3.2)

Subalgebra	$\langle X_1+X_4, X_1+2X_4-X_5\rangle$
Generator	$\left\langle\frac{\partial}{\partial t}+\frac{\partial}{\partial z},\frac{\partial}{\partial t}+2\frac{\partial}{\partial z}-u\frac{\partial}{\partial u}\right\rangle$
Reduced equation	$\frac{x^3}{x-2M}+(2x-2M)W_x+(x^2-2Mx)W_{xx}-(x^2-2Mx)W_x^2+\cot(s)W_s$ $+W_{ss}-W_s^2-\csc^2(s)=0$
First Solution	$u_1(t,r,y,z)=k_1(2M-r)^{2M}e^{r+t-z}\cos\left[\ln\lvert\csc(y)-\cot(y)\rvert\right]H_1(M,r)$ where $H_1(M,r)=HeunC\left(-4M,4M,0,8M^2,-8M^2,1-\frac{r}{2M}\right)$
Second Solution	$u_2(t,r,y,z)=k_2(2M-r)^{-2M}e^{r+t-z}\cos\left[\ln\lvert\csc(y)-\cot(y)\rvert\right]H_2(M,r)$ where $H_2(M,r)=HeunC\left(-4M,-4M,0,8M^2,-8M^2,1-\frac{r}{2M}\right)$
Combined Solution	$u(t,r,y,z)=k_1(2M-r)^{2M}e^{r+t-z}\cos\left[\ln\lvert\csc(y)-\cot(y)\rvert\right]H_1(M,r)$ $+k_2(2M-r)^{-2M}e^{r+t-z}\cos\left[\ln\lvert\csc(y)-\cot(y)\rvert\right]H_2(M,r)$

Table 5.7.2 : Exact solutions of the wave equation (5.3.2)

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