DYNAMIC MODELING OF ROTATING FLEXIBLE PIPE CONVEYING FLUID AND SUBJECTED TO EXTERNAL AXIAL FLOW

BY

Fadi Abdelhadi Ghaith

A Dissertation Presented to the

DEANSHIP OF GRADUATE STUDIES

KING FAHD UNIVERSITY OF PETROLEUM & MINERALS

DHAHRAN, SAUDI ARABIA

In Partial Fulfillment of the
Requirements for the Degree of

DOCTOR OF PHILOSOPHY

In

MECHANICAL ENGINEERING

December, 2009

KING FAHD UNIVERSITY OF PETROLEUM & MINERALS

DHAHRAN 31261, SAUDI ARABIA

DEANSHIP OF GRADUATE STUDIES

This dissertation, written by **Fadi Abdelhadi Ghaith**

Under the directions of his thesis advisor and approved by his thesis committee, has been presented to and accepted by the Dean of graduate Studies, in partial fulfillment of the requirements for the degree of **DOCTOR OF PHILOSOPHY IN MECHANICAL ENGINEERING.**

Dissertation Committee

Dr. Yehia Khulief
(Dissertation Advisor)

Dr. Faleh Al-Sulaiman
(Member)

Dr. Amro Al-Qutub
(Department Chairman)

Dr. Muhammad Hawwa
(Member)

Dr. Salam A. Zummo
(Dean of graduate Studies)

Dr. Ali Al-Gadhib
(Member)

1/1/2010
Date

Dr. Rached Ben-Mansour
(Member)

This work is dedicated to my Parents,
Brother and Sisters.

ACKNOWLEDGEMENTS

It is with great pleasure that I acknowledge the help of many people who guided me in the accomplishment and assisted me in the preparation of this work.

In the first place, I would like to express my indebtedness and heartfelt gratitude to my advisor Prof. Yehia Khulief for his guidance and precious advices throughout this work.

Most importantly, deep thanks to my parents, brother and sisters for their sacrifices, encouragements and support.

Special thanks to all friends who shared me their ideas and experience, particularly, Dr. Mohammad Jarrar and Eng. Fadi Al-Badour.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

NOMENCLATURE

ENGLISH SYMBOLS

A	Nominal cross sectional area of the pipe, m^2.
A_f	Inner cross sectional area of the pipe, m^2.
A_o	Outer cross sectional area of the pipe, m^2.
A_1	Cross sectional area of the pipe after deformation, m^2.
A_i^u	Coefficients of the equations of motion in the axial direction, which are associated with the quadratic nonlinearity terms, where *i=1, 2… 28.*
a	Kelvin-Voigt coefficient of the material damping.
B_i^u	Coefficients of the equations of motion in the axial direction, which are associated with the cubic nonlinearity terms, where *i=1, 2… 28.*
b_z	The body forces, N.
C	Radial clearance.
$\overline{C}$	Enlargement coefficient at the lower end of the pipe.
C_b	Base drag coefficient.
C_{DP}	The form- drag coefficient due to the normal component.
C_f	Frictional damping coefficient.
C_N	Friction coefficient in the normal direction.
C_T	Friction coefficient in the tangential direction.
C_i^u	Coefficients of the equations of motion in the axial direction which represent the linear damping terms, where *i=1, 2…6.*
C_i^v	Coefficients of the equations of motion in the lateral direction in *X-Y* plane, which represent the linear damping terms, where *i=1, 2…6.*
C_i^w	Coefficients of the equations of motion in the lateral direction in *X-Z* plane, which represent the linear damping terms, where *i=1, 2…6.*
c_d	The dimensionless form- drag coefficient due to the normal component.
c_N	Dimensionless friction coefficient in the normal direction.

c_T	Dimensionless friction coefficient in the tangential direction.
D_{ch}	Inner diameter of the outer cylinder, m.
D_h	The hydraulic diameter of the outer cylinder, m.
D_o	Outer diameter of the pipe, m.
D_i	Inner diameter of the pipe, m.
D_i^v	Coefficients of the equations of motion in the lateral direction in *X-Y* plane, corresponding to the extensible condition and associated with the quadratic coupling nonlinearity terms, where *i=1, 2… 26.*
D_i^w	Coefficients of the equations of motion in the lateral direction in *X-Z* plane, corresponding to the extensible condition and associated with the quadratic coupling nonlinearity terms, where *i=1, 2… 29.*
E	Modulus of elasticity, N/m^2.
e	Average wall roughness height.
$\overline{E}$	Error function.
E_i^v	Coefficients of the equations of motion in the lateral direction in *X-Y* plane, corresponding to the extensible condition and associated with the quadratic nonlinearity terms, where *i=1, 2… 20.*
E_i^w	Coefficients of the equations of motion in the lateral direction in *X-Z* plane, corresponding to extensible condition and associated with the quadratic nonlinearity terms, where *i=1, 2… 20.*
F_A	The inviscid hydrodynamic force, N.
F_{eff}	Effective forces, N.
F_H	Hydrostatic force due to the pipe rotation.
F_L	Longitudinal frictional force, N.
F_N	Normal frictional force, N.
F_o	Normal hydrodynamic force induced by the pipe rotation.
F_{px}	Hydrostatic pressure forces in the *x* direction, N.
F_{py}	Hydrostatic pressure forces in the *y* direction, N.
F_{pz}	Hydrostatic pressure forces in the *z* direction, N.
F_Q	Tangential hydrodynamic force induced by the pipe rotation.
F_Y	Hydrodynamic force induced by the pipe rotation in Y-direction.
F_Z	Hydrodynamic force induced by the pipe rotation in Z-direction.

F_i^v	Coefficients of the equations of motion in the lateral direction in *X-Y* plane, corresponding to the extensible condition and associated with the cubic nonlinearity terms, where *i=1, 2… 26.*
F_i^w	Coefficients of the equations of motion in the lateral direction in *X-Z* plane, corresponding to the extensible condition and associated with the cubic nonlinearity terms, where *i=1, 2…34.*
f_r	Friction coefficient between the internal fluid and inner pipe's wall.
f_t	Friction factor between the annular flow and pipe's wall.
G	Modulus of rigidity, N/m^2.
G_i^v	Coefficients of the equations of motion in the lateral direction in *X-Y* plane, corresponding to the inextensible condition and associated with the quadratic coupling nonlinearity terms, where *i=1, 2…4.*
G_i^w	Coefficients of the equations of motion in the lateral direction in *X-Z* plane, corresponding to the inextensible condition and associated with the quadratic coupling nonlinearity terms, where *i=1, 2…6.*
g	Gravitational acceleration, m/s^2.
$\check{g}(t^*)$	Dimensionless time dependant vector of modal coordinates associated with the dimensionless torsion.
H_c	Angular momentum.
H_i^v	Coefficients of the equations of motion in the lateral direction in *X-Y* plane, corresponding to the inextensible condition and associated with the quadratic nonlinearity terms, where *i=1, 2…8.*
H_i^w	Coefficients of the equations of motion in the lateral direction in *X-Z* plane, corresponding to the inextensible condition and associated with the quadratic nonlinearity terms, where *i=1, 2…8.*
h	Minimum film thickness.
$\overline{h}$	Hydraulic coefficient.
h_o	Loss of head due to the sudden flow enlargement.
I	Area moment of inertia of the pipe, m^4.
$\widetilde{I}$	Varaitional function.
I_D	Diameteral mass moment of inertia of the pipe per unit length, kg.m.
$I_{D,f}$	Diameteral mass moment of inertia of the internal fluid per unit length, kg.m.

I_p	Polar mass moment of inertia of the pipe per unit length, kg.m.
$I_{p,f}$	Polar mass moment of inertia of the internal fluid per unit length, kg.m.
I_{xx}	Mass moment of inertia about *x*-axis, kg.m^2.
I_{yy}	Mass moment of inertia about *y*-axis, kg.m^2.
I_{zz}	Mass moment of inertia about *z*-axis, kg.m^2.
I_i^v	Coefficients of the equations of motion in the lateral direction in *X-Y* plane, corresponding to the inextensible condition and associated with the cubic coupling nonlinearity terms, where *i=1, 2…14.*
I_i^w	Coefficients of the equations of motion in the lateral direction in *X-Z* plane, corresponding to the inextensible condition and associated with the cubic coupling nonlinearity terms, where *i=1, 2…18.*
J	Polar area moment of inertia of the pipe, m^4.
K_i^u	Coefficients of the equations of motion in the axial direction which represent the linear stiffness terms, where *i=1, 2…6.*
K_i^v	Coefficients of the equations of motion in the lateral direction in *X-Y* plane, which represent the linear stiffness terms, where *i=1, 2…6.*
K_i^w	Coefficients of the equations of motion in the lateral direction in *X-Z* plane, which represent the linear stiffness terms, where *i=1, 2…6.*
L	Length of the pipe, m.
$\widetilde{L}$	Lagrangian function.
L_i^v	Coefficients of the equations of motion in the lateral direction in *X-Y* plane, corresponding to the inextensible condition and associated with the cubic nonlinearity terms, where *i=1, 2…10.*
L_i^w	Coefficients of the equations of motion in the lateral direction in *X-Z* plane, corresponding to the inextensible condition and associated with the cubic nonlinearity terms, where *i=1, 2…10.*
M	Mass of the internal fluid per unit length, kg/m.
M_a	The added hydrodynamic mass per unit length of the annular flow, kg/m.
M_i^u	Coefficients of the equations of motion in the axial direction which represent the linear inertia terms, where *i=1, 2…6.*
M_i^v	Coefficients of the equations of motion in the lateral direction in *X-Y* plane, which represent the linear inertia terms, where *i=1, 2…6.*

M_i^w	Coefficients of the equations of motion in the lateral direction in *X-Z* plane, which represent the linear inertia terms, where *i=1, 2…6.*
m	Mass of the pipe per unit length, kg/m.
m_H	The hydrodynamic mass.
N	Number of modes.
$\tilde{O}$	Order of magnitude.
P	Fluid pressurization force, N.
$\tilde{P}$	Pressure distribution, Pa.
$p(t^*)$	Dimensionless time dependant vector of modal coordinates associated with the dimensionless axial deflection.
p_i	The resultants of the steady state pressure acting on the inner surface of the pipe, Pa.
p_o	The resultants of the steady state pressure acting on the outer surface of the pipe, Pa.
$\bar{p}_o$	The value of the steady state pressure acting on the outer surface of the pipe at $X=\frac{L}{2}$, Pa.
$q(t^*)$	Dimensionless time dependant vector of modal coordinates associated with the dimensionless lateral deflection in *X-Y* plane.
R	Radius of the inner pipe, m.
R_p	Global position vector of the point mass p.
$r(t^*)$	Dimensionless time dependant vector of modal coordinates associated with the dimensionless lateral deflection in *X-Z* plane.
r_{ch}	Inside radius of the outer stationary cylinder, m.
r_L	The position vector at the free end of the pipe.
r_o	Outer radius of the pipe, m.
Re	Reynolds Number.
s	Curvilinear Coordinate which represents the pipe arc length.
T	Total kinetic energy of both pipe and fluid.
$\breve{T}$	Tension in the pipe either externally applied or associated with frictional forces, N.
$\bar{T}$	Externally imposed uniform tension, N.
T_f	Total kinetic energy of the fluid, J.
$T_{f,\ t}$	Kinetic Energy of the internal fluid due to translation.
$T_{f,\ r}$	Kinetic Energy of the internal fluid due to the pipe flexure.
T_p	Total kinetic energy of the pipe, J.
$T_{p,\ t}$	Kinetic Energy of the pipe due to translation.
$T_{p,\ r}$	Kinetic Energy of the pipe due to rotation.
t	Time.

t_o	Initial time instant.
t_f	Final time instant.
t^*	Dimensionless time.
t_1^*	Dimensionless time corresponding to the non rotating pipe configuration.
t_2^*	Dimensionless time corresponding to the rotating pipe configuration.
U_f	Axial flow velocity relative to the axially deformed pipe, m/s.
U_i	Internal flow velocity, m/s.
U_o	External flow velocity, m/s.
U_1	Flow velocity after pipe deformation.
u	Axial displacement of the pipe, m.
$u(L)$	Axial displacement at the end of the pipe, m.
$\bar{u}$	Velocity of the fluid in the annular thickness, h.
$u_{i,1}$	Dimensionless velocity of the internal flow corresponding to the non rotating pipe configuration.
$u_{i,2}$	Dimensionless velocity of the internal flow corresponding to the rotating pipe configuration.
$u_{o,1}$	Dimensionless velocity of the external flow corresponding to non rotating pipe configuration.
$u_{o,2}$	Dimensionless velocity of the external flow corresponding to the rotating pipe configuration.
$\bar{u}_p$	Velocity component due to the pressure difference of the annular thickness, h.
V	Total potential energy, J.
V_a	Potential energy due to axial deformation.
V_b	Potential energy due to bending.
V_f	Absolute velocity of the fluid, m/s.
V_g	Potential energy due to gravity.
V_p	Velocity of the pipe, m/s.
V_{rf}	The relative fluid-body velocity, m/s.
V_t	Potential energy due to torsion.
v	Lateral displacement of the pipe in X-Y plane, m.
W_{nc}	Work done by non-conservative forces, J.
w	Lateral displacement of the pipe in X-Z plane, m.
x-y	Rectangular Cartesian coordinate axes after deformation.
X-Y	Rectangular Cartesian coordinate axes prior to deformation.

GREEK SYMBOLS

α	Location of minimum film thickness.
α_1	Dimensionless Kelvin-Voigt coefficient of material damping corresponding to the non rotating pipe configuration.
α_2	Dimensionless Kelvin-Voigt coefficient of material damping corresponding to the rotating pipe configuration.
$\overline{\alpha}$	Modal frequency.
β	Angular coordinate from *h* and the center of the pipe.
β_i	Mass ratio of the internal fluid corresponding to the non rotating pipe configuration.
$\beta_{i,1}$	Mass ratio of the internal fluid corresponding to the rotating pipe configuration.
$\beta_{i,2}$	Diametral mass moment of inertia ratio of the pipe and the internal fluid.
$\beta_{i,3}$	Polar mass moment of inertia ratio of the pipe.
$\beta_{i,4}$	Polar mass moment of inertia ratio of the internal fluid.
$\beta_{m,1}$	Flexural flexibility coefficient corresponding to the non rotating pipe configuration.
$\beta_{m,2}$	Flexural rigidity coefficient corresponding to the rotating pipe configuration.
$\beta_{o,1}$	Mass ratio of the external fluid corresponding to the non rotating pipe configuration.
$\beta_{o,2}$	Mass ratio of the external fluid corresponding to the rotating pipe configuration.
γ_1	Dimensionless gravity coefficient of the pipe and the internal fluid.
γ_2	Dimensionless gravity coefficient of the external fluid.
η_1	Dimensionless lateral deflection of the pipe in *X-Y* plane.
η_2	Dimensionless lateral deflection of the pipe in *X-Z* plane.
η_3	Dimensionless angle of torsion.
$\overline{\eta}$	The flow profile parameter.
ξ_1	Dimensionless length corresponding to the extensible configurations, $\left(\frac{X}{L}\right)$.
ξ_2	Dimensionless length corresponding to the inextensible configurations, $\left(\frac{s}{L}\right)$.

ζ	Dimensionless axial displacement of the pipe.
ε	Axial strain along the centerline of the pipe.
ϵ	Order of magnitud
E	Slenderness ratio.
φ	Angle of torsion, rad.
ϕ	Elastic slope at a position ζ or X.
ϕ_1	Elastic slope at a position *s* or *X* in *X*-*Z* plane.
ϕ_2	Elastic slope at a position *s* or *X* in *X*-*Y* plane.
θ	The angle between relative fluid-body velocity V_{rf} and the *X*-axis, rad.
θ_y	Angle of rotation about y_2 axis, rad.
θ_z	Angle of rotation about z_1 axis, rad.
ψ	Angle of attack, rad.
$\rho_{f,i}$	Density of the internal fluid, kg/m^3.
$\rho_{f,o}$	Density of the external fluid, kg/m^3.
λ	Angular coordinate of the annulus position which rotates with the pipe.
λ_{ch}	Ratio of the inner cylinder diameter.
λ_i	Ratio of the inner pipe diameter.
τ	Unit vector tangential to the pipe.
τ_L	The tangential unit vector at the free end of the pipe.
$\kappa(\zeta)$	Curvature of the centerline at ζ.
δ	Maximum deflection of rotating pipe.
δW_{nc}	Infinitesimal work done by non-conservative force.
p	Material point.
Δ	Ratio of maximum deflection to the annular clearance.
ν	Poisson ratio.
χ	Dimensionless added mass.
ω	Instantaneous angular velocity, rad/s.
Ω	Angular speed, rad/s.
Ψ	The vector of assumed mode shapes, corresponding to the dimensionless axial deflection.
Φ	The vector of assumed mode shapes, corresponding to the dimensionless transverse deflection in the *X*-*Y* plane.
Θ	The vector of assumed mode shapes, corresponding to the dimensionless transverse deflection in the *X*-*Z* plane.
Ξ	The vector of assumed mode shapes, corresponding to torsion.
Γ	Torque exerted on the pipe due to the fluid-induced frictional forces of the rotating pipe, N.

$\overline{\Gamma}_1$	Dimensionless tension corresponding to the extensible non rotating pipe.
$\overline{\Gamma}_2$	Dimensionless tension corresponding to the extensible rotating pipe.
$\Pi_{iL,1}$	Dimensionless internal pressure corresponding to the non rotating pipe configuration.
$\Pi_{iL,2}$	Dimensionless internal pressure corresponding to the rotating pipe configuration.
$\Pi_{oL,1}$	Dimensionless external pressure corresponding to the non rotating pipe configuration.
$\Pi_{oL,2}$	Dimensionless external pressure corresponding to the rotating pipe configuration.
$\Pi_{0,1}$	Dimensionless axial flexibility corresponding to the non rotating pipe configuration.
$\Pi_{0,2}$	Dimensionless axial flexibility corresponding to the rotating pipe configuration.
$\overline{\Pi}_1$	Dimensionless pressurization corresponding to the extensible non rotating pipe.
$\overline{\Pi}_2$	Dimensionless pressurization corresponding to the extensible rotating pipe.
$\overline{\Pi}_3$	Dimensionless pressurization associated with the flow cross sectional area and corresponding to the extensible non rotating pipe.
$\aleph_1$	Gravity coefficient of the external fluid.
$\aleph_2$	Mass coefficient of the external fluid.
$\aleph_3$	Mass coefficient of the internal fluid.

DISSERTATION ABSTRACT

Name: Fadi Abdelhadi Zuhdi Ghaith

Title of Study: Dynamic Modeling of Rotating Flexible Pipe Conveying Fluid and Subjected to External Axial Flow

Major Field: Mechanical Engineering

Date of Degree: December, 2009.

In the present work, the nonlinear equations of motion which describe the dynamics of both rotating and non rotating slender flexible pipe conveying fluid and subjected to external axial flow, are derived using the Lagrangian approach together with the variational technique. The formulation takes into account all types of inertia and elastic coupling between axial, lateral and torsional deflections. This formulation is also applicable to both extensible and inextensible conditions associated with different end constraints.

The obtained nonlinear equations of motion are discretized via unimodal Galerkin's method by utilizing mode shapes of the linear model.

The derived models were employed to investigate the dynamics of selected engineering applications, as manifested by the vibrations of a tube in a double pipe heat exchanger, and the flow induced vibrations of rotating drillstring.

Based on the numerical simulation results of an extensible non rotating flexible pipe conveying fluid and subjected to external axial flow, which may represent the vibrations of a double pipe heat exchanger, it was found that both internal and external flows play a dominant role in softening the lateral and axial natural frequencies of the system. Also, it

was found that when the annular spacing decreased, a noticeable drop of the lateral natural frequency occurred.

The developed three-dimensional mathematical model was extended to simulate the flow induced vibration of rotating drillstring under the inextensibility condition. The governing equations of motion showed that the most dominant nonlinear terms of the lateral directions in *X*-*Y* and *X*-*Z* planes are associated with a cubic nonlinearity. Upon performing the numerical simulations for selected realistic parameters, it was found that both external and internal flows play dominant role in softening the lateral and torsional natural frequencies of the system. This study has demonstrated that the stability of the drill pipe is highly dependant on the external flow velocity and annular spacing. Moreover, the three-dimensional analysis was found necessary for predicting the accurate vibrational behavior of a rotating flexible pipe conveying fluid and subjected to external axial flow, particularly at high rotational speeds. The three-dimensional model is capable of capturing the non-symmetrical nature of the gyroscopic forces in *X*-*Y* and *X*-*Z* planes.

Another nonlinear mathematical model is derived for the flow induced vibration of a rotating drillstring, by considering fixed-simply supported-sliding end conditions and ignoring the inextensibility condition. This study showed that the lateral natural frequencies of this model are higher than the cantilevered configuration. This comparison reveals the importance of selecting appropriate end conditions for achieving reliable and accurate model of the drill pipe system.

DOCTOR OF PHILOSOPHY IN MECHANICAL ENGINEERING
KING FAHD UNIVERSITY OF PETROLEUM AND MINERALS
DHAHRAN, SAUDI ARABIA

ملخص بحث

درجة الدكتوراة في الفلسفة

الاسم : فادي عبدالهادي زهدي غيث

العنوان : النمذجة الديناميكية لأنبوب مرن دوار يحمل سائلا في داخله و يتعرض لتدفق سائل خارجه.

التخصص : الهندسة الميكانيكية

تاريخ التخرج : ديسمبر 2009

لقد تم في هذه الدراسة اشتقاق معادلات الحركة اللاخطية التي تصف الديناميكا المتعلقة بأنبوب مرن (في كلا وضعي الثبات و الدوران) و يتعرض لتدفق سائل خارجه باستعمال نظام لانجرانج وطريقة التغير الزمني في الحركة المفترضة معا. إن النمذجة الحالية تأخذ بعين الاعتبار جميع أنواع الاقتران ما بين قوى الجمود و المرونة في كل من الاتجاه الأفقي و العمودي و الدوراني للإهتزازات الناتجة. كما أن النمذجة الحالية تصلح لكلتا الحالتين القابلة و الغير قابلة للإستطالة وفق محددات مختلفة للحركة.لقد تم تجزئة المعادلات اللاخطية باستعمال أشكال الاهتزازات المتعلقة بالنظام الخطي.

لقد تم توظيف الأنظمة المشتقة لدراسة ديناميكية بعض التطبيقات الهندسية المنتقاة و التي تظهر جليا بالإهتزازات في الأنبوب الداخلي للمبادل الحراري ذو الأنبوب المزدوج و الاهتزازات في أنبوب حفر البترول الدوار.

بالاعتماد على المحاكاة العددية للأنبوب المرن- القابل للاستطالة- الذي يحمل سائلا بداخله و يتعرض لسائل آخر خارجه و الذي بالإمكان أن يمثل الاهتزازات في المبادل الحراري ذو الأنبوب المزدوج، فقد وجد أن وجود كلا السائلين الخارجي و الداخلي يلعب دورا مهما في تقليل قيمة

الترددات الطبيعية الأفقية و العمودية.كما وجد انه عندما تقل المسافة المحصورة ما بين الأنبوبين، يكون هناك هبوط ملحوظ في التردد الطبيعي العمودي.

لقد تم استطراد النظام الثلاثي الأبعاد المطور لمحاكاة الاهتزازات المتعلقة بالتدفق لأنبوب الحفر الدوار الغير قابل للاستطالة. في حين أظهرت المعادلات الحركية أن أكثر العناصر المؤثرة في الحركة العمودية في كلا المستويين تكون مرتبطة باللاخطية ذات الدرجة الثالثة.

بإجراء المحاكاة الرقمية لهذا النظام و باستعمال قيم واقعية تصف النظام الحالي، وجد أن وجود كلا السائلين الداخلي و الخارجي يلعب دورا مهما في تقليل التردد الطبيعي للاهتزازات العمودية و الدورانية. لقد أظهرت هذه الدراسة أن إتزان النظام يعتمد بشكل كبير على سرعة تدفق السائل الخارجي و قيمة المسافة المحصورة. كما وجد أن طريقة التحليل ثلاثي الأبعاد ضرورية للتنبؤ بسلوك الاهتزازات في الأنبوب المرن الدوار الذي يحمل سائلا بداخله و بتعرض لتدفق سائل خارجه خاصة عند السرعات الدورانية العالية .ان النظام ثلاثي الابعاد يعد قادرا على التقاط الطبيعة الغير متطابقة للقوى الدورانية في كلا المستويين.

لقد تم اشتقاق نظام رياضي آخر يصف الاهتزازات المتعلقة بالتدفق في أنبوب الحفر الدوار القابل للاستطالة المثبت كليا من إحدى طرفيه في حين ان الطرف الاخر مثبت بواسطة دعامة منزلقة. و لقد أظهرت هذه الدراسة أن التردد الطبيعي للاهتزازات العمودية في هذا النظام أعلى من النظام المعلق، كما أن هذه المقارنة أظهرت أهمية اختيار المقيدات المناسبة للحركة للحصول على نظام دقيق و معتمد لأنبوب الحفر الدوار.

درجة الدكتوراة في الفلسفة في الهندسة الميكانيكية

جامعة الملك فهد للبترول والمعادن

الظهران –المملكة العربية السعودية

CHAPTER ONE

INTRODUCTION

1.1 Motivation

Flow-induced vibration is a term used to denote a phenomenon associated with the response of structures immersed in or conveying fluid. The term covers cases in which an interaction develops between fluid-dynamic forces and elastodynamic forces in the structure. The study of such phenomenon extends to three disciplines: structural mechanics, mechanical vibrations and fluid dynamics.

Flow induced vibrations could be classified in terms of the source of excitation as extraneously induced excitation (EIE) due to pressure fluctuation, instability-induced excitation (IIE) which is associated with flow instability such as lock in phenomena, and movement induced excitation (MIE) due to self excited vibrations such as fluttering of cantilever pipe conveying fluid, Naudascher and Rockwell [1].

The fluid-structure interaction (FSI) in pipe conveying fluid systems has been investigated extensively, because of its relevance to mechanical, civil, nuclear and aeronautical engineering applications.

Cylindrical structures subjected to external or annular flows belong to another class of FSI systems, which are found in many engineering applications, particularly in the power-generating, chemical and petrochemical industries. One of the most challenging FSI problems that have not been addressed until now is rotating pipe conveying fluid and

subjected to external axial flow. The complexity of this model arise from the direct effects of rotation on the flow induced vibration and stability of the system.

Understanding and prediction of the dynamic behavior of such FSI systems are of prime importance to the design and trouble-free operation of heat exchangers, nuclear fuel elements, and drilling applications. On the other hand, it is crucial to understand the complex vibrational mechanisms experienced by similar systems in order to better control its functional operation and improve its performance.

Motivated by the quest of understanding the fundamentals of fluid-structure interactions, as well as their relevance to applications in several areas of engineering, the dynamical behavior of rotating pipe conveying fluid and subjected to external axial flow was investigated by this research work.

1.2 Literature Review

The current model of this study consists of a rotating flexible pipe conveying fluid and subjected to external axial flow. This problem belongs to a major subject area within the general realm of fluid-structure interaction. In the light of several schemes of fluid-structure configurations associated with the proposed model, in addition to the rotation of flexible pipe, the literature survey will be divided into five sections. At the end of this survey, the current status of this problem is outlined.

1.2.1 Flexible pipe conveying fluid

The dynamics of pipes conveying fluid, either continuously flexible or articulated, have been studied extensively in the literature. The stability of hanging or

standing cantilevers conveying fluid was first examined theoretically and experimentally by Paidoussis [2].

Semler et al. [3] derived the nonlinear equations of motion of pipes conveying fluid by both energy and Newtonian methods for cantilevered pipes and for pipes with fixed ends. The fluid was assumed to be incompressible, turbulent, fully developed and approximated as plug flow. Two nonlinear equations of motion for pipe conveying fluid with fixed ends were obtained in terms of lateral deflection and axial shortening. By employing the inextensibility condition for the cantilevered pipe conveying fluid, the system was reduced in terms of lateral deflection only. The planar dynamics of a fluid-conveying cantilevered pipe with a small mass attached at the free end were examined theoretically and experimentally by Paidoussis and Semler [4]. The mass at the end was modeled as a lumped point-mass and the nonlinear equation of motion was formulated in terms of lateral deflection. The system was discretized using conventional Galerkin's method. Jump phenomena as well as chaotic oscillations were observed in the experiment and have been illustrated theoretically, revealing the importance of even small mass at the end.

Stability, Double degeneracy and Chaos in cantilevered pipes conveying fluid were examined by Paidoussis and Li [5]. The nonlinear dynamics of planar motions of cantilevered pipes conveying fluid were studied via a two mode discretization of the governing partial differential equation, after replacing the non-linear inertial terms by equivalent ones through a perturbation procedure. It was shown that for a standing cantilever, the pipe is statically unstable at lower velocities if the pipe is sufficiently long, while it has Hopf bifurcation followed by pitchfork bifurcation at higher flow velocities.

Centre Manifold theory was used to show the existence of chaotic behavior. The numerical results are presented in the form of phase portraits, bifurcation diagrams and Lyapunov exponents.

Szabo [6] performed bifurcation analysis of a clamped-free flexible pipe conveying pulsate flow. The fluid was assumed to be incompressible, frictionless with periodic pulsating velocity. Equation of motion was derived using Hamilton's principle and geometric nonlinearity was involved. The system was discretized using Galerkin's method. Stability charts show that the harmonic perturbation of the fluid velocity with certain values of perturbation amplitude and forcing frequency could destabilize the equilibrium of the pipe even for values less than the critical ones. It was shown also that appropriate harmonic perturbation could stabilize unstable equilibrium, and this will be useful when great fluid mass has to be transported in thin elastic pipe.

A low-dimensional model for the planar nonlinear dynamics of a fluid-conveying pipe was constructed by Sarkar and Paidoussis [7] using the proper orthogonal decomposition method (PODM) in the post flutter region. Equation of motion obtained by Semler et al. [3], which has been illustrated previously, was employed in this work. Firstly, the nonlinear PDE of motion is converted into a finite set of coupled ordinary differential equations by Galerkin's method using the cantilever beam modes as basis.

Wadham-Gagnon et al. [8] investigated the three-dimensional nonlinear dynamics of unrestrained and restrained cantilevered pipes conveying fluid. The equations of motion were derived, presented in the dimensionless form and then discretized via Galerkin's method.

It is clear that the dynamic modeling and stability analysis tackled in previous papers were limited to flexible pipes with internal flow only, without considering external flow and pipe rotation.

1.2.2 Flexible cylinder subjected to axial flow

Understanding and prediction of the dynamics of cylinders in axial flow is considered as a fundamental requirement in order to develop a comprehensive dynamic model. Historically, the first specific study on the dynamics of a slender flexible cylinder subjected to axial flow was addressed by Hawthorne [9] and it was concerned with stability of Dracone barge. The Dracone is a long flexible towed tubular container with tapering ends, which was designed to carry oil and other liquids lighter than sea-water. This analysis was extended and generalized for different boundary conditions and supported experimentally by Paidoussis [10]. Later, a more general and corrected linear equation of motion was derived by Paidoussis [11].

Lopes et al. [12] derived the nonlinear equations of motion of a slender cantilevered cylinder in axial flow via Hamilton's principle. It was assumed that the flow velocity is constant and the cylinder is slender and obeys Euler-Bernoulli beam theory. Invicid forces were modeled via slender-body potential flow theory by following closely the Lighthill formulation [13], wherein the viscous, hydrostatic and gravity-related terms were derived separately.

Semler et al. [14] explored the equations of motion for the dynamics of a cantilevered cylinder in axial flow using numerical tools such as the Finite Difference Method and AUTO software, which was developed by Doedel and Kerneves [15] in

order to solve the discretized ordinary differential equations. The linear dynamics is considered first, focusing on the effect of flow velocity, free end conditions and frictional coefficients on stability. Also nonlinear dynamics were examined by means of bifurcation diagrams, phase-plane plots and Poincare maps. The obtained theoretical results showed good agreement with experimental work done by Paidoussis et al. [16].

A weakly nonlinear equation of motion of an extensible slender cylinder subjected to axial flow was derived by Modarres-Sadeghi et al. [17], corrected to third-order of magnitude, using Hamilton's principle. Lateral deflections were assumed to be of first order magnitude while treating axial deflections as second order. For convenience, inviscid, hydrostatic and viscous forces were determined by direct application of Navier-Stokes equations. The derived equation was not the definitive nonlinear equation of motion for this system, since it was not obtained by a unified nonlinear treatment of fluid mechanics. Based on the derived nonlinear equations of motion, the dynamics of the system was studied from a nonlinear point of view, and the existence of post divergence instabilities of the cylinder was established.

1.2.3 Dynamic analysis of flexible rotors

Although modeling of flexible rotor is not the core subject of the current research, it is found necessary to go through the most related and recent literature in this area, because the proposed model consists of a rotating flexible pipe.

Khulief et al. [18] presented a dynamic model of the drillstring. The equation of motion of the rotating drillstring was derived using Lagrangian approach together with finite element method. The formulated model accounts for the gyroscopic effect, the torsional /

bending inertia coupling, effect of the gravitational force field and stick-slip interaction forces. Explicit expressions of the finite element coefficient matrices are derived using consistent mass formulation. Modal transformations are invoked to obtain a reduced order modal form of the dynamic equations. It is important to mention that internal damping of the pipe and flow induced forces were neglected in this work.

Sampaio et al. [19] studied the coupling of axial and torsional vibrations on drillstring. The model has been described as a vertical slender beam under axial rotation. It is shown that geometrical nonlinearity plays significant role in the stiffening of the beam. The model was analyzed using nonlinear finite element approximation, in which large rotations and nonlinear strain displacements are taken into account. The effect of structural damping was included also in this model.

The aforementioned models did not include the effects of fluid-structure interaction related to internal and external flow.

1.2.4 Fluid-structure interaction of flexible rotors

Modeling of the fluid-structure interaction in flexible rotors has been studied extensively in the last two decades, due to its importance in drilling applications.

Berlioz et al. [20] studied the dynamic behavior of drillstring. This work focused on the laboratory tests concerned with the lateral behavior of a rod representative of the part of a drillstring in the area of rotary oil drilling. Theoretical Finite Element model was proposed to present the vibrations of the drillstring by using a substantial part of the rotor dynamics theory presented in Lalanne and Ferraris [21]. The beam element which represents the drillstring was presented as two nodes and six degrees of freedom at each

node, including three translations and three rotations. The stabilizers were considered as bearing elements and the drill hole as a radial gap element. The mud in the annulus between the drill-string and the well borehole was modeled with mass, damping-gyroscopic and stiffness matrices, while the mud inside the pipe was modeled with a mass matrix only following the formulation presented by Axisa and Antunes [22]. The effects of gravity and buoyancy were taken into account. It is important to mention that the theoretical model was suggested only in this paper, while nothing was provided regarding the formulation of the mass, damping-gyroscopic and stiffness matrices. The experimental results highlighted the different kinds of lateral instabilities and they were compared with existing experimental results.

A theoretical model was developed by Antunes and axisa [23] for predicting the dynamic behavior and stability of rotating shaft immersed in both a concentric and eccentric fluid annulus. This study was concerned with a moderate flow confinement for both steady and fluctuating flows. This work investigated the importance of the confinement level on the dynamic behavior of the rotor and showed that rotor eccentricity is very important parameter, leading to important qualitative differences in the system dynamic responses.

Khulief and Al-Sulaiman [24] presented an experimental investigation for the drillstring vibrations using a specially designed drilling test rig. The test rig was constructed to simulate the drillstring vibrational response due to various excitation mechanisms, which include stick-slip, well-borehole contact, and drilling fluid interaction. The test rig was driven by a variable speed motor which allows for testing different drilling speeds, while a magnetic tension brake is used to simulate stick-slip. In

addition, shaker is employed to excite the drillstring axially in order to simulate the weight-on-bit (WOB). The drillstring was instrumented for vibration measurements. The experimental measurements for the fluid friction factor were utilized to tune such values obtained via the simulation of the dynamic model. This work demonstrated the significance of the experimental procedures in tuning and validation of the finite element multibody model of the drillstring to accommodate the effect of the drilling fluid. In this study, only the effect of the internal fluid was considered.

Drilling Dynamics in the presence of the mud flow was investigated by Schmalhorst et al. [25]. The interaction between the drillstring and instationary mud flow circulation were taken into account by considering the shear forces inside the pipe due to the relative velocity between the drilling fluid and the pipe. The equations of motion for the drillstring were derived by applying the theory of virtual work. The model was formulated as continuous model in terms of axial, lateral and torsional freedoms in addition to the flow rate and the pressure of the mud. The Galerkin's method in combination with a nonlinear Finite Element method were applied to discretize the set of the governing partial differential equations. Coupled axial, torsional, lateral and pressure vibrations were estimated numerically and the influence of the pressure pulses of the mud on the drillstring vibrations has been investigated. In spite of the importance of the invicid hydrodynamic forces, gravity and frictional forces of the external mud flow on the vibrations of the drillstring, their effect was not considered in this work.

The fully developed laminar flow of non-Newtonian liquids through annuli has been studied by Escudier et al. [26]. They studied the velocity profiles in the annulus for many different combinations of concentric and eccentric annuli with and without center

body rotation. Continuity and momentum partial differential equations were formulated based on generalized Newtonian fluid assumptions, in which the viscosity is dependant only on the second invariant of the stress-strain tensor. Numerical analysis has been carried to solve governing equations using a second order differencing scheme. Velocity components of the flow for different radii ratios were obtained and compared with experiments.

On the other hand, preliminary and simple discrete models were provided in literature that also considered the effect of flow induced vibrations in rotating drillstrings. Chrisoforou and Yigit [27] presented dynamic model of rotating drillstrings with borehole interactions. The axial excitation due to bit/formation interactions and hydrodynamic damping were included. By neglecting the torsional vibration of the drillstring, equations of motion were obtained by using assumed mode method and applying the Lagrangian approach. The equivalent system parameters for the lumped model such as mass, stiffness and damping were derived from the associated continuous model of the drillstring. Cull and Tucker [28] modeled the damping of drillstring mud by a dashpot parallel to equivalent torsional spring of the drillstring. The equations of motion were obtained using a simplified lumped parameter model with two degrees of torsional freedom, one for the rotary table and other for Bottom-Hole-Assemply (BHA).

Zamanian et al. [29] studied the self-exited stick slip oscillations of rotary drill string with a drag bit. A more realistic model was considered including the effects of rotary table and damping of drillstring mud. The lumped parameters of the system were obtained by equating the equivalent lumped parameters system to the associated continuous model using Energy method. The model was formulated as discrete model

with two degrees of torsional freedom and one degree of axial freedom. It was assumed that drill collars and the bit behave as rigid bodies and the moment of inertia of the drill pipe was neglected in comparison with the moment of inertia of rotary table and drill collar. The equations of motion were obtained by using a simplified lumped parameter model and have been solved using Euler-forward Finite Difference technique.

1.2.5 Flexible pipe conveying fluid and subjected to external axial flow.

The problem of a tubular beam subjected to both internal and external flows has been studied previously by several investigators. Hannoyer and Paidoussis [30] combined theory and experiments on the linear dynamics and stability analysis of a cylinder with supported or fixed ends, subjected to both internal and external axial flow. Equation of motion was derived in terms of lateral deflection and flow was modeled as steady flow. The study revealed a rich dynamical behavior, involving multiple divergence and flutter instabilities. Theory and experiments were in quite good agreement.

Paidoussis and Besancon [31] discussed various aspects of the dynamics and stability of clusters of tubular beams containing internally flowing fluid and surrounded by a bounded external axial flow. Equation of motion was formulated as one degree of freedom in terms of lateral deflection and flow was assumed to be steady and fully developed. The obtained equation of motion was discretized using the conventional Galerkin’s method. The general character of free motions was established by obtaining the eigenfrequencies of the system and studying their evolution with increasing internal or external flow. Stability diagrams were obtained for the critical flow velocities, beyond

which the system would loose stability by buckling, under the combined effect of internal and external flow.

Wang and Bloom [32] derived a mathematical model to study the dynamics of a submerged and inclined concentric pipe system with internal and external flows. The addressed problem was inspired by the geometry of silo-pipe system design (Silo water mixing unit). The main function of this unit is to dilute fiber stock with water. Two degrees of freedom model was extracted by assuming fully developed turbulent flow with negligible viscoelastic damping and by ignoring the axial inertia effects. The obtained partial differential equations were solved numerically using the finite difference scheme. It is shown that the inclination angle and gravity are not important design parameters for the adopted model. This work didn’t include nonlinearity terms while the authors paid much effort in selecting optimal design parameters of the pipes such as radius and length, which achieve stability.

A recent comprehensive study was carried out by Paidoussis et al. [33]. They developed a theoretical model for the dynamics of a long tubular cantilever conveying fluid downwards, which then flows upwards around the cantilever as a confined annular flow.

The discussed problem was inspired by the flow-powered drill-string with a floating drill-bit. The system consists of a drill pipe and a floating drill-bit. Sludge with the debris flows upwards around the drill pipe to the surface. The mathematical model was obtained by assuming fully developed turbulent flow with negligible viscoelastic damping of the pipe. The system was reduced to one partial differential equation in terms

of lateral deflection only. The obtained equation of motion was solved using the hyprid Galerkin-Fourier method and the conventional Galerkin's method.

It was shown that as the annular space is widened, the dynamics is dominated by the inside flow and the system loses stability by flutter. At low internal flow velocities, the flow adds damping to the system. For narrower annuli (less than 20 % of the pipe outer diameter), the dynamics is dominated by the outside flow, which destabilize the system, inducing flutter at low velocities.

1.2.6 Current status

In the light of the above literature survey, it was shown that earlier studies within the general realm of fluid-structure interaction area focused on the linear dynamics of pipe conveying fluid, partly because of its relevance to mechanical, civil, nuclear and aeronautical engineering applications and partly because it was considered one of the simplest FSI systems. In the last two decades, more investigations were directed to the nonlinear formulations in order to examine stability and chaotic behavior of pipes conveying fluid.

The dynamics of a flexible cylinder subjected to axial flow was studied extensively in the literature from the linear analysis point of view. Most of recent studies, however, focused on the nonlinear behavior of such systems by considering geometrical nonlinearity and using models of several inviscid forces.

By combining the hydrodynamic forces of a pipe conveying fluid and the flexible cylinder subjected to axial external flow, the problem of tubular beam subjected to both internal and external flow was explored by several investigators. Although many

studies were performed in this area, all reported investigations were restricted to the linear formulations and ignored the geometric nonlinearities. Some comprehensive dynamic models of fluid-free flexible rotors were reported in the literature, which took into account gyroscopic effect, torsional/bending inertia coupling and effect of gravitational forces. However, these models didn't consider the effects of fluid-structure interaction associated with internal and external flows; as well as internal damping of the rotor.

The increasing demand for gaining more insight into the fluid interaction systems in rotating equipment such as drilling operations has oriented researchers and investigators to search for more accurate dynamic models that combine elastic and hydrodynamic forces.

Some investigators presented dynamic models for similar systems which could represent rotating pipe conveying fluid and subjected to external axial flow such as drillstring systems. However, a comprehensive model has not been achieved and the understanding of all vibration phenomena involved is still lacking. In the previously cited investigations, some ignored the moment of inertia of the pipe, others included internal flow and neglected the effects of external hydrodynamic forces, and some studies were restricted to discrete models using a simplified lumped parameter approach.

Due to the complex dynamics of a rotating pipe conveying fluid and subjected to external axial flow, some investigators performed experimental studies in order to come up with better understanding of such a system. Although experimental investigations are crucial to grasp what happens in the actual application, they are time consuming, costly, and can only be performed on simplified laboratory test rigs.

At this point, one can conclude that a comprehensive model of rotating flexible pipe conveying fluid and subjected to external axial flow has not been addressed in the available literature, in particular when a system is subjected to a reference rotation. This finding has motivated the current proposed research work that is expected to contribute to the area of flow induced vibration of rotating flexible pipe conveying fluid especially in those applications associated with drilling operation.

1.3 Research Work Description

1.3.1 Problem statement

In the current research work, a nonlinear mathematical model for a rotating flexible pipe conveying fluid and subjected to external axial flow will be developed. The model consists of a rotating tubular pipe, conveying fluid downwards, while subjected to external flow in reverse direction in the annulus formed by the inner pipe and an outer rigid cylinder, as shown diagrammatically in figure 1.1. The main aim of this work is to analyze the flow-induced vibration and stability of this system.

1.3.2 Research objectives

The main objectives of the current research are summarized as follow:

1. Derivation of the nonlinear equations of motion of the following dynamic models:
    - Extensible flexible pipe conveying fluid and subjected to external axial flow.
    - Inextensible rotating flexible pipe conveying fluid and subjected to external axial flow.

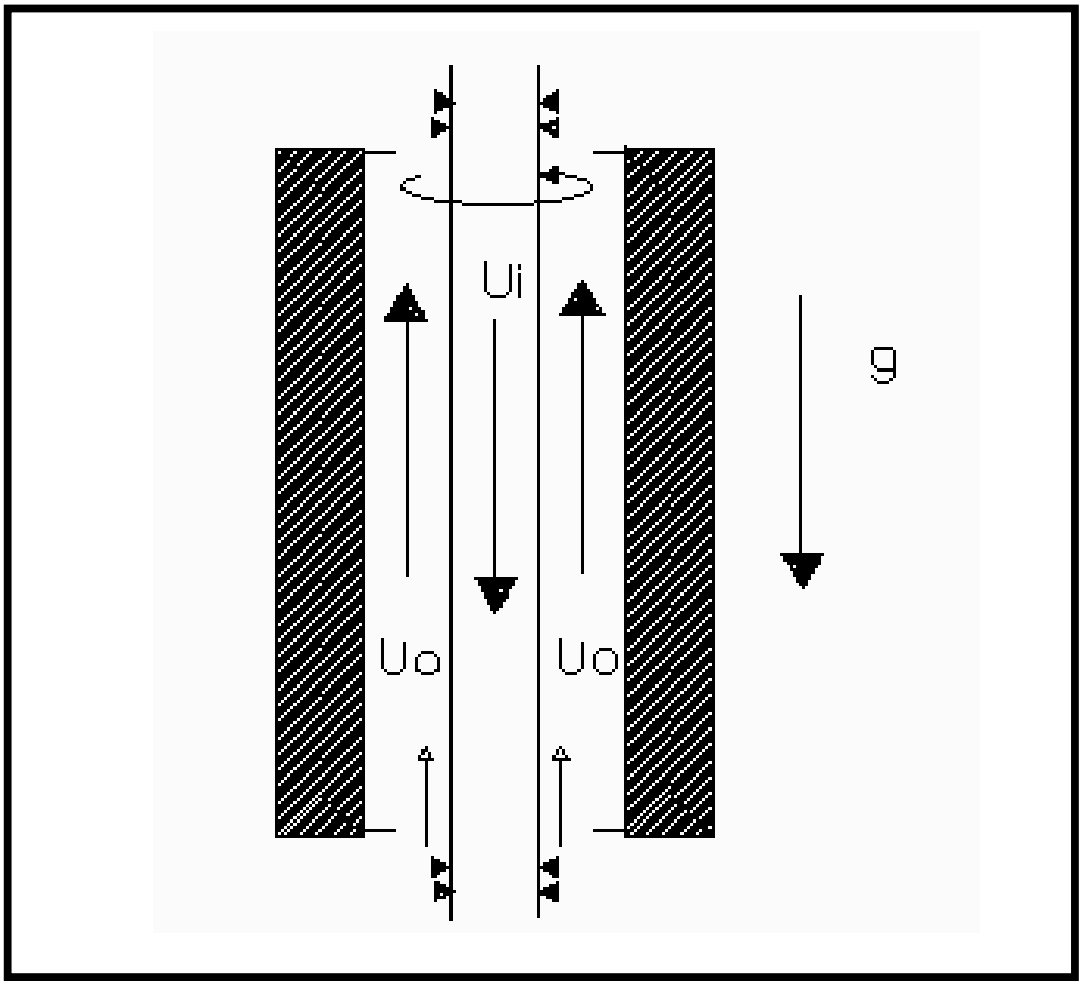


Figure 1.1: Rotating pipe conveying fluid and subjected to external axial flow.

- Extensible rotating flexible pipe conveying fluid and subjected to external axial flow.

Energy method and principle of virtual work are used to identify hydrodynamic forces exerted by both internal and external flows on the pipe, and the extended Hamilton's principle is applied to formulate the governing equations of motion. The main assumptions underlying the formulation include the following:

- The internal pipe is slender, and obeys Euler-Bernoulli theory.
- The material of the internal pipe is elastic, homogeneous and isotropic.
- The outer cylinder is rigid.
- The internal and external fluids are Newtonian, incompressible and fully developed.
- Steady Flow.

2. Performing dimensionless analysis for the models described above.
3. Exploring different end conditions, e.g. free, pinned, clamped, etc.
4. Invoking modal coordinates in dicretizing the obtained partial differential equations.
5. Solving the model numerically using MATLAB® .
6. Applying the current model to the two following different case studies by adopting appropriate assumptions and boundary conditions:
    a) Vibration of a tube in a double-pipe (double concentric tube) heat exchanger.

        In this type of heat exchangers, one fluid flows through the inside pipe (tube-side flow) and the second fluid through the annular space between

the outside and the inside pipe (shell-side flow). In this case, pipe rotation is not included. It will be assumed that the velocities of the tube-side and shell-side flows are independent.

b) Vibration of a rotating drillstring.

The proposed model in this work may represent the system inspired by drilling applications. The system consists of a hollow rotating drill pipe containing circulation fluid, which flow downwards through the bit, and then upwards through the borehole to the surface. It will be assumed that velocities of inside and outside fluids are dependent.

7. Estimation of the axial, lateral and torsional natural frequencies of the system.
8. Investigating the effect of selected design parameters on the natural frequencies and stability of the system.

1.4 Organization

In this dissertation, comprehensive nonlinear dynamic models for both stationary and rotating flexible pipe conveying fluid and subjected to external axial are derived for different end conditions.

In chapter 2, the nonlinear dynamic model of extensible flexible pipe conveying fluid and subjected to external axial flow is formulated.

Derivation of the governing equations of motion of inextensible flexible rotating pipe and subjected to external axial flow is performed in chapter 3. Also many case studies are presented in this chapter. Another study is performed in chapter 4 to formulate

the equations of motion of extensible flexible rotating pipe and subjected to external axial flow.

Chapter 5 is devoted to present the governing equations of motion of all models derived in chapters 2, 3 and 4 in the dimensionless form.

Solution methodology and adopted discretization technique are discussed extensively in chapter 6.

Numerical results for different case studies are carried out in chapter 7. This chapter includes several parametric and comparison studies.

Finally, conclusions and recommendations for future work are presented in chapter 8.

CHAPTER TWO

NONLINEAR DYNAMIC MODELING OF EXTENSIBLE FLEXIBLE PIPE CONVEYING FLUID AND SUBJECTED TO EXTERNAL AXIAL FLOW

In this chapter, the nonlinear equations of motion of a flexible pipe conveying fluid and subjected to external flow are derived based on Energy methods.

2.1 The Problem Statement

Consider the planar motion of the flexible pipe shown in figure 2.1.The model consists of a uniform tubular pipe of length L, cross sectional area A, mass per unit length m and flexural rigidity EI. The pipe is conveying fluid of mass per unit length M, flowing axially with velocity U_i, while an external fluid of mass density $\rho_{f,\,o}$, flows in reverse direction in the annulus formed by the inner pipe and an outer rigid cylinder with velocity U_o. The main assumptions underlying this formulation are:

- The internal pipe is slender, and obeys Euler-Bernoulli theory.
- The material of the internal pipe is elastic, homogeneous and isotropic.
- The outer cylinder is rigid.
- The internal and external fluids are Newtonian, incompressible and fully developed.

- The internal flow is fully developed turbulent and may be approximated by a plug flow (i.e. as if it were an infinitely flexible rod traveling through the pipe , then all points of the fluid having the same velocity relative to the pipe).
- Steady flow such that the mean velocities of the internal and external flows are constants and free from pulsation.
- The motion is planar due to symmetry in *X-Y* and *X-Z* planes.
- The strain in the pipe is considered small while deformation could be large.
- The pipe is assumed to be initially lying horizontally in the *X-Y* plane.
- Pipe ends are assumed to be fixed, therefore; inextensibility condition is not applicable.

2.2 Order of Magnitude Considerations

Although the deflection of the pipe could be large, only cubic nonlinear terms will be retained in the final equations of motion; thus, an order of magnitude analysis will be useful. The lateral displacement v is assumed to be small relative to the length of the pipe, while the longitudinal displacement u is much smaller than v and should have higher order of magnitude [34], accordingly, one may state that:

$$v \sim \tilde{O}(\epsilon) \quad , \quad u \sim \tilde{O}(\epsilon^2) \tag{2.1}$$

where $(\epsilon << 1)$.

One should consider that the various expressions of kinetic and potential energies are kept at $\tilde{O}(\epsilon^4)$, since the variational technique always requires one order higher than the order of final equations.

2.3 Kinetic Energy Expression

The kinetic energy of the aforementioned system is made up of the following:

- Kinetic Energy of the pipe, T_p.
- Kinetic Energy of the internal fluid, T_f.

2.3.1 Kinetic energy of the pipe

The deformed configuration of the pipe in *X-Y* plane is shown in figure 2.2. The Lagrangian coordinates are introduced here as (*X, Y),* corresponding to original equilibrium state of the pipe, (i.e. the material point p_o). The Eulerian coordinate system is denoted by (*x, y*), corresponding to the current configuration of the pipe, (i.e. the material point *p*). The displacements (*u, v*) are defined as follow:

$$u = x - X, \qquad v = y - Y \tag{2.2}$$

The global position of the material point *p* after deformation can be written as

$$R_p = x\ i + y\ j = \left(u + X\right) i + \left(v + Y\right)\ j \tag{2.3}$$

Thus

$$\dot{R}_p = \dot{u}\,(x,t)\,i + \dot{v}\,(x,t)\ \ j \tag{2.4}$$

where

u (x,t) : axial shortening due to bending.

v(x,t): lateral deformation of the pipe in *X-Y* plane at a distance *x* measured from the fixed end of the pipe along the neutral axis in the undeformed configuration and at time *t*.

s: curvilinear coordinate measured along the centerline of the pipe.

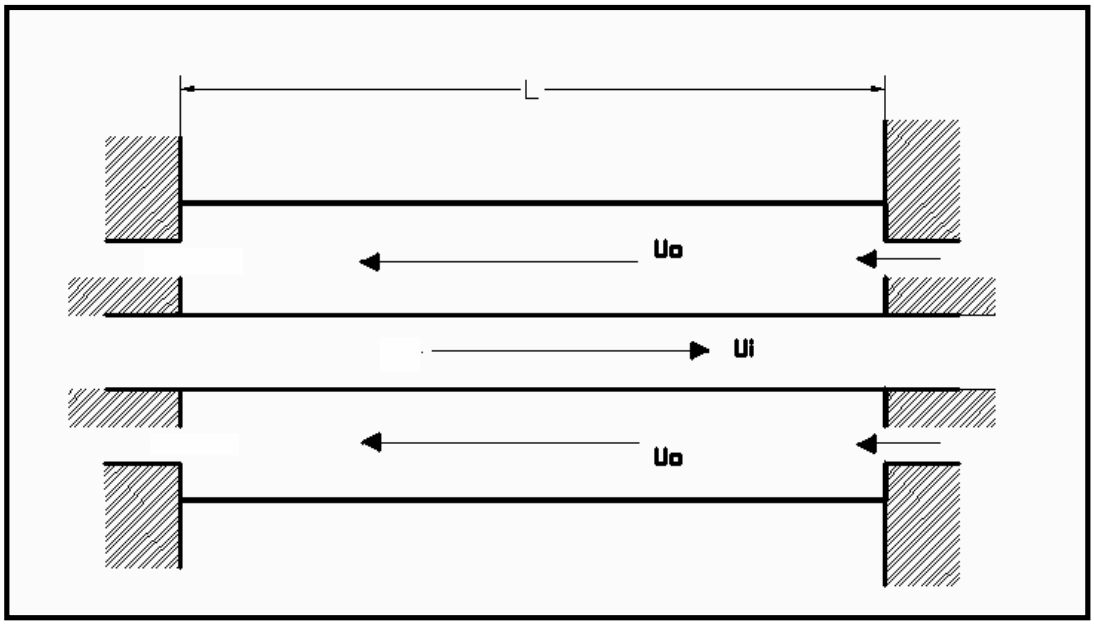


Figure 2.1: Fixed ends pipe conveying fluid and subjected to external axial flow

Then the kinetic energy of the pipe can be written as

$$T_p = \frac{1}{2}\int_0^L m\left[\dot{u}^2 + \dot{v}^2\right] dX \tag{2.5}$$

where

m: mass per unit length of the pipe.

L: length of the pipe.

2.3.2 Kinetic energy of the internal fluid

The basic assumption underlying estimation of kinetic energy of the internal flow is approximating the internal flow by a plug flow, i.e. as if it were an infinitely flexible rod traveling through the pipe, and therefore all points of the fluid having a velocity U_i relative to the pipe. This is an adopted approximation for a fully developed turbulent flow [34].

This implies that

$$V_f = V_p + U_1(X)\tau \tag{2.6}$$

where

V_f: absolute velocity of the fluid.

V_p: velocity of the pipe.

$U_1(X)$: flow velocity of the internal flow after pipe deformation.

τ: unit vector tangential to the pipe, shown on figure 2.2.

The unit vector tangential to the pipe can be defined as

$$\tau = \frac{\partial x}{\partial s} i + \frac{\partial y}{\partial s} j \tag{2.7}$$

where s is the curvilinear coordinate along the pipe

Note that, for extensible pipe δX and δs are not equal, but they are related as [3]

$$\frac{\partial X}{\partial s} = \frac{1}{1+\varepsilon} \tag{2.8}$$

where ε is the axial strain along the centerline of the pipe.

On the other hand, the flow velocity after deformation U_1 could be related to the average velocity before deformation U_i in a manner such that

$$U_1(X) = U_i\left(\frac{A}{A_1}\right) = U_i(1+\varepsilon) \tag{2.9}$$

where

A: cross sectional area of the pipe before deformation.

A_1: cross sectional area of the pipe after deformation.

By substituting equations (2.8) into equation (2.7), one obtains

$$\tau = \left(\frac{x'}{1+\varepsilon}\, i + \frac{y'}{1+\varepsilon}\, j\right) = \left(\frac{(u'+1)}{1+\varepsilon}\, i + \frac{v'}{1+\varepsilon}\, j\right) \tag{2.10}$$

where $(') = \partial(\ )/\partial X$

Using equations (2.9) and (2.10), the kinetic energy of the internal flow can be written as

$$\begin{aligned} T_f = {} & \frac{1}{2}\int_0^L M\left[\dot{u}^2 + \dot{v}^2\right]dX + U_i\int_0^L M\left[\dot{u} + \dot{u}\,u' + \dot{v}\,v'\right]dX \\ & + \frac{1}{2}\int_0^L M\,U_i^{\,2}\left(1 + u'^2 + v'^2 + 2u'\right)dX \end{aligned} \tag{2.11}$$

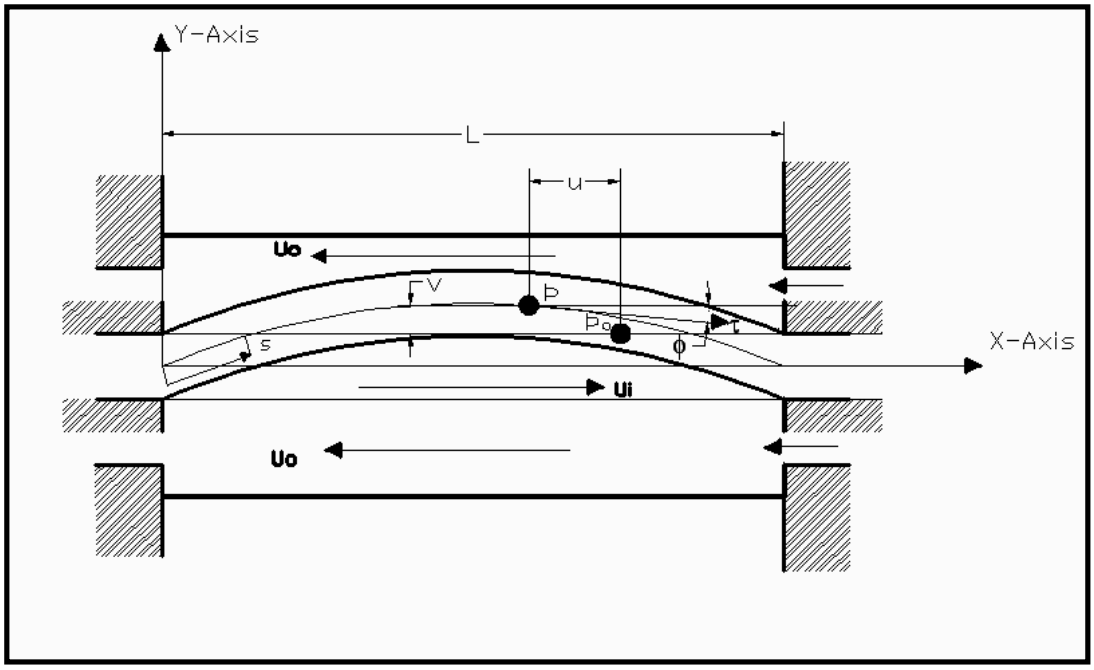


Figure 2.2: Deflected configuration of the pipe in *X-Y* plane.

where M is the mass of the internal fluid per unit length of the pipe and prime denotes differentiation with respect to X.

The total kinetic energy of the system can be determined by adding equations (2.5) and (2.11), i.e.

$$T = \frac{m+M}{2}\int_0^L \left[\dot{u}^2 + \dot{v}^2\right]dX + U_i \int_0^L M\left[\dot{u} + \dot{u}\,u' + \dot{v}\,v'\right]dX + \frac{U_i^2}{2}\int_0^L M\left(1 + u'^2 + v'^2 + 2u'\right)dX \tag{2.12}$$

2.4 Potential Energy Expression

The system's potential energy is made up of the elastic pipe-bending strain energy V_b and strain energy due to axial deformation V_a.

2.4.1 *Strain energy due to bending*

Following the analysis presented by Stocker [35], the strain energy due to bending can be expressed by

$$V_b = \frac{E\,I}{2}\int_0^L \left[\left(1+\varepsilon\right)^2 \kappa^2\right] dX \tag{2.13}$$

where κ is the curvature of the centerline of the pipe.

Let ϕ be the inclination angle of the pipe and the X axis at a general point; as shown in figure 2.2, and s is the curvilinear coordinate along the pipe. Now, for a pipe undergoining planar motion, either extensible or inextensible, the curvature κ is given by

$$\kappa = \frac{\partial \phi}{\partial s} \tag{2.14}$$

Using equation (2.8), the curvature κ in terms of X coordinate can be stated as

$$\kappa = \frac{1}{1+\varepsilon}\frac{\partial \phi}{\partial X} \tag{2.15}$$

Based on the pipe deformed geometry, the angle ϕ can be defined as

$$\cos\ \phi = \frac{1+(\partial u/\partial X)}{1+\varepsilon(X)} \tag{2.16}$$

$$\sin\ \phi = \frac{(\partial v/\partial X)}{1+\varepsilon(X)} \tag{2.17}$$

Using equations (2.16) and (2.17), one can write that

$$\frac{\partial \phi}{\partial X} = \frac{\left[\frac{\partial^2 v}{\partial X^2}\left(1+\frac{\partial u}{\partial X}\right)-\frac{\partial v}{\partial X}\frac{\partial^2 u}{\partial X^2}\right]}{(1+\varepsilon)^2} \tag{2.18}$$

By substituting equation (2.18) into equation (2.15), we get

$$\kappa = \frac{\left[\frac{\partial^2 v}{\partial X^2}\left(1+\frac{\partial u}{\partial X}\right)-\frac{\partial v}{\partial X}\frac{\partial^2 u}{\partial X^2}\right]}{(1+\varepsilon)^3} \tag{2.19}$$

The strain is stated by [3] as

$$\varepsilon = u' + \frac{1}{2}v'^2 + \tilde{O}(\epsilon^4) \tag{2.20}$$

Substituting equations (2.19) and (2.20) into equation (2.13) and retaining terms up to fourth order, one obtains the following expression for the beam bending potential energy V_b

$$V_b = \frac{EI}{2}\int_0^L \left[v''^2 - 2v''^2u' - 2v''^2v'^2 - 2v'v''u''\right]dX \tag{2.21}$$

2.4.2 Strain energy due axial deformation

Based on the stress-strain relationship, the strain energy according to axial deformation V_a is expressed by

$$V_a = \frac{E\,A}{2}\int_0^L \varepsilon^2\, dX \tag{2.22}$$

Other components of strain energy due to axial deformation are associated with the tension $\breve{T}$ and differential pressurization force P.

For generalization, all these components are considered in this analysis. Now, the potential energy due to axial deformation can be rewritten as

$$V_a = \frac{E\,A}{2}\int_0^L \left(\frac{\breve{T} - P}{EA} + \varepsilon\right)^2 dX \tag{2.23}$$

where

P: fluid pressurization force.

$\breve{T}$: Tension in the pipe either externally applied or associated with frictional forces.

By substituting equation (2.20) into equation (2.23) and retaining terms up to fourth order yields

$$\begin{aligned} V_a = {} & \frac{E\,A}{2}\int_0^L \left[u'^2 + \frac{1}{4}v'^4 + u'v'^2\right] dX + \int_0^L \left(\breve{T} - P\right)\left(u' + \frac{1}{2}v'^2 - \frac{u'v'^2}{2} - \frac{1}{8}v'^4\right) dX \\ & + \int_0^L \left[\frac{\left(\breve{T} - P\right)^2}{2\,E\,A}\right] dX \end{aligned} \tag{2.24}$$

Hence, the total potential energy expression can be written as

$$
\begin{aligned}
V \;=\; & \frac{EI}{2}\int_0^L \left[v''^2 - 2v''^2u' - 2v''^2v'^2 - 2v'v''u''\right] dX \\
& + \frac{EA}{2}\int_0^L \left[u'^2 + \frac{1}{4}v'^4 + u'v'^2\right] dX \\
& + \int_0^L \left(\tilde{T} - P\right)\left(u' + \frac{1}{2}v'^2 - \frac{u'v'^2}{2} - \frac{1}{8}v'^4\right) dX + \int_0^L \left[\frac{\left(\tilde{T} - P\right)^2}{2EA}\right] dX
\end{aligned}
\tag{2.25}
$$

2.5 The Lagrangian Function

The Lagrangian function $\tilde{L}$ is defined as

$$
\tilde{L} \;=\; T \;-\; V \tag{2.26}
$$

where

T: Total kinetic energy of the system.

V: Total potential energy of the system

Using equations (2.12) and (2.25), the Lagrangian function can be written as

$$
\begin{aligned}
\tilde{L} \;=\; & \frac{m}{2}\int_0^L \left[\dot{u}^2 + \dot{v}^2\right] dX + \frac{M}{2}\int_0^L \left[\dot{u}^2 + \dot{v}^2\right] dX + U_i M \int_0^L \left[\dot{u} + \dot{u}u' + \dot{v}v'\right] dX \\
& + \frac{MU_i^2}{2}\int_0^L \left(1 + u'^2 + v'^2 + 2u'\right) dX \\
& - \frac{EI}{2}\int_0^L \left[v''^2 - 2v''^2u' - 2v''^2v'^2 - 2v'v''u''\right] dX \\
& - \frac{EA}{2}\int_0^L \left[u'^2 + \frac{1}{4}v'^4 + u'v'^2\right] dX \\
& - \int_0^L \left(\tilde{T} - P\right)\left(u' + \frac{1}{2}v'^2 - \frac{u'v'^2}{2} - \frac{1}{8}v'^4\right) dX - \int_0^L \left[\frac{\left(\tilde{T} - P\right)^2}{2EA}\right] dX
\end{aligned}
\tag{2.27}
$$

2.6 Hydrodynamic Forces

Referring to figure 2.3, which shows an element of the pipe, the hydrodynamic forces acting on the pipe are

- The inviscid hydrodynamic force, F_A.
- Normal frictional force F_N.
- Longitudinal frictional forces, F_L.
- Hydrostatic pressure forces in the x and y directions, F_{px} and F_{py} respectively.

2.6.1 Inviscid hydrodynamic forces

To determine the inviscid hydrodynamic forces, Lighthill's model of slender body theory [36] is adopted in this formulation.

Referring to figure 2.4, we can define the relative fluid-body velocity by

$$V_{rf} = \dot{y} + \dot{x} - U_f \tag{2.28}$$

where U_f represents the axial flow velocity relative to the axially deformed pipe. This relative velocity is related to the axial velocity of the external flow in same manner presented in [12]. However by reversing the sign of U_o to account for the direction of the external flow, one obtains that:

$$U_f = -U_o\left(1 - \frac{\partial u}{\partial X}\right) \tag{2.29}$$

Let's consider figure 2.4 which shows an element δX subjected to deformation by the fluid. This representation enables one to define the angles required in determination of forces.

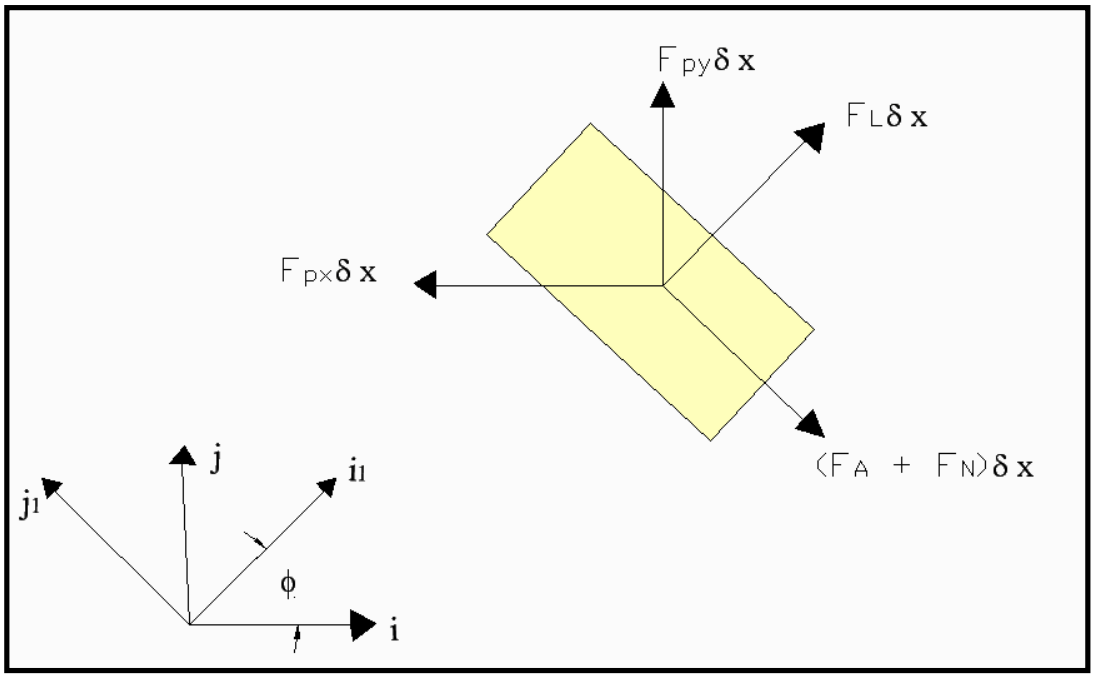


Figure 2.3: Hydrodynamic forces exerted by external fluid on the pipe.

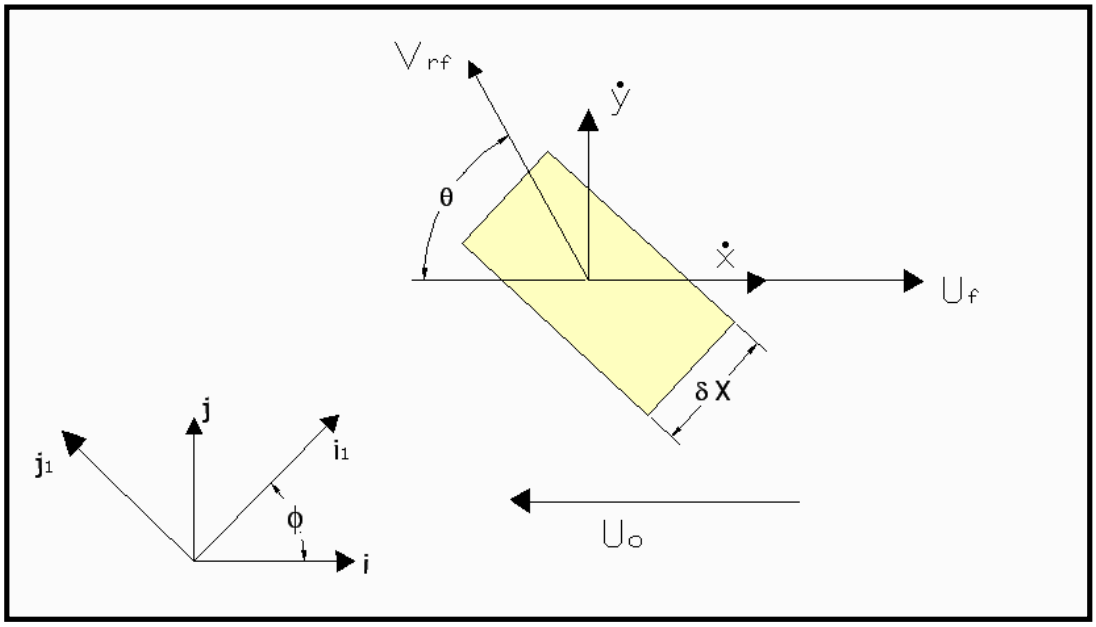


Figure 2.4: Element of the pipe shows velocities and angles

The angle ϕ represents the angle between longitudinal axis of the element and X-axis, while θ represents the angle between relative fluid-body velocity V_{rf} and the X-axis. Projecting the relative fluid-body velocity on j_1, one can find that

$$V_{rf} = \dot{y}\cos\phi + \left(U_f - \dot{x}\right)\sin\phi \tag{2.30}$$

Referring to figure 2.4, the angle between the longitudinal axis of the element and X-axis can be found as

$$\phi = \tan^{-1}\left(\frac{\partial y/\partial X}{\partial x/\partial X}\right) = \tan^{-1}\left(\frac{\partial y/\partial X}{1-\partial u/\partial X}\right) \tag{2.31}$$

Using series expansion, we can write

$$\cos\phi = 1 - \frac{1}{2}v'^2 + \tilde{O}\,(\epsilon^4) \quad , \quad \sin\phi = v' - u'v' - \frac{1}{2}v'^3 + \tilde{O}\,(\epsilon^5) \tag{2.32}$$

By substituting equations (2.29) and (2.32) into equation (2.30) and retaining terms up to fourth order, we get

$$V_{rf} = \dot{v} - U_o v' - \frac{1}{2}\dot{v}v'^2 + 2U_o u'v' + \frac{1}{2}U_o v'^3 - \dot{u}v' \tag{2.33}$$

The next step involves the extension of Lighthill's linear potential flow theory to a third order nonlinear formulation. It is important to recognize that the inviscid hydrodynamic force is equal to the lift force in magnitude, but acts in opposite direction. Lopes et al. [12] obtained linear and nonlinear expressions for the inviscid hydrodynamic force based on the following assumptions:

- The fluid doesn't penetrate the cylinder.
- The fluid velocity of the external flow U_o normal to the outer cylinder is zero.

In this work, a nonlinear expression for this force is used, and therefore, the inviscid hydrodynamic force can be obtained as

$$F_A = \left\{ \frac{\partial}{\partial t} + \left[-U_o\left(1-u'\right) - \left(\dot{u} - U_o u'\right) \right] \frac{\partial}{\partial X} \right\}$$
$$\times \left[\left(\dot{v} - U_o v'\right) - \left(\dot{u}v' - 2U_o u'v'\right) - \frac{1}{2}\left(\dot{v} - U_o v'\right)v'^2 \right] M_a \tag{2.34}$$
$$+ \frac{1}{2} M_a \left(\dot{v} - U_o v'\right) v' \left(\dot{v} - U_o v'\right)'$$

where M_a is the added hydrodynamic mass per unit length of the annular flow.

Noting that U_o is constant and simplifying equation (2.34) yields

$$F_A = M_a \left(\ddot{v} - \ddot{u}v' - 2\dot{u}\dot{v}' - \frac{1}{2}\ddot{v}v'^2 - \frac{3}{2}\dot{v}\dot{v}'v' \right)$$
$$- M_a U_o \left(2\dot{v}' - 3\dot{u}'v' - 4u'\dot{v}' - \frac{5}{2}\dot{v}'v'^2 - 2\dot{u}v'' - \frac{3}{2}\dot{v}v'v'' \right) \tag{2.35}$$
$$+ M_a U_o^{\,2} \left(v'' - 2u''v' - 4u'v'' - 2v'^2 v'' \right)$$

Based on the geometry provided in figure 2.1, the added hydrodynamic mass of the annular flow can be expressed as [33]

$$M_a = \frac{\left(D_{ch}/D_o\right)^2 + 1}{\left(D_{ch}/D_o\right)^2 - 1} \rho_{f,o} A_o \tag{2.36}$$

where

D_{ch} : inner diameter of the outer cylinder.

D_o: outer diameter of the pipe.

A_o: external cross sectional area of the pipe.

$\rho_{f,o}$:the density of the fluid of external flow per unit length.

2.6.2 Frictional forces

The frictional forces are formulated along the same lines proposed by Taylor [37], which stated that

$$F_N = \frac{1}{2}\rho D_o U_o^{\,2}\left(C_N \sin\psi + C_{DP}\sin^2\psi\right) \tag{2.37}$$

$$F_L = \frac{1}{2} D_o U_o^{\,2} C_T \cos\psi \tag{2.38}$$

where

C_{DP}: The form- drag coefficient due to the normal component.

C_N: Friction coefficient in normal direction.

C_T: Friction coefficient in tangential direction.

ψ :Angle of attack such that $\psi = \phi + \theta$.

By referring to figure (2.4), the angle between the relative fluid-body velocity V_{rf} and X can be defined as

$$\theta = \tan^{-1}\left(\frac{\partial y/\partial t}{U_f - \partial x/\partial t}\right) \tag{2.39}$$

Or,

$$\theta = \frac{\dot{v}}{U_f} + \frac{\dot{x}\dot{v}}{U_f^{\,2}} - \frac{1}{3}\frac{\dot{v}^3}{U_f^{\,3}} + \tilde{O}\,(\epsilon^5) \tag{2.40}$$

Then, one can write

$$\cos\theta = 1 - \frac{1}{2}\frac{\dot{v}^2}{U_f^{\,2}} + \tilde{O}\,(\epsilon^4) \tag{2.41}$$

$$\sin\theta = \frac{\dot{v}}{U_f} + \frac{\dot{u}\dot{v}}{U_f^{\,2}} - \frac{1}{2}\frac{\dot{v}^3}{U_f^{\,3}} + \tilde{O}\,(\epsilon^5) \tag{2.42}$$

By combining equations (2.31), (2.40) and (2.41), one can obtain

$$\cos\psi = 1 - \frac{1}{2}\left(v'^2 + \frac{2v'\dot{v}}{U_f} + \frac{\dot{v}^2}{U_f^{\,2}} \right) + \tilde{O}\,(\epsilon^4) \tag{2.43}$$

$$\sin\psi = v' + \frac{\dot{v}}{U_f} - u'v' + \frac{\dot{u}\,\dot{v}}{U_f^{\,2}} - \frac{1}{2}\left(v'^3 + \frac{\dot{v}^3}{U_f^{\,3}} + \frac{v'^2\dot{v}}{U_f} + \frac{v'\dot{v}^2}{U_f^{\,2}} \right) + \tilde{O}\,(\epsilon^5) \tag{2.44}$$

By substituting equations (2.43) and (2.44) into equations (2.37) and (2.38) and relating U_f to U_o through equation (2.29), one obtains

$$F_N = \frac{1}{2}\rho_{f,o}\, D_o U_o^{\,2} \begin{bmatrix} C_N\left(-v' + \frac{\dot{v}}{U_o} + \frac{\dot{v}u'}{U_o} + u'v' + \frac{\dot{u}\dot{v}}{U_o^{\,2}} \right) \\ -\frac{C_N}{2}\left(-v'^3 + \frac{\dot{v}^3}{U_o^{\,3}} + \frac{v'^2\dot{v}}{U_o} + \frac{v'\dot{v}^2}{U_o^{\,2}} \right) \\ +C_{DP}\left(-v'|v'| + \frac{v'|\dot{v}| + |v'|\dot{v}}{U_o} + \frac{\dot{v}|\dot{v}|}{U_o^{\,2}} \right) \end{bmatrix} \tag{2.45}$$

$$F_L = \frac{1}{2}\rho_{f,o}\, D_o\, U_o^{\,2}\, C_T \left(1 - \frac{1}{2}\left(-v'^2 + \frac{2v'\dot{v}}{U_o} + \frac{\dot{v}^2}{U_o^{\,2}} \right) \right) \tag{2.46}$$

In some analyses, distinct values of C_N and C_T are used; however, the simplified form $C_N = C_T = C_f$ is frequently used. The value of the frictional damping coefficient can be estimated based on Swamee and Jain formula [38], in which the frictional coefficient C_T can be obtained directly in terms of the Reynolds number Re, and relative roughness of the pipe $\frac{e}{D}$ such that: $C_T = 1.325\left\{ \ln\left[0.27\left(\frac{e}{D}\right) + 5.74\left(\frac{1}{\mathrm{Re}}\right)^{0.9} \right] \right\}^{-2}$.

On the other hand, the value of the frictional damping coefficient was estimated semi-empirically by Hannoyer and Paidoussis [30] as *0.0125*. This value is adopted in the numerical simulations presented in Chapter 7.

2.6.3 The hydrostatic pressure forces

Following the procedure in Lopes et al. [12], the hydrostatic pressure forces F_{px} and F_{py} shown in figure 2.3, which are the resultants of the steady state pressure p_o acting on the outer surface of the pipe element, and noting that p_o varies linearly with X, such forces are found for a constant section of the pipe to be

$$-F_{px} = \frac{\partial p_o}{\partial x} A_o \left(-\frac{1}{2} v'^2 + u' \right) - v'v'' \, p_o \, A_o \tag{2.47}$$

$$F_{py} = \frac{\partial p_o}{\partial x} A_o \left(-\frac{1}{2} v'^3 + v' \right) + p_o \, A_o \left(v'' - u''v' - u'v'' - \frac{3}{2} v'^2 v'' \right) \tag{2.48}$$

Furthermore, by assuming the lateral movement of the pipe to have a negligible effect on the axial pressure distribution [11], one may write

$$A_o \frac{\partial p_o}{\partial x} = \frac{1}{2} \rho_{f,o} \, D_o \, U_o^{\,2} \, C_T \, \frac{D_o}{D_h} \tag{2.49}$$

where D_h is the hydraulic diameter of the outer cylinder, which can be defined as

$$D_h = \frac{\left(D_{ch}^{\,2} - D_o^{\,2} \right)}{\left(D_{ch} + D_o \right)} \tag{2.50}$$

By substituting equation (2.49) into equations (2.47) and (2.48), one obtains

$$-F_{px} = \left(-\frac{1}{2} v'^2 + u' \right) \left(\frac{1}{2} \rho_{f,o} \, D_o \, U_o^{\,2} \, C_T \, \frac{D_o}{D_h} \right) - v'v'' \, p_o \, A_o \tag{2.51}$$

$$\begin{aligned} F_{py} = & \left(-\frac{1}{2} v'^3 + v' \right) \left(\frac{1}{2} \rho_{f,o} \, D_o \, U_o^{\,2} \, C_T \, \frac{D_o}{D_h} \right) \\ & + p_o \, A_o \left(v'' - u''v' - u'v'' - \frac{3}{2} v'^2 v'' \right) \end{aligned} \tag{2.52}$$

In order to estimate the pressure distribution p_o, following the same procedure introduced in [11], one notes that

$$A_o \frac{\partial p_o}{\partial X} = A_o \frac{\partial p_o}{\partial x}\left(1 + u'\right) \tag{2.53}$$

Now, integrating equation (2.49) from X to L, one obtains

$$A_o\, p_o\,(X) = A_o\, p_o\,(L) - \left(\frac{1}{2}\rho_{f,o}\, D_o\, U_o^{\,2}\, C_T \frac{D_o}{D_h}\right)\left(L - X - u\right) \tag{2.54}$$

where $p_o(L)$ is the pressure at the downstream end of the pipe, which may be represented by

$$A_o\, p_o\,(L) = \left(1 - 2\nu\right)\bar{p}_o A_o \tag{2.55}$$

where

ν : Poisson ratio.

$\bar{p}_o$: the value of p_o at $X = \frac{L}{2}$.

2.7 Equations of Motion

The equations of motion and the boundary conditions for a flexible pipe conveying fluid and subjected to external axial flow are derived using extended Hamilton's principle.

2.7.1 Hamilton's principle

The energy method is based on Hamilton's principle, which states that, of all the varied paths satisfying the prescribed initial and final configuration, the actual (true) path extremizes the function $\tilde{I} = \int_{t_o}^{t_f} \tilde{L}\, dt$, where t_o and t_f denote the initial and final time

instants. By including work done by the non-conservative forces within the integrand, we get the extended Hamilton's principle, which states

$$\delta \int_{t_o}^{t_f} \widetilde{L}\ dt\ + \int_{t_o}^{t_f} \delta W_{nc}\, dt = 0 \tag{2.56}$$

Using the variation of the functional $\widetilde{I}$ and noting the fact that the variation and integral operators commute, then the extended Hamilton's principle can be written as

$$\int_{t_o}^{t_f} \left(\delta \widetilde{L} + \delta W_{nc}\right) dt\ = 0 \tag{2.57}$$

where δW_{nc} denotes the virtual work done by non-conservative forces and not included in the Lagrangain function.

It is important to note that even if there are no explicit external forces applied to the pipe conveying fluid, δW_{nc} in equation (2.56) does not vanish if one or both ends of the pipe were not fixed [3]. This issue will be discussed in more details in chapter three.

2.7.2 Variation of the Lagrangian function

In order to perform the variation of the Lagrangian function, it is essential to specify the boundary conditions, which can be stated as

$$u\left(x=0\right)=v\left(x=0\right)=0 \tag{2.58}$$

$$u\left(x=L\right)=v\left(x=L\right)=0 \tag{2.59}$$

$$v'\left(x=0\right)=v'\left(x=L\right)=0 \tag{2.60}$$

Substituting equation (2.27) and (2.58-2.60) in the variational formula, and after algebraic manipulations, including several integrations by parts, one can obtain

$$\int_{t_o}^{t_1} \delta \widetilde{L}\, dt = -\int_{t_o}^{t_f}\int_0^L \left\{ \begin{array}{l} (m+M)\ddot{u} + (2M\,U_i)\,\dot{u}' + \left(M\,U_i^{\,2}\right)u'' \\ -(E\,A)u'' + \left(\breve{T} - P - E\,A\right)v'\,v'' - E\,I\left(v''''\,v' + v''\,v'''\right) \end{array} \right\} \delta u\, dX\, dt$$

$$-\int_{t_o}^{t_f}\int_0^L \left\{ \begin{array}{l} (m+M)\ddot{v} + (2M\,U_i)\,\dot{v}' + \left(M\,U_i^{\,2}\right)v'' \\ -\left(\breve{T} - P\right)v'' + (E\,I)v'''' \\ -EI\begin{pmatrix} 3u'''\,v'' + 4u''\,v''' + 2u'\,v'''' + v'\,u'''' \\ +2v'^2\,v'''' + 8v'\,v''\,v''' + 2v''^3 \end{pmatrix} \\ +\left(\breve{T} - P - EA\right)\left(u''\,v' + u'\,v'' + \dfrac{3}{2}v'^2\,v''\right) \end{array} \right\} \delta v\, dX\, dt \tag{2.61}$$

2.7.3 Total virtual work of external fluid hydrodynamic forces

Referring to figure 2.3, the virtual work of the external hydrodynamic forces in *x*-direction can be expressed as

$$\delta\, W_x = \int_{t_o}^{t_f}\int_0^L \left\{ -F_{\,px} + F_L \cos\phi + \left(F_A + F_N\right)\sin\phi \right\} \delta x\, dX\, dt \tag{2.62}$$

Then, using equations (2.32), (2.35) and (2.45-2.46), and after some mathematical manipulations, the following expressions can be obtained

$$F_L \cos\phi = \frac{\rho_{f,o}\; D_o\, U_o C_T}{2}\left(1 + v'^2 - \frac{v'\,\dot{v}}{U_o} - \frac{\dot{v}^2}{2U_o^{\,2}}\right) \tag{2.63}$$

$$\begin{aligned} \left(F_A + F_N\right)\sin\phi = {} & M_a\left(\ddot{v}\,v'\right) + M_a\,U_o\left(-2\dot{v}'\,v'\right) + M_a\,U_o^{\,2}\left(v'\,v''\right) \\ & + \left(\frac{\rho_{f,o}\; D_o U_o^{\,2} C_N}{2}\right)\left(-v'^2 + \frac{\dot{v}\,v'}{U_o}\right) \\ & + \left[\frac{\rho_{f,o}\; D_o\, U_o\, C_{DP}}{2}\right]\left(-v'^2\left|v'\right| + \frac{\left|\dot{v}\right|\,v'^2 + \left|v'\right|v'\,\dot{v}}{U_o} + \frac{\dot{v}\left|\dot{v}\right|v'}{U_o^{\,2}}\right) \end{aligned} \tag{2.64}$$

Substituting equations (2.51), (2.54-2.55), (2.63) and (2.64) into equation (2.62), and retaining the terms up to 3rd order of magnitude, one can obtain the virtual work in *x*-direction as

$$\delta W_x = -\int_{t_o}^{t_f}\int_0^L \left\{ \begin{array}{l} -M_a(\ddot{v}v') - M_a U_o(-2\dot{v}'v') - M_a U_o^{\,2}(v'v'') \\ -\left(\dfrac{\rho_{f,o}\ D_o U_o^{\,2} C_T}{2}\dfrac{D_o}{D_h}\right)\left(-\dfrac{1}{2}v'^2 + u' + (L-X)(v'v'')\right) \\ +[(1-2\nu)\bar{p}_o\ A_o\ ](v'v'') \\ -\left(\dfrac{\rho_{f,o}\ D_o U_o^{\,2} C_T}{2}\right)\left(1 + v'^2 - \dfrac{v'\dot{v}}{U_o} - \dfrac{\dot{v}^2}{2U_o^{\,2}}\right) \\ -\left(\dfrac{\rho_{f,o}\ D_o U_o^{\,2} C_N}{2}\right)\left(-v'^2 + \dfrac{\dot{v}v'}{U_o}\right) \\ -\left(\dfrac{\rho_{f,o}\ D_o U_o\ C_{DP}}{2}\right)\left(-v'^2|v'| + \dfrac{|\dot{v}|v'^2 + |v'|v'\dot{v}}{U_o} + \dfrac{\dot{v}|\dot{v}|v'}{U_o^{\,2}}\right) \end{array} \right\} \delta u\, dX dt \tag{2.65}$$

Referring to figure 2.3, the virtual work of the external hydrodynamic forces in *y*-direction can be expressed as

$$\delta\ W_y = \int_{t_o}^{t_f}\int_0^L \left\{F_{\ py} + F_L \sin\phi - (F_A + F_N)\cos\phi\right\}\delta\, y\, dX\, dt \tag{2.66}$$

Now, using equations (2.32), (2.35), and (2.45-2.46), and after some mathematical manipulations, the following expressions can be obtained

$$F_L \sin\phi = \left(\frac{\rho_{f,o}\ D_o\, U_o^{\,2} C_T}{2}\right)\left(v' - u'v' - \frac{\dot{v}v'^2}{U_o} - \frac{\dot{v}^2\, v'}{2U_o^{\,2}}\right) \tag{2.67}$$

$$\begin{aligned} -(F_A + F_N)\cos\phi = & -\left(\frac{\rho_{f,o}\ D_o\, U_o^{\,2} C_N}{2}\right)\left(\begin{array}{l} -v' + \dfrac{\dot{v}}{U_o} + \dfrac{\dot{v}u'}{U_o} + u'v' + \dfrac{\dot{u}\,\dot{v}}{U_o^{\,2}} \\ +v'^3 - \dfrac{\dot{v}v'^2}{U_o} - \dfrac{\dot{v}^3}{2U_o^{\,3}} - \dfrac{\dot{v}^2 v'}{2U_o^{\,2}} \end{array}\right) \\ & -\left(\frac{\rho_{f,o}\ D_o\, U_o^{\,2} C_{DP}}{2}\right)\left(-v'|v'| + \frac{|\dot{v}|\ v' + |v'|\ \dot{v}}{U_o} + \frac{\dot{v}|\dot{v}|}{U_o^{\,2}}\right) \\ & -M_a\left(\ddot{v} - \ddot{u}v' - 2\dot{u}\dot{v}' - \ddot{v}v'^2 - \frac{3}{2}\dot{v}\dot{v}'v'\right) \\ & -M_a U_o\left(-2\dot{v}' + 3\dot{u}'v' + 4u'\dot{v}' + \frac{7}{2}\dot{v}'v'^2 + 2\dot{u}v'' + \frac{3}{2}\dot{v}v'v''\right) \\ & -M_a U_o^{\,2}\left(v'' - 2u''v' - 4u'v'' - \frac{5}{2}v''v'^2\right) \end{aligned} \tag{2.68}$$

By substituting equations (2.52), (2.54-2.55), (2.67) and (2.68) into equation (2.66), one can obtain the virtual work in y-direction as

$$\delta W_y = -\int_{t_o}^{t_f}\int_0^L \left\{ \begin{array}{l} -\left(\dfrac{\rho_{f,o}\ D_o U_o^{\,2} C_T}{2}\dfrac{D_o}{D_h}\right)\left(\begin{array}{l}\left(v' - \dfrac{1}{2}v'^3 + u\,v''\right) \\ -(L-X)\left(v'' - u''v' - u'v'' - \dfrac{3}{2}v'^2\,v''\right)\end{array}\right) \\ -\left[(1-2\nu)\,\overline{p}_o\ A_o\ \right]\left(v'' - u''v' - u'v'' - \dfrac{3}{2}v'^2\,v''\right) \\ -\left(\dfrac{\rho_{f,o}\ D_o\,U_o^{\,2} C_T}{2}\right)\left(v' - u'v' - \dfrac{\dot{v}\,v'^2}{U_o} - \dfrac{\dot{v}^2\,v'}{2U_o^{\,2}}\right) \\ +\left(\dfrac{\rho_{f,o}\ D_o\,U_o^{\,2} C_N}{2}\right)\left(\begin{array}{l} -v' + \dfrac{\dot{v}}{U_o} + \dfrac{\dot{v}\,u'}{U_o} + u'v' + \dfrac{\dot{u}\,\dot{v}}{U_o^{\,2}} \\ +v'^3 - \dfrac{\dot{v}\,v'^2}{U_o} - \dfrac{\dot{v}^3}{2U_o^{\,3}} - \dfrac{\dot{v}^2 v'}{2U_o^{\,2}}\end{array}\right) \\ +\left(\dfrac{\rho_{f,o}\ D_o\,U_o^{\,2} C_{DP}}{2}\right)\left(-v'|v'| + \dfrac{|\dot{v}|\ v' + |v'|\ \dot{v}}{U_o} + \dfrac{\dot{v}|\dot{v}|}{U_o^{\,2}}\right) \\ +M_a\left(\ddot{v} - \ddot{u}\,v' - 2\,\dot{u}\,\dot{v}' - \ddot{v}v'^2 - \dfrac{3}{2}\dot{v}\,\dot{v}'\,v'\right) \\ +M_a\,U_o\left(-2\dot{v}' + 3\,\dot{u}'v' + 4\,u'\dot{v}' + \dfrac{7}{2}\dot{v}'v'^2 + 2\,\dot{u}\,v'' + \dfrac{3}{2}\dot{v}v'v''\right) \\ +M_a\,U_o^{\,2}\left(v'' - 2\,u''v' - 4\,u'v'' - \dfrac{5}{2}v''\,v'^2\right) \end{array}\right\} \delta\,v\,dX\,dt \tag{2.69}$$

One may observe that the total virtual work of the external fluid hydrodynamic forces is the summation of equations (2.65) and (2.69).

2.7.4 Formulation of equations of motion

Substituting equations (2.61), (2.65) and (2.69) into the extended Hamilton's principle given by equation (2.57), one eventually finds the following two coupled equations of motion; one in the X coordinate and the other in Y coordinate, which

describe the motion of an extensible pipe conveying fluid and subjected to counter external axial flow :

$$
\begin{aligned}
&(m+M)\ddot{u}+(2MU_i)\dot{u}'+\left(MU_i^{\,2}\right)u''-(EA)u''+\left(\breve{T}-P-EA\right)v'v'' \\
&-EI\left(v''''v'+v''v'''\right)-M_a\left(\ddot{v}v'\right)-M_aU_o\left(-2\dot{v}'v'\right)-M_aU_o^{\,2}\left(v'v''\right) \\
&-\left(\frac{\rho_{f,o}\ D_oU_o^{\,2}C_T}{2}\frac{D_o}{D_h}\right)\left(-\frac{1}{2}v'^2+u'+(L-X)(v'v'')\right) \\
&+\left[(1-2\nu)\bar{p}_o\,A_o\right](v'v'') \\
&-\left(\frac{\rho_{f,o}\ D_oU_o^{\,2}C_T}{2}\right)\left(1+v'^2-\frac{v'\dot{v}}{U_o}-\frac{\dot{v}^2}{2U_o^{\,2}}\right) \\
&-\left(\frac{\rho_{f,o}\ D_oU_o^{\,2}C_N}{2}\right)\left(-v'^2+\frac{\dot{v}v'}{U_o}\right) \\
&-\left(\frac{\rho_{f,o}\ D_oU_oC_{DP}}{2}\right)\left(-v'^2|v'|+\frac{|\dot{v}|v'^2+|v'|v'\dot{v}}{U_o}+\frac{\dot{v}|\dot{v}|v'}{U_o^{\,2}}\right)=0
\end{aligned}
\tag{2.70}
$$

$$
\begin{aligned}
&(m+M)\ddot{v}+(2MU_i)\dot{v}'+\left(MU_i^{\,2}\right)v''-\left(\breve{T}-P\right)v''+(EI)v'''' \\
&-EI\begin{pmatrix}3u'''v''+4u''v'''+2u'v''''+v'u'''' \\ +2v'^2v''''+8v'v''v'''+2v''^3\end{pmatrix}+\left(\breve{T}-P-EA\right)\left(u''v'+u'v''+\frac{3}{2}v'^2v''\right) \\
&-\left(\frac{\rho_{f,o}\ D_oU_o^{\,2}C_T}{2}\frac{D_o}{D_h}\right)\left(\left(v'-\frac{1}{2}v'^3+uv''\right)-(L-X)\left(v''-u''v'-u'v''-\frac{3}{2}v'^2v''\right)\right) \\
&-\left[(1-2\nu)\bar{p}_o\,A_o\right]\left(v''-u''v'-u'v''-\frac{3}{2}v'^2v''\right)-\left(\frac{\rho_{f,o}\ D_oU_o^{\,2}C_T}{2}\right)\begin{pmatrix}v'-u'v' \\ -\dfrac{\dot{v}v'^2}{U_o}-\dfrac{\dot{v}^2v'}{2U_o^{\,2}}\end{pmatrix} \\
&+\left(\frac{\rho_{f,o}\ D_oU_o^{\,2}C_N}{2}\right)\left(-v'+\frac{\dot{v}}{U_o}+\frac{\dot{v}u'}{U_o}+u'v'+\frac{\dot{u}\dot{v}}{U_o^{\,2}}+v'^3-\frac{\dot{v}v'^2}{U_o}-\frac{\dot{v}^3}{2U_o^{\,3}}-\frac{\dot{v}^2v'}{2U_o^{\,2}}\right) \\
&+\left(\frac{\rho_{f,o}\ D_oU_o^{\,2}C_{DP}}{2}\right)\left(-v'|v'|+\frac{|\dot{v}|v'+|v'|\dot{v}}{U_o}+\frac{\dot{v}|\dot{v}|}{U_o^{\,2}}\right) \\
&+M_a\left(\ddot{v}-\ddot{u}v'-2\dot{u}\dot{v}'-\ddot{v}v'^2-\frac{3}{2}\dot{v}\dot{v}'v'\right) \\
&+M_aU_o\left(-2\dot{v}'+3\dot{u}'v'+4u'\dot{v}'+\frac{7}{2}\dot{v}'v'^2+2\dot{u}v''+\frac{3}{2}\dot{v}v'v''\right) \\
&+M_aU_o^{\,2}\left(v''-2u''v'-4u'v''-\frac{5}{2}v''v'^2\right)=0
\end{aligned}
\tag{2.71}
$$

It is important to recognize that P represents the net differential pressure force, which can be related to the internal flow pressure p_i and external flow pressure p_o at $X=L$ such that

$$P = A_f\, p_i(L) - A_o\, p_o(L) \tag{2.72}$$

where A_f is the internal cross sectional flow area.

Utilizing equation (2.72), the governing equations of motion can be rewritten as

$$\begin{aligned}
&(m+M)\ddot{u} + (2M\,U_i)\,\dot{u}' + \left(M\,U_i^{\,2}\right)u'' - (E\,A)u'' \\
&+\left(\tilde{T} - A_f\, p_i(L) + A_o\, p_o(L) - E\,A\right)v'\,v'' \\
&-E\,I\left(v''''\,v' + v''\,v'''\right) - M_a\left(\ddot{v}\,v'\right) - M_a\,U_o\left(-2\,\dot{v}'\,v'\right) - M_a\,U_o^{\,2}\left(v'\,v''\right) \\
&-\left(\frac{\rho_{f,o}\;D_o U_o^{\,2} C_T}{2}\frac{D_o}{D_h}\right)\left(-\frac{1}{2}v'^2 + u' + (L-X)(v'\,v'')\right) \\
&+\left[(1-2\nu)\,\bar{p}_o\,A_o\right](v'\,v'') \\
&-\left(\frac{\rho_{f,o}\;D_o U_o^{\,2} C_T}{2}\right)\left(1 + v'^2 - \frac{v'\dot{v}}{U_o} - \frac{\dot{v}^2}{2U_o^{\,2}}\right) \\
&-\left(\frac{\rho_{f,o}\;D_o U_o^{\,2} C_N}{2}\right)\left(-v'^2 + \frac{\dot{v}\,v'}{U_o}\right) \\
&-\left(\frac{\rho_{f,o}\;D_o\,U_o\,C_{DP}}{2}\right)\left(-v'^2\,|v'| + \frac{|\dot{v}|\,v'^2 + |v'|\,v'\,\dot{v}}{U_o} + \frac{\dot{v}\,|\dot{v}|\,v'}{U_o^{\,2}}\right) = 0
\end{aligned} \tag{2.73}$$

$$
\begin{aligned}
&(m+M)\ddot{v}+(2M\,U_i)\,\dot{v}'+\left(M\,U_i^{\,2}\right)v''-\left(\breve{T}-A_f\;p_i(L)+A_o\;p_o(L)\right)v'' \\
&+(E\,I)v''''-EI\begin{pmatrix}3u'''v''+4u''v'''+2u'v''''+v'u'''' \\ +2v'^2\,v''''+8v'v''v'''+2v''^3\end{pmatrix} \\
&+\left(\breve{T}-A_f\;p_i(L)+A_o\;p_o(L)-EA\right)\left(u''v'+u'v''+\frac{3}{2}v'^2\,v''\right) \\
&-\left(\frac{\rho_{f,o}\;\;D_oU_o^{\,2}C_T}{2}\frac{D_o}{D_h}\right)\begin{pmatrix}\left(v'-\frac{1}{2}v'^3+u\,v''\right) \\ -(L-X)\left(v''-u''v'-u'v''-\frac{3}{2}v'^2\,v''\right)\end{pmatrix} \\
&-\left[(1-2\nu)\,\overline{p}_o\;A_o\;\right]\left(v''-u''v'-u'v''-\frac{3}{2}v'^2\,v''\right) \\
&-\left(\frac{\rho_{f,o}\;D_o\,U_o^{\,2}C_T}{2}\right)\left(v'-u'v'-\frac{\dot{v}v'^2}{U_o}-\frac{\dot{v}^2\,v'}{2U_o^{\,2}}\right) \\
&+\left(\frac{\rho_{f,o}\;D_o\,U_o^{\,2}C_N}{2}\right)\begin{pmatrix}-v'+\frac{\dot{v}}{U_o}+\frac{\dot{v}u'}{U_o}+u'v'+\frac{\dot{u}\,\dot{v}}{U_o^{\,2}} \\ +v'^3-\frac{\dot{v}v'^2}{U_o}-\frac{\dot{v}^3}{2U_o^{\,3}}-\frac{\dot{v}^2v'}{2U_o^{\,2}}\end{pmatrix} \\
&+\left(\frac{\rho_{f,o}\;D_o\,U_o^{\,2}C_{DP}}{2}\right)\left(-v'|v'|+\frac{|\dot{v}|\;v'+|v'|\;\dot{v}}{U_o}+\frac{\dot{v}|\dot{v}|}{U_o^{\,2}}\right) \\
&+M_a\left(\ddot{v}-\ddot{u}\,v'-2\dot{u}\,\dot{v}'-\ddot{v}v'^2-\frac{3}{2}\dot{v}\,\dot{v}'\,v'\right) \\
&+M_a\,U_o\left(-2\dot{v}'+3\dot{u}'v'+4u'\dot{v}'+\frac{7}{2}\dot{v}'v'^2+2\dot{u}\,v''+\frac{3}{2}\dot{v}v'v''\right) \\
&+M_a\,U_o^{\,2}\left(v''-2u''v'-4u'v''-\frac{5}{2}v''\,v'^2\right)=0
\end{aligned}
\tag{2.74}
$$

On the other hand, by following a procedure similar to that introduced in [11], the tension $\breve{T}$ can be expressed as

$$
\frac{\partial\breve{T}}{\partial x}=-\frac{\rho_{f,o}\;D_oU_o^{\,2}C_T}{2} \tag{2.75}
$$

Using the relation derived previously in equation (2.10), one may write

$$\frac{\partial \breve{T}}{\partial X} = \frac{\partial \breve{T}}{\partial x}\left(1 + u'\right) \tag{2.76}$$

Substituting equation (2.76) into equation (2.75), and integrating from X to L, one obtains

$$\breve{T}(X) = \breve{T}(L) + \left(\frac{1}{2}\rho_{f,o}\, D_o\, U_o^{\,2}\, C_T\right)\left(L - X - u\right) \tag{2.77}$$

where $\breve{T}(L)$ is the tension at the downstream end of the pipe, and can be represented by

$$\breve{T}(L) = \overline{T} - \frac{\rho_{f,o}\;\; D_o U_o^{\,2} C_T\, L}{2}\left(1 - \frac{D_o}{D_h}\right) \tag{2.78}$$

where $\overline{T}$ is the externally imposed uniform tension.

Finally, one may express the tension $\breve{T}$ as

$$\breve{T}(X) = \overline{T} + \frac{\rho_{f,o}\;\; D_o U_o^{\,2} C_T\, L}{2}\left(\frac{D_o}{D_h}\right) + \left(\frac{1}{2}\rho_{f,o}\, D_o\, U_o^{\,2}\, C_T\right)\left(\frac{L}{2} - X - u\right) \tag{2.79}$$

By substituting equation (2.79) into the Lagranginan function given by equation (2.27), performing the variational analysis for all terms including $\breve{T}$, and after mathematical manipulations, the governing equations of motion can be expressed as

$$\begin{aligned}
&\left(m + M\right)\ddot{u} + \left(2M\,U_i\right)\dot{u}' + \left(M\,U_i^{\,2}\right)u'' - \left(E\,A\right)u'' + \left(\overline{T} - A_f\; p_i(L) + A_o\; p_o(L) - E\,A\right)v'\,v'' \\
&- E\,I\left(v'''\,v' + v''\,v'''\right) - M_a\left(\ddot{v}\,v'\right) - M_a\,U_o\left(-2\,\dot{v}'\,v'\right) - M_a\,U_o^{\,2}\left(v'\,v''\right) \\
&- \left(\frac{\rho_{f,o}\;\; D_o U_o^{\,2} C_T}{2}\frac{D_o}{D_h}\right)\left(-\frac{1}{2}v'^2 + u' + \left(\frac{L}{2} - X\right)\left(v'\,v''\right)\right) + \left[\left(1 - 2\nu\right)\overline{p}_o\; A_o\;\right]\left(v'\,v''\right) \\
&- \left(\frac{\rho_{f,o}\;\; D_o U_o^{\,2} C_T}{2}\right)\left(\frac{3}{2}v'^2 - u' - \frac{v'\dot{v}}{U_o} - \frac{\dot{v}^2}{2U_o^{\,2}} - \left(\frac{L}{2} - X\right)\left(v'\,v''\right)\right) \\
&- \left(\frac{\rho_{f,o}\;\; D_o U_o^{\,2} C_N}{2}\right)\left(-v'^2 + \frac{\dot{v}\,v'}{U_o}\right) \\
&- \left(\frac{\rho_{f,o}\;\; D_o\, U_o\, C_{DP}}{2}\right)\left(-v'^2\left|v'\right| + \frac{\left|\dot{v}\right|v'^2 + \left|v'\right|v'\,\dot{v}}{U_o} + \frac{\dot{v}\left|\dot{v}\right|v'}{U_o^{\,2}}\right) = 0
\end{aligned} \tag{2.80}$$

$$
\begin{aligned}
&(m+M)\ddot{v}+(2M\,U_i)\,\dot{v}'+\left(M\,U_i^{\,2}\right)v''-\left(\overline{T}-A_f\;p_i(L)+A_o\;p_o(L)\right)v'' \\
&+(E\,I)v''''-EI\left(3u'''v''+4u''v'''+2u'v''''+v'u''''+2v'^2\,v''''+8v'v''v'''+2v''^3\right) \\
&+\left(\overline{T}-A_f\;p_i(L)+A_o\;p_o(L)-EA\right)\left(u''v'+u'v''+\frac{3}{2}v'^2\,v''\right) \\
&-\left(\frac{\rho_{f,o}\;\;D_oU_o^{\,2}C_T}{2}\frac{D_o}{D_h}\right)\left(\left(v'-\frac{1}{2}v'^3+u\,v''\right)-\left(\frac{L}{2}-X\right)\begin{pmatrix} v''-u''v' \\ -u'v''-\frac{3}{2}v'^2\,v'' \end{pmatrix}\right) \\
&-\left[(1-2\nu)\overline{p}_o\;A_o\;\right]\left(v''-u''v'-u'v''-\frac{3}{2}v'^2\,v''\right) \\
&-\left(\frac{\rho_{f,o}\;\;D_o\,U_o^{\,2}C_T}{2}\right)\begin{pmatrix} \frac{1}{2}v'^3-uv''-u'v'-\frac{\dot{v}\,v'^2}{U_o}-\frac{\dot{v}^2\,v'}{2U_o^{\,2}} \\ +\left(\frac{L}{2}-X\right)\left(v''-u''v'-u'v''-\frac{3}{2}v'^2\,v''\right) \end{pmatrix} \\
&+\left(\frac{\rho_{f,o}\;\;D_o\,U_o^{\,2}C_N}{2}\right)\begin{pmatrix} -v'+\frac{\dot{v}}{U_o}+\frac{\dot{v}u'}{U_o}+u'v'+\frac{\dot{u}\,\dot{v}}{U_o^{\,2}} \\ +v'^3-\frac{\dot{v}\,v'^2}{U_o}-\frac{\dot{v}^3}{2U_o^{\,3}}-\frac{\dot{v}^2v'}{2U_o^{\,2}} \end{pmatrix} \\
&+\left(\frac{\rho_{f,o}\;\;D_o\,U_o^{\,2}C_{DP}}{2}\right)\left(-v'|v'|+\frac{|\dot{v}|\;v'+|v'|\;\dot{v}}{U_o}+\frac{\dot{v}|\dot{v}|}{U_o^{\,2}}\right) \\
&+M_a\left(\ddot{v}-\ddot{u}\,v'-2\dot{u}\,\dot{v}'-\ddot{v}v'^2-\frac{3}{2}\dot{v}\,\dot{v}'\,v'\right) \\
&+M_a\,U_o\left(-2\dot{v}'+3\dot{u}'v'+4u'\dot{v}'+\frac{7}{2}\dot{v}'v'^2+2\dot{u}\,v''+\frac{3}{2}\dot{v}v'v''\right) \\
&+M_a\,U_o^{\,2}\left(v''-2u''v'-4u'v''-\frac{5}{2}v''\,v'^2\right)=0
\end{aligned}
\tag{2.81}
$$

A remarkable feature of the governing equations of motion is the absence of the fluid-frictional force of the internal flow. It should be recognized that these forces play no role in the dynamics of the system, since they were accounted by its twin effects; a tension on the pipe and a pressure drop of the fluid. More investigations of the nonlinear equations of motion are provided in Chapters 6 and 7.

2.7 Dissipative Forces Due to Material Damping

One of the simplest models used to simulate the damping of materials is the Kelven-Voigt model [34]. In this model, the relationship between stress and strain is written as

$$\sigma = E\,\varepsilon + E^{*}\frac{d\varepsilon}{dt} \tag{2.82}$$

where σ is the stress and ε is the strain, as defined in equation (2.20). The same model (i.e. Kelvin-Voigt) can also be used to represent dissipative forces in the material during bending of the pipe. Following the approach of Stoker [35], such dissipative force can be implemented directly into the equations of motion by performing the following substitution into the equation of motion

$$E \rightarrow E\left[1 + a\left(\frac{\partial}{\partial t}\right)\right] \tag{2.83}$$

where *a* is the coefficient of Kelvin-Voigt material damping.

By substituting equation (2.83) into the obtained equations of motion, the equations of motion can be written as follows:

$$\begin{aligned}
&(m+M)\ddot{u} + (2M\,U_i)\,\dot{u}' + \left(M\,U_i^{\,2}\right)u'' - (E\,A)u'' - (E\,A\,a)\,\dot{u}'' \\
&+\left(\overline{T} - A_f\,p_i(L) + A_o\,p_o(L) - E\,A\right)v'\,v'' - (E\,A\,a)(v'\,\dot{v}'' + v''\,\dot{v}') - E\,I\,(v'''\,v' + v''\,v''') \\
&- E\,I\,a\,(v'''\,\dot{v}' + v'\,\dot{v}''' + v''\,\dot{v}''' + v'''\,\dot{v}'') - M_a\,(\ddot{v}\,v') - M_a\,U_o\,(-2\,\dot{v}'\,v') - M_a\,U_o^{\,2}\,(v'\,v'') \\
&-\left(\frac{\rho_{f,o}\;D_o U_o^{\,2} C_T}{2}\frac{D_o}{D_h}\right)\left(-\frac{1}{2}v'^2 + u' + \left(\frac{L}{2} - X\right)(v'\,v'')\right) \\
&+\left[(1-2\nu)\,\overline{p}_o\,A_o\right](v'\,v'') - \left(\frac{\rho_{f,o}\;D_o U_o^{\,2} C_T}{2}\right)\left(\frac{3}{2}v'^2 - u' - \frac{v'\dot{v}}{U_o} - \frac{\dot{v}^2}{2U_o^{\,2}} - \left(\frac{L}{2} - X\right)(v'\,v'')\right) \\
&-\left(\frac{\rho_{f,o}\;D_o U_o^{\,2} C_N}{2}\right)\left(-v'^2 + \frac{\dot{v}\,v'}{U_o}\right) \\
&-\left(\frac{\rho_{f,o}\;D_o\,U_o\,C_{DP}}{2}\right)\left(-v'^2\,|v'| + \frac{|\dot{v}|\,v'^2 + |v'|\,v'\,\dot{v}}{U_o} + \frac{\dot{v}\,|\dot{v}|\,v'}{U_o^{\,2}}\right) = 0
\end{aligned} \tag{2.84}$$

$$
\begin{aligned}
&(m+M)\ddot{v}+(2M\,U_i)\,\dot{v}'+\left(M\,U_i^{\,2}\right)v''-\left(\overline{T}-A_f\;p_i(L)+A_o\;p_o(L)\right)v'' \\
&+(E\,I)v''''+(E\,I)\dot{v}''''-EI\begin{pmatrix}3u'''v''+4u''v'''+2u'v''''+v'u'''' \\ +2v'^2\,v''''+8v'v''v'''+2v''^3\end{pmatrix} \\
&-E\,I\,a\begin{pmatrix}3u'''\dot{v}''+3v''\dot{u}'''+4u''\dot{v}'''+4v'''\dot{u}''+2u'\dot{v}''''+2v''''\dot{u}' \\ +v'\dot{u}''''+u''''\dot{v}'+2v'^2\dot{v}''''+4\dot{v}'v''''+8v'v''\dot{v}''' \\ +8v'''\left(v'\dot{v}''+v''\dot{v}'\right)+6v''^2\dot{v}''\end{pmatrix} \\
&+\left(\overline{T}-A_f\;p_i(L)+A_o\;p_o(L)-EA\right)\left(u''v'+u'v''+\frac{3}{2}v'^2\,v''\right) \\
&-E\,A\,a\left(u''\dot{v}'+v'\dot{u}''+u'\dot{v}''+v''\dot{u}'+\frac{3}{2}v'^2\,\dot{v}''+3\dot{v}'v''\right) \\
&-\left(\frac{\rho_{f,o}\;\;D_oU_o^{\,2}C_T}{2}\frac{D_o}{D_h}\right)\begin{pmatrix}\left(v'-\frac{1}{2}v'^3+u\,v''\right) \\ -\left(\frac{L}{2}-X\right)\left(v''-u''v'-u'v''-\frac{3}{2}v'^2\,v''\right)\end{pmatrix} \\
&-\left[(1-2\nu)\,\overline{p}_o\;A_o\right]\left(v''-u''v'-u'v''-\frac{3}{2}v'^2\,v''\right) \\
&-\left(\frac{\rho_{f,o}\;\;D_o\,U_o^{\,2}C_T}{2}\right)\begin{pmatrix}\frac{1}{2}v'^3-uv''-u'v'-\frac{\dot{v}\,v'^2}{U_o}-\frac{\dot{v}^2\,v'}{2U_o^{\,2}} \\ +\left(\frac{L}{2}-X\right)\left(v''-u''v'-u'v''-\frac{3}{2}v'^2\,v''\right)\end{pmatrix} \\
&+\left(\frac{\rho_{f,o}\;\;D_o\,U_o^{\,2}C_N}{2}\right)\begin{pmatrix}-v'+\frac{\dot{v}}{U_o}+\frac{\dot{v}u'}{U_o}+u'v'+\frac{\dot{u}\,\dot{v}}{U_o^{\,2}} \\ +v'^3-\frac{\dot{v}\,v'^2}{U_o}-\frac{\dot{v}^3}{2U_o^{\,3}}-\frac{\dot{v}^2v'}{2U_o^{\,2}}\end{pmatrix} \\
&+\left(\frac{\rho_{f,o}\;\;D_o\,U_o^{\,2}C_{DP}}{2}\right)\left(-v'|v'|+\frac{|\dot{v}|\;v'+|v'|\;\dot{v}}{U_o}+\frac{\dot{v}|\dot{v}|}{U_o^{\,2}}\right) \\
&+M_a\left(\ddot{v}-\ddot{u}\,v'-2\dot{u}\,\dot{v}'-\ddot{v}v'^2-\frac{3}{2}\dot{v}\,\dot{v}'\,v'\right) \\
&+M_a\,U_o\left(-2\dot{v}'+3\dot{u}'v'+4u'\dot{v}'+\frac{7}{2}\dot{v}'v'^2+2\dot{u}\,v''+\frac{3}{2}\dot{v}v'v''\right) \\
&+M_a\,U_o^{\,2}\left(v''-2u''v'-4u'v''-\frac{5}{2}v''\,v'^2\right)=0
\end{aligned}
\tag{2.85}
$$

CHAPTER THREE

NONLINEAR DYNAMIC MODELING OF INEXTENSIBLE ROTATING FLEXIBLE PIPE CONVEYING FLUID AND SUBJECTED TO EXTERNAL AXIAL FLOW

In this chapter, the nonlinear equations of motion of a cantilevered flexible rotating pipe conveying fluid and subjected to external axial flow are derived using Lagrangian approach.

3.1 The Problem Statement

Consider the motion of a flexible pipe shown in figure 3.1.The model consists of rotating uniform tubular pipe of length L, cross sectional area A, mass per unit length m and flexural rigidity EI. The pipe is conveying fluid downwards. The fluid of mass per unit length M, flowing axially with velocity U_i, while the external fluid of density $\rho_{f,o}$ is flowing in reverse direction with velocity U_o in the annulus formed by the inner pipe and an outer rigid cylinder . The main assumptions underlying the formulation may include the following:

- The internal pipe is slender, and obeys Euler-Bernoulli theory.
- The material of the internal pipe is elastic, homogeneous and isotropic
- The outer cylinder is rigid.

- The internal and external fluids are Newtonian, incompressible and fully developed.
- The internal flow is fully developed turbulent and may be approximated by a plug flow, (i.e. as if it were an infinitely flexible rod traveling through the pipe, then all points of the fluid having the same velocity relative to the pipe).
- Steady flow conditions such that the mean velocities of internal and external flows are constants and free from pulsation.
- Fixed-free end conditions are assumed, and therefore; inextensibility condition is utilized.
- External flow is represented by the induced hydrodynamic forces.

3.2 Order of Magnitude Considerations

Although large deflections of the pipe are considered, only cubic nonlinear terms will be retained in the final equations of motion; thus, an order of magnitude analysis will be useful. The lateral displacements v, w and the angle of rotation φ are considered small relative to the length of the pipe, while the longitudinal displacement u is much smaller than v and w, therefore u should have higher order of magnitude [3]. Accordingly, one may state that:

$$v \sim \tilde{O}(\epsilon) \quad , w \sim \tilde{O}(\epsilon) \quad , \quad \varphi \sim \tilde{O}(\epsilon) \quad , \quad u \sim \tilde{O}(\epsilon^2) \tag{3.1}$$

where $(\epsilon << 1)$.

One should note that the various expressions of kinetic and potential energies should be kept at $\tilde{O}(\epsilon^4)$, since the variational technique always requires one order higher than the order of final equations.

3.3 Kinetic Energy Expressions

The kinetic energy of the above described pipe system is composed of the following:

- Kinetic Energy of the pipe due to translation, $T_{p,\,t}$.
- Kinetic Energy of the pipe due to rotation, $T_{p,\,r}$.
- Kinetic Energy of the internal fluid due to translation, $T_{f,\,t}$.
- Kinetic Energy of the internal fluid due to the pipe flexure , $T_{f,\,r}$.

3.3.1 Kinetic energy of the pipe due to translation

The deformed configurations of the pipe in both *X-Z* and *X-Y* planes are shown in figures 3.2 and 3.3, respectively. The Lagrangian coordinate is introduced here as (*X, Y, Z)* corresponding to the original equilibrium state of the pipe, (i.e. the material point p_o). The Eulerian coordinate system is denoted as (*x, y, z*) corresponding to the current configuration of the pipe, (i.e. the material point *p*). The displacements (*u, v, w*) are defined as follows:

$$u = x - X, \qquad v = y - Y, \qquad w = z - Z \tag{3.2}$$

The global position of the material point *p* after deformation can be written as

$$R_p = \left(X + u\,(z,t)\right) i + \left(v + Y\right) j + \left(w + Z\right) k \tag{3.3}$$

Thus

$$\dot{R}_p = \dot{u}\; i + \dot{v}\;\; j + \dot{w}\; k \tag{3.4}$$

where

u (x, t) : axial shortening due to bending.

w (x, t): Lateral deformation of the pipe in *X-Z* plane at a distance *x* measured from the fixed end of the pipe along the neutral axis in the undeformed configuration and at time *t*.

v (x, t): Lateral deformation of the pipe in *X-Y* plane at a distance *x* measured from the fixed end of the pipe along the neutral axis in the undeformed configuration and at time *t*.

By noting that $X = s$ under inextensibility condition, then the kinetic energy of the pipe due to translation can be written as

$$T_{p,\,t} = \frac{1}{2}\int_0^L m\left[\dot{w}^2 + \dot{v}^2 + \dot{u}^2\right] ds \tag{3.5}$$

where

m: Mass per unit length of the pipe.

s: Curvilinear coordinates measured at the centerline of he pipe.

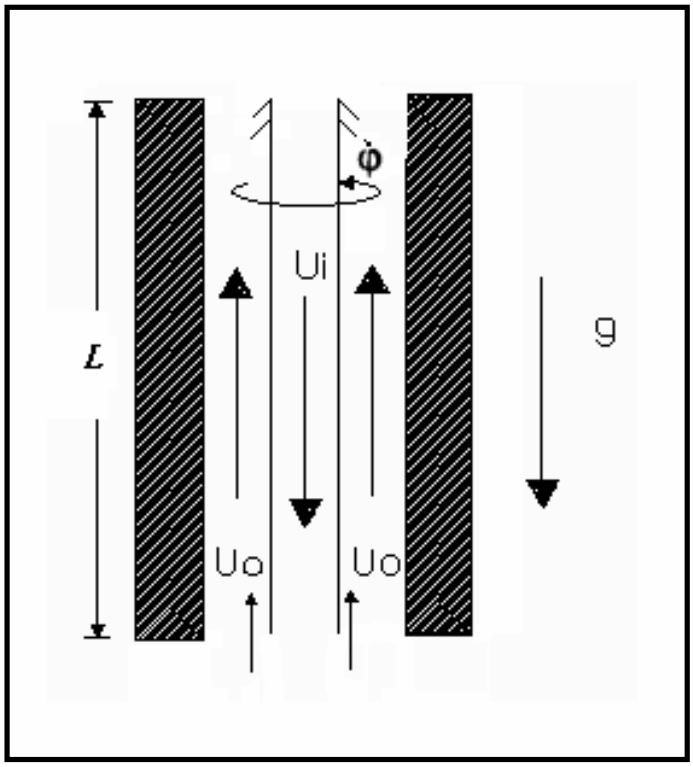


Figure 3.1: Cantilevered rotating pipe conveying fluid and subjected to external axial flow

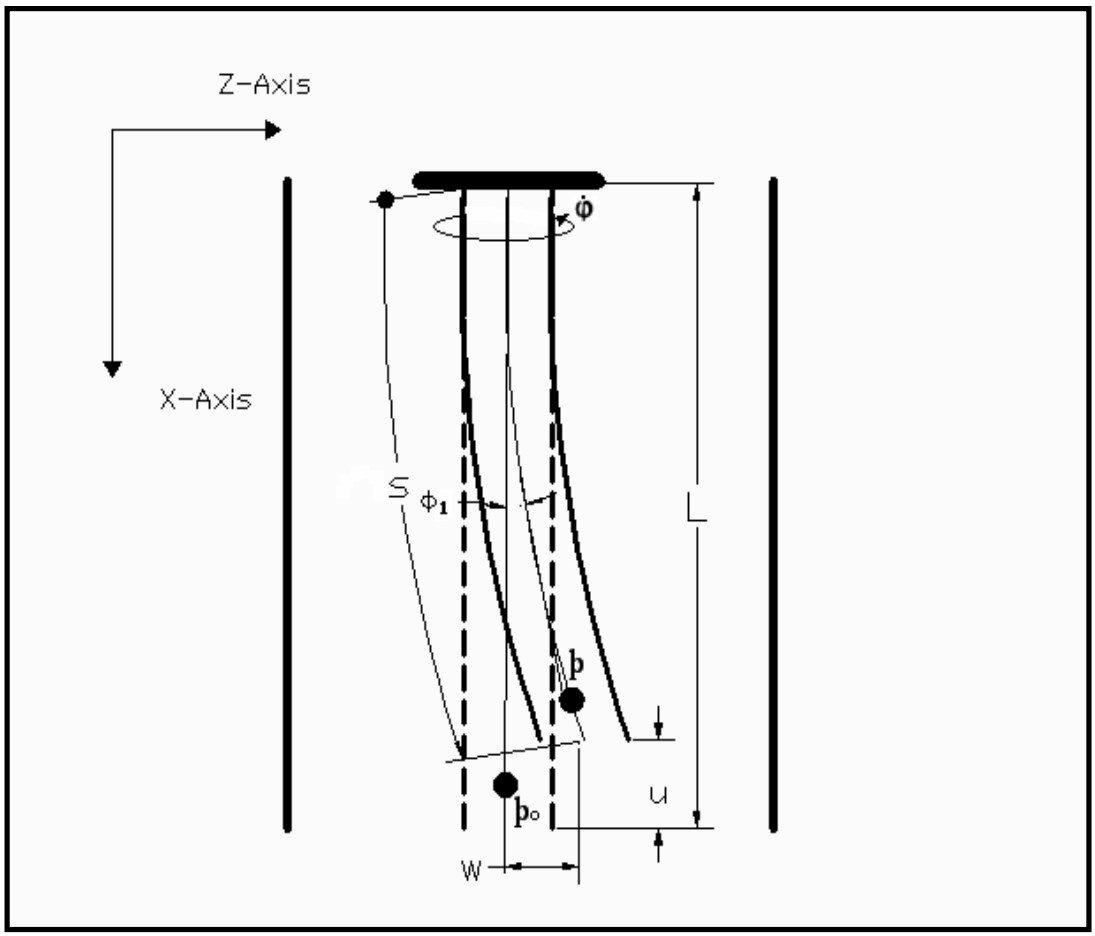


Figure 3.2: Deflected configuration of the pipe in *X-Z* plane.

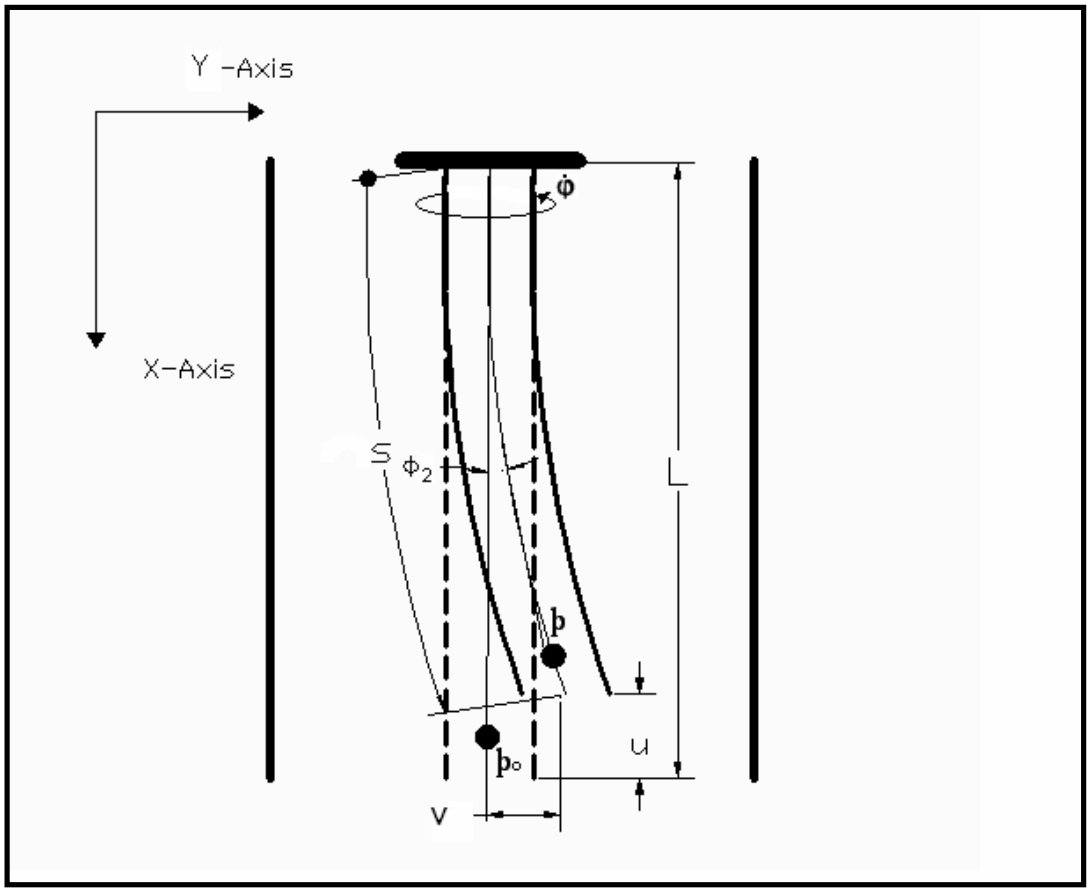


Figure 3.3: Deflected configuration of the pipe in *X-Y* plane.

3.3.2 Kinetic energy of the pipe due to rotation

The *x y z* coordinate system (corresponding to the Cartesian coordinate after deformation) is rotated with respect to *X Y Z* coordinate system (corresponding to Cartesian coordinate prior to deformation) through a set of angles as shown in figure 3.4. The general orientation of the pipe element cross-section can be described by first rotating it by an angle φ around the *X* axis, then an angle θ_z around the new *z* axis (z_1), and then by an angle θ_y around the final *y* axis (y_2). The instantaneous angular velocity vector ω of the *x y z* frame can be expressed as

$$\omega = \dot{\varphi}\ I + \dot{\theta}_z\ k_1 + \dot{\theta}_y\ j_2 \tag{3.6}$$

where I, k_1 and j_2 are unit vectors along X, z_1 and y_2 axes respectively.

Transforming equation (3.6) into *X Y Z* coordinate system and assuming θ_y and θ_z to be small angles, one gets

$$\omega = \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} = \begin{bmatrix} \dot{\varphi} - \dot{\theta}_y \theta_z \\ \dot{\theta}_y \cos(\varphi) + \dot{\theta}_z \sin(\varphi) \\ \dot{\theta}_z \cos(\varphi) - \dot{\theta}_y \sin(\varphi) \end{bmatrix} \tag{3.7}$$

Then kinetic energy expression can be written as follows

$$T_{p,r} = \frac{1}{2}\ \omega\ .\ H_C \tag{3.8}$$

where H_C is the angular momentum relative to the center of the mass and given as :

$$H_C = (I_{xx}\omega_x - I_{xy}\omega_y - I_{xz}\omega_z)i + (I_{yy}\omega_y - I_{xy}\omega_x - I_{yz}\omega_z)j + (I_{zz}\omega_z - I_{xz}\omega_y - I_{yz}\omega_y)k \tag{3.9}$$

Since the three coordinate planes are planes of symmetry of the pipe, then all products of inertia vanish. Therefore,

$$H_C = (I_{xx}\omega_x)i + (I_{yy}\omega_y)j + (I_{zz}\omega_z)k \tag{3.10}$$

where

I_{xx} : mass moment of inertia about x-axis.

I_{yy} : mass moment of inertia about y-axis.

I_{zz} : mass moment of inertia about z-axis.

Note that $I_{yy} = I_{zz}$ for the circular pipe.

By defining $(I_{yy} / L) = (I_{zz} / L) = I_D$ and $(I_{xx} / L) = I_p$, and substituting equation (3.10) into equation (3.8) one gets

$$T_{p,r} = \frac{1}{2} I_p \int_0^L \left[\dot{\varphi}^2 + \dot{\theta}_y{}^2 \theta_z{}^2 - 2\dot{\varphi}\dot{\theta}_y \theta_z\right] ds + \frac{1}{2} I_D \int_0^L \left[\dot{\theta}_y{}^2 + \dot{\theta}_z{}^2\right] ds \tag{3.11}$$

3.3.3 Kinetic energy of the internal flow due to translation

The basic assumption that will be used in estimating the kinetic energy is that the internal flow is approximated by a plug flow, i.e. as if it were an infinitely flexible rod traveling through the pipe, while all points of the fluid have velocity U_i relative to the pipe. This is an acceptable approximation for a fully developed turbulent flow [34], which implies

$$V_f = V_p + U_i \tau \tag{3.12}$$

where

V_f: translational velocity of the fluid.

V_p: translational velocity of the pipe.

U_i : average velocity of the internal flow.

τ : unit vector tangential to the pipe.

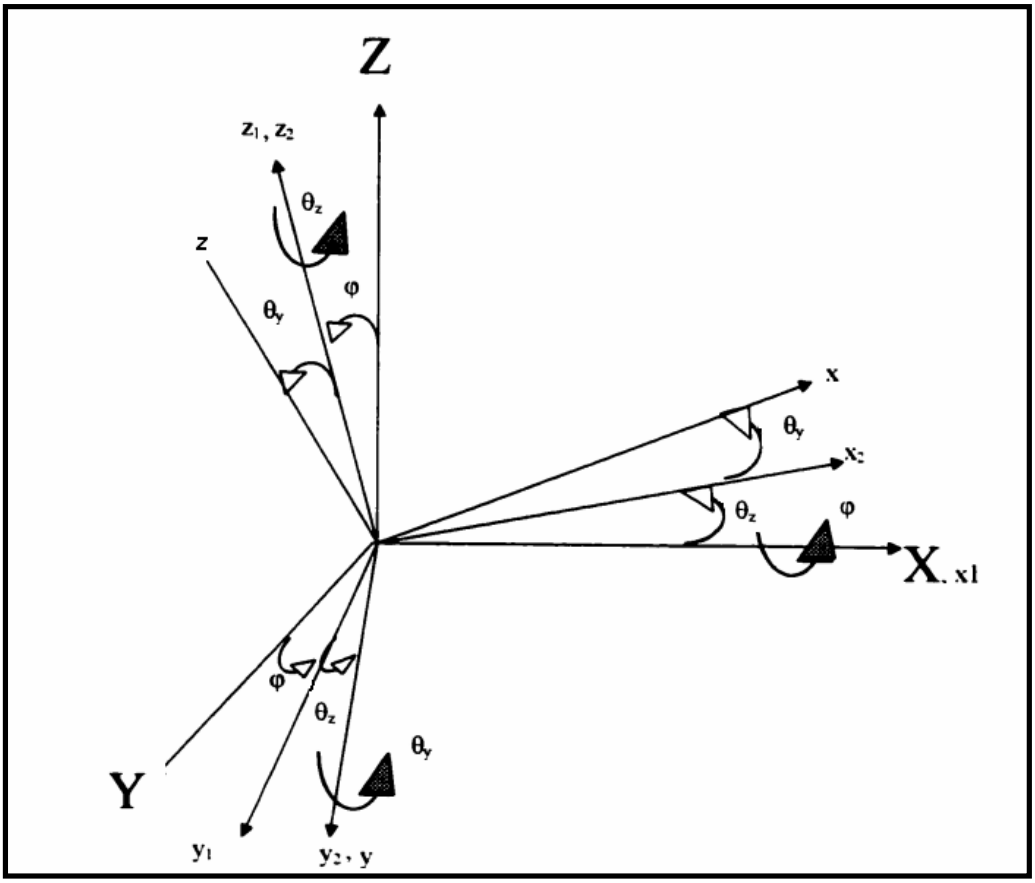


Figure 3.4: Rotation angles.

The unit vector τ tangential to the pipe can be defined as

$$\tau = \frac{\partial x}{\partial s} i + \frac{\partial y}{\partial s} j + \frac{\partial z}{\partial s} k = \left(1 + \frac{\partial u}{\partial s}\right) i + \frac{\partial v}{\partial s} j + \frac{\partial w}{\partial s} k \tag{3.13}$$

By substituting equations (3.4) and (3.13) into equation (3.12), the kinetic energy of the internal flow due translation can be expressed as

$$\begin{aligned} T_{f,t} = & \frac{1}{2}\int_0^L M\left[\dot{w}^2 + \dot{v}^2 + \dot{u}^2\right] ds + U_i \int_0^L M\left[\dot{u} + \dot{u}u' + \dot{v}v' + \dot{w}w'\right] ds \\ & + \frac{1}{2}\int_0^L M U_i^2 \left(1 + u'^2 + v'^2 + w'^2 + 2u'\right) ds \end{aligned} \tag{3.14}$$

where M is the mass of internal fluid per unit length of the pipe and prime ($'$) denotes the derivative with respect to the curvilinear coordinate s.

The associated inextensibility condition [8] can be stated as

$$\left(\frac{\partial x}{\partial s}\right)^2 + \left(\frac{\partial y}{\partial s}\right)^2 + \left(\frac{\partial z}{\partial s}\right)^2 = \left(1 + \frac{\partial u}{\partial s}\right)^2 + \left(\frac{\partial v}{\partial s}\right)^2 + \left(\frac{\partial w}{\partial s}\right)^2 = 1 \tag{3.15}$$

Equation (3.14) can be simplified using the inextensible condition as

$$\begin{aligned} T_{f,t} = & \frac{1}{2}\int_0^L M\left[\dot{w}^2 + \dot{v}^2 + \dot{u}^2\right] ds + U_i \int_0^L M\left[\dot{u} + \dot{u}u' + \dot{v}v' + \dot{w}w'\right] ds \\ & + \frac{1}{2}\int_0^L M U_i^2 . ds \end{aligned} \tag{3.16}$$

3.3.4 Kinetic energy of the internal flow due to the pipe flexure

The kinetic energy of the internal fluid due to rotation is limited to those rotations about y and z axis [39], which are associated with the pipe motion in lateral directions, while the effect of fluid friction forces induced by the inner wall of the pipe

due to its rotation will be taken into account in the potential energy expression. Accordingly, one may express the kinetic energy of the fluid due to rotation as follows:

$$T_{f,r} = \frac{1}{2} I_{D,f} \int_0^L \left[\dot{\theta}_y^{\,2} + \dot{\theta}_z^{\,2}\right] ds \tag{3.17}$$

where $I_{D,f}$ is mass moment of inertia of the fluid per unit length about y and z axis.

3.3.5 Total Kinetic Energy Expression

The total kinetic energy expression is the summation of the kinetic energies of the pipe and fluid due to the translation and rotation. Combining equations (3.5), (3.11), (3.16) and (3.17), we obtain

$$\begin{aligned} T = & \frac{1}{2}(m+M)\int_0^L \left[\dot{w}^2 + \dot{v}^2 + \dot{u}^2\right] ds + U_i \int_0^L M\left[\dot{u} + \dot{u}\,u' + \dot{v}\,v' + \dot{w}\,w'\right] ds \\ & + \frac{1}{2}\int_0^L M\,U_i^{\,2}.ds + \frac{1}{2} I_p \int_0^L \left[\dot{\varphi}^2 + \dot{\theta}_y^{\,2}\theta_z^{\,2} - 2\dot{\varphi}\dot{\theta}_y\theta_z\right] ds \\ & + \frac{1}{2}\left(I_D + I_{D,f}\right)\int_0^L \left[\dot{\theta}_y^{\,2} + \dot{\theta}_z^{\,2}\right] ds \end{aligned} \tag{3.18}$$

By recognizing that bending deformations are related to rotations as

$$\theta_z = \frac{\partial v(x,t)}{\partial s} \tag{3.19}$$

$$\theta_y = \frac{-\partial w(x,t)}{\partial s} \tag{3.20}$$

Substituting equations (3.19-3.20) into equation (3.18), it is found that

$$
\begin{aligned}
T = & \frac{1}{2}\,(m+M)\int_0^L \left[\dot{w}^2+\dot{v}^2+\dot{u}^2\right]ds + U_i\int_0^L M\left[\dot{u}+\dot{u}\,u'+\dot{v}\,v'+\dot{w}\,w'\right]ds \\
& + \frac{1}{2}\int_0^L M\,U_i^{\,2}.ds \\
& + \frac{1}{2}\left(I_p\right)\int_0^L \left[\begin{array}{l} \dot{\varphi}^2 + \left(\dfrac{-\partial^2 w(x,t)}{\partial s\,\partial t}\right)^2\left(\dfrac{\partial v(x,t)}{\partial s}\right)^2 \\ +2\dot{\varphi}\left(\dfrac{\partial^2 w(x,t)}{\partial s\,\partial t}\dfrac{\partial v(x,t)}{\partial s}\right) \end{array}\right]ds \\
& + \frac{1}{2}\left(I_D+I_{D,f}\right)\int_0^L \left[\left(\frac{-\partial^2 w(x,t)}{\partial s\,\partial t}\right)^2+\left(\frac{\partial v(x,t)}{\partial s\,\partial t}\right)^2\right]ds
\end{aligned}
\tag{3.21}
$$

Equation (3.21) can be written in a more compact form such

$$
\begin{aligned}
T = & \frac{1}{2}\,(m+M)\int_0^L \left[\dot{w}^2+\dot{v}^2+\dot{u}^2\right]ds + U_i\int_0^L M\left[\dot{u}+\dot{u}\,u'+\dot{v}\,v'+\dot{w}\,w'\right]ds \\
& + \frac{1}{2}\int_0^L M\,U_i^{\,2}.ds \\
& + \frac{1}{2}\left(I_p\right)\int_0^L \left[\dot{\varphi}^2 + \dot{w}'^2 v'^2 + 2\dot{\varphi}\,\dot{w}'v'\right]ds \\
& + \frac{1}{2}\left(I_D+I_{D,f}\right)\int_0^L \left[\dot{w}'^2+\dot{v}'^2\right]ds
\end{aligned}
\tag{3.22}
$$

where prime $(')$ denotes the derivative with respect to the curvilinear coordinate s.

3.4 Potential Energy Expression

The system potential energy is made up of the elastic pipe-bending strain energy V_b, strain energy due to torsion V_t and strain energy due to gravity V_g. One should recognize that according to conventional inextensible theory, the centerline of the pipe is assumed to be unstretched, and the steady flow-induced forces due to pressure and tension forces

are entirely neglected [34]. On the other hand, the steady flow-related forces should be taken into account in the case of extensible condition.

3.4.1 Strain energy due to bending

Similar to the analysis presented in chapter two, and by considering the assumed inextensible condition, the strain energy due to bending can be written as

$$V_b = \frac{EI}{2}\int_0^L \kappa^2(s,t).ds \tag{3.23}$$

where

$\kappa(s)$: Curvature of the centerline of the pipe at s .

I : Area moment of inertia of the pipe.

Note that, for the three-dimensional analysis, it is important to consider the bending in both *X-Y* and *X-Z* planes.

Following the analysis presented in [40], the curvature may expressed as

$$\kappa(s,t) = \frac{\phi_1'}{L} + \frac{\phi_2'}{L} \tag{3.24}$$

Referring to figure 3.2, one notes that

$$\sin\ \phi_1 = \frac{w'}{L} \quad , \quad \sin\ \phi_2 = \frac{v'}{L} \tag{3.25}$$

where ϕ_1 and ϕ_2 are the elastic slopes at a position s in the *X-Z* and *X-Y* planes respectively, and the prime denotes a derivative with respect to the arc length s. By differentiating equation (3.25), and using the following trigonometric relations

$$\cos\phi_1 = \sqrt{1-\sin^2\phi_1} = \sqrt{1-\left(\frac{w'}{L}\right)^2} \;,\; \cos\phi_2 = \sqrt{1-\sin^2\phi_2} = \sqrt{1-\left(\frac{v'}{L}\right)^2} \tag{3.26}$$

Then, one gets

$$\phi_1{}' = \frac{w''}{L}\left[1-\left(\frac{w'}{L}\right)^2\right]^{-1/2} \;,\; \phi_2{}' = \frac{v''}{L}\left[1-\left(\frac{v'}{L}\right)^2\right]^{-1/2} \tag{3.27}$$

Upon substituting equation (3.27) into equation (3.24), noting that $\frac{w'}{L}$ *and* $\frac{v'}{L}<1$ and expanding the terms $\left[1-\left(\frac{w'}{L}\right)^2\right]^{-1/2}$ and $\left[1-\left(\frac{v'}{L}\right)^2\right]^{-1/2}$ in the resulting equation into power series , while retaining terms up to fourth order and substituting the result into equation (3.23), yields the following expression for the pipe bending potential energy

$$V_b = \frac{EI}{2}\int_0^L \left[v''^2 + \left(v'v''\right)^2 + w''^2 + \left(w'w''\right)^2 + 2v'v''w'w''\right]ds \tag{3.28}$$

3.4.2 Strain energy due to torsion

The strain energy V_t due to torsion is given by

$$V_t = \frac{G\ J}{2}\int_0^L \varphi'^2\, ds \tag{3.29}$$

where

G: Modulus of rigidity for the pipe.

J: polar moment of inertia for the pipe.

Other component of strain energy due to torsion should be considered as associated with the frictional torque Γ. By considering such component, then one can write strain energy as

$$V_t = \frac{G\ J}{2}\int_0^L \varphi'^2\, ds - \int_0^L \Gamma\, \varphi'\, ds + \int_0^L \frac{\Gamma^2}{2\,G\,J} ds \tag{3.30}$$

where Γ is the torque exerted on the pipe due to the fluid-induced frictional forces of the rotating pipe.

3.4.3 Strain energy due to gravitational field

The gravitational energy depends on the distribution of mass, and can be expressed in x direction as

$$V_g = -(m+M)\, g \int_0^L x\, dx \tag{3.31}$$

where g is the gravitational acceleration.

3.4.4 Total potential energy expression

The total potential energy expression based on the conventional inextensibility condition is the summation of the potential energy due to bending, torsion, and gravitational potential energy. Using equations (3.28- 3.31), the total potential energy can be expressed as

$$\begin{aligned} V\ &= \frac{EI}{2}\int_0^L \left[v''^2 + \left(v'\,v''\right)^2 + w''^2 + \left(w'\,w''\right)^2 + 2v'\,v''w'\,w''\right] ds \\ &+ \frac{G\ J}{2}\int_0^L \varphi'^2\, ds - \int_0^L \Gamma\, \varphi'\, ds + \int_0^L \frac{\Gamma^2}{2GJ} ds - (m+M)\, g \int_0^L x\, dx \end{aligned} \tag{3.32}$$

If the modified inextensible condition is used, then one should consider the steady state forces associated with the differential change between internal and external pressures [34]. For the modified inextensible theory, equation (3.32) can be rewritten as

$$
\begin{aligned}
V &= \frac{EI}{2}\int_0^L \left[v''^2 + \left(v'v''\right)^2 + w''^2 + \left(w'w''\right)^2 + 2v'v''w'w''\right]ds \\
&+ \frac{GJ}{2}\int_0^L \varphi'^2\, ds - \int_0^L \Gamma\varphi'\, ds + \int_0^L \frac{\Gamma^2}{2GJ}\, ds - (m+M)\, g\int_0^L x\, dx \\
&- \int_0^L P\left(u' + \frac{1}{2}v'^2 - \frac{u'v'^2}{2} - \frac{1}{8}v'^4\right)ds \\
&- \int_0^L P\left(u' + \frac{1}{2}w'^2 - \frac{u'w'^2}{2} - \frac{1}{8}w^4\right)ds \\
&+ \int_0^L \left[\frac{P^2}{EA}\right]ds
\end{aligned}
\tag{3.33}
$$

where P is the force due to pressure difference between internal and external flow, as given by equation (2.72). The analysis performed hereafter is based on the conventional inextensible theory, while the issue of pressure difference will be treated separately in case of fixed-pinned end conditions.

3.5 The Lagrangian Function

The Lagrangian function $\widetilde{L}$ is defined as

$$\widetilde{L} = T - V \tag{3.34}$$

where

T: Total kinetic energy of the system.

V: Total potential energy of the system

Using equations (3.22) and (3.32), then the Lagrangian function can be rewritten as

$$
\begin{aligned}
\widetilde{L} = & \frac{1}{2}(m+M)\int_0^L \left[\dot{w}^2+\dot{v}^2+\dot{u}^2\right]ds + U_i\int_0^L M\left[\dot{u}+\dot{u}\,u'+\dot{v}\,v'+\dot{w}\,w'\right]ds \\
& + \frac{1}{2}\int_0^L M\,U_i^{\,2}\,ds + \frac{1}{2}\left(I_p\right)\int_0^L\left[\dot{\varphi}^2+\dot{w}'^2 v'^2+2\dot{\varphi}\,\dot{w}'v'\right]ds \\
& + \frac{1}{2}\left(I_D+I_{D,f}\right)\int_0^L\left[\dot{w}'^2+\dot{v}'^2\right]ds \\
& - \frac{EI}{2}\int_0^L\left[v''^2+\left(v'v''\right)^2+w''^2+\left(w'w''\right)^2+2v'v''w'w''\right]ds \\
& - \frac{G\,J}{2}\int_0^L \varphi'^2\,ds + \int_0^L \Gamma\,\varphi'\,ds - \int_0^L \frac{\Gamma^2}{2GJ}\,ds \\
& + (m+M)\,g\int_0^L x\,dx
\end{aligned}
\tag{3.35}
$$

3.6 The Inextensibility Condition

The dependence of the kinetic energy on the axial shortening u and axial velocity $\dot{u}$ can be eliminated by assuming that the pipe is undergoing inextensible 3-dimentional bending motion. The axial shortening u corresponding to the lateral deflections w and v of this inextensible bending motion is given by [40] as

$$
u = -\frac{1}{2}\int_0^s\left[w'^2+\frac{1}{4}w'^4+v'^2+\frac{1}{4}v'^4\right]d\eta \tag{3.36}
$$

Differentiating equation (3.36) with respect to time, leads to the following expression for axial velocity $\dot{u}$ in terms of bending transverse deflections

$$
\dot{u} = -\frac{1}{2}\frac{d}{dt}\left[\int_0^s\left[w'^2+\frac{1}{4}w'^4+v'^2+\frac{1}{4}v'^4\right]d\eta\right] \tag{3.37}
$$

Now, equations (3.36) and (3.37) can be used to eliminate the axial shortening term from the governing equations of motion.

3.7 Hydrodynamic Forces Exerted on the Pipe due to the Axial Motion of the External Flow

The hydrodynamic forces acting on the pipe may be summarized as follow:

- The inviscid hydrodynamic force, F_A .
- Normal frictional force F_N.
- Longitudinal frictional forces, F_L.
- Hydrostatic pressure forces in the x, y and z directions, F_{px}, F_{py} and F_{pz} respectively.

3.7.1 Inviscid hydrodynamic forces

The inviscid hydrodynamic forces are evaluated using Lighthill's model of slender body theory [36]. The derivation obtained previously in section 2.6.1 is used here, but the current analysis is extended in *X-Z* plane.

The nonlinear expression for the inviscid hydrodynamic force can be expressed as

$$\begin{aligned}
F_A = {} & M_a\left(\ddot{v} - \ddot{u}v' - 2\dot{u}\dot{v}' - \frac{1}{2}\ddot{v}v'^2 - \frac{3}{2}\dot{v}\dot{v}'v'\right) \\
& - M_a U_o\left(2\dot{v}' - 3\dot{u}'v' - 4u'\dot{v}' - \frac{5}{2}\dot{v}'v'^2 - 2\dot{u}v'' - \frac{3}{2}\dot{v}v'v''\right) \\
& + M_a U_o^2\left(v'' - 2u''v' - 4u'v'' - 2v'^2v''\right) \\
& + M_a\left(\ddot{w} - \ddot{u}w' - 2\dot{u}\dot{w}' - \frac{1}{2}\ddot{w}w'^2 - \frac{3}{2}\dot{w}\dot{w}'w'\right) \\
& - M_a U_o\left(2\dot{w}' - 3\dot{u}'w' - 4u'\dot{w}' - \frac{5}{2}\dot{w}'w'^2 - 2\dot{u}w'' - \frac{3}{2}\dot{w}w'w''\right) \\
& + M_a U_o^2\left(w'' - 2u''w' - 4u'w'' - 2w'^2w''\right)
\end{aligned} \tag{3.38}$$

where M_a is defined by equation (2.36).

3.7.2 Hydrodynamic frictional forces

Based on the analysis presented in section 2.6.2, the normal and longitudinal frictional forces can be expressed, respectively, as

$$F_N = \frac{1}{2}\rho_{f,o} D_o U_o^{\,2} \left[\begin{array}{l} C_N \begin{pmatrix} -v' + \dfrac{\dot{v}}{U_o} + \dfrac{\dot{v}u'}{U_o} + u'v' + \dfrac{\dot{u}\dot{v}}{U_o^{\,2}} \\ -w' + \dfrac{\dot{w}}{U_o} + \dfrac{\dot{w}u'}{U_o} + u'w' + \dfrac{\dot{u}\dot{w}}{U_o^{\,2}} \end{pmatrix} \\ -\dfrac{C_N}{2} \begin{pmatrix} -v'^3 + \dfrac{\dot{v}^3}{U_o^{\,3}} + \dfrac{v'^2\dot{v}}{U_o} + \dfrac{v'\dot{v}^2}{U_o^{\,2}} \\ -w'^3 + \dfrac{\dot{w}^3}{U_o^{\,3}} + \dfrac{w'^2\dot{w}}{U_o} + \dfrac{w'\dot{w}^2}{U_o^{\,2}} \end{pmatrix} \\ +C_{DP} \begin{pmatrix} -v'|v'| + \dfrac{v'|\dot{v}| + |v'|\dot{v}}{U_o} + \dfrac{\dot{v}|\dot{v}|}{U_o^{\,2}} \\ -w'|w'| + \dfrac{w'|\dot{w}| + |w'|\dot{w}}{U_o} + \dfrac{\dot{w}|\dot{w}|}{U_o^{\,2}} \end{pmatrix} \end{array} \right] \tag{3.39}$$

$$F_L = \frac{1}{2}\rho_{f,o} D_o U_o^{\,2} C_T \left(1 - \frac{1}{2}\left(-v'^2 + \frac{2v'\dot{v}}{U_o} + \frac{\dot{v}^2}{U_o^{\,2}} \right) - \frac{1}{2}\left(-w'^2 + \frac{2w'\dot{w}}{U_o} + \frac{\dot{w}^2}{U_o^{\,2}} \right) \right) \tag{3.40}$$

3.7.3 The hydrostatic pressure forces

Based on the analysis presented in section 2.6.3, the hydrostatic pressure forces in *x*, *y* and *z* directions can be expressed, respectively, as

$$-F_{px} = \left(-\frac{1}{2}v'^2 - \frac{1}{2}w'^2 + u' \right)\left(\frac{1}{2}\rho_{f,o} D_o U_o^{\,2} C_T \frac{D_o}{D_h} + \rho_{f.o}\, g\, A_o \right) - \left(v'v'' + w'w'' \right) p_o A_o \tag{3.41}$$

$$\begin{aligned} F_{py} = & \left(-\frac{1}{2}v'^3 + v' \right)\left(\frac{1}{2}\rho_{f,o} D_o U_o^{\,2} C_T \frac{D_o}{D_h} + \rho_{f.o}\, g\, A_o \right) \\ & + p_o A_o \left(v'' - u''v' - u'v'' - \frac{3}{2}v'^2 v'' \right) \end{aligned} \tag{3.42}$$

$$F_{pz} = \left(-\frac{1}{2} w'^3 + w' \right) \left(\frac{1}{2} \rho_{f,o} D_o U_o^{\,2} C_T \frac{D_o}{D_h} + \rho_{f.o} g A_o \right) + p_o A_o \left(w'' - u''w' - u'w'' - \frac{3}{2} w'^2 w'' \right) \tag{3.43}$$

Also one should consider that

$$A_o p_o(s) = A_o p_o(L) - \left(\frac{1}{2} \rho_{f,o} D_o U_o^{\,2} C_T \frac{D_o}{D_h} + \rho_{f.o} g A_o \right) (L - s - u) \tag{3.44}$$

3.8 Hydrodynamic Forces Exerted on the Pipe due to the Pipe Rotation.

In this section, the hydrodynamic forces exerted by the external flow on the pipe due to the pipe rotation are investigated.

3.8.1 System description and modeling assumptions

Figure 3.5 shows the geometry of a rotating pipe in a fluid annulus [41]. The following simplifying assumptions in deriving the equations of motion will be used:

- The flow modeled as being two-dimensional in *Y-Z* plane.
- The external flow is assumed to be incompressible and fully developed turbulent flow with Reynold's Number *Re* larger than 5000, [41].
- The flow is assumed to be free of vortices [23].
- The radial gradients in the velocity and pressure fields are neglected.
- The viscosity is constant throughout the film.
- There is no slip at the wall (i.e.; between the fluid-solid boundaries).
- The values of the annular clearance C to the rotor radius R, and the maximum deflection of the rotating pipe center δ to the annular clearance are within moderate values (up to $C/R \sim 0.3$ and $\delta/C \sim 0.2$). Noting that for large clearance values, the effect of the hydrodynamic forces becomes less significant.

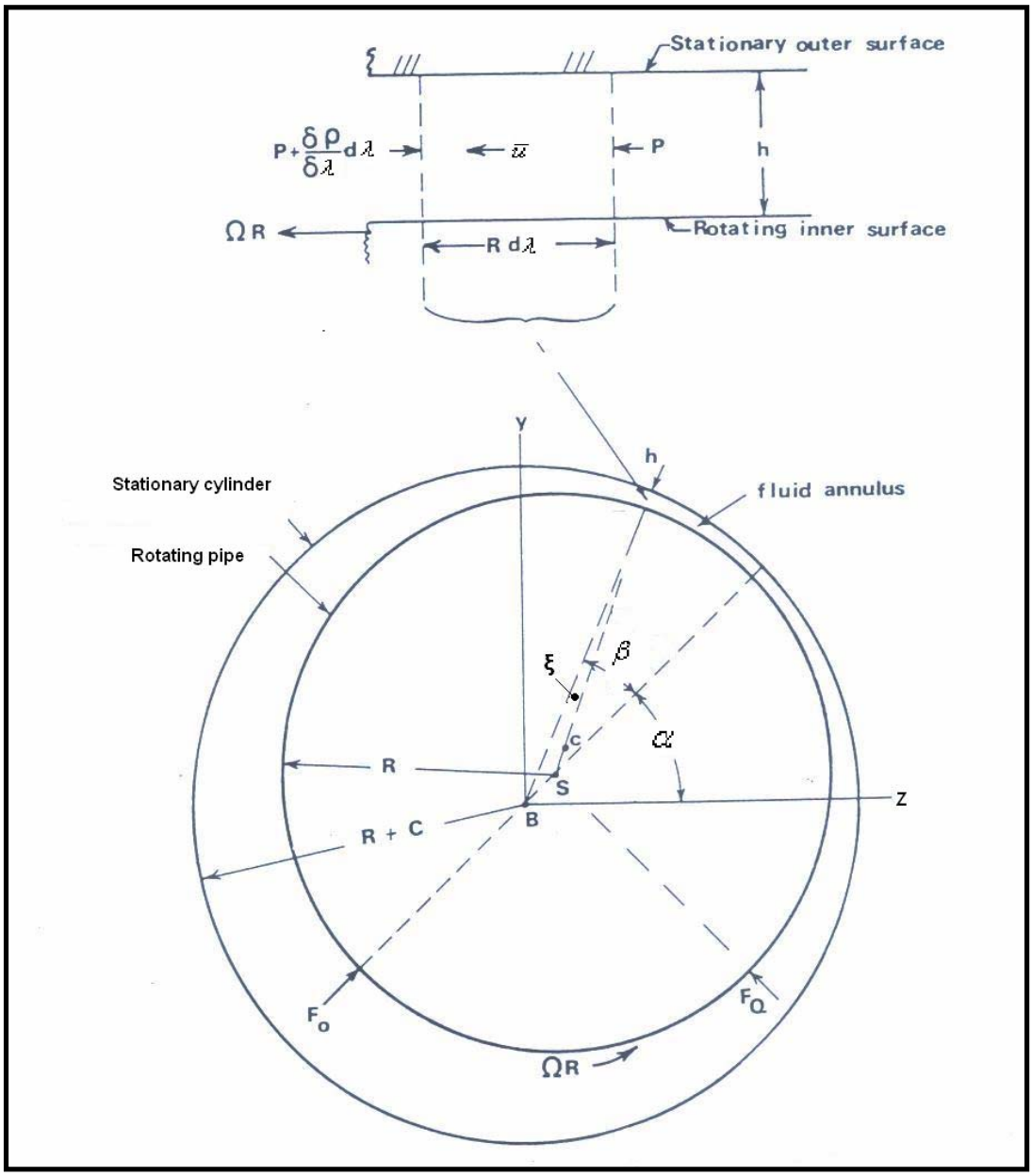


Figure 3.5: Geometry of a rotating pipe in a fluid annulus, [41].

3.8.2 The hydrodynamic mass

Due to accelerated rotation of pipe inside the stationary outer cylinder, pressure will be developed causing fluid force on the inner pipe as

$$F_H = -\left(\frac{r_{ch}^{\ 2} + r_o^{\ 2}}{r_{ch}^{\ 2} - r_o^{\ 2}}\right)\pi\rho_{f,o}\ L r_o^{\ 2}\ddot{Z} \tag{3.45}$$

Also the hydrodynamics force associated with the hydrodynamic mass is given by

$$F_H = -m_H\ \ddot{Z} \tag{3.46}$$

where,

F_H: hydrostatic force

r_{ch}: inside radius of outer stationary cylinder.

r_o: outside radius of inner rotating pipe.

C: radial clearance, such $C = r_{ch} - r_o$.

L: length of pipe.

$\rho_{f,o}$: fluid density of the external flow

Utilizing equations (3.45) and (3.46), one can write

$$\frac{m_H}{\pi\rho_{f,o}\ L} = \left(\frac{(r_{ch} + r_o)^2 - 2r_{ch}\ r_o}{C(r_o + r_{ch})}\right) r_o^{\ 2} \tag{3.47}$$

By considering that the radial clearance C is small compared to the pipe radius R, and considering small thickness of the pipe, then the hydrodynamic mass can be defined as

$$m_H = \frac{\pi R^3 \rho_{f,o} L}{C} \tag{3.48}$$

It is noted that this definition of m_H agrees with that presented by Antunes et al. [23] for moderate values of the gap between the pipe and the outer cylinder, i.e. C/R ≤ 0.1. It is

important to recognize that the definition of m_H presented in equation (3.48) is equivalent as definition to the integral of equation (2.36) over the length of the pipe for C/R ≤ 0.1. On the other hand, equation (2.36) is more accurate because it accounts of the pipe thickness and accommodates larger flow confinement. Therefore, equation (2.36) is used hereafter, such that $(m_H/L) = M_a$.

3.8.3 Minimum film thickness

Annuls thickness and fluid velocity can be evaluated by applying the continuity equation. Referring to figure 3.5, we can define the following parameters:

h: minimum film thickness.

α : location of minimum film thickness h.

β: angular coordinate of annular position which rotates with the pipe, and measured from the location of minimum film thickness h.

λ: angular coordinate of annular position, such that $\lambda = \alpha + \beta$.

δ : maximum deflection of the center of the rotating pipe.

Referring to figure 3.5, the annulus thickness h can be defined as

$$h = C(1 - \Delta \cos \beta) \tag{3.49}$$

where Δ is the ratio of the maximum deflection to the annular clearance , i.e. $\Delta = \dfrac{\delta}{C}$.

3.8.4 The continuity equation

The three dimensional continuity equation in the Cartesian coordinates is defined as

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \tag{3.50}$$

Let us consider the continuum with respect to h in the Z-direction, and then equation (3.50) may be rewritten as:

$$\frac{\partial h}{\partial t}+\frac{\partial(h\bar{u})}{\partial z}=0 \tag{3.51}$$

Upon using polar transformation, and equating equation (3.49) to equation (3.51), one obtains the following nonlinear equation

$$\frac{\partial h}{\partial t}=\dot{h}=-C\dot{\Delta}\cos\beta-C\Delta\dot{\alpha}\sin\beta=-\frac{1}{R}\frac{\partial(h\bar{u})}{\partial\lambda} \tag{3.52}$$

By integrating equation (3.52) with respect to λ, the product of the annulus thickness and the fluid velocity $\bar{u}$ can be expressed as

$$h\bar{u}=RC\dot{\Delta}\sin\beta-RC\Delta\dot{\alpha}\cos\beta+m(t) \tag{3.53}$$

where *m(t)* is the constant of integration.

By applying boundary conditions (at $h=C\Rightarrow\delta=0$ *and* $\Delta=\dot{\Delta}=0$) to equation (3.53), which can be rewritten as

$$\bar{u}=\frac{\Omega RC}{2h}+\frac{RC\dot{\Delta}\sin\beta}{h}-\frac{RC\Delta\dot{\alpha}\cos\beta}{h} \tag{3.54}$$

where Ω denotes the angular speed. By substituting equation (3.49) into equation (3.54), one may write

$$\bar{u}=\frac{\Omega R}{2(1-\Delta\cos\beta)}+\frac{R\dot{\Delta}\sin\beta}{1-\Delta\cos\beta}-\frac{R\Delta\dot{\alpha}\cos\beta}{1-\Delta\cos\beta} \tag{3.55}$$

In the analysis that follows, Δ is treated as a small quantity ($\Delta<<1$). Multiplying equation (3.55) by $\frac{1+\Delta\cos\beta}{1+\Delta\cos\beta}$, dropping nonlinear terms such that $\Delta^2\cong 0$ *and* $\Delta\dot{\Delta}=0$, and after some mathematical manipulations one obtains

$$\bar{u}=\frac{\Omega R}{2}(1+\Delta\cos\beta)+R\dot{\Delta}\sin\beta-R\Delta\dot{\alpha}\cos\beta \tag{3.56}$$

By differentiating equation (3.56) noting that $\dot{\beta} = -\dot{\alpha}$, one may finds that

$$\bar{\dot{u}} = \frac{\Omega R}{2}(\Delta\dot{\alpha}\sin\beta + \dot{\Delta}\cos\beta) + R\ddot{\Delta}\sin\beta - 2R\dot{\Delta}\dot{\alpha}\cos\beta - \left(R\Delta\left(\dot{\alpha}^2\sin\beta + \ddot{\alpha}\cos\beta\right)\right) \tag{3.57}$$

3.8.5 Conservation of linear momentum

Applying the general equation of conservation of linear momentum in Z direction, the pressure distribution can be expressed as

$$\frac{1}{\rho_{f,o}R}\frac{\partial\widetilde{P}}{\partial\lambda} - b_z = -\frac{\bar{u}}{R}\frac{\partial\bar{u}}{\partial\lambda} - \frac{\partial\bar{u}}{\partial t} \tag{3.58}$$

where b_z is the body forces.

By considering that the effective forces F_{eff} relates proportionally to the velocity component due to pressure difference $\bar{u}_p$, we may write

$$F_{eff} = -b_z = f_t\,\bar{u}_p = \frac{\Omega R\ f_t}{2}\Delta\cos\beta + R\dot{\Delta}f_t\sin\beta - R\Delta\dot{\alpha}\,f_t\cos\beta \tag{3.59}$$

where f_t represents the friction factor, which is defined for fully developed turbulent flow as, [41]

$$f_t = 0.0556\frac{\Omega R\eta}{C\,\mathrm{Re}} \tag{3.60}$$

where $\bar{\eta}$ represents the profile parameter, which is given by [40] as $\bar{\eta} = 1.14$.

Now, differentiating $\bar{u}$ with respect to λ, and substituting into equation (3.58), one finds after few mathematical manipulation including neglecting high order terms, that

$$\frac{1}{\rho_{f,o}R}\frac{\partial\widetilde{P}}{\partial\lambda} = \left\{\begin{array}{l}\cos\beta\left(-\Omega R\dot{\Delta} + 2R\dot{\Delta}\dot{\alpha} + R\Delta\ddot{\alpha} - \frac{\Omega}{2}R\Delta f_t + R\Delta\dot{\alpha}\,f_t\right)\\ +\sin\beta\left(-\Omega R\,\Delta\dot{\alpha} - R\ddot{\Delta} + R\Delta\dot{\alpha}^2 + \frac{\Omega^2 R\Delta}{4} - R\dot{\Delta}\,f_t\right)\end{array}\right\} \tag{3.61}$$

In order to evaluate the pressure distribution, equation (3.61) is integrated with respect to λ to obtain

$$\widetilde{P} = \left\{ \begin{array}{l} \rho_{f,o} R^2 \sin\beta \left(-\Omega\dot{\Delta} + 2\dot{\Delta}\dot{\alpha} + \Delta\ddot{\alpha} - \frac{\Omega}{2}\Delta f_t + \Delta\dot{\alpha} f_t \right) \\ + \rho_{f,o} R^2 \cos\beta \left(\Omega\Delta\dot{\alpha} + \ddot{\Delta} - \Delta\dot{\alpha}^2 - \frac{\Omega^2\Delta}{4} + \dot{\Delta} f_t \right) \end{array} \right\} \tag{3.62}$$

3.8.6 Hydrodynamic forces in the rotating polar Coordinates

The fluid forces whose directions are shown in figure 3.5 are evaluated by performing the following integrations:

$$F_o = -\int_0^{2\pi} L R \widetilde{P} \cos\beta \, d\beta \tag{3.63}$$

$$F_Q = -\int_0^{2\pi} L R \widetilde{P} \sin\beta \, d\beta \tag{3.64}$$

where F_o and F_Q are normal and tangential hydrodynamic forces induced by the pipe rotation.

Substituting equation (3.62) into equations (3.63) and (3.64), the hydrodynamic fluid forces are obtained as follow:

$$-F_o = m_H C \left(\Omega\Delta\dot{\alpha} + \ddot{\Delta} - \Delta\dot{\alpha}^2 - \frac{\Omega^2\Delta}{4} + \dot{\Delta} f_t \right) \tag{3.65}$$

$$-F_Q = m_H C \left(-\Omega\dot{\Delta} + 2\dot{\Delta}\dot{\alpha} + \Delta\ddot{\alpha} - \frac{\Omega}{2}\Delta f_t + \Delta\dot{\alpha} f_t \right) \tag{3.66}$$

By recalling that $\Delta = \dfrac{\delta}{C}$, then equations (3.65) and (3.66) can be expressed as

$$-F_o = m_H \left(\Omega \delta \dot{\alpha} + \ddot{\delta} - \delta \dot{\alpha}^2 - \frac{\Omega^2 \delta}{4} + \dot{\delta} f_t \right) \tag{3.67}$$

$$-F_Q = m_H C \left(-\Omega \dot{\delta} + 2\dot{\delta}\dot{\alpha} + \delta \ddot{\alpha} - \frac{\Omega}{2} \delta f_t + \delta \dot{\alpha} f_t \right) \tag{3.68}$$

3.8.7 Hydrodynamic forces in the fixed Cartesian coordinates

The fixed Cartesian coordinates can be related to the rotating coordinate as

$$w = \delta \cos\alpha \qquad , \qquad v = \delta \sin\alpha \tag{3.69}$$

Now, using equation (3.69) and its derivatives, and after some mathematical manipulations, equations (3.67) and (3.68) can be expressed as

$$-F_Z = m_H \left(\ddot{w} + \dot{w} f_t - \frac{\Omega^2}{4} w + \Omega \dot{v} + \frac{\Omega f_t}{2} v \right) \tag{3.70}$$

$$-F_Y = m_H \left(\ddot{v} + \dot{v} f_t - \frac{\Omega^2}{4} v + \Omega \dot{w} + \frac{\Omega f_t}{2} w \right) \tag{3.71}$$

It is noted that when comparing equations (3.70) and (3.71) with the work presented by Antunes et al. [23], that both formulations are similar, except for the absence of higher order terms of Δ in the current model. One should note that the absence of these higher terms have minor effects on the hydrodynamic forces since Δ is treated as a small quantity ($\Delta \leq 0.2$). On the other hand, the current model accounts for the dissipative forces while the presented in [23] neglected these dissipative effects. Accordingly, one may consider that dropping of higher order terms of Δ is considered acceptable approximation.

It is important to note that the inertial terms, which appeared in equations (3.70) and (3.71), are the same as those obtained within the hydrodynamic forces due to axial motion. Accordingly, as superposition is applied, they should be considered once in formulating the total hydrodynamic forces.

3.9 Equations of Motion

The equations of motion of a rotating flexible pipe conveying fluid and subjected to external axial flow are derived using extended Hamilton's principle.

3.9.1 Hamilton's principle

The extended Hamilton's principle was defined previously in chapter 2, which states

$$\delta \int_{t_o}^{t_f} \widetilde{L}\ dt\ + \int_{t_o}^{t_f} \delta W_{nc}\, dt = 0 \tag{3.72}$$

It is important to note that, even if there are no explicit external forces applied to the pipe conveying fluid, δW_{nc} in equation (3.72) does not vanish if one or both ends of the pipe were not fixed [3]. For the cantilevered pipe, which discharges fluid at the free end, one should consider that the fluid transfers energy to the pipe due to the motion at the free end. The virtual work done by the discharged fluid is equal to the product of the virtual displacement and the change of momentum of the fluid. In this case, equation (2.56) can be rewritten as [3]:

$$\delta \int_{t_o}^{t_f} \widetilde{L}\ dt\ + \int_{t_o}^{t_f} \delta W_{nc}\, dt = \int_{t_o}^{t_f} \left\{ M\, U_i \left[\frac{\partial r_L}{\partial t} + U_i\, \tau_L \right] . \delta r_L \right\} dt \tag{3.73}$$

where

r_L: The position vector at the free end of the pipe.

τ_L: The tangential unit vector at the free end of the pipe.

3.9.2 Variation of the Lagrangian function

In order to perform the variation of the Lagrangian function, it is essential to specify the boundary conditions, which can be stated based on the geometry of the problem as

$$v(s=0)=v'(s=0)=w(s=0)=w'(s=0)=0 \tag{3.74}$$

$$v''(s=L)=v'''(s=L)=w''(s=L)=w'''(s=L)=0 \tag{3.75}$$

Firstly, let's perform the variational analysis of the inextensibility condition, as given by equation (3.36), which can be written as

$$\delta u=\left\{-\left(v'+\frac{1}{2}v'^3\right)+\int_0^s\left(v''+\frac{3}{2}v'^2v''\right)ds\right\}\delta v+\left\{-\left(w'+\frac{1}{2}w'^3\right)+\int_0^s\left(w''+\frac{3}{2}w'^2w''\right)ds\right\}\delta w \tag{3.76}$$

For simplicity, the variational analysis is performed separately for both kinetic and potential energy expressions.

3.9.2.1 Variation of the kinetic energy expression

Upon applying Hamilton's variational operator to equation (3.22), and after some manipulations, one obtains

$$
\begin{aligned}
\delta \int_{t_o}^{t_f} T\, dt = & -\int_{t_o}^{t_f}\int_0^L \left[(m+M)\ddot{u} + (2M\,U_i)\,\dot{u}' \right] \delta u\, ds\, dt \\
& -\int_{t_o}^{t_f}\int_0^L \begin{bmatrix} (m+M)\ddot{v} + (2M\,U_i)\,\dot{v}' \\ +(I_D + I_{D,f})(2\,\ddot{v}'') \\ +(I_p)(v''\dot{w}'^2 + 2v'\dot{w}'\dot{w}'' + \dot{\varphi}\,\dot{w}'' + \dot{w}'\dot{\varphi}') \end{bmatrix} \delta v\, ds\, dt \\
& -\int_{t_o}^{t_f}\int_0^L \begin{bmatrix} (m+M)\ddot{w} + (2M\,U_i)\,\dot{w}' \\ +(I_D + I_{D,f})(2\,\ddot{w}'') \\ +(I_p)\begin{pmatrix} \dot{w}'(v'\dot{v}'' + \dot{v}'v'') + 2v'^2\ddot{w}'' + 3\ddot{w}'v'v'' \\ +2\dot{w}''v'\dot{v}' + \dfrac{\dot{\varphi}(v'\ddot{w}' + \dot{w}''\dot{v}')}{\dot{w}'} \\ -\dfrac{2\dot{\varphi}v'\dot{w}''\ddot{w}'}{\dot{w}'^2} + \dfrac{\ddot{\varphi}v'\dot{w}''}{\dot{w}'} + \dfrac{\dot{\varphi}'v'\ddot{w}'}{\dot{w}'} \\ +\dfrac{\dot{\varphi}(v'\ddot{w}'' + \ddot{w}'v'')}{\dot{w}'} + \dot{\varphi}\dot{v}'' + v''\ddot{\varphi} \\ +v'\ddot{\varphi}' + \dot{\varphi}'\dot{v}' \end{pmatrix} \end{bmatrix} \delta w\, ds\, dt \\
& -\int_{t_o}^{t_f}\int_0^L \left[(I_p)(\ddot{\varphi} + \dot{w}'\dot{v}' + v'\ddot{w}') \right] \delta\varphi\, ds\, dt \\
& +M\,U_i \int_{t_o}^{t_f} \left[\dot{u}_L\,\delta u_L + \dot{v}_L\,\delta v_L + \dot{w}_L\,\delta w_L \right] dt
\end{aligned}
\tag{3.77}
$$

In order to eliminate u and its derivative, the inextensibility condition given by equation (3.36) can be used. In addition to equation (3.76), the following relation is introduced:

$$
\int_0^L g(s) \left[\int_0^s f(s)\,\delta v\, ds \right] ds = \int_0^L \left[\int_s^L g(s)\, ds \right] f(s)\,\delta v\, ds \tag{3.78}
$$

Upon using equations (3.36), (3.76) and the relation given by equation (3.78), the variation of kinetic energy can be expressed as follows:

$$
\begin{aligned}
\delta \int_{t_o}^{t_f} T\,dt = \; & -\int_{t_o}^{t_f}\int_0^L \left[\begin{array}{l}
(m+M)\left(\ddot{v}+v'\int_0^s\left(\dot{v}'^2+v'\ddot{v}'+\dot{w}'^2+w'\ddot{w}'\right)ds\right) \\
+\left(2M\,U_i\right)\left(\dot{v}'\left(1+v'^2\right)+v'w'\dot{w}'-v''\int_s^L\left(v'\dot{v}'+w'\dot{w}'\right)ds\right) \\
-(m+M)\left(v''\int_0^s\int_s^L\left(\dot{v}'^2+v'\ddot{v}'+\dot{w}'^2+w'\ddot{w}'\right)ds\;ds\right) \\
+\left(I_D+I_{D,f}\right)\left(2\ddot{v}''\right)+\left(I_p\right)\left(v''\dot{w}'^2+2v'\dot{w}'\dot{w}''+\dot{\varphi}\dot{w}''+\dot{w}'\dot{\varphi}'\right)
\end{array}\right]\delta v\,ds\,dt \\
& -\int_{t_o}^{t_f}\int_0^L \left[\begin{array}{l}
(m+M)\left(\ddot{w}+w'\int_0^s\left(\dot{w}'^2+w'\ddot{w}'+\dot{v}'^2+v'\ddot{v}'\right)ds\right) \\
+\left(2M\,U_i\right)\left(\dot{w}'\left(1+w'^2\right)+w'v'\dot{v}'-w''\int_s^L\left(w'\dot{w}'+v'\dot{v}'\right)ds\right) \\
-(m+M)\left(w''\int_0^s\int_s^L\left(\dot{w}'^2+w'\ddot{w}'+\dot{v}'^2+v'\ddot{v}'\right)ds\;ds\right) \\
+\left(I_D+I_{D,f}\right)\left(2\ddot{w}''\right) \\
+\left(I_p\right)\left(\begin{array}{l}
\dot{w}'\left(v'\dot{v}''+\dot{v}'v''\right)+2v'^2\ddot{w}''+3\ddot{w}'v'v'' \\
+2\dot{w}''v'\dot{v}'+\dfrac{\dot{\varphi}\left(v'\ddot{w}'+\dot{w}''\dot{v}'\right)}{\dot{w}'} \\
-\dfrac{2\dot{\varphi}v'\dot{w}''\ddot{w}'}{\dot{w}'^2}+\dfrac{\ddot{\varphi}v'\dot{w}''}{\dot{w}'}+\dfrac{\dot{\varphi}'v'\ddot{w}'}{\dot{w}'} \\
+\dfrac{\dot{\varphi}\left(v'\ddot{w}''+\ddot{w}'v''\right)}{\dot{w}'}+\dot{\varphi}\dot{v}''+v''\ddot{\varphi} \\
+v'\ddot{\varphi}'+\dot{\varphi}'\dot{v}'
\end{array}\right)
\end{array}\right]\delta w\,ds\,dt \\
& -\int_{t_o}^{t_f}\int_0^L\left[\left(I_p\right)\left(\ddot{\varphi}+\dot{w}'\dot{v}'+v'\ddot{w}'\right)\right]\;\delta\varphi\,ds\,dt \\
& +M\,U_i\int_{t_o}^{t_f}\left[\dot{u}_L\,\delta u_L+\dot{v}_L\,\delta v_L+\dot{w}_L\,\delta w_L\right]dt
\end{aligned}
\tag{3.79}
$$

3.9.2.2 Variation of the potential energy expression

Before performing the variational analysis of the potential energy expression, it is noted that Γ is a function of the angular velocity $\dot{\varphi}$, while the associated shear stress is given by [23], as

$$\tau = \frac{1}{2}\rho_{f,o}\,\dot{\varphi}^2\, f_r\; R^2 \tag{3.80}$$

where R is the internal radius of the pipe, and f_r is the friction coefficient which is defined emperically.

Now, we can evaluate the torque associated with frictional forces of the internal fluid due to rotation as

$$\Gamma = 2\, I_{p,f}\; L\; f_r \dot{\varphi}^2 \tag{3.81}$$

where $I_{p,f}$ is the polar mass moment of inertia of the internal flow per unit length.

Applying Hamilton's variational operator to equation (3.32), and after some mathematical manipulations, one obtains

$$\begin{aligned}
\delta \int_{t_o}^{t_f} V\, dt = & \int_{t_o}^{t_f}\int_0^L \begin{bmatrix} EI\left(v'''' + v''^3 + 4v'v''v''' + v'^2 v''''\right) \\ + EI\left(v''w''^2 + v''w'w''' + 3v'w''w''' + v'w'w''''\right) \\ + (m+M)g\left(v' + \frac{1}{2}v'^3 + \frac{1}{2}v'\,w'^2\right) \\ -(m+M)g(L-s)\left(v'' + \frac{3}{2}v'^2 v'' + \frac{1}{2}w'^2 v'' + v'w'w''\right) \end{bmatrix} \delta v\, ds\, dt \\
& + \int_{t_o}^{t_f}\int_0^L \begin{bmatrix} EI\left(w'''' + w''^3 + 4w'w''w''' + w'^2 w''''\right) \\ + EI\left(w''v''^2 + w''v'v''' + 3w'v''v''' + w'v'v''''\right) \\ + (m+M)g\left(w' + \frac{1}{2}w'^3 + \frac{1}{2}w'\,v'^2\right) \\ -(m+M)g(L-s)\left(w'' + \frac{3}{2}w'^2 w'' + \frac{1}{2}v'^2 w'' + w'v'v''\right) \end{bmatrix} \delta w\, ds\, dt \\
& - \int_{t_o}^{t_f}\int_0^L \left[G\,J\,\varphi'' - 4\, I_{p,f}\; L\, f_r\left(\varphi'\ddot{\varphi} + 2\,\dot{\varphi}\,\dot{\varphi}'\right)\right]\delta\varphi\, ds\, dt
\end{aligned} \tag{3.82}$$

3.9.2.3 Variation of the Lagrangian function

Utilizing equations (3.79) and (3.82), the variation of the Lagrangian function can be expressed as

$$
\begin{aligned}
&\delta \int_{t_o}^{t_f} \widetilde{L}\, dt = \\
&-\int_{t_o}^{t_f}\int_{0}^{L}
\left[
\begin{array}{l}
(m+M)\left(\ddot{v} + v' \int_0^s \left(\dot{v}'^2 + v'\ddot{v}' + \dot{w}'^2 + w'\ddot{w}'\right) ds \right) \\
+(2M\,U_i)\left(\dot{v}'\left(1+v'^2\right) + v'w'\dot{w}' - v'' \int_s^L \left(v'\dot{v}' + w'\dot{w}'\right) ds \right) \\
-(m+M)\left(v'' \int_0^s \int_s^L \left(\dot{v}'^2 + v'\ddot{v}' + \dot{w}'^2 + w'\ddot{w}'\right) ds\; ds \right) \\
+\left(I_D + I_{D,f}\right)\left(2\,\ddot{v}''\right) + \left(I_p\right)\left(v''\dot{w}'^2 + 2v'\dot{w}'\dot{w}'' + \dot{\varphi}\,\dot{w}'' + \dot{w}'\dot{\varphi}'\right) \\
+EI\left(v'''' + v''^3 + 4v'v''v''' + v'^2 v''''\right) + EI\left(v''w''^2 + v''w'w''' + 3v'w''w''' + v'w'w''''\right) \\
+(m+M)\,g\left(v' + \frac{1}{2}v'^3 + \frac{1}{2}v'\,w'^2\right) - (m+M)\,g(L-s)\left(v'' + \frac{3}{2}v'^2 v'' + \frac{1}{2}w'^2 v'' + v'w'w''\right)
\end{array}
\right]
\delta v\, ds\, dt \\
&-\int_{t_o}^{t_f}\int_{0}^{L}
\left[
\begin{array}{l}
(m+M)\left(\ddot{w} + w' \int_0^s \left(\dot{w}'^2 + w'\ddot{w}' + \dot{v}'^2 + v'\ddot{v}'\right) ds \right) \\
+(2M\,U_i)\left(\dot{w}'\left(1+w'^2\right) + w'v'\dot{v}' - w'' \int_s^L \left(w'\dot{w}' + v'\dot{v}'\right) ds \right) \\
-(m+M)\left(w'' \int_0^s \int_s^L \left(\dot{w}'^2 + w'\ddot{w}' + \dot{v}'^2 + v'\ddot{v}'\right) ds\; ds \right) + \left(I_D + I_{D,f}\right)\left(2\,\ddot{w}''\right) \\
+\left(I_p\right)\left(
\begin{array}{l}
\dot{w}'\left(v'\dot{v}'' + \dot{v}'v''\right) + 2v'^2\ddot{w}'' + 3\,\ddot{w}'v'v'' + 2\dot{w}''v'\dot{v}' + \dfrac{\dot{\varphi}\left(v'\ddot{w}' + \dot{w}''\dot{v}'\right)}{\dot{w}'} - \dfrac{2\,\dot{\varphi}\,v'\dot{w}''\ddot{w}'}{\dot{w}'^2} \\
+\dfrac{\ddot{\varphi}\,v'\dot{w}''}{\dot{w}'} + \dfrac{\dot{\varphi}'\,v'\ddot{w}'}{\dot{w}'} + \dfrac{\dot{\varphi}\left(v'\ddot{w}'' + \ddot{w}'v''\right)}{\dot{w}'} + \dot{\varphi}\,\dot{v}'' + v''\ddot{\varphi} + v'\ddot{\varphi}' + \dot{\varphi}'\,\dot{v}'
\end{array}
\right) \\
+EI\left(w'''' + w''^3 + 4w'w''w''' + w'^2 w''''\right) \\
+EI\left(w''v''^2 + w''v'v''' + 3w'v''v''' + w'v'v''''\right) + (m+M)\,g\left(w' + \frac{1}{2}w'^3 + \frac{1}{2}w'\,v'^2\right) \\
-(m+M)\,g(L-s)\left(w'' + \frac{3}{2}w'^2 w'' + \frac{1}{2}v'^2 w'' + w'v'v''\right)
\end{array}
\right]
\delta w\, ds\, dt \\
&-\int_{t_o}^{t_f}\int_{0}^{L}\left[\left(I_p\right)\left(\ddot{\varphi} + \dot{w}'\dot{v}' + v'\ddot{w}'\right) - \left(G\,J\,\varphi'' - 4\,I_{p,f}\,L\,f_r\left(\varphi'\ddot{\varphi} + 2\,\dot{\varphi}\,\dot{\varphi}'\right)\right)\right]\;\delta\varphi\, ds\, dt \\
&+M\,U_i \int_{t_o}^{t_f}\left[\dot{u}_L\,\delta u_L + \dot{v}_L\,\delta v_L + \dot{w}_L\,\delta w_L\right] dt
\end{aligned}
\tag{3.83}
$$

3.9.2.4 Virtual work done by the discharged fluid

The terms associated with the end effects due to the open nature of the system are developed by substituting equations (3.3) and (3.13) into the right side of equation (3.73), thus yielding

$$\int_{t_o}^{t_f}\left\{M\,U_i\left[\frac{\partial r_L}{\partial t}+U_i\,\tau_L\right].\delta r_L\right\}dt = M\,U_i\int_{t_o}^{t_f}\left(\dot{u}_L\,\delta u_L+\dot{v}_L\,\delta v_L+\dot{w}_L\,\delta w_L\right)dt$$
$$+M\,U_i^{\,2}\int_{t_o}^{t_f}\left(u'_L\,\delta u_L+v'_L\,\delta v_L+w'_L\,\delta w_L\right)dt \tag{3.84}$$

The first term of equation (3.84) cancels out with the last term of equation (3.83). Regarding the second term, it can be simplified by considering the inextensibility condition such that: $u'\,\delta u\;+v'\,\delta v+w'\,\delta w\;=0$, and utilizing equations (3.76) and (3.78).Now, equation (3.84) can be simplified as

$$\int_{t_o}^{t_f}\left\{M\,U_i\left[\frac{\partial r_L}{\partial t}+U_i\,\tau_L\right].\delta r_L\right\}dt = M\,U_i^{\,2}\int_{t_o}^{t_f}\int_0^L\left(u''\,\delta u+v''\,\delta v\;+w''\,\delta w\right)dt$$
$$=M\,U_i^{\,2}\int_{t_o}^{t_f}\int_0^L\begin{bmatrix} v''\left(1+v'^2\right)+w'w''v' \\ -v''\int_s^L v'v''ds-v''\int_s^L w'w''\,ds\end{bmatrix}\delta v\,ds\,dt$$
$$+M\,U_i^{\,2}\int_{t_o}^{t_f}\int_0^L\begin{bmatrix} w''\left(1+w'^2\right)+v'v''w' \\ -w''\int_s^L w'w''ds-w''\int_s^L v'v''\,ds\end{bmatrix}\delta w\,ds\,dt$$

(3.85)

3.9.3 *Total virtual work of the external fluid hydrodynamic forces*

The total virtual work of hydrodynamic forces associated with external fluid is due to the axial and rotational motion of the external flow. It is noted that the virtual work of these forces in *x*-direction is due axial motion only. Referring to figure 2.3, the virtual work of the external hydrodynamic forces in *x*-direction can be expressed as

$$\delta W_x = \int_{t_o}^{t_f}\int_0^L \left\{ -F_{px} + F_L \cos\phi + \left(F_A + F_N\right)\sin\phi \right\} \delta x \, ds \, dt \tag{3.86}$$

Substituting equations (3.38-3.41) and equation (3.44) into equation (3.86), performing some mathematical manipulations, and retaining the terms up to 3rd order, one can obtain the virtual work in *x*-direction as

$$\delta W_x = -\int_{t_o}^{t_f}\int_0^L \left\{ \begin{array}{l} -M_a\left(\ddot{v}v' + \ddot{w}w'\right) - M_a U_o\left(-2\dot{v}'v' - 2\dot{w}'w'\right) \\ -M_a U_o^2\left(v'v'' + w'w''\right) \\ -\left(\dfrac{\rho_{f,o}\ D_o U_o^2 C_T}{2}\dfrac{D_o}{D_h}\right)\left(-\dfrac{1}{2}\left(v'^2 + w'^2\right) + u' + (L-s)\left(v'v'' + w'w''\right)\right) \\ +A_o\, p_o(L)\left(v'v'' + w'w''\right) \\ -\left(\rho_{f,o}\, A_o\, g\right)\left(-\dfrac{1}{2}\left(v'^2 + w'^2\right) + u' + (L-s)\left(v'v'' + w'w''\right)\right) \\ -\left(\dfrac{\rho_{f,o}\ D_o U_o^2 C_T}{2}\right)\left(1 + \left(v'^2 + w'^2\right) - \dfrac{\left(v'\dot{v} + w'\dot{w}\right)}{U_o} - \dfrac{\left(\dot{v}^2 + \dot{w}^2\right)}{2U_o^2}\right) \\ -\left(\dfrac{\rho_{f,o}\ D_o U_o^2 C_N}{2}\right)\left(-\left(v'^2 + w'^2\right) + \dfrac{\left(\dot{v}v' + \dot{w}w'\right)}{U_o}\right) \\ -\left(\dfrac{\rho_{f,o}\ D_o U_o C_{DP}}{2}\right)\left(\begin{array}{l} -v'^2|v'| + \dfrac{|\dot{v}|v'^2 + |v'|v'\dot{v}}{U_o} + \dfrac{\dot{v}|\dot{v}|v'}{U_o^2} \\ -w'^2|w'| + \dfrac{|\dot{w}|w'^2 + |w'|w'\dot{w}}{U_o} + \dfrac{\dot{w}|\dot{w}|w'}{U_o^2} \end{array}\right) \end{array} \right\} \delta u \, ds dt$$

(3.87)

Referring to figures 2.3 and 3.5, the virtual work of the external hydrodynamic forces in y-direction can be expressed as the superposition of the hydrodynamic forces due to axial and rotational motions, keeping in mind that repeated inertial terms should be used once as mentioned previously in section 3.8.7. Accordingly, the virtual work of the total external hydrodynamic forces in y-direction can be expressed as

$$\delta W_y = \int_{t_o}^{t_f}\int_0^L \left\{ F_Y + F_{py} + F_L \sin\phi - \left(F_A + F_N\right)\cos\phi \right\} \delta y \, dX \, dt \tag{3.88}$$

Substituting equations (3.38-3.42), (3.44) and (3.71) into equation (3.88), performing several mathematical manipulations and retaining terms up to 3rd order, one can obtain the virtual work in y-direction as

$$\delta W_y = -\int_{t_o}^{t_f}\int_0^L \left\{ \begin{array}{l} -\left(\dfrac{\rho_{f,o}\ D_o U_o^{\,2} C_T}{2}\dfrac{D_o}{D_h} + \rho_{f,o}\, A_o\, g\right)\left(\begin{array}{l}\left(v' - \dfrac{1}{2}v'^3 + u\,v''\right) \\ -(L-s)\left(\begin{array}{l} v'' - u''v' \\ -u'v'' - \dfrac{3}{2}v'^2\, v'' \end{array}\right)\end{array}\right) \\ -A_o\, p_o(L)\left(v'' - u''v' - u'v'' - \dfrac{3}{2}v'^2\, v''\right) \\ -\left(\dfrac{\rho_{f,o}\ D_o\, U_o^{\,2} C_T}{2}\right)\left(v' - u'v' - \dfrac{\dot{v}\, v'^2}{U_o} - \dfrac{\dot{v}^2\, v'}{2U_o^{\,2}}\right) \\ +\left(\dfrac{\rho_{f,o}\ D_o\, U_o^{\,2} C_N}{2}\right)\left(\begin{array}{l} -v' + \dfrac{\dot{v}}{U_o} + \dfrac{\dot{v}u'}{U_o} + u'v' + \dfrac{\dot{u}\,\dot{v}}{U_o^{\,2}} \\ +v'^3 - \dfrac{\dot{v}\, v'^2}{U_o} - \dfrac{\dot{v}^3}{2U_o^{\,3}} - \dfrac{\dot{v}^2 v'}{2U_o^{\,2}} \end{array}\right) \\ +\left(\dfrac{\rho_{f,o}\ D_o\, U_o^{\,2} C_{DP}}{2}\right)\left(-v'|v'| + \dfrac{|\dot{v}|\, v' + |v'|\, \dot{v}}{U_o} + \dfrac{\dot{v}|\dot{v}|}{U_o^{\,2}}\right) \\ +M_a\left(\ddot{v} - \ddot{u}\, v' - 2\dot{u}\,\dot{v}' - \ddot{v}v'^2 - \dfrac{3}{2}\dot{v}\,\dot{v}'\, v' + \dot{v}\, f_t - \dfrac{\Omega^2}{4}v + \Omega\,\dot{w} + \dfrac{\Omega\, f_t}{2}w\right) \\ +M_a\, U_o\left(-2\dot{v}' + 3\dot{u}'v' + 4u'\dot{v}' + \dfrac{7}{2}\dot{v}'v'^2 + 2\dot{u}\, v'' + \dfrac{3}{2}\dot{v}v'v''\right) \\ +M_a\, U_o^{\,2}\left(v'' - 2u''v' - 4u'v'' - \dfrac{5}{2}v''\, v'^2\right) \end{array} \right\} \delta v \, ds \, dt \tag{3.89}$$

Using similar procedure, as described above in y-direction, one can obtain the total virtual work of the hydrodynamic forces in z-direction as

$$\delta W_z = -\int_{t_o}^{t_f}\int_0^L \left\{ \begin{array}{l} -\left(\dfrac{\rho_{f,o}\ D_o U_o^{\,2} C_T}{2}\dfrac{D_o}{D_h} + \rho_{f,o}\ A_o\ g\right)\left(\begin{array}{l}\left(w' - \dfrac{1}{2}w'^3 + u\,w''\right) \\ -(L-s)\left(\begin{array}{l} w'' - u''w' \\ -u'w'' - \dfrac{3}{2}w'^2\ w''\end{array}\right)\end{array}\right) \\ -A_o\ p_o(L)\left(w'' - u''w' - u'w'' - \dfrac{3}{2}w'^2\ w''\right) \\ -\left(\dfrac{\rho_{f,o}\ D_o\ U_o^{\,2} C_T}{2}\right)\left(w' - u'w' - \dfrac{\dot{w}\,w'^2}{U_o} - \dfrac{\dot{w}^2\ w'}{2U_o^{\,2}}\right) \\ +\left(\dfrac{\rho_{f,o}\ D_o\ U_o^{\,2} C_N}{2}\right)\left(\begin{array}{l} -w' + \dfrac{\dot{w}}{U_o} + \dfrac{\dot{w}u'}{U_o} + u'w' + \dfrac{\dot{u}\,\dot{w}}{U_o^{\,2}} \\ +w'^3 - \dfrac{\dot{w}\,w'^2}{U_o} - \dfrac{\dot{w}^3}{2U_o^{\,3}} - \dfrac{\dot{w}^2 w'}{2U_o^{\,2}}\end{array}\right) \\ +\left(\dfrac{\rho_{f,o}\ D_o\ U_o^{\,2} C_{DP}}{2}\right)\left(-w'|w'| + \dfrac{|\dot{w}|\ w' + |w'|\ \dot{w}}{U_o} + \dfrac{\dot{w}|\dot{w}|}{U_o^{\,2}}\right) \\ +M_a\left(\begin{array}{l}\ddot{w} - \ddot{u}\,w' - 2\dot{u}\,\dot{w}' - \ddot{w}w'^2 - \dfrac{3}{2}\dot{w}\,\dot{w}'\,w' \\ +\dot{w}\,f_t - \dfrac{\Omega^2}{4}w + \Omega\,\dot{v} + \dfrac{\Omega\,f_t}{2}v\end{array}\right) \\ +M_a\,U_o\left(-2\dot{w}' + 3\dot{u}'w' + 4u'\dot{w}' + \dfrac{7}{2}\dot{w}'w'^2 + 2\dot{u}\,w'' + \dfrac{3}{2}\dot{w}w'w''\right) \\ +M_a\,U_o^{\,2}\left(w'' - 2u''w' - 4u'w'' - \dfrac{5}{2}w''\,w'^2\right)\end{array}\right\}\delta\,w\,ds\,dt$$

(3.90)

In order to eliminate the axial displacement u and its derivative, the inextensibility condition introduced by equation (3.36) is used. Upon substituting equations (3.36), (3.76) and (3.78), replacing Ω with $\dot{\varphi}$ for convenience, the total virtual work of hydrodynamic forces in y and z directions, respectively, can be expressed as follow:

$$
\delta W_y = -\int_{t_o}^{t_f}\int_0^L \left\{
\begin{array}{l}
-\left(\dfrac{\rho_{f,o}\ D_o U_o{}^2 C_T}{2}\dfrac{D_o}{D_h} + \rho_{f,o}\ A_o\ g\right)\left(\begin{array}{l} v' - \dfrac{1}{2}v'^3 \\ -(L-s)\left(v'' - \dfrac{3}{2}v'^2\ v''\right)\end{array}\right) \\
-A_o\ p_o(L)\left(v'' - \dfrac{3}{2}v'^2\ v'' - v''\displaystyle\int_s^L \left(v'v'' + w'w''\right)ds\right) \\
-\left(\dfrac{\rho_{f,o}\ D_o\ U_o{}^2 C_T}{2}\right)\left(\begin{array}{l} (L-s)\left(v'' + \dfrac{3}{2}v'^2\ v''\right) - \dfrac{3}{2}v'^3 - w'^2 v' \\ + \dfrac{w'\dot{w}v'}{U_o} + \dfrac{\dot{w}^2 v'}{2U_o{}^2} - v''\displaystyle\int_s^L \dfrac{v'\dot{v} + w'\dot{w}}{U_o}ds \\ - v''\displaystyle\int_s^L \dfrac{\dot{v}^2 + \dot{w}^2}{2U_o{}^2}ds + v''\displaystyle\int_s^L \left(v'^2 + w'^2\right)ds \end{array}\right) \\
+\left(\dfrac{\rho_{f,o}\ D_o U_o{}^2 C_N}{2}\right)\left(\begin{array}{l} -v' + \dfrac{\dot{v}}{U_o} + \dfrac{\dot{w}w'v'}{U_o} - v'w'^2 - \dfrac{\dot{v}^3}{2U_o{}^3} - \dfrac{\dot{v}^2 v'}{2U_o{}^2} \\ + v''\displaystyle\int_s^L \left(v'^2 + w'^2\right) \\ - \dot{v}\displaystyle\int_0^s \left(\dfrac{v'\dot{v}' + w'\dot{w}'}{U_o{}^2}\right)ds - v''\displaystyle\int_s^L \left(\dfrac{\dot{v}v' + \dot{w}w'}{U_o}\right)ds \end{array}\right) \\
+\left(\dfrac{\rho_{f,o}\ D_o U_o{}^2 C_{DP}}{2}\right)\left(-v'|v'| + \dfrac{|\dot{v}|\ v' + |v'|\ \dot{v}}{U_o} + \dfrac{\dot{v}|\dot{v}|}{U_o{}^2}\right) \\
+M_a\left(\begin{array}{l} \ddot{v} + v'\displaystyle\int_0^s \left(\dot{v}'^2 + \dot{w}'^2 + v'\ddot{v}' + w'\ddot{w}'\right)ds + 2\dot{v}'\displaystyle\int_0^s \left(v'\dot{v}' + w'\dot{w}'\right)ds \\ - v''\displaystyle\int_s^L \left(\ddot{v}v' + \ddot{w}w'\right)ds - \dfrac{3}{2}\dot{v}\dot{v}'v' + \dot{v}f_t - \dfrac{\dot{\varphi}^2}{4}v + \dot{\varphi}\dot{w} + \dfrac{\dot{\varphi}f_t}{2}w \end{array}\right) \\
+M_a U_o\left(\begin{array}{l} -2\dot{v}' + \dfrac{7}{2}\dot{v}'v'^2 - 2v''\displaystyle\int_0^s \left(v'\dot{v}' + w'\dot{w}'\right)ds + \dfrac{3}{2}\dot{v}v'v'' \\ + 2v''\displaystyle\int_s^L \left(v'\dot{v}' + w'\dot{w}'\right)ds \end{array}\right) \\
+M_a U_o{}^2\left(v'' - \dfrac{5}{2}v''v'^2 - v''\displaystyle\int_s^L \left(v'v'' + w'w''\right)ds\right)
\end{array}
\right\}\delta v\, ds\, dt
$$

(3.91)

$$
\delta W_z = -\int_{t_o}^{t_f}\int_0^L \left\{
\begin{array}{l}
-\left(\dfrac{\rho_{f,o}\ D_o U_o{}^2 C_T}{2}\dfrac{D_o}{D_h} + \rho_{f,o}\ A_o\ g\right)\left(\begin{array}{l} w' - \dfrac{1}{2}w'^3 \\ -(L-s)\left(w'' - \dfrac{3}{2}w'^2\ w''\right)\end{array}\right) \\
-A_o\ p_o(L)\left(w'' - \dfrac{3}{2}w'^2\ w'' - w''\displaystyle\int_s^L \left(v'v'' + w'w''\right)ds\right) \\
-\left(\dfrac{\rho_{f,o}\ D_o\ U_o{}^2 C_T}{2}\right)\left(\begin{array}{l} (L-s)\left(w'' + \dfrac{3}{2}w'^2\ w''\right) - \dfrac{3}{2}w'^3 - v'^2 w' \\ + \dfrac{v'\dot{v}w'}{U_o} + \dfrac{\dot{v}^2 w'}{2U_o{}^2} - w''\displaystyle\int_s^L \dfrac{v'\dot{v} + w'\dot{w}}{U_o}ds \\ - w''\displaystyle\int_s^L \dfrac{\dot{v}^2 + \dot{w}^2}{2U_o{}^2}ds + w''\displaystyle\int_s^L \left(v'^2 + w'^2\right)ds \end{array}\right) \\
+\left(\dfrac{\rho_{f,o}\ D_o\ U_o{}^2 C_N}{2}\right)\left(\begin{array}{l} -w' + \dfrac{\dot{w}}{U_o} + \dfrac{\dot{v}v'w'}{U_o} - w'v'^2 - \dfrac{\dot{w}^3}{2U_o{}^3} - \dfrac{\dot{w}^2 w'}{2U_o{}^2} \\ + w''\displaystyle\int_s^L \left(v'^2 + w'^2\right) \\ - \dot{w}\displaystyle\int_0^s \left(\dfrac{v'\dot{v}' + w'\dot{w}'}{U_o{}^2}\right)ds - w''\displaystyle\int_s^L \left(\dfrac{\dot{v}v' + \dot{w}w'}{U_o}\right)ds \end{array}\right) \\
+\left(\dfrac{\rho_{f,o}\ D_o\ U_o{}^2 C_{DP}}{2}\right)\left(-w'|w'| + \dfrac{|\dot{w}|\ w' + |w'|\ \dot{w}}{U_o} + \dfrac{\dot{w}|\dot{w}|}{U_o{}^2}\right) \\
+M_a\left(\begin{array}{l} \ddot{w} + w'\displaystyle\int_0^s \left(\dot{v}'^2 + \dot{w}'^2 + v'\ddot{v}' + w'\ddot{w}'\right)ds + 2\dot{w}'\displaystyle\int_0^s \left(v'\dot{v}' + w'\dot{w}'\right)ds \\ - w''\displaystyle\int_s^L \left(\ddot{v}\,v' + \ddot{w}\,w'\right)ds - \dfrac{3}{2}\dot{w}\,\dot{w}'\,w' + \dot{w}\,f_t - \dfrac{\dot{\varphi}^2}{4}w + \dot{\varphi}\,\dot{v} + \dfrac{\dot{\varphi}\,f_t}{2}v \end{array}\right) \\
+M_a\,U_o\left(\begin{array}{l} -2\dot{w}' + \dfrac{7}{2}\dot{w}'w'^2 - 2w''\displaystyle\int_0^s \left(w'\dot{w}' + v'\dot{v}'\right)ds + \dfrac{3}{2}\dot{w}w'w'' \\ + 2w''\displaystyle\int_s^L \left(v'\dot{v}' + w'\dot{w}'\right)ds \end{array}\right) \\
+M_a\,U_o{}^2\left(w'' - \dfrac{5}{2}w''\,w'^2 - w''\displaystyle\int_s^L \left(v'v'' + w'w''\right)ds\right)
\end{array}
\right\}\delta\, w\, ds\, dt
$$

(3.92)

3.9.4 Formulation of equations of motion

By substituting equations (3.83), (3.85), (3.91) and (3.92) into the extended Hamilton's principle given by equation (3.73), one eventually finds the following three coupled equations of motion in terms of v, w and φ which describe the motion of an inextensible rotating pipe conveying fluid and subjected to counter external axial flow :

$$
\begin{aligned}
&(m+M)\begin{pmatrix} \ddot{v}+v'\int_0^s\left(\dot{v}'^2+v'\ddot{v}'+\dot{w}'^2+w'\ddot{w}'\right)ds \\ -v''\int_0^s\int_s^L\left(\dot{v}'^2+v'\ddot{v}'+\dot{w}'^2+w'\ddot{w}'\right)ds\ ds \end{pmatrix} \\
&+(2M\,U_i)\left(\dot{v}'\left(1+v'^2\right)+v'w'\dot{w}'-v''\int_s^L\left(v'\dot{v}'+w'\dot{w}'\right)ds\right) \\
&+M\,U_i^{\,2}\left[v''\left(1+v'^2\right)+w'w''v'-v''\int_s^L v'v''ds-v''\int_s^L w'w''\,ds\right] \\
&+\left(I_D+I_{D,f}\right)\left(2\ddot{v}''\right)+\left(I_p\right)\left(v''\dot{w}'^2+2v'\dot{w}'\dot{w}''+\dot{\varphi}\,\dot{w}''+\dot{w}'\dot{\varphi}'\right) \\
&+EI\left(v''''+v''^3+4v'v''v'''+v'^2v''''\right)+EI\left(v''w''^2+v''w'w'''+3v'w''w'''+v'w'w''''\right) \\
&-A_o\,p_o(L)\left(v''-\frac{3}{2}v'^2\,v''-v''\int_s^L\left(v'v''+w'w''\right)ds\right) \\
&+M_a\begin{pmatrix} \ddot{v}+v'\int_0^s\left(\dot{v}'^2+\dot{w}'^2+v'\ddot{v}'+w'\ddot{w}'\right)ds+2\dot{v}'\int_0^s\left(v'\dot{v}'+w'\dot{w}'\right)ds \\ -v''\int_s^L\left(\ddot{v}\,v'+\ddot{w}\,w'\right)ds-\frac{3}{2}\dot{v}\,\dot{v}'\,v'+\dot{v}\,f_t-\frac{\dot{\varphi}^2}{4}v+\dot{\varphi}\,\dot{w}+\frac{\dot{\varphi}\,f_t}{2}w \end{pmatrix} \\
&+M_a\,U_o\begin{pmatrix} -2\dot{v}'+\frac{7}{2}\dot{v}'v'^2-2v''\int_0^s\left(v'\dot{v}'+w'\dot{w}'\right)ds+\frac{3}{2}\dot{v}v'v'' \\ +2v''\int_s^L\left(v'\dot{v}'+w'\dot{w}'\right)ds \end{pmatrix} \\
&+M_a\,U_o^{\,2}\left(v''-\frac{5}{2}v''v'^2-v''\int_s^L\left(v'v''+w'w''\right)ds\right)
\end{aligned}
$$

$$
\begin{aligned}
&+(m+M)g\left(v'+\frac{1}{2}v'^3+\frac{1}{2}v'w'^2-(L-s)\left(v''+\frac{3}{2}v'^2v''+\frac{1}{2}w'^2v''+v'w'w''\right)\right)\\
&-\left(\frac{\rho_{f,o}\ D_oU_o{}^2C_T}{2}\frac{D_o}{D_h}+\rho_{f,o}\,A_o\,g\right)\left(v'-\frac{1}{2}v'^3-(L-s)\left(v''-\frac{3}{2}v'^2\,v''\right)\right)\\
&-\left(\frac{\rho_{f,o}\ D_oU_o{}^2C_T}{2}\right)\begin{pmatrix}(L-s)\left(v''+\frac{3}{2}v'^2\,v''\right)-\frac{3}{2}v'^3-w'^2v'\\+\frac{w'\dot{w}v'}{U_o}+\frac{\dot{w}^2v'}{2U_o{}^2}-v''\int_s^L\frac{v'\dot{v}+w'\dot{w}}{U_o}ds\\-v''\int_s^L\frac{\dot{v}^2+\dot{w}^2}{2U_o{}^2}ds+v''\int_s^L\left(v'^2+w'^2\right)ds\end{pmatrix}\\
&+\left(\frac{\rho_{f,o}\ D_oU_o{}^2C_N}{2}\right)\begin{pmatrix}-v'+\frac{\dot{v}}{U_o}+\frac{\dot{w}w'v'}{U_o}-v'w'^2-\frac{\dot{v}^3}{2U_o{}^3}-\frac{\dot{v}^2v'}{2U_o{}^2}\\+v''\int_s^L\left(v'^2+w'^2\right)\\-\dot{v}\int_0^s\left(\frac{v'\dot{v}'+w'\dot{w}'}{U_o{}^2}\right)ds-v''\int_s^L\left(\frac{\dot{v}v'+\dot{w}w'}{U_o}\right)ds\end{pmatrix}\\
&+\left(\frac{\rho_{f,o}\ D_oU_o{}^2C_{DP}}{2}\right)\left(-v'|v'|+\frac{|\dot{v}|\,v'+|v'|\,\dot{v}}{U_o}+\frac{\dot{v}|\dot{v}|}{U_o{}^2}\right)\ =\ 0
\end{aligned}
\tag{3.93}
$$

$$
\begin{aligned}
&(m+M)\begin{pmatrix}\ddot{w}+w'\int_0^s\left(\dot{w}'^2+w'\ddot{w}'+\dot{v}'^2+v'\ddot{v}'\right)ds\\-w''\int_0^s\int_s^L\left(\dot{w}'^2+w'\ddot{w}'+\dot{v}'^2+v'\ddot{v}'\right)ds\ ds\end{pmatrix}\\
&+(2M\,U_i)\left(\dot{w}'\left(1+w'^2\right)+w'v'\dot{v}'-w''\int_s^L\left(w'\dot{w}'+v'\dot{v}'\right)ds\right)\\
&+M\,U_i{}^2\left[w''\left(1+w'^2\right)+v'v''w'-w''\int_s^L w'w''ds-w''\int_s^L v'v''\,ds\right]+\left(I_D+I_{D,f}\right)\left(2\,\ddot{w}''\right)\\
&+EI\left(w''''+w''^3+4w'w''w'''+w'^2w''''\right)+EI\left(w''v''^2+w''v'v'''+3w'v''v'''+w'v'v''''\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\left(I_p\right)\left(\begin{array}{l}\dot{w}'\left(v'\dot{v}''+\dot{v}'v''\right)+2v'^2\ddot{w}''+3\ddot{w}'v'v''+2\dot{w}''v'\dot{v}'+\dfrac{\dot{\varphi}\left(v'\ddot{w}'+\dot{w}''\dot{v}'\right)}{\dot{w}'}\\ -\dfrac{2\dot{\varphi}v'\dot{w}''\ddot{w}'}{\dot{w}'^2}+\dfrac{\ddot{\varphi}v'\dot{w}''}{\dot{w}'}+\dfrac{\dot{\varphi}'v'\ddot{w}'}{\dot{w}'}+\dfrac{\dot{\varphi}\left(v'\ddot{w}''+\ddot{w}'v''\right)}{\dot{w}'}+\dot{\varphi}\dot{v}''+v''\ddot{\varphi}+v'\ddot{\varphi}'+\dot{\varphi}'\dot{v}'\end{array}\right)\\
&+\left(m+M\right)g\left(w'+\frac{1}{2}w'^3+\frac{1}{2}w'v'^2-\left(L-s\right)\left(w''+\frac{3}{2}w'^2w''+\frac{1}{2}v'^2w''+w'v'v''\right)\right)\\
&-\left(\frac{\rho_{f,o}\ D_oU_o{}^2C_T}{2}\frac{D_o}{D_h}+\rho_{f,o}\,A_o\,g\right)\left(w'-\frac{1}{2}w'^3-\left(L-s\right)\left(w''-\frac{3}{2}w'^2\,w''\right)\right)\\
&-A_o\,p_o\left(L\right)\left(w''-\frac{3}{2}w'^2\,w''-w''\int_s^L\left(v'v''+w'w''\right)ds\right)\\
&-\left(\frac{\rho_{f,o}\ D_oU_o{}^2C_T}{2}\right)\left(\begin{array}{l}\left(L-s\right)\left(w''+\dfrac{3}{2}w'^2\,w''\right)-\dfrac{3}{2}w'^3-v'^2w'\\ +\dfrac{v'\dot{v}w'}{U_o}+\dfrac{\dot{v}^2w'}{2U_o{}^2}-w''\displaystyle\int_s^L\dfrac{v'\dot{v}+w'\dot{w}}{U_o}ds\\ -w''\displaystyle\int_s^L\dfrac{\dot{v}^2+\dot{w}^2}{2U_o{}^2}ds+w''\int_s^L\left(v'^2+w'^2\right)ds\end{array}\right)\\
&+\left(\frac{\rho_{f,o}\ D_oU_o{}^2C_N}{2}\right)\left(\begin{array}{l}-w'+\dfrac{\dot{w}}{U_o}+\dfrac{\dot{v}v'w'}{U_o}-w'v'^2-\dfrac{\dot{w}^3}{2U_o{}^3}-\dfrac{\dot{w}^2w'}{2U_o{}^2}\\ +w''\displaystyle\int_s^L\left(v'^2+w'^2\right)\\ -\dot{w}\displaystyle\int_0^s\left(\dfrac{v'\dot{v}'+w'\dot{w}'}{U_o{}^2}\right)ds-w''\int_s^L\left(\dfrac{\dot{v}v'+\dot{w}w'}{U_o}\right)ds\end{array}\right)\\
&+\left(\frac{\rho_{f,o}\ D_oU_o{}^2C_{DP}}{2}\right)\left(-w'\left|w'\right|+\frac{\left|\dot{w}\right|\,w'+\left|w'\right|\,\dot{w}}{U_o}+\frac{\dot{w}\left|\dot{w}\right|}{U_o{}^2}\right)\\
&+M_a\left(\begin{array}{l}\ddot{w}+w'\displaystyle\int_0^s\left(\dot{v}'^2+\dot{w}'^2+v'\ddot{v}'+w'\ddot{w}'\right)ds+2\dot{w}'\int_0^s\left(v'\dot{v}'+w'\dot{w}'\right)ds\\ -w''\displaystyle\int_s^L\left(\ddot{v}\,v'+\ddot{w}\,w'\right)ds-\dfrac{3}{2}\dot{w}\,\dot{w}'\,w'+\dot{w}\,f_t-\dfrac{\dot{\varphi}^2}{4}w+\dot{\varphi}\,\dot{v}+\dfrac{\dot{\varphi}\,f_t}{2}v\end{array}\right)\\
&+M_a\,U_o\left(-2\dot{w}'+\frac{7}{2}\dot{w}'w'^2-2w''\int_0^s\left(w'\dot{w}'+v'\dot{v}'\right)ds+\frac{3}{2}\dot{w}w'w''+2w''\int_s^L\left(v'\dot{v}'+w'\dot{w}'\right)ds\right)\\
&+M_a\,U_o{}^2\left(w''-\frac{5}{2}w''\,w'^2-w''\int_s^L\left(v'v''+w'w''\right)ds\right)\ =\ 0
\end{aligned}
$$

(3.94)

and,

$$\left(I_p\right)\left(\ddot{\varphi}+\dot{w}'\dot{v}'+v'\ddot{w}'\right)-\left(G\,J\,\varphi''-4\,I_{p,f}\,L\,f_r\left(\varphi'\ddot{\varphi}+2\,\dot{\varphi}\,\dot{\varphi}'\right)\right)=0 \tag{3.95}$$

Full investigations of the emerged nonlinear terms are presented in Chapters 6 and 7.

3.10 Dissipative Forces Due to Material Damping

By considering the Kelven-Voigt model presented in chapter two and utilizing equation (2.83), the equations of motion, including the internal damping of the pipe, can be expressed as

$$\begin{aligned}
&\left(m+M\right)\begin{pmatrix}\ddot{v}+v'\int_0^s\left(\dot{v}'^2+v'\ddot{v}'+\dot{w}'^2+w'\ddot{w}'\right)ds\\ -v''\int_0^s\int_s^L\left(\dot{v}'^2+v'\ddot{v}'+\dot{w}'^2+w'\ddot{w}'\right)ds\;ds\end{pmatrix}\\
&+\left(2M\,U_i\right)\left(\dot{v}'\left(1+v'^2\right)+v'w'\dot{w}'-v''\int_s^L\left(v'\dot{v}'+w'\dot{w}'\right)ds\right)\\
&+M\,U_i^{\,2}\left[v''\left(1+v'^2\right)+w'w''v'-v''\int_s^L v'v''ds-v''\int_s^L w'w''\,ds\right]\\
&+\left(I_D+I_{D,f}\right)\left(2\,\ddot{v}''\right)+\left(I_p\right)\left(v''\dot{w}'^2+2v'\dot{w}'\dot{w}''+\dot{\varphi}\,\dot{w}''+\dot{w}'\dot{\varphi}'\right)\\
&+EI\left(v''''+v''^3+4v'v''v'''+v'^2v''''\right)+EI\,a\begin{pmatrix}\dot{v}''''+3v''^2\dot{v}''+4v'v''\dot{v}'''+4v'v'''\dot{v}''\\ +4\dot{v}'v''v'''+v'^2\dot{v}''''+2v'''v'\dot{v}'\end{pmatrix}\\
&+EI\left(v''w''^2+v''w'w'''+3v'w''w'''+v'w'w''''\right)\\
&+EI\,a\begin{pmatrix}2v''w''\dot{w}''+\dot{v}''w''^2+v''w'\dot{w}'''+v''w'''\dot{w}'+\dot{v}''w'w'''+3v'w''\dot{w}'''\\ +3v'w'''\dot{w}''+3\dot{v}'w''w'''+v'w'\dot{w}''''+v'w''''\dot{w}'+\dot{v}'w'w''''\end{pmatrix}\\
&-A_o\,p_o\left(L\right)\left(v''-\frac{3}{2}v'^2\,v''-v''\int_s^L\left(v'v''+w'w''\right)ds\right)\\
&+M_a\begin{pmatrix}\ddot{v}+v'\int_0^s\left(\dot{v}'^2+\dot{w}'^2+v'\ddot{v}'+w'\ddot{w}'\right)ds+2\dot{v}'\int_0^s\left(v'\dot{v}'+w'\dot{w}'\right)ds\\ -v''\int_s^L\left(\ddot{v}\,v'+\ddot{w}\,w'\right)ds-\frac{3}{2}\dot{v}\,\dot{v}'\,v'+\dot{v}\,f_t-\frac{\dot{\varphi}^2}{4}v+\dot{\varphi}\,\dot{w}+\frac{\dot{\varphi}\,f_t}{2}w\end{pmatrix}
\end{aligned}$$

$$
\begin{aligned}
&+M_a U_o \left(\begin{array}{l} -2\dot{v}' + \frac{7}{2}\dot{v}'v'^2 - 2v''\int_0^s \left(v'\dot{v}' + w'\dot{w}'\right)ds + \frac{3}{2}\dot{v}v'v'' \\ +2v''\int_s^L \left(v'\dot{v}' + w'\dot{w}'\right)ds \end{array} \right) \\
&+M_a U_o^{\,2}\left(v'' - \frac{5}{2}v''v'^2 - v''\int_s^L \left(v'v'' + w'w''\right)ds \right) \\
&+(m+M)g\left(v' + \frac{1}{2}v'^3 + \frac{1}{2}v'w'^2 - (L-s)\left(v'' + \frac{3}{2}v'^2v'' + \frac{1}{2}w'^2v'' + v'w'w'' \right)\right) \\
&-\left(\frac{\rho_{f,o}\ D_o U_o^{\,2} C_T}{2}\frac{D_o}{D_h} + \rho_{f,o}\, A_o\, g \right)\left(v' - \frac{1}{2}v'^3 - (L-s)\left(v'' - \frac{3}{2}v'^2\, v'' \right)\right) \\
&-\left(\frac{\rho_{f,o}\ D_o U_o^{\,2} C_T}{2} \right)\left(\begin{array}{l} (L-s)\left(v'' + \frac{3}{2}v'^2\, v'' \right) - \frac{3}{2}v'^3 - w'^2v' \\ +\frac{w'\dot{w}v'}{U_o} + \frac{\dot{w}^2v'}{2U_o^{\,2}} - v''\int_s^L \frac{v'\dot{v} + w'\dot{w}}{U_o}ds \\ -v''\int_s^L \frac{\dot{v}^2 + \dot{w}^2}{2U_o^{\,2}}ds + v''\int_s^L \left(v'^2 + w'^2\right)ds \end{array} \right) \\
&+\left(\frac{\rho_{f,o}\ D_o U_o^{\,2} C_N}{2} \right)\left(\begin{array}{l} -v' + \frac{\dot{v}}{U_o} + \frac{\dot{w}w'v'}{U_o} - v'w'^2 - \frac{\dot{v}^3}{2U_o^{\,3}} - \frac{\dot{v}^2v'}{2U_o^{\,2}} \\ +v''\int_s^L \left(v'^2 + w'^2\right)ds \\ -\dot{v}\int_0^s \left(\frac{v'\dot{v}' + w'\dot{w}'}{U_o^{\,2}} \right)ds - v''\int_s^L \left(\frac{\dot{v}v' + \dot{w}w'}{U_o} \right)ds \end{array} \right) \\
&+\left(\frac{\rho_{f,o}\ D_o U_o^{\,2} C_{DP}}{2} \right)\left(-v'|v'| + \frac{|\dot{v}|\ v' + |v'|\ \dot{v}}{U_o} + \frac{\dot{v}|\dot{v}|}{U_o^{\,2}} \right) = 0
\end{aligned}
\tag{3.96}
$$

$$
(m+M)\begin{pmatrix} \ddot{w}+w'\int_0^s \left(\dot{w}'^2+w'\ddot{w}'+\dot{v}'^2+v'\ddot{v}'\right)ds \\ -w''\int_0^s\int_s^L \left(\dot{w}'^2+w'\ddot{w}'+\dot{v}'^2+v'\ddot{v}'\right)ds\ ds \end{pmatrix}
$$

$$
+(2M\,U_i)\left(\dot{w}'\left(1+w'^2\right)+w'v'\dot{v}'-w''\int_s^L \left(w'\dot{w}'+v'\dot{v}'\right)ds\right)
$$

$$
+M\,U_i^{\,2}\begin{bmatrix} w''\left(1+w'^2\right)+v'v''w' \\ -w''\int_s^L w'w''ds-w''\int_s^L v'v''\,ds \end{bmatrix}
$$

$$
+\left(I_D+I_{D,f}\right)\left(2\,\ddot{w}''\right)+\left(I_p\right)\begin{pmatrix} \dot{w}'\left(v'\dot{v}''+\dot{v}'v''\right)+2v'^2\ddot{w}''+3\,\ddot{w}'v'v''+2\dot{w}''v'\dot{v}' \\ +\dfrac{\dot{\varphi}\left(v'\ddot{w}'+\dot{w}''\dot{v}'\right)}{\dot{w}'}-\dfrac{2\,\dot{\varphi}\,v'\dot{w}''\ddot{w}'}{\dot{w}'^2}+\dfrac{\ddot{\varphi}\,v'\dot{w}''}{\dot{w}'} \\ +\dfrac{\dot{\varphi}'\,v'\ddot{w}'}{\dot{w}'}+\dfrac{\dot{\varphi}\left(v'\ddot{w}''+\ddot{w}'v''\right)}{\dot{w}'} \\ +\dot{\varphi}\,\dot{v}''+v''\ddot{\varphi}+v'\ddot{\varphi}'+\dot{\varphi}'\,\dot{v}' \end{pmatrix}
$$

$$
+EI\left(w''''+w''^3+4w'w''w'''+w'^2w'''''\right)+EI\,a\begin{pmatrix} \dot{w}''''+3w''^2\dot{w}''+4w'w''\dot{w}'''+4w'w'''\dot{w}'' \\ +4\dot{w}'w''w'''+w'^2\dot{w}'''''+2w'''''w'\dot{w}' \end{pmatrix}
$$

$$
+EI\left(w''v''^2+w''v'v'''+3w'v''v'''+w'v'v'''''\right)
$$

$$
+EI\,a\begin{pmatrix} 2w''v''\dot{v}''+\dot{w}''v''^2+w''v'\dot{v}'''+w''v'''\dot{v}'+\dot{w}''v'v'''+3w'v''\dot{v}''' \\ +3w'v'''\dot{v}''+3\dot{w}'v''v'''+w'v'\dot{v}'''''+w'v'''''\dot{v}'+\dot{w}'v'v''''' \end{pmatrix}
$$

$$
+(m+M)g\left(w'+\frac{1}{2}w'^3+\frac{1}{2}w'v'^2-(L-s)\left(w''+\frac{3}{2}w'^2w''+\frac{1}{2}v'^2w''+w'v'v''\right)\right)
$$

$$
-\left(\frac{\rho_{f,o}\ \ D_oU_o^{\,2}C_T}{2}\frac{D_o}{D_h}+\rho_{f,o}\,A_o\,g\right)\begin{pmatrix} w'-\dfrac{1}{2}w'^3 \\ -(L-s)\left(w''-\dfrac{3}{2}w'^2\,w''\right) \end{pmatrix}
$$

$$
-A_o\,p_o(L)\left(w''-\frac{3}{2}w'^2\,w''-w''\int_s^L \left(v'v''+w'w''\right)ds\right)
$$

$$
-\left(\frac{\rho_{f,o}\ \ D_oU_o^{\,2}C_T}{2}\right)\begin{pmatrix} (L-s)\left(w''+\dfrac{3}{2}w'^2\,w''\right)-\dfrac{3}{2}w'^3-v'^2w' \\ +\dfrac{v'\dot{v}w'}{U_o}+\dfrac{\dot{v}^2w'}{2U_o^{\,2}}-w''\int_s^L \dfrac{v'\dot{v}+w'\dot{w}}{U_o}ds \\ -w''\int_s^L \dfrac{\dot{v}^2+\dot{w}^2}{2U_o^{\,2}}ds+w''\int_s^L \left(v'^2+w'^2\right)ds \end{pmatrix}
$$

$$
\begin{aligned}
&+\left(\frac{\rho_{f,o}\ D_o U_o{}^2 C_N}{2}\right)\left(\begin{array}{l}
-w'+\dfrac{\dot{w}}{U_o}+\dfrac{\dot{v}v'w'}{U_o}-w'v'^2-\dfrac{\dot{w}^3}{2U_o{}^3}-\dfrac{\dot{w}^2 w'}{2U_o{}^2}\\
+w''\displaystyle\int_s^L\left(v'^2+w'^2\right)ds\\
-\dot{w}\displaystyle\int_0^s\left(\frac{v'\dot{v}'+w'\dot{w}'}{U_o{}^2}\right)ds-w''\int_s^L\left(\frac{\dot{v}v'+\dot{w}w'}{U_o}\right)ds
\end{array}\right)\\
&+\left(\frac{\rho_{f,o}\ D_o U_o{}^2 C_{DP}}{2}\right)\left(-w'|w'|+\frac{|\dot{w}|\ w'+|w'|\ \dot{w}}{U_o}+\frac{\dot{w}|\dot{w}|}{U_o{}^2}\right)\\
&+M_a\left(\begin{array}{l}
\ddot{w}+w'\displaystyle\int_0^s\left(\dot{v}'^2+\dot{w}'^2+v'\ddot{v}'+w'\ddot{w}'\right)ds+2\dot{w}'\int_0^s\left(v'\dot{v}'+w'\dot{w}'\right)ds\\
-w''\displaystyle\int_s^L\left(\ddot{v}v'+\ddot{w}w'\right)ds-\frac{3}{2}\dot{w}\dot{w}'w'+\dot{w}f_t-\frac{\dot{\varphi}^2}{4}w+\dot{\varphi}\dot{v}+\frac{\dot{\varphi}f_t}{2}v
\end{array}\right)\\
&+M_a U_o\left(\begin{array}{l}
-2\dot{w}'+\dfrac{7}{2}\dot{w}'w'^2-2w''\displaystyle\int_0^s\left(w'\dot{w}'+v'\dot{v}'\right)ds+\frac{3}{2}\dot{w}w'w''\\
+2w''\displaystyle\int_s^L\left(v'\dot{v}'+w'\dot{w}'\right)ds
\end{array}\right)\\
&+M_a U_o{}^2\left(w''-\frac{5}{2}w''w'^2-w''\int_s^L\left(v'v''+w'w''\right)ds\right)\ =\ 0
\end{aligned}
\tag{3.97}
$$

and,

$$
\left(I_p\right)\left(\ddot{\varphi}+\dot{w}'\dot{v}'+v'\ddot{w}'\right)-\left(GJ\varphi''-4I_{p,f}\ L\ f_r\left(\varphi'\ddot{\varphi}+2\dot{\varphi}\dot{\varphi}'\right)\right)=0 \tag{3.98}
$$

3.11 Case Studies

The obtained mathematical model for inextensible rotating pipe conveying fluid and subjected to external axial flow is a rich and flexible dynamical model, which can be modified to match several engineering applications as discussed hereafter in this section.

3.11.1 Fluid-free cantilevered flexible rotating pipe

The equations of motion (3.96-3.98) can be reduced to describe the motion of fluid-free flexible rotating pipe by eliminating both internal and external flows. Accordingly, the equations of motion of a cantilevered inextensible flexible rotating pipe mounted vertically can be expressed as:

$$
\begin{aligned}
& m\begin{pmatrix} \ddot{v} + v'\int_0^s \left(\dot{v}'^2 + v'\ddot{v}' + \dot{w}'^2 + w'\ddot{w}'\right)ds \\ - v''\int_0^s\int_s^L \left(\dot{v}'^2 + v'\ddot{v}' + \dot{w}'^2 + w'\ddot{w}'\right)ds\ ds \end{pmatrix} \\
& + I_D\left(2\ddot{v}''\right) + \left(I_p\right)\left(v''\dot{w}'^2 + 2v'\dot{w}'\dot{w}'' + \dot{\varphi}\dot{w}'' + \dot{w}'\dot{\varphi}'\right) \\
& + EI\left(v'''' + v''^3 + 4v'v''v''' + v'^2v''''\right) + EI\,a\begin{pmatrix} \dot{v}'''' + 3v''^2\dot{v}'' + 4v'v''\dot{v}''' + 4v'v'''\dot{v}'' \\ + 4\dot{v}'v''v''' + v'^2\dot{v}'''' + 2v''''v'\dot{v}' \end{pmatrix} \\
& + EI\left(v''w''^2 + v''w'w''' + 3v'w''w''' + v'w'w''''\right) \\
& + EI\,a\begin{pmatrix} 2v''w''\dot{w}'' + \dot{v}''w''^2 + v''w'\dot{w}''' + v''w'''\dot{w}' + \dot{v}''w'w''' + 3v'w''\dot{w}''' \\ + 3v'w'''\dot{w}'' + 3\dot{v}'w''w''' + v'w'\dot{w}'''' + v'w''''\dot{w}' + \dot{v}'w'w'''' \end{pmatrix} \\
& + m\,g\left(v' + \frac{1}{2}v'^3 + \frac{1}{2}v'w'^2 - \left(L - s\right)\left(v'' + \frac{3}{2}v'^2v'' + \frac{1}{2}w'^2v'' + v'w'w''\right)\right) = 0
\end{aligned}
\tag{3.99}
$$

$$
\begin{aligned}
& m\left(\ddot{w}+w'\int_0^s\left(\dot{w}'^2+w'\ddot{w}'+\dot{v}'^2+v'\ddot{v}'\right)ds-w''\int_0^s\int_s^L\left(\dot{w}'^2+w'\ddot{w}'+\dot{v}'^2+v'\ddot{v}'\right)ds\ ds\right) \\
& +I_D\left(2\ddot{w}''\right)+\left(I_p\right)\begin{pmatrix}\dot{w}'\left(v'\dot{v}''+\dot{v}'v''\right)+2v'^2\ddot{w}''+3\ddot{w}'v'v''+2\dot{w}''v'\dot{v}' \\ +\dfrac{\dot{\varphi}\left(v'\ddot{w}'+\dot{w}''\dot{v}'\right)}{\dot{w}'}-\dfrac{2\dot{\varphi}v'\dot{w}''\ddot{w}'}{\dot{w}'^2}+\dfrac{\ddot{\varphi}v'\dot{w}''}{\dot{w}'} \\ +\dfrac{\dot{\varphi}'v'\ddot{w}'}{\dot{w}'}+\dfrac{\dot{\varphi}\left(v'\ddot{w}''+\ddot{w}'v''\right)}{\dot{w}'} \\ +\dot{\varphi}\dot{v}''+v''\ddot{\varphi}+v'\ddot{\varphi}'+\dot{\varphi}'\dot{v}'\end{pmatrix} \\
& +EI\left(w''''+w''^3+4w'w''w'''+w'^2w''''\right)+EI\,a\begin{pmatrix}\dot{w}''''+3w''^2\dot{w}''+4w'w''\dot{w}'''+4w'w'''\dot{w}'' \\ +4\dot{w}'w''w'''+w'^2\dot{w}''''+2w''''w'\dot{w}'\end{pmatrix} \\
& +EI\left(w''v''^2+w''v'v'''+3w'v''v'''+w'v'v''''\right) \\
& +EI\,a\begin{pmatrix}2w''v''\dot{v}''+\dot{w}''v''^2+w''v'\dot{v}'''+w''v'''\dot{v}'+\dot{w}''v'v'''+3w'v''\dot{v}''' \\ +3w'v'''\dot{v}''+3\dot{w}'v''v'''+w'v'\dot{v}''''+w'v''''\dot{v}'+\dot{w}'v'v''''\end{pmatrix} \\
& +m\ g\left(w'+\frac{1}{2}w'^3+\frac{1}{2}w'v'^2-\left(L-s\right)\left(w''+\frac{3}{2}w'^2w''+\frac{1}{2}v'^2w''+w'v'v''\right)\right)=0
\end{aligned}
\tag{3.100}
$$

and,

$$
I_p\left(\ddot{\varphi}+\dot{w}'\dot{v}'+v'\ddot{w}'\right)-GJ\varphi''=0 \tag{3.101}
$$

3.11.2 Cantilevered rotating flexible pipe conveying fluid

The current model described by equations (3.96-3.98) may represent the dynamics of inextensible flexible rotating pipe by eliminating the effect of the external flow. It is found that such model has not been addressed in the literature. All previous studies in the literature tackled the dynamics of flexible pipe conveying fluid without considering the pipe rotation. The equations of motion which describe the dynamics of inextensible flexible rotating pipe can be stated as

$$
\begin{aligned}
&(m+M)\left(\ddot{v}+v'\int_0^s\left(\dot{v}'^2+v'\ddot{v}'+\dot{w}'^2+w'\ddot{w}'\right)ds-v''\int_0^s\int_s^L\left(\dot{v}'^2+v'\ddot{v}'+\dot{w}'^2+w'\ddot{w}'\right)ds\,ds\right)\\
&+(2M\,U_i)\left(\dot{v}'\left(1+v'^2\right)+v'w'\dot{w}'-v''\int_s^L\left(v'\dot{v}'+w'\dot{w}'\right)ds\right)\\
&+M\,U_i^{\,2}\left[v''\left(1+v'^2\right)+w'w''v'-v''\int_s^L v'v''ds-v''\int_s^L w'w''\,ds\right]\\
&+\left(I_D+I_{D,f}\right)\left(2\ddot{v}''\right)+\left(I_p\right)\left(v''\dot{w}'^2+2v'\dot{w}'\dot{w}''+\dot{\varphi}\dot{w}''+\dot{w}'\dot{\varphi}'\right)\\
&+EI\left(v''''+v''^3+4v'v''v'''+v'^2v''''\right)+EI\,a\begin{pmatrix}\dot{v}''''+3v''^2\dot{v}''+4v'v''\dot{v}'''+4v'v'''\dot{v}''\\+4\dot{v}'v''v'''+v'^2\dot{v}''''+2v''''v'\dot{v}'\end{pmatrix}\\
&+EI\left(v''w''^2+v''w'w'''+3v'w''w'''+v'w'w''''\right)\\
&+EI\,a\begin{pmatrix}2v''w''\dot{w}''+\dot{v}''w''^2+v''w'\dot{w}'''+v''w'''\dot{w}'+\dot{v}''w'w'''+3v'w''\dot{w}'''\\+3v'w'''\dot{w}''+3\dot{v}'w''w'''+v'w'\dot{w}''''+v'w''''\dot{w}'+\dot{v}'w'w''''\end{pmatrix}\\
&+(m+M)g\left(v'+\frac{1}{2}v'^3+\frac{1}{2}v'w'^2-(L-s)\left(v''+\frac{3}{2}v'^2v''+\frac{1}{2}w'^2v''+v'w'w''\right)\right)=0
\end{aligned}
\tag{3.102}
$$

$$
\begin{aligned}
&(m+M)\left(\ddot{w}+w'\int_0^s\left(\dot{w}'^2+w'\ddot{w}'+\dot{v}'^2+v'\ddot{v}'\right)ds-w''\int_0^s\int_s^L\left(\dot{w}'^2+w'\ddot{w}'+\dot{v}'^2+v'\ddot{v}'\right)ds\ ds\right)\\
&+(2M\,U_i)\left(\dot{w}'\left(1+w'^2\right)+w'v'\dot{v}'-w''\int_s^L\left(w'\dot{w}'+v'\dot{v}'\right)ds\right)\\
&+M\,U_i^{\,2}\left[w''\left(1+w'^2\right)+v'v''w'-w''\int_s^L w'w''ds-w''\int_s^L v'v''\,ds\right]\\
&+\left(I_D+I_{D,f}\right)\left(2\,\ddot{w}''\right)+\left(I_p\right)\begin{pmatrix}\dot{w}'\left(v'\dot{v}''+\dot{v}'v''\right)+2v'^2\ddot{w}''+3\,\ddot{w}'v'v''+2\dot{w}''v'\dot{v}'\\+\dfrac{\dot{\varphi}\left(v'\ddot{w}'+\dot{w}''\dot{v}'\right)}{\dot{w}'}-\dfrac{2\,\dot{\varphi}\,v'\dot{w}''\ddot{w}'}{\dot{w}'^2}+\dfrac{\dot{\varphi}'\,v'\dot{w}''}{\dot{w}'}\\+\dfrac{\ddot{\varphi}\,v'\ddot{w}'}{\dot{w}'}+\dfrac{\dot{\varphi}\left(v'\ddot{w}''+\ddot{w}'v''\right)}{\dot{w}'}\\+\dot{\varphi}\,\dot{v}''+v''\ddot{\varphi}+v'\ddot{\varphi}'+\dot{\varphi}'\,\dot{v}'\end{pmatrix}\\
&+EI\left(w''''+w''^3+4w'w''w'''+w'^2w''''\right)+EI\,a\begin{pmatrix}\dot{w}''''+3w''^2\dot{w}''+4w'w''\dot{w}'''+4w'w'''\dot{w}''\\+4\dot{w}'w''w'''+w'^2\dot{w}''''+2w''''w'\dot{w}'\end{pmatrix}\\
&+EI\left(w''v''^2+w''v'v'''+3w'v''v'''+w'v'v''''\right)\\
&+EI\,a\begin{pmatrix}2w''v''\dot{v}''+\dot{w}''v''^2+w''v'\dot{v}'''+w''v'''\dot{v}'+\dot{w}''v'v'''+3w'v''\dot{v}'''\\+3w'v'''\dot{v}''+3\dot{w}'v''v'''+w'v'\dot{v}''''+w'v''''\dot{v}'+\dot{w}'v'v''''\end{pmatrix}\\
&+(m+M)g\left(w'+\frac{1}{2}w'^3+\frac{1}{2}w'v'^2-(L-s)\left(w''+\frac{3}{2}w'^2w''+\frac{1}{2}v'^2w''+w'v'v''\right)\right)=0
\end{aligned}
\tag{3.103}
$$

and

$$
\left(I_p\right)\left(\ddot{\varphi}+\dot{w}'\dot{v}'+v'\ddot{w}'\right)-\left(G\,J\,\varphi''-4\,I_{p,f}\,L\,f_r\left(\varphi'\ddot{\varphi}+2\,\dot{\varphi}\,\dot{\varphi}'\right)\right)=0 \tag{3.104}
$$

3.11.3 Cantilevered flexible pipe conveying fluid and subjected to external axial flow

The equations of motion of non rotating pipe conveying fluid and subjected to external axial flow can be extracted from equations (3.96-3.97) .Similar case has been tackled by Paidoussis et al. [33], but that study was limited to the linear analysis and didn't consider the inextensibility condition and the effects of the material damping. Full

nonlinear three dimensional model which accounts the dissipative forces due to material damping are described by the following equations of motion:

$$
\begin{aligned}
&(m+M)\begin{pmatrix} \ddot{v}+v'\int_0^s\left(\dot{v}'^2+v'\ddot{v}'+\dot{w}'^2+w'\ddot{w}'\right)ds \\ -v''\int_0^s\int_s^L\left(\dot{v}'^2+v'\ddot{v}'+\dot{w}'^2+w'\ddot{w}'\right)ds\ ds \end{pmatrix} \\
&+(2M\,U_i)\left(\dot{v}'\left(1+v'^2\right)+v'w'\dot{w}'-v''\int_s^L\left(v'\dot{v}'+w'\dot{w}'\right)ds\right) \\
&+M\,U_i^{\,2}\left[v''\left(1+v'^2\right)+w'w''v'-v''\int_s^L v'v''ds-v''\int_s^L w'w''\,ds\right] \\
&+EI\left(v''''+v''^3+4v'v''v'''+v'^2v''''\right)+EI\,a\begin{pmatrix}\dot{v}''''+3v''^2\dot{v}''+4v'v''\dot{v}'''+4v'v'''\dot{v}'' \\ +4\dot{v}'v''v'''+v'^2\dot{v}''''+2v''''v'\dot{v}'\end{pmatrix} \\
&+EI\left(v''w''^2+v''w'w'''+3v'w''w'''+v'w'w''''\right) \\
&+EI\,a\begin{pmatrix}2v''w''\dot{w}''+\dot{v}''w''^2+v''w'\dot{w}'''+v''w'''\dot{w}'+\dot{v}''w'w'''+3v'w''\dot{w}''' \\ +3v'w'''\dot{w}''+3\dot{v}'w''w'''+v'w'\dot{w}''''+v'w''''\dot{w}'+\dot{v}'w'w''''\end{pmatrix} \\
&-A_o\,p_o(L)\left(v''-\frac{3}{2}v'^2\,v''-v''\int_s^L\left(v'v''+w'w''\right)ds\right) \\
&+M_a\begin{pmatrix}\ddot{v}+v'\int_0^s\left(\dot{v}'^2+\dot{w}'^2+v'\ddot{v}'+w'\ddot{w}'\right)ds+2\dot{v}'\int_0^s\left(v'\dot{v}'+w'\dot{w}'\right)ds \\ -v''\int_s^L\left(\ddot{v}\,v'+\ddot{w}\,w'\right)ds-\frac{3}{2}\dot{v}\,\dot{v}'\,v'\end{pmatrix} \\
&+M_a\,U_o\begin{pmatrix}-2\dot{v}'+\frac{7}{2}\dot{v}'v'^2-2v''\int_0^s\left(v'\dot{v}'+w'\dot{w}'\right)ds+\frac{3}{2}\dot{v}v'v'' \\ +2v''\int_s^L\left(v'\dot{v}'+w'\dot{w}'\right)ds\end{pmatrix} \\
&+M_a\,U_o^{\,2}\left(v''-\frac{5}{2}v''v'^2-v''\int_s^L\left(v'v''+w'w''\right)ds\right) \\
&+(m+M)g\left(v'+\frac{1}{2}v'^3+\frac{1}{2}v'w'^2-(L-s)\left(v''+\frac{3}{2}v'^2v''+\frac{1}{2}w'^2v''+v'w'w''\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&-\left(\frac{\rho_{f,o}\ D_o U_o^{\,2} C_T}{2}\frac{D_o}{D_h}+\rho_{f,o}\,A_o\,g\right)\left(v'-\frac{1}{2}v'^3-(L-s)\left(v''-\frac{3}{2}v'^2\,v''\right)\right)\\
&-\left(\frac{\rho_{f,o}\ D_o U_o^{\,2} C_T}{2}\right)\left(\begin{array}{l}(L-s)\left(v''+\frac{3}{2}v'^2\,v''\right)-\frac{3}{2}v'^3-w'^2v'\\ +\frac{w'\dot{w}v'}{U_o}+\frac{\dot{w}^2v'}{2U_o^{\,2}}-v''\int_s^L\frac{v'\dot{v}+w'\dot{w}}{U_o}ds\\ -v''\int_s^L\frac{\dot{v}^2+\dot{w}^2}{2U_o^{\,2}}ds+v''\int_s^L\left(v'^2+w'^2\right)ds\end{array}\right)\\
&+\left(\frac{\rho_{f,o}\ D_o U_o^{\,2} C_N}{2}\right)\left(\begin{array}{l}-v'+\frac{\dot{v}}{U_o}+\frac{\dot{w}w'v'}{U_o}-v'w'^2-\frac{\dot{v}^3}{2U_o^{\,3}}-\frac{\dot{v}^2v'}{2U_o^{\,2}}\\ +v''\int_s^L\left(v'^2+w'^2\right)ds\\ -\dot{v}\int_0^s\left(\frac{v'\dot{v}'+w'\dot{w}'}{U_o^{\,2}}\right)ds-v''\int_s^L\left(\frac{\dot{v}v'+\dot{w}w'}{U_o}\right)ds\end{array}\right)\\
&+\left(\frac{\rho_{f,o}\ D_o U_o^{\,2} C_{DP}}{2}\right)\left(-v'|v'|+\frac{|\dot{v}|\ v'+|v'|\ \dot{v}}{U_o}+\frac{\dot{v}|\dot{v}|}{U_o^{\,2}}\right)\ =\ 0
\end{aligned}
\tag{3.105}
$$

$$
\begin{aligned}
&(m+M)\left(\ddot{w}+w'\int_0^s\left(\dot{w}'^2+w'\ddot{w}'+\dot{v}'^2+v'\ddot{v}'\right)ds-w''\int_0^s\int_s^L\left(\dot{w}'^2+w'\ddot{w}'+\dot{v}'^2+v'\ddot{v}'\right)ds\ ds\right)\\
&+(2M\,U_i)\left(\dot{w}'\left(1+w'^2\right)+w'v'\dot{v}'-w''\int_s^L\left(w'\dot{w}'+v'\dot{v}'\right)ds\right)\\
&+M\,U_i^{\,2}\left[w''\left(1+w'^2\right)+v'v''w'-w''\int_s^L w'w''ds-w''\int_s^L v'v''\,ds\right]\\
&+EI\left(w''''+w''^3+4w'w''w'''+w'^2w''''\right)+EI\,a\left(\begin{array}{l}\dot{w}''''+3w''^2\dot{w}''+4w'w''\dot{w}'''+4w'w'''\dot{w}''\\ +4\dot{w}'w''w'''+w'^2\dot{w}''''+2w''''w'\dot{w}'\end{array}\right)\\
&+EI\left(w''v''^2+w''v'v'''+3w'v''v'''+w'v'v''''\right)\\
&+EI\,a\left(\begin{array}{l}2w''v''\dot{v}''+\dot{w}''v''^2+w''v'\dot{v}'''+w''v'''\dot{v}'+\dot{w}''v'v'''+3w'v''\dot{v}'''\\ +3w'v'''\dot{v}''+3\dot{w}'v''v'''+w'v'\dot{v}''''+w'v''''\dot{v}'+\dot{w}'v'v''''\end{array}\right)\\
&+(m+M)g\left(w'+\frac{1}{2}w'^3+\frac{1}{2}w'v'^2-(L-s)\left(w''+\frac{3}{2}w'^2w''+\frac{1}{2}v'^2w''+w'v'v''\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&-\left(\frac{\rho_{f,o}\ D_o U_o{}^2 C_T}{2}\frac{D_o}{D_h}+\rho_{f,o}\,A_o\,g\right)\begin{pmatrix} w'-\frac{1}{2}w'^3 \\ -(L-s)\left(w''-\frac{3}{2}w'^2\,w''\right)\end{pmatrix} \\
&-A_o\,p_o(L)\left(w''-\frac{3}{2}w'^2\,w''-w''\int_s^L\ (v'v''+w'w'')ds\right) \\
&-\left(\frac{\rho_{f,o}\ D_o U_o{}^2 C_T}{2}\right)\begin{pmatrix} (L-s)\left(w''+\frac{3}{2}w'^2\,w''\right)-\frac{3}{2}w'^3-v'^2w' \\ +\frac{v'\dot{v}w'}{U_o}+\frac{\dot{v}^2w'}{2U_o{}^2}-w''\int_s^L\ \frac{v'\dot{v}+w'\dot{w}}{U_o}ds \\ -w''\int_s^L\ \frac{\dot{v}^2+\dot{w}^2}{2U_o{}^2}ds+w''\int_s^L\ \left(v'^2+w'^2\right)ds\end{pmatrix} \\
&+\left(\frac{\rho_{f,o}\ D_o U_o{}^2 C_N}{2}\right)\begin{pmatrix} -w'+\frac{\dot{w}}{U_o}+\frac{\dot{v}v'w'}{U_o}-w'v'^2-\frac{\dot{w}^3}{2U_o{}^3}-\frac{\dot{w}^2w'}{2U_o{}^2} \\ +w''\int_s^L\ \left(v'^2+w'^2\right)ds \\ -\dot{w}\int_0^s\left(\frac{v'\dot{v}'+w'\dot{w}'}{U_o{}^2}\right)ds-w''\int_s^L\ \left(\frac{\dot{v}v'+\dot{w}w'}{U_o}\right)ds\end{pmatrix} \\
&+\left(\frac{\rho_{f,o}\ D_o U_o{}^2 C_{DP}}{2}\right)\left(-w'|w'|+\frac{|\dot{w}|\ w'+|w'|\ \dot{w}}{U_o}+\frac{\dot{w}|\dot{w}|}{U_o{}^2}\right) \\
&+M_a\begin{pmatrix} \ddot{w}+w'\int_0^s\ \left(\dot{v}'^2+\dot{w}'^2+v'\ddot{v}'+w'\ddot{w}'\right)ds+2\dot{w}'\int_0^s\ (v'\dot{v}'+w'\dot{w}')ds \\ -w''\int_s^L\ (\ddot{v}\,v'+\ddot{w}\,w')ds-\frac{3}{2}\dot{w}\,\dot{w}'\,w'\end{pmatrix} \\
&+M_a\,U_o\begin{pmatrix} -2\dot{w}'+\frac{7}{2}\dot{w}'w'^2-2w''\int_0^s\ (w'\dot{w}'+v'\dot{v}')ds+\frac{3}{2}\dot{w}w'w'' \\ +2w''\int_s^L\ (v'\dot{v}'+w'\dot{w}')ds\end{pmatrix} \\
&+M_a\,U_o{}^2\left(w''-\frac{5}{2}w''w'^2-w''\int_s^L\ (v'v''+w'w'')ds\right)\ =\ 0
\end{aligned}
$$

(3.106)

It is noted that equations (3.105) and (3.106) are identical for the stationary pipe, this is reasonable as all gyroscopic forces are absent. Accordingly, the system is symmetry in X-Y and X-Z planes.

3.11.4 Cantilevered rotating flexible pipe conveying fluid downwards, which then flows upwards as a confined annular flow.

The equations of motion obtained in section 3.7 are sufficient to describe the motion of a rotating flexible pipe conveying fluid downwards, which then flows upwards as a confined annular flow. Such a model may represent the system inspired by drilling applications. The system consists of a hollow rotating drill pipe containing circulation fluid, which flow downwards through the bit, and then upwards through the borehole to the surface as shown in figure 3.6. It is clear that velocities of the inside and outside flows are dependent. The relation between the inside and outside flows can be established based on the continuity of the flow, such that

$$U_i \, A_f = U_o \, A_{ch} \tag{3.107}$$

From equation (3.107), one may express the velocity of the external fluid U_o in terms of U_i as

$$U_o = \left(\frac{D_i^{\,2}}{D_{ch}^{\,2} - D_o^{\,2}} \right) U_i \tag{3.108}$$

where

D_i : inner diameter of the pipe.

D_o: outer diameter of the pipe.

D_{ch} : inner diameter of the channel.

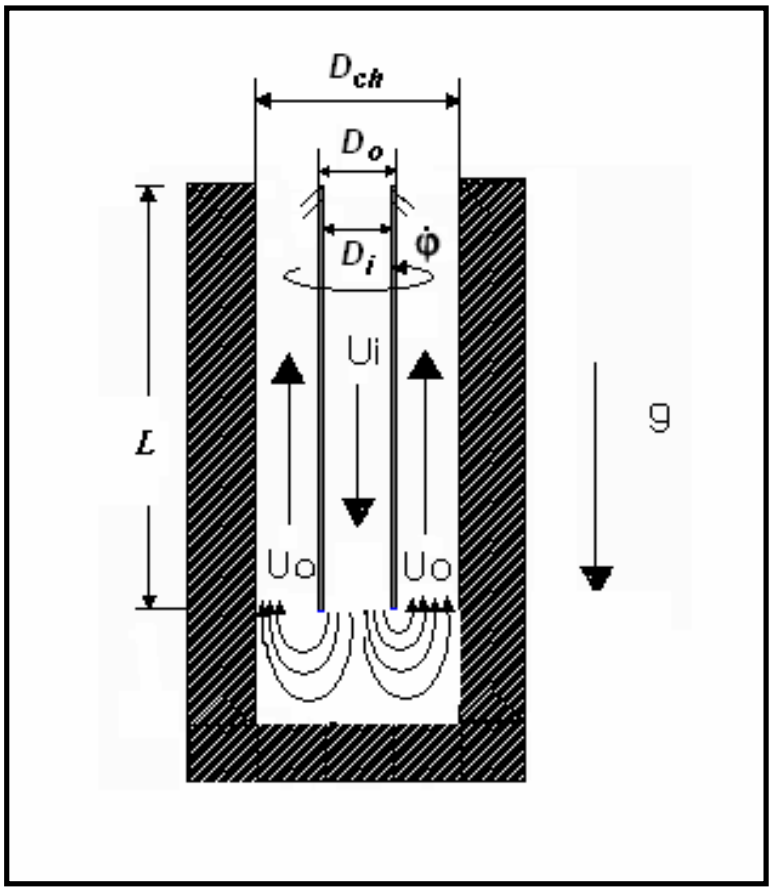


Figure 3.6: Cantilevered rotating flexible pipe conveying fluid downwards, which then flows upwards in the outer annulus.

Also, it is noted that the pressure at the end of the pipe $p_o(L)$ would normally arise from the base drag at the free end of the pipe as defined in [12], such that

$$A_o\, p_o\,(L) = \frac{1}{2}\rho_{f,o}\, U_o^{\,2}\, C_b = \frac{1}{2}\rho_{f,o} C_b \left(\frac{D_i^{\,2}}{D_{ch}^{\,2} - D_o^{\,2}} \right)^2 U_i^{\,2} \tag{3.109}$$

where C_b is the base drag coefficient.

For generality, the density of internal fluid $\rho_{f,i}$ and the density of the fluid in the annulus $\rho_{f,o}$ are assumed to have different values since in drilling operations the internal fluid represents the circulation mud while the annular flow represents the mud together with the cutting debris, which is expected to have higher density.

By substituting equations (3.99 -3.101) into the equations of motion (i.e. 3.96-3.98), then the equations of motion that represent the motion of the drill string can be expressed as

$$\begin{aligned}
&(m+M)\left(\ddot{v} + v'\int_0^s \left(\dot{v}'^2 + v'\ddot{v}' + \dot{w}'^2 + w'\ddot{w}'\right)ds - v''\int_0^s\int_s^L \left(\dot{v}'^2 + v'\ddot{v}' + \dot{w}'^2 + w'\ddot{w}'\right)ds\; ds \right) \\
&+ (2M\,U_i)\left(\dot{v}'\left(1+v'^2\right) + v'w'\dot{w}' - v''\int_s^L \left(v'\dot{v}' + w'\dot{w}'\right)ds \right) \\
&+ M\,U_i^{\,2}\left[v''\left(1+v'^2\right) + w'w''v' - v''\int_s^L v'v''ds - v''\int_s^L w'w''\,ds \right] \\
&+ \left(I_D + I_{D,f}\right)\left(2\,\ddot{v}''\right) + \left(I_p\right)\left(v''\dot{w}'^2 + 2v'\dot{w}'\dot{w}'' + \dot{\varphi}\,\dot{w}'' + \dot{w}'\dot{\varphi}'\right) \\
&+ EI\left(v''''' + v''^3 + 4v'v''v''' + v'^2 v'''''\right) + EI\,a\begin{pmatrix} \dot{v}''''' + 3v''^2\dot{v}'' + 4v'v''\dot{v}''' + 4v'v'''\dot{v}'' \\ + 4\dot{v}'v''v''' + v'^2\dot{v}''''' + 2v'''''v'\dot{v}' \end{pmatrix} \\
&+ EI\left(v''w''^2 + v''w'w''' + 3v'w''w''' + v'w'w'''''\right) \\
&+ EI\,a\begin{pmatrix} 2v''w''\dot{w}'' + \dot{v}''w''^2 + v''w'\dot{w}''' + v''w'''\dot{w}' + \dot{v}''w'w''' + 3v'w''\dot{w}''' \\ + 3v'w'''\dot{w}'' + 3\dot{v}'w''w''' + v'w'\dot{w}''''' + v'w'''''\dot{w}' + \dot{v}'w'w''''' \end{pmatrix} \\
&- \frac{1}{2}\rho_{f,o} C_b \left(\frac{D_i^{\,2}}{D_{ch}^{\,2} - D_o^{\,2}} \right)^2 U_i^{\,2} \left(v'' - \frac{3}{2}v'^2 v'' - v''\int_s^L \left(v'v'' + w'w''\right)ds \right) \\
&+ M_a \begin{pmatrix} \ddot{v} + v'\int_0^s \left(\dot{v}'^2 + \dot{w}'^2 + v'\ddot{v}' + w'\ddot{w}'\right)ds + 2\dot{v}'\int_0^s \left(v'\dot{v}' + w'\dot{w}'\right)ds \\ - v''\int_s^L \left(\ddot{v}\,v' + \ddot{w}\,w'\right)ds - \frac{3}{2}\dot{v}\,\dot{v}'\,v' + \dot{v}\,f_t - \frac{\dot{\varphi}^2}{4}v + \dot{\varphi}\,\dot{w} + \frac{\dot{\varphi}\,f_t}{2}w \end{pmatrix}
\end{aligned}$$

$$
\begin{aligned}
&+M_a\left(\frac{D_i^{\,2}}{D_{ch}^{\,2}-D_o^{\,2}}\right)U_i\left(-2\dot{v}'+\frac{7}{2}\dot{v}'v'^2-2v''\int_0^s\left(v'\dot{v}'+w'\dot{w}'\right)ds+\frac{3}{2}\dot{v}v'v''+2v''\int_s^L\left(v'\dot{v}'+w'\dot{w}'\right)ds\right)\\
&+M_a\left(\frac{D_i^{\,2}}{D_{ch}^{\,2}-D_o^{\,2}}\right)^2U_i^{\,2}\left(v''-\frac{5}{2}v''v'^2-v''\int_s^L\left(v'v''+w'w''\right)ds\right)\\
&+(m+M)g\left(v'+\frac{1}{2}v'^3+\frac{1}{2}v'w'^2-(L-s)\left(v''+\frac{3}{2}v'^2v''+\frac{1}{2}w'^2v''+v'w'w''\right)\right)\\
&-\left(\frac{\rho_{f,o}\;D_o\left(\frac{D_i^{\,2}}{D_{ch}^{\,2}-D_o^{\,2}}\right)^2U_i^{\,2}C_T}{2}\frac{D_o}{D_h}+\rho_{f,o}\,A_o\,g\right)\left(v'-\frac{1}{2}v'^3-(L-s)\left(v''-\frac{3}{2}v'^2\,v''\right)\right)\\
&-\left(\frac{\rho_{f,o}\;D_o\left(\frac{D_i^{\,2}}{D_{ch}^{\,2}-D_o^{\,2}}\right)^2U_i^{\,2}C_T}{2}\right)\left(\begin{aligned}
&(L-s)\left(v''+\frac{3}{2}v'^2\,v''\right)-\frac{3}{2}v'^3-w'^2v'\\
&+\left(\frac{D_{ch}^{\,2}-D_o^{\,2}}{D_i^{\,2}}\right)\frac{w'\dot{w}v'}{U_i}+\left(\frac{D_{ch}^{\,2}-D_o^{\,2}}{D_i^{\,2}}\right)^2\frac{\dot{w}^2v'}{2U_i^{\,2}}\\
&-\left(\frac{D_{ch}^{\,2}-D_o^{\,2}}{D_i^{\,2}}\right)v''\int_s^L\frac{v'\dot{v}+w'\dot{w}}{U_i}ds\\
&-v''\int_s^L\left(\frac{D_{ch}^{\,2}-D_o^{\,2}}{D_i^{\,2}}\right)^2\frac{\dot{v}^2+\dot{w}^2}{2U_i^{\,2}}ds+v''\int_s^L\left(v'^2+w'^2\right)ds
\end{aligned}\right)\\
&+\left(\frac{\rho_{f,o}\;D_o\left(\frac{D_i^{\,2}}{D_{ch}^{\,2}-D_o^{\,2}}\right)^2U_i^{\,2}C_N}{2}\right)\left(\begin{aligned}
&-v'+\left(\frac{D_{ch}^{\,2}-D_o^{\,2}}{D_i^{\,2}}\right)\frac{\dot{v}}{U_i}+\left(\frac{D_{ch}^{\,2}-D_o^{\,2}}{D_i^{\,2}}\right)\frac{\dot{w}w'v'}{U_i}\\
&-v'w'^2-\left(\frac{D_{ch}^{\,2}-D_o^{\,2}}{D_i^{\,2}}\right)^3\frac{\dot{v}^3}{2U_i^{\,3}}-\left(\frac{D_{ch}^{\,2}-D_o^{\,2}}{D_i^{\,2}}\right)^2\frac{\dot{v}^2v'}{2U_i^{\,2}}\\
&+v''\int_s^L\left(v'^2+w'^2\right)-v''\int_s^L\left(\frac{D_{ch}^{\,2}-D_o^{\,2}}{D_i^{\,2}}\right)\left(\frac{\dot{v}v'+\dot{w}w'}{U_i}\right)ds\\
&-\dot{v}\int_0^s\left(\frac{D_{ch}^{\,2}-D_o^{\,2}}{D_i^{\,2}}\right)^2\left(\frac{v'\dot{v}'+w'\dot{w}'}{U_i^{\,2}}\right)ds
\end{aligned}\right)\\
&+\left(\frac{\rho_{f,o}\;D_o\left(\frac{D_i^{\,2}}{D_{ch}^{\,2}-D_o^{\,2}}\right)^2U_i^{\,2}C_{DP}}{2}\right)\left(\begin{aligned}
&-v'|v'|+\left(\frac{D_{ch}^{\,2}-D_o^{\,2}}{D_i^{\,2}}\right)\frac{|\dot{v}|\;v'+|v'|\;\dot{v}}{U_i}\\
&+\left(\frac{D_{ch}^{\,2}-D_o^{\,2}}{D_i^{\,2}}\right)^2\frac{\dot{v}|\dot{v}|}{U_i^{\,2}}
\end{aligned}\right)=0
\end{aligned}
$$

(3.110)

$$
\begin{aligned}
&(m+M)\begin{pmatrix} \ddot{w}+w'\int_0^s \left(\dot{w}'^2+w'\ddot{w}'+\dot{v}'^2+v'\ddot{v}'\right)ds \\ -w''\int_0^s\int_s^L \left(\dot{w}'^2+w'\ddot{w}'+\dot{v}'^2+v'\ddot{v}'\right)ds\ ds \end{pmatrix} \\
&+(2M\,U_i)\left(\dot{w}'\left(1+w'^2\right)+w'v'\dot{v}'-w''\int_s^L \left(w'\dot{w}'+v'\dot{v}'\right)ds\right) \\
&+M\,U_i^{\,2}\begin{bmatrix} w''\left(1+w'^2\right)+v'v''w' \\ -w''\int_s^L w'w''ds-w''\int_s^L v'v''\,ds \end{bmatrix}+\left(I_D+I_{D,f}\right)\left(2\,\ddot{w}''\right) \\
&+\left(I_p\right)\begin{pmatrix} \dot{w}'\left(v'\dot{v}''+\dot{v}'v''\right)+2v'^2\ddot{w}''+3\,\ddot{w}'v'v''+2\dot{w}''v'\dot{v}'+\dfrac{\dot{\varphi}\left(v'\ddot{w}'+\dot{w}''\dot{v}'\right)}{\dot{w}'} \\ -\dfrac{2\,\dot{\varphi}\,v'\dot{w}''\ddot{w}'}{\dot{w}'^2}+\dfrac{\ddot{\varphi}\,v'\dot{w}''}{\dot{w}'}+\dfrac{\dot{\varphi}'\,v'\ddot{w}'}{\dot{w}'}+\dfrac{\dot{\varphi}\left(v'\ddot{w}''+\ddot{w}'v''\right)}{\dot{w}'}+\dot{\varphi}\,\dot{v}''+v''\ddot{\varphi}+v'\ddot{\varphi}'+\dot{\varphi}'\,\dot{v}' \end{pmatrix} \\
&+EI\left(w''''+w''^3+4w'w''w'''+w'^2w''''\right)+EI\begin{pmatrix} w''v''^2+w''v'v''' \\ +3w'v''v'''+w'v'v'''' \end{pmatrix} \\
&+\ EI\,a\begin{pmatrix} \dot{w}''''+3w''^2\dot{w}''+4w'w''\dot{w}'''+4w'w'''\dot{w}'' \\ +4\dot{w}'w''w'''+w'^2\dot{w}''''+2w''''w'\dot{w}' \end{pmatrix} \\
&+EI\,a\begin{pmatrix} 2w''v''\dot{v}''+\dot{w}''v''^2+w''v'\dot{v}'''+w''v'''\dot{v}'+\dot{w}''v'v'''+3w'v''\dot{v}''' \\ +3w'v'''\dot{v}''+3\dot{w}'v''v'''+w'v'\dot{v}''''+w'v''''\dot{v}'+\dot{w}'v'v'''' \end{pmatrix} \\
&+(m+M)g\left(w'+\frac{1}{2}w'^3+\frac{1}{2}w'v'^2-(L-s)\left(w''+\frac{3}{2}w'^2w''+\frac{1}{2}v'^2w''+w'v'v''\right)\right) \\
&-\left(\frac{\rho_{f,o}\quad D_o\left(\dfrac{D_i^{\,2}}{D_{ch}^{\,2}-D_o^{\,2}}\right)^2 U_i^{\,2}C_T}{2}\frac{D_o}{D_h}+\rho_{f,o}\,A_o\,g\right)\begin{pmatrix} w'-\dfrac{1}{2}w'^3 \\ -(L-s)\left(w''-\dfrac{3}{2}w'^2\,w''\right) \end{pmatrix} \\
&-\frac{1}{2}\rho_{f,o}C_b\left(\frac{D_i^{\,2}}{D_{ch}^{\,2}-D_o^{\,2}}\right)^2 U_i^{\,2}\left(w''-\frac{3}{2}w'^2\,w''-w''\int_s^L \left(v'v''+w'w''\right)ds\right)
\end{aligned}
$$

$$
-\left(\frac{\rho_{f,o}\ D_o\left(\frac{D_i^{\,2}}{D_{ch}^{\,2}-D_o^{\,2}}\right)^2 U_i^{\,2} C_T}{2}\right)\left(\begin{array}{l}
(L-s)\left(w''+\frac{3}{2}w'^2\ w''\right)-\frac{3}{2}w'^3-v'^2 w' \\
+\left(\frac{D_{ch}^{\,2}-D_o^{\,2}}{D_i^{\,2}}\right)\frac{v'\dot{v}w'}{U_i}+\left(\frac{D_{ch}^{\,2}-D_o^{\,2}}{D_i^{\,2}}\right)^2\frac{\dot{v}^2 w'}{2U_i^{\,2}} \\
-w''\int_s^L\left(\frac{D_{ch}^{\,2}-D_o^{\,2}}{D_i^{\,2}}\right)\frac{v'\dot{v}+w'\dot{w}}{U_i}ds \\
-w''\int_s^L\left(\frac{D_{ch}^{\,2}-D_o^{\,2}}{D_i^{\,2}}\right)^2\frac{\dot{v}^2+\dot{w}^2}{2U_i^{\,2}}ds+w''\int_s^L\left(v'^2+w'^2\right)ds
\end{array}\right)
$$

$$
+\left(\frac{\rho_{f,o}\ D_o\left(\frac{D_i^{\,2}}{D_{ch}^{\,2}-D_o^{\,2}}\right)^2 U_i^{\,2} C_N}{2}\right)\left(\begin{array}{l}
-w'+\left(\frac{D_{ch}^{\,2}-D_o^{\,2}}{D_i^{\,2}}\right)\frac{\dot{w}}{U_i}+\left(\frac{D_{ch}^{\,2}-D_o^{\,2}}{D_i^{\,2}}\right)\frac{\dot{v}v'w'}{U_i} \\
-w'v'^2-\left(\frac{D_{ch}^{\,2}-D_o^{\,2}}{D_i^{\,2}}\right)^3\frac{\dot{w}^3}{2U_i^{\,3}}-\left(\frac{D_{ch}^{\,2}-D_o^{\,2}}{D_i^{\,2}}\right)^2\frac{\dot{w}^2 w'}{2U_i^{\,2}} \\
-w''\int_s^L\left(\frac{D_{ch}^{\,2}-D_o^{\,2}}{D_i^{\,2}}\right)\left(\frac{\dot{v}v'+\dot{w}w'}{U_i}\right)ds \\
+w''\int_s^L\left(v'^2+w'^2\right)-\dot{w}\int_0^s\left(\frac{D_{ch}^{\,2}-D_o^{\,2}}{D_i^{\,2}}\right)^2\left(\frac{v'\dot{v}'+w'\dot{w}'}{U_i^{\,2}}\right)ds
\end{array}\right)
$$

$$
+\left(\frac{\rho_{f,o}\ D_o\left(\frac{D_i^{\,2}}{D_{ch}^{\,2}-D_o^{\,2}}\right)^2 U_i^{\,2} C_{DP}}{2}\right)\left(\begin{array}{l}
-w'|w'|+\left(\frac{D_{ch}^{\,2}-D_o^{\,2}}{D_i^{\,2}}\right)\frac{|\dot{w}|\ w'+|w'|\ \dot{w}}{U_i} \\
+\left(\frac{D_{ch}^{\,2}-D_o^{\,2}}{D_i^{\,2}}\right)^2\frac{\dot{w}|\dot{w}|}{U_i^{\,2}}
\end{array}\right)
$$

$$
+M_a\left(\begin{array}{l}
\ddot{w}+w'\int_0^s\left(\dot{v}'^2+\dot{w}'^2+v'\ddot{v}'+w'\ddot{w}'\right)ds+2\dot{w}'\int_0^s\left(v'\dot{v}'+w'\dot{w}'\right)ds \\
-w''\int_s^L\left(\ddot{v}\,v'+\ddot{w}\,w'\right)ds-\frac{3}{2}\dot{w}\,\dot{w}'\,w'+\dot{w}\,f_t-\frac{\dot{\varphi}^2}{4}w+\dot{\varphi}\,\dot{v}+\frac{\dot{\varphi}\,f_t}{2}v
\end{array}\right)
$$

$$
+M_a\left(\frac{D_i^{\,2}}{D_{ch}^{\,2}-D_o^{\,2}}\right)U_i\left(\begin{array}{l}
-2\dot{w}'+\frac{7}{2}\dot{w}'w'^2-2w''\int_0^s\left(w'\dot{w}'+v'\dot{v}'\right)ds+\frac{3}{2}\dot{w}w'w'' \\
+2w''\int_s^L\left(v'\dot{v}'+w'\dot{w}'\right)ds
\end{array}\right)
$$

$$
+M_a\left(\frac{D_i^{\,2}}{D_{ch}^{\,2}-D_o^{\,2}}\right)^2 U_i^{\,2}\left(w''-\frac{5}{2}w''\,w'^2-w''\int_s^L\left(v'v''+w'w''\right)ds\right)\ =\ 0
$$

(3.111)

and,

$$\left(I_p\right)\left(\ddot{\varphi}+\dot{w}'\dot{v}'+v'\ddot{w}'\right)-\left(G\,J\,\varphi''-4\,I_{p,f}\,L\,f_r\left(\varphi'\ddot{\varphi}+2\,\dot{\varphi}\,\dot{\varphi}'\right)\right)=0 \qquad (3.112)$$

The obtained equations of motion are function of lateral displacements and angle of rotation, while the axial deformation is absent due to the inextensibility condition. On the other hand, some experimental studies, such as [18], showed that the fixed-simple supported model may describe the motion of drill string more accurately than the cantilevered configuration. This issue will be discussed in details in chapter four.

CHAPTER FOUR

NONLINEAR DYNAMIC MODELING OF EXTENSIBLE ROTATING FLEXIBLE PIPE CONVEYING FLUID AND SUBJECTED TO EXTERNAL AXIAL FLOW

In this chapter, the nonlinear equations of motion of extensible flexible rotating pipe conveying fluid and subjected to external axial flow are derived using Energy method.

4.1 The Problem Statement

Consider the motion of the flexible pipe shown in figure 4.1.The model consists of rotating uniform tubular pipe of length L, cross sectional area A, mass per unit length m and flexural rigidity EI. The pipe is conveying fluid of mass per unit length M downwards, which flows axially with velocity U_i, while external fluid of density $\rho_{f,o}$, flows in reverse direction with velocity U_o in the annulus formed by the inner pipe and an outer cylinder . The main assumptions underlying the formulation include the following:

- The internal pipe is slender, and obeys Euler-Bernoulli theory.
- The material of the internal pipe is elastic, homogeneous and isotropic
- The outer cylinder is rigid.
- The internal and external fluids are Newtonian, incompressible and fully developed.

- The internal flow is fully developed turbulent and approximated by a plug flow, (i.e. as if it were an infinitely flexible rod traveling through the pipe, then all points of the fluid having a velocity relative to the pipe).
- Steady flow such that the mean velocities of internal and external flows are constant and free from pulsation.
- Fixed-simply supported end conditions are assumed, and therefore; inextensibility condition is not applicable.
- External flow is represented by the induced hydrodynamic forces.

4.2 Order of Magnitude Considerations

Although large deflections of the pipe are considered, only cubic nonlinear terms will be retained in the final equations of motion; thus, an order of magnitude analysis will be useful. The lateral displacements v, w and the angle of rotation φ are considered small relative to the length of the pipe, while the longitudinal displacement u is much smaller than v and w, therefore u should have higher order of magnitude [3]. Accordingly, one may state that:

$$v \sim \tilde{O}(\epsilon) \quad , w \sim \tilde{O}(\epsilon) \quad , \quad \varphi \sim \tilde{O}(\epsilon) \quad , \quad u \sim \tilde{O}(\epsilon^2) \tag{4.1}$$

where $(\epsilon << 1)$.

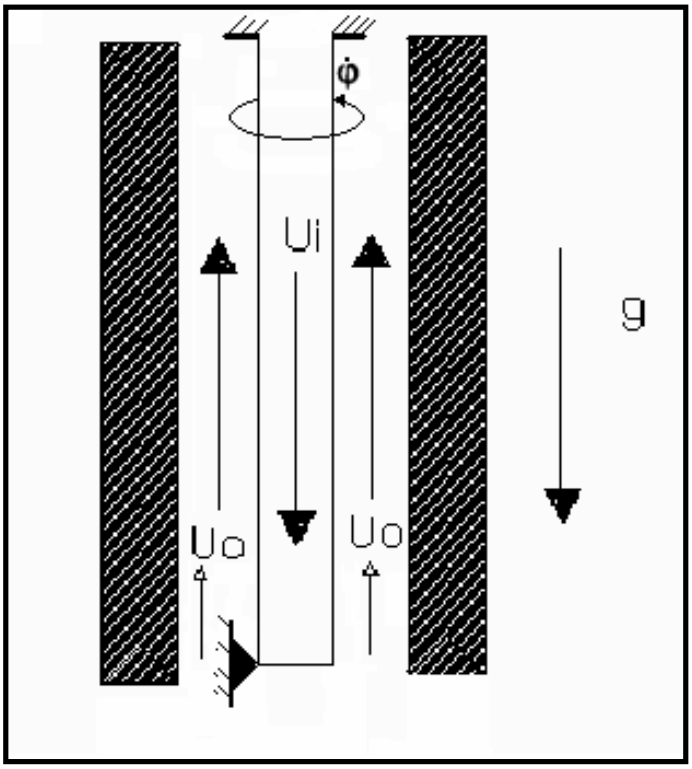


Figure 4.1: Fixed-simply supported rotating pipe conveying fluid and subjected to external axial flow

One should note that the various expressions of kinetic and potential energies should be kept at $\tilde{O}(\epsilon^4)$, since the variational technique always requires one order higher than the order of final equations.

4.3 Kinetic Energy Expression

The kinetic energy of the above described model consists of the following:

- Kinetic Energy of the pipe due to translation, $T_{p,t}$.
- Kinetic Energy of the pipe due to rotation, $T_{p,r}$.
- Kinetic Energy of the internal fluid due to translation, $T_{f,t}$.
- Kinetic Energy of the internal fluid due to the pipe flexure, $T_{f,r}$.

4.3.1 Kinetic energy of the pipe due to translation

The deformed configurations of the pipe in both *X-Z* and *X-Y* planes are shown in figures 4.2 and 4.3, respectively. Based on the analysis performed in section 3.3.1 and noting that $X \neq s$ for extensible pipe, the kinetic energy of the pipe due to translation can be written as

$$T_{p,t} = \frac{1}{2}\int_0^L m\left[\dot{w}^2 + \dot{v}^2 + \dot{u}^2\right] dX \tag{4.2}$$

where

u (x, t) : axial shortening due to bending.

w (x, t): lateral deformation of the pipe in *X-Z* plane at a distance *x* measured from the fixed end of the beam along the neutral axis in the undeformed configuration and at time *t*.

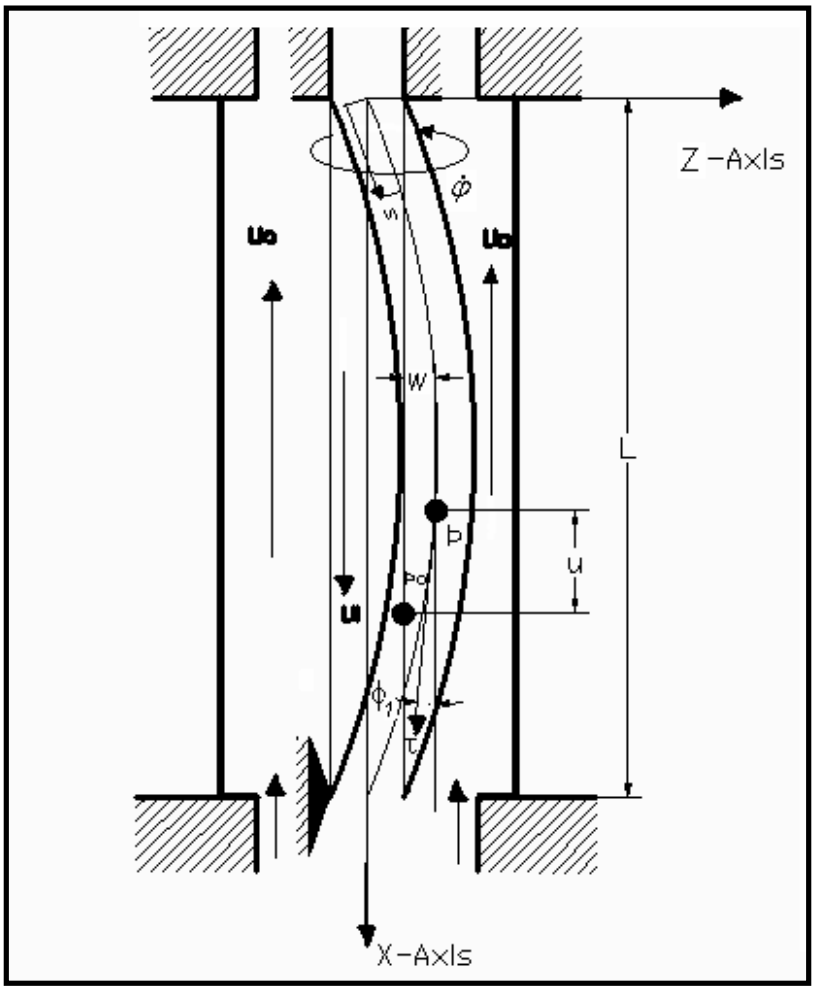


Figure 4.2: Deformed configuration of the pipe in *X-Z* plane.

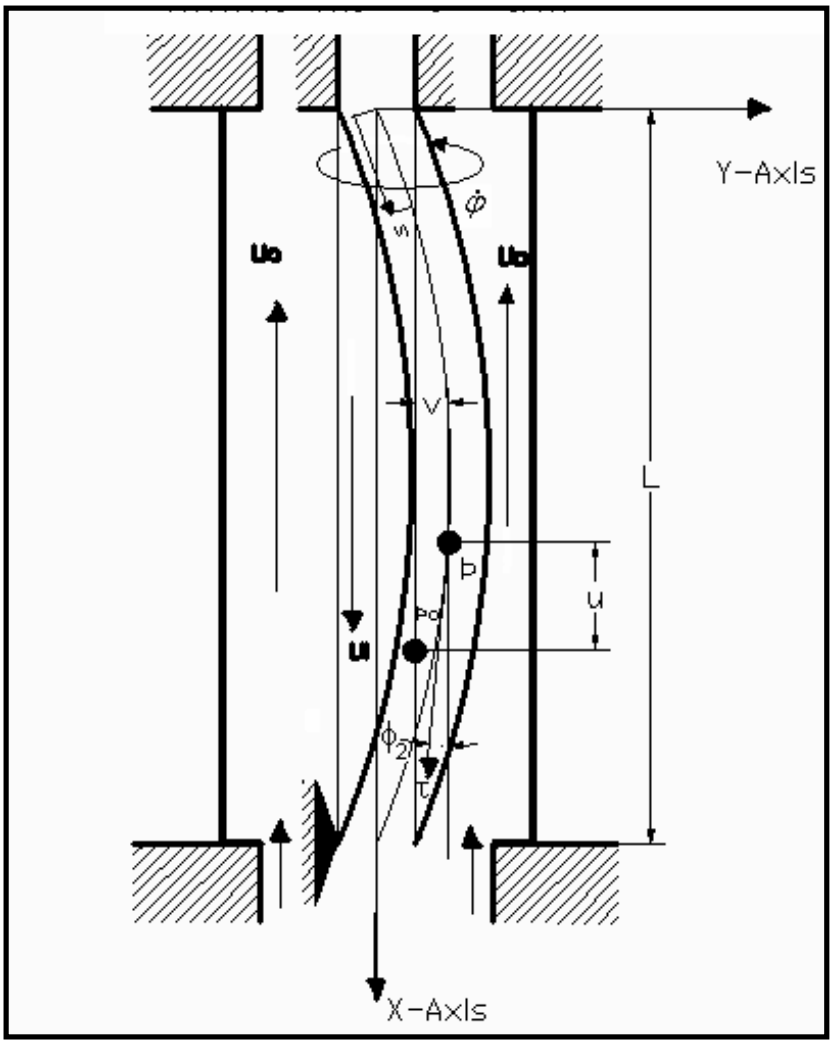


Figure 4.3: Deformed configuration of the pipe in *X*-*Y* plane.

v (x, t): lateral deformation of the pipe in *X-Y* plane at a distance *x* measured from the fixed end of the beam along the neutral axis in the undeformed configuration and at time *t*.

m: mass per unit length of the pipe.

4.3.2 Kinetic energy of the pipe due to rotation

The kinetic energy of the pipe due to rotation is derived similar to that derivation performed in section 3.3.2, hence, we can express the kinetic energy of the pipe due to rotation as

$$T_{p,r} = \frac{1}{2} I_p \int_0^L \left[\dot{\varphi}^2 + \dot{\theta}_y{}^2 \theta_z{}^2 - 2\dot{\varphi}\dot{\theta}_y \theta_z\right] dX + \frac{1}{2} I_D \int_0^L \left[\dot{\theta}_y{}^2 + \dot{\theta}_z{}^2\right] dX \tag{4.3}$$

where

I_p : mass moment of inertia per unit length of the pipe about *x*-axis.

I_D: mass moment of inertia per unit length of the pipe about *y* and *z* axes.

4.3.3 Kinetic energy of the internal flow due to translation

The basic assumption that will be used in estimating the kinetic energy is that the internal flow is approximated by a plug flow, i.e. as if it were an infinitely flexible rod traveling through the pipe, while all points of the fluid have velocity U_i relative to the pipe. This is an acceptable approximation for a fully developed turbulent flow [30], which implies

$$V_f = V_p + U_i \tau \tag{4.4}$$

where

V_f: translational velocity of the fluid.

V_p: translational velocity of the pipe.

U_i : average velocity of the internal flow.

τ : unit vector tangential to the pipe.

The unit vector tangential to the pipe τ could be defined as

$$\tau = \frac{\partial x}{\partial s} i + \frac{\partial y}{\partial s} j + \frac{\partial z}{\partial s} k = \left(1 + \frac{\partial u}{\partial s}\right) i + \frac{\partial v}{\partial s} j + \frac{\partial w}{\partial s} k \tag{4.5}$$

Following the procedure described in chapter 2, and by substituting the velocity of the pipe V_p presented in equation (3.4), and equation (4.5) into equation (4.4), we obtain the following expression for the kinetic energy of the internal flow due translation

$$\begin{aligned} T_{f,t} = & \frac{1}{2}\int_0^L M\left[\dot{w}^2 + \dot{v}^2 + \dot{u}^2\right] dX + U_i \int_0^L M\left[\dot{u} + \dot{u}\,u' + \dot{v}\,v' + \dot{w}\,w'\right] dX \\ & + \frac{1}{2}\int_0^L M\,U_i^{\,2}\left(1 + u'^2 + v'^2 + w'^2 + 2u'\right) dX \end{aligned} \tag{4.6}$$

where M is the mass of internal fluid per unit length of the pipe and the prime $(')$ denotes the derivative with respect to X.

4.3.4 Kinetic energy of the internal flow due to the pipe flexure

The kinetic energy of the internal fluid due to rotation is limited to those rotations about y and z axis [39], which are associated with the pipe motion in lateral directions, while the effect of fluid friction forces induced by the inner wall of the pipe due to its rotation will be taken into account in the potential energy expression. Accordingly, one may express the kinetic energy of the fluid due to rotation as follows:

$$T_{f,r} = \frac{1}{2} I_{D,f} \int_0^L \left[\dot{\theta}_y^{\,2} + \dot{\theta}_z^{\,2}\right] ds \tag{4.7}$$

where $I_{D,f}$ is mass moment of inertia of the fluid per unit length about y and z axis.

4.3.5 Total kinetic energy expression

The total kinetic energy expression is the summation of the kinetic energies of the pipe and fluid due to the translation and rotation. Using equations (4.2), (4.3), and (4.6-4.7), one obtains

$$\begin{aligned} T = & \frac{1}{2}(m+M)\int_0^L \left[\dot{w}^2 + \dot{v}^2 + \dot{u}^2\right] dX + U_i \int_0^L M\left[\dot{u} + \dot{u}u' + \dot{v}v' + \dot{w}w'\right] dX \\ & + \frac{1}{2}\int_0^L M U_i^{\,2}\left(1 + u'^2 + v'^2 + w'^2 + 2u'\right) dX \\ & + \frac{1}{2}(I_p)\int_0^L \left[\dot{\varphi}^2 + \dot{\theta}_y^{\,2}\theta_z^{\,2} - 2\dot{\varphi}\dot{\theta}_y\theta_z\right] dX \\ & + \frac{1}{2}\left(I_D + I_{D,f}\right)\int_0^1 \left[\dot{\theta}_y^{\,2} + \dot{\theta}_z^{\,2}\right] dX \end{aligned} \tag{4.8}$$

By recognizing that bending deformations are related to the rotations as

$$\theta_z = \frac{\partial v(x,t)}{\partial x} \tag{4.9}$$

$$\theta_y = \frac{-\partial w(x,t)}{\partial x} \tag{4.10}$$

Substituting equations (4.9-4.10) into equation, one obtains

$$
\begin{aligned}
T = & \frac{1}{2}\left(m+M\right)\int_0^L \left[\dot{w}^2+\dot{v}^2+\dot{u}^2\right]dX + U_i\int_0^L M\left[\dot{u}+\dot{u}\,u'+\dot{v}\,v'+\dot{w}\,w'\right]dX \\
& + \frac{1}{2}\int_0^L M\,U_i^{\,2}\left(1+u'^2+v'^2+w'^2+2u'\right).dX \\
& + \frac{1}{2}\left(I_p\right)\int_0^L \left[\dot{\varphi}^2+\dot{w}'^2v'^2+2\dot{\varphi}\,\dot{w}'v'\right]dX \\
& + \frac{1}{2}\left(I_D+I_{D,f}\right)\int_0^L \left[\dot{w}'^2+\dot{v}'^2\right]dX
\end{aligned}
\tag{4.11}
$$

where prime $(')$ denotes the derivative with respect to X.

By comparing equations (4.11) and (3.22), the only difference is that U^2 term associated with the extensible case survive in the kinetic energy expression. This term vanishes in the case of inextensibility condition.

4.4 Potential Energy Expression

The system potential energy is made up of the elastic pipe-bending strain energy V_b, strain energy due to axial deformation V_a, strain energy due to torsion V_t, and strain energy due gravitational force V_g.

4.4.1 Strain energy due to bending

Following the analysis presented by Stocker [39], the strain energy due to bending can be expressed by

$$
V_b = \frac{E\,I}{2}\int_0^L \left[\left(1+\varepsilon\right)^2\kappa^2\right]dX \tag{4.12}
$$

where

$\kappa(X)$: is the curvature of the centerline of the pipe.

I : Area moment of inertia of the pipe.

Note that, for the three-dimensional analysis, it is necessary to consider the bending in both *X*-*Y* and *X*-*Z* planes.

Referring to figures 4.2 and 4.3, let ϕ_1 and ϕ_2 are the angles between the position of the pipe and the *X* axis in both *X*-*Z* and *X*-*Y* planes, respectively, and *s* is the curvilinear coordinate along the pipe. For a pipe undergoining planar motion, either extensible or inextensible, the curvature κ is given by

$$\kappa = \frac{\partial \phi_1}{\partial s} + \frac{\partial \phi_2}{\partial s} \tag{4.13}$$

Upon using equation (2.8), the curvature κ in terms of *X* coordinate can be obtained as

$$\kappa = \frac{1}{1+\varepsilon}\frac{\partial \phi_1}{\partial X} + \frac{1}{1+\varepsilon}\frac{\partial \phi_2}{\partial X} \tag{4.14}$$

Based on the geometry, the angles ϕ_1 and ϕ_2 can be defined as

$$\cos\ \phi_1 = \cos\ \phi_2 = \frac{1+(\partial u/\partial X)}{1+\varepsilon(X)} \tag{4.15}$$

and

$$\sin\ \phi_1 = \frac{(\partial v/\partial X)}{1+\varepsilon(X)}\ ,\ \sin\ \phi_2 = \frac{(\partial w/\partial X)}{1+\varepsilon(X)} \tag{4.16}$$

Using equations (4.15) and (4.16), one gets

$$\frac{\partial \phi_1}{\partial X} = \frac{\left[\frac{\partial^2 v}{\partial X^2}\left(1+\frac{\partial u}{\partial X}\right) - \frac{\partial v}{\partial X}\frac{\partial^2 u}{\partial X^2}\right]}{(1+\varepsilon)^2}\ ,\ \frac{\partial \phi_2}{\partial X} = \frac{\left[\frac{\partial^2 w}{\partial X^2}\left(1+\frac{\partial u}{\partial X}\right) - \frac{\partial w}{\partial X}\frac{\partial^2 u}{\partial X^2}\right]}{(1+\varepsilon)^2} \tag{4.17}$$

By substituting equation (4.17) into equation (4.14), yields

$$\kappa = \frac{\left[\frac{\partial^2 v}{\partial X^2}\left(1+\frac{\partial u}{\partial X}\right) - \frac{\partial v}{\partial X}\frac{\partial^2 u}{\partial X^2}\right]}{(1+\varepsilon)^3} + \frac{\left[\frac{\partial^2 w}{\partial X^2}\left(1+\frac{\partial u}{\partial X}\right) - \frac{\partial w}{\partial X}\frac{\partial^2 u}{\partial X^2}\right]}{(1+\varepsilon)^3} \tag{4.18}$$

The strain is given by [3] as

$$\varepsilon = u' + \frac{1}{2} v'^2 + \tilde{O}(\epsilon^4) \tag{4.19}$$

Substituting equations (4.18) and (4.19) into equation (4.12), retaining terms up to fourth order, yields the following expression for the beam bending potential energy:

$$V_b = \frac{EI}{2} \int_0^L \begin{bmatrix} v''^2 - 2v''^2 u' - 2v''^2 v'^2 - 2v'v''u'' \\ + w''^2 - 2w''^2 u' - 2w''^2 w'^2 - 2w'w''u'' \\ + 4v''^2 v'^2 \; w''^2 w'^2 \end{bmatrix} dX \tag{4.20}$$

4.4.2 Strain energy due axial deformation

Based on the stress-strain relationship, the strain energy associated with axial deformation V_a is expressed by

$$V_a = \frac{EA}{2} \int_0^L \varepsilon^2 \, dX \tag{4.21}$$

Other components of strain energy related to the axial deformation are associated with the tension $\breve{T}$ and differential pressurization force P.

For generalization, all these components are considered in this formulation. The potential energy due to axial deformation can be rewritten as

$$V_a = \frac{EA}{2} \int_0^L \left(\frac{\breve{T} - P}{EA} + \varepsilon \right)^2 dX \tag{4.22}$$

where

P: Fluid pressurization force.

$\breve{T}$: Tension in the pipe either externally applied or associated with frictional forces.

Substituting equation (4.19) into equation (4.22) for both *X*-*Y* and *X*-*Z* planes, and retaining terms up to fourth order yields

$$V_a = \frac{E\,A}{2}\int_0^L \begin{bmatrix} u'^2 + \frac{1}{4}v'^4 + u'v'^2 \\ \frac{1}{4}w'^4 + u'w'^2 \end{bmatrix} dX + \int_0^L \left(\breve{T} - P\right) \begin{pmatrix} u' + \frac{1}{2}v'^2 - \frac{u'v'^2}{2} - \frac{1}{8}v'^4 \\ + \frac{1}{2}w'^2 - \frac{u'w'^2}{2} - \frac{1}{8}w^4 \end{pmatrix} dX + \int_0^L \left[\frac{\left(\breve{T} - P\right)^2}{E\,A}\right] dX \tag{4.23}$$

4.4.3 Strain energy due to torsion

The strain energy V_t due to torsion is given by

$$V_t = \frac{G\ J}{2}\int_0^L \varphi'^2\, ds \tag{4.24}$$

where

G: Modulus of rigidity for the pipe.

J: polar moment of inertia for the pipe.

Other component of strain energy due to torsion should be considered as associated with the frictional torque *Γ*. By considering such component, then one can write strain energy as

$$V_t = \frac{G\ J}{2}\int_0^L \varphi'^2\, ds - \int_0^L \Gamma\, \varphi'\, ds + \int_0^L \frac{\Gamma^2}{2\,G\,J}\, ds \tag{4.25}$$

where *Γ* is the torque exerted on the pipe due to the fluid-induced frictional forces of the rotating pipe.

4.4.4 Strain energy due to gravitational field

The gravitational energy depends on the distribution of mass, and can be expressed in x direction as

$$V_g = -(m+M)\,g\int_0^L x\,dx \tag{4.26}$$

where g is the gravitational acceleration.

4.4.5 Total potential energy expression

The total potential energy expression is the summation of the potential energies due to bending, torsion, and axial deformations in addition to gravitational potential energy. Using equations (4.20), (4.23), (4.25-4.26), the total potential energy can be expressed by

$$\begin{aligned} V \;=\; & \frac{EI}{2}\int_0^L \begin{bmatrix} v''^2 - 2v''^2u' - 2v''^2v'^2 - 2v'v''u'' \\ + w''^2 - 2w''^2u' - 2w''^2w'^2 - 2w'w''u'' \\ + 4v''^2v'^2\; w''^2w'^2 \end{bmatrix} dX + \frac{EA}{2}\int_0^L \begin{bmatrix} u'^2 + \frac{1}{4}v'^4 + u'v'^2 \\ + \frac{1}{4}w'^4 + u'w'^2 \end{bmatrix} dX \\ & + \int_0^L \left(\breve{T} - P\right)\begin{pmatrix} u' + \frac{1}{2}v'^2 - \frac{u'v'^2}{2} - \frac{1}{8}v'^4 \\ + \frac{1}{2}w'^2 - \frac{u'w'^2}{2} - \frac{1}{8}w'^4 \end{pmatrix} dX + \int_0^L \left[\frac{\left(\breve{T} - P\right)^2}{EA}\right] dX + \frac{GJ}{2}\int_0^L \varphi'^2\,dX - \int_0^L \Gamma\varphi'\,ds \\ & + \int_0^L \frac{\Gamma^2}{2GJ}\,ds - (m+M)\,g\int_0^L \left(X+u\right)dX \end{aligned} \tag{4.27}$$

By comparing this equation with equation (3.32), which is associated with the inextensible condition, we note that current formulation of the potential energy expression accounts for the forces associated with the differential change between

internal and external pressures; also it accommodates the tension in the pipe either externally applied or associated with the frictional forces. Such forces vanish in equation (3.32) due to the inextensibility condition.

4.5 The Lagrangian Function

Based on the definition of Lagrangian function presented in equation (3.34), and by using equations (4.11) and (4.27), the Lagrangian function $\tilde{L}$ can be written as

$$
\begin{aligned}
\tilde{L} = & \frac{1}{2}(m+M)\int_0^L \left[\dot{w}^2+\dot{v}^2+\dot{u}^2\right]dX + U_i\int_0^L M\left[\dot{u}+\dot{u}\,u'+\dot{v}\,v'+\dot{w}\,w'\right]dX \\
& + \frac{1}{2}\int_0^L M\,U_i^{\,2}\left(1+u'^2+v'^2+w'^2+2u'\right)dX \\
& + \frac{1}{2}\left(I_p\right)\int_0^L\left[\dot{\varphi}^2+\dot{w}'^2v'^2+2\dot{\varphi}\,\dot{w}'v'\right]dX + \frac{1}{2}\left(I_D+I_{D,f}\right)\int_0^L\left[\dot{w}'^2+\dot{v}'^2\right]dX \\
& - \frac{E\,I}{2}\int_0^L \begin{bmatrix} v''^2-2v''^2u'-2v''^2v'^2-2v'v''u'' \\ +w''^2-2w''^2u'-2w''^2w'^2-2w'w''u'' \\ +4v''^2v'^2\;w''^2w'^2 \end{bmatrix} dX \\
& - \frac{E\,A}{2}\int_0^L\left[u'^2+\frac{1}{4}v'^4+u'v'^2+\frac{1}{4}w'^4+u'w'^2\right]dX \\
& - \int_0^L\left(\breve{T}-P\right)\begin{pmatrix} u'+\frac{1}{2}v'^2-\frac{u'v'^2}{2}-\frac{1}{8}v'^4 \\ +\frac{1}{2}w'^2-\frac{u'w'^2}{2}-\frac{1}{8}w'^4 \end{pmatrix} dX \\
& - \int_0^L\left[\frac{\left(\breve{T}-P\right)^2}{E\,A}\right]dX - \frac{G\,J}{2}\int_0^L \varphi'^2\,dX + \int_0^L \Gamma\,\varphi'\,dX - \int_0^L \frac{\Gamma^2}{2\,G\,J}\,dX \\
& + (m+M)\,g\int_0^L\left(X+u\right)dX
\end{aligned}
\tag{4.28}
$$

4.6 Hydrodynamic Forces Exerted on the Pipe

The hydrodynamic forces acting on the pipe may be summarized as follow:

- The inviscid hydrodynamic force, F_A due to the axial motion of the external flow. This term is identical to the corresponding term derived for inextensible pipe, as given by equation (3.38).
- Normal frictional force F_N due to the axial motion of the external flow, this term is identical for extensible and inextensible pipes and given as equation (3.39).
- Longitudinal frictional forces, F_L which is stated as equation (3.40).
- Hydrostatic pressure forces in the x, y and z directions, F_{px}, F_{py} and F_{pz} respectively. The expressions of these forces are given by equations (3.41-3.43).
- Hydrodynamic forces exerted on the pipe due to the pipe rotation, F_Z and F_Y. Based on the same assumptions and analysis presented in section 3.8, these forces are defined as equations (3.70) and (3.71), respectively.

4.7 Equations of Motion

The equations of motion of a flexible pipe conveying fluid and subjected to external axial flow are derived using the extended Hamilton's principle.

4.7.1 Hamilton's principle

The extended Hamilton's principle is defined as

$$\delta \int_{t_o}^{t_f} L \, dt + \int_{t_o}^{t_f} \delta W_{nc} \, dt = 0 \tag{4.29}$$

It is important to note that even if there are no explicit external forces applied to the pipe conveying fluid, δW_{nc} in equation (4.29) will not vanish if one end of the pipe is free, as discussed in subsection 3.9.1.

4.7.2 Variation of the Lagrangian function

In order to perform the variation of the Lagrangian function, it is essential to specify the boundary conditions, which can be stated based on the geometry of the problem as

$$u(X=0)=v(X=0)=w(X=0)=0 \tag{4.30}$$

$$v(X=L)=w(X=L)=u(X=L)=0 \tag{4.31}$$

$$v'(X=0)=v''(X=L)=w'(X=0)=w''(X=L)=0 \tag{4.32}$$

For simplicity, the variational analysis is performed separately for both kinetic and potential energy expressions.

4.7.2.1 Variation of the kinetic energy expression

Upon applying Hamilton's variational operator to equation (4.11), and after some mathematical manipulations, we obtains

$$
\begin{aligned}
\delta \int_{t_o}^{t_f} T\, dt = & -\int_{t_o}^{t_f}\int_{0}^{L} \left[(m+M)\ddot{u} + (2M\,U_i)\,\dot{u}' + M\,U_i^{\,2}\,u'' \right] \delta u\, dX\, dt \\
& -\int_{t_o}^{t_f}\int_{0}^{L} \left[\begin{array}{l} (m+M)\ddot{v} + (2M\,U_i)\,\dot{v}' + M\,U_i^{\,2}\,v'' + \left(I_D + I_{D,f}\right)\left(2\,\ddot{v}''\right) \\ +\left(I_p\right)\left(v''\dot{w}'^{\,2} + 2v'\dot{w}'\dot{w}'' + \dot{\varphi}\,\dot{w}'' + \dot{w}'\,\dot{\varphi}'\right) \end{array} \right] \delta v\, dX\, dt \\
& -\int_{t_o}^{t_f}\int_{0}^{L} \left[\begin{array}{l} (m+M)\ddot{w} + (2M\,U_i)\,\dot{w}' + M\,U_i^{\,2}\,w'' + \left(I_D + I_{D,f}\right)\left(2\,\ddot{w}''\right) \\ +\left(I_p\right)\left(\begin{array}{l} \dot{w}'\left(v'\dot{v}'' + \dot{v}'v''\right) + 2v'^{\,2}\ddot{w}'' + 3\ddot{w}'v'v'' + 2\dot{w}''v'\dot{v}' \\ + \dfrac{\dot{\varphi}\left(v'\ddot{w}' + \dot{w}''\dot{v}'\right)}{\dot{w}'} - \dfrac{2\dot{\varphi}v'\dot{w}''\ddot{w}'}{\dot{w}'^{\,2}} + \dfrac{\ddot{\varphi}v'\dot{w}''}{\dot{w}'} + \dfrac{\dot{\varphi}'v'\ddot{w}'}{\dot{w}'} \\ + \dfrac{\dot{\varphi}\left(v'\ddot{w}'' + \ddot{w}'v''\right)}{\dot{w}'} + \dot{\varphi}\dot{v}'' + v''\ddot{\varphi} + v'\ddot{\varphi}' + \dot{\varphi}'\dot{v}' \end{array} \right) \end{array} \right] \delta w\, dX\, dt \\
& -\int_{t_o}^{t_f}\int_{0}^{L} \left[\left(I_p\right)\left(\ddot{\varphi} + \dot{w}'\dot{v}' + v'\ddot{w}'\right) \right] \delta\varphi\, dX\, dt
\end{aligned}
\tag{4.33}
$$

4.7.2.2 Variation of the potential energy expression

By considering equation (3.81), which defines the torque associated with the frictional forces exerted on the pipe due to the internal flow rotation, and upon applying Hamilton's variational operator to equation (4.27), and after many mathematical manipulations, one obtains

$$
\begin{aligned}
\int_{t_o}^{t_1} \delta V\, dt = \quad & \int_{t_o}^{t_f}\int_0^L \left\{ \begin{array}{l} (E\,A)u'' - (\breve{T} - P - E\,A)v'\,v'' \\ + E\,I\,(v''''\,v' + v''\,v''') + (m+M)\,g \\ - (\breve{T} - P - E\,A)w'\,w'' \\ + E\,I\,(w''''\,w' + w''\,w''') \end{array} \right\} \delta u\, dX\, dt \\
& + \int_{t_o}^{t_f}\int_0^L \left\{ \begin{array}{l} (\breve{T} - P)v'' - (E\,I)v'''' \\ + EI \begin{pmatrix} 3u'''\,v'' + 4u''\,v''' + 2u'\,v'''' + v'\,u'''' \\ + 2v'^2\,v'''' + 8v'\,v''\,v''' + 2v''^3 \end{pmatrix} \\ - (\breve{T} - P - EA)\left(u''\,v' + u'\,v'' + \frac{3}{2}v'^2\,v'' \right) \\ - (m+M)\,g \begin{pmatrix} v' - \frac{1}{2}v'^3 + uv'' \\ - (L-X)\left(v'' - u''v' - u'v'' - \frac{3}{2}v'^2\,v'' \right) \end{pmatrix} \end{array} \right\} \delta v\, dX\, dt \\
& + \int_{t_o}^{t_f}\int_0^L \left\{ \begin{array}{l} (\breve{T} - P)w'' - (E\,I)w'''' \\ + EI \begin{pmatrix} 3u'''\,w'' + 4u''\,w''' + 2u'\,w'''' + w'\,u'''' \\ + 2w'^2\,w'''' + 8w'\,w''\,w''' + 2w''^3 \end{pmatrix} \\ - (\breve{T} - P - EA)\left(u''\,w' + u'\,w'' + \frac{3}{2}w'^2\,w'' \right) \\ - (m+M)\,g \begin{pmatrix} w' - \frac{1}{2}w'^3 + uw'' \\ - (L-X)\left(w'' - u''w' - u'w'' - \frac{3}{2}w'^2\,w'' \right) \end{pmatrix} \end{array} \right\} \delta w\, dX\, dt \qquad (4.34) \\
& - \int_{t_o}^{t_f}\int_0^L \left[G\,J\,\varphi'' - 4\,I_{p,f}\,L\;\; f_r\,(\varphi'\ddot{\varphi} + 2\dot{\varphi}\dot{\varphi}') \right] \delta\varphi\, ds\, dt
\end{aligned}
$$

4.7.2.3 Variation of the Lagrangian function

Upon substituting equations (4.33) and (4.34) in the Lagrangian formula, one obtains the expression of variation of the Lagrangian function as

$$
\begin{aligned}
&\delta \int_{t_o}^{t_f} \tilde{L}\, dt = \\
&-\int_{t_o}^{t_f}\int_{0}^{L} \begin{bmatrix} \left[(m+M)\ddot{u} + (2M\,U_i)\,\dot{u}' + M\,U_i^{\,2}\,u''\right] - (E\,A)u'' + \left(\breve{T} - P - E\,A\right)v'\,v'' \\ -E\,I\left(v''''\,v' + v''\,v'''\right) - (m+M)\,g + \left(\breve{T} - P - E\,A\right)w'\,w'' - E\,I\left(w''''\,w' + w''\,w'''\right) \end{bmatrix} \delta u\, dX\, dt \\
&-\int_{t_o}^{t_f}\int_{0}^{L} \begin{bmatrix} (m+M)\ddot{v} + (2M\,U_i)\,\dot{v}' + M\,U_i^{\,2}\,v'' - \left(\breve{T} - P\right)v'' + (E\,I)v'''' \\ + \left(I_D + I_{D,f}\right)\left(2\,\ddot{v}''\right) + \left(I_p\right)\left(v''\dot{w}'^2 + 2v'\dot{w}'\dot{w}'' + \dot{\varphi}\,\dot{w}'' + \dot{w}'\,\dot{\varphi}'\right) \\ - EI\begin{pmatrix} 3u''''\,v'' + 4u''\,v''' + 2u'\,v'''' + v'\,u'''' \\ + 2v'^2\,v'''' + 8v'\,v''\,v''' + 2v''^3 \end{pmatrix} + \left(\breve{T} - P - EA\right)\left(u''\,v' + u'\,v'' + \frac{3}{2}v'^2\,v''\right) \\ + (m+M)\,g\left(v' - \frac{1}{2}v'^3 + uv'' - (L-X)\left(v'' - u''v' - u'v'' - \frac{3}{2}v'^2\,v''\right)\right) \end{bmatrix} \delta v\, dX\, dt \\
&-\int_{t_o}^{t_f}\int_{0}^{L} \begin{bmatrix} (m+M)\ddot{w} + (2M\,U_i)\,\dot{w}' + M\,U_i^{\,2}\,w'' + \left(I_D + I_{D,f}\right)\left(2\,\ddot{w}''\right) \\ + \left(I_p\right)\begin{pmatrix} \dot{w}'\left(v'\dot{v}'' + \dot{v}'v''\right) + 2v'^2\ddot{w}'' + 3\ddot{w}'v'v'' + 2\dot{w}''v'\dot{v}' + \dfrac{\dot{\varphi}\left(v'\ddot{w}' + \dot{w}''\dot{v}'\right)}{\dot{w}'} \\ - \dfrac{2\dot{\varphi}\,v'\dot{w}''\ddot{w}'}{\dot{w}'^2} + \dfrac{\ddot{\varphi}\,v'\dot{w}''}{\dot{w}'} + \dfrac{\dot{\varphi}'\,v'\ddot{w}'}{\dot{w}'} + \dfrac{\dot{\varphi}\left(v'\ddot{w}'' + \ddot{w}'v''\right)}{\dot{w}'} + \dot{\varphi}\,\dot{v}'' + v''\ddot{\varphi} \\ + v'\ddot{\varphi}' + \dot{\varphi}'\,\dot{v}' \end{pmatrix} \\ - \left(\breve{T} - P\right)w'' + (E\,I)w'''' - EI\begin{pmatrix} 3u''''\,w'' + 4u''\,w''' + 2u'\,w'''' + w'\,u'''' \\ + 2w'^2\,w'''' + 8w'\,w''\,w''' + 2w''^3 \end{pmatrix} \\ + \left(\breve{T} - P - EA\right)\left(u''\,w' + u'\,w'' + \frac{3}{2}w'^2\,w''\right) \\ + (m+M)\,g\left(w' - \frac{1}{2}w'^3 + uw'' - (L-X)\left(w'' - u''w' - u'w'' - \frac{3}{2}w'^2\,w''\right)\right) \end{bmatrix} \delta w\, dX\, dt \\
&-\int_{t_o}^{t_f}\int_{0}^{L} \left[\left(I_p\right)\left(\ddot{\varphi} + \dot{w}'\dot{v}' + v'\ddot{w}'\right) - \left(G\,J\,\varphi'' - 4\,I_{p,f}\,L\;\; f_r\left(\varphi'\ddot{\varphi} + 2\dot{\varphi}\,\dot{\varphi}'\right)\right)\right]\;\; \delta\varphi\, dX\, dt
\end{aligned}
\tag{4.35}
$$

4.7.3 *Total virtual work of the external fluid hydrodynamic forces*

The total virtual work of hydrodynamic forces associated with the external fluid is due to the axial and rotational motions of the external flow. Based on the analysis presented in chapter 3, and after some mathematical manipulations, which include retaining the terms up to 3rd order, the following expressions of virtual work in *x*, *y* and *z* directions are obtained:

$$\delta W_x = -\int_{t_o}^{t_f}\int_0^L \left\{ \begin{array}{l} -M_a\left(\ddot{v}\,v' + \ddot{w}\,w'\right) - M_a\,U_o\left(-2\,\dot{v}'\,v' - 2\,\dot{w}'w'\right) \\ -M_a\,U_o^{\,2}\left(v'\,v'' + w'w''\right) \\ -\left(\dfrac{\rho_{f,o}\;\;D_o U_o^{\,2} C_T}{2}\,\dfrac{D_o}{D_h}\right)\left(\begin{array}{l} -\dfrac{1}{2}\left(v'^2 + w'^2\right) + u' \\ +\left(L - X\right)\left(v'\,v'' + w'w''\right) \end{array}\right) \\ +\left(1 - 2\upsilon\right)A_o\;\overline{p}_o\left(v'\,v'' + w'w''\right) \\ -\left(\rho_{f,o}\;A_o\;g\right)\left(\begin{array}{l} -\dfrac{1}{2}\left(v'^2 + w'^2\right) + u' \\ +\left(\dfrac{L}{2} - X\right)\left(v'\,v'' + w'w''\right) \end{array}\right) \\ -\left(\dfrac{\rho_{f,o}\;\;D_o U_o^{\,2} C_T}{2}\right)\left(\begin{array}{l} 1 + \left(v'^2 + w'^2\right) - \dfrac{\left(v'\dot{v} + w'\dot{w}\right)}{U_o} \\ -\dfrac{\left(\dot{v}^2 + \dot{w}^2\right)}{2U_o^{\,2}} \end{array}\right) \\ -\left(\dfrac{\rho_{f,o}\;\;D_o U_o^{\,2} C_N}{2}\right)\left(-\left(v'^2 + w'^2\right) + \dfrac{\left(\dot{v}\,v' + \dot{w}w'\right)}{U_o}\right) \\ -\left(\dfrac{\rho_{f,o}\;\;D_o\,U_o\,C_{DP}}{2}\right)\left(\begin{array}{l} -v'^2\left|v'\right| + \dfrac{\left|\dot{v}\right|v'^2 + \left|v'\right|v'\dot{v}}{U_o} + \dfrac{\dot{v}\left|\dot{v}\right|v'}{U_o^{\,2}} \\ -w'^2\left|w'\right| + \dfrac{\left|\dot{w}\right|w'^2 + \left|w'\right|w'\dot{w}}{U_o} \\ +\dfrac{\dot{w}\left|\dot{w}\right|w'}{U_o^{\,2}} \end{array}\right) \end{array} \right\} \delta u\,dXdt \qquad (4.36)$$

$$
\delta W_y = -\int_{t_o}^{t_f}\int_0^L \left\{
\begin{array}{l}
\left(\dfrac{\rho_{f,o}\ D_o U_o{}^2 C_T}{2}\dfrac{D_o}{D_h}\right)\left(\begin{array}{l}\left(v' - \dfrac{1}{2}v'^3 + u\,v''\right)\\ -(L-X)\left(\begin{array}{l} v'' - u''v' \\ -u'v'' - \dfrac{3}{2}v'^2\,v''\end{array}\right)\end{array}\right)\\
-\rho_{f,o}\,A_o\,g\left(\begin{array}{l}\left(v' - \dfrac{1}{2}v'^3 + u\,v''\right)\\ -\left(\dfrac{L}{2} - X\right)\left(\begin{array}{l} v'' - u''v' \\ -u'v'' - \dfrac{3}{2}v'^2\,v''\end{array}\right)\end{array}\right)\\
-(1-2\upsilon)A_o\,\overline{p}_o\left(v'' - u''v' - u'v'' - \dfrac{3}{2}v'^2\,v''\right)\\
-\left(\dfrac{\rho_{f,o}\ D_o\,U_o{}^2 C_T}{2}\right)\left(v' - u'v' - \dfrac{\dot{v}\,v'^2}{U_o} - \dfrac{\dot{v}^2\,v'}{2U_o{}^2}\right)\\
+\left(\dfrac{\rho_{f,o}\ D_o\,U_o{}^2 C_N}{2}\right)\left(\begin{array}{l} -v' + \dfrac{\dot{v}}{U_o} + \dfrac{\dot{v}u'}{U_o} + u'v' + \dfrac{\dot{u}\,\dot{v}}{U_o{}^2} \\ +v'^3 - \dfrac{\dot{v}\,v'^2}{U_o} - \dfrac{\dot{v}^3}{2U_o{}^3} - \dfrac{\dot{v}^2 v'}{2U_o{}^2}\end{array}\right)\\
+\left(\dfrac{\rho_{f,o}\ D_o\,U_o{}^2 C_{DP}}{2}\right)\left(-v'|v'| + \dfrac{|\dot{v}|\ v' + |v'|\ \dot{v}}{U_o} + \dfrac{\dot{v}|\dot{v}|}{U_o{}^2}\right)\\
+M_a\left(\begin{array}{l}\ddot{v} - \ddot{u}\,v' - 2\dot{u}\,\dot{v}' - \ddot{v}v'^2 - \dfrac{3}{2}\dot{v}\,\dot{v}'\,v' \\ +\dot{v}\,f_t - \dfrac{\dot{\varphi}^2}{4}v + \dot{\varphi}\,\dot{w} + \dfrac{\dot{\varphi}\,f_t}{2}w\end{array}\right)\\
+M_a\,U_o\left(-2\dot{v}' + 3\dot{u}'v' + 4u'\dot{v}' + \dfrac{7}{2}\dot{v}'v'^2 + 2\dot{u}\,v'' + \dfrac{3}{2}\dot{v}v'v''\right)\\
+M_a\,U_o{}^2\left(v'' - 2u''v' - 4u'v'' - \dfrac{5}{2}v''\,v'^2\right)
\end{array}\right\}\delta v\,dX\,dt
$$

(4.37)

and,

$$\delta W_z = -\int_{t_o}^{t_f}\int_0^L \left\{ \begin{array}{l} -\left(\dfrac{\rho_{f,o}\ D_o U_o{}^2 C_T}{2}\dfrac{D_o}{D_h}\right)\left(\begin{array}{l}\left(w' - \dfrac{1}{2}w'^3 + u\,w''\right) \\ -(L-X)\left(\begin{array}{l} w'' - u''w' \\ -u'w'' - \dfrac{3}{2}w'^2\,w''\end{array}\right)\end{array}\right) \\ -\rho_{f,o}\,A_o\,g\left(\begin{array}{l}\left(w' - \dfrac{1}{2}w'^3 + u\,w''\right) \\ -\left(\dfrac{L}{2} - X\right)\left(\begin{array}{l} w'' - u''w' \\ -u'w'' - \dfrac{3}{2}w'^2\,w''\end{array}\right)\end{array}\right) \\ -(1-2\upsilon)A_o\,\overline{p}_o\left(w'' - u''w' - u'w'' - \dfrac{3}{2}w'^2\,w''\right) \\ -\left(\dfrac{\rho_{f,o}\ D_o U_o{}^2 C_T}{2}\right)\left(w' - u'w' - \dfrac{\dot{w}\,w'^2}{U_o} - \dfrac{\dot{w}^2\,w'}{2U_o{}^2}\right) \\ +\left(\dfrac{\rho_{f,o}\ D_o U_o{}^2 C_N}{2}\right)\left(\begin{array}{l} -w' + \dfrac{\dot{w}}{U_o} + \dfrac{\dot{w}u'}{U_o} + u'w' + \dfrac{\dot{u}\,\dot{w}}{U_o{}^2} \\ +w'^3 - \dfrac{\dot{w}\,w'^2}{U_o} - \dfrac{\dot{w}^3}{2U_o{}^3} - \dfrac{\dot{w}^2 w'}{2U_o{}^2}\end{array}\right) \\ +\left(\dfrac{\rho_{f,o}\ D_o U_o{}^2 C_{DP}}{2}\right)\left(-w'|w'| + \dfrac{|\dot{w}|\ w' + |w'|\ \dot{w}}{U_o} + \dfrac{\dot{w}|\dot{w}|}{U_o{}^2}\right) \\ +M_a\left(\begin{array}{l}\ddot{w} - \ddot{u}\,w' - 2\dot{u}\,\dot{w}' - \ddot{w}w'^2 - \dfrac{3}{2}\dot{w}\,\dot{w}'\,w' \\ +\dot{w}\,f_t - \dfrac{\dot{\varphi}^2}{4}w + \dot{\varphi}\,\dot{v} + \dfrac{\dot{\varphi}\,f_t}{2}v\end{array}\right) \\ +M_a\,U_o\left(-2\dot{w}' + 3\dot{u}'w' + 4u'\dot{w}' + \dfrac{7}{2}\dot{w}'w'^2 + 2\dot{u}\,w'' + \dfrac{3}{2}\dot{w}w'w''\right) \\ +M_a\,U_o{}^2\left(w'' - 2u''w' - 4u'w'' - \dfrac{5}{2}w''\,w'^2\right)\end{array}\right\}\delta\,w\,dX\,dt$$

(4.38)

4.7.4 Formulation of equations of motion

By substituting equations (4.35-4.38), into the extended Hamilton's principle given by equation (4.29), one eventually finds the following four coupled equations of motion in terms of *u*, *v*, *w* and *φ*, which describe the motion of an extensible rotating pipe conveying fluid and subjected to counter external axial flow:

$$
\begin{aligned}
&\left[(m+M)\ddot{u}+(2M\,U_i)\,\dot{u}'+M\,U_i^{\,2}\,u''\right]-(E\,A)u''+\left(\breve{T}-P-E\,A\right)v'\,v'' \\
&-E\,I\,(v''''\,v'+v''\,v''')-(m+M)\,g\;+\left(\breve{T}-P-E\,A\right)w'\,w''-E\,I\,(w''''\,w'+w''\,w''') \\
&-M_a\left(\ddot{v}\,v'+\ddot{w}\,w'\right)-M_a\,U_o\left(-2\,\dot{v}'\,v'-2\,\dot{w}'w'\right)-M_a\,U_o^{\,2}\left(v'\,v''+w'w''\right) \\
&-\left(\frac{\rho_{f,o}\;\;D_oU_o^{\,2}C_T}{2}\frac{D_o}{D_h}\right)\begin{pmatrix}-\frac{1}{2}\left(v'^2+w'^2\right)+u' \\ +(L-X)\left(v'\,v''+w'w''\right)\end{pmatrix} \\
&+(1-2\upsilon)A_o\,\overline{p}_o\left(v'\,v''+w'w''\right)-\left(\rho_{f,o}\;A_o\,g\right)\begin{pmatrix}-\frac{1}{2}\left(v'^2+w'^2\right)+u' \\ +\left(\frac{L}{2}-X\right)\left(v'\,v''+w'w''\right)\end{pmatrix} \\
&-\left(\frac{\rho_{f,o}\;\;D_oU_o^{\,2}C_T}{2}\right)\begin{pmatrix}1+\left(v'^2+w'^2\right)-\frac{\left(v'\dot{v}+w'\dot{w}\right)}{U_o} \\ -\frac{\left(\dot{v}^2+\dot{w}^2\right)}{2U_o^{\,2}}\end{pmatrix} \\
&-\left(\frac{\rho_{f,o}\;\;D_oU_o^{\,2}C_N}{2}\right)\left(-\left(v'^2+w'^2\right)+\frac{\left(\dot{v}\,v'+\dot{w}w'\right)}{U_o}\right) \\
&-\left(\frac{\rho_{f,o}\;\;D_o\,U_o\,C_{DP}}{2}\right)\begin{pmatrix}-v'^2\left|v'\right|+\frac{\left|\dot{v}\right|v'^2+\left|v'\right|v'\dot{v}}{U_o}+\frac{\dot{v}\left|\dot{v}\right|v'}{U_o^{\,2}} \\ -w'^2\left|w'\right|+\frac{\left|\dot{w}\right|w'^2+\left|w'\right|w'\dot{w}}{U_o} \\ +\frac{\dot{w}\left|\dot{w}\right|w'}{U_o^{\,2}}\end{pmatrix} \;=\; 0
\end{aligned}
\tag{4.39}
$$

$$
\begin{aligned}
&(m+M)\ddot{v}+(2M\,U_i)\,\dot{v}'+M\,U_i^{\,2}\,v''+\left(I_D+I_{D,f}\right)(2\,\ddot{v}'')\\
&+\left(I_p\right)\left(v''\dot{w}'^2+2v'\dot{w}'\dot{w}''+\dot{\varphi}\,\dot{w}''+\dot{w}'\dot{\varphi}'\right)-\left(\breve{T}-P\right)v''+(E\,I)v''''\\
&-EI\begin{pmatrix}3u'''\,v''+4u''\,v'''+2u'\,v''''+v'\,u''''\\+2\,v'^2\,v''''+8\,v'\,v''\,v'''+2\,v''^3\end{pmatrix}+\left(\breve{T}-P-EA\right)\left(u''\,v'+u'\,v''+\frac{3}{2}v'^2\,v''\right)\\
&+(m+M)\,g\left(v'-\frac{1}{2}v'^3+uv''-(L-X)\left(v''-u''v'-u'v''-\frac{3}{2}v'^2\,v''\right)\right)\\
&-\left(\frac{\rho_{f,o}\;\;D_oU_o^{\,2}C_T}{2}\frac{D_o}{D_h}\right)\begin{pmatrix}\left(v'-\frac{1}{2}v'^3+u\,v''\right)\\-(L-X)\begin{pmatrix}v''-u''v'\\-u'v''-\frac{3}{2}v'^2\,v''\end{pmatrix}\end{pmatrix}\\
&-\rho_{f,o}\,A_o\,g\begin{pmatrix}\left(v'-\frac{1}{2}v'^3+u\,v''\right)\\-\left(\frac{L}{2}-X\right)\begin{pmatrix}v''-u''v'\\-u'v''-\frac{3}{2}v'^2\,v''\end{pmatrix}\end{pmatrix}\\
&-(1-2\upsilon)A_o\,\overline{p}_o\left(v''-u''v'-u'v''-\frac{3}{2}v'^2\,v''\right)\\
&-\left(\frac{\rho_{f,o}\;\;D_o\,U_o^{\,2}C_T}{2}\right)\left(v'-u'v'-\frac{\dot{v}\,v'^2}{U_o}-\frac{\dot{v}^2\,v'}{2U_o^{\,2}}\right)\\
&+\left(\frac{\rho_{f,o}\;\;D_o\,U_o^{\,2}C_N}{2}\right)\begin{pmatrix}-v'+\frac{\dot{v}}{U_o}+\frac{\dot{v}\,u'}{U_o}+u'v'+\frac{\dot{u}\,\dot{v}}{U_o^{\,2}}\\+v'^3-\frac{\dot{v}\,v'^2}{U_o}-\frac{\dot{v}^3}{2U_o^{\,3}}-\frac{\dot{v}^2v'}{2U_o^{\,2}}\end{pmatrix}\\
&+\left(\frac{\rho_{f,o}\;\;D_o\,U_o^{\,2}C_{DP}}{2}\right)\left(-v'\left|v'\right|+\frac{\left|\dot{v}\right|\,v'+\left|v'\right|\,\dot{v}}{U_o}+\frac{\dot{v}\left|\dot{v}\right|}{U_o^{\,2}}\right)\\
&+M_a\begin{pmatrix}\ddot{v}-\ddot{u}\,v'-2\dot{u}\,\dot{v}'-\ddot{v}v'^2-\frac{3}{2}\dot{v}\,\dot{v}'\,v'\\+\dot{v}\,f_t-\frac{\dot{\varphi}^2}{4}v+\dot{\varphi}\,\dot{w}+\frac{\dot{\varphi}\,f_t}{2}w\end{pmatrix}\\
&+M_a\,U_o\left(-2\dot{v}'+3\dot{u}'v'+4u'\dot{v}'+\frac{7}{2}\dot{v}'v'^2+2\dot{u}\,v''+\frac{3}{2}\dot{v}v'v''\right)\\
&+M_a\,U_o^{\,2}\left(v''-2u''v'-4u'v''-\frac{5}{2}v''\,v'^2\right)\quad=\quad 0
\end{aligned}
\tag{4.40}
$$

$$
\begin{aligned}
&(m+M)\ddot{w}+(2M\,U_i)\,\dot{w}'+M\,U_i^{\,2}\,w''+\left(I_D+I_{D,f}\right)\left(2\,\ddot{w}''\right)\\
&+\left(I_p\right)\begin{pmatrix}\dot{w}'\left(v'\dot{v}''+\dot{v}'v''\right)+2v'^2\ddot{w}''+3\,\ddot{w}'v'v''+2\dot{w}''v'\dot{v}'+\dfrac{\dot{\varphi}\left(v'\ddot{w}'+\dot{w}''\dot{v}'\right)}{\dot{w}'}\\
-\dfrac{2\,\dot{\varphi}\,v'\dot{w}''\ddot{w}'}{\dot{w}'^2}+\dfrac{\ddot{\varphi}\,v'\dot{w}''}{\dot{w}'}+\dfrac{\dot{\varphi}'\,v'\ddot{w}'}{\dot{w}'}+\dfrac{\dot{\varphi}\left(v'\ddot{w}''+\ddot{w}'v''\right)}{\dot{w}'}+\dot{\varphi}\,\dot{v}''+v''\ddot{\varphi}+v'\ddot{\varphi}'+\dot{\varphi}'\,\dot{v}'\end{pmatrix}\\
&-\left(\breve{T}-P\right)w''+\left(E\,I\right)w''''-EI\begin{pmatrix}3u'''\,w''+4u''\,w'''+2u'\,w''''+w'u''''\\+2\,w'^2\,w''''+8\,w'\,w''\,w'''+2\,w''^3\end{pmatrix}\\
&+\left(\breve{T}-P-EA\right)\left(u''\,w'+u'\,w''+\frac{3}{2}w'^2\,w''\right)+(m+M)\,g\begin{pmatrix}w'-\dfrac{1}{2}w'^3+uw''\\-(L-X)\begin{pmatrix}w''-u''w'\\-u'w''-\dfrac{3}{2}w'^2\,w''\end{pmatrix}\end{pmatrix}\\
&-\left(\frac{\rho_{f,o}\;\;D_oU_o^{\,2}C_T}{2}\frac{D_o}{D_h}\right)\left(\left(w'-\frac{1}{2}w'^3+u\,w''\right)-(L-X)\left(w''-u''w'-u'w''-\frac{3}{2}w'^2\,w''\right)\right)\\
&-\rho_{f,o}\,A_o\,g\left(\left(w'-\frac{1}{2}w'^3+u\,w''\right)-\left(\frac{L}{2}-X\right)\left(w''-u''w'-u'w''-\frac{3}{2}w'^2\,w''\right)\right)\\
&-(1-2\upsilon)A_o\,\overline{p}_o\left(w''-u''w'-u'w''-\frac{3}{2}w'^2\,w''\right)-\left(\frac{\rho_{f,o}\;\;D_o\,U_o^{\,2}C_T}{2}\right)\begin{pmatrix}w'-u'w'\\-\dfrac{\dot{w}\,w'^2}{U_o}-\dfrac{\dot{w}^2\,w'}{2U_o^{\,2}}\end{pmatrix}\\
&+\left(\frac{\rho_{f,o}\;\;D_o\,U_o^{\,2}C_N}{2}\right)\left(-w'+\frac{\dot{w}}{U_o}+\frac{\dot{w}u'}{U_o}+u'w'+\frac{\dot{u}\,\dot{w}}{U_o^{\,2}}+w'^3-\frac{\dot{w}\,w'^2}{U_o}-\frac{\dot{w}^3}{2U_o^{\,3}}-\frac{\dot{w}^2w'}{2U_o^{\,2}}\right)\\
&+\left(\frac{\rho_{f,o}\;\;D_o\,U_o^{\,2}C_{DP}}{2}\right)\left(-w'\left|w'\right|+\frac{\left|\dot{w}\right|\,w'+\left|w'\right|\,\dot{w}}{U_o}+\frac{\dot{w}\left|\dot{w}\right|}{U_o^{\,2}}\right)\\
&+M_a\left(\ddot{w}-\ddot{u}\,w'-2\dot{u}\,\dot{w}'-\ddot{w}w'^2-\frac{3}{2}\dot{w}\,\dot{w}'\,w'+\dot{w}\,f_t-\frac{\dot{\varphi}^2}{4}w+\dot{\varphi}\,\dot{v}+\frac{\dot{\varphi}\,f_t}{2}v\right)\\
&+M_a\,U_o\left(-2\dot{w}'+3\dot{u}'w'+4u'\dot{w}'+\frac{7}{2}\dot{w}'w'^2+2\dot{u}\,w''+\frac{3}{2}\dot{w}w'w''\right)\\
&+M_a\,U_o^{\,2}\left(w''-2u''w'-4u'w''-\frac{5}{2}w''\,w'^2\right)\;=\;0
\end{aligned}
\tag{4.41}
$$

and,

$$
\left(I_p\right)\left(\ddot{\varphi}+\dot{w}'\dot{v}'+v'\ddot{w}'\right)-\left(G\,J\,\varphi''-4\,I_{p,f}\;L\,f_r\,\left(\varphi'\ddot{\varphi}+2\,\dot{\varphi}\,\dot{\varphi}'\right)\right)=0 \tag{4.42}
$$

It is important to recognize that P represents the net differential pressurization force, and it can be related to the internal flow pressure p_i and external flow pressure p_o such that

$$P = A_f \, p_i(L) - A_o \, p_o(L) \tag{4.43}$$

where A_f is the internal cross sectional flow area.

On the other hand, by following a procedure similar to that introduced in [17], the tension $\breve{T}$ can be expressed as

$$\frac{\partial T}{\partial x} = -\left[\frac{\rho_{f,o} \; D_o U_o^{\,2} C_T}{2} + (m+M) g \right] \tag{4.44}$$

Utilizing equation (2.8), one can write

$$\frac{\partial T}{\partial X} = \frac{\partial T}{\partial x}(1+u') \tag{4.45}$$

Substituting equation (4.45) into equation (4.44) and integrating from X to L, one obtains

$$\breve{T}(X) = \breve{T}(L) + \left(\frac{1}{2} \rho_{f,o} \, D_o \, U_o^{\,2} \, C_T + (m+M) g \right)(L - X - u) \tag{4.46}$$

where $\breve{T}(L)$ is the tension at the downstream end of the pipe, and is given by [17] as

$$\breve{T}(L) = \overline{T} + \frac{\rho_{f,o} \;\; D_o U_o^{\,2} C_T \, L}{2} \left(\frac{D_o}{D_h} - 1 \right) - \frac{(m+M) g \, L}{2} \tag{4.47}$$

where $\overline{T}$ is the externally imposed uniform tension.

Finally, one may express the tension $\breve{T}$ as

$$\breve{T}(X) = \overline{T} + \frac{\rho_{f,o} \;\; D_o U_o^{\,2} C_T \, L}{2} \left(\frac{D_o}{D_h} \right) + \left(\frac{1}{2} \rho_{f,o} \, D_o \, U_o^{\,2} \, C_T + (m+M) g \right) \left(\frac{L}{2} - X - u \right) \tag{4.48}$$

By substituting equation (4.48) into the Lagranginan function given by equation (4.35), performing the variational analysis for all terms including $\breve{T}$, and carrying out mathematical manipulations, the governing equations of motion can be expressed as

$$
\begin{aligned}
&(m+M)\ddot{u}+(2M\,U_i)\,\dot{u}'+M\,U_i^{\,2}\,u''-(E\,A)u'' \\
&+\left(\overline{T}-A_f\;p_i(L)+A_o\;p_o(L)-E\,A\right)(v'\,v''+w'\,w'') \\
&-E\,I\,(v''''\,v'+v''\,v'''+w''''\,w'+w''\,w''')+(m+M)g\begin{pmatrix}-\frac{1}{2}v'^2-\frac{1}{2}w'^2+u' \\ +\left(\frac{L}{2}-X\right)(v'\,v''+w'w'')\end{pmatrix} \\
&-M_a(\ddot{v}\,v'+\ddot{w}\,w')-M_a\,U_o\,(-2\,\dot{v}'\,v'-2\,\dot{w}'w')-M_a\,U_o^{\,2}\,(v'\,v''+w'w'') \\
&-\left(\frac{\rho_{f,o}\;\;D_oU_o^{\,2}C_T}{2}\frac{D_o}{D_h}\right)\left(-\frac{1}{2}(v'^2+w'^2)+u'+\left(\frac{L}{2}-X\right)(v'\,v''+w'w'')\right) \\
&+(1-2\upsilon)A_o\;\overline{p}_o\;(v'\,v''+w'w'')-(\rho_{f,o}\;A_o\;g)\left(-\frac{1}{2}(v'^2+w'^2)+u'+\left(\frac{L}{2}-X\right)(v'\,v''+w'w'')\right) \\
&-\left(\frac{\rho_{f,o}\;\;D_oU_o^{\,2}C_T}{2}\right)\left(\frac{3}{2}(v'^2+w'^2)-u'-\frac{(v'\dot{v}+w'\dot{w})}{U_o}-\frac{(\dot{v}^2+\dot{w}^2)}{2U_o^{\,2}}-\left(\frac{L}{2}-X\right)(v'\,v''+w'w'')\right) \\
&-\left(\frac{\rho_{f,o}\;\;D_oU_o^{\,2}C_N}{2}\right)\left(-(v'^2+w'^2)+\frac{(\dot{v}\,v'+\dot{w}w')}{U_o}\right) \\
&-\left(\frac{\rho_{f,o}\;\;D_o\,U_o^{\,2}\,C_{DP}}{2}\right)\begin{pmatrix}-v'^2|v'|+\frac{|\dot{v}|v'^2+|v'|v'\dot{v}}{U_o}+\frac{\dot{v}|\dot{v}|v'}{U_o^{\,2}} \\ -w'^2|w'|+\frac{|\dot{w}|w'^2+|w'|w'\dot{w}}{U_o}+\frac{\dot{w}|\dot{w}|w'}{U_o^{\,2}}\end{pmatrix} \;=\; 0
\end{aligned}
\tag{4.49}
$$

$$
\begin{aligned}
&(m+M)\ddot{v}+(2M\,U_i)\,\dot{v}'+M\,U_i^{\,2}\,v''+\left(I_D+I_{D,f}\right)(2\,\ddot{v}'')+\left(I_p\right)\left(v''\dot{w}'^2+2v'\dot{w}'\dot{w}''+\dot{\varphi}\,\dot{w}''+\dot{w}'\dot{\varphi}'\right)\\
&-\left(\overline{T}-A_f\;p_i(L)+A_o\;p_o(L)\right)v''+(E\,I)v''''\\
&-EI\left(3u'''v''+4u''v'''+2u'v''''+v'u''''+2v'^2\,v''''+8v'v''v'''+2v''^3\right)\\
&+\left(\overline{T}-A_f\;p_i(L)+A_o\;p_o(L)-EA\right)\left(u''v'+u'v''+\frac{3}{2}v'^2\,v''\right)\\
&+(m+M)\,g\left(v'-\frac{1}{2}v'^3+uv''-\left(\frac{L}{2}-X\right)\left(v''-u''v'-u'v''-\frac{3}{2}v'^2\,v''\right)\right)\\
&-\left(\frac{\rho_{f,o}\;\;D_oU_o^{\,2}C_T}{2}\frac{D_o}{D_h}\right)\left(\left(v'-\frac{1}{2}v'^3+u\,v''\right)-\left(\frac{L}{2}-X\right)\begin{pmatrix}v''-u''v'\\-u'v''-\frac{3}{2}v'^2\,v''\end{pmatrix}\right)\\
&-\rho_{f,o}\;A_o\;g\begin{pmatrix}\left(v'-\frac{1}{2}v'^3+u\,v''\right)\\-\left(\frac{L}{2}-X\right)\begin{pmatrix}v''-u''v'\\-u'v''-\frac{3}{2}v'^2\,v''\end{pmatrix}\end{pmatrix}-(1-2\upsilon)A_o\;\overline{p}_o\left(v''-u''v'-u'v''-\frac{3}{2}v'^2\,v''\right)\\
&-\left(\frac{\rho_{f,o}\;\;D_o\,U_o^{\,2}C_T}{2}\right)\begin{pmatrix}\frac{1}{2}v'^3-uv''-u'v'-\frac{\dot{v}\,v'^2}{U_o}-\frac{\dot{v}^2\,v'}{2U_o^{\,2}}\\+\left(\frac{L}{2}-X\right)\left(v''-u''v'-u'v''-\frac{3}{2}v'^2\,v''\right)\end{pmatrix}\\
&+\left(\frac{\rho_{f,o}\;\;D_o\,U_o^{\,2}C_N}{2}\right)\left(-v'+\frac{\dot{v}}{U_o}+\frac{\dot{v}u'}{U_o}+u'v'+\frac{\dot{u}\,\dot{v}}{U_o^{\,2}}+v'^3-\frac{\dot{v}\,v'^2}{U_o}-\frac{\dot{v}^3}{2U_o^{\,3}}-\frac{\dot{v}^2v'}{2U_o^{\,2}}\right)\\
&+\left(\frac{\rho_{f,o}\;\;D_o\,U_o^{\,2}C_{DP}}{2}\right)\left(-v'|v'|+\frac{|\dot{v}|\;v'+|v'|\;\dot{v}}{U_o}+\frac{\dot{v}|\dot{v}|}{U_o^{\,2}}\right)\\
&+M_a\left(\ddot{v}-\ddot{u}\,v'-2\dot{u}\,\dot{v}'-\ddot{v}v'^2-\frac{3}{2}\dot{v}\,\dot{v}'\,v'+\dot{v}\,f_t-\frac{\dot{\varphi}^2}{4}v+\dot{\varphi}\,\dot{w}+\frac{\dot{\varphi}\,f_t}{2}w\right)\\
&+M_a\,U_o\left(-2\dot{v}'+3\dot{u}'v'+4u'\dot{v}'+\frac{7}{2}\dot{v}'v'^2+2\dot{u}\,v''+\frac{3}{2}\dot{v}v'v''\right)\\
&+M_a\,U_o^{\,2}\left(v''-2u''v'-4u'v''-\frac{5}{2}v''\,v'^2\right)\quad=\quad 0
\end{aligned}
\tag{4.50}
$$

$$
\begin{aligned}
&(m+M)\ddot{w}+(2M\,U_i)\,\dot{w}'+M\,U_i^{\,2}\,w''+\left(I_D+I_{D,f}\right)(2\,\ddot{w}'') \\
&-\left(\overline{T}-A_f\;p_i(L)+A_o\;p_o(L)\right)w''+(E\,I)\,w''''' \\
&+\left(I_p\right)\begin{pmatrix}\dot{w}'\left(v'\dot{v}''+\dot{v}'v''\right)+2v'^2\ddot{w}''+3\,\ddot{w}'v'v''+2\dot{w}''v'\dot{v}'+\dfrac{\dot{\varphi}\left(v'\ddot{w}'+\dot{w}''\dot{v}'\right)}{\dot{w}'} \\ -\dfrac{2\,\dot{\varphi}\,v'\dot{w}''\ddot{w}'}{\dot{w}'^2}+\dfrac{\ddot{\varphi}\,v'\dot{w}''}{\dot{w}'}+\dfrac{\dot{\varphi}'\,v'\ddot{w}'}{\dot{w}'}+\dfrac{\dot{\varphi}\left(v'\ddot{w}''+\ddot{w}'v''\right)}{\dot{w}'}+\dot{\varphi}\,\dot{v}''+v''\ddot{\varphi}+v'\ddot{\varphi}'+\dot{\varphi}'\,\dot{v}'\end{pmatrix} \\
&-EI\left(3u'''\,w''+4u''\,w'''+2u'\,w'''''+w'\,u'''''+2\,w'^2\;w'''''+8\,w'\,w''\,w'''+2\,w''^3\right) \\
&+\left(\overline{T}-A_f\;p_i(L)+A_o\;p_o(L)-EA\right)\left(u''\,w'+u'\,w''+\frac{3}{2}w'^2\;w''\right) \\
&+(m+M)\,g\left(w'-\frac{1}{2}w'^3+uw''-(L-X)\left(w''-u''w'-u'w''-\frac{3}{2}w'^2\;w''\right)\right) \\
&-\left(\frac{\rho_{f,o}\;\;D_oU_o^{\,2}C_T}{2}\frac{D_o}{D_h}\right)\left(\left(w'-\frac{1}{2}w'^3+u\,w''\right)-\left(\frac{L}{2}-X\right)\begin{pmatrix}w''-u''w' \\ -u'w''-\dfrac{3}{2}w'^2\;w''\end{pmatrix}\right) \\
&-\rho_{f,o}\;A_o\;g\begin{pmatrix}\left(w'-\dfrac{1}{2}w'^3+u\,w''\right) \\ -\left(\dfrac{L}{2}-X\right)\begin{pmatrix}w''-u''w' \\ -u'w''-\dfrac{3}{2}w'^2\;w''\end{pmatrix}\end{pmatrix}-(1-2\upsilon)A_o\;\overline{p}_o\left(w''-u''w'-u'w''-\frac{3}{2}w'^2\;w''\right) \\
&-\left(\frac{\rho_{f,o}\;\;D_o\,U_o^{\,2}C_T}{2}\right)\begin{pmatrix}\dfrac{1}{2}w'^3-uw''-u'w'-\dfrac{\dot{v}\,w'^2}{U_o}-\dfrac{\dot{v}^2\;w'}{2U_o^{\,2}} \\ +\left(\dfrac{L}{2}-X\right)\left(w''-u''w'-u'w''-\dfrac{3}{2}w'^2\;w''\right)\end{pmatrix} \\
&+\left(\frac{\rho_{f,o}\;\;D_o\,U_o^{\,2}C_N}{2}\right)\left(-w'+\frac{\dot{w}}{U_o}+\frac{\dot{w}u'}{U_o}+u'w'+\frac{\dot{u}\,\dot{w}}{U_o^{\,2}}+w'^3-\frac{\dot{w}\,w'^2}{U_o}-\frac{\dot{w}^3}{2U_o^{\,3}}-\frac{\dot{w}^2w'}{2U_o^{\,2}}\right) \\
&+\left(\frac{\rho_{f,o}\;\;D_o\,U_o^{\,2}C_{DP}}{2}\right)\left(-w'\left|w'\right|+\frac{\left|\dot{w}\right|\;w'+\left|w'\right|\;\dot{w}}{U_o}+\frac{\dot{w}\left|\dot{w}\right|}{U_o^{\,2}}\right) \\
&+M_a\begin{pmatrix}\ddot{w}-\ddot{u}\,w'-2\,\dot{u}\,\dot{w}'-\ddot{w}w'^2-\dfrac{3}{2}\dot{w}\,\dot{w}'\,w' \\ +\dot{w}\,f_t-\dfrac{\dot{\varphi}^2}{4}w+\dot{\varphi}\,\dot{v}+\dfrac{\dot{\varphi}\,f_t}{2}v\end{pmatrix} \\
&+M_a\,U_o\left(-2\dot{w}'+3\dot{u}'w'+4u'\dot{w}'+\frac{7}{2}\dot{w}'w'^2+2\dot{u}\,w''+\frac{3}{2}\dot{w}w'w''\right) \\
&+M_a\,U_o^{\,2}\left(w''-2u''w'-4u'w''-\frac{5}{2}w''\,w'^2\right)\;=\;0
\end{aligned}
\tag{4.51}
$$

and,

$$\left(I_p\right)\left(\ddot{\varphi}+\dot{w}'\dot{v}'+v'\ddot{w}'\right)-\left(G\,J\,\varphi''-4\,I_{p,f}\;L\;\;f_r\left(\varphi'\ddot{\varphi}+2\,\dot{\varphi}\,\dot{\varphi}'\right)\right)=0 \tag{4.52}$$

4.8 Dissipative Forces Due to Material Damping

By considering the Kelven-Voigt model presented in chapter 2 and utilizing equation (2.83), the equations of motion which takes into account the internal damping of the pipe material can be expressed as

$$\begin{aligned}
&\left(m+M\right)\ddot{u}+\left(2M\,U_i\right)\dot{u}'+M\,U_i^{\,2}\,u''-\left(E\,A\right)u''-EA\,a\,\ddot{u}''\\
&+\left(\overline{T}-A_f\;p_i\left(L\right)+A_o\;p_o\left(L\right)-E\,A\right)\left(v'\,v''+w'\,w''\right)\\
&-EA\,a\;\left(v'\,\dot{v}''+\dot{v}'v''+w'\dot{w}''+\dot{w}'w''\right)\\
&-E\,I\left(v''''\,v'+v''\,v'''+w''''\,w'+w''\,w'''\right)-EI\,a\left(v''''\dot{v}'+v'\dot{v}''''+w''''\dot{w}'+w'\dot{w}''''\right)\\
&+\left(m+M\right)g\left(-\frac{1}{2}v'^2-\frac{1}{2}w'^2+u'+\left(\frac{L}{2}-X\right)\left(v'\,v''+w'w''\right)\right)\\
&-M_a\left(\ddot{v}\,v'+\ddot{w}\,w'\right)-M_a\,U_o\left(-2\,\dot{v}'\,v'-2\,\dot{w}'w'\right)-M_a\,U_o^{\,2}\left(v'\,v''+w'w''\right)\\
&-\left(\frac{\rho_{f,o}\;\;D_oU_o^{\,2}C_T}{2}\frac{D_o}{D_h}\right)\left(-\frac{1}{2}\left(v'^2+w'^2\right)+u'+\left(\frac{L}{2}-X\right)\left(v'\,v''+w'w''\right)\right)\\
&+\left(1-2\upsilon\right)A_o\;\overline{p}_o\;\left(v'\,v''+w'w''\right)-\left(\rho_{f,o}\;A_o\;g\right)\begin{pmatrix}-\frac{1}{2}\left(v'^2+w'^2\right)+u'\\+\left(\frac{L}{2}-X\right)\left(v'\,v''+w'w''\right)\end{pmatrix}\\
&-\left(\frac{\rho_{f,o}\;\;D_oU_o^{\,2}C_T}{2}\right)\begin{pmatrix}\frac{3}{2}\left(v'^2+w'^2\right)-u'-\frac{\left(v'\dot{v}+w'\dot{w}\right)}{U_o}\\-\frac{\left(\dot{v}^2+\dot{w}^2\right)}{2U_o^{\,2}}-\left(\frac{L}{2}-X\right)\left(v'\,v''+w'w''\right)\end{pmatrix}\\
&-\left(\frac{\rho_{f,o}\;\;D_oU_o^{\,2}C_N}{2}\right)\left(-\left(v'^2+w'^2\right)+\frac{\left(\dot{v}\,v'+\dot{w}w'\right)}{U_o}\right)\\
&-\left(\frac{\rho_{f,o}\;\;D_o\,U_o\,C_{DP}}{2}\right)\begin{pmatrix}-v'^2\left|v'\right|+\frac{\left|\dot{v}\right|v'^2+\left|v'\right|v'\dot{v}}{U_o}+\frac{\dot{v}\left|\dot{v}\right|v'}{U_o^{\,2}}\\-w'^2\left|w'\right|+\frac{\left|\dot{w}\right|w'^2+\left|w'\right|w'\dot{w}}{U_o}+\frac{\dot{w}\left|\dot{w}\right|w'}{U_o^{\,2}}\end{pmatrix}\;=\;0
\end{aligned} \tag{4.53}$$

$$
\begin{aligned}
&(m+M)\ddot{v}+(2M\,U_i)\,\dot{v}'+M\,U_i^{\,2}\,v''+\left(I_D+I_{D,f}\right)\left(2\,\ddot{v}''\right)\\
&+\left(I_p\right)\left(v''\dot{w}'^2+2v'\dot{w}'\dot{w}''+\dot{\varphi}\,\dot{w}''+\dot{w}'\dot{\varphi}'\right)-\left(\overline{T}-A_f\;p_i(L)+A_o\;p_o(L)\right)v''+(E\,I)v''''\\
&+EI\,a\,\dot{v}''''-EI\left(3u'''v''+4u''v'''+2u'v''''+v'u''''+2v'^2\,v''''+8v'v''v'''+2v''^3\right)\\
&-EI\,a\begin{pmatrix}3u'''\dot{v}''+3v''\dot{u}'''+4u''\dot{v}'''+4v'''\dot{u}''+2u'\dot{v}''''+2v''''\dot{u}'+v'\dot{u}''''+u''''v'\\+2v'^2\dot{v}''''+4v''''v'\dot{v}'+8v'v''\dot{v}'''+8v'v'''\dot{v}''+8\dot{v}'v''v'''+6v''^2\dot{v}''\end{pmatrix}\\
&+\left(\overline{T}-A_f\;p_i(L)+A_o\;p_o(L)-EA\right)\left(u''v'+u'v''+\frac{3}{2}v'^2\,v''\right)\\
&-EA\,a\left(u''\dot{v}'+v'\dot{u}''+u'\dot{v}''+v''\dot{u}'+\frac{3}{2}v'^2\,\dot{v}''+3v''v'\dot{v}'\right)\\
&+(m+M)\,g\left(v'-\frac{1}{2}v'^3+uv''-\left(\frac{L}{2}-X\right)\left(v''-u''v'-u'v''-\frac{3}{2}v'^2\,v''\right)\right)\\
&-\left(\frac{\rho_{f,o}\;\;D_oU_o^{\,2}C_T}{2}\frac{D_o}{D_h}\right)\left(\left(v'-\frac{1}{2}v'^3+u\,v''\right)-\left(\frac{L}{2}-X\right)\begin{pmatrix}v''-u''v'\\-u'v''-\frac{3}{2}v'^2\,v''\end{pmatrix}\right)\\
&-\rho_{f,o}\;A_o\;g\begin{pmatrix}\left(v'-\frac{1}{2}v'^3+u\,v''\right)\\-\left(\frac{L}{2}-X\right)\begin{pmatrix}v''-u''v'\\-u'v''-\frac{3}{2}v'^2\,v''\end{pmatrix}\end{pmatrix}-(1-2\upsilon)A_o\;\overline{p}_o\left(v''-u''v'-u'v''-\frac{3}{2}v'^2\,v''\right)\\
&-\left(\frac{\rho_{f,o}\;\;D_o\,U_o^{\,2}C_T}{2}\right)\begin{pmatrix}\frac{1}{2}v'^3-uv''-u'v'-\frac{\dot{v}\,v'^2}{U_o}-\frac{\dot{v}^2\,v'}{2U_o^{\,2}}\\+\left(\frac{L}{2}-X\right)\left(v''-u''v'-u'v''-\frac{3}{2}v'^2\,v''\right)\end{pmatrix}\\
&+\left(\frac{\rho_{f,o}\;\;D_o\,U_o^{\,2}C_N}{2}\right)\left(-v'+\frac{\dot{v}}{U_o}+\frac{\dot{v}u'}{U_o}+u'v'+\frac{\dot{u}\,\dot{v}}{U_o^{\,2}}+v'^3-\frac{\dot{v}\,v'^2}{U_o}-\frac{\dot{v}^3}{2U_o^{\,3}}-\frac{\dot{v}^2v'}{2U_o^{\,2}}\right)\\
&+\left(\frac{\rho_{f,o}\;\;D_o\,U_o^{\,2}C_{DP}}{2}\right)\left(-v'\left|v'\right|+\frac{\left|\dot{v}\right|\,v'+\left|v'\right|\,\dot{v}}{U_o}+\frac{\dot{v}\left|\dot{v}\right|}{U_o^{\,2}}\right)\\
&+M_a\left(\ddot{v}-\ddot{u}\,v'-2\dot{u}\,\dot{v}'-\ddot{v}v'^2-\frac{3}{2}\dot{v}\,\dot{v}'\,v'+\dot{v}\,f_t-\frac{\dot{\varphi}^2}{4}v+\dot{\varphi}\,\dot{w}+\frac{\dot{\varphi}\,f_t}{2}w\right)\\
&+M_a\,U_o\left(-2\dot{v}'+3\dot{u}'v'+4u'\dot{v}'+\frac{7}{2}\dot{v}'v'^2+2\dot{u}\,v''+\frac{3}{2}\dot{v}v'v''\right)\\
&+M_a\,U_o^{\,2}\left(v''-2u''v'-4u'v''-\frac{5}{2}v''v'^2\right)\quad=\quad 0
\end{aligned}
\tag{4.54}
$$

$$
\begin{aligned}
&(m+M)\ddot{w}+(2M\,U_i)\,\dot{w}'+M\,U_i^{\,2}\,w''+\left(I_D+I_{D,f}\right)(2\,\ddot{w}'') \\
&+\left(I_p\right)\begin{pmatrix}\dot{w}'\left(v'\dot{v}''+\dot{v}'v''\right)+2v'^2\ddot{w}''+3\,\ddot{w}'v'v''+2\dot{w}''v'\dot{v}'+\dfrac{\dot{\varphi}\left(v'\ddot{w}'+\dot{w}''\dot{v}'\right)}{\dot{w}'} \\ -\dfrac{2\,\dot{\varphi}\,v'\dot{w}''\ddot{w}'}{\dot{w}'^2}+\dfrac{\ddot{\varphi}\,v'\dot{w}''}{\dot{w}'}+\dfrac{\dot{\varphi}'\,v'\ddot{w}'}{\dot{w}'}+\dfrac{\dot{\varphi}\left(v'\ddot{w}''+\ddot{w}'v''\right)}{\dot{w}'}+\dot{\varphi}\,\dot{v}''+v''\ddot{\varphi}+v'\ddot{\varphi}'+\dot{\varphi}'\,\dot{v}'\end{pmatrix} \\
&-\left(\overline{T}-A_f\;p_i(L)+A_o\;p_o(L)\right)w''+(E\,I)\,w''''+EI\,a\,\dot{w}''''-EI\begin{pmatrix}3u'''\,w''+4u''\,w'''+2u'\,w''''' \\ +w'\,u''''+2\,w'^2\;w'''' \\ +8\,w'\,w''\,w'''+2\,w''^3\end{pmatrix} \\
&-EI\,a\begin{pmatrix}3u'''\dot{w}''+3w''\dot{u}'''+4\,u''\dot{w}'''+4w'''\dot{u}''+2u'\dot{w}''''+2w''''\dot{u}'+w'\dot{u}''''+u''''w' \\ +2w'^2\dot{w}''''+4w''''w'\dot{w}'+8w'w''\dot{w}'''+8w'w'''\dot{w}''+8\dot{w}'w''w'''+6\,w''^2\dot{w}''\end{pmatrix} \\
&+\left(\overline{T}-A_f\;p_i(L)+A_o\;p_o(L)-EA\right)\left(u''\,w'+u'\,w''+\frac{3}{2}w'^2\;w''\right) \\
&-EA\,a\left(u''\dot{w}'+w'\dot{u}''+u'\dot{w}''+w''\dot{u}'+\frac{3}{2}w'^2\;\dot{w}''+3w''w'\dot{w}'\right) \\
&+(m+M)\,g\left(w'-\frac{1}{2}w'^3+uw''-(L-X)\left(w''-u''w'-u'w''-\frac{3}{2}w'^2\;w''\right)\right) \\
&-\left(\frac{\rho_{f,o}\;\;D_oU_o^{\,2}C_T}{2}\frac{D_o}{D_h}\right)\left(\left(w'-\frac{1}{2}w'^3+u\,w''\right)-\left(\frac{L}{2}-X\right)\left(w''-u''w'-u'w''-\frac{3}{2}w'^2\;w''\right)\right) \\
&-\rho_{f,o}\;A_o\;g\left(\left(w'-\frac{1}{2}w'^3+u\,w''\right)-\left(\frac{L}{2}-X\right)\left(w''-u''w'-u'w''-\frac{3}{2}w'^2\;w''\right)\right) \\
&-(1-2\upsilon)A_o\;\overline{p}_o\;\left(w''-u''w'-u'w''-\frac{3}{2}w'^2\;w''\right) \\
&-\left(\frac{\rho_{f,o}\;\;D_o\,U_o^{\,2}C_T}{2}\right)\left(\frac{1}{2}w'^3-uw''-u'w'-\frac{\dot{v}\,w'^2}{U_o}-\frac{\dot{v}^2\;w'}{2U_o^{\,2}}+\left(\frac{L}{2}-X\right)\begin{pmatrix}w''-u''w' \\ -u'w''-\frac{3}{2}w'^2\;w''\end{pmatrix}\right) \\
&+\left(\frac{\rho_{f,o}\;\;D_o\,U_o^{\,2}C_N}{2}\right)\left(-w'+\frac{\dot{w}}{U_o}+\frac{\dot{w}u'}{U_o}+u'w'+\frac{\dot{u}\,\dot{w}}{U_o^{\,2}}+w'^3-\frac{\dot{w}\,w'^2}{U_o}-\frac{\dot{w}^3}{2U_o^{\,3}}-\frac{\dot{w}^2w'}{2U_o^{\,2}}\right) \\
&+\left(\frac{\rho_{f,o}\;\;D_o\,U_o^{\,2}C_{DP}}{2}\right)\left(-w'\left|w'\right|+\frac{\left|\dot{w}\right|\;w'+\left|w'\right|\;\dot{w}}{U_o}+\frac{\dot{w}\left|\dot{w}\right|}{U_o^{\,2}}\right) \\
&+M_a\left(\ddot{w}-\ddot{u}\,w'-2\dot{u}\,\dot{w}'-\ddot{w}w'^2-\frac{3}{2}\dot{w}\,\dot{w}'\,w'+\dot{w}\,f_t-\frac{\dot{\varphi}^2}{4}w+\dot{\varphi}\,\dot{v}+\frac{\dot{\varphi}\,f_t}{2}v\right) \\
&+M_a\,U_o\left(-2\dot{w}'+3\dot{u}'w'+4u'\dot{w}'+\frac{7}{2}\dot{w}'w'^2+2\dot{u}\,w''+\frac{3}{2}\dot{w}w'w''\right) \\
&+M_a\,U_o^{\,2}\left(w''-2u''w'-4u'w''-\frac{5}{2}w''\,w'^2\right)\;\;=\;\;0
\end{aligned}
\tag{4.55}
$$

and,

$$\left(I_p\right)\left(\ddot{\varphi}+\dot{w}'\dot{v}'+v'\ddot{w}'\right)-\left(G\,J\,\varphi''-4\,I_{p,f}\,L\;\;f_r\left(\varphi'\ddot{\varphi}+2\,\dot{\varphi}\,\dot{\varphi}'\right)\right)=0 \tag{4.56}$$

4.9 Case Studies

The obtained mathematical model is a considerably comprehensive dynamical model and can be adapted to match several engineering applications, particularly in the drilling applications by either changing design parameters or the end conditions. The following case studies can be tackled:

4.9.1 Fixed-simply supported rotating flexible pipe conveying fluid downwards, which then flows upwards as a confined annular flow

The motion of the drill string could be idealized as rotating flexible pipe, fixed at one end and simply supported at the another end in both *X-Y* and *X-Z* planes, that is conveying fluid downwards, which then flows upwards as a confined annular flow as shown in figure 4.4. The additional features of this model over that described in section 3.11.4 accounts of the following:

- Steady state forces such as pressurization and tension in the pipe.
- The axial deformation u associated with the axial strain in the pipe.

By assuming smooth transition between internal and annular flow at the lower end of the pipe, one may write

$$p_i\left(L\right)+\frac{1}{2}\rho_{f,i}\,U_i^{\,2}=p_o\left(L\right)+\frac{1}{2}\rho_{f,o}\,U_o^{\,2}+\rho_{f,i}\,g\,h_o \tag{4.57}$$

where $\rho_{f,\,i}$ is the density of the internal fluid and h_o is the loss of head that arises due to the sudden enlargement in the flow from A_f to A_{ch}, which is given by [33] as

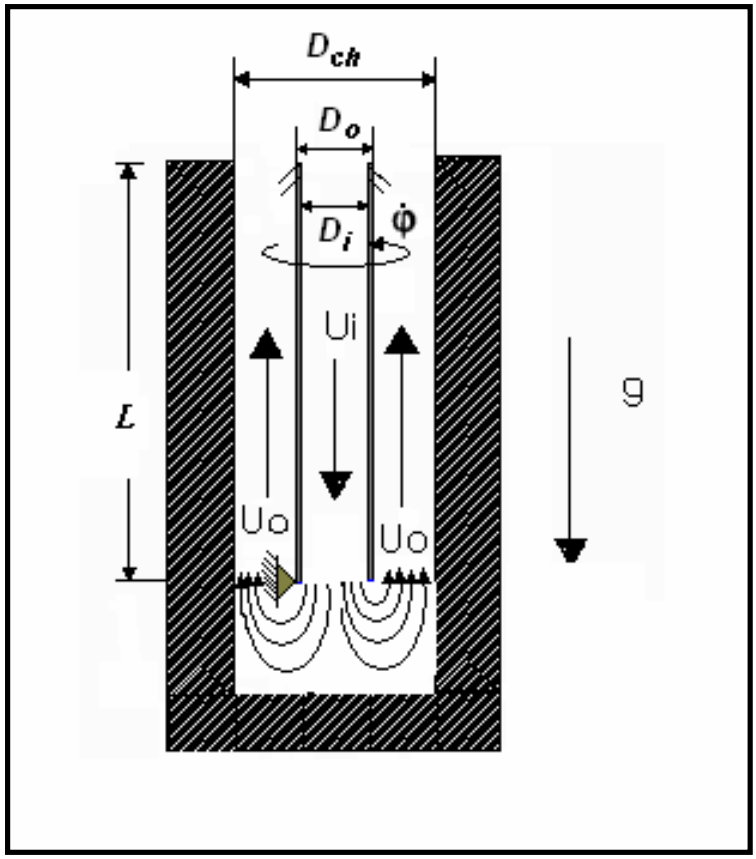


Figure 4.4: Fixed-simply supported rotating flexible pipe conveying fluid downwards, which then flows upwards in the outer annulus.

$$h_o = \frac{1}{2g}\overline{C}\left(U_i - U_o\right)^2 \tag{4.58}$$

where $\overline{C} = 1$ as given by [42].

Based on the analysis performed in section 2.6.3, and by considering the weight of the pipe, one may represent the pressure at the downstream end of the extensible pipe $p_o(L)$, as given by [17]

$$A_o\, p_o\,(L) = \left(1 - 2\nu\right)\overline{p}_o A_o + \rho_{f,o}\, g\, A_o \frac{L}{2} \tag{4.59}$$

By substituting equations (4.58) and (4.59) into equation (4.57), one obtains

$$p_i\left(L\right) = \left(1 - 2\upsilon\right)\overline{p}_o + \frac{\rho_{f,o}}{2}\left(g\,L + U_i^{\,2}\left(\frac{D_i^{\,2}}{D_{ch}^{\,2} - D_o^{\,2}}\right)^2\right) + \frac{\rho_{f,i}\,U_i^{\,2}}{2}\left(\overline{C}\left(1 - \frac{D_i^{\,2}}{D_{ch}^{\,2} - D_o^{\,2}}\right)^2 - 1\right) \tag{4.60}$$

Upon substituting equation (4.60) into the equations of motion obtained for the extensible rotating pipe, one obtains the equations of motion that describe the dynamics of a simply supported drill string as

$$
\begin{aligned}
&(m+M)\ddot{u}+(2M\,U_i)\,\dot{u}'+M\,U_i^{\,2}\,u''-(E\,A)u''-EA\,a\,\dot{u}'' \\
&+\left(\overline{T}-A_f\left(\begin{array}{l}(1-2\upsilon)\,\overline{p}_o+\dfrac{\rho_{f,o}}{2}\left(g\,L+U_i^{\,2}\left(\dfrac{D_i^{\,2}}{D_{ch}^{\,2}-D_o^{\,2}}\right)^2\right)\\ +\dfrac{\rho_{f,i}\,U_i^{\,2}}{2}\left(\overline{C}\left(1-\dfrac{D_i^{\,2}}{D_{ch}^{\,2}-D_o^{\,2}}\right)^2-1\right)\end{array}\right)-E\,A\right)(v'\,v''+w'\,w'') \\
&-EA\,a\,\left(v'\,\dot{v}''+\dot{v}'v''+w'\dot{w}''+\dot{w}'w''\right) \\
&-E\,I\left(v'''\,v'+v''\,v'''+w'''\,w'+w''\,w'''\right)-EI\,a\left(v'''\dot{v}'+v'\dot{v}'''+w'''\dot{w}'+w'\dot{w}'''\right) \\
&+(m+M)\,g\left(-\frac{1}{2}v'^2-\frac{1}{2}w'^2+u'+\left(\frac{L}{2}-X\right)(v'\,v''+w'w'')\right)-M_a\left(\ddot{v}\,v'+\ddot{w}\,w'\right) \\
&-M_a\left(\frac{D_i^{\,2}}{D_{ch}^{\,2}-D_o^{\,2}}\right)\,U_i\,\left(-2\,\dot{v}'\,v'-2\,\dot{w}'w'\right)-M_a\left(\frac{D_i^{\,2}}{D_{ch}^{\,2}-D_o^{\,2}}\right)^2\,U_i^{\,2}\,(v'\,v''+w'w'') \\
&-\left(\frac{\rho_{f,o}\;\;D_o\left(\dfrac{D_i^{\,2}}{D_{ch}^{\,2}-D_o^{\,2}}\right)^2\,U_i^{\,2}C_T}{2}\,\frac{D_o}{D_h}\right)\left(-\frac{1}{2}\left(v'^2+w'^2\right)+u'+\left(\frac{L}{2}-X\right)(v'\,v''+w'w'')\right) \\
&+2\,(1-2\upsilon)A_o\;\overline{p}_o\,(v'\,v''+w'w'')-\left(\rho_{f,o}\;A_o\;g\right)\left(-\frac{1}{2}\left(v'^2+w'^2\right)+u'-X\,(v'\,v''+w'w'')\right) \\
&-\left(\frac{\rho_{f,o}\;\;D_o\left(\dfrac{D_i^{\,2}}{D_{ch}^{\,2}-D_o^{\,2}}\right)^2\,U_i^{\,2}C_T}{2}\right)\left(\begin{array}{l}\dfrac{3}{2}\left(v'^2+w'^2\right)-u'-\left(\dfrac{D_{ch}^{\,2}-D_o^{\,2}}{D_i^{\,2}}\right)\dfrac{(v'\dot{v}+w'\dot{w})}{U_i}\\ -\left(\dfrac{D_{ch}^{\,2}-D_o^{\,2}}{D_i^{\,2}}\right)^2\dfrac{\left(\dot{v}^2+\dot{w}^2\right)}{2U_i^{\,2}}-\left(\dfrac{L}{2}-X\right)(v'\,v''+w'w'')\end{array}\right) \\
&-\left(\frac{\rho_{f,o}\;\;D_o\left(\dfrac{D_i^{\,2}}{D_{ch}^{\,2}-D_o^{\,2}}\right)^2\,U_i^{\,2}C_N}{2}\right)\left(-\left(v'^2+w'^2\right)+\left(\frac{D_{ch}^{\,2}-D_o^{\,2}}{D_i^{\,2}}\right)\frac{(\dot{v}\,v'+\dot{w}w')}{U_i}\right) \\
&-\left(\frac{\rho_{f,o}\;\;D_o\left(\dfrac{D_i^{\,2}}{D_{ch}^{\,2}-D_o^{\,2}}\right)^2\,U_i^{\,2}C_{DP}}{2}\right)\left(\begin{array}{l}-v'^2|v'|+\left(\dfrac{D_{ch}^{\,2}-D_o^{\,2}}{D_i^{\,2}}\right)\dfrac{|\dot{v}|v'^2+|v'|v'\dot{v}}{U_i}\\ +\left(\dfrac{D_{ch}^{\,2}-D_o^{\,2}}{D_i^{\,2}}\right)^2\dfrac{\dot{v}|\dot{v}|v'}{U_i^{\,2}}+\left(\dfrac{D_{ch}^{\,2}-D_o^{\,2}}{D_i^{\,2}}\right)^2\dfrac{\dot{w}|\dot{w}|w'}{U_i^{\,2}}\\ -w'^2|w'|+\left(\dfrac{D_{ch}^{\,2}-D_o^{\,2}}{D_i^{\,2}}\right)\dfrac{|\dot{w}|w'^2+|w'|w'\dot{w}}{U_i}\end{array}\right) \;=\; 0
\end{aligned}
$$

(4.61)

$$
\begin{aligned}
&(m+M)\ddot{v}+(2M\,U_i)\,\dot{v}'+M\,U_i^{\,2}\,v''+\left(I_D+I_{D,f}\right)(2\,\ddot{v}'')+\left(I_p\right)\left(v''\dot{w}'^2+2v'\dot{w}'\dot{w}''+\dot{\varphi}\,\dot{w}''+\dot{w}'\dot{\varphi}'\right)\\
&+(E\,I)v''''-EA\left(u''v'+u'v''+\frac{3}{2}v'^2\,v''\right)-EA\,a\left(u''\dot{v}'+v'\dot{u}''+u'\dot{v}''+v''\dot{u}'+\frac{3}{2}v'^2\,\dot{v}''+3v''v'\dot{v}'\right)\\
&+EI\,a\,\dot{v}''''-EI\left(3u'''v''+4u''v'''+2u'v''''+v'u''''+2v'^2\,v''''+8v'v''v'''+2v''^3\right)\\
&-EI\,a\begin{pmatrix}3u'''\dot{v}''+3v''\dot{u}'''+4u''\dot{v}'''+4v'''\dot{u}''+2u'\dot{v}''''+2v''''\dot{u}'+v'\dot{u}''''+u''''v'\\+2v'^2\dot{v}''''+4v''''v'\dot{v}'+8v'v''\dot{v}'''+8v'v'''\dot{v}''+8\dot{v}'v''v'''+6v''^2\dot{v}''\end{pmatrix}\\
&+\left(\overline{T}-A_f\begin{pmatrix}(1-2\upsilon)\overline{p}_o+\dfrac{\rho_{f,o}}{2}\left(g\,L+U_i^{\,2}\left(\dfrac{D_i^{\,2}}{D_{ch}^{\,2}-D_o^{\,2}}\right)^2\right)\\+\dfrac{\rho_{f,i}\,U_i^{\,2}}{2}\left(\overline{C}\left(1-\dfrac{D_i^{\,2}}{D_{ch}^{\,2}-D_o^{\,2}}\right)^2-1\right)\end{pmatrix}\right)\left(-v''+u''v'+u'v''+\frac{3}{2}v'^2\,v''\right)\\
&+(m+M)g\left(v'-\frac{1}{2}v'^3+uv''-\left(\frac{L}{2}-X\right)\left(v''-u''v'-u'v''-\frac{3}{2}v'^2\,v''\right)\right)\\
&-\left(\frac{\rho_{f,o}\;D_o\left(\dfrac{D_i^{\,2}}{D_{ch}^{\,2}-D_o^{\,2}}\right)^2U_i^{\,2}C_T}{2}\,\frac{D_o}{D_h}\right)\left(\left(v'-\frac{1}{2}v'^3+u\,v''\right)-\left(\frac{L}{2}-X\right)\begin{pmatrix}v''-u''v'\\-u'v''-\dfrac{3}{2}v'^2\,v''\end{pmatrix}\right)\\
&-\rho_{f,o}\,A_o\,g\begin{pmatrix}\left(v'-\dfrac{1}{2}v'^3+u\,v''\right)\\+X\begin{pmatrix}v''-u''v'\\-u'v''-\dfrac{3}{2}v'^2\,v''\end{pmatrix}\end{pmatrix}-2(1-2\upsilon)A_o\,\overline{p}_o\left(v''-u''v'-u'v''-\frac{3}{2}v'^2\,v''\right)\\
&-\left(\frac{\rho_{f,o}\;D_o\left(\dfrac{D_i^{\,2}}{D_{ch}^{\,2}-D_o^{\,2}}\right)^2U_i^{\,2}C_T}{2}\right)\begin{pmatrix}\dfrac{1}{2}v'^3-uv''-u'v'-\dfrac{\dot{v}\,v'^2}{U_o}-\dfrac{\dot{v}^2\,v'}{2U_o^{\,2}}\\+\left(\dfrac{L}{2}-X\right)\left(v''-u''v'-u'v''-\dfrac{3}{2}v'^2\,v''\right)\end{pmatrix}\\
&+\left(\frac{\rho_{f,o}\;D_o\left(\dfrac{D_i^{\,2}}{D_{ch}^{\,2}-D_o^{\,2}}\right)^2U_i^{\,2}C_N}{2}\right)\begin{pmatrix}-v'+\left(\dfrac{D_{ch}^{\,2}-D_o^{\,2}}{D_i^{\,2}}\right)\dfrac{\dot{v}}{U_i}+\left(\dfrac{D_{ch}^{\,2}-D_o^{\,2}}{D_i^{\,2}}\right)\dfrac{\dot{v}\,u'}{U_i}+u'v'\\+\left(\dfrac{D_{ch}^{\,2}-D_o^{\,2}}{D_i^{\,2}}\right)^2\dfrac{\dot{u}\,\dot{v}}{U_i^{\,2}}+v'^3-\left(\dfrac{D_{ch}^{\,2}-D_o^{\,2}}{D_i^{\,2}}\right)\dfrac{\dot{v}\,v'^2}{U_i}\\-\left(\dfrac{D_{ch}^{\,2}-D_o^{\,2}}{D_i^{\,2}}\right)^3\dfrac{\dot{v}^3}{2U_i^{\,3}}-\left(\dfrac{D_{ch}^{\,2}-D_o^{\,2}}{D_i^{\,2}}\right)^2\dfrac{\dot{v}^2v'}{2U_i^{\,2}}\end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\frac{\rho_{f,o}\ D_o\left(\frac{D_i^{\,2}}{D_{ch}^{\,2}-D_o^{\,2}}\right)^2 U_i^{\,2} C_{DP}}{2}\right)\left(\begin{array}{l}-v'|v'|+\left(\frac{D_{ch}^{\,2}-D_o^{\,2}}{D_i^{\,2}}\right)\frac{|\dot{v}|\ v'+|v'|\ \dot{v}}{U_i}\\ +\left(\frac{D_{ch}^{\,2}-D_o^{\,2}}{D_i^{\,2}}\right)^2\frac{\dot{v}|\dot{v}|}{U_i^{\,2}}\end{array}\right)\\
&+M_a\left(\begin{array}{l}\ddot{v}-\ddot{u}\,v'-2\dot{u}\,\dot{v}'-\ddot{v}v'^2-\frac{3}{2}\dot{v}\,\dot{v}'\,v'\\ +\dot{v}\,f_t-\frac{\dot{\varphi}^2}{4}v+\dot{\varphi}\,\dot{w}+\frac{\dot{\varphi}\,f_t}{2}w\end{array}\right)\\
&+M_a\left(\frac{D_i^{\,2}}{D_{ch}^{\,2}-D_o^{\,2}}\right)U_i\left(-2\dot{v}'+3\dot{u}'v'+4u'\dot{v}'+\frac{7}{2}\dot{v}'v'^2+2\dot{u}\,v''+\frac{3}{2}\dot{v}v'v''\right)\\
&+M_a\left(\frac{D_i^{\,2}}{D_{ch}^{\,2}-D_o^{\,2}}\right)^2 U_i^{\,2}\left(v''-2u''v'-4u'v''-\frac{5}{2}v''v'^2\right)\quad=\quad 0
\end{aligned}
\tag{4.62}
$$

$$
\begin{aligned}
&(m+M)\ddot{w}+(2M\,U_i)\,\dot{w}'+M\,U_i^{\,2}\,w''+\left(I_D+I_{D,f}\right)(2\,\ddot{w}'')\\
&+\left(I_p\right)\left(\begin{array}{l}\dot{w}'(v'\dot{v}''+\dot{v}'v'')+2v'^2\ddot{w}''+3\ddot{w}'v'v''+2\dot{w}''v'\dot{v}'+\frac{\dot{\varphi}\left(v'\ddot{w}'+\dot{w}''\dot{v}'\right)}{\dot{w}'}\\ -\frac{2\dot{\varphi}\,v'\dot{w}''\ddot{w}'}{\dot{w}'^2}+\frac{\ddot{\varphi}\,v'\dot{w}''}{\dot{w}'}+\frac{\dot{\varphi}'\,v'\ddot{w}'}{\dot{w}'}+\frac{\dot{\varphi}\left(v'\ddot{w}''+\ddot{w}'v''\right)}{\dot{w}'}+\dot{\varphi}\,\dot{v}''+v''\ddot{\varphi}+v'\ddot{\varphi}'+\dot{\varphi}'\,\dot{v}'\end{array}\right)\\
&-EI\left(\begin{array}{l}3u'''\,w''+4u''\,w'''+2u'\,w''''+w'\,u''''\\ +2\,w'^2\,w''''+8\,w'\,w''\,w'''+2\,w''^3\end{array}\right)+(E\,I)w''''+EI\,a\,\dot{w}''''\\
&-EI\,a\left(\begin{array}{l}3u'''\dot{w}''+3w''\dot{u}'''+4u''\dot{w}'''+4w'''\dot{u}''+2u'\dot{w}''''+2w''''\dot{u}'+w'\dot{u}''''+u''''w'\\ +2w'^2\dot{w}''''+4w''''w'\dot{w}'+8w'w''\dot{w}'''+8w'w'''\dot{w}''+8\dot{w}'w''w'''+6\,w''^2\dot{w}''\end{array}\right)\\
&+\left(\overline{T}-A_f\left(\begin{array}{l}(1-2\upsilon)\overline{p}_o+\frac{\rho_{f,o}}{2}\left(g\,L+U_i^{\,2}\left(\frac{D_i^{\,2}}{D_{ch}^{\,2}-D_o^{\,2}}\right)^2\right)\\ +\frac{\rho_{f,i}\,U_i^{\,2}}{2}\left(C\left(1-\frac{D_i^{\,2}}{D_{ch}^{\,2}-D_o^{\,2}}\right)^2-1\right)\end{array}\right)\right)\left(\begin{array}{l}-w''+u''\,w'\\ +u'\,w''+\frac{3}{2}w'^2\,w''\end{array}\right)\\
&-EA\left(u''\,w'+u'\,w''+\frac{3}{2}w'^2\,w''\right)-EA\,a\left(u''\dot{w}'+w'\dot{u}''+u'\dot{w}''+w''\dot{u}'+\frac{3}{2}w'^2\,\dot{w}''+3w''w'\dot{w}'\right)\\
&+(m+M)g\left(w'-\frac{1}{2}w'^3+uw''-\left(\frac{L}{2}-X\right)\left(w''-u''w'-u'w''-\frac{3}{2}w'^2\,w''\right)\right)
\end{aligned}
$$

$$
-\left(\frac{\rho_{f,o}\;\; D_o\left(\frac{D_i^{\,2}}{D_{ch}^{\,2}-D_o^{\,2}}\right)^2 U_i^{\,2} C_T}{2}\,\frac{D_o}{D_h}\right)\left(\begin{array}{l}\left(w'-\frac{1}{2}w'^3+u\,w''\right)\\ -\left(\frac{L}{2}-X\right)\left(\begin{array}{l}w''-u''w'\\ -u'w''-\frac{3}{2}w'^2\,w''\end{array}\right)\end{array}\right)
$$

$$
-\rho_{f,o}\,A_o\,g\left(\begin{array}{l}\left(w'-\frac{1}{2}w'^3+u\,w''\right)\\ +X\left(\begin{array}{l}w''-u''w'\\ -u'w''-\frac{3}{2}w'^2\,w''\end{array}\right)\end{array}\right)-2(1-2\upsilon)A_o\,\overline{p}_o\left(w''-u''w'-u'w''-\frac{3}{2}w'^2\,w''\right)
$$

$$
-\left(\frac{\rho_{f,o}\;\; D_o\left(\frac{D_i^{\,2}}{D_{ch}^{\,2}-D_o^{\,2}}\right)^2 U_i^{\,2} C_T}{2}\right)\left(\begin{array}{l}\frac{1}{2}w'^3-uw''-u'w'-\left(\frac{D_{ch}^{\,2}-D_o^{\,2}}{D_i^{\,2}}\right)\frac{\dot{v}\,w'^2}{U_i}\\ -\left(\frac{D_{ch}^{\,2}-D_o^{\,2}}{D_i^{\,2}}\right)^2\frac{\dot{v}^2\,w'}{2U_i^{\,2}}\\ +\left(\frac{L}{2}-X\right)\left(w''-u''w'-u'w''-\frac{3}{2}w'^2\,w''\right)\end{array}\right)
$$

$$
+\left(\frac{\rho_{f,o}\;\; D_o\left(\frac{D_i^{\,2}}{D_{ch}^{\,2}-D_o^{\,2}}\right)^2 U_i^{\,2} C_N}{2}\right)\left(\begin{array}{l}-w'+\left(\frac{D_{ch}^{\,2}-D_o^{\,2}}{D_i^{\,2}}\right)\frac{\dot{w}}{U_i}+\left(\frac{D_{ch}^{\,2}-D_o^{\,2}}{D_i^{\,2}}\right)\frac{\dot{w}u'}{U_i}+u'w'\\ +\left(\frac{D_{ch}^{\,2}-D_o^{\,2}}{D_i^{\,2}}\right)^2\frac{\dot{u}\,\dot{w}}{U_i^{\,2}}+w'^3-\left(\frac{D_{ch}^{\,2}-D_o^{\,2}}{D_i^{\,2}}\right)\frac{\dot{w}\,w'^2}{U_i}\\ -\left(\frac{D_{ch}^{\,2}-D_o^{\,2}}{D_i^{\,2}}\right)^3\frac{\dot{w}^3}{2U_i^{\,3}}-\left(\frac{D_{ch}^{\,2}-D_o^{\,2}}{D_i^{\,2}}\right)^2\frac{\dot{w}^2w'}{2U_i^{\,2}}\end{array}\right)
$$

$$
+\left(\frac{\rho_{f,o}\;\; D_o\left(\frac{D_i^{\,2}}{D_{ch}^{\,2}-D_o^{\,2}}\right)^2 U_i^{\,2} C_{DP}}{2}\right)\left(\begin{array}{l}-w'|w'|+\left(\frac{D_{ch}^{\,2}-D_o^{\,2}}{D_i^{\,2}}\right)\frac{|\dot{w}|\;w'+|w'|\;\dot{w}}{U_i}\\ +\left(\frac{D_{ch}^{\,2}-D_o^{\,2}}{D_i^{\,2}}\right)^2\frac{\dot{w}|\dot{w}|}{U_i^{\,2}}\end{array}\right)
$$

$$
+M_a\left(\ddot{w}-\ddot{u}\,w'-2\dot{u}\,\dot{w}'-\ddot{w}w'^2-\frac{3}{2}\dot{w}\,\dot{w}'\,w'+\dot{w}\,f_t-\frac{\dot{\varphi}^2}{4}w+\dot{\varphi}\,\dot{v}+\frac{\dot{\varphi}\,f_t}{2}v\right)
$$

$$
+M_a\left(\frac{D_i^{\,2}}{D_{ch}^{\,2}-D_o^{\,2}}\right)U_i\left(-2\dot{w}'+3\dot{u}'w'+4u'\dot{w}'+\frac{7}{2}\dot{w}'w'^2+2\dot{u}\,w''+\frac{3}{2}\dot{w}w'w''\right)
$$

$$
+M_a\left(\frac{D_i^{\,2}}{D_{ch}^{\,2}-D_o^{\,2}}\right)^2 U_i^{\,2}\left(w''-2u''w'-4u'w''-\frac{5}{2}w''\,w'^2\right)\;=\;0
$$

(4.63)

and,

$$\left(I_p\right)\left(\ddot{\varphi}+\dot{w}'\dot{v}'+v'\ddot{w}'\right)-\left(GJ\ \varphi''-4I_{p,f}\ L\ f_r\left(\varphi'\ddot{\varphi}+2\dot{\varphi}\dot{\varphi}'\right)\right)=0 \qquad (4.64)$$

In the current model, one notes that the drill string is not permitted to slide axially in *X*-direction, which doesn't mimic the actual practice of drilling. The next subsection is devoted to produce more reliable and accurate model which takes into consideration the axial sliding of the inner pipe.

4.9.2 Extensible rotating flexible pipe with one sliding end conveying fluid downwards, which then flows upwards as a confined annular flow

Another idealization for the motion of drill string can be made by considering the drill string as rotating flexible pipe, fixed at one end while it is restricted to deflect laterally in both *X*-*Y* and *X*-*Z* planes at the another end by using sliding support, conveying fluid downwards, which then flows upwards as a confined annular flow, as shown in figure 4.5.

In order to come up with the equations of motion which describe this model, one may consider the following modifications, as proposed by [17]:

$$A_o\ p_o\,(X)=-\left(\frac{1}{2}\rho_{f,o}\ D_o\ U_o^{\ 2}\ C_T\ \frac{D_o}{D_h}+\rho_{f,o}\ g\ A_o\right)\left(L-X+u\left(L\right)-u\right) \qquad (4.65)$$

where *u (L)* is the axial displacement at the end of the pipe. The tension can be expressed as

$$\breve{T}\,(X)=\frac{1}{2}\rho_{f,o}\ D_o\ U_o^{\ 2}C_b+\left(\frac{1}{2}\rho_{f,o}\ D_o\ U_o^{\ 2}\ C_T+\left(M+m\right)g\right)\left(L-X+u(L)-u\right) \qquad (4.66)$$

where C_b is the base pressure coefficient [17]. Using above expressions (i.e. equations 4.65 and 4.66) in formulating the equations of motion, one can write the equations of motion in the following form:

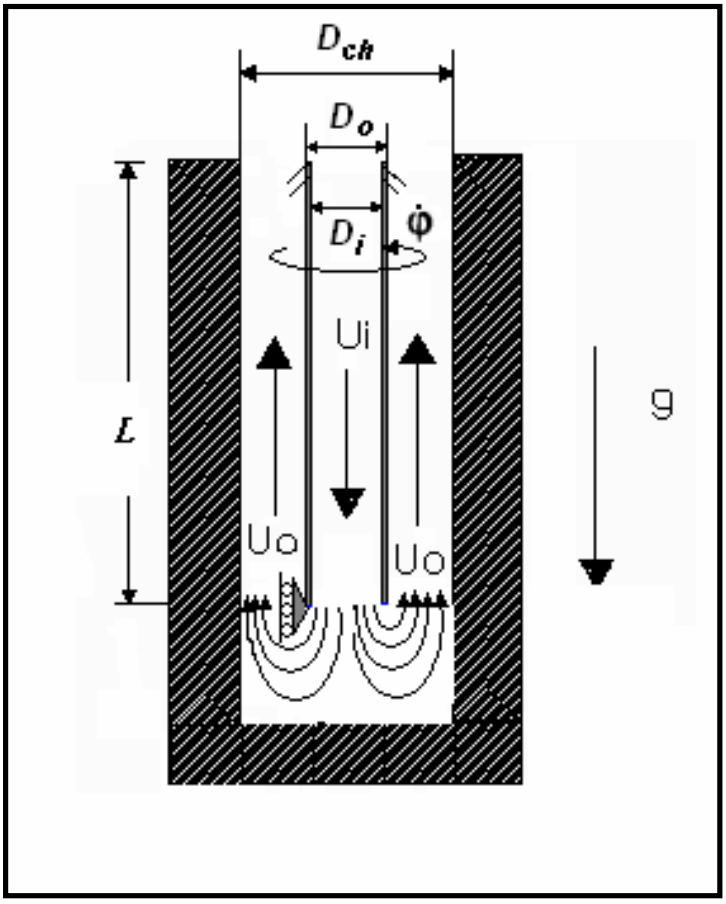


Figure 4.5: Rotating flexible pipe with one sliding end conveying fluid downwards, which then flows upwards in the outer annulus.

$$
\begin{aligned}
&(m+M)\ddot{u}+(2M\,U_i)\,\dot{u}'+M\,U_i^{\,2}\,u''-(E\,A)u''-(E\,Aa)\dot{u}'' \\
&+\left(-A_f\left(\frac{\rho_{f,o}}{2}\left(g\,L+U_i^{\,2}\left(\frac{D_i^{\,2}}{D_{ch}^{\,2}-D_o^{\,2}}\right)^2\right)+\frac{\rho_{f,i}\,U_i^{\,2}}{2}\left(\overline{C}\left(1-\frac{D_i^{\,2}}{D_{ch}^{\,2}-D_o^{\,2}}\right)^2-1\right)\right)-E\,A\right)(v'\,v''+w'\,w'') \\
&-EA\,a\,\left(v'\,\dot{v}''+\dot{v}'v''+w'\dot{w}''+\dot{w}'w''\right) \\
&-E\,I\left(v''''\,v'+v''\,v''''+w''''\,w'+w''\,w''''\right)-EI\,a\left(v''''\dot{v}'+v'\dot{v}''''+w''''\dot{w}'+w'\dot{w}''''\right) \\
&+(m+M)g\left(-\frac{1}{2}v'^2-\frac{1}{2}w'^2+u'+(L-X)(v'\,v''+w'w'')\right) \\
&-M_a\left(\ddot{v}\,v'+\ddot{w}\,w'\right)-M_a\left(\frac{D_i^{\,2}}{D_{ch}^{\,2}-D_o^{\,2}}\right)\,U_i\,\left(-2\,\dot{v}'\,v'-2\,\dot{w}'w'\right)-M_a\left(\frac{D_i^{\,2}}{D_{ch}^{\,2}-D_o^{\,2}}\right)^2 U_i^{\,2}\left(v'\,v''+w'w''\right) \\
&-\left(\frac{\rho_{f,o}\;\;D_o\left(\frac{D_i^{\,2}}{D_{ch}^{\,2}-D_o^{\,2}}\right)^2 U_i^{\,2}C_T}{2}\,\frac{D_o}{D_h}\right)\left(-\frac{1}{2}\left(v'^2+w'^2\right)+u'+(L-X)(v'\,v''+w'w'')\right) \\
&-\left(\rho_{f,o}\,A_o\,g\right)\begin{pmatrix}-\frac{1}{2}\left(v'^2+w'^2\right)+u' \\ +(L-X)(v'\,v''+w'w'')\end{pmatrix}+\left(\frac{\rho_{f,o}\;\;D_o\left(\frac{D_i^{\,2}}{D_{ch}^{\,2}-D_o^{\,2}}\right)^2 U_i^{\,2}C_b}{2}\right)(v'\,v''+w'\,w'') \\
&-\left(\frac{\rho_{f,o}\;\;D_o\left(\frac{D_i^{\,2}}{D_{ch}^{\,2}-D_o^{\,2}}\right)^2 U_i^{\,2}C_T}{2}\right)\begin{pmatrix}\frac{3}{2}\left(v'^2+w'^2\right)-u'-\left(\frac{D_{ch}^{\,2}-D_o^{\,2}}{D_i^{\,2}}\right)\frac{(v'\dot{v}+w'\dot{w})}{U_i} \\ -\left(\frac{D_{ch}^{\,2}-D_o^{\,2}}{D_i^{\,2}}\right)^2\frac{\left(\dot{v}^2+\dot{w}^2\right)}{2U_i^{\,2}}-(L-X)(v'\,v''+w'w'')\end{pmatrix} \\
&-\left(\frac{\rho_{f,o}\;\;D_o\left(\frac{D_i^{\,2}}{D_{ch}^{\,2}-D_o^{\,2}}\right)^2 U_i^{\,2}C_N}{2}\right)\left(-\left(v'^2+w'^2\right)+\left(\frac{D_{ch}^{\,2}-D_o^{\,2}}{D_i^{\,2}}\right)\frac{(\dot{v}\,v'+\dot{w}w')}{U_i}\right) \\
&-\left(\frac{\rho_{f,o}\;\;D_o\left(\frac{D_i^{\,2}}{D_{ch}^{\,2}-D_o^{\,2}}\right)^2 U_i^{\,2}C_{DP}}{2}\right)\begin{pmatrix}-v'^2|v'|+\left(\frac{D_{ch}^{\,2}-D_o^{\,2}}{D_i^{\,2}}\right)\frac{|\dot{v}|v'^2+|v'|v'\dot{v}}{U_i} \\ +\left(\frac{D_{ch}^{\,2}-D_o^{\,2}}{D_i^{\,2}}\right)^2\frac{\dot{v}|\dot{v}|v'}{U_i^{\,2}}+\left(\frac{D_{ch}^{\,2}-D_o^{\,2}}{D_i^{\,2}}\right)^2\frac{\dot{w}|\dot{w}|w'}{U_i^{\,2}} \\ -w'^2|w'|+\left(\frac{D_{ch}^{\,2}-D_o^{\,2}}{D_i^{\,2}}\right)\frac{|\dot{w}|w'^2+|w'|w'\dot{w}}{U_i}\end{pmatrix} \;=\; 0
\end{aligned}
\tag{4.67}
$$

$$
\begin{aligned}
&(m+M)\ddot{v}+(2M\,U_i)\,\dot{v}'+M\,U_i^{\,2}\,v''+\left(I_D+I_{D,f}\right)(2\,\ddot{v}'')+\left(I_p\right)\left(v''\dot{w}'^2+2v'\dot{w}'\dot{w}''+\dot{\varphi}\,\dot{w}''+\dot{w}'\,\dot{\varphi}'\right)\\
&+(E\,I)v''''-EA\left(u''\,v'+u'\,v''+\frac{3}{2}v'^2\,v''\right)-EA\,a\left(u''\dot{v}'+v'\dot{u}''+u'\dot{v}''+v''\dot{u}'+\frac{3}{2}v'^2\,\dot{v}''+3v''v'\dot{v}'\right)\\
&+EI\,a\,\dot{v}''''-EI\left(3u'''\,v''+4u''\,v'''+2u'\,v''''+v'\,u''''+2v'^2\,v''''+8v'\,v''\,v'''+2v''^3\right)\\
&-EI\,a\begin{pmatrix}3u'''\dot{v}''+3v''\dot{u}'''+4u''\dot{v}'''+4v'''\dot{u}''+2u'\dot{v}''''+2v''''\dot{u}'+v'\dot{u}''''+u''''v'\\+2v'^2\dot{v}''''+4v''''v'\dot{v}'+8v'v''\dot{v}'''+8v'v'''\dot{v}''+8\dot{v}'v''v'''+6\,v''^2\dot{v}''\end{pmatrix}\\
&+\left(-A_f\begin{pmatrix}\dfrac{\rho_{f,o}}{2}\left(g\,L+U_i^{\,2}\left(\dfrac{D_i^{\,2}}{D_{ch}^{\,2}-D_o^{\,2}}\right)^2\right)\\+\dfrac{\rho_{f,i}\,U_i^{\,2}}{2}\left(\overline{C}\left(1-\dfrac{D_i^{\,2}}{D_{ch}^{\,2}-D_o^{\,2}}\right)^2-1\right)\end{pmatrix}\right)\left(-v''+u''\,v'+u'\,v''+\frac{3}{2}v'^2\,v''\right)\\
&+\left(\frac{\rho_{f,o}\;\;D_o\left(\dfrac{D_i^{\,2}}{D_{ch}^{\,2}-D_o^{\,2}}\right)^2U_i^{\,2}C_b}{2}\right)\left(-v''+u''\,v'+u'\,v''+\frac{3}{2}v'^2\,v''\right)\\
&+(m+M)\,g\left(v'-\frac{1}{2}v'^3+uv''-u(L)v''-(L-X)\left(v''-u''v'-u'v''-\frac{3}{2}v'^2\,v''\right)\right)\\
&-\left(\frac{\rho_{f,o}\;\;D_o\left(\dfrac{D_i^{\,2}}{D_{ch}^{\,2}-D_o^{\,2}}\right)^2U_i^{\,2}C_T}{2}\frac{D_o}{D_h}\right)\left(\begin{pmatrix}v'-\dfrac{1}{2}v'^3+u\,v''-u(L)v''\end{pmatrix}\\-(L-X)\begin{pmatrix}v''-u''v'\\-u'v''-\dfrac{3}{2}v'^2\,v''\end{pmatrix}\right)\\
&-\rho_{f,o}\,A_o\,g\left(\left(v'-\frac{1}{2}v'^3+u\,v''-u(L)v''\right)-(L-X)\begin{pmatrix}v''-u''v'\\-u'v''-\dfrac{3}{2}v'^2\,v''\end{pmatrix}\right)\\
&-\left(\frac{\rho_{f,o}\;\;D_o\left(\dfrac{D_i^{\,2}}{D_{ch}^{\,2}-D_o^{\,2}}\right)^2U_i^{\,2}C_T}{2}\right)\begin{pmatrix}\dfrac{1}{2}v'^3-uv''-u'v'+u(L)v''-\left(\dfrac{D_{ch}^{\,2}-D_o^{\,2}}{D_i^{\,2}}\right)\dfrac{\dot{v}\,v'^2}{U_i}\\-\left(\dfrac{D_{ch}^{\,2}-D_o^{\,2}}{D_i^{\,2}}\right)^2\dfrac{\dot{v}^2\,v'}{2U_2^{\,2}}\\+(L-X)\left(v''-u''v'-u'v''-\dfrac{3}{2}v'^2\,v''\right)\end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\frac{\rho_{f,o}\ D_o\left(\frac{D_i^{\,2}}{D_{ch}^{\,2}-D_o^{\,2}}\right)^2 U_i^{\,2} C_N}{2}\right)\left(\begin{array}{l}
-v'+\left(\frac{D_{ch}^{\,2}-D_o^{\,2}}{D_i^{\,2}}\right)\frac{\dot{v}}{U_i}+\left(\frac{D_{ch}^{\,2}-D_o^{\,2}}{D_i^{\,2}}\right)\frac{\dot{v}u'}{U_i}+u'v' \\
+\left(\frac{D_{ch}^{\,2}-D_o^{\,2}}{D_i^{\,2}}\right)^2\frac{\dot{u}\dot{v}}{U_i^{\,2}}+v'^3-\left(\frac{D_{ch}^{\,2}-D_o^{\,2}}{D_i^{\,2}}\right)\frac{\dot{v}v'^2}{U_i} \\
-\left(\frac{D_{ch}^{\,2}-D_o^{\,2}}{D_i^{\,2}}\right)^3\frac{\dot{v}^3}{2U_i^{\,3}}-\left(\frac{D_{ch}^{\,2}-D_o^{\,2}}{D_i^{\,2}}\right)^2\frac{\dot{v}^2v'}{2U_i^{\,2}}
\end{array}\right) \\
&+\left(\frac{\rho_{f,o}\ D_o\left(\frac{D_i^{\,2}}{D_{ch}^{\,2}-D_o^{\,2}}\right)^2 U_i^{\,2} C_{DP}}{2}\right)\left(\begin{array}{l}
-v'|v'|+\left(\frac{D_{ch}^{\,2}-D_o^{\,2}}{D_i^{\,2}}\right)\frac{|\dot{v}|\ v'+|v'|\ \dot{v}}{U_i} \\
+\left(\frac{D_{ch}^{\,2}-D_o^{\,2}}{D_i^{\,2}}\right)^2\frac{\dot{v}|\dot{v}|}{U_i^{\,2}}
\end{array}\right) \\
&+M_a\left(\ddot{v}-\ddot{u}v'-2\dot{u}\dot{v}'-\ddot{v}v'^2-\frac{3}{2}\dot{v}\dot{v}'v'+\dot{v}f_t-\frac{\dot{\varphi}^2}{4}v+\dot{\varphi}\dot{w}+\frac{\dot{\varphi}f_t}{2}w\right) \\
&+M_a\left(\frac{D_i^{\,2}}{D_{ch}^{\,2}-D_o^{\,2}}\right)U_i\left(-2\dot{v}'+3\dot{u}'v'+4u'\dot{v}'+\frac{7}{2}\dot{v}'v'^2+2\dot{u}v''+\frac{3}{2}\dot{v}v'v''\right) \\
&+M_a\left(\frac{D_i^{\,2}}{D_{ch}^{\,2}-D_o^{\,2}}\right)^2 U_i^{\,2}\left(v''-2u''v'-4u'v''-\frac{5}{2}v''v'^2\right) \quad = \quad 0
\end{aligned}
\tag{4.68}
$$

$$
\begin{aligned}
&(m+M)\ddot{w}+(2M\,U_i)\,\dot{w}'+M\,U_i^{\,2}\,w''+\left(I_D+I_{D,f}\right)\left(2\,\ddot{w}''\right)\\
&+\left(I_p\right)\begin{pmatrix}\dot{w}'\left(v'\dot{v}''+\dot{v}'v''\right)+2v'^2\ddot{w}''+3\,\ddot{w}'v'v''+2\dot{w}''v'\dot{v}'+\dfrac{\dot{\varphi}\left(v'\ddot{w}'+\dot{w}''\dot{v}'\right)}{\dot{w}'}\\
-\dfrac{2\,\dot{\varphi}\,v'\dot{w}''\ddot{w}'}{\dot{w}'^2}+\dfrac{\ddot{\varphi}\,v'\dot{w}''}{\dot{w}'}+\dfrac{\dot{\varphi}'\,v'\ddot{w}'}{\dot{w}'}+\dfrac{\dot{\varphi}\left(v'\ddot{w}''+\ddot{w}'v''\right)}{\dot{w}'}+\dot{\varphi}\,\dot{v}''+v''\ddot{\varphi}+v'\ddot{\varphi}'+\dot{\varphi}'\,\dot{v}'\end{pmatrix}\\
&-EI\begin{pmatrix}3u'''w''+4u''w'''+2u'w''''+w'u'''''\\+2w'^2\,w''''+8w'w''w'''+2w''^3\end{pmatrix}+(E\,I)w''''+EI\,a\,\dot{w}''''\\
&-EI\,a\begin{pmatrix}3u'''\dot{w}''+3w''\dot{u}'''+4u''\dot{w}'''+4w'''\dot{u}''+2u'\dot{w}''''+2w''''\dot{u}'+w'\dot{u}''''+u''''w'\\+2w'^2\dot{w}''''+4w''''w'\dot{w}'+8w'w''\dot{w}'''+8w'w'''\dot{w}''+8\dot{w}'w''w'''+6\,w''^2\dot{w}''\end{pmatrix}\\
&+\left(-A_f\begin{pmatrix}\dfrac{\rho_{f,o}}{2}\left(g\,L+U_i^{\,2}\left(\dfrac{D_i^{\,2}}{D_{ch}^{\,2}-D_o^{\,2}}\right)^2\right)\\+\dfrac{\rho_{f,i}\,U_i^{\,2}}{2}\left(\overline{C}\left(1-\dfrac{D_i^{\,2}}{D_{ch}^{\,2}-D_o^{\,2}}\right)^2-1\right)\end{pmatrix}\right)\left(-w''+u''w'+u'w''+\frac{3}{2}w'^2\,w''\right)\\
&+\left(\frac{\rho_{f,o}\;\;D_o\left(\dfrac{D_i^{\,2}}{D_{ch}^{\,2}-D_o^{\,2}}\right)^2U_i^{\,2}C_b}{2}\right)\left(-w''+u''w'+u'w''+\frac{3}{2}w'^2\,w''\right)\\
&-EA\left(u''w'+u'w''+\frac{3}{2}w'^2\,w''\right)-EA\,a\left(u''\dot{w}'+w'\dot{u}''+u'\dot{w}''+w''\dot{u}'+\frac{3}{2}w'^2\,\dot{w}''+3w''w'\dot{w}'\right)\\
&+(m+M)g\left(w'-\frac{1}{2}w'^3+uw''-u(L)w''-(L-X)\left(w''-u''w'-u'w''-\frac{3}{2}w'^2\,w''\right)\right)\\
&-\left(\frac{\rho_{f,o}\;\;D_o\left(\dfrac{D_i^{\,2}}{D_{ch}^{\,2}-D_o^{\,2}}\right)^2U_i^{\,2}C_T}{2}\frac{D_o}{D_h}\right)\begin{pmatrix}\left(w'-\dfrac{1}{2}w'^3+u\,w''-u(L)w''\right)\\-(L-X)\begin{pmatrix}w''-u''w'\\-u'w''-\dfrac{3}{2}w'^2\,w''\end{pmatrix}\end{pmatrix}\\
&-\rho_{f,o}\,A_o\,g\left(\left(w'-\frac{1}{2}w'^3+u\,w''-u(L)w''\right)-(L-X)\begin{pmatrix}w''-u''w'\\-u'w''-\dfrac{3}{2}w'^2\,w''\end{pmatrix}\right)
\end{aligned}
$$

$$
\begin{aligned}
&-\left(\frac{\rho_{f,o}\ D_o\left(\frac{D_i^{\,2}}{D_{ch}^{\,2}-D_o^{\,2}}\right)^2 U_i^{\,2} C_T}{2}\right)\left(\begin{array}{l}\frac{1}{2}w'^3-uw''+u(L)w''-u'w'-\left(\frac{D_{ch}^{\,2}-D_o^{\,2}}{D_i^{\,2}}\right)\frac{\dot{v}\,w'^2}{U_i}\\ -\left(\frac{D_{ch}^{\,2}-D_o^{\,2}}{D_i^{\,2}}\right)^2\frac{\dot{v}^2\,w'}{2U_i^{\,2}}\\ +(L-X)\left(w''-u''w'-u'w''-\frac{3}{2}w'^2\,w''\right)\end{array}\right)\\
&+\left(\frac{\rho_{f,o}\ D_o\left(\frac{D_i^{\,2}}{D_{ch}^{\,2}-D_o^{\,2}}\right)^2 U_i^{\,2} C_N}{2}\right)\left(\begin{array}{l}-w'+\left(\frac{D_{ch}^{\,2}-D_o^{\,2}}{D_i^{\,2}}\right)\frac{\dot{w}}{U_i}+\left(\frac{D_{ch}^{\,2}-D_o^{\,2}}{D_i^{\,2}}\right)\frac{\dot{w}u'}{U_i}+u'w'\\ +\left(\frac{D_{ch}^{\,2}-D_o^{\,2}}{D_i^{\,2}}\right)^2\frac{\dot{u}\,\dot{w}}{U_i^{\,2}}+w'^3-\left(\frac{D_{ch}^{\,2}-D_o^{\,2}}{D_i^{\,2}}\right)\frac{\dot{w}\,w'^2}{U_i}\\ -\left(\frac{D_{ch}^{\,2}-D_o^{\,2}}{D_i^{\,2}}\right)^3\frac{\dot{w}^3}{2U_i^{\,3}}-\left(\frac{D_{ch}^{\,2}-D_o^{\,2}}{D_i^{\,2}}\right)^2\frac{\dot{w}^2w'}{2U_i^{\,2}}\end{array}\right)\\
&+\left(\frac{\rho_{f,o}\ D_o\left(\frac{D_i^{\,2}}{D_{ch}^{\,2}-D_o^{\,2}}\right)^2 U_i^{\,2} C_{DP}}{2}\right)\left(\begin{array}{l}-w'|w'|+\left(\frac{D_{ch}^{\,2}-D_o^{\,2}}{D_i^{\,2}}\right)\frac{|\dot{w}|\ w'+|w'|\ \dot{w}}{U_i}\\ +\left(\frac{D_{ch}^{\,2}-D_o^{\,2}}{D_i^{\,2}}\right)^2\frac{\dot{w}|\dot{w}|}{U_i^{\,2}}\end{array}\right)\\
&+M_a\left(\ddot{w}-\ddot{u}\,w'-2\dot{u}\,\dot{w}'-\ddot{w}w'^2-\frac{3}{2}\dot{w}\,\dot{w}'\,w'+\dot{w}\,f_t-\frac{\dot{\varphi}^2}{4}w+\dot{\varphi}\,\dot{v}+\frac{\dot{\varphi}\,f_t}{2}v\right)\\
&+M_a\left(\frac{D_i^{\,2}}{D_{ch}^{\,2}-D_o^{\,2}}\right)U_i\left(-2\dot{w}'+3\dot{u}'w'+4u'\dot{w}'+\frac{7}{2}\dot{w}'w'^2+2\dot{u}\,w''+\frac{3}{2}\dot{w}w'w''\right)\\
&+M_a\left(\frac{D_i^{\,2}}{D_{ch}^{\,2}-D_o^{\,2}}\right)^2 U_i^{\,2}\left(w''-2u''w'-4u'w''-\frac{5}{2}w''\,w'^2\right)\ =\ 0
\end{aligned}
\tag{4.69}
$$

and,

$$
(I_p)(\ddot{\varphi}+\dot{w}'\dot{v}'+v'\ddot{w}')-\left(G\,J\,\varphi''-4\,I_{p,f}\,L\ \ f_r\,(\varphi'\ddot{\varphi}+2\,\dot{\varphi}\,\dot{\varphi}')\right)=0 \tag{4.70}
$$

CHAPTER FIVE

DIMENSIONLESS ANALYSIS

In this chapter, the nonlinear equations of motion, which were presented in chapters 2, 3 and 4, will be expressed in the most general dimensionless form by defining appropriate dimensionless parameters. The dimensionless analysis is considered as powerful tool to study the qualitative behavior of the system. On the other hand, such analysis is useful to avoid dealing with extremely high or small numeric values of the system parameters; this can be performed by relating such quantities to each other using suitable dimensionless groups. Also, it is noted that such analysis is necessary to check the dimensional compatibility of the bulky equations which were derived previously. In the next subsections, the dimensionless analysis is carried out for the derived models obtained in chapter 2, 3 and 4, in addition to some selected case studies.

5.1 Extensible Flexible Pipe Conveying Fluid and Subjected to External Axial Flow.

In this section, the dimensionless forms of equations (2.84-2.85) will be established. Equations of motion can be rendered dimensionless by defining the following dimensionless quantities:

$$\xi_1 = \frac{X}{L} \tag{5.1}$$

$$\zeta = \frac{u}{L} \tag{5.2}$$

$$\eta_1 = \frac{v}{L} \tag{5.3}$$

$$t_1^{\ *} = \left(\frac{E\,I}{m + M + \rho_{f,o}\,A_o} \right)^{\frac{1}{2}} \frac{t}{L^2} \tag{5.4}$$

$$\alpha_1 = \left(\frac{E\,I}{m + M + \rho_{f,o}\,A_o} \right)^{\frac{1}{2}} \frac{a}{L^2} \tag{5.5}$$

$$\beta_i = \left(\frac{M}{m + M + \rho_{f,o}\,A_o} \right) \tag{5.6}$$

$$\beta_{o,1} = \left(\frac{\rho_{f,o}\,A_o}{m + M + \rho_{f,o}\,A_o} \right) \tag{5.7}$$

$$u_{i,1} = \left(\frac{M}{EI} \right)^{\frac{1}{2}} U_i\,L \tag{5.8}$$

$$u_{o,1} = \left(\frac{\rho_{f,o}\,A_o}{EI} \right)^{\frac{1}{2}} U_o\,L \tag{5.9}$$

$$c_N = \frac{4}{\pi} C_N \quad , \quad c_T = \frac{4}{\pi} C_T \quad , \quad c_d = \frac{4}{\pi} C_{DP} \tag{5.10}$$

$$\chi = \frac{M_a}{\rho_{f,o}\,A_o} \tag{5.11}$$

$$\bar{h} = \frac{D_o}{D_h} \tag{5.12}$$

$$\mathrm{E} = \frac{L}{D_o} \tag{5.13}$$

$$\Pi_{0,1} = \frac{EAL^2}{EI} \tag{5.14}$$

$$\Pi_{iL,1} = \frac{p_i(L) A_f \, L^2}{EI} \tag{5.15}$$

$$\Pi_{oL,1} = \frac{p_o(L) A_o \, L^2}{EI} \tag{5.16}$$

$$\overline{\Pi}_1 = \frac{\overline{p}_o A_o \, L^2}{EI} \tag{5.17}$$

$$\overline{\Gamma}_1 = \frac{\overline{T} \, L^2}{EI} \tag{5.18}$$

Substituting equations (5.1-5.18) into equations (2.84) and (2.85) yields the following forms of the dimensionless equations

$$\begin{aligned}
&\left(1-\beta_{o,1}\right)\ddot{\xi} + 2u_{i,1}\sqrt{\beta_i}\,\dot{\xi}' + u_{i,1}{}^2\,\zeta'' - \Pi_{0,1}\,\zeta'' - \left(\Pi_{0,1}\,\alpha_1\right)\dot{\xi}'' \\
&+\left(\overline{\Gamma}_1 - \Pi_{iL,1} + \Pi_{oL,1} - \Pi_{0,1}\right)\eta_1'\,\eta_1'' - \left(\Pi_{0,1}\,\alpha_1\right)\left(\eta_1'\dot{\eta}_1'' + \eta_1''\dot{\eta}_1'\right) \\
&-\left(\eta_1{}^{(4)}\eta_1' + \eta_1''\eta_1'''\right) - \alpha_1\left(\eta_1{}^{(4)}\,\dot{\eta}_1' + \eta_1'\,\dot{\eta}_1{}^{(4)} + \eta_1''\dot{\eta}_1''' + \eta_1'''\dot{\eta}_1''\right) \\
&-\chi\,\beta_{o,1}\left(\ddot{\eta}_1\eta_1'\right) + 2\,\chi\,u_{o,1}\sqrt{\beta_{o,1}}\,\dot{\eta}_1'\eta_1' - \chi\,u_{o,1}{}^2\left(\eta_1'\,\eta_1''\right) \\
&-\left(\frac{\mathrm{E}\,c_T\,u_{o,1}{}^2}{2}\,\overline{h}\right)\left(-\frac{1}{2}\eta_1'^2 + \zeta' + \left(\frac{1}{2}-\xi_1\right)\left(\eta_1'\,\eta_1''\right)\right) \\
&+\left(1-2\nu\right)\overline{\Pi}_1\left(\eta_1'\,\eta_1''\right) \\
&-\left(\frac{\mathrm{E}\,c_T\,u_{o,1}{}^2}{2}\right)\left(\frac{3}{2}\eta_1'^2 - \zeta' - \frac{\sqrt{\beta_{o,1}}}{u_{o,1}}\dot{\eta}_1\eta_1' - \frac{\beta_{o,1}}{2u_{o,1}{}^2}\dot{\eta}_1{}^2 - \left(\frac{1}{2}-\xi_1\right)\left(\eta_1'\,\eta_1''\right)\right) \\
&-\left(\frac{\mathrm{E}\,c_N\,u_{o,1}{}^2}{2}\right)\left(-\eta_1'^2 + \frac{\sqrt{\beta_{o,1}}}{u_{o,1}}\dot{\eta}_1\eta_1'\right) \\
&-\left(\frac{\mathrm{E}\,c_d\,u_{o,1}{}^2}{2}\right)\left(-\eta_1'^2\left|\eta_1'\right| + \sqrt{\beta_{o,1}}\,\frac{\left|\dot{\eta}_1\right|\eta_1'^2 + \left|\eta_1'\right|\eta_1'\,\dot{\eta}_1}{u_{o,1}} + \beta_{o,1}\,\frac{\dot{\eta}_1\left|\dot{\eta}_1\right|\eta_1'}{u_{o,1}{}^2}\right) = 0
\end{aligned} \tag{5.19}$$

$$
\begin{aligned}
&\left(1-\beta_{o,1}\right)\ddot{\eta}_1+2u_{i,1}\sqrt{\beta_i}\,\dot{\eta}_1'+u_{i,1}{}^2\,\eta_1''-\left(\overline{\Gamma}_1-\Pi_{iL,1}+\Pi_{oL,1}\right)\eta_1'' \\
&+\eta_1{}^{(4)}+\alpha_1\dot{\eta}_1{}^{(4)}-\begin{pmatrix}3\zeta'''\eta_1''+4\zeta''\eta_1'''+2\zeta'\eta_1{}^{(4)}+\eta_1'\zeta^{(4)} \\ +2\eta_1'^2\,\eta_1{}^{(4)}+8\eta_1'\eta_1''\,\eta_1'''+2\left(\eta_1''\right)^3\end{pmatrix} \\
&-\alpha_1\begin{pmatrix}3\zeta'''\dot{\eta}_1''+3\dot{\zeta}'''\eta_1''+4\zeta''\dot{\eta}_1'''+4\dot{\zeta}''\eta_1'''+2\zeta'\dot{\eta}_1{}^{(4)}+2\dot{\zeta}'\eta_1{}^{(4)} \\ +\eta_1'\dot{\zeta}^{(4)}+\dot{\eta}_1'\zeta^{(4)}+2\eta_1'^2\,\dot{\eta}_1{}^{(4)}+4\dot{\eta}_1'\,\eta_1{}^{(4)}+8\eta_1'\eta_1''\dot{\eta}_1''' \\ +8\eta_1'''\left(\eta_1'\dot{\eta}_1''+\eta_1''\dot{\eta}_1'\right)+6\left(\eta_1''\right)^2\dot{\eta}_1''\end{pmatrix} \\
&+\left(\overline{\Gamma}_1-\Pi_{iL,1}+\Pi_{oL,1}-\Pi_{0,1}\right)\left(\zeta''\eta_1'+\zeta'\eta_1''+\frac{3}{2}\eta_1'^2\,\eta_1''\right) \\
&-\left(\Pi_{0,1}\,\alpha_1\right)\left(\zeta''\dot{\eta}_1'+\eta_1'\dot{\zeta}''+\zeta'\dot{\eta}_1''+\eta_1''\dot{\zeta}'+\frac{3}{2}\eta_1'^2\,\dot{\eta}_1''+3\dot{\eta}_1'\eta_1''\right) \\
&-\left(\frac{\mathrm{E}\,c_T\,u_{o,1}{}^2}{2}\bar{h}\right)\begin{pmatrix}\left(\eta_1'-\frac{1}{2}\eta_1'^3+\zeta\,\eta_1''\right) \\ -\left(\frac{1}{2}-\xi_1\right)\left(\eta_1''-\zeta''\eta_1'-\zeta'\eta_1''-\frac{3}{2}\eta_1'^2\,\eta_1''\right)\end{pmatrix} \\
&-\left(1-2\nu\right)\overline{\Pi}_1\;\left(\eta_1''-\zeta''\eta_1'-\zeta'\eta_1''-\frac{3}{2}\eta_1'^2\,\eta_1''\right) \\
&-\left(\frac{\mathrm{E}\,c_T\,u_{o,1}{}^2}{2}\right)\begin{pmatrix}\frac{1}{2}\eta_1'^3-\zeta\,\eta_1''-\zeta'\eta_1'-\sqrt{\beta_{o,1}}\,\frac{\dot{\eta}_1\eta_1'^2}{u_{o,1}}-\beta_{o,1}\,\frac{\dot{\eta}_1{}^2\,\eta_1'}{2u_{o,1}{}^2} \\ +\left(\frac{1}{2}-\xi_1\right)\left(\eta_1''-\zeta''\eta_1'-\zeta'\eta_1''-\frac{3}{2}\eta_1'^2\,\eta_1''\right)\end{pmatrix} \\
&+\left(\frac{\mathrm{E}\,c_N\,u_{o,1}{}^2}{2}\right)\begin{pmatrix}-\eta_1'+\sqrt{\beta_{o,1}}\,\frac{\dot{\eta}_1}{u_{o,1}}+\sqrt{\beta_{o,1}}\,\frac{\dot{\eta}_1\zeta'}{u_{o,1}}+\zeta'\eta_1'+\beta_{o,1}\,\frac{\dot{\zeta}\,\dot{\eta}_1}{u_{o,1}{}^2}+\eta_1'^3 \\ -\frac{\beta_{o,1}}{2u_{o,1}{}^2}\dot{\eta}_1{}^2\eta_1'-\frac{\sqrt{\beta_{o,1}}}{u_{o,1}}\dot{\eta}_1\,\eta_1'^2-\frac{\beta_{o,1}{}^{\frac{3}{2}}}{2u_{o,1}{}^3}\dot{\eta}_1{}^3\end{pmatrix} \\
&+\left(\frac{\mathrm{E}\,c_d\,u_{o,1}{}^2}{2}\right)\left(-\eta_1'\,\left|\eta_1'\right|+\sqrt{\beta_{o,1}}\,\frac{\left|\dot{\eta}_1\right|\eta_1'+\left|\eta_1'\right|\dot{\eta}_1}{u_{o,1}}+\beta_{o,1}\,\frac{\dot{\eta}_1\left|\dot{\eta}_1\right|}{u_{o,1}{}^2}\right) \\
&+\chi\,\beta_{o,1}\left(\ddot{\eta}_1-\ddot{\zeta}\,\eta_1'-2\dot{\zeta}\,\dot{\eta}_1'-\ddot{\eta}_1\eta_1'^2-\frac{3}{2}\dot{\eta}_1\,\dot{\eta}_1'\eta_1'\right) \\
&+\chi\,u_{o,1}\sqrt{\beta_{o,1}}\;\left(-2\dot{\eta}_1'+3\dot{\zeta}'\eta_1'+4\zeta'\dot{\eta}_1'+\frac{7}{2}\dot{\eta}_1'\eta_1'^2+2\dot{\zeta}\,\eta_1''+\frac{3}{2}\dot{\eta}_1\eta_1'\eta_1''\right) \\
&+\chi\,u_{o,1}{}^2\left(\eta_1''-2\zeta''\eta_1'-4\zeta'\eta_1''-\frac{5}{2}\eta_1''\eta_1'^2\right)=0
\end{aligned}
\tag{5.20}
$$

where $(')=\partial(\;)/\partial\xi_1$, $(\dot{\cdot})=\partial(\;)/\partial t_1{}^*$.

5.2 Inextensible Rotating Flexible Pipe Conveying Fluid and Subjected to External Axial Flow

In this section, the dimensionless forms of the equations (3.96-3.98) will be established. Those equations maybe rendered dimensionless through the use following dimensionless quantities:

$$\xi_2 = \frac{s}{L} \tag{5.21}$$

$$\eta_1 = \frac{v}{L} \tag{5.22}$$

$$\eta_2 = \frac{w}{L} \tag{5.23}$$

$$\eta_3 = \varphi \tag{5.24}$$

$$t_2^{\ *} = \left(\frac{E\,I + G\,J}{mL^2 + ML^2 + \rho_{f,o}\,A_o L^2 + I_p + I_D + I_{D,f}} \right)^{\frac{1}{2}} \frac{t}{L} \tag{5.25}$$

$$\alpha_2 = \left(\frac{E\,I + G\,J}{mL^2 + ML^2 + \rho_{f,o}\,A_o L^2 + I_p + I_D + I_{D,f}} \right)^{\frac{1}{2}} \frac{a}{L} \tag{5.26}$$

$$\beta_{i,1} = \left(\frac{ML^2}{mL^2 + ML^2 + \rho_{f,o}\,A_o L^2 + I_p + I_D + I_{D,f}} \right) \tag{5.27}$$

$$\beta_{i,2} = \left(\frac{I_D + I_{D,f}}{mL^2 + ML^2 + \rho_{f,o}\,A_o L^2 + I_p + I_D + I_{D,f}} \right) \tag{5.28}$$

$$\beta_{i,3} = \left(\frac{I_p}{mL^2 + ML^2 + \rho_{f,o}\,A_o L^2 + I_p + I_D + I_{D,f}} \right) \tag{5.29}$$

$$\beta_{i,4} = \left(\frac{I_{p,f}\ L}{mL^3 + ML^3 + \rho_{f,o}\ A_o L^3 + I_p\ L + I_D\ L\ + I_{D,f}\ L} \right) \tag{5.30}$$

$$\beta_{m,1} = \frac{EI}{EI + GJ} \tag{5.31}$$

$$\beta_{m,2} = \frac{GJ}{EI + GJ} \tag{5.32}$$

$$\beta_{o,2} = \left(\frac{\rho_{f,o}\ A_o\ L^2}{mL^2 + ML^2 + \rho_{f,o}\ A_o L^2 + I_p + I_D + I_{D,f}} \right) \tag{5.33}$$

$$u_{i,2} = \left(\frac{M}{EI + GJ} \right)^{\frac{1}{2}} U_i\ L \tag{5.34}$$

$$u_{o,2} = \left(\frac{\rho_{f,o}\ A_o}{EI + GJ} \right)^{\frac{1}{2}} U_o\ L \tag{5.35}$$

$$c_N = \frac{4}{\pi} C_N \quad , \quad c_T = \frac{4}{\pi} C_T \quad , \quad c_d = \frac{4}{\pi} C_{DP} \tag{5.36}$$

$$\chi = \frac{M_a}{\rho_{f,o}\ A_o} \tag{5.37}$$

$$\bar{h} = \frac{D_o}{D_h} \tag{5.38}$$

$$\mathrm{E} = \frac{L}{D_o} \tag{5.39}$$

$$\Pi_{oL,2} = \frac{p_o(L) A_o\ L^2}{EI + G\ J} \tag{5.40}$$

$$\gamma_1 = \frac{(M + m) g\ L^3}{EI + GJ} \tag{5.41}$$

$$\gamma_2 = \frac{\rho_{f,o}\, A_o\, g\, L^3}{EI + GJ} \tag{5.42}$$

Utilizing equations (5.21-5.42), we can express the dimensionless forms of the governing equations of motion as

$$
\begin{aligned}
&\left(1-\beta_{o,2}-\beta_{i,2}-\beta_{i,3}\right)\begin{pmatrix} \ddot{\eta}_1 + \eta_1' \int_0^{\xi_2} \left(\dot{\eta}_1'^2 + \eta_1'\ddot{\eta}_1' + \dot{\eta}_2'^2 + \eta_2'\ddot{\eta}_2'\right) d\xi_2 \\ - \eta_1'' \int_0^{\xi_2}\int_{\xi_2}^{1} \left(\dot{\eta}_1'^2 + \eta_1'\ddot{\eta}_1' + \dot{\eta}_2'^2 + \eta_2'\ddot{\eta}_2'\right) d\xi_2\, d\xi_2 \end{pmatrix} \\
&+ \left(2\sqrt{\beta_{i,1}}\, u_{i,2}\right)\left(\dot{\eta}_1'\left(1+\eta_1'^2\right) + \eta_1'\eta_2'\dot{\eta}_2' - \eta_1'' \int_{\xi_2}^{1} \left(\eta_1'\dot{\eta}_1' + \eta_2'\dot{\eta}_2'\right) d\xi_2 \right) \\
&+ u_{i,2}^{\,2}\left[\eta_1''\left(1+\eta_1'^2\right) + \eta_2'\eta_2''\eta_1' - \eta_1'' \int_{\xi_2}^{1} \eta_1'\eta_1'' d\xi_2 - \eta_1'' \int_{\xi_2}^{1} \eta_2'\eta_2''\, d\xi_2 \right] \\
&+ \beta_{i,2}\left(2\ddot{\eta}_1''\right) + \beta_{i,3}\left(\eta_1''\dot{\eta}_2'^2 + 2\eta_1'\dot{\eta}_2'\dot{\eta}_2'' + \dot{\eta}_3\dot{\eta}_2'' + \dot{\eta}_2'\dot{\eta}_3'\right) \\
&+ \beta_{m,1}\left(\eta_1^{(4)} + \left(\eta_1''\right)^3 + 4\eta_1'\eta_1''\eta_1''' + \eta_1'^2\eta_1^{(4)}\right) + \beta_{m,1}\,\alpha_2 \begin{pmatrix} \dot{\eta}_1^{(4)} + 3\left(\eta_1''\right)^2\dot{\eta}_1'' + 4\eta_1'\eta_1''\dot{\eta}_1''' + 4\eta_1'\eta_1'''\dot{\eta}_1'' \\ + 4\dot{\eta}_1'\eta_1''\eta_1''' + \eta_1'^2\dot{\eta}_1^{(4)} + 2\eta_1^{(4)}\eta_1'\dot{\eta}_1' \end{pmatrix} \\
&+ \beta_{m,1}\left(\eta_1''\eta_2''^2 + \eta_1''\eta_2'\eta_2''' + 3\eta_1'\eta_2''\eta_2''' + \eta_1'\eta_2'\eta_2^{(4)}\right) \\
&+ \beta_{m,1}\,\alpha_2 \begin{pmatrix} 2\eta_1''\eta_2''\dot{\eta}_2'' + \dot{\eta}_1''\eta_2''^2 + \eta_1''\eta_2'\dot{\eta}_2''' + \eta_1''\eta_2'''\dot{\eta}_2' + \dot{\eta}_1''\eta_2'\eta_2''' + 3\eta_1'\eta_2''\dot{\eta}_2''' \\ + 3\eta_1'\eta_2'''\dot{\eta}_2'' + 3\dot{\eta}_1'\eta_2''\eta_2''' + \eta_1'\eta_2'\dot{\eta}_2^{(4)} + \eta_1'\eta_2^{(4)}\dot{\eta}_2' + \dot{\eta}_1'\eta_2'\eta_2^{(4)} \end{pmatrix} \\
&- \Pi_{oL,2}\left(\eta_1'' - \frac{3}{2}\eta_1'^2\eta_1'' - \eta_1'' \int_{\xi_2}^{1} \left(\eta_1'\eta_1'' + \eta_2'\eta_2''\right) d\xi_2 \right) \\
&+ \chi\,\beta_{o,2}\begin{pmatrix} \ddot{\eta}_1 + \eta_1' \int_0^{\xi_2} \left(\dot{\eta}_1'^2 + \dot{\eta}_2'^2 + \eta_1'\ddot{\eta}_1' + \eta_2'\ddot{\eta}_2'\right) d\xi_2 + 2\dot{\eta}_1' \int_0^{\xi_2} \left(\eta_1'\dot{\eta}_1' + \eta_2'\dot{\eta}_2'\right) d\xi_2 \\ - \eta_1'' \int_{\xi_2}^{1} \left(\ddot{\eta}_1\eta_1' + \ddot{\eta}_2\eta_2'\right) d\xi_2 - \frac{3}{2}\dot{\eta}_1\dot{\eta}_1'\eta_1' + \dot{\eta}_1 f_t - \frac{\dot{\eta}_3^{\,2}}{4}\eta_1 + \dot{\eta}_3\dot{\eta}_2 + \frac{\dot{\eta}_3 f_t}{2}\eta_2 \end{pmatrix} \\
&+ \chi\sqrt{\beta_{o,2}}\, u_{o,2}\begin{pmatrix} -2\dot{\eta}_1' + \frac{7}{2}\dot{\eta}_1'\eta_1'^2 - 2\eta_1'' \int_0^{\xi_2} \left(\eta_1'\dot{\eta}_1' + \eta_2'\dot{\eta}_2'\right) d\xi_2 + \frac{3}{2}\dot{\eta}_1\eta_1'\eta_1'' \\ + 2\eta_1'' \int_{\xi_2}^{1} \left(\eta_1'\dot{\eta}_1' + \eta_2'\dot{\eta}_2'\right) d\xi_2 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&+\chi\, u_{o,2}{}^{2}\left(\eta_1''-\frac{5}{2}\eta_1''\eta_1'^{2}-\eta_1''\int_{\xi_2}^{1}\left(\eta_1'\eta_1''+\eta_2'\eta_2''\right)d\xi_2\right)\\
&+\gamma_1\left(\eta_1'+\frac{1}{2}\left(\eta_1'\right)^3+\frac{1}{2}\eta_1'\eta_2'^{2}-\left(1-\xi_2\right)\left(\eta_1''+\frac{3}{2}\eta_1'^{2}\eta_1''+\frac{1}{2}\eta_2'^{2}\eta_1''+\eta_1'\eta_2'\eta_2''\right)\right)\\
&-\gamma_2\left(\eta_1'-\frac{1}{2}\left(\eta_1'\right)^3-\left(1-\xi_2\right)\left(\eta_1''-\frac{3}{2}\eta_1'^{2}\eta_1''\right)\right)\\
&-\left(\frac{\mathrm{E}\,u_{o,2}{}^{2}c_T}{2}\bar{h}\right)\left(\eta_1'-\frac{1}{2}\left(\eta_1'\right)^3-\left(1-\xi_2\right)\left(\eta_1''-\frac{3}{2}\eta_1'^{2}\eta_1''\right)\right)\\
&-\left(\frac{\mathrm{E}\,u_{o,2}{}^{2}c_T}{2}\right)\left(\begin{aligned}
&\left(1-\xi_2\right)\left(\eta_1''+\frac{3}{2}\eta_1'^{2}\eta_1''\right)-\frac{3}{2}\eta_1'^{3}-\eta_2'^{2}\eta_1'\\
&+\sqrt{\beta_{o,2}}\,\frac{\eta_2'\,\dot{\eta}_2\,\eta_1'}{u_{o,2}}+\beta_{o,2}\,\frac{\dot{\eta}_2{}^{2}\eta_1'}{2u_{o,2}{}^{2}}-\eta_1''\int_{\xi_2}^{1}\sqrt{\beta_{o,2}}\,\frac{\eta_1'\dot{\eta}_1+\eta_2'\dot{\eta}_2}{u_{o,2}}\,d\xi_2\\
&-\eta_1''\int_{\xi_2}^{1}\beta_{o,2}\,\frac{\dot{\eta}_1{}^{2}+\dot{\eta}_2{}^{2}}{2u_{o,2}{}^{2}}\,d\xi_2+\eta_1''\int_{\xi_2}^{1}\left(\eta_1'^{2}+\eta_2'^{2}\right)d\xi_2
\end{aligned}\right)\\
&+\left(\frac{\mathrm{E}\,u_{o,2}{}^{2}c_N}{2}\right)\left(\begin{aligned}
&-\eta_1'+\sqrt{\beta_{o,2}}\,\frac{\dot{\eta}_1}{u_{o,2}}+\sqrt{\beta_{o,2}}\,\frac{\eta_2'\,\dot{\eta}_2\,\eta_1'}{u_{o,2}}-\eta_1'\eta_2'^{2}-\beta_{o,2}{}^{\frac{3}{2}}\,\frac{\dot{\eta}_1{}^{3}}{2u_{o,2}{}^{3}}\\
&-\beta_{o,2}\,\frac{\dot{\eta}_1{}^{2}\eta_1'}{2u_{o,2}{}^{2}}+\eta_1''\int_{\xi_2}^{1}\left(\eta_1'^{2}+\eta_2'^{2}\right)d\xi_2\\
&-\dot{\eta}_1\beta_{o,2}\int_{0}^{\xi_2}\left(\frac{\eta_1'\dot{\eta}_1'+\eta_2'\dot{\eta}_2'}{u_{o,2}{}^{2}}\right)d\xi_2-\eta_1''\int_{\xi_2}^{1}\sqrt{\beta_{o,2}}\left(\frac{\dot{\eta}_1\eta_1'+\dot{\eta}_2\eta_2'}{u_{o,2}}\right)d\xi_2
\end{aligned}\right)\\
&+\left(\frac{\mathrm{E}\,u_{o,2}{}^{2}c_d}{2}\right)\left(-\eta_1'\,\left|\eta_1'\right|+\sqrt{\beta_{o,2}}\,\frac{\left|\dot{\eta}_1\right|\eta_1'+\left|\eta_1'\right|\dot{\eta}_1}{u_{o,2}}+\beta_{o,2}\,\frac{\dot{\eta}_1\left|\dot{\eta}_1\right|}{u_{o,2}{}^{2}}\right)=0
\end{aligned}
$$

(5.43)

$$
\left(1-\beta_{o,2}-\beta_{i,2}-\beta_{i,3}\right)\begin{pmatrix}\ddot{\eta}_2+\eta_2'\int_0^{\xi_2}\left(\dot{\eta}_2'^{\,2}+\eta_2'\ddot{\eta}_2'+\dot{\eta}_1'^{\,2}+\eta_1'\ddot{\eta}_1'\right)d\xi_2\\ -\eta_2''\int_0^{\xi_2}\int_{\xi_2}^{1}\left(\dot{\eta}_2'^{\,2}+\eta_2'\ddot{\eta}_2'+\dot{\eta}_1'^{\,2}+\eta_1'\ddot{\eta}_1'\right)d\xi_2\;d\xi_2\end{pmatrix}
$$

$$
+\left(2\sqrt{\beta_{i,1}}\,u_{i,2}\right)\left(\dot{\eta}_2'\left(1+\eta_2'^{\,2}\right)+\eta_2'\eta_1'\dot{\eta}_1'-\eta_2''\int_{\xi_2}^{1}\left(\eta_2'\dot{\eta}_2'+\eta_1'\dot{\eta}_1'\right)d\xi_2\right)
$$

$$
+u_{i,2}^{\;2}\left[\eta_2''\left(1+\eta_2'^{\,2}\right)+\eta_1'\eta_1''\eta_2'-\eta_2''\int_{\xi_2}^{1}\eta_1'\eta_1''d\xi_2-\eta_2''\int_{\xi_2}^{1}\eta_2'\eta_2''\,d\xi_2\right]
$$

$$
+\beta_{i,2}\left(2\ddot{\eta}_2''\right)+\beta_{i,3}\begin{pmatrix}\dot{\eta}_2'\left(\eta_1'\dot{\eta}_1''+\dot{\eta}_1'\eta_1''\right)+2\eta_1'^{\,2}\ddot{\eta}''_{\,2}+3\ddot{\eta}_2'\,\eta_1'\eta_1''+2\dot{\eta}_2''\eta_1'\dot{\eta}_1'\\ +\dfrac{\dot{\eta}_3\left(\eta_1'\ddot{\eta}_2'+\dot{\eta}_1'\,\dot{\eta}_2''\right)}{\dot{\eta}_2'}-\dfrac{2\,\dot{\eta}_3\,\eta_1'\dot{\eta}_2''\ddot{\eta}_2'}{\dot{\eta}_2'^{\,2}}+\dfrac{\ddot{\eta}_3\,\eta_1'\dot{\eta}_2''}{\dot{\eta}_2'}\\ +\dfrac{\dot{\eta}_3'\,\eta_1'\,\ddot{\eta}_2'}{\dot{\eta}_2'}+\dfrac{\dot{\eta}_3\left(\eta_1'\ddot{\eta}_2''+\ddot{\eta}_2'\eta_1''\right)}{\dot{\eta}_2'}+\dot{\eta}_3\,\dot{\eta}_1''+\eta_1''\ddot{\eta}_3+\eta_1'\ddot{\eta}_3'+\dot{\eta}_3'\,\dot{\eta}_1'\end{pmatrix}
$$

$$
+\beta_{m,1}\left(\eta_2^{\,(4)}+\left(\eta_2''\right)^3+4\eta_2'\eta_2''\,\eta_2'''+\eta_2'^{\,2}\eta_2^{\,(4)}\right)+\beta_{m,1}\,\alpha_2\begin{pmatrix}\dot{\eta}_2^{\,(4)}+3\left(\eta_2''\right)^2\dot{\eta}_2''+4\eta_2'\eta_2''\dot{\eta}_2'''+4\eta_2'\eta_2'''\dot{\eta}_2''\\ +4\dot{\eta}_2'\eta_2''\eta_2'''+\eta_2'^{\,2}\dot{\eta}_2^{\,(4)}+2\eta_2^{\,(4)}\eta_2'\dot{\eta}_2'\end{pmatrix}
$$

$$
+\beta_{m,1}\left(\eta_2''\eta_1''^{\,2}+\eta_2''\,\eta_1'\eta_1'''+3\eta_2'\eta_1''\eta_1'''+\eta_2'\eta_1'\eta_1^{\,(4)}\right)
$$

$$
+\beta_{m,1}\,\alpha_2\begin{pmatrix}2\eta_2''\,\eta_1''\dot{\eta}_1''+\dot{\eta}_2''\,\eta_1''^{2}+\eta_2''\,\eta_1'\dot{\eta}_1'''+\eta_2''\;\eta_1'''\dot{\eta}_1'+\dot{\eta}_2''\,\eta_1'\eta_1'''+3\eta_2'\eta_1''\dot{\eta}_1'''\\ +3\eta_2'\,\eta_1'''\dot{\eta}_1''+3\dot{\eta}_2'\eta_1''\eta_1'''+\eta_2'\eta_1'\dot{\eta}_1^{\,(4)}+\eta_2'\eta_1^{\,(4)}\dot{\eta}_1'+\dot{\eta}_2'\,\eta_1'\eta_1^{\,(4)}\end{pmatrix}
$$

$$
-\Pi_{oL,2}\left(\eta_2''-\frac{3}{2}\eta_2'^{\,2}\,\eta_2''-\eta_2''\int_{\xi_2}^{1}\left(\eta_1'\eta_1''+\eta_2'\eta_2''\right)d\xi_2\right)
$$

$$
+\chi\,\beta_{o,2}\begin{pmatrix}\ddot{\eta}_2+\eta_2'\int_0^{\xi_2}\left(\dot{\eta}_1'^{\,2}+\dot{\eta}_2'^{\,2}+\eta_1'\ddot{\eta}_1'+\eta_2'\ddot{\eta}_2'\right)d\xi_2+2\dot{\eta}_2'\int_0^{\xi_2}\left(\eta_1'\dot{\eta}_1'+\eta_2'\dot{\eta}_2'\right)d\xi_2\\ -\eta_2''\int_{\xi_2}^{1}\left(\ddot{\eta}_1\,\eta_1'+\ddot{\eta}_2\,\eta_2'\right)d\xi_2-\dfrac{3}{2}\dot{\eta}_2\dot{\eta}_2'\,\eta_2'+\dot{\eta}_2\;f_t-\dfrac{\dot{\eta}_3^{\;2}}{4}\eta_2+\dot{\eta}_3\dot{\eta}_1+\dfrac{\dot{\eta}_1\;f_t}{2}\eta_1\end{pmatrix}
$$

$$
+\chi\sqrt{\beta_{o,2}}\,u_{o,2}\begin{pmatrix}-2\dot{\eta}_2'+\dfrac{7}{2}\dot{\eta}_2'\eta_2'^{\,2}-2\eta_2''\int_0^{\xi_2}\left(\eta_1'\dot{\eta}_1'+\eta_2'\dot{\eta}_2'\right)d\xi_2\\ +\dfrac{3}{2}\dot{\eta}_2\eta_2'\eta_2''+2\eta_2''\int_{\xi_2}^{1}\left(\eta_1'\dot{\eta}_1'+\eta_2'\dot{\eta}_2'\right)d\xi_2\end{pmatrix}
$$

$$+\chi\, u_{o,2}{}^{2}\left(\eta_2'' - \frac{5}{2}\eta_2''\eta_2'^{\,2} - \eta_2''\int_{\xi_2}^{1}\left(\eta_1'\eta_1'' + \eta_2'\eta_2''\right)d\xi_2\right)$$

$$+\gamma_1\left(\eta_2' + \frac{1}{2}(\eta_2')^3 + \frac{1}{2}\eta_2'\eta_1'^{2} - (1-\xi_2)\left(\eta_2'' + \frac{3}{2}\eta_2'^{\,2}\eta_2'' + \frac{1}{2}\eta_1'^{2}\eta_2'' + \eta_2'\eta_1'\eta_1''\right)\right)$$

$$-\gamma_2\left(\eta_2' - \frac{1}{2}(\eta_2')^3 - (1-\xi_2)\left(\eta_2'' - \frac{3}{2}\eta_2'^{\,2}\eta_2''\right)\right)$$

$$-\left(\frac{\mathrm{E}u_{o,2}{}^{2}c_T}{2}\bar{h}\right)\left(\eta_2' - \frac{1}{2}(\eta_2')^3 - (1-\xi_2)\left(\eta_2'' - \frac{3}{2}\eta_2'^{\,2}\eta_2''\right)\right)$$

$$-\left(\frac{\mathrm{E}u_{o,2}{}^{2}c_T}{2}\right)\left(\begin{array}{l}(1-\xi_2)\left(\eta_2'' + \frac{3}{2}\eta_2'^{\,2}\eta_2''\right) - \frac{3}{2}\eta_2'^{\,3} - \eta_1'^{2}\eta_2' \\ + \sqrt{\beta_{o,2}}\,\frac{\eta_1'\dot{\eta}_1\eta_2'}{u_{o,2}} + \beta_{o,2}\,\frac{\dot{\eta}_1{}^{2}\eta_2'}{2u_{o,2}{}^{2}} - \eta_2''\int_{\xi_2}^{1}\sqrt{\beta_{o,2}}\,\frac{\eta_1'\dot{\eta}_1 + \eta_2'\dot{\eta}_2}{u_{o,2}}d\xi_2 \\ - \eta_2''\int_{\xi_2}^{1}\beta_{o,2}\,\frac{\dot{\eta}_1{}^{2} + \dot{\eta}_2{}^{2}}{2u_{o,2}{}^{2}}d\xi_2 + \eta_2''\int_{\xi_2}^{1}\left(\eta_1'^{2} + \eta_2'^{\,2}\right)d\xi_2\end{array}\right)$$

$$+\left(\frac{\mathrm{E}u_{o,2}{}^{2}c_N}{2}\right)\left(\begin{array}{l}-\eta_2' + \sqrt{\beta_{o,2}}\,\frac{\dot{\eta}_2}{u_{o,2}} + \sqrt{\beta_{o,2}}\,\frac{\eta_1'\dot{\eta}_1\eta_2'}{u_{o,2}} - \eta_2'\eta_1'^{2} - \beta_{o,2}{}^{\frac{3}{2}}\,\frac{\dot{\eta}_2{}^{3}}{2u_{o,2}{}^{3}} \\ - \beta_{o,2}\,\frac{\dot{\eta}_2{}^{2}\eta_2'}{2u_{o,2}{}^{2}} + \eta_2''\int_{\xi_2}^{1}\left(\eta_1'^{2} + \eta_2'^{\,2}\right)d\xi_2 \\ - \dot{\eta}_2\beta_{o,2}\int_{0}^{\xi_2}\left(\frac{\eta_1'\dot{\eta}_1' + \eta_2'\dot{\eta}_2'}{u_{o,2}{}^{2}}\right)d\xi_2 - \eta_2''\int_{\xi_2}^{1}\sqrt{\beta_{o,2}}\left(\frac{\dot{\eta}_1\eta_1' + \dot{\eta}_2\eta_2'}{u_{o,2}}\right)d\xi_2\end{array}\right) \tag{5.44}$$

$$+\left(\frac{\mathrm{E}u_{o,2}{}^{2}c_d}{2}\right)\left(-\eta_2'\,\left|\eta_2'\right| + \sqrt{\beta_{o,2}}\,\frac{\left|\dot{\eta}_2\right|\eta_2' + \left|\eta_2'\right|\dot{\eta}_2}{u_{o,2}} + \beta_{o,2}\,\frac{\dot{\eta}_2\left|\dot{\eta}_2\right|}{u_{o,2}{}^{2}}\right) = 0$$

and,

$$\left(\beta_{i,3}\right)\left(\ddot{\eta}_3 + \dot{\eta}_2'\dot{\eta}_1' + \eta_1'\ddot{\eta}_2'\right) - \left(\beta_{m,2}\,\eta_3'' - 4\beta_{i,4}\;f_r\left(\eta_3'\ddot{\eta}_3 + 2\dot{\eta}_3\dot{\eta}_3'\right)\right) = 0 \tag{5.45}$$

where $(') = \partial(\;)/\partial\xi_2$, $(\dot{\cdot}) = \partial(\;)/\partial t_2{}^{*}$.

5.3 Inextensible Rotating Flexible Pipe Conveying Fluid downwards, which then flows Upwards as A confined Annular Flow

In this section, the dimensionless forms of the equations (3.110-3.112) are established. Governing equations of motion can be rendered dimensionless by defining the following dimensionless quantities:

$$\lambda_i = \frac{D_i}{D_o} \tag{5.46}$$

$$\lambda_{ch} = \frac{D_{ch}}{D_o} \tag{5.47}$$

$$u_{o,2} = \frac{\lambda_i^{\,2}}{\lambda_{ch}^{\,2} - 1}\, u_{i,2} \tag{5.48}$$

$$c_b = \frac{4}{\pi} C_b \tag{5.49}$$

Utilizing equations (5.21-5.34) and (5.36-5.42), and (5.46-5.49), the dimensionless form of the equations of motion of an inextensible rotating flexible pipe conveying fluid downwards, which then flows upwards as a confined annular flow can be expressed as

$$
\left(1-\beta_{o,2}-\beta_{i,2}-\beta_{i,3}\right)\begin{pmatrix}\ddot{\eta}_1+\eta_1'\int_0^{\xi_2}\left(\dot{\eta}_1'^2+\eta_1'\ddot{\eta}_1'+\dot{\eta}_2'^2+\eta_2'\ddot{\eta}_2'\right)d\xi_2 \\ -\eta_1''\int_0^{\xi_2}\int_{\xi_2}^{1}\left(\dot{\eta}_1'^2+\eta_1'\ddot{\eta}_1'+\dot{\eta}_2'^2+\eta_2'\ddot{\eta}_2'\right)d\xi_2\ d\xi_2\end{pmatrix}
$$

$$
+\left(2\sqrt{\beta_{i,1}}\,u_{i,2}\right)\left(\dot{\eta}_1'\left(1+\eta_1'^2\right)+\eta_1'\eta_2'\dot{\eta}_2'-\eta_1''\int_{\xi_2}^{1}\left(\eta_1'\dot{\eta}_1'+\eta_2'\dot{\eta}_2'\right)d\xi_2\right)
$$

$$
+u_{i,2}{}^2\left[\eta_1''\left(1+\eta_1'^2\right)+\eta_2'\eta_2''\eta_1'-\eta_1''\int_{\xi_2}^{1}\eta_1'\eta_1''d\xi_2-\eta_1''\int_{\xi_2}^{1}\eta_2'\eta_2''\,d\xi_2\right]
$$

$$
+\beta_{i,2}\left(2\ddot{\eta}_1''\right)+\beta_{i,3}\left(\eta_1''\dot{\eta}_2'^2+2\eta_1'\dot{\eta}_2'\dot{\eta}_2''+\dot{\eta}_3\dot{\eta}_2''+\dot{\eta}_2'\dot{\eta}_3'\right)
$$

$$
+\beta_{m,1}\left(\eta_1^{(4)}+\left(\eta_1''\right)^3+4\eta_1'\eta_1''\eta_1'''+\eta_1'^2\eta_1^{(4)}\right)+\beta_{m,1}\,\alpha_2\begin{pmatrix}\dot{\eta}_1^{(4)}+3\left(\eta_1''\right)^2\dot{\eta}_1''+4\eta_1'\eta_1''\dot{\eta}_1'''+4\eta_1'\eta_1'''\dot{\eta}_1'' \\ +4\dot{\eta}_1'\eta_1''\eta_1'''+\eta_1'^2\dot{\eta}_1^{(4)}+2\eta_1^{(4)}\eta_1'\dot{\eta}_1'\end{pmatrix}
$$

$$
+\beta_{m,1}\left(\eta_1''\eta_2''^2+\eta_1''\eta_2'\eta_2'''+3\eta_1'\eta_2''\eta_2'''+\eta_1'\eta_2'\eta_2^{(4)}\right)
$$

$$
+\beta_{m,1}\,\alpha_2\begin{pmatrix}2\eta_1''\eta_2''\dot{\eta}_2''+\dot{\eta}_1''\eta_2''^2+\eta_1''\eta_2'\dot{\eta}_2'''+\eta_1''\eta_2'''\dot{\eta}_2'+\dot{\eta}_1''\eta_2'\eta_2'''+3\eta_1'\eta_2''\dot{\eta}_2''' \\ +3\eta_1'\eta_2'''\dot{\eta}_2''+3\dot{\eta}_1'\eta_2''\eta_2'''+\eta_1'\eta_2'\dot{\eta}_2^{(4)}+\eta_1'\eta_2^{(4)}\dot{\eta}_2'+\dot{\eta}_1'\eta_2'\eta_2^{(4)}\end{pmatrix}
$$

$$
-\left(\frac{\lambda_i^{\,2}}{\lambda_{ch}^{\,2}-1}\right)^2\frac{c_b\,u_{i,2}{}^2}{2}\left(\eta_1''-\frac{3}{2}\eta_1'^2\eta_1''-\eta_1''\int_{\xi_2}^{1}\left(\eta_1'\eta_1''+\eta_2'\eta_2''\right)d\xi_2\right)
$$

$$
+\chi\,\beta_{o,2}\begin{pmatrix}\ddot{\eta}_1+\eta_1'\int_0^{\xi_2}\left(\dot{\eta}_1'^2+\dot{\eta}_2'^2+\eta_1'\ddot{\eta}_1'+\eta_2'\ddot{\eta}_2'\right)d\xi_2+2\dot{\eta}_1'\int_0^{\xi_2}\left(\eta_1'\dot{\eta}_1'+\eta_2'\dot{\eta}_2'\right)d\xi_2 \\ -\eta_1''\int_{\xi_2}^{1}\left(\ddot{\eta}_1\eta_1'+\ddot{\eta}_2\eta_2'\right)d\xi_2-\frac{3}{2}\dot{\eta}_1\dot{\eta}_1'\eta_1'+\dot{\eta}_1\,f_t-\frac{\dot{\eta}_3^{\,2}}{4}\eta_1+\dot{\eta}_3\dot{\eta}_2+\frac{\dot{\eta}_3\,f_t}{2}\eta_2\end{pmatrix}
$$

$$
+\chi\sqrt{\beta_{o,2}}\left(\frac{\lambda_i^{\,2}}{\lambda_{ch}^{\,2}-1}\right)u_{i,2}\begin{pmatrix}-2\dot{\eta}_1'+\frac{7}{2}\dot{\eta}_1'\eta_1'^2-2\eta_1''\int_0^{\xi_2}\left(\eta_1'\dot{\eta}_1'+\eta_2'\dot{\eta}_2'\right)d\xi_2+\frac{3}{2}\dot{\eta}_1\eta_1'\eta_1'' \\ +2\eta_1''\int_{\xi_2}^{1}\left(\eta_1'\dot{\eta}_1'+\eta_2'\dot{\eta}_2'\right)d\xi_2\end{pmatrix}
$$

$$
+\chi\left(\frac{\lambda_i^{\,2}}{\lambda_{ch}^{\,2}-1}\right)^2u_{i,2}{}^2\left(\eta_1''-\frac{5}{2}\eta_1''\eta_1'^2-\eta_1''\int_{\xi_2}^{1}\left(\eta_1'\eta_1''+\eta_2'\eta_2''\right)d\xi_2\right)
$$

$$
+\gamma_1\left(\eta_1'+\frac{1}{2}\left(\eta_1'\right)^3+\frac{1}{2}\eta_1'\eta_2'^2-\left(1-\xi_2\right)\left(\eta_1''+\frac{3}{2}\eta_1'^2\eta_1''+\frac{1}{2}\eta_2'^2\eta_1''+\eta_1'\eta_2'\eta_2''\right)\right)
$$

$$
-\gamma_2\left(\eta_1'-\frac{1}{2}\left(\eta_1'\right)^3-\left(1-\xi_2\right)\left(\eta_1''-\frac{3}{2}\eta_1'^2\eta_1''\right)\right)
$$

$$
\begin{aligned}
&-\left(\frac{\mathrm{E}u_{i,2}{}^{2}c_T}{2}\left(\frac{\lambda_i{}^{2}}{\lambda_{ch}{}^{2}-1}\right)^{2}\bar{h}\right)\left(\eta_1'-\frac{1}{2}(\eta_1')^{3}-(1-\xi_2)\left(\eta_1''-\frac{3}{2}\eta_1'^{2}\eta_1''\right)\right)\\
&-\left(\frac{\mathrm{E}u_{i,2}{}^{2}c_T}{2}\left(\frac{\lambda_i{}^{2}}{\lambda_{ch}{}^{2}-1}\right)^{2}\right)\left(\begin{aligned}
&(1-\xi_2)\left(\eta_1''+\frac{3}{2}\eta_1'^{2}\eta_1''\right)-\frac{3}{2}\eta_1'^{3}-\eta_2'^{2}\eta_1'\\
&+\sqrt{\beta_{o,2}}\left(\frac{\lambda_{ch}{}^{2}-1}{\lambda_i{}^{2}}\right)^{2}\frac{\eta_2'\,\dot{\eta}_2\,\eta_1'}{u_{i,2}}+\beta_{o,2}\left(\frac{\lambda_{ch}{}^{2}-1}{\lambda_i{}^{2}}\right)^{2}\frac{\dot{\eta}_2{}^{2}\eta_1'}{2u_{i,2}{}^{2}}\\
&-\eta_1''\int_{\xi_2}^{1}\sqrt{\beta_{o,2}}\left(\frac{\lambda_{ch}{}^{2}-1}{\lambda_i{}^{2}}\right)\frac{\eta_1'\dot{\eta}_1+\eta_2'\dot{\eta}_2}{u_{i,2}}d\xi_2\\
&-\eta_1''\int_{\xi_2}^{1}\beta_{o,2}\left(\frac{\lambda_{ch}{}^{2}-1}{\lambda_i{}^{2}}\right)^{2}\frac{\dot{\eta}_1{}^{2}+\dot{\eta}_2{}^{2}}{2u_{i,2}{}^{2}}d\xi_2+\eta_1''\int_{\xi_2}^{1}\left(\eta_1'^{2}+\eta_2'^{2}\right)d\xi_2
\end{aligned}\right)\\
&+\left(\frac{\mathrm{E}u_{i,2}{}^{2}c_N}{2}\left(\frac{\lambda_i{}^{2}}{\lambda_{ch}{}^{2}-1}\right)^{2}\right)\left(\begin{aligned}
&-\eta_1'+\sqrt{\beta_{o,2}}\left(\frac{\lambda_{ch}{}^{2}-1}{\lambda_i{}^{2}}\right)\frac{\dot{\eta}_1}{u_{i,2}}+\sqrt{\beta_{o,2}}\left(\frac{\lambda_{ch}{}^{2}-1}{\lambda_i{}^{2}}\right)\frac{\eta_2'\,\dot{\eta}_2\,\eta_1'}{u_{i,2}}\\
&-\eta_1'\eta_2'^{2}-\beta_{o,2}{}^{\frac{3}{2}}\left(\frac{\lambda_{ch}{}^{2}-1}{\lambda_i{}^{2}}\right)^{3}\frac{\dot{\eta}_1{}^{3}}{2u_{i,2}{}^{3}}\\
&-\beta_{o,2}\left(\frac{\lambda_{ch}{}^{2}-1}{\lambda_i{}^{2}}\right)^{2}\frac{\dot{\eta}_1{}^{2}\eta_1'}{2u_{i,2}{}^{2}}+\eta_1''\int_{\xi_2}^{1}\left(\eta_1'^{2}+\eta_2'^{2}\right)d\xi_2\\
&-\eta_1''\int_{\xi_2}^{1}\sqrt{\beta_{o,2}}\left(\frac{\lambda_{ch}{}^{2}-1}{\lambda_i{}^{2}}\right)\left(\frac{\dot{\eta}_1\eta_1'+\dot{\eta}_2\eta_2'}{u_{i,2}}\right)d\xi_2\\
&-\dot{\eta}_1\beta_{o,2}\left(\frac{\lambda_{ch}{}^{2}-1}{\lambda_i{}^{2}}\right)^{2}\int_{0}^{\xi_2}\left(\frac{\eta_1'\dot{\eta}_1'+\eta_2'\dot{\eta}_2'}{u_{i,2}{}^{2}}\right)d\xi_2
\end{aligned}\right)\\
&+\left(\frac{\mathrm{E}u_{i,2}{}^{2}c_d}{2}\left(\frac{\lambda_i{}^{2}}{\lambda_{ch}{}^{2}-1}\right)^{2}\right)\left(\begin{aligned}
&-\eta_1'\,|\eta_1'|+\sqrt{\beta_{o,2}}\left(\frac{\lambda_{ch}{}^{2}-1}{\lambda_i{}^{2}}\right)\frac{|\dot{\eta}_1|\eta_1'+|\eta_1'|\,\dot{\eta}_1}{u_{i,2}}\\
&+\beta_{o,2}\left(\frac{\lambda_{ch}{}^{2}-1}{\lambda_i{}^{2}}\right)^{2}\frac{\dot{\eta}_1|\dot{\eta}_1|}{u_{i,2}{}^{2}}
\end{aligned}\right)=0
\end{aligned}
$$

(5.50)

$$
\left(1-\beta_{o,2}-\beta_{i,2}-\beta_{i,3}\right)\begin{pmatrix}\ddot{\eta}_2+\eta_2'\int_0^{\xi_2}\left(\dot{\eta}_2'^{\,2}+\eta_2'\ddot{\eta}_2'+\dot{\eta}_1'^{\,2}+\eta_1'\ddot{\eta}_1'\right)d\xi_2\\ -\eta_2''\int_0^{\xi_2}\int_{\xi_2}^{1}\left(\dot{\eta}_2'^{\,2}+\eta_2'\ddot{\eta}_2'+\dot{\eta}_1'^{\,2}+\eta_1'\ddot{\eta}_1'\right)d\xi_2\;d\xi_2\end{pmatrix}
$$

$$
+\left(2\sqrt{\beta_{i,1}}\,u_{i,2}\right)\left(\dot{\eta}_2'\left(1+\eta_2'^{\,2}\right)+\eta_2'\eta_1'\dot{\eta}_1'-\eta_2''\int_{\xi_2}^{1}\left(\eta_2'\dot{\eta}_2'+\eta_1'\dot{\eta}_1'\right)d\xi_2\right)
$$

$$
+\,u_{i,2}^{\;2}\left[\eta_2''\left(1+\eta_2'^{\,2}\right)+\eta_1'\eta_1''\eta_2'-\eta_2''\int_{\xi_2}^{1}\eta_1'\eta_1''d\xi_2-\eta_2''\int_{\xi_2}^{1}\eta_2'\eta_2''\,d\xi_2\right]
$$

$$
+\beta_{i,2}\left(2\ddot{\eta}_2''\right)+\beta_{i,3}\begin{pmatrix}\dot{\eta}_2'\left(\eta_1'\dot{\eta}_1''+\dot{\eta}_1'\eta_1''\right)+2\eta_1'^{\,2}\ddot{\eta}''_{\,2}+3\ddot{\eta}_2'\,\eta_1'\eta_1''+2\dot{\eta}_2''\eta_1'\dot{\eta}_1'\\ +\dfrac{\dot{\eta}_3\left(\eta_1'\ddot{\eta}_2'+\dot{\eta}_1'\,\dot{\eta}_2''\right)}{\dot{\eta}_2'}-\dfrac{2\,\dot{\eta}_3\,\eta_1'\dot{\eta}_2''\ddot{\eta}_2'}{\dot{\eta}_2'^{\,2}}+\dfrac{\ddot{\eta}_3\,\eta_1'\dot{\eta}_2''}{\dot{\eta}_2'}\\ +\dfrac{\dot{\eta}_3'\,\eta_1'\,\ddot{\eta}_2'}{\dot{\eta}_2'}+\dfrac{\dot{\eta}_3\left(\eta_1'\ddot{\eta}_2''+\ddot{\eta}_2'\eta_1''\right)}{\dot{\eta}_2'}+\dot{\eta}_3\,\dot{\eta}_1''+\eta_1''\ddot{\eta}_3+\eta_1'\ddot{\eta}_3'+\dot{\eta}_3'\,\dot{\eta}_1'\end{pmatrix}
$$

$$
+\beta_{m,1}\left(\eta_2^{\,(4)}+\left(\eta_2''\right)^3+4\eta_2'\eta_2''\,\eta_2'''+\eta_2'^{\,2}\eta_2^{\,(4)}\right)+\beta_{m,1}\,\alpha_2\begin{pmatrix}\dot{\eta}_2^{\,(4)}+3\left(\eta_2''\right)^2\dot{\eta}_2''+4\eta_2'\eta_2''\dot{\eta}_2'''+4\eta_2'\eta_2'''\dot{\eta}_2''\\ +4\dot{\eta}_2'\eta_2''\eta_2'''+\eta_2'^{\,2}\dot{\eta}_2^{\,(4)}+2\eta_2^{\,(4)}\eta_2'\dot{\eta}_2'\end{pmatrix}
$$

$$
+\beta_{m,1}\left(\eta_2''\eta_1''^{2}+\eta_2''\,\eta_1'\eta_1'''+3\eta_2'\eta_1''\eta_1'''+\eta_2'\eta_1'\eta_1^{\,(4)}\right)
$$

$$
+\beta_{m,1}\,\alpha_2\begin{pmatrix}2\eta_2''\,\eta_1''\dot{\eta}_1''+\dot{\eta}_2''\,\eta_1''^{2}+\eta_2''\,\eta_1'\dot{\eta}_1'''+\eta_2''\;\eta_1'''\dot{\eta}_1'+\dot{\eta}_2''\,\eta_1'\eta_1'''+3\eta_2'\eta_1''\dot{\eta}_1'''\\ +3\eta_2'\,\eta_1'''\dot{\eta}_1''+3\dot{\eta}_2'\eta_1''\eta_1'''+\eta_2'\eta_1'\dot{\eta}_1^{\,(4)}+\eta_2'\eta_1^{\,(4)}\dot{\eta}_1'+\dot{\eta}_2'\,\eta_1'\eta_1^{\,(4)}\end{pmatrix}
$$

$$
-\left(\frac{\lambda_i^{\,2}}{\lambda_{ch}^{\,2}-1}\right)^2\frac{c_b\,u_{i,2}^{\;2}}{2}\left(\eta_2''-\frac{3}{2}\eta_2'^{\,2}\,\eta_2''-\eta_2''\int_{\xi_2}^{1}\left(\eta_1'\eta_1''+\eta_2'\eta_2''\right)d\xi_2\right)
$$

$$
+\chi\,\beta_{o,2}\begin{pmatrix}\ddot{\eta}_2+\eta_2'\int_0^{\xi_2}\left(\dot{\eta}_1'^{\,2}+\dot{\eta}_2'^{\,2}+\eta_1'\ddot{\eta}_1'+\eta_2'\ddot{\eta}_2'\right)d\xi_2+2\dot{\eta}_2'\int_0^{\xi_2}\left(\eta_1'\dot{\eta}_1'+\eta_2'\dot{\eta}_2'\right)d\xi_2\\ -\eta_2''\int_{\xi_2}^{1}\left(\ddot{\eta}_1\,\eta_1'+\ddot{\eta}_2\,\eta_2'\right)d\xi_2-\frac{3}{2}\dot{\eta}_2\dot{\eta}_2'\,\eta_2'+\dot{\eta}_2\;f_t-\frac{\dot{\eta}_3^{\,2}}{4}\eta_2+\dot{\eta}_3\dot{\eta}_1+\frac{\dot{\eta}_1\;f_t}{2}\eta_1\end{pmatrix}
$$

$$
+\chi\sqrt{\beta_{o,2}}\,u_{i,2}\left(\frac{\lambda_i^{\,2}}{\lambda_{ch}^{\,2}-1}\right)^2\begin{pmatrix}-2\dot{\eta}_2'+\dfrac{7}{2}\dot{\eta}_2'\eta_2'^{\,2}-2\eta_2''\displaystyle\int_0^{\xi_2}\left(\eta_1'\dot{\eta}_1'+\eta_2'\dot{\eta}_2'\right)d\xi_2\\ +\dfrac{3}{2}\dot{\eta}_2\eta_2'\eta_2''+2\eta_2''\displaystyle\int_{\xi_2}^{1}\left(\eta_1'\dot{\eta}_1'+\eta_2'\dot{\eta}_2'\right)d\xi_2\end{pmatrix}
$$

$$
+\chi\;u_{i,2}^{\;2}\left(\frac{\lambda_i^{\,2}}{\lambda_{ch}^{\,2}-1}\right)^2\left(\eta_2''-\frac{5}{2}\eta_2''\,\eta_2'^{\,2}-\eta_2''\int_{\xi_2}^{1}\left(\eta_1'\eta_1''+\eta_2'\eta_2''\right)d\xi_2\right)
$$

$$
+\gamma_1\left(\eta_2'+\frac{1}{2}\left(\eta_2'\right)^3+\frac{1}{2}\eta_2'\,\eta_1'^{\,2}-\left(1-\xi_2\right)\left(\eta_2''+\frac{3}{2}\eta_2'^{\,2}\eta_2''+\frac{1}{2}\eta_1'^{\,2}\eta_2''+\eta_2'\,\eta_1'\eta_1''\right)\right)
$$

$$
\begin{aligned}
&-\gamma_2\left(\eta_2' - \frac{1}{2}(\eta_2')^3 - (1-\xi_2)\left(\eta_2'' - \frac{3}{2}\eta_2'^2\eta_2''\right)\right) \\
&-\left(\frac{\mathrm{E}u_{i,2}^{\ 2}c_T}{2}\left(\frac{\lambda_i^{\ 2}}{\lambda_{ch}^{\ 2}-1}\right)^2 \bar{h}\right)\left(\eta_2' - \frac{1}{2}(\eta_2')^3 - (1-\xi_2)\left(\eta_2'' - \frac{3}{2}\eta_2'^2\eta_2''\right)\right) \\
&-\left(\frac{\mathrm{E}u_{i,2}^{\ 2}c_T}{2}\left(\frac{\lambda_i^{\ 2}}{\lambda_{ch}^{\ 2}-1}\right)^2\right)\left(\begin{array}{l}
(1-\xi_2)\left(\eta_2'' + \frac{3}{2}\eta_2'^2\eta_2''\right) - \frac{3}{2}\eta_2'^3 - \eta_1'^2\eta_2' \\
+\sqrt{\beta_{o,2}}\left(\frac{\lambda_{ch}^{\ 2}-1}{\lambda_i^{\ 2}}\right)\frac{\eta_1'\dot{\eta}_1\eta_2'}{u_{i,2}} + \beta_{o,2}\left(\frac{\lambda_{ch}^{\ 2}-1}{\lambda_i^{\ 2}}\right)^2\frac{\dot{\eta}_1^{\ 2}\eta_2'}{2u_{i,2}^{\ 2}} \\
-\eta_2''\int_{\xi_2}^{1}\sqrt{\beta_{o,2}}\left(\frac{\lambda_{ch}^{\ 2}-1}{\lambda_i^{\ 2}}\right)\frac{\eta_1'\dot{\eta}_1 + \eta_2'\dot{\eta}_2}{u_{i,2}}d\xi_2 \\
-\eta_2''\int_{\xi_2}^{1}\beta_{o,2}\left(\frac{\lambda_{ch}^{\ 2}-1}{\lambda_i^{\ 2}}\right)^2\frac{\dot{\eta}_1^{\ 2}+\dot{\eta}_2^{\ 2}}{2u_{i,2}^{\ 2}}d\xi_2 + \eta_2''\int_{\xi_2}^{1}\left(\eta_1'^2+\eta_2'^2\right)d\xi_2
\end{array}\right) \\
&+\left(\frac{\mathrm{E}u_{i,2}^{\ 2}c_N}{2}\left(\frac{\lambda_i^{\ 2}}{\lambda_{ch}^{\ 2}-1}\right)^2\right)\left(\begin{array}{l}
-\eta_2' + \sqrt{\beta_{o,2}}\left(\frac{\lambda_{ch}^{\ 2}-1}{\lambda_i^{\ 2}}\right)\frac{\dot{\eta}_2}{u_{i,2}} + \sqrt{\beta_{o,2}}\left(\frac{\lambda_{ch}^{\ 2}-1}{\lambda_i^{\ 2}}\right)\frac{\eta_1'\dot{\eta}_1\eta_2'}{u_{i,2}} \\
-\eta_2'\eta_1'^2 - \beta_{o,2}^{\frac{3}{2}}\left(\frac{\lambda_{ch}^{\ 2}-1}{\lambda_i^{\ 2}}\right)^3\frac{\dot{\eta}_2^{\ 3}}{2u_{i,2}^{\ 3}} \\
-\beta_{o,2}\left(\frac{\lambda_{ch}^{\ 2}-1}{\lambda_i^{\ 2}}\right)^2\frac{\dot{\eta}_2^{\ 2}\eta_2'}{2u_{i,2}^{\ 2}} + \eta_2''\int_{\xi_2}^{1}\left(\eta_1'^2+\eta_2'^2\right)d\xi_2 \\
-\eta_2''\int_{\xi_2}^{1}\sqrt{\beta_{o,2}}\left(\frac{\lambda_{ch}^{\ 2}-1}{\lambda_i^{\ 2}}\right)\left(\frac{\dot{\eta}_1\eta_1' + \dot{\eta}_2\eta_2'}{u_{i,2}}\right)d\xi_2 \\
-\dot{\eta}_2\beta_{o,2}\int_0^{\xi_2}\left(\frac{\lambda_{ch}^{\ 2}-1}{\lambda_i^{\ 2}}\right)^2\left(\frac{\eta_1'\dot{\eta}_1' + \eta_2'\dot{\eta}_2'}{u_{i,2}^{\ 2}}\right)d\xi_2
\end{array}\right) \\
&+\left(\frac{\mathrm{E}u_{i,2}^{\ 2}c_d}{2}\left(\frac{\lambda_i^{\ 2}}{\lambda_{ch}^{\ 2}-1}\right)^2\right)\left(\begin{array}{l}
-\eta_2'\ |\eta_2'| + \sqrt{\beta_{o,2}}\left(\frac{\lambda_{ch}^{\ 2}-1}{\lambda_i^{\ 2}}\right)\frac{|\dot{\eta}_2|\eta_2' + |\eta_2'|\dot{\eta}_2}{u_{i,2}} \\
+\beta_{o,2}\left(\frac{\lambda_{ch}^{\ 2}-1}{\lambda_i^{\ 2}}\right)^2\frac{\dot{\eta}_2|\dot{\eta}_2|}{u_{i,2}^{\ 2}}
\end{array}\right) = 0
\end{aligned}
\tag{5.51}
$$

$$
\left(\beta_{i,3}\right)\left(\ddot{\eta}_3 + \dot{\eta}_2'\dot{\eta}_1' + \eta_1'\ddot{\eta}_2'\right) - \left(\beta_{m,2}\,\eta_3'' - 4\beta_{i,4}\ f_r\left(\eta_3'\ddot{\eta}_3 + 2\dot{\eta}_3\dot{\eta}_3'\right)\right) = 0 \tag{5.52}
$$

5.4 Extensible Rotating Flexible Pipe Conveying Fluid and Subjected to External Axial Flow.

The dimensionless form of the equations (4.53-4.56) will be established in this section. Equations of motion can be rendered dimensionless by defining following dimensionless quantities:

$$\zeta = \frac{u}{L} \tag{5.53}$$

$$\Pi_{0,2} = \frac{EAL^2}{EI + GJ} \tag{5.54}$$

$$\Pi_{iL,2} = \frac{p_i(L) A_f \, L^2}{EI + G\,J} \tag{5.55}$$

$$\Pi_{oL,2} = \frac{p_o(L) A_o \, L^2}{EI + GJ} \tag{5.56}$$

$$\overline{\Pi}_2 = \frac{\overline{p}_o A_o \, L^2}{EI + GJ} \tag{5.57}$$

$$\overline{\Gamma}_2 = \frac{\overline{T}\, L^2}{EI + GJ} \tag{5.58}$$

Substituting equations (5.1), (5.22-5.42) and (5.53-5.58), yields the following dimensionless equations of motion of an extensible rotating flexible pipe conveying fluid and subjected to external axial flow

$$
\begin{aligned}
&\left(1-\beta_{o,2}-\beta_{i,2}-\beta_{i,3}\right)\ddot{\zeta}+2u_{i,2}\sqrt{\beta_{i,1}}\,\dot{\zeta}'+u_{i,2}{}^{2}\;\zeta''-\Pi_{0,2}\,\zeta''-\left(\Pi_{0,2}\,\alpha_2\right)\dot{\zeta}'' \\
&+\left(\overline{\Gamma_2}-\Pi_{iL,2}+\Pi_{oL,2}-\Pi_{0,2}\right)\left(\eta_1'\,\eta_1''+\eta_2'\,\eta_2''\right)\;-\left(\Pi_{0,2}\,\alpha_2\right)\left(\eta_1'\dot{\eta}_1''+\eta_1''\,\dot{\eta}_1'+\eta_2'\dot{\eta}_2''+\eta_2''\,\dot{\eta}_2'\right) \\
&-\beta_{m,1}\left(\eta_1{}^{(4)}\eta_1'\;+\eta_1''\eta_1'''+\eta_2{}^{(4)}\eta_2'\;+\eta_2''\,\eta_2'''\right)-\beta_{m,1}\alpha_2\begin{pmatrix}\eta_1{}^{(4)}\,\dot{\eta}_1'+\eta_1'\,\dot{\eta}_1{}^{(4)}+\eta_1''\dot{\eta}_1'''+\eta_1'''\dot{\eta}_1'' \\ +\eta_2{}^{(4)}\,\dot{\eta}_2'+\eta_2'\,\dot{\eta}_2{}^{(4)}+\eta_2''\,\dot{\eta}_2'''+\eta_2'''\dot{\eta}_2''\end{pmatrix} \\
&+\gamma_1\left(-\frac{1}{2}\eta_1'^{2}-\frac{1}{2}\eta_2'^{2}+\zeta'+\left(\frac{1}{2}-\xi_1\right)\left(\eta_1'\,\eta_1''+\eta_2'\,\eta_2''\right)\right) \\
&-\gamma_2\left(-\frac{1}{2}\eta_1'^{2}-\frac{1}{2}\eta_2'^{2}+\zeta'+\left(\frac{1}{2}-\xi_1\right)\left(\eta_1'\,\eta_1''+\eta_2'\,\eta_2''\right)\right) \\
&-\chi\,\beta_{o,2}\left(\ddot{\eta}_1\eta_1'+\ddot{\eta}_2\eta_2'\right)+2\,\chi\,u_{o,2}\sqrt{\beta_{o,2}}\left(\dot{\eta}_1'\eta_1'+\dot{\eta}_2'\eta_2'\right)-\chi\;u_{o,2}{}^{2}\left(\eta_1'\,\eta_1''+\eta_2'\,\eta_2''\right) \\
&-\left(\frac{\mathrm{E}\,c_T\,u_{o,2}{}^{2}}{2}\,\bar{h}\right)\left(-\frac{1}{2}\eta_1'^{2}-\frac{1}{2}\eta_2'^{2}+\zeta'+\left(\frac{1}{2}-\xi_1\right)\left(\eta_1'\,\eta_1''+\eta_2'\,\eta_2''\right)\right) \\
&+\left(1-2\nu\right)\overline{\Pi}_2\left(\eta_1'\,\eta_1''+\eta_2'\,\eta_2''\right) \\
&-\left(\frac{\mathrm{E}\,c_T\,u_{o,2}{}^{2}}{2}\right)\begin{pmatrix}\frac{3}{2}\left(\eta_1'^{2}+\eta_2'^{2}\right)-\zeta'-\frac{\sqrt{\beta_{o,2}}}{u_{o,2}}\left(\dot{\eta}_1\eta_1'+\dot{\eta}_2\eta_2'\right) \\ -\frac{\beta_{o,2}}{2u_{o,2}{}^{2}}\left(\dot{\eta}_1{}^{2}+\dot{\eta}_2{}^{2}\right)-\left(\frac{1}{2}-\xi_1\right)\left(\eta_1'\,\eta_1''+\eta_2'\,\eta_2''\right)\end{pmatrix} \\
&-\left(\frac{\mathrm{E}\,c_N\,u_{o,2}{}^{2}}{2}\right)\left(-\eta_1'^{2}-\eta_2'^{2}+\frac{\sqrt{\beta_{o,2}}}{u_{o,2}}\left(\dot{\eta}_1\eta_1'+\dot{\eta}_2\eta_2'\right)\right) \\
&-\left(\frac{\mathrm{E}\,c_d\,u_{o,2}{}^{2}}{2}\right)\begin{pmatrix}-\eta_1'^{2}\left|\eta_1'\right|+\sqrt{\beta_{o,2}}\,\frac{\left|\dot{\eta}_1\right|\eta_1'^{2}+\left|\eta_1'\right|\eta_1'\,\dot{\eta}_1}{u_{o,2}}+\beta_{o,2}\,\frac{\dot{\eta}_1\left|\dot{\eta}_1\right|\eta_1'}{u_{o,2}{}^{2}} \\ -\eta_2'^{2}\left|\eta_2'\right|+\sqrt{\beta_{o,2}}\,\frac{\left|\dot{\eta}_2\right|\eta_2'^{2}+\left|\eta_2'\right|\eta_2'\,\dot{\eta}_2}{u_{o,2}}+\beta_{o,2}\,\frac{\dot{\eta}_2\left|\dot{\eta}_2\right|\eta_2'}{u_{o,2}{}^{2}}\end{pmatrix}=0
\end{aligned}
\tag{5.59}
$$

$$
\begin{aligned}
&\left(1-\beta_{o,2}-\beta_{i,2}-\beta_{i,3}\right)\ddot{\eta}_1 + 2u_{i,2}\sqrt{\beta_{i,1}}\,\dot{\eta}_1' + u_{i,2}{}^2\ \eta_1'' - \left(\overline{\Gamma_2} - \Pi_{iL,2} + \Pi_{oL,2}\right)\eta_1'' \\
&+\beta_{i,2}\left(2\ddot{\eta}_1''\right) + \beta_{i,3}\left(\eta_1''\dot{\eta}_2'^2 + 2\eta_1'\dot{\eta}_2'\dot{\eta}_2'' + \dot{\eta}_3\dot{\eta}_2'' + \dot{\eta}_2'\dot{\eta}_3'\right) + \beta_{m,1}\eta_1{}^{(4)} + \alpha_2\beta_{m,1}\dot{\eta}_1{}^{(4)} \\
&-\beta_{m,1}\left(3\zeta'''\eta_1'' + 4\zeta''\eta_1''' + 2\zeta'\eta_1{}^{(4)} + \eta_1'\zeta^{(4)} + 2\eta_1'^2\,\eta_1{}^{(4)} + 8\eta_1'\eta_1''\,\eta_1''' + 2\left(\eta_1''\right)^3\right) \\
&-\beta_{m,1}\alpha_2\begin{pmatrix} 3\zeta'''\dot{\eta}_1'' + 3\dot{\zeta}'''\eta_1'' + 4\zeta''\dot{\eta}_1''' + 4\dot{\zeta}''\eta_1''' + 2\zeta'\dot{\eta}_1{}^{(4)} + 2\dot{\zeta}'\eta_1{}^{(4)} + \eta_1'\dot{\zeta}^{(4)} + \dot{\eta}_1'\zeta^{(4)} \\ + 2\eta_1'^2\,\dot{\eta}_1{}^{(4)} + 4\dot{\eta}_1'\ \eta_1{}^{(4)} + 8\eta_1'\eta_1''\dot{\eta}_1''' + 8\eta_1'''\left(\eta_1'\dot{\eta}_1'' + \eta_1''\dot{\eta}_1'\right) + 6\left(\eta_1''\right)^2\dot{\eta}_1'' \end{pmatrix} \\
&+\left(\overline{\Gamma_2} - \Pi_{iL,2} + \Pi_{oL,2} - \Pi_{0,2}\right)\left(\zeta''\eta_1' + \zeta'\eta_1'' + \frac{3}{2}\eta_1'^2\,\eta_1''\right) \\
&-\left(\Pi_{0,2}\,\alpha_2\right)\left(\zeta''\dot{\eta}_1' + \eta_1'\dot{\zeta}'' + \zeta'\dot{\eta}_1'' + \eta_1''\dot{\zeta}' + \frac{3}{2}\eta_1'^2\,\dot{\eta}_1'' + 3\dot{\eta}_1'\eta_1''\right) \\
&+\gamma_1\left(\eta_1' + \frac{1}{2}\left(\eta_1'\right)^3 + \zeta\,\eta_1'' - \left(\frac{1}{2} - \xi_1\right)\left(\eta_1'' - \frac{3}{2}\eta_1'^2\eta_1'' - \zeta''\eta_1' - \zeta'\eta_1''\right)\right) \\
&-\gamma_2\left(\left(\eta_1' - \frac{1}{2}\eta_1'^3 + \zeta\,\eta_1''\right) - \left(\frac{1}{2} - \xi_1\right)\left(\eta_1'' - \zeta''\eta_1' - \zeta'\eta_1'' - \frac{3}{2}\eta_1'^2\,\eta_1''\right)\right) \\
&-\left(\frac{\mathrm{E}\,c_T\,u_{o,2}{}^2}{2}\bar{h}\right)\left(\left(\eta_1' - \frac{1}{2}\eta_1'^3 + \zeta\,\eta_1''\right) - \left(\frac{1}{2} - \xi_1\right)\left(\eta_1'' - \zeta''\eta_1' - \zeta'\eta_1'' - \frac{3}{2}\eta_1'^2\,\eta_1''\right)\right) \\
&-\left(1-2\nu\right)\overline{\Pi}_2\ \left(\eta_1'' - \zeta''\eta_1' - \zeta'\eta_1'' - \frac{3}{2}\eta_1'^2\,\eta_1''\right) \\
&-\left(\frac{\mathrm{E}\,c_T\,u_{o,2}{}^2}{2}\right)\begin{pmatrix} \frac{1}{2}\eta_1'^3 - \zeta\,\eta_1'' - \zeta'\eta_1' - \sqrt{\beta_{o,2}}\,\frac{\dot{\eta}_1\eta_1'^2}{u_{o,2}} - \beta_{o,2}\,\frac{\dot{\eta}_1{}^2\,\eta_1'}{2u_{o,2}{}^2} \\ + \left(\frac{1}{2} - \xi_1\right)\left(\eta_1'' - \zeta''\eta_1' - \zeta'\eta_1'' - \frac{3}{2}\eta_1'^2\,\eta_1''\right) \end{pmatrix} \\
&+\left(\frac{\mathrm{E}\,c_N\,u_{o,2}{}^2}{2}\right)\begin{pmatrix} -\eta_1' + \sqrt{\beta_{o,2}}\,\frac{\dot{\eta}_1}{u_{o,2}} + \sqrt{\beta_{o,2}}\,\frac{\dot{\eta}_1\zeta'}{u_{o,2}} + \zeta'\eta_1' + \beta_{o,2}\,\frac{\dot{\zeta}\,\dot{\eta}_1}{u_{o,2}{}^2} + \eta_1'^3 \\ -\frac{\beta_{o,2}}{2u_{o,2}{}^2}\dot{\eta}_1{}^2\eta_1' - \frac{\sqrt{\beta_{o,2}}}{u_{o,2}}\dot{\eta}_1\,\eta_1'^2 - \frac{\beta_{o,2}{}^{\frac{3}{2}}}{2u_{o,2}{}^3}\dot{\eta}_1{}^3 \end{pmatrix} \\
&+\left(\frac{\mathrm{E}\,c_d\,u_{o,2}{}^2}{2}\right)\left(-\eta_1'\ \left|\eta_1'\right| + \sqrt{\beta_{o,2}}\,\frac{\left|\dot{\eta}_1\right|\eta_1' + \left|\eta_1'\right|\dot{\eta}_1}{u_{o,2}} + \beta_{o,2}\,\frac{\dot{\eta}_1\left|\dot{\eta}_1\right|}{u_{o,2}{}^2}\right) \\
&+\chi\,\beta_{o,2}\left(\ddot{\eta}_1 - \ddot{\zeta}\,\eta_1' - 2\dot{\zeta}\,\dot{\eta}_1' - \ddot{\eta}_1\eta_1'^2 - \frac{3}{2}\dot{\eta}_1\,\dot{\eta}_1'\eta_1' + \dot{\eta}_1\ f_t - \frac{\dot{\eta}_3{}^2}{4}\eta_1 + \dot{\eta}_3\dot{\eta}_2 + \frac{\dot{\eta}_3\ f_t}{2}\eta_2\right) \\
&+\chi\,u_{o,2}\sqrt{\beta_{o,2}}\ \left(-2\dot{\eta}_1' + 3\dot{\zeta}'\eta_1' + 4\zeta'\dot{\eta}_1' + \frac{7}{2}\dot{\eta}_1'\eta_1'^2 + 2\dot{\zeta}\,\eta_1'' + \frac{3}{2}\dot{\eta}_1\eta'\eta_1''\right) \\
&+\chi\ u_{o,2}{}^2\left(\eta_1'' - 2\zeta''\eta_1' - 4\zeta'\eta_1'' - \frac{5}{2}\eta_1''\eta_1'^2\right) = 0
\end{aligned}
\tag{5.60}
$$

$$
\begin{aligned}
&\left(1-\beta_{o,2}-\beta_{i,2}-\beta_{i,3}\right)\ddot{\eta}_2 + 2u_{i,2}\sqrt{\beta_{i,1}}\,\dot{\eta}_2' + u_{i,2}{}^2\,\eta_2'' - \left(\overline{\Gamma_2} - \Pi_{iL,2} + \Pi_{oL,2}\right)\eta_2'' + \beta_{i,2}\left(2\ddot{\eta}_2''\right) \\
&+\beta_{i,3}\begin{pmatrix} \dot{\eta}_2'\left(\eta_1'\dot{\eta}_1'' + \dot{\eta}_1'\eta_1''\right) + 2\eta_1'^2\ddot{\eta}''_2 + 3\ddot{\eta}_2'\,\eta_1'\eta_1'' + 2\dot{\eta}_2''\eta_1'\dot{\eta}_1' + \dfrac{\dot{\eta}_3\left(\eta_1'\ddot{\eta}_2' + \dot{\eta}_1'\dot{\eta}_2''\right)}{\dot{\eta}_2'} \\ -\dfrac{2\dot{\eta}_3\,\eta_1'\dot{\eta}_2''\ddot{\eta}_2'}{\dot{\eta}_2'^2} + \dfrac{\ddot{\eta}_3\,v'\dot{\eta}_2''}{\dot{\eta}_2'} + \dfrac{\dot{\eta}_3'\,\eta_1'\,\ddot{\eta}_2'}{\dot{\eta}_2'} + \dfrac{\dot{\eta}_3\left(\eta_1'\ddot{\eta}_2'' + \ddot{\eta}_2'\eta_1''\right)}{\dot{\eta}_2'} \\ +\dot{\eta}_3\,\dot{\eta}_1'' + \eta_1''\ddot{\eta}_3 + \eta_1'\ddot{\eta}_3' + \dot{\eta}_3'\,\dot{\eta}_1' \end{pmatrix} \\
&+\beta_{m,1}\eta_2{}^{(4)} - \beta_{m,1}\begin{pmatrix} 3\zeta'''\eta_2'' + 4\zeta''\eta_2''' + 2\zeta'\eta_2{}^{(4)} + \eta_2'\,\zeta^{(4)} \\ +2\eta_2'^2\,\eta_2{}^{(4)} + 8\eta_2'\eta_2''\,\eta_2''' + 2\left(\eta_2''\right)^3 \end{pmatrix} \\
&-\beta_{m,1}\alpha_2\begin{pmatrix} 3\zeta'''\dot{\eta}_2'' + 3\dot{\zeta}'''\eta_2'' + 4\zeta''\dot{\eta}_2''' + 4\dot{\zeta}''\eta_2''' + 2\zeta'\dot{\eta}_2{}^{(4)} + 2\dot{\zeta}'\eta_2{}^{(4)} \\ +\eta_2'\,\dot{\zeta}^{(4)} + \dot{\eta}_2'\,\zeta^{(4)} + 2\eta_2'^2\,\dot{\eta}_2{}^{(4)} + 4\dot{\eta}_2'\,\eta_2{}^{(4)} \\ +8\eta_2'\eta_2''\dot{\eta}_2''' + 8\eta_2'''\left(\eta_2'\dot{\eta}_2'' + \eta_2''\dot{\eta}_2'\right) + 6\left(\eta_2''\right)^2\dot{\eta}_2'' \end{pmatrix} \\
&+\alpha_2\beta_{m,1}\dot{\eta}_2{}^{(4)} + \left(\overline{\Gamma_2} - \Pi_{iL,2} + \Pi_{oL,2} - \Pi_{0,2}\right)\left(\zeta''\eta_2' + \zeta'\eta_2'' + \frac{3}{2}\eta_2'^2\,\eta_2''\right) \\
&-\left(\Pi_{0,2}\,\alpha_2\right)\left(\zeta''\dot{\eta}_2' + \eta_2'\,\dot{\zeta}'' + \zeta'\dot{\eta}_2'' + \eta_2''\dot{\zeta}' + \frac{3}{2}\eta_2'^2\,\dot{\eta}_2'' + 3\dot{\eta}_2'\,\eta_2''\right) \\
&+\gamma_1\left(\eta_2' + \frac{1}{2}\left(\eta_2'\right)^3 + \zeta\,\eta_2'' - \left(\frac{1}{2} - \xi_1\right)\left(\eta_2'' - \frac{3}{2}\eta_2'^2\eta_2'' - \zeta''\eta_2' - \zeta'\eta_2''\right)\right) \\
&-\left(\frac{\mathrm{E}\,c_T\,u_{o,1}{}^2}{2}\bar{h} + \gamma_2\right)\left(\left(\eta_2' - \frac{1}{2}\eta_2'^3 + \zeta\,\eta_2''\right) - \left(\frac{1}{2} - \xi_1\right)\left(\eta_2'' - \zeta''\eta_2' - \zeta'\eta_2'' - \frac{3}{2}\eta_2'^2\,\eta_2''\right)\right) \\
&-\left(1-2\nu\right)\overline{\Pi}_2\left(\eta_2'' - \zeta''\eta_2' - \zeta'\eta_2'' - \frac{3}{2}\eta_2'^2\,\eta_2''\right) \\
&-\left(\frac{\mathrm{E}\,c_T\,u_{o,2}{}^2}{2}\right)\begin{pmatrix} \dfrac{1}{2}\eta_2'^3 - \zeta\,\eta_2'' - \zeta'\eta_2' - \sqrt{\beta_{o,2}}\,\dfrac{\dot{\eta}_2\eta_2'^2}{u_{o,2}} - \beta_{o,2}\,\dfrac{\dot{\eta}_2{}^2\,\eta_2'}{2u_{o,2}{}^2} \\ +\left(\dfrac{1}{2} - \xi_1\right)\left(\eta_2'' - \zeta''\eta_2' - \zeta'\eta_2'' - \dfrac{3}{2}\eta_2'^2\,\eta_2''\right) \end{pmatrix} \\
&+\left(\frac{\mathrm{E}\,c_N\,u_{o,2}{}^2}{2}\right)\begin{pmatrix} -\eta_2' + \sqrt{\beta_{o,2}}\,\dfrac{\dot{\eta}_2}{u_{o,2}} + \sqrt{\beta_{o,2}}\,\dfrac{\dot{\eta}_2\,\zeta'}{u_{o,2}} + \zeta'\eta_1' + \beta_{o,2}\,\dfrac{\dot{\zeta}\,\dot{\eta}_2}{u_{o,2}{}^2} + \eta_2'^3 \\ -\dfrac{\beta_{o,2}}{2u_{o,2}{}^2}\dot{\eta}_2{}^2\eta_2' - \dfrac{\sqrt{\beta_{o,2}}}{u_{o,2}}\dot{\eta}_2\,\eta_2'^2 - \dfrac{\beta_{o,2}{}^{\frac{3}{2}}}{2u_{o,2}{}^3}\dot{\eta}_2{}^3 \end{pmatrix}
\end{aligned}
$$

$$+\left(\frac{\mathrm{E}\,c_d\,u_{o,2}{}^2}{2}\right)\left(-\eta_2'\;\left|\eta_2'\right|+\sqrt{\beta_{o,2}}\,\frac{\left|\dot{\eta}_2\right|\eta_2'\;+\left|\eta_2'\right|\,\dot{\eta}_2}{u_{o,2}}+\beta_{o,2}\,\frac{\dot{\eta}_2\left|\dot{\eta}_2\right|}{u_{o,2}{}^2}\right)$$
$$+\chi\,\beta_{o,2}\left(\ddot{\eta}_2-\ddot{\zeta}\,\eta_2'-2\,\dot{\zeta}\,\dot{\eta}_2'-\ddot{\eta}_2\eta_2'^{\,2}-\frac{3}{2}\dot{\eta}_2\,\dot{\eta}_2'\,\eta_2'+\dot{\eta}_2\;f_t-\frac{\dot{\eta}_3{}^2}{4}\eta_1+\dot{\eta}_3\dot{\eta}_1+\frac{\dot{\eta}_3\;f_t}{2}\eta_1\right)$$
$$+\chi\,u_{o,2}\sqrt{\beta_{o,2}}\;\left(-2\dot{\eta}_2'+3\dot{\zeta}'\,\eta_2'+4\,\zeta'\,\dot{\eta}_2'+\frac{7}{2}\dot{\eta}_2'\eta_2'^{\,2}+2\,\dot{\zeta}\,\eta_2''+\frac{3}{2}\dot{\eta}_2\eta_2'\eta_2''\right)$$
$$+\chi\;u_{o,2}{}^2\left(\eta_2''-2\,\zeta''\,\eta_2'-4\,\zeta'\,\eta_2''-\frac{5}{2}\eta_2''\,\eta_2'^{\,2}\right)=0 \tag{5.61}$$

and,

$$\left(\beta_{i,3}\right)\left(\ddot{\eta}_3+\dot{\eta}_2'\dot{\eta}_1'+\eta_1'\ddot{\eta}_2'\right)-\left(\beta_{m,2}\,\eta_3''-4\,\beta_{i,4}\;f_r\left(\eta_3'\ddot{\eta}_3+2\,\dot{\eta}_3\,\dot{\eta}_3'\right)\right)=0 \tag{5.62}$$

5.5 Extensible Rotating Flexible Pipe Conveying Fluid Downwards, which then flows Upwards as A confined Annular Flow.

In order to express the dimensionless form of the equations (4.61 -4.64), the following dimensionless quantities are defined

$$\overline{\Pi}_3=\frac{\overline{p}_o\,A_f\,L^2}{EI+GJ} \tag{5.63}$$

$$\aleph_1=\frac{\rho_{f,o}\,g\,L^3\,A_f}{2\left(EI+GJ\right)} \tag{5.64}$$

$$\aleph_2=\frac{\rho_{f,o}\;A_f}{2\,M} \tag{5.65}$$

$$\aleph_3=\frac{\rho_{f,i}\;A_f}{2\,M} \tag{5.66}$$

Utilizing equations (5.1), (5.22-5.34), (5.36-5.42) and (5.63-5.66), and after some mathematical manipulations, yields the following dimensionless equations of motion

$$
\begin{aligned}
&\left(1-\beta_{o,2}-\beta_{i,2}-\beta_{i,3}\right)\ddot{\zeta}+2u_{i,2}\sqrt{\beta_{i,1}}\,\dot{\zeta}'+u_{i,2}{}^{2}\;\zeta''-\Pi_{0,2}\,\zeta''-\left(\Pi_{0,2}\,\alpha_2\right)\dot{\zeta}'' \\
&+\begin{pmatrix}\overline{\Gamma_2}-(1-2\nu)\overline{\Pi}_3-\aleph_1-\aleph_2\,u_{i,2}{}^{2}\left(\dfrac{\lambda_i{}^{2}}{\lambda_{ch}{}^{2}-1}\right)^{2} \\ -\aleph_3\,u_{i,2}{}^{2}\left(\overline{C}\left(1-\dfrac{\lambda_i{}^{2}}{\lambda_{ch}{}^{2}-1}\right)^{2}-1\right)-\Pi_{0,2}\end{pmatrix}\left(\eta_1'\,\eta_1''+\eta_2'\,\eta_2''\right) \\
&-\left(\Pi_{0,2}\,\alpha_2\right)\left(\eta_1'\dot{\eta}_1''+\eta_1''\,\dot{\eta}_1'+\eta_2'\dot{\eta}_2''+\eta_2''\,\dot{\eta}_2'\right) \\
&-\beta_{m,1}\left(\eta_1{}^{(4)}\eta_1'+\eta_1''\,\eta_1'''+\eta_2{}^{(4)}\eta_2'+\eta_2''\,\eta_2'''\right)-\beta_{m,1}\alpha_2\begin{pmatrix}\eta_1{}^{(4)}\,\dot{\eta}_1'+\eta_1'\,\dot{\eta}_1{}^{(4)}+\eta_1''\dot{\eta}_1'''+\eta_1'''\dot{\eta}_1'' \\ +\eta_2{}^{(4)}\,\dot{\eta}_2'+\eta_2'\,\dot{\eta}_2{}^{(4)}+\eta_2''\,\dot{\eta}_2'''+\eta_2'''\dot{\eta}_2''\end{pmatrix} \\
&+\gamma_1\left(-\frac{1}{2}\eta_1'^{2}-\frac{1}{2}\eta_2'^{2}+\zeta'+\left(\frac{1}{2}-\xi_1\right)\left(\eta_1'\,\eta_1''+\eta_2'\,\eta_2''\right)\right) \\
&-\gamma_2\left(-\frac{1}{2}\eta_1'^{2}-\frac{1}{2}\eta_2'^{2}+\zeta'-\xi_1\left(\eta_1'\,\eta_1''+\eta_2'\,\eta_2''\right)\right)+2(1-2\nu)\overline{\Pi}_2\left(\eta_1'\,\eta_1''+\eta_2'\,\eta_2''\right) \\
&-\chi\,\beta_{o,2}\left(\ddot{\eta}_1\eta_1'+\ddot{\eta}_2\eta_2'\right)+2\,\chi\,u_{i,2}\,\frac{\lambda_i{}^{2}}{\lambda_{ch}{}^{2}-1}\sqrt{\beta_{o,2}}\left(\dot{\eta}_1'\eta_1'+\dot{\eta}_2'\eta_2'\right) \\
&-\chi\;u_{i,2}{}^{2}\left(\frac{\lambda_i{}^{2}}{\lambda_{ch}{}^{2}-1}\right)^{2}\left(\eta_1'\,\eta_1''+\eta_2'\,\eta_2''\right) \\
&-\left(\frac{\mathrm{E}\,c_T\,u_{i,2}{}^{2}}{2}\left(\frac{\lambda_i{}^{2}}{\lambda_{ch}{}^{2}-1}\right)^{2}\overline{h}\right)\left(-\frac{1}{2}\eta_1'^{2}-\frac{1}{2}\eta_2'^{2}+\zeta'+\left(\frac{1}{2}-\xi_1\right)\left(\eta_1'\,\eta_1''+\eta_2'\,\eta_2''\right)\right) \\
&-\left(\frac{\mathrm{E}\,c_T\,u_{i,2}{}^{2}}{2}\right)\left(\frac{\lambda_i{}^{2}}{\lambda_{ch}{}^{2}-1}\right)^{2}\begin{pmatrix}\dfrac{3}{2}\left(\eta_1'^{2}+\eta_2'^{2}\right)-\zeta'-\left(\dfrac{\lambda_{ch}{}^{2}-1}{\lambda_i{}^{2}}\right)\dfrac{\sqrt{\beta_{o,2}}}{u_{i,2}}\left(\dot{\eta}_1\eta_1'+\dot{\eta}_2\eta_2'\right) \\ -\left(\dfrac{\lambda_{ch}{}^{2}-1}{\lambda_i{}^{2}}\right)^{2}\dfrac{\beta_{o,2}}{2\,u_{i,2}{}^{2}}\left(\dot{\eta}_1{}^{2}+\dot{\eta}_2{}^{2}\right)-\left(\dfrac{1}{2}-\xi_1\right)\left(\eta_1'\,\eta_1''+\eta_2'\,\eta_2''\right)\end{pmatrix} \\
&-\left(\frac{\mathrm{E}\,c_N\,u_{i,2}{}^{2}}{2}\right)\left(\frac{\lambda_i{}^{2}}{\lambda_{ch}{}^{2}-1}\right)^{2}\left(-\eta_1'^{2}-\eta_2'^{2}+\left(\frac{\lambda_{ch}{}^{2}-1}{\lambda_i{}^{2}}\right)\frac{\sqrt{\beta_{o,2}}}{u_{i,2}}\left(\dot{\eta}_1\eta_1'+\dot{\eta}_2\eta_2'\right)\right) \\
&-\left(\frac{\mathrm{E}\,c_d\,u_{i,2}{}^{2}}{2}\right)\left(\frac{\lambda_i{}^{2}}{\lambda_{ch}{}^{2}-1}\right)^{2}\begin{pmatrix}-\eta_1'^{2}\left|\eta_1'\right|+\sqrt{\beta_{o,2}}\left(\dfrac{\lambda_{ch}{}^{2}-1}{\lambda_i{}^{2}}\right)\dfrac{\left|\dot{\eta}_1\right|\eta_1'^{2}+\left|\eta_1'\right|\eta_1'\,\dot{\eta}_1}{u_{i,2}} \\ +\beta_{o,2}\left(\dfrac{\lambda_{ch}{}^{2}-1}{\lambda_i{}^{2}}\right)^{2}\dfrac{\dot{\eta}_1\left|\dot{\eta}_1\right|\eta_1'}{u_{i,2}{}^{2}}+\beta_{o,2}\left(\dfrac{\lambda_{ch}{}^{2}-1}{\lambda_i{}^{2}}\right)^{2}\dfrac{\dot{\eta}_2\left|\dot{\eta}_2\right|\eta_2'}{u_{i,2}{}^{2}} \\ -\eta_2'^{2}\left|\eta_2'\right|+\sqrt{\beta_{o,2}}\left(\dfrac{\lambda_{ch}{}^{2}-1}{\lambda_i{}^{2}}\right)\dfrac{\left|\dot{\eta}_2\right|\eta_2'^{2}+\left|\eta_2'\right|\eta_2'\,\dot{\eta}_2}{u_{i,2}}\end{pmatrix}=0
\end{aligned}
\tag{5.67}
$$

$$
\begin{aligned}
&\left(1-\beta_{o,2}-\beta_{i,2}-\beta_{i,3}\right)\ddot{\eta}_1 + 2u_{i,2}\sqrt{\beta_{i,1}}\,\dot{\eta}_1' + u_{i,2}^{\;2}\;\eta_1'' \\
&+\beta_{i,2}\left(2\ddot{\eta}_1''\right)+\beta_{i,3}\left(\eta_1''\dot{\eta}_2'^{\,2}+2\eta_1'\dot{\eta}_2'\dot{\eta}_2''+\dot{\eta}_3\dot{\eta}_2''+\dot{\eta}_2'\dot{\eta}_3'\right)+\beta_{m,1}\eta_1^{(4)}+\alpha_2\beta_{m,1}\dot{\eta}_1^{(4)} \\
&-\beta_{m,1}\left(3\zeta'''\eta_1''+4\zeta''\eta_1'''+2\zeta'\eta_1^{(4)}+\eta_1'\zeta^{(4)}+2\eta_1'^{2}\eta_1^{(4)}+8\eta_1'\eta_1''\eta_1'''+2\left(\eta_1''\right)^3\right) \\
&-\beta_{m,1}\alpha_2\begin{pmatrix}3\zeta'''\dot{\eta}_1''+3\dot{\zeta}'''\eta_1''+4\zeta''\dot{\eta}_1'''+4\dot{\zeta}''\eta_1'''+2\zeta'\dot{\eta}_1^{(4)}+2\dot{\zeta}'\eta_1^{(4)}+\eta_1'\dot{\zeta}^{(4)}+\dot{\eta}_1'\zeta^{(4)} \\ +2\eta_1'^{2}\dot{\eta}_1^{(4)}+4\dot{\eta}_1'\;\eta_1^{(4)}+8\eta_1'\eta_1''\dot{\eta}_1'''+8\eta_1'''\left(\eta_1'\dot{\eta}_1''+\eta_1''\dot{\eta}_1'\right)+6\left(\eta_1''\right)^2\dot{\eta}_1''\end{pmatrix} \\
&+\begin{pmatrix}\overline{\Gamma_2}-\left(1-2\nu\right)\overline{\Pi}_3-\aleph_1-\aleph_2\,u_{i,2}^{\;2}\left(\dfrac{\lambda_i^{\;2}}{\lambda_{ch}^{\;2}-1}\right)^2 \\ -\aleph_3\,u_{i,2}^{\;2}\left(\overline{C}\left(1-\dfrac{\lambda_i^{\;2}}{\lambda_{ch}^{\;2}-1}\right)^2-1\right)\end{pmatrix}\left(-\eta_1''+\zeta''\eta_1'+\zeta'\eta_1''+\frac{3}{2}\eta_1'^{2}\,\eta_1''\right) \\
&-\Pi_{0,2}\left(\zeta''\eta_1'+\zeta'\eta_1''+\frac{3}{2}\eta_1'^{2}\,\eta_1''\right)-\left(\Pi_{0,2}\,\alpha_2\right)\left(\zeta''\dot{\eta}_1'+\eta_1'\dot{\zeta}''+\zeta'\dot{\eta}_1''+\eta_1''\dot{\zeta}'+\frac{3}{2}\eta_1'^{2}\,\dot{\eta}_1''+3\dot{\eta}_1'\eta_1''\right) \\
&+\gamma_1\left(\eta_1'+\frac{1}{2}\left(\eta_1'\right)^3+\zeta\,\eta_1''-\left(\frac{1}{2}-\xi_1\right)\left(\eta_1''-\frac{3}{2}\eta_1'^{2}\eta_1''-\zeta''\eta_1'-\zeta'\eta_1''\right)\right) \\
&-\gamma_2\left(\left(\eta_1'-\frac{1}{2}\eta_1'^{3}+\zeta\,\eta_1''\right)+\xi_1\left(\eta_1''-\zeta''\eta_1'-\zeta'\eta_1''-\frac{3}{2}\eta_1'^{2}\,\eta_1''\right)\right) \\
&-\left(\frac{\mathrm{E}\,c_T\,u_{i,2}^{\;2}}{2}\left(\frac{\lambda_i^{\;2}}{\lambda_{ch}^{\;2}-1}\right)^2\overline{h}\right)\left(\left(\eta_1'-\frac{1}{2}\eta_1'^{3}+\zeta\,\eta_1''\right)-\left(\frac{1}{2}-\xi_1\right)\left(\eta_1''-\zeta''\eta_1'-\zeta'\eta_1''-\frac{3}{2}\eta_1'^{2}\,\eta_1''\right)\right) \\
&-2\left(1-2\nu\right)\overline{\Pi}_2\;\left(\eta_1''-\zeta''\eta_1'-\zeta'\eta_1''-\frac{3}{2}\eta_1'^{2}\,\eta_1''\right) \\
&-\left(\frac{\mathrm{E}\,c_T\,u_{i,2}^{\;2}}{2}\left(\frac{\lambda_i^{\;2}}{\lambda_{ch}^{\;2}-1}\right)^2\right)\begin{pmatrix}\dfrac{1}{2}\eta_1'^{3}-\zeta\,\eta_1''-\zeta'\eta_1'-\sqrt{\beta_{o,2}}\left(\dfrac{\lambda_{ch}^{\;2}-1}{\lambda_i^{\;2}}\right)\dfrac{\dot{\eta}_1\eta_1'^{2}}{u_{i,2}} \\ -\beta_{o,2}\left(\dfrac{\lambda_{ch}^{\;2}-1}{\lambda_i^{\;2}}\right)^2\dfrac{\dot{\eta}_1^{\;2}\,\eta_1'}{2u_{i,2}^{\;2}} \\ +\left(\dfrac{1}{2}-\xi_1\right)\left(\eta_1''-\zeta''\eta_1'-\zeta'\eta_1''-\dfrac{3}{2}\eta_1'^{2}\,\eta_1''\right)\end{pmatrix} \\
&+\left(\frac{\mathrm{E}\,c_N\,u_{i,2}^{\;2}}{2}\right)\left(\frac{\lambda_i^{\;2}}{\lambda_{ch}^{\;2}-1}\right)^2\begin{pmatrix}-\eta_1'+\sqrt{\beta_{o,2}}\left(\dfrac{\lambda_{ch}^{\;2}-1}{\lambda_i^{\;2}}\right)\dfrac{\dot{\eta}_1}{u_{i,2}}+\sqrt{\beta_{o,2}}\left(\dfrac{\lambda_{ch}^{\;2}-1}{\lambda_i^{\;2}}\right)\dfrac{\dot{\eta}_1\,\zeta'}{u_{i,2}}+\zeta'\eta_1' \\ +\beta_{o,2}\left(\dfrac{\lambda_{ch}^{\;2}-1}{\lambda_i^{\;2}}\right)^2\dfrac{\dot{\zeta}\,\dot{\eta}_1}{u_{i,2}^{\;2}}+\eta_1'^{3}-\dfrac{\beta_{o,2}}{2u_{i,2}^{\;2}}\left(\dfrac{\lambda_{ch}^{\;2}-1}{\lambda_i^{\;2}}\right)^2\dot{\eta}_1^{\;2}\eta_1' \\ -\dfrac{\sqrt{\beta_{o,2}}}{u_{i,2}}\left(\dfrac{\lambda_{ch}^{\;2}-1}{\lambda_i^{\;2}}\right)\dot{\eta}_1\;\eta_1'^{2}-\dfrac{\beta_{o,2}^{\;\frac{3}{2}}}{2u_{i,2}^{\;3}}\left(\dfrac{\lambda_{ch}^{\;2}-1}{\lambda_i^{\;2}}\right)^3\dot{\eta}_1^{\;3}\end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\frac{\mathrm{E}\,c_d\,u_{i,2}{}^2}{2}\right)\left(\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2\left(\begin{array}{l}-\eta_1'\ \left|\eta_1'\right|+\sqrt{\beta_{o,2}}\left(\frac{\lambda_{ch}{}^2-1}{\lambda_i{}^2}\right)\frac{\left|\dot\eta_1\right|\eta_1'\ +\left|\eta_1'\right|\dot\eta_1}{u_{i,2}}\\ +\beta_{o,2}\left(\frac{\lambda_{ch}{}^2-1}{\lambda_i{}^2}\right)^2\frac{\dot\eta_1\left|\dot\eta_1\right|}{u_{i,2}{}^2}\end{array}\right)\\
&+\chi\,\beta_{o,2}\left(\ddot\eta_1-\ddot\zeta\,\eta_1'-2\dot\zeta\,\dot\eta_1'-\ddot\eta_1\eta_1'^2-\frac{3}{2}\dot\eta_1\,\dot\eta_1'\eta_1'+\dot\eta_1\ f_t-\frac{\dot\eta_3{}^2}{4}\eta_1+\dot\eta_3\dot\eta_2+\frac{\dot\eta_3\ f_t}{2}\eta_2\right)\\
&+\chi\,u_{i,2}\sqrt{\beta_{o,2}}\,\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\left(-2\dot\eta_1'+3\dot\zeta'\eta_1'+4\zeta'\dot\eta_1'+\frac{7}{2}\dot\eta_1'\eta_1'^2+2\dot\zeta\,\eta_1''+\frac{3}{2}\dot\eta_1\eta'\eta_1''\right)\\
&+\chi\ u_{i,2}{}^2\left(\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2\left(\eta_1''-2\zeta''\eta_1'-4\zeta'\eta_1''-\frac{5}{2}\eta_1''\eta_1'^2\right)=0
\end{aligned}
\tag{5.68}
$$

$$
\begin{aligned}
&\left(1-\beta_{o,2}-\beta_{i,2}-\beta_{i,3}\right)\ddot\eta_2\ +2u_{i,2}\sqrt{\beta_{i,1}}\,\dot\eta_2'+u_{i,2}{}^2\ \eta_2''+\beta_{i,2}\left(2\ddot\eta_2''\right)\\
&+\beta_{i,3}\left(\begin{array}{l}\dot\eta_2'\left(\eta_1'\dot\eta_1''+\dot\eta_1'\eta_1''\right)+2\eta_1'^2\ddot\eta''_2+3\ddot\eta_2'\,\eta_1'\eta_1''+2\dot\eta_2''\eta_1'\dot\eta_1'+\frac{\dot\eta_3\left(\eta_1'\ddot\eta_2'+\dot\eta_1'\,\dot\eta_2''\right)}{\dot\eta_2'}\\ -\frac{2\dot\eta_3\,\eta_1'\dot\eta_2''\ddot\eta_2'}{\dot\eta_2'^2}+\frac{\ddot\eta_3\,v'\dot\eta_2''}{\dot\eta_2'}+\frac{\dot\eta_3'\,\eta_1'\,\ddot\eta_2'}{\dot\eta_2'}+\frac{\dot\eta_3\left(\eta_1'\ddot\eta_2''+\ddot\eta_2'\eta_1''\right)}{\dot\eta_2'}+\dot\eta_3\,\dot\eta_1''+\eta_1''\ddot\eta_3+\eta_1'\ddot\eta_3'+\dot\eta_3'\,\dot\eta_1'\end{array}\right)\\
&+\ \beta_{m,1}\eta_2{}^{(4)}-\beta_{m,1}\left(3\zeta'''\eta_2''+4\zeta''\eta_2'''+2\zeta'\eta_2{}^{(4)}+\eta_2'\,\zeta^{(4)}+2\eta_2'^2\,\eta_2{}^{(4)}+8\eta_2'\eta_2''\ \eta_2'''+2\left(\eta_2''\right)^3\right)\\
&-\beta_{m,1}\alpha_2\left(\begin{array}{l}3\zeta'''\dot\eta_2''+3\dot\zeta'''\eta_2''+4\zeta''\dot\eta_2'''+4\dot\zeta''\eta_2'''+2\zeta'\dot\eta_2{}^{(4)}+2\dot\zeta'\eta_2{}^{(4)}+\eta_2'\,\dot\zeta^{(4)}\\ +\dot\eta_2'\,\zeta^{(4)}+2\eta_2'^2\,\dot\eta_2{}^{(4)}+4\dot\eta_2'\ \ \eta_2{}^{(4)}+8\eta_2'\eta_2''\dot\eta_2'''+8\eta_2'''\left(\eta_2'\dot\eta_2''+\eta_2''\dot\eta_2'\right)+6\left(\eta_2''\right)^2\dot\eta_2''\end{array}\right)\\
&+\ \alpha_2\beta_{m,1}\dot\eta_2{}^{(4)}+\left(\begin{array}{l}\overline{\Gamma_2}-\left(1-2\nu\right)\overline{\Pi}_3-\aleph_1-\aleph_2\,u_{i,2}{}^2\left(\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2\\ -\aleph_3\,u_{i,2}{}^2\left(\overline{C}\left(1-\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2-1\right)\end{array}\right)\left(-\eta_2''+\zeta''\eta_2'+\zeta'\eta_2''+\frac{3}{2}\eta_2'^2\,\eta_2''\right)\\
&-\Pi_{0,2}\left(\zeta''\eta_2'+\zeta'\eta_2''+\frac{3}{2}\eta_2'^2\,\eta_2''\right)-\left(\Pi_{0,2}\,\alpha_2\right)\left(\zeta''\dot\eta_2'+\eta_2'\,\dot\zeta''+\zeta'\dot\eta_2''+\eta_2''\dot\zeta'+\frac{3}{2}\eta_2'^2\,\dot\eta_2''+3\dot\eta_2'\,\eta_2''\right)\\
&+\gamma_1\left(\eta_2'+\frac{1}{2}\left(\eta_2'\right)^3+\zeta\,\eta_2''-\left(\frac{1}{2}-\xi_1\right)\left(\eta_2''-\frac{3}{2}\eta_2'^2\eta_2''-\zeta''\eta_2'-\zeta'\eta_2''\right)\right)\\
&-\left(\frac{\mathrm{E}\,c_T\,u_{i,2}{}^2}{2}\bar h\right)\left(\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2\left(\left(\eta_2'-\frac{1}{2}\eta_2'^3+\zeta\,\eta_2''\right)-\left(\frac{1}{2}-\xi_1\right)\left(\eta_2''-\zeta''\eta_2'-\zeta'\eta_2''-\frac{3}{2}\eta_2'^2\,\eta_2''\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&-\gamma_2\left(\left(\eta_2' - \frac{1}{2}\eta_2'^{\,3} + \zeta\,\eta_2''\right) + \xi_1\left(\eta_2'' - \zeta''\eta_2' - \zeta'\eta_2'' - \frac{3}{2}\eta_2'^{\,2}\,\eta_2''\right)\right) \\
&-2(1-2\nu)\overline{\Pi}_2 \quad \left(\eta_2'' - \zeta''\eta_2' - \zeta'\eta_2'' - \frac{3}{2}\eta_2'^{\,2}\,\eta_2''\right) \\
&-\left(\frac{\mathrm{E}\,c_T\,u_{i,2}^{\;2}}{2}\right)\left(\frac{\lambda_i^{\;2}}{\lambda_{ch}^{\;2}-1}\right)^2 \left(\begin{array}{l}
\frac{1}{2}\eta_2'^{\,3} - \zeta\,\eta_2'' - \zeta'\eta_2' - \sqrt{\beta_{o,2}}\left(\frac{\lambda_{ch}^{\;2}-1}{\lambda_i^{\;2}}\right)\frac{\dot{\eta}_2\eta_2'^{\,2}}{u_{i,2}} \\
-\beta_{o,2}\left(\frac{\lambda_{ch}^{\;2}-1}{\lambda_i^{\;2}}\right)^2 \frac{\dot{\eta}_2^{\;2}\,\eta_2'}{2u_{i,2}^{\;2}} + \left(\frac{1}{2}-\xi_1\right)\left(\eta_2'' - \zeta''\eta_2' - \zeta'\eta_2'' - \frac{3}{2}\eta_2'^{\,2}\,\eta_2''\right)
\end{array}\right) \\
&\left(\frac{\mathrm{E}\,c_N\,u_{i,2}^{\;2}}{2}\right)\left(\frac{\lambda_i^{\;2}}{\lambda_{ch}^{\;2}-1}\right)^2 \left(\begin{array}{l}
-\eta_2' + \sqrt{\beta_{o,2}}\left(\frac{\lambda_{ch}^{\;2}-1}{\lambda_i^{\;2}}\right)\frac{\dot{\eta}_2}{u_{i,2}} + \sqrt{\beta_{o,2}}\left(\frac{\lambda_{ch}^{\;2}-1}{\lambda_i^{\;2}}\right)\frac{\dot{\eta}_2\,\zeta'}{u_{i,2}} \\
+\zeta'\eta_1' + \beta_{o,2}\left(\frac{\lambda_{ch}^{\;2}-1}{\lambda_i^{\;2}}\right)^2 \frac{\dot{\zeta}\,\dot{\eta}_2}{u_{i,2}^{\;2}} + \eta_2'^{\,3} \\
-\frac{\beta_{o,2}}{2u_{i,2}^{\;2}}\left(\frac{\lambda_{ch}^{\;2}-1}{\lambda_i^{\;2}}\right)^2 \dot{\eta}_2^{\;2}\eta_2' - \frac{\sqrt{\beta_{o,2}}}{u_{i,2}}\left(\frac{\lambda_{ch}^{\;2}-1}{\lambda_i^{\;2}}\right)\dot{\eta}_2\,\eta_2'^{\,2} \\
-\frac{\beta_{o,2}^{\;\frac{3}{2}}}{2u_{i,2}^{\;3}}\left(\frac{\lambda_{ch}^{\;2}-1}{\lambda_i^{\;2}}\right)^3 \dot{\eta}_2^{\;3}
\end{array}\right) \\
&+\left(\frac{\mathrm{E}\,c_d\,u_{i,2}^{\;2}}{2}\right)\left(\frac{\lambda_i^{\;2}}{\lambda_{ch}^{\;2}-1}\right)^2 \left(\begin{array}{l}
-\eta_2'\,\left|\eta_2'\right| + \sqrt{\beta_{o,2}}\left(\frac{\lambda_{ch}^{\;2}-1}{\lambda_i^{\;2}}\right)\frac{\left|\dot{\eta}_2\right|\eta_2' + \left|\eta_2'\right|\dot{\eta}_2}{u_{i,2}} \\
+\beta_{o,2}\left(\frac{\lambda_{ch}^{\;2}-1}{\lambda_i^{\;2}}\right)^2 \frac{\dot{\eta}_2\left|\dot{\eta}_2\right|}{u_{i,2}^{\;2}}
\end{array}\right) \\
&+\chi\,\beta_{o,2}\left(\ddot{\eta}_2 - \ddot{\zeta}\,\eta_2' - 2\dot{\zeta}\,\dot{\eta}_2' - \ddot{\eta}_2\eta_2'^{\,2} - \frac{3}{2}\dot{\eta}_2\,\dot{\eta}_2'\,\eta_2' + \dot{\eta}_2\,f_t - \frac{\dot{\eta}_3^{\;2}}{4}\eta_1 + \dot{\eta}_3\dot{\eta}_1 + \frac{\dot{\eta}_3\,f_t}{2}\eta_1\right) \\
&+\chi\,u_{i,2}\,\frac{\lambda_i^{\;2}}{\lambda_{ch}^{\;2}-1}\sqrt{\beta_{o,2}}\quad\left(-2\dot{\eta}_2' + 3\dot{\zeta}'\eta_2' + 4\zeta'\dot{\eta}_2' + \frac{7}{2}\dot{\eta}_2'\eta_2'^{\,2} + 2\dot{\zeta}\,\eta_2'' + \frac{3}{2}\dot{\eta}_2\eta_2'\eta_2''\right) \\
&+\chi\,u_{i,2}^{\;2}\left(\frac{\lambda_i^{\;2}}{\lambda_{ch}^{\;2}-1}\right)^2\left(\eta_2'' - 2\zeta''\eta_2' - 4\zeta'\eta_2'' - \frac{5}{2}\eta_2''\,\eta_2'^{\,2}\right) = \;0
\end{aligned}
\tag{5.69}
$$

and,

$$
\left(\beta_{i,3}\right)\left(\ddot{\eta}_3 + \dot{\eta}_2'\dot{\eta}_1' + \eta_1'\ddot{\eta}_2'\right) - \left(\beta_{m,2}\,\eta_3'' - 4\,\beta_{i,4}\;\;f_r\left(\eta_3'\ddot{\eta}_3 + 2\,\dot{\eta}_3\,\dot{\eta}_3'\right)\right) = 0 \tag{5.70}
$$

5.6 Extensible Rotating Flexible Pipe with One Sliding End Conveying Fluid Downwards, which then flows Upwards as A confined Annular Flow.

Utilizing the dimensionless quantities defined by equations (5.1), (5.22-5.34), (5.36-5.42) and (5.63-5.66), the dimensionless form of equations (4.67-4.70) can be expressed as

$$
\begin{aligned}
&\left(1-\beta_{o,2}-\beta_{i,2}-\beta_{i,3}\right)\ddot{\zeta}+2u_{i,2}\sqrt{\beta_{i,1}}\,\dot{\zeta}'+u_{i,2}^{\;2}\;\zeta''-\Pi_{0,2}\,\zeta''-\left(\Pi_{0,2}\,\alpha_2\right)\dot{\zeta}'' \\
&-\left(\aleph_1+\aleph_2\,u_{i,2}^{\;2}\left(\frac{\lambda_i^{\;2}}{\lambda_{ch}^{\;2}-1}\right)^2+\aleph_3\,u_{i,2}^{\;2}\left(\overline{C}\left(1-\frac{\lambda_i^{\;2}}{\lambda_{ch}^{\;2}-1}\right)^2-1\right)+\Pi_{0,2}\right)\left(\eta_1'\,\eta_1''+\eta_2'\,\eta_2''\right) \\
&-\left(\Pi_{0,2}\,\alpha_2\right)\left(\eta_1'\dot{\eta}_1''+\eta_1''\dot{\eta}_1'+\eta_2'\dot{\eta}_2''+\eta_2''\,\dot{\eta}_2'\right) \\
&-\beta_{m,1}\left(\eta_1^{(4)}\eta_1'+\eta_1''\eta_1'''+\eta_2^{(4)}\eta_2'+\eta_2''\,\eta_2'''\right)-\beta_{m,1}\alpha_2\begin{pmatrix}\eta_1^{(4)}\,\dot{\eta}_1'+\eta_1'\,\dot{\eta}_1^{(4)}+\eta_1''\dot{\eta}_1'''+\eta_1'''\dot{\eta}_1'' \\ +\eta_2^{(4)}\,\dot{\eta}_2'+\eta_2'\,\dot{\eta}_2^{(4)}+\eta_2''\,\dot{\eta}_2'''+\eta_2'''\dot{\eta}_2''\end{pmatrix} \\
&+\left(\gamma_1-\gamma_2\right)\left(-\frac{1}{2}\eta_1'^{2}-\frac{1}{2}\eta_2'^{\,2}+\zeta'+\left(1-\xi_1\right)\left(\eta_1'\,\eta_1''+\eta_2'\,\eta_2''\right)\right)-\chi\,\beta_{o,2}\left(\ddot{\eta}_1\eta_1'+\ddot{\eta}_2\eta_2'\right) \\
&+2\,\chi\,u_{i,2}\frac{\lambda_i^{\;2}}{\lambda_{ch}^{\;2}-1}\sqrt{\beta_{o,2}}\left(\dot{\eta}_1'\eta_1'+\dot{\eta}_2'\eta_2'\right)-\chi\;u_{i,2}^{\;2}\left(\frac{\lambda_i^{\;2}}{\lambda_{ch}^{\;2}-1}\right)^2\left(\eta_1'\,\eta_1''+\eta_2'\,\eta_2''\right) \\
&-\left(\frac{\mathrm{E}\,c_T\,u_{i,2}^{\;2}}{2}\left(\frac{\lambda_i^{\;2}}{\lambda_{ch}^{\;2}-1}\right)^2\overline{h}\right)\left(-\frac{1}{2}\eta_1'^{2}-\frac{1}{2}\eta_2'^{\,2}+\zeta'+\left(1-\xi_1\right)\left(\eta_1'\,\eta_1''+\eta_2'\,\eta_2''\right)\right) \\
&+\left(\frac{c_b\,u_{i,2}^{\;2}}{2}\left(\frac{\lambda_i^{\;2}}{\lambda_{ch}^{\;2}-1}\right)^2\right)\left(\eta_1'\,\eta_1''+\eta_2'\,\eta_2''\right) \\
&-\left(\frac{\mathrm{E}\,c_T\,u_{i,2}^{\;2}}{2}\right)\left(\frac{\lambda_i^{\;2}}{\lambda_{ch}^{\;2}-1}\right)^2\begin{pmatrix}\frac{3}{2}\left(\eta_1'^{2}+\eta_2'^{\,2}\right)-\zeta'-\frac{\sqrt{\beta_{o,2}}}{u_{i,2}}\left(\frac{\lambda_{ch}^{\;2}-1}{\lambda_i^{\;2}}\right)\left(\dot{\eta}_1\eta_1'+\dot{\eta}_2\eta_2'\right) \\ -\frac{\beta_{o,2}}{2u_{i,2}^{\;2}}\left(\frac{\lambda_{ch}^{\;2}-1}{\lambda_i^{\;2}}\right)^2\left(\dot{\eta}_1^{\;2}+\dot{\eta}_2^{\;2}\right)-\left(1-\xi_1\right)\left(\eta_1'\,\eta_1''+\eta_2'\,\eta_2''\right)\end{pmatrix} \\
&-\left(\frac{\mathrm{E}\,c_N\,u_{i,2}^{\;2}}{2}\right)\left(\frac{\lambda_i^{\;2}}{\lambda_{ch}^{\;2}-1}\right)^2\left(-\eta_1'^{2}-\eta_2'^{\,2}+\frac{\sqrt{\beta_{o,2}}}{u_{i,1}}\left(\frac{\lambda_{ch}^{\;2}-1}{\lambda_i^{\;2}}\right)\left(\dot{\eta}_1\eta_1'+\dot{\eta}_2\eta_2'\right)\right)
\end{aligned}
$$

$$
-\left(\frac{\mathrm{E}\,c_d\,u_{i,2}{}^2}{2}\right)\left(\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2\left(\begin{array}{l}
-\eta_1'^2\,|\eta_1'|+\sqrt{\beta_{o,2}}\left(\frac{\lambda_{ch}{}^2-1}{\lambda_i{}^2}\right)\frac{|\dot\eta_1|\,\eta_1'^2+|\eta_1'|\,\eta_1'\,\dot\eta_1}{u_{i,2}}\\
+\beta_{o,2}\left(\frac{\lambda_{ch}{}^2-1}{\lambda_i{}^2}\right)\frac{\dot\eta_1\,|\dot\eta_1|\,\eta_1'}{u_{i,2}{}^2}+\beta_{o,2}\left(\frac{\lambda_{ch}{}^2-1}{\lambda_i{}^2}\right)\frac{\dot\eta_2\,|\dot\eta_2|\,\eta_2'}{u_{i,2}{}^2}\\
-\eta_2'^2\,|\eta_2'|+\sqrt{\beta_{o,2}}\left(\frac{\lambda_{ch}{}^2-1}{\lambda_i{}^2}\right)\frac{|\dot\eta_2|\,\eta_2'^2+|\eta_2'|\,\eta_2'\,\dot\eta_2}{u_{i,2}}
\end{array}\right)=0 \qquad (5.71)
$$

$$
\begin{aligned}
&\left(1-\beta_{o,2}-\beta_{i,2}-\beta_{i,3}\right)\ddot\eta_1+2u_{i,2}\sqrt{\beta_{i,1}}\,\dot\eta_1'+u_{i,2}{}^2\,\eta_1''+\beta_{i,2}\left(2\ddot\eta_1''\right)\\
&+\beta_{i,3}\left(\eta_1''\dot\eta_2'^2+2\eta_1'\dot\eta_2'\dot\eta_2''+\dot\eta_3\dot\eta_2''+\dot\eta_2'\dot\eta_3'\right)+\beta_{m,1}\eta_1{}^{(4)}+\alpha_2\beta_{m,1}\dot\eta_1{}^{(4)}\\
&-\beta_{m,1}\left(3\zeta'''\eta_1''+4\zeta''\eta_1'''+2\zeta'\eta_1{}^{(4)}+\eta_1'\zeta^{(4)}+2\eta_1'^2\,\eta_1{}^{(4)}+8\eta_1'\eta_1''\,\eta_1'''+2\left(\eta_1''\right)^3\right)\\
&-\beta_{m,1}\alpha_2\left(\begin{array}{l}3\zeta'''\dot\eta_1''+3\dot\zeta'''\eta_1''+4\zeta''\dot\eta_1'''+4\dot\zeta''\eta_1'''+2\zeta'\dot\eta_1{}^{(4)}+2\dot\zeta'\eta_1{}^{(4)}+\eta_1'\dot\zeta^{(4)}\\
+\dot\eta_1'\zeta^{(4)}+2\eta_1'^2\,\dot\eta_1{}^{(4)}+4\dot\eta_1'\,\eta_1{}^{(4)}+8\eta_1'\eta_1''\dot\eta_1'''+8\eta_1'''\left(\eta_1'\dot\eta_1''+\eta_1''\dot\eta_1'\right)+6\left(\eta_1''\right)^2\dot\eta_1''\end{array}\right)\\
&-\left(\aleph_1+\aleph_2\,u_{i,2}{}^2\left(\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2+\aleph_3\,u_{i,2}{}^2\left(\overline{C}\left(1-\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2-1\right)\right)\left(\begin{array}{l}-\eta_1''+\zeta''\eta_1'\\+\zeta'\eta_1''+\frac{3}{2}\eta_1'^2\,\eta_1''\end{array}\right)\\
&+\left(\frac{c_b\,u_{i,2}{}^2}{2}\left(\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2\right)\left(-\eta_1''+\zeta''\eta_1'+\zeta'\eta_1''+\frac{3}{2}\eta_1'^2\,\eta_1''\right)\\
&-\Pi_{0,2}\left(\zeta''\eta_1'+\zeta'\eta_1''+\frac{3}{2}\eta_1'^2\,\eta_1''\right)-\left(\Pi_{0,2}\,\alpha_2\right)\left(\begin{array}{l}\zeta''\dot\eta_1'+\eta_1'\dot\zeta''+\zeta'\dot\eta_1''\\+\eta_1''\dot\zeta'+\frac{3}{2}\eta_1'^2\,\dot\eta_1''+3\dot\eta_1'\eta_1''\end{array}\right)\\
&+\gamma_1\left(\eta_1'+\frac{1}{2}\left(\eta_1'\right)^3+\zeta\,\eta_1''-\zeta(1)\,\eta_1''-\left(1-\xi_1\right)\left(\eta_1''-\frac{3}{2}\eta_1'^2\eta_1''-\zeta''\eta_1'-\zeta'\eta_1''\right)\right)\\
&-\gamma_2\left(\left(\eta_1'-\frac{1}{2}\eta_1'^3+\zeta\,\eta_1''-\zeta(1)\,\eta_1''\right)-\left(1-\xi_1\right)\left(\eta_1''-\zeta''\eta_1'-\zeta'\eta_1''-\frac{3}{2}\eta_1'^2\,\eta_1''\right)\right)\\
&-\left(\frac{\mathrm{E}\,c_T\,u_{i,2}{}^2}{2}\left(\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2\overline{h}\right)\left(\begin{array}{l}\left(\eta_1'-\frac{1}{2}\eta_1'^3+\zeta\,\eta_1''-\zeta(1)\,\eta_1''\right)\\-\left(1-\xi_1\right)\left(\eta_1''-\zeta''\eta_1'-\zeta'\eta_1''-\frac{3}{2}\eta_1'^2\,\eta_1''\right)\end{array}\right)\\
&-\left(\frac{\mathrm{E}\,c_T\,u_{i,2}{}^2}{2}\left(\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2\right)\left(\begin{array}{l}\frac{1}{2}\eta_1'^3-\zeta\,\eta_1''+\zeta(1)\,\eta_1''-\zeta'\eta_1'-\sqrt{\beta_{o,2}}\left(\frac{\lambda_{ch}{}^2-1}{\lambda_i{}^2}\right)\frac{\dot\eta_1\eta_1'^2}{u_{i,2}}\\-\beta_{o,2}\left(\frac{\lambda_{ch}{}^2-1}{\lambda_i{}^2}\right)^2\frac{\dot\eta_1{}^2\,\eta_1'}{2u_{i,2}{}^2}+\left(1-\xi_1\right)\left(\eta_1''-\zeta''\eta_1'-\zeta'\eta_1''-\frac{3}{2}\eta_1'^2\,\eta_1''\right)\end{array}\right)
\end{aligned}
$$

$$+\left(\frac{\mathrm{E}\,c_N\,u_{i,2}{}^2}{2}\right)\left(\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2\left(\begin{array}{l}-\eta_1'+\sqrt{\beta_{o,2}}\left(\frac{\lambda_{ch}{}^2-1}{\lambda_i{}^2}\right)\frac{\dot\eta_1}{u_{i,2}}+\sqrt{\beta_{o,2}}\left(\frac{\lambda_{ch}{}^2-1}{\lambda_i{}^2}\right)\frac{\dot\eta_1\,\zeta'}{u_{i,2}}\\ +\zeta'\eta_1'+\beta_{o,2}\left(\frac{\lambda_{ch}{}^2-1}{\lambda_i{}^2}\right)^2\frac{\dot\zeta\,\dot\eta_1}{u_{i,2}{}^2}+\eta_1'^3-\frac{\beta_{o,2}}{2u_{i,2}{}^2}\left(\frac{\lambda_{ch}{}^2-1}{\lambda_i{}^2}\right)^2\dot\eta_1{}^2\eta_1'\\ -\frac{\sqrt{\beta_{o,2}}}{u_{i,2}}\left(\frac{\lambda_{ch}{}^2-1}{\lambda_i{}^2}\right)\dot\eta_1\,\eta_1'^2-\frac{\beta_{o,2}{}^{\frac{3}{2}}}{2u_{i,2}{}^3}\left(\frac{\lambda_{ch}{}^2-1}{\lambda_i{}^2}\right)^3\dot\eta_1{}^3\end{array}\right)$$

$$+\left(\frac{\mathrm{E}\,c_d\,u_{i,2}{}^2}{2}\right)\left(\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2\left(\begin{array}{l}-\eta_1'\,|\eta_1'|+\sqrt{\beta_{o,2}}\left(\frac{\lambda_{ch}{}^2-1}{\lambda_i{}^2}\right)\frac{|\dot\eta_1|\eta_1'+|\eta_1'|\,\dot\eta_1}{u_{i,2}}\\ +\beta_{o,2}\left(\frac{\lambda_{ch}{}^2-1}{\lambda_i{}^2}\right)^2\frac{\dot\eta_1|\dot\eta_1|}{u_{i,2}{}^2}\end{array}\right)$$

$$+\chi\,\beta_{o,2}\left(\ddot\eta_1-\ddot\zeta\,\eta_1'-2\dot\zeta\,\dot\eta_1'-\ddot\eta_1\eta_1'^2-\frac{3}{2}\dot\eta_1\,\dot\eta_1'\,\eta_1'+\dot\eta_1\,f_t-\frac{\dot\eta_3{}^2}{4}\eta_1+\dot\eta_3\dot\eta_2+\frac{\dot\eta_3\,f_t}{2}\eta_2\right)$$

$$+\chi\,u_{i,2}\sqrt{\beta_{o,2}}\,\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\left(-2\dot\eta_1'+3\dot\zeta'\eta_1'+4\zeta'\dot\eta_1'+\frac{7}{2}\dot\eta_1'\eta_1'^2+2\dot\zeta\,\eta_1''+\frac{3}{2}\dot\eta_1\eta'\eta_1''\right)$$

$$+\chi\,u_{i,2}{}^2\left(\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2\left(\eta_1''-2\zeta''\eta_1'-4\zeta'\eta_1''-\frac{5}{2}\eta_1''\eta_1'^2\right)=0$$

(5.72)

$$\left(1-\beta_{o,2}-\beta_{i,2}-\beta_{i,3}\right)\ddot\eta_2+2u_{i,2}\sqrt{\beta_{i,1}}\,\dot\eta_2'+u_{i,2}{}^2\,\eta_2''+\beta_{i,2}\left(2\ddot\eta_2''\right)$$

$$+\beta_{i,3}\left(\begin{array}{l}\dot\eta_2'\left(\eta_1'\dot\eta_1''+\dot\eta_1'\eta_1''\right)+2\eta_1'^2\ddot\eta''_2+3\ddot\eta_2'\,\eta_1'\eta_1''+2\dot\eta_2''\eta_1'\dot\eta_1'+\frac{\dot\eta_3\left(\eta_1'\ddot\eta_2'+\dot\eta_1'\,\dot\eta_2''\right)}{\dot\eta_2'}\\ -\frac{2\dot\eta_3\,\eta_1'\dot\eta_2''\ddot\eta_2'}{\dot\eta_2'^2}+\frac{\ddot\eta_3\,v'\dot\eta_2''}{\dot\eta_2'}+\frac{\dot\eta_3'\,\eta_1'\,\ddot\eta_2'}{\dot\eta_2'}+\frac{\dot\eta_3\left(\eta_1'\ddot\eta_2''+\ddot\eta_2'\eta_1''\right)}{\dot\eta_2'}\\ +\dot\eta_3\,\dot\eta_1''+\eta_1''\ddot\eta_3+\eta_1'\ddot\eta_3'+\dot\eta_3'\,\dot\eta_1'\end{array}\right)$$

$$+\beta_{m,1}\eta_2{}^{(4)}-\beta_{m,1}\left(3\zeta'''\eta_2''+4\zeta''\eta_2'''+2\zeta'\eta_2{}^{(4)}+\eta_2'\,\zeta^{(4)}+2\eta_2'^2\,\eta_2{}^{(4)}+8\eta_2'\eta_2''\,\eta_2'''+2\left(\eta_2''\right)^3\right)$$

$$-\beta_{m,1}\alpha_2\left(\begin{array}{l}3\zeta'''\dot\eta_2''+3\dot\zeta'''\eta_2''+4\zeta''\dot\eta_2'''+4\dot\zeta''\eta_2'''+2\zeta'\dot\eta_2{}^{(4)}+2\dot\zeta'\eta_2{}^{(4)}+\eta_2'\,\dot\zeta^{(4)}\\ +\dot\eta_2'\,\zeta^{(4)}+2\eta_2'^2\,\dot\eta_2{}^{(4)}+4\dot\eta_2'\,\eta_2{}^{(4)}+8\eta_2'\eta_2''\dot\eta_2'''+8\eta_2'''\left(\eta_2'\dot\eta_2''+\eta_2''\dot\eta_2'\right)+6\left(\eta_2''\right)^2\dot\eta_2''\end{array}\right)$$

$$+\alpha_2\beta_{m,1}\dot\eta_2{}^{(4)}-\left(\begin{array}{l}\aleph_1+\aleph_2\,u_{i,2}{}^2\left(\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2\\ +\aleph_3\,u_{i,2}{}^2\left(\overline{C}\left(1-\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2-1\right)\end{array}\right)\left(-\eta_2''+\zeta''\eta_2'+\zeta'\eta_2''+\frac{3}{2}\eta_2'^2\,\eta_2''\right)$$

$$
\begin{aligned}
&+\left(\frac{c_b\, u_{i,2}{}^2}{2}\left(\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2\right)\left(-\eta_1''+\zeta''\eta_1'+\zeta'\eta_1''+\frac{3}{2}\eta_1'^2\,\eta_1''\right)\\
&-\Pi_{0,2}\left(\zeta''\eta_2'+\zeta'\eta_2''+\frac{3}{2}\eta_2'^2\,\eta_2''\right)-\left(\Pi_{0,2}\,\alpha_2\right)\left(\zeta''\dot\eta_2'+\eta_2'\,\dot\zeta''+\zeta'\dot\eta_2''+\eta_2''\dot\zeta'+\frac{3}{2}\eta_2'^2\,\dot\eta_2''+3\dot\eta_2'\,\eta_2''\right)\\
&+\left(\gamma_1-\gamma_2\right)\left(\eta_2'+\frac{1}{2}\left(\eta_2'\right)^3+\zeta\,\eta_2''-\zeta(1)\eta_2''-\left(1-\xi_1\right)\left(\eta_2''-\frac{3}{2}\eta_2'^2\eta_2''-\zeta''\eta_2'-\zeta'\eta_2''\right)\right)\\
&-\left(\frac{\mathrm{E}\,c_T\,u_{i,2}{}^2}{2}\bar h\right)\left(\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2\left(\left(\eta_2'-\frac{1}{2}\eta_2'^3+\zeta\,\eta_2''\,\zeta(1)\eta_2''\right)-\left(1-\xi_1\right)\begin{pmatrix}\eta_2''-\zeta''\eta_2'\\ -\zeta'\eta_2''-\frac{3}{2}\eta_2'^2\,\eta_2''\end{pmatrix}\right)\\
&-\left(\frac{\mathrm{E}\,c_T\,u_{i,2}{}^2}{2}\right)\left(\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2\begin{pmatrix}\frac{1}{2}\eta_2'^3-\zeta\,\eta_2''+\zeta(1)\eta_2''-\zeta'\eta_2'-\sqrt{\beta_{o,2}}\left(\frac{\lambda_{ch}{}^2-1}{\lambda_i{}^2}\right)\frac{\dot\eta_2\eta_2'^2}{u_{i,2}}\\ -\beta_{o,2}\left(\frac{\lambda_{ch}{}^2-1}{\lambda_i{}^2}\right)^2\frac{\dot\eta_2{}^2\,\eta_2'}{2u_{i,2}{}^2}\\ +\left(1-\xi_1\right)\left(\eta_2''-\zeta''\eta_2'-\zeta'\eta_2''-\frac{3}{2}\eta_2'^2\,\eta_2''\right)\end{pmatrix}\\
&\left(\frac{\mathrm{E}\,c_N\,u_{i,2}{}^2}{2}\right)\left(\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2\begin{pmatrix}-\eta_2'+\sqrt{\beta_{o,2}}\left(\frac{\lambda_{ch}{}^2-1}{\lambda_i{}^2}\right)\frac{\dot\eta_2}{u_{i,2}}+\sqrt{\beta_{o,2}}\left(\frac{\lambda_{ch}{}^2-1}{\lambda_i{}^2}\right)\frac{\dot\eta_2\,\zeta'}{u_{i,2}}+\zeta'\eta_1'\\ +\beta_{o,2}\left(\frac{\lambda_{ch}{}^2-1}{\lambda_i{}^2}\right)^2\frac{\dot\zeta\,\dot\eta_2}{u_{i,2}{}^2}+\eta_2'^3-\frac{\beta_{o,2}{}^{\frac{3}{2}}}{2u_{i,2}{}^3}\left(\frac{\lambda_{ch}{}^2-1}{\lambda_i{}^2}\right)^3\dot\eta_2{}^3\\ -\frac{\beta_{o,2}}{2u_{i,2}{}^2}\left(\frac{\lambda_{ch}{}^2-1}{\lambda_i{}^2}\right)^2\dot\eta_2{}^2\eta_2'-\frac{\sqrt{\beta_{o,2}}}{u_{i,2}}\left(\frac{\lambda_{ch}{}^2-1}{\lambda_i{}^2}\right)\dot\eta_2\,\eta_2'^2\end{pmatrix}\\
&+\left(\frac{\mathrm{E}\,c_d\,u_{i,2}{}^2}{2}\right)\left(\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2\begin{pmatrix}-\eta_2'\,\left|\eta_2'\right|+\sqrt{\beta_{o,2}}\left(\frac{\lambda_{ch}{}^2-1}{\lambda}\right)\frac{\left|\dot\eta_2\right|\eta_2'+\left|\eta_2'\right|\dot\eta_2}{u_{i,2}}\\ +\left(\frac{\lambda_{ch}{}^2-1}{\lambda_i{}^2}\right)^2\beta_{o,2}\frac{\dot\eta_2\left|\dot\eta_2\right|}{u_{i,2}{}^2}\end{pmatrix}\\
&+\chi\,\beta_{o,2}\left(\ddot\eta_2-\ddot\zeta\,\eta_2'-2\dot\zeta\,\dot\eta_2'-\ddot\eta_2\eta_2'^2-\frac{3}{2}\dot\eta_2\,\dot\eta_2'\,\eta_2'+\dot\eta_2\,f_t-\frac{\dot\eta_3{}^2}{4}\eta_1+\dot\eta_3\dot\eta_1+\frac{\dot\eta_3\,f_t}{2}\eta_1\right)\\
&+\chi\,u_{i,2}\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\sqrt{\beta_{o,2}}\left(-2\dot\eta_2'+3\dot\zeta'\eta_2'+4\zeta'\dot\eta_2'+\frac{7}{2}\dot\eta_2'\eta_2'^2+2\dot\zeta\,\eta_2''+\frac{3}{2}\dot\eta_2\eta_2'\eta_2''\right)\\
&+\chi\,u_{i,2}{}^2\left(\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2\left(\eta_2''-2\zeta''\eta_2'-4\zeta'\eta_2''-\frac{5}{2}\eta_2''\eta_2'^2\right)=0
\end{aligned}
\tag{5.73}
$$

$$
\left(\beta_{i,3}\right)\left(\ddot\eta_3+\dot\eta_2'\dot\eta_1'+\eta_1'\ddot\eta_2'\right)-\left(\beta_{m,2}\,\eta_3''-4\beta_{i,4}\;f_r\left(\eta_3'\ddot\eta_3+2\dot\eta_3\,\dot\eta_3'\right)\right)=0 \tag{5.74}
$$

CHAPTER SIX

SOLUTION METHODOLOGY

This chapter is devoted to outline the proposed procedure for solving the dimensionless coupled nonlinear partial differential equations presented in chapter 5. This task can be performed by applying a discretization scheme with appropriate comparison functions as will be discussed hereafter with more details.

6.1 Discretization via Galerkin's Method

The governing nonlinear partial differential equations described in chapter 5 don't admit a closed form of solution. However, such equations may be reduced to a set of ordinary differential equations by applying one of the discretization techniques such as Galerkin's method.

The Galerkin's method is utilized to approximate the dimensionless transverse deflections $\eta_1(\xi,t^*)$ and $\eta_2(\xi,t^*)$, dimensionless axial shortening $\zeta(\xi,t^*)$ and torsional rotation $\eta_3(\xi,t^*)$, by expanding them in time and spatial domains. Accordingly, one can write

$$\zeta(\xi,t^*)=\sum_{i=1}^{N}\Psi_j(\xi)\ p_j(t^*) \tag{6.1}$$

$$\eta_1(\xi,t^*)=\sum_{i=1}^{N}\Phi_j(\xi)\ q_j(t^*) \tag{6.2}$$

$$\eta_2(\xi,t^*)=\sum_{i=1}^{N}\Theta_j(\xi)\ r_j(t^*) \tag{6.3}$$

$$\eta_3(\xi,t^*)=\sum_{i=1}^{N}\Xi_j(\xi)\ \breve{g}_j(t^*) \tag{6.4}$$

where

N: Number of modes.

p_j: The time dependant vector of modal coordinates associated with the dimensionless axial deflection ζ.

Ψ : The vector of assumed mode shapes, corresponding to the dimensionless axial deflection.

q_j: The time dependant vector of modal coordinates associated with the dimensionless lateral deflection η_1 in the X-Y plane.

Φ : The vector of assumed mode shapes, corresponding to the dimensionless transverse deflection in the X-Y plane.

r_j: The time dependant vector of modal coordinates associated with the dimensionless lateral deflection η_2 in the X-Z plane.

Θ : The vector of assumed mode shapes, corresponding to the dimensionless transverse deflection in the X-Z plane.

$\breve{g}_j$: The time dependant vector of modal coordinates associated with the dimensionless torsional rotation η_3.

Ξ : The vector of assumed mode shapes, corresponding to torsion.

Now, by assuming that the pipe vibration is denominated by a single mode, then one may write

$$\zeta(\xi,t^*)=\Psi(\xi)\ p(t^*) \tag{6.5}$$

$$\eta_1(\xi,t^*)=\Phi(\xi)\ q(t^*) \tag{6.6}$$

$$\eta_2(\xi,t^*)=\Theta(\xi)\ r(t^*) \tag{6.7}$$

$$\eta_3(\xi,t^*)=\Xi(\xi)\ \breve{g}(t^*) \tag{6.8}$$

The vectors of assumed mode shapes Ψ, Φ, Θ, and Ξ can be estimated by solving the associated frequency equations of the corresponding linear model for each of vibration mode, as will be discussed in section 6.2.

One should consider that when the approximations given by equations (6.5-6.8) are substituted into the left side of the governing equations of motion, the result will generally not be zero [34], but equal to an error function, which maybe denoted by $\overline{E}$.

Finlayson and Scriven [43] showed that such error could be eliminated by weighting the error $\overline{E}$ with the vector of the single assumed mode and integrate it over the spatial domain. This can be expressed mathematically as

$$\int_0^1 \overline{E}\Psi_i\, d\xi=\int_0^1 \overline{E}\Phi_i\, d\xi=\int_0^1 \overline{E}\Theta_i\, d\xi=\int_0^1 \overline{E}\Xi_i\, d\xi=0 \tag{6.9}$$

On the other hand, in the view of the mode shapes orthogonality [12], one may write that

$$\int_0^1 \Psi_i\Psi_j d\xi=\int_0^1 \Phi_i\Phi_j d\xi=\int_0^1 \Theta_i\Theta_j d\xi=\int_0^1 \Xi_i\Xi_j d\xi=\delta_{ij} \tag{6.10}$$

Where δ_{ij} is the Kronecker delta. One should keep in mind that equation (6.10) is valid only for the normalized mode shapes. For generality, not normalized mode shapes are used in the current analysis.

In the next subsections, the dimensionless equations of motion obtained in chapter 5 are discretized via unimodal Galerkin's method.

6.1.1 Extensible flexible pipe conveying fluid and subjected to external axial flow

In this subsection, the dimensionless equations of motion (5.19) and (5.20), which describe the dynamics of extensible flexible pipe with both ends are fixed, conveying fluid and subjected to external axial flow, are discretized via unimodal Galerkin's method. Substituting expressions (6.5) and (6.6) into equations (5.19) and (5.20), multiplying equation (5.19) by $\Psi(\xi_1)$and equation (5.20) by $\Phi(\xi_1)$ and integrating over the dimensionless domain, and after long mathematical manipulation, we obtain the following coupled ordinary nonlinear differential equations in terms of the modal coordinates p and q:

$$\begin{pmatrix} [M_1{}^u]\ddot{p}+[C_1{}^u]\dot{p}+[K_1{}^u]p+[A_1{}^u]q^2+[A_2{}^u]q\,\dot{q}+[A_3{}^u]\,\dot{q}^2+[A_4{}^u]q\,\ddot{q} \\ +[B_1{}^u]q^2|q|+[B_2{}^u]q|q|\,\dot{q}\ +[B_3{}^u]q^2|\dot{q}|+[B_4{}^u]q\,\dot{q}|\dot{q}| \end{pmatrix}=0 \qquad (6.11)$$

$$\begin{pmatrix} [M_1{}^v]\ddot{q}+[C_1{}^v]\dot{q}+[K_1{}^v]q+[D_1{}^v]p\,q+[D_2{}^v]\dot{p}\,q+[D_3{}^v]p\,\dot{q}+[D_4{}^v]\dot{p}\,\dot{q} \\ +[D_5{}^v]\ddot{p}\,q+[E_1{}^v]q\dot{q}\ +[E_2{}^v]q|q|+[E_3{}^v]\dot{q}|q|+[E_4{}^v]q|\dot{q}|+[E_5{}^v]\dot{q}|\dot{q}| \\ +[F_1{}^v]q^3+[F_2{}^v]q^2\,\dot{q}+[F_3{}^v]q\,\dot{q}^2+[F_4{}^v]\,\dot{q}^3+[F_5{}^v]q^2\,\ddot{q} \end{pmatrix}=0 \qquad (6.12)$$

where the values of coefficients are found to be

$$M_1{}^u=(1-\beta_{o,1})\int_0^1\Psi^2 d\xi_1 \qquad (6.13)$$

$$C_1^{\,u} = 2u_{i,1}\sqrt{\beta_i}\int_0^1 \Psi\Psi' d\xi_1 - \Pi_{0,1}\alpha_1\int_0^1 \Psi\Psi'' d\xi_1 \tag{6.14}$$

$$K_1^{\,u} = \left(u_{i,1}^{\;2} - \Pi_{0,1}\right)\int_0^1 \Psi''\Psi . d\xi_1 + \frac{\mathrm{E}\, c_T\, u_{o,1}^{\;2}}{2}\left(1-\bar{h}\right)\int_0^1 \Psi\Psi' d\xi_1 \tag{6.15}$$

$$\begin{aligned} A_1^{\,u} = & \left(\begin{array}{l} \overline{\Gamma_1} - \Pi_{iL,1} + \Pi_{oL,1} - \Pi_{0,1} - \chi\, u_{o,1}^{\;2} \\ + \left(1-2\nu\right)\overline{\Pi}_1 + \left(\frac{1}{2} - \xi_1\right)\frac{\mathrm{E}\, c_T\, u_{o,1}^{\;2}}{2}\left(1-\bar{h}\right) \end{array}\right)\int_0^1 \Psi\Phi'\Phi'' d\xi_1 \\ & + \frac{\mathrm{E}\, u_{o,1}^{\;2}}{2}\left(c_N - c_T\left(\frac{3}{2} - \frac{\bar{h}}{2}\right)\right)\int_0^1 \Psi\Phi'^2 d\xi_1 \\ & - \left(\int_0^1 \Psi\Phi^{(4)}\Phi' d\xi_1 + \int_0^1 \Psi\Phi''\Phi''' d\xi_1\right) \end{aligned} \tag{6.16}$$

$$\begin{aligned} A_2^{\,u} = & \left(\frac{\mathrm{E}\, u_{o,1}\sqrt{\beta_{o,1}}}{2}\left(c_T - c_N\right)\right)\int_0^1 \Psi\Phi\Phi' d\xi_1 - \left(2\Pi_{0,1}\,\alpha_1\right)\int_0^1 \Psi\Phi'\Phi'' d\xi_1 \\ & + 2\,\chi\, u_{o,1}\sqrt{\beta_{o,1}}\int_0^1 \Psi\Phi'^2 d\xi_1 - 2\alpha_1\left(\int_0^1 \Psi\Phi^{(4)}\Phi' dX + \int_0^1 \Psi\Phi''\Phi''' d\xi_1\right) \end{aligned} \tag{6.17}$$

$$A_3^{\,u} = \frac{\mathrm{E}\, c_T\, \beta_{o,1}}{4}\int_0^1 \Psi\Phi^2 d\xi_1 \tag{6.18}$$

$$A_4^{\,u} = -\chi\,\beta_{o,1}\int_0^1 \Psi\Phi\Phi' d\xi_1 \tag{6.19}$$

$$B_1^{\,u} = \frac{\mathrm{E}\, c_d\, u_{o,1}^{\;2}}{2}\int_0^1 \Psi\Phi'^2\left|\Phi'\right| d\xi_1 \tag{6.20}$$

$$B_2^{\,u} = -\frac{\mathrm{E}\, c_d\, u_{o,1}}{2}\sqrt{\beta_{o,1}}\int_0^1 \Psi\Phi'\left|\Phi'\right|\Phi\, d\xi_1 \tag{6.21}$$

$$B_3^{\,u} = -\frac{\mathrm{E}\, c_d\, u_{o,1}}{2}\sqrt{\beta_{o,1}}\int_0^1 \Psi\Phi'^2\left|\Phi\right| d\xi_1 \tag{6.22}$$

$$B_4{}^u = -\frac{\mathrm{E}\,c_d}{2}\beta_{o,1}\int_0^1 \Psi\Phi'|\Phi|\Phi\,d\xi_1 \tag{6.23}$$

$$M_1{}^v = \left(\left(1-\beta_{o,1}\right)+\chi\,\beta_{o,1}\right)\int_0^1 \Phi^2 d\xi_1 \tag{6.24}$$

$$\begin{aligned} C_1{}^v &= \left(2u_{i,1}\sqrt{\beta_i}-2\chi\,u_{o,1}\sqrt{\beta_{o,1}}\right)\int_0^1 \Phi\Phi' d\xi_1 \\ &\quad +\alpha_1\int_0^1 \Phi\Phi^{(4)} d\xi_1 + \frac{\mathrm{E}\,c_N\,u_{o,1}}{2}\sqrt{\beta_{o,1}}\int_0^1 \Phi^2 d\xi_1 \end{aligned} \tag{6.25}$$

$$\begin{aligned} K_1{}^v &= \left(\begin{array}{l} u_{i,1}{}^2-\left(\overline{\Gamma_1}-\Pi_{iL,1}+\Pi_{oL,1}\right)+\chi\,u_{o,1}{}^2-\dfrac{\mathrm{E}\,c_T\,u_{o,1}{}^2}{2}\left(1-\overline{h}\right)\left(\dfrac{1}{2}-\xi_1\right) \\ -\left(1-2\nu\right)\overline{\Pi}_1 \end{array}\right)\int_0^1 \Phi\Phi''\,d\xi_1 \\ &\quad -\left(\frac{\mathrm{E}\,c_N\,u_{o,1}{}^2}{2}+\frac{\mathrm{E}\,c_T\,\overline{h}\,u_{o,1}{}^2}{2}\right)\int_0^1 \Phi\Phi'\,d\xi_1+\int_0^1 \Phi\Phi^{(4)}\,d\xi_1 \end{aligned} \tag{6.26}$$

$$\begin{aligned} D_1{}^v &= \left(\begin{array}{l} \dfrac{\mathrm{E}\,c_T\,u_{o,1}{}^2}{2}\left(1-\overline{h}\right)\left(\dfrac{1}{2}-\xi_1\right) \\ +\left(\overline{\Gamma_1}-\Pi_{iL,1}+\Pi_{oL,1}-\Pi_{0,1}\right)+\left(1-2\nu\right)\overline{\Pi}_1 \end{array}\right)\int_0^1 \left(\Phi\Psi''\Phi'+\Phi\Psi'\Phi''\right)d\xi_1 \\ &\quad +\frac{\mathrm{E}\,u_{o,1}{}^2\left(c_N+c_T\right)}{2}\int_0^1 \Phi\Psi'\Phi'\,d\xi_1+\frac{\mathrm{E}\,c_T\,u_{o,1}{}^2\left(1-\overline{h}\right)}{2}\int_0^1 \Phi\Psi\Phi''\,d\xi_1-3\int_0^1 \Phi\Psi'''\Phi''\,d\xi_1 \\ &\quad -4\int_0^1 \Phi\Psi''\Phi'''\,d\xi_1-2\int_0^1 \Phi\Psi'\Phi^{(4)}\,d\xi_1-\int_0^1 \Phi\Psi^{(4)}\Phi'\,d\xi_1-\chi\,u_{o,1}{}^2\int_0^1 \left(2\Phi\Psi''\Phi'+4\Phi\Psi'\Phi''\right)d\xi_1 \end{aligned} \tag{6.27}$$

$$\begin{aligned} D_2{}^v &= -\alpha_1\left(3\int_0^1 \Phi\Psi'''\Phi''\,d\xi_1+4\int_0^1 \Phi\Psi''\Phi'''\,d\xi_1+2\int_0^1 \Phi\Psi'\Phi^{(4)}\,d\xi_1+\int_0^1 \Phi\Psi^{(4)}\Phi'\,d\xi_1\right) \\ &\quad +3\chi\,u_{o,1}\sqrt{\beta_{o,1}}\int_0^1 \Phi\Psi'\Phi'\,d\xi_1+2\chi\,u_{o,1}\sqrt{\beta_{o,1}}\int_0^1 \Phi\Psi\Phi''\,d\xi_1-\Pi_{0,1}\alpha_1\int_0^1 \left(\Phi\Psi''\Phi'+\Phi\Psi'\Phi''\right)d\xi_1 \end{aligned} \tag{6.28}$$

$$D_3^{\,v} = -\alpha_1\left(3\int_0^1 \Phi\Psi'''\Phi''\,d\xi_1 + 4\int_0^1 \Phi\Psi''\Phi'''\,d\xi_1 + 2\int_0^1 \Phi\Psi'\Phi^{(4)}\,d\xi_1 + \int_0^1 \Phi\Psi^{(4)}\Phi'\,d\xi_1\right)$$
$$-\Pi_{0,1}\alpha_1\int_0^1 \left(\Phi\Psi''\Phi' + \Phi\Psi'\Phi''\right)d\xi_1 + \frac{\mathrm{E}\,c_N\,u_{o,1}\sqrt{\beta_{o,1}}}{2}\int_0^1 \Phi^2\Psi'\,d\xi_1 \tag{6.29}$$
$$+4\chi\,u_{o,1}\sqrt{\beta_{o,1}}\int_0^1 \Phi\Psi'\Phi'\,d\xi_1$$

$$D_4^{\,v} = \frac{\mathrm{E}\,c_N}{2}\beta_{o,1}\int_0^1 \Phi^2\Psi\,d\xi_1 - 2\chi\beta_{o,1}\int_0^1 \Phi\Psi\,\Phi' d\xi_1 \tag{6.30}$$

$$D_5^{\,v} = -\chi\,\beta_{o,1}\int_0^1 \Phi\Psi\Phi'\,d\xi_1 \tag{6.31}$$

$$E_1^{\,v} = -4\alpha_1\int_0^1 \Phi\Phi'\Phi^{(4)}\,d\xi_1 - 3\alpha_1\Pi_{0,1}\int_0^1 \Phi\Phi'\Phi''\,d\xi_1 \tag{6.32}$$

$$E_2^{\,v} = -\frac{\mathrm{E}\,c_d\,u_{o,1}^{\;2}}{2}\int_0^1 \Phi\Phi'\left|\Phi'\right|d\xi_1 \tag{6.33}$$

$$E_3^{\,v} = \frac{\mathrm{E}\,c_d\,u_{o,1}}{2}\sqrt{\beta_{o,1}}\int_0^1 \Phi\left|\Phi'\right|\Phi\,d\xi_1 \tag{6.34}$$

$$E_4^{\,v} = \frac{\mathrm{E}\,c_d\,u_{o,1}}{2}\sqrt{\beta_{o,1}}\int_0^1 \Phi\Phi'\left|\Phi\right|d\xi_1 \tag{6.35}$$

$$E_5^{\,v} = \frac{\mathrm{E}\,c_d}{2}\beta_{o,1}\int_0^1 \Phi^2\left|\Phi\right|\,d\xi_1 \tag{6.36}$$

$$F_1^{\,v} = \frac{3}{2}\begin{pmatrix}\dfrac{\mathrm{E}\,c_T\,u_{o,1}^{\;2}}{2}\left(1-\bar{h}\right)\left(\dfrac{1}{2}-\xi_1\right) + \left(1-2\nu\right)\overline{\Pi}_1 \\ +\left(\overline{\Gamma}_1 - \Pi_{iL,1} + \Pi_{oL,1} - \Pi_{0,1}\right)\end{pmatrix}\int_0^1 \Phi\Phi'^2\Phi''\,d\xi_1$$
$$-\frac{5}{2}\chi\,u_{o,1}^{\;2}\int_0^1 \Phi\Phi''\Phi'^2 d\xi_1 + \left(\frac{\mathrm{E}\,c_T\,u_{o,1}^{\;2}}{4}\left(\bar{h}-1\right) + \frac{\mathrm{E}\,c_N\,u_{o,1}^{\;2}}{2}\right)\int_0^1 \Phi\left(\Phi'\right)^3 d\xi_1 \tag{6.37}$$
$$-2\int_0^1 \Phi\Phi'^2\Phi^{(4)}d\xi_1 - 8\int_0^1 \Phi\Phi'\Phi''\Phi'''d\xi_1 - 2\int_0^1 \Phi\left(\Phi''\right)^3 d\xi_1$$

$$F_2^{\,v} = -\alpha_1\left(2\int_0^1 \Phi\Phi'^2\Phi^{(4)}d\xi_1 + 24\int_0^1 \Phi\Phi'\Phi''\Phi'''d\xi_1 + 6\int_0^1 \Phi\left(\Phi''\right)^3 d\xi_1\right)$$
$$-\frac{3}{2}\Pi_{0,1}\alpha_1\int_0^1 \Phi\Phi'^2\Phi''\,d\xi_1 + \frac{\mathrm{E}\,u_{o,1}}{2}\sqrt{\beta_{o,1}}\left(c_T - c_N\right)\int_0^1 \Phi^2\Phi'^2 d\xi_1 \tag{6.38}$$
$$+\chi\,u_{o,1}\sqrt{\beta_{o,1}}\left(\frac{7}{2}\int_0^1 \Phi\left(\Phi'\right)^3 d\xi_1 + \frac{3}{2}\int_0^1 \Phi^2\,\Phi'\Phi''\,d\xi_1\right)$$

$$F_3^{\,v} = \frac{\mathrm{E}\,\beta_{o,1}}{4}\left(c_T - c_N\right)\int_0^1 \Phi^3\Phi' d\xi_1 - \frac{3}{2}\chi\beta_{o,1}\int_0^1 \Phi^2\Phi'^2 d\xi_1 \tag{6.39}$$

$$F_4^{\,v} = -\frac{\mathrm{E}\,c_N}{4u_{o,1}}\beta_{o,1}^{\frac{3}{2}}\int_0^1 \Phi^4\,d\xi_1 \tag{6.40}$$

$$F_5^{\,v} = -\chi\,\beta_{o,1}\int_0^1 \Phi^2\Phi'^2\,d\xi_1 \tag{6.41}$$

6.1.2 Inextensible rotating flexible pipe conveying fluid and subjected to external axial flow.

In this subsection, the dimensionless equations of motion (5.43-5.45), which describe the vibration of a cantilevered pipe conveying fluid and subjected to external axial flow, are discretized via unimodal Galerkin's method. Following the the same procedure described in subsection 6.1.1, and after tedious mathematical manipulations, we obtain the following coupled ordinary nonlinear differential equations in terms of the modal coordinates *q, r* and *ǧ*:

$$\left(\begin{array}{l}\left[M_2{}^v\right]\ddot{q}+\left[C_2{}^v\right]\dot{q}+\left[K_2{}^v\right]q+\left[G_1{}^v\right]\dot{\breve{g}}\,r+\left[G_2{}^v\right]\dot{\breve{g}}\,\dot{r}+\left[H_1{}^v\right]q\,|q|+\left[H_2{}^v\right]\dot{q}\,|q| \\ +\left[H_3{}^v\right]q\,|\dot{q}|+\left[H_4{}^v\right]\dot{q}\,|\dot{q}|+\left[I_1{}^v\right]q\,r^2+\left[I_2{}^v\right]\dot{q}\,r^2+\left[I_3{}^v\right]q\,\dot{r}^2+\left[I_4{}^v\right]q\,r\,\dot{r}+\left[I_5{}^v\right]q\,\ddot{r}\,r \\ +\left[I_6{}^v\right]\dot{q}r\,\dot{r}+\left[I_7{}^v\right]q\,\dot{\breve{g}}^2+\left[L_1{}^v\right]q^3+\left[L_2{}^v\right]q^2\,\dot{q}+\left[L_3{}^v\right]q\,\dot{q}^2+\left[L_4{}^v\right]\dot{q}^3+\left[L_5{}^v\right]q^2\,\ddot{q}\end{array}\right)=0$$

(6.42)

$$\left(\begin{array}{l}\left[M_2{}^w\right]\ddot{r}+\left[C_2{}^w\right]\dot{r}+\left[K_2{}^w\right]r+\left[G_1{}^w\right]\dot{\breve{g}}\,q+\left[G_2{}^w\right]\dot{\breve{g}}\,\dot{q}+\left[G_3{}^w\right]\ddot{\breve{g}}\,q+\left[H_1{}^w\right]r\,|r| \\ +\left[H_2{}^w\right]\dot{r}\,|r|+\left[H_3{}^w\right]r\,|\dot{r}|+\left[H_4{}^w\right]\dot{r}\,|\dot{r}|+\left[I_1{}^w\right]r\,q^2+\left[I_2{}^w\right]\dot{r}\,q^2+\left[I_3{}^w\right]r\,\dot{q}^2 \\ +\left[I_4{}^w\right]r\,q\,\dot{q}+\left[I_5{}^w\right]r\,\ddot{q}\,q+\left[I_6{}^w\right]\dot{r}\,q\,\dot{q}+\left[I_7{}^w\right]r\,\dot{\breve{g}}^2+\left[I_8{}^w\right]\ddot{r}\,q^2+\left[I_9{}^w\right]\dfrac{\ddot{r}\,\dot{\breve{g}}\,q}{\dot{r}} \\ +\left[L_1{}^w\right]r^3+\left[L_2{}^w\right]r^2\,\dot{r}+\left[L_3{}^w\right]r\,\dot{r}^2+\left[L_4{}^w\right]\dot{r}^3+\left[L_5{}^w\right]r^2\,\ddot{r}\end{array}\right)=0 \qquad (6.43)$$

and,

$$\left[M_2{}^\varphi\right]\ddot{\breve{g}}+\left[K_2{}^\varphi\right]\breve{g}+\left[T_1{}^\varphi\right]\dot{\breve{g}}^2+\left[T_2{}^\varphi\right]\ddot{\breve{g}}\,\breve{g}+\left[T_3{}^\varphi\right]\dot{r}\,\dot{q}+\left[T_4{}^\varphi\right]q\,\ddot{r}=0 \qquad (6.44)$$

where the values of coefficients are found to be:

$$M_2{}^v=\left(1-\beta_{o,2}-\beta_{i,2}-\beta_{i,3}+\chi\,\beta_{o,2}\right)\int_0^1\Phi^{\,2}d\xi_2+2\beta_{i,2}\int_0^1\Phi\,\Phi''\,d\xi_2 \qquad (6.45)$$

$$\begin{aligned}C_2{}^v=&\left(2u_{i,2}\sqrt{\beta_{i,1}}-2\chi\,u_{o,2}\sqrt{\beta_{o,2}}\right)\int_0^1\Phi\Phi'd\xi_2+\alpha_2\beta_{m,1}\int_0^1\Phi\Phi^{(4)}d\xi_2 \\ &+\left(\frac{\mathrm{E}\,c_N\,u_{o,2}}{2}\sqrt{\beta_{o,2}}+\chi\,\beta_{o,2}\,f_t\right)\int_0^1\Phi^{\,2}d\xi_2\end{aligned} \qquad (6.46)$$

$$K_2^{\,v} = \left(u_{i,2}^{\;2} - \Pi_{oL,2} + \chi\, u_{o,2}^{\;2} - \left(\frac{\mathrm{E}\, c_T\, u_{o,2}^{\;2}}{2} \left(1-\bar{h}\right) + \left(\gamma_1 - \gamma_2\right) \right) \left(1-\xi_2\right) \right) \int_0^1 \Phi\Phi''\, d\xi_2$$
$$- \left(\frac{\mathrm{E}\left(c_N + c_T\,\bar{h}\right) u_{o,2}^{\;2}}{2} - \left(\gamma_1 - \gamma_2\right) \right) \int_0^1 \Phi\Phi'\, d\xi_2 + \beta_{m,1} \int_0^1 \Phi\Phi^{(4)}\, d\xi_2 \tag{6.47}$$

$$G_1^{\,v} = \frac{\chi\, \beta_{o,2}\, f_t}{2} \int_0^1 \Phi\,\Theta\,\Xi\, d\xi_2 \tag{6.48}$$

$$G_2^{\,v} = \beta_{i,3} \left(\int_0^1 \Phi\,\Theta''\,\Xi\, d\xi_2 + \int_0^1 \Phi\,\Theta'\,\Xi'\, d\xi_2 \right) + \chi\, \beta_{o,2} \int_0^1 \Phi\,\Theta\,\Xi\, d\xi_2 \tag{6.49}$$

$$H_1^{\,v} = -\frac{\mathrm{E}\, c_d\, u_{o,2}^{\;2}}{2} \int_0^1 \Phi\Phi' \left|\Phi'\right| d\xi_2 \tag{6.50}$$

$$H_2^{\,v} = \frac{\mathrm{E}\, c_d\, u_{o,2}}{2} \sqrt{\beta_{o,2}} \int_0^1 \Phi \left|\Phi'\right| \Phi\, d\xi_2 \tag{6.51}$$

$$H_3^{\,v} = \frac{\mathrm{E}\, c_d\, u_{o,2}}{2} \sqrt{\beta_{o,2}} \int_0^1 \Phi\Phi' \left|\Phi\right| d\xi_2 \tag{6.52}$$

$$H_4^{\,v} = \frac{\mathrm{E}\, c_d}{2} \beta_{o,2} \int_0^1 \Phi^2 \left|\Phi\right|\, d\xi_2 \tag{6.53}$$

$$\begin{aligned}
I_1^{\,v} = {} & u_{i,2}^{\;2} \left(\int_0^1 \Phi\,\Theta'\Theta''\Phi'\, d\xi_2 - \int_0^1 \int_{\xi_2}^1 \Phi\Phi''\Theta'\Theta''\, d\xi_2\, d\xi_2 \right) \\
& + \left(\Pi_{oL,2} - \chi\, u_{o,2}^{\;2} \right) \int_0^1 \int_{\xi_2}^1 \Phi\Phi''\Theta'\Theta''\, d\xi_2\, d\xi_2 \\
& + \beta_{m,1} \left(\int_0^1 \Phi\Phi''\Theta''^{\,2}\, d\xi_2 + \int_0^1 \Phi\Phi''\Theta'\Theta'''\, d\xi_2 + 3\int_0^1 \Phi\Phi'\Theta''\Theta'''\, d\xi_2 + \int_0^1 \Phi\Phi'\Theta'\Theta^{(4)}\, d\xi_2 \right) \\
& + \frac{\gamma_1}{2} \left(\int_0^1 \Phi\,\Phi'\Theta'^{\,2}\, d\xi_2 - \left(1-\xi_2\right) \left(\int_0^1 \Phi\,\Phi''\Theta'^{\,2}\, d\xi_2 + 2\int_0^1 \Phi\,\Phi'\Theta'\Theta''\, d\xi_2 \right) \right) \\
& + \frac{\mathrm{E}\, u_{o,2}^{\;2} \left(c_T - c_N\right)}{2} \left(\int_0^1 \Phi\Phi'\Theta'^{\,2}\, d\xi_2 - \int_0^1 \int_{\xi_2}^1 \Phi\Phi''\Theta'^{\,2}\, d\xi_2\, d\xi_2 \right)
\end{aligned} \tag{6.54}$$

$$I_2^{\,v} = \beta_{m,1}\alpha_2\left(\left(\int_0^1 \Phi\Phi''\Theta''^2 d\xi_2 + \int_0^1 \Phi\Phi''\Theta'\Theta''' d\xi_2 + 3\int_0^1 \Phi\Phi'\Theta''\Theta''' d\xi_2 + \int_0^1 \Phi\Phi'\Theta'\Theta^{(4)} d\xi_2\right)\right) \tag{6.55}$$

$$\begin{aligned} I_3^{\,v} = {} & \left(1-\beta_{o,2}-\beta_{i,2}-\beta_{i,3}+\chi\,\beta_{o,2}\right)\left(\int_0^1\int_0^{\xi_2} \Phi\Phi'\Theta'^2 d\xi_2\, d\xi_2\right) \\ & -\frac{\beta_{o,2}\,\mathrm{E}\,c_T}{4}\int_0^1 \Phi\,\Phi'\Theta^2 d\xi_2 - \left(1-\beta_{o,2}-\beta_{i,2}-\beta_{i,3}\right)\left(\int_0^1\int_0^{\xi_2}\int_{\xi_2}^1 \Phi\Phi''\Theta'^2 d\xi_2 d\xi_2 d\xi_2\right) \\ & +\frac{\beta_{o,2}\,\mathrm{E}}{4}c_T\int_0^1\int_{\xi_2}^1 \Phi\,\Phi''\Theta^2 d\xi_2 d\xi_2 + \beta_{i,3}\left(\int_0^1 \Phi\,\Phi''\,\Theta'^2 d\xi_2 + 2\int_0^1 \Phi\,\Phi'\,\Theta'\Theta'' d\xi_2\right) \end{aligned} \tag{6.56}$$

$$\begin{aligned} I_4^{\,v} = {} & 2\sqrt{\beta_{i,1}}\,u_{i,2}\left(\int_0^1 \Phi\,\Phi'\Theta'^2 d\xi_2 - \int_0^1\int_{\xi_2}^1 \Phi\Phi''\Theta'^2 d\xi_2\, d\xi_2\right) \\ & +2\chi\sqrt{\beta_{o,2}}\,u_{o,2}\left(\int_0^1\int_{\xi_2}^1 \Phi\Phi''\Theta'^2 d\xi_2\, d\xi_2 - \int_0^1\int_0^{\xi_2} \Phi\Phi''\Theta'^2 d\xi_2\, d\xi_2\right) \\ & +\beta_{m,1}\,\alpha_2\left(\begin{array}{l} \int_0^1 \Phi\,\Phi''\Theta'\Theta''' d\xi_2 + 6\int_0^1 \Phi\,\Phi'\Theta''\Theta''' d\xi_2 + 2\int_0^1 \Phi\,\Phi'\Theta'\Theta^{(4)} d\xi_2 \\ +2\int_0^1 \Phi\,\Phi''\Theta''^2 d\xi_2 + \int_0^1 \Phi\,\Phi''\Theta'\Theta^{(4)} d\xi_2 \end{array}\right) \\ & +\frac{u_{o,2}\,\mathrm{E}}{4}\sqrt{\beta_{o,2}}\left(c_N - c_T\right)\left(\int_0^1 \Phi\,\Phi'\Theta\Theta' d\xi_2 - \int_0^1\int_{\xi_2}^1 \Phi\Phi''\Theta\Theta' d\xi_2\, d\xi_2\right) \end{aligned} \tag{6.57}$$

$$\begin{aligned} I_5^{\,v} = {} & \left(1-\beta_{o,2}-\beta_{i,2}-\beta_{i,3}+\chi\,\beta_{o,2}\right)\left(\int_0^1\int_0^{\xi_2} \Phi\Phi'\Theta'^2 d\xi_2\, d\xi_2\right) \\ & -\left(1-\beta_{o,2}-\beta_{i,2}-\beta_{i,3}\right)\int_0^1\int_0^{\xi_2}\int_{\xi_2}^1 \Phi\,\Phi''\Theta'^2 d\xi_2\, d\xi_2 d\xi_2 \\ & -\chi\,\beta_{o,2}\int_0^1\int_{\xi_2}^1 \Phi\Phi''\Theta\,\Theta' d\xi_2\, d\xi_2 \end{aligned} \tag{6.58}$$

$$I_6^{\,v} = 2\chi\,\beta_{o,2}\int_0^1\int_0^{\xi_2} \Phi\Phi'\Theta'^2 d\xi_2\, d\xi_2 - \frac{\mathrm{E}\,c_N}{2}\beta_{o,2}\int_0^1\int_0^{\xi_2} \Phi^2\Theta'^2 d\xi_2\, d\xi_2 \tag{6.59}$$

$$I_7{}^{v} = -\frac{\chi\,\beta_{o,2}}{4}\int_0^1 \Phi^2\,\Xi^2 d\xi_2 \tag{6.60}$$

$$\begin{aligned}
L_1{}^{v} =& -\frac{3}{2}\left(\left(\frac{\mathrm{E}\,c_T\,u_{o,2}{}^2}{2}\left(\bar{h}+1\right)+\gamma_1+\gamma_2\right)\left(1-\xi_2\right)-\Pi_{oL,2}-\frac{2}{3}u_{i,2}{}^2+\frac{5}{3}\chi\,u_{o,2}{}^2\right)\int_0^1 \Phi\Phi'^2\Phi'' d\xi_2 \\
&+\frac{1}{2}\left(\frac{\mathrm{E}\,c_T\,u_{o,2}{}^2}{2}\left(\bar{h}+3\right)+\gamma_1+\gamma_2\right)\int_0^1 \Phi\left(\Phi'\right)^3 d\xi_2+\left(\frac{\mathrm{E}\,u_{o,2}{}^2}{2}\left(c_N-c_T\right)\right)\int_0^1\int_{\xi_2}^1 \Phi\Phi'^2\Phi''\,d\xi_2 \\
&+\left(\Pi_{oL,2}-\chi\,u_{o,2}{}^2-u_{i,2}{}^2\right)\int_0^1\int_{\xi_2}^1 \Phi\Phi''^2\Phi'\,d\xi_2 \\
&+\beta_{m,1}\left(\int_0^1 \Phi\left(\Phi''\right)^3 d\xi_2+4\int_0^1 \Phi\,\Phi'\Phi''\Phi'''d\xi_2+\int_0^1 \Phi\,\Phi'^2\Phi^{(4)}d\xi_2\right)
\end{aligned} \tag{6.61}$$

$$\begin{aligned}
L_2{}^{v} =& \left(-2\sqrt{\beta_{i,2}}u_{i,2}+2\chi\sqrt{\beta_{o,2}}u_{o,2}\right)\int_0^1\int_{\xi_2}^1 \Phi\,\Phi'^2\Phi''d\xi_2 d\xi_2 \\
&+\beta_{m,1}\alpha_2\left(3\int_0^1 \Phi\left(\Phi''\right)^3 d\xi_2+3\int_0^1 \Phi\Phi'^2\Phi^{(4)}d\xi_2+12\int_0^1 \Phi\Phi'\Phi''\Phi'''d\xi_2\right) \\
&+\left(2\sqrt{\beta_{i,2}}u_{i,2}+\frac{7}{2}\chi\sqrt{\beta_{o,2}}u_{o,2}\right)\int_0^1 \Phi\;\Phi'^3 d\xi_2 \\
&-\frac{\mathrm{E}\,\left(c_N-c_T\right)u_{o,2}}{2}\sqrt{\beta_{o,2}}\int_0^1\int_{\xi_2}^1 \Phi^2\,\Phi'\;\Phi''d\xi_2 d\xi_2 \\
&+\chi\sqrt{\beta_{o,2}}u_{o,2}\left(\frac{3}{2}\int_0^1 \Phi^2\Phi'\;\Phi''d\xi_2-2\int_0^1\int_0^{\xi_2} \Phi\,\Phi'^2\Phi''d\xi_2 d\xi_2\right)
\end{aligned} \tag{6.62}$$

$$\begin{aligned}
L_3{}^{v} =& \frac{\mathrm{E}\,c_T}{4}\beta_{o,2}\int_0^1\int_{\xi_2}^1 \Phi^3\,\Phi''d\xi_2 d\xi_2-\frac{\mathrm{E}\,c_N}{4}\beta_{o,2}\left(\int_0^1\int_0^{\xi_2} \Phi^2\,\Phi'^2 d\xi_2 d\xi_2-\int_0^1 \Phi^3\Phi'\,d\xi_2\right) \\
&+\left(1-\beta_{o,2}-\beta_{i,2}-\beta_{i,3}+3\chi\,\beta_{o,2}\right)\int_0^1\int_0^{\xi_2} \Phi\,\Phi'^3 d\xi_2 d\xi_2 \\
&-\left(1-\beta_{o,2}-\beta_{i,2}-\beta_{i,3}\right)\int_0^1\int_0^{\xi_2}\int_{\xi_2}^1 \Phi\,\Phi'^2\Phi''d\xi_2 d\xi_2 d\xi_2-\frac{3}{2}\chi\,\beta_{o,2}\int_0^1 \Phi^2\,\Phi'^2 d\xi_2
\end{aligned} \tag{6.63}$$

$$L_4{}^{v} = -\frac{\mathrm{E}\,c_N}{4u_{o,2}}\beta_{o,2}{}^{\frac{3}{2}}\int_0^1 \Phi^4 d\xi_2 \tag{6.64}$$

$$L_5{}^{v} = \left(1-\beta_{o,2}-\beta_{i,2}-\beta_{i,3}+\chi\,\beta_{o,2}\right)\int_0^1\int_0^{\xi_2} \Phi\,\Phi'^{3} d\xi_2 d\xi_2 \\ -\left(1-\beta_{o,2}-\beta_{i,2}-\beta_{i,3}\right)\int_0^1\int_0^{\xi_2}\int_{\xi_2}^1 \Phi\,\Phi'^{2}\Phi'' d\xi_2 d\xi_2 d\xi_2 - \chi\,\beta_{o,2}\int_0^1\int_{\xi_2}^1 \Phi^2\,\Phi'\Phi'' d\xi_2 \tag{6.65}$$

$$M_2{}^{w} = \left(1-\beta_{o,2}-\beta_{i,2}-\beta_{i,3}+\chi\,\beta_{o,2}\right)\int_0^1 \Theta^{\,2}\, d\xi_2 + 2\beta_{i,2}\int_0^1 \Theta\,\Theta''\, d\xi_2 \tag{6.66}$$

$$C_2{}^{w} = \left(2u_{i,2}\sqrt{\beta_{i,1}} - 2\chi\,u_{o,2}\sqrt{\beta_{o,2}}\right)\int_0^1 \Theta\Theta' d\xi_2 + \alpha_2\beta_{m,1}\int_0^1 \Theta\Theta^{(4)} d\xi_2 \\ +\left(\frac{\mathrm{E}\,c_N\,u_{o,2}}{2}\sqrt{\beta_{o,2}} + \chi\,\beta_{o,2}\,f_t\right)\int_0^1 \Theta^{\,2}\, d\xi_2 \tag{6.67}$$

$$K_2{}^{w} = \left(u_{i,2}{}^{2} - \Pi_{oL,2} + \chi\,u_{o,2}{}^{2} - \left(\frac{\mathrm{E}\,c_T\,u_{o,2}{}^{2}}{2}\left(1-\bar{h}\right) + \left(\gamma_1-\gamma_2\right)\right)\left(1-\xi_2\right)\right)\int_0^1 \Theta\Theta''\, d\xi_2 \\ -\left(\frac{\mathrm{E}\left(c_N + c_T\,\bar{h}\right)u_{o,2}{}^{2}}{2} - \left(\gamma_1-\gamma_2\right)\right)\int_0^1 \Theta\Theta'\, d\xi_2 + \beta_{m,1}\int_0^1 \Theta\Theta^{(4)}\, d\xi_2 \tag{6.68}$$

$$G_1{}^{w} = \frac{\chi\,\beta_{o,2}\,f_t}{2}\int_0^1 \Phi\,\Theta\,\Xi\, d\xi_2 \tag{6.69}$$

$$G_2{}^{w} = \beta_{i,3}\left(\int_0^1 \frac{\Theta\Xi\Phi'\Theta''}{\Theta'} d\xi_2 + \int_0^1 \Theta\,\Xi\Phi'' d\xi_2 + \int_0^1 \Theta\,\Xi'\Phi' d\xi_2\right) + \chi\,\beta_{o,2}\int_0^1 \Phi\,\Theta\,\Xi\, d\xi_2 \tag{6.70}$$

$$G_3{}^{w} = \beta_{i,3}\left(\int_0^1 \frac{\Theta\Xi\Phi'\Theta''}{\Theta'} d\xi_2 + \int_0^1 \Theta\,\Xi\Phi'' d\xi_2 + \int_0^1 \Theta\,\Xi'\Phi' d\xi_2\right) \tag{6.71}$$

$$H_1{}^{w} = -\frac{\mathrm{E}\,c_d\,u_{o,2}{}^{2}}{2}\int_0^1 \Theta\,\Theta'\left|\Theta'\right| d\xi_2 \tag{6.72}$$

$$H_2{}^{w} = \frac{\mathrm{E}\,c_d\,u_{o,2}}{2}\sqrt{\beta_{o,2}}\int_0^1 \Theta\left|\Theta'\right|\Theta\, d\xi_2 \tag{6.73}$$

$$H_3{}^{w} = \frac{\mathrm{E}\,c_d\,u_{o,2}}{2}\sqrt{\beta_{o,2}}\int_0^1 \Theta\Theta'\left|\Theta\right| d\xi_2 \tag{6.74}$$

$$H_4{}^{w} = \frac{\mathrm{E}\, c_d}{2} \beta_{o,2} \int_0^1 \Theta^2 \left|\Theta\right| \, d\xi_2 \tag{6.75}$$

$$\begin{aligned}
I_1{}^{w} = {} & u_{i,2}{}^2 \left(\int_0^1 \Theta\, \Phi' \Phi'' \Theta' d\xi_2 - \int_0^1 \int_{\xi_2}^1 \Theta \Theta'' \Phi' \Phi'' d\xi_2 d\xi_2 \right) \\
& + \left(\Pi_{oL,2} - \chi\, u_{o,2}{}^2 \right) \int_0^1 \int_{\xi_2}^1 \Phi' \Phi'' \Theta\, \Theta'' d\xi_2 d\xi_2 \\
& + \beta_{m,1} \left(\int_0^1 \Theta \Theta'' \Phi''^2 d\xi_2 + \int_0^1 \Theta \Theta'' \Phi' \Phi''' d\xi_2 + 3 \int_0^1 \Theta \Theta' \Phi'' \Phi''' d\xi_2 + \int_0^1 \Phi^{(4)} \Phi' \Theta' \Theta d\xi_2 \right) \\
& + \frac{\gamma_1}{2} \left(\int_0^1 \Theta \Theta' \Phi'^2 d\xi_2 - \left(1 - \xi_2\right) \left(\int_0^1 \Theta\, \Theta'' \Phi'^2 d\xi_2 + 2 \int_0^1 \Theta\, \Theta' \Phi' \Phi'' d\xi_2 \right) \right) \\
& + \frac{\mathrm{E}\, u_{o,2}{}^2 \left(c_T - c_N\right)}{2} \left(\int_0^1 \Theta \Theta' \Phi'^2 d\xi_2 - \int_0^1 \int_{\xi_2}^1 \Theta \Theta'' \Phi'^2 d\xi_2 d\xi_2 \right)
\end{aligned} \tag{6.76}$$

$$I_2{}^{w} = \beta_{m,1} \alpha_2 \left(\left(\int_0^1 \Theta \Theta'' \Phi''^2 d\xi_2 + \int_0^1 \Theta \Theta'' \Phi' \Phi''' d\xi_2 + 3 \int_0^1 \Theta \Theta' \Phi'' \Phi''' d\xi_2 + \int_0^1 \Theta \Theta' \Phi' \Phi^{(4)} d\xi_2 \right) \right) \tag{6.77}$$

$$\begin{aligned}
I_3{}^{w} = {} & \left(1 - \beta_{o,2} - \beta_{i,2} - \beta_{i,3} + \chi\, \beta_{o,2}\right) \left(\int_0^1 \int_0^{\xi_2} \Theta \Theta' \Phi'^2 d\xi_2\, d\xi_2 \right) \\
& - \frac{\beta_{o,2}\, \mathrm{E}\, c_T}{4} \int_0^1 \Theta\, \Theta' \Phi^2 d\xi_2 - \left(1 - \beta_{o,2} - \beta_{i,2} - \beta_{i,3}\right) \left(\int_0^1 \int_0^{\xi_2} \int_{\xi_2}^1 \Theta \Theta'' \Phi'^2 d\xi_2 d\xi_2 d\xi_2 \right) \\
& + \frac{\beta_{o,2}\, \mathrm{E}}{4} c_T \int_0^1 \int_{\xi_2}^1 \Theta \Theta'' \Phi^2 d\xi_2 d\xi_2
\end{aligned} \tag{6.78}$$

$$
\begin{aligned}
I_4{}^w &= 2\sqrt{\beta_{i,1}}\,u_{i,2}\left(\int_0^1 \Theta\,\Theta'\Phi'^2 d\xi_2 - \int_0^1\int_{\xi_2}^1 \Theta\,\Theta''\Phi'^2 d\xi_2\,d\xi_2\right) \\
&\quad + 2\chi\sqrt{\beta_{o,2}}\,u_{o,2}\left(\int_0^1\int_{\xi_2}^1 \Theta\,\Theta''\Phi'^2 d\xi_2\,d\xi_2 - \int_0^1\int_0^{\xi_2} \Theta\,\Theta''\Phi'^2 d\xi_2\,d\xi_2\right) \\
&\quad + \beta_{m,1}\,\alpha_2\left(\begin{array}{l}\int_0^1 \Theta\,\Theta''\Phi'\Phi''' d\xi_2 + 6\int_0^1 \Theta\Theta'\Phi''\Phi''' d\xi_2 + 2\int_0^1 \Theta\Theta'\Phi'\Phi^{(4)} d\xi_2 \\ + 2\int_0^1 \Theta\,\Theta''\Phi''^2 d\xi_2 + \int_0^1 \Theta\,\Theta''\Phi'\Phi^{(4)} d\xi_2\end{array}\right) \\
&\quad + \frac{u_{o,2}\,\mathrm{E}}{4}\sqrt{\beta_{o,2}}\,(c_N - c_T)\left(\int_0^1 \Phi\,\Phi'\Theta\Theta' d\xi_2 - \int_0^1\int_{\xi_2}^1 \Theta\Theta''\Phi\Phi' d\xi_2\,d\xi_2\right)
\end{aligned}
\tag{6.79}
$$

$$
\begin{aligned}
I_5{}^w &= \left(1 - \beta_{o,2} - \beta_{i,2} - \beta_{i,3} + \chi\,\beta_{o,2}\right)\left(\int_0^1\int_0^{\xi_2} \Theta\Theta'\Phi'^2 d\xi_2\,d\xi_2\right) \\
&\quad - \left(1 - \beta_{o,2} - \beta_{i,2} - \beta_{i,3}\right)\int_0^1\int_0^{\xi_2}\int_{\xi_2}^1 \Theta\Theta''\Phi'^2 d\xi_2\,d\xi_2 d\xi_2 \\
&\quad - \chi\,\beta_{o,2}\int_0^1\int_{\xi_2}^1 \Theta\Theta''\Phi\Phi' d\xi_2\,d\xi_2
\end{aligned}
\tag{6.80}
$$

$$
\begin{aligned}
I_6{}^w &= 2\chi\,\beta_{o,2}\int_0^1\int_0^{\xi_2} \Theta\Theta'\Phi'^2 d\xi_2\,d\xi_2 - \frac{\mathrm{E}\,c_N}{2}\beta_{o,2}\int_0^1\int_0^{\xi_2} \Theta^2\Phi'^2 d\xi_2\,d\xi_2 \\
&\quad + 2\,\beta_{i,3}\int_0^1 \Theta\Theta'\Phi'\Phi'' d\xi_2 + 2\beta_{i,3}\int_0^1 \Theta\,\Theta''\Phi'^2 d\xi_2
\end{aligned}
\tag{6.81}
$$

$$
I_7{}^w = -\frac{\chi\,\beta_{o,2}}{4}\int_0^1 \Theta^2\,\Xi^2 d\xi_2 \tag{6.82}
$$

$$
I_8{}^w = \beta_{i,3}\left(2\int_0^1 \Theta\,\Phi'^2\Theta'' d\xi_2 + 3\int_0^1 \Theta\,\Theta'\Phi'\Phi'' d\xi_2\right) \tag{6.83}
$$

$$
I_9{}^w = \beta_{i,3}\left(\int_0^1 \Theta\,\Xi\,\Phi' d\xi_2 + \int_0^1 \Theta\,\Xi'\,\Phi' d\xi_2 + \int_0^1 \Theta\,\Xi\,\Phi'' d\xi_2 - \int_0^1 \frac{\Theta\Xi\,\Theta''\Phi'}{\Theta'} d\xi_2\right) \tag{6.84}
$$

$$
\begin{aligned}
L_1^{\,w} = & -\frac{3}{2}\left(\begin{array}{l}\left(\dfrac{\mathrm{E}\,c_T\,u_{o,2}^{\;2}}{2}\left(\bar{h}+1\right)+\gamma_1+\gamma_2\right)\left(1-\xi_2\right)\\ -\Pi_{oL,2}-\dfrac{2}{3}u_{i,2}^{\;2}+\dfrac{5}{3}\chi\,u_{o,2}^{\;2}\end{array}\right)\int_0^1\Theta\,\Theta'^2\Theta''d\xi_2\\
& +\frac{1}{2}\left(\frac{\mathrm{E}\,c_T\,u_{o,2}^{\;2}}{2}\left(\bar{h}+3\right)+\gamma_1+\gamma_2\right)\int_0^1\Theta\left(\Theta'\right)^3d\xi_2\\
& +\left(\frac{\mathrm{E}\,u_{o,2}^{\;2}}{2}\left(c_N-c_T\right)\right)\int_0^1\int_{\xi_2}^1\Theta\,\Theta'^2\Theta''\,d\xi_2\\
& +\left(\Pi_{oL,2}-\chi\,u_{o,2}^{\;2}-u_{i,2}^{\;2}\right)\int_0^1\int_{\xi_2}^1\Theta\,\Theta''^2\Theta'\,d\xi_2\\
& +\beta_{m,1}\left(\int_0^1\Theta\left(\Theta''\right)^3d\xi_2+4\int_0^1\Theta\,\Theta'\Theta''\Theta'''d\xi_2+\int_0^1\Theta\,\Theta'^2\Theta^{(4)}d\xi_2\right)
\end{aligned}
\tag{6.85}
$$

$$
\begin{aligned}
L_2^{\,w} = & \left(-2\sqrt{\beta_{i,2}}u_{i,2}+2\chi\sqrt{\beta_{o,2}}u_{o,2}\right)\int_0^1\int_{\xi_2}^1\Theta\,\Theta'^2\Theta''d\xi_2d\xi_2\\
& +\beta_{m,1}\alpha_2\left(3\int_0^1\Theta\left(\Theta''\right)^3d\xi_2+3\int_0^1\Theta\,\Theta'^2\Theta^{(4)}d\xi_2+12\int_0^1\Theta\,\Theta'\Theta''\Theta'''d\xi_2\right)\\
& +\left(2\sqrt{\beta_{i,2}}u_{i,2}+\frac{7}{2}\chi\sqrt{\beta_{o,2}}u_{o,2}\right)\int_0^1\Theta\,\Theta'^3d\xi_2\\
& -\frac{\mathrm{E}\left(c_N-c_T\right)u_{o,2}}{2}\sqrt{\beta_{o,2}}\int_0^1\int_{\xi_2}^1\Theta^2\,\Theta'\Theta''d\xi_2d\xi_2\\
& +\chi\sqrt{\beta_{o,2}}u_{o,2}\left(\frac{3}{2}\int_0^1\Theta^2\Theta'\Theta''d\xi_2-2\int_0^1\int_0^{\xi_2}\Theta\,\Theta'^2\Theta''d\xi_2d\xi_2\right)
\end{aligned}
\tag{6.86}
$$

$$
\begin{aligned}
L_3^{\,w} = & \frac{\mathrm{E}\,c_T}{4}\beta_{o,2}\int_0^1\int_{\xi_2}^1\Theta^3\,\Theta''d\xi_2d\xi_2-\frac{\mathrm{E}\,c_N}{4}\beta_{o,2}\left(\int_0^1\int_0^{\xi_2}\Theta^2\,\Theta'^2d\xi_2d\xi_2-\int_0^1\Theta^3\Theta'd\xi_2\right)\\
& +\left(1-\beta_{o,2}-\beta_{i,2}-\beta_{i,3}+3\chi\,\beta_{o,2}\right)\int_0^1\int_0^{\xi_2}\Theta\,\Theta'^3d\xi_2d\xi_2\\
& -\left(1-\beta_{o,2}-\beta_{i,2}-\beta_{i,3}\right)\int_0^1\int_0^{\xi_2}\int_{\xi_2}^1\Theta\,\Theta'^2\Theta''d\xi_2d\xi_2d\xi_2-\frac{3}{2}\chi\,\beta_{o,2}\int_0^1\Theta^2\,\Theta'^2d\xi_2
\end{aligned}
\tag{6.87}
$$

$$
L_4^{\,w} = -\frac{\mathrm{E}\,c_N}{4u_{o,2}}\beta_{o,2}^{\frac{3}{2}}\int_0^1\Theta^4d\xi_2 \tag{6.88}
$$

$$L_5^{\ w} = \left(1-\beta_{o,2}-\beta_{i,2}-\beta_{i,3}+\chi\,\beta_{o,2}\right)\int_0^1\int_0^{\xi_2} \Theta\Theta'^3 d\xi_2 d\xi_2$$
$$-\left(1-\beta_{o,2}-\beta_{i,2}-\beta_{i,3}\right)\int_0^1\int_0^{\xi_2}\int_{\xi_2}^1 \Theta\Theta'^2\Theta'' d\xi_2 d\xi_2 d\xi_2 - \chi\,\beta_{o,2}\int_0^1\int_{\xi_2}^1 \Theta^2\,\Theta'\Theta'' d\xi_2 \tag{6.89}$$

$$M_2^{\ \varphi} = \beta_{i,3}\int_0^1 \Xi^{\ 2}\, d\xi_2 \tag{6.90}$$

$$K_2^{\ \varphi} = -\,\beta_{m,2}\int_0^1 \Xi\,\Xi''\, d\xi_2 \tag{6.91}$$

$$T_1^{\ \varphi} = 8\,\beta_{i,4}\, f_r \int_0^1 \Xi^2\,\Xi'\, d\xi_2 \tag{6.92}$$

$$T_2^{\ \varphi} = 4\,\beta_{i,4}\, f_r \int_0^1 \Xi^2\,\Xi'\, d\xi_2 \tag{6.93}$$

$$T_3^{\ \varphi} = \beta_{i,3}\int_0^1 \Xi\,\Theta'\Phi'\, d\xi_2 \tag{6.94}$$

$$T_4^{\ \varphi} = \beta_{i,3}\int_0^1 \Xi\,\Theta'\Phi'\, d\xi_2 \tag{6.95}$$

6.1.3 Inextensible rotating flexible pipe conveying fluid downwards, which then flows upwards as a confined annular flow.

In this subsection, the dimensionless equations of motion (5.50 -5.52), that describe the vibration of a cantilevered pipe, conveying fluid downwards, which then flows upwards as a confined flow, are discretized via unimodal Galerkin's method. Following similar procedure described in section 6.1.2, we obtain the following coupled ordinary nonlinear differential equations in terms of the modal coordinates *q, r* and *ǧ*:

$$\left(\begin{array}{l}\left[M_3{}^v\right]\ddot{q}+\left[C_3{}^v\right]\dot{q}+\left[K_3{}^v\right]q+\left[G_3{}^v\right]\dot{\breve{g}}\,r+\left[G_4{}^v\right]\dot{\breve{g}}\,\dot{r}+\left[H_5{}^v\right]q\,|q|+\left[H_6{}^v\right]\dot{q}\,|q| \\ +\left[H_7{}^v\right]q\,|\dot{q}|+\left[H_8{}^v\right]\dot{q}\,|\dot{q}|+\left[I_8{}^v\right]q\,r^2+\left[I_9{}^v\right]\dot{q}\,r^2+\left[I_{10}{}^v\right]q\,\dot{r}^2+\left[I_{11}{}^v\right]q\,r\,\dot{r}+\left[I_{12}{}^v\right]q\,\ddot{r}\,r \\ +\left[I_{13}{}^v\right]\dot{q}r\,\dot{r}+\left[I_{14}{}^v\right]q\,\dot{\breve{g}}^2+\left[L_6{}^v\right]q^3+\left[L_7{}^v\right]q^2\,\dot{q}+\left[L_8{}^v\right]q\,\dot{q}^2+\left[L_9{}^v\right]\dot{q}^3+\left[L_{10}{}^v\right]q^2\,\ddot{q}\end{array}\right)=0$$

(6.96)

$$\left(\begin{array}{l}\left[M_3{}^w\right]\ddot{r}+\left[C_3{}^w\right]\dot{r}+\left[K_3{}^w\right]r+\left[G_4{}^w\right]\dot{\breve{g}}\,q+\left[G_5{}^w\right]\dot{\breve{g}}\,\dot{q}+\left[G_6{}^w\right]\ddot{\breve{g}}\,q+\left[H_5{}^w\right]r\,|r| \\ +\left[H_6{}^w\right]\dot{r}\,|r|+\left[H_7{}^w\right]r\,|\dot{r}|+\left[H_8{}^w\right]\dot{r}\,|\dot{r}|+\left[I_{10}{}^w\right]r\,q^2+\left[I_{11}{}^w\right]\dot{r}\,q^2+\left[I_{12}{}^w\right]\ddot{r}\,q^2 \\ +\left[I_{13}{}^w\right]r\,\dot{q}^2+\left[I_{14}{}^w\right]r\,q\,\dot{q}+\left[I_{15}{}^w\right]r\,\ddot{q}\,q+\left[I_{16}{}^w\right]\dot{r}\,q\,\dot{q}+\left[I_{17}{}^w\right]r\,\dot{\breve{g}}^2+\left[I_{18}{}^w\right]\dfrac{\ddot{r}\,\dot{\breve{g}}\,q}{\dot{r}} \\ +\left[L_6{}^w\right]r^3+\left[L_7{}^w\right]r^2\,\dot{r}+\left[L_8{}^w\right]r\,\dot{r}^2+\left[L_9{}^w\right]\dot{r}^3+\left[L_{10}{}^w\right]r^2\,\ddot{r}\end{array}\right)=0$$

(6.97)

and,

$$\left[M_3{}^\varphi\right]\ddot{\breve{g}}+\left[K_3{}^\varphi\right]\breve{g}+\left[T_5{}^\varphi\right]\dot{\breve{g}}^2+\left[T_6{}^\varphi\right]\ddot{\breve{g}}\,\breve{g}+\left[T_7{}^\varphi\right]\dot{r}\,\dot{q}+\left[T_8{}^\varphi\right]q\,\ddot{r}=0 \tag{6.98}$$

where the values of coefficients are found as follow

$$M_3{}^v=\left(1-\beta_{o,2}-\beta_{i,2}-\beta_{i,3}+\chi\,\beta_{o,2}\right)\int_0^1\Phi^2\;d\xi_2+2\beta_{i,2}\int_0^1\Phi\,\Phi''\,d\xi_2 \tag{6.99}$$

$$\begin{aligned}C_3{}^v=&\left(2u_{i,2}\sqrt{\beta_{i,1}}-2\chi\,u_{i,2}\frac{\lambda_i}{\lambda_{ch}{}^2-1}\sqrt{\beta_{o,2}}\right)\int_0^1\Phi\Phi'd\xi_2\\&+\alpha_2\beta_{m,1}\int_0^1\Phi\Phi^{(4)}d\xi_2+\left(\frac{\mathrm{E}\,c_N\,u_{i,2}}{2}\frac{\lambda_i}{\lambda_{ch}{}^2-1}\sqrt{\beta_{o,2}}+\chi\,\beta_{o,2}\,f_t\right)\int_0^1\Phi\,\Phi''\,d\xi_2\end{aligned} \tag{6.100}$$

$$K_3^{\,v} = \left(\begin{array}{l} u_{i,2}^{\;2} - \dfrac{c_b\, u_{i,2}^{\;2}}{2} \left(\dfrac{\lambda_i^{\;2}}{\lambda_{ch}^{\;2} - 1} \right)^2 + \chi\, u_{i,2}^{\;2} \left(\dfrac{\lambda_i^{\;2}}{\lambda_{ch}^{\;2} - 1} \right)^2 \\ - \left(\dfrac{\mathrm{E}\, c_T\, u_{i,2}^{\;2}}{2} \left(\dfrac{\lambda_i^{\;2}}{\lambda_{ch}^{\;2} - 1} \right)^2 \left(1 - \bar{h}\right) + \left(\gamma_1 - \gamma_2\right) \right) \left(1 - \xi_2\right) \end{array} \right) \int_0^1 \Phi \Phi''\, d\xi_2$$

$$- \left(\frac{\mathrm{E}\left(c_N + c_T \bar{h}\right) u_{i,2}^{\;2}}{2} \left(\frac{\lambda_i^{\;2}}{\lambda_{ch}^{\;2} - 1} \right)^2 - \left(\gamma_1 - \gamma_2\right) \right) \int_0^1 \Phi \Phi'\, d\xi_2 + \beta_{m,1} \int_0^1 \Phi \Phi^{(4)}\, d\xi_2 \tag{6.101}$$

$$G_3^{\,v} = \frac{\chi\, \beta_{o,2}\, f_t}{2} \int_0^1 \Phi\, \Theta\, \Xi\, d\xi_2 \tag{6.102}$$

$$G_4^{\,v} = \beta_{i,3} \left(\int_0^1 \Phi\, \Theta''\, \Xi\, d\xi_2 + \int_0^1 \Phi\, \Theta'\, \Xi'\, d\xi_2 \right) + \chi\, \beta_{o,2} \int_0^1 \Phi\, \Theta\, \Xi\, d\xi_2 \tag{6.103}$$

$$H_5^{\,v} = -\frac{\mathrm{E}\, c_d\, u_{i,2}^{\;2}}{2} \left(\frac{\lambda_i^{\;2}}{\lambda_{ch}^{\;2} - 1} \right)^2 \int_0^1 \Phi \Phi' \left|\Phi'\right| d\xi_2 \tag{6.104}$$

$$H_6^{\,v} = \frac{\mathrm{E}\, c_d\, u_{i,2}}{2} \frac{\lambda_i^{\;2}}{\lambda_{ch}^{\;2} - 1} \sqrt{\beta_{o,2}} \int_0^1 \Phi \left|\Phi'\right| \Phi\, d\xi_2 \tag{6.105}$$

$$H_7^{\,v} = \frac{\mathrm{E}\, c_d\, u_{i,2}}{2} \frac{\lambda_i^{\;2}}{\lambda_{ch}^{\;2} - 1} \sqrt{\beta_{o,2}} \int_0^1 \Phi \Phi' \left|\Phi\right| d\xi_2 \tag{6.106}$$

$$H_8^{\,v} = \frac{\mathrm{E}\, c_d}{2} \beta_{o,2} \int_0^1 \Phi^2 \left|\Phi\right|\, d\xi_2 \tag{6.107}$$

$$
\begin{aligned}
I_8{}^{v} = {} & u_{i,2}{}^{2}\left(\int_0^1 \Phi\,\Theta'\Theta''\Phi' d\xi_2 - \int_0^1\int_{\xi_2}^1 \Phi\Phi''\Theta'\Theta'' d\xi_2 d\xi_2\right) \\
& + \left(\frac{c_b u_{i,2}{}^{2}}{2} - \chi u_{i,2}{}^{2}\right)\left(\frac{\lambda_i{}^{2}}{\lambda_{ch}{}^{2}-1}\right)^2 \int_0^1\int_{\xi_2}^1 \Phi\Phi''\Theta'\Theta'' d\xi_2 d\xi_2 \\
& + \beta_{m,1}\left(\int_0^1 \Phi\Phi''\Theta''^2 d\xi_2 + \int_0^1 \Phi\Phi''\Theta'\Theta''' d\xi_2 + 3\int_0^1 \Phi\Phi'\Theta''\Theta''' d\xi_2 + \int_0^1 \Phi\Phi'\Theta'\Theta^{(4)} d\xi_2\right) \\
& + \frac{\gamma_1}{2}\left(\int_0^1 \Phi\,\Phi'\Theta'^2 d\xi_2 - (1-\xi_2)\left(\int_0^1 \Phi\,\Phi''\Theta'^2 d\xi_2 + 2\int_0^1 \Phi\,\Phi'\Theta'\Theta'' d\xi_2\right)\right) \\
& + \frac{\mathrm{E}\, u_{i,2}{}^{2}(c_T - c_N)}{2}\left(\frac{\lambda_i{}^{2}}{\lambda_{ch}{}^{2}-1}\right)^2\left(\int_0^1 \Phi\Phi'\Theta'^2 d\xi_2 - \int_0^1\int_{\xi_2}^1 \Phi\Phi''\Theta'^2 d\xi_2 d\xi_2\right)
\end{aligned}
\tag{6.108}
$$

$$
I_9{}^{v} = \beta_{m,1}\alpha_2\left(\left(\int_0^1 \Phi\Phi''\Theta''^2 d\xi_2 + \int_0^1 \Phi\Phi''\Theta'\Theta''' d\xi_2 + 3\int_0^1 \Phi\Phi'\Theta''\Theta''' d\xi_2 + \int_0^1 \Phi\Phi'\Theta'\Theta^{(4)} d\xi_2\right)\right)
\tag{6.109}
$$

$$
\begin{aligned}
I_{10}{}^{v} = {} & (1 - \beta_{o,2} - \beta_{i,2} - \beta_{i,3} + \chi\,\beta_{o,2})\left(\int_0^1\int_0^{\xi_2} \Phi\Phi'\Theta'^2 d\xi_2\, d\xi_2\right) \\
& - \frac{\beta_{o,2}\,\mathrm{E}\, c_T}{4}\int_0^1 \Phi\,\Phi'\Theta^2 d\xi_2 - (1 - \beta_{o,2} - \beta_{i,2} - \beta_{i,3})\left(\int_0^1\int_0^{\xi_2}\int_{\xi_2}^1 \Phi\Phi''\Theta'^2 d\xi_2 d\xi_2 d\xi_2\right) \\
& + \frac{\beta_{o,2}\,\mathrm{E}}{4} c_T \int_0^1\int_{\xi_2}^1 \Phi\,\Phi''\Theta^2 d\xi_2 d\xi_2 + \beta_{i,3}\left(\int_0^1 \Phi\,\Phi''\,\Theta'^2 d\xi_2 + 2\int_0^1 \Phi\,\Phi'\Theta'\,\Theta'' d\xi_2\right)
\end{aligned}
\tag{6.110}
$$

$$
\begin{aligned}
I_{11}{}^{v} = {} & 2\sqrt{\beta_{i,1}}\, u_{i,2}\left(\int_0^1 \Phi\,\Phi'\Theta'^2 d\xi_2 - \int_0^1\int_{\xi_2}^1 \Phi\Phi''\Theta'^2 d\xi_2\, d\xi_2\right) \\
& + 2\chi\sqrt{\beta_{o,2}}\, u_{i,2}\,\frac{\lambda_i{}^{2}}{\lambda_{ch}{}^{2}-1}\left(\int_0^1\int_{\xi_2}^1 \Phi\Phi''\Theta'^2 d\xi_2\, d\xi_2 - \int_0^1\int_0^{\xi_2} \Phi\Phi''\Theta'^2 d\xi_2\, d\xi_2\right) \\
& + \beta_{m,1}\,\alpha_2\left(\begin{array}{l}\int_0^1 \Phi\,\Phi''\Theta'\Theta''' d\xi_2 + 6\int_0^1 \Phi\,\Phi'\Theta''\Theta''' d\xi_2 + 2\int_0^1 \Phi\,\Phi'\Theta'\Theta^{(4)} d\xi_2 \\ + 2\int_0^1 \Phi\,\Phi''\Theta''^2 d\xi_2 + \int_0^1 \Phi\,\Phi''\Theta'\Theta^{(4)} d\xi_2\end{array}\right) \\
& + \frac{u_{i,2}\,\mathrm{E}}{4}\,\frac{\lambda_i{}^{2}}{\lambda_{ch}{}^{2}-1}\sqrt{\beta_{o,2}}\,(c_N - c_T)\left(\int_0^1 \Phi\,\Phi'\Theta\Theta' d\xi_2 - \int_0^1\int_{\xi_2}^1 \Phi\Phi''\Theta\Theta' d\xi_2\, d\xi_2\right)
\end{aligned}
\tag{6.111}
$$

$$
\begin{aligned}
I_{12}{}^{v} = {} & \left(1-\beta_{o,2}-\beta_{i,2}-\beta_{i,3}+\chi\,\beta_{o,2}\right)\left(\int_0^1\int_0^{\xi_2}\Phi\Phi'\Theta'^{\,2}d\xi_2\,d\xi_2\right) \\
& -\left(1-\beta_{o,2}-\beta_{i,2}-\beta_{i,3}\right)\int_0^1\int_0^{\xi_2}\int_{\xi_2}^1 \Phi\,\Phi''\Theta'^{\,2}d\xi_2\,d\xi_2 d\xi_2 \\
& -\chi\,\beta_{o,2}\int_0^1\int_{\xi_2}^1\Phi\Phi''\Theta\,\Theta' d\xi_2\,d\xi_2
\end{aligned}
\tag{6.112}
$$

$$
I_{13}{}^{v} = 2\chi\,\beta_{o,2}\int_0^1\int_0^{\xi_2}\Phi\Phi'\Theta'^{\,2}d\xi_2\,d\xi_2 - \frac{\mathrm{E}\,c_N}{2}\beta_{o,2}\int_0^1\int_0^{\xi_2}\Phi^2\Theta'^{\,2}d\xi_2\,d\xi_2 \tag{6.113}
$$

$$
I_{14}{}^{v} = -\frac{\chi\,\beta_{o,2}}{4}\int_0^1\Phi^2\,\Xi^2 d\xi_2 \tag{6.114}
$$

$$
\begin{aligned}
L_6{}^{v} = {} & -\frac{3}{2}\left(\begin{array}{l}\left(\dfrac{\mathrm{E}\,c_T\,u_{i,2}{}^2}{2}\left(\dfrac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2\left(\bar{h}+1\right)+\gamma_1+\gamma_2\right)\left(1-\xi_2\right) \\ -\dfrac{\mathrm{E}\,c_b\,u_{i,2}{}^2}{2}\left(\dfrac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2-\dfrac{2}{3}u_{i,2}{}^2+\dfrac{5}{3}\chi\left(\dfrac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2 u_{i,2}{}^2\end{array}\right)\int_0^1\Phi\Phi'^{\,2}\Phi''d\xi_2 \\
& +\frac{1}{2}\left(\frac{\mathrm{E}\,c_T\,u_{i,2}{}^2}{2}\left(\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2\left(\bar{h}+3\right)+\gamma_1+\gamma_2\right)\int_0^1\Phi\left(\Phi'\right)^3 d\xi_2 \\
& +\left(\frac{\mathrm{E}\,u_{i,2}{}^2}{2}\left(\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2\left(c_N-c_T\right)\right)\int_0^1\int_{\xi_2}^1 \Phi\Phi'^{\,2}\Phi''\,d\xi_2 \\
& +\left(\frac{c_b\,u_{i,2}{}^2}{2}\left(\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2-\chi\left(\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2 u_{i,2}{}^2-u_{i,2}{}^2\right)\int_0^1\int_{\xi_2}^1 \Phi\Phi''^{\,2}\Phi'\,d\xi_2 \\
& +\beta_{m,1}\left(\int_0^1\Phi\left(\Phi''\right)^3 d\xi_2+4\int_0^1\Phi\,\Phi'\,\Phi''\Phi'''d\xi_2+\int_0^1\Phi\,\Phi'^{\,2}\Phi^{(4)}d\xi_2\right)
\end{aligned}
\tag{6.115}
$$

$$L_7{}^v = \left(-2\sqrt{\beta_{i,2}}u_{i,2} + 2\chi\sqrt{\beta_{o,2}}\,\frac{\lambda_i{}^2}{\lambda_{ch}{}^2 - 1}u_{i,2} \right)\int_0^1\int_{\xi_2}^1 \Phi\,\Phi'^2\Phi''d\xi_2 d\xi_2$$
$$+ \beta_{m,1}\alpha_2\left(3\int_0^1 \Phi\left(\Phi''\right)^3 d\xi_2 + 3\int_0^1 \Phi\Phi'^2\Phi^{(4)}d\xi_2 + 12\int_0^1 \Phi\Phi'\Phi''\Phi'''d\xi_2 \right)$$
$$+ \left(2\sqrt{\beta_{i,2}}u_{i,2} + \frac{7}{2}\chi\sqrt{\beta_{o,2}}u_{i,2}\,\frac{\lambda_i{}^2}{\lambda_{ch}{}^2 - 1} \right)\int_0^1 \Phi\;\Phi'^3 d\xi_2$$
$$- \frac{\mathrm{E}\left(c_N - c_T\right)u_{i,2}}{2}\,\frac{\lambda_i{}^2}{\lambda_{ch}{}^2 - 1}\sqrt{\beta_{o,2}}\int_0^1\int_{\xi_2}^1 \Phi^2\,\Phi'\;\Phi''d\xi_2 d\xi_2$$
$$+ \chi\sqrt{\beta_{o,2}}u_{i,2}\,\frac{\lambda_i{}^2}{\lambda_{ch}{}^2 - 1}\left(\frac{3}{2}\int_0^1 \Phi^2\Phi'\;\Phi''d\xi_2 - 2\int_0^1\int_0^{\xi_2} \Phi\,\Phi'^2\Phi''d\xi_2 d\xi_2 \right)$$

(6.116)

$$L_8{}^v = \frac{\mathrm{E}\,c_T}{4}\beta_{o,2}\int_0^1\int_{\xi_2}^1 \Phi^3\,\Phi''d\xi_2 d\xi_2 - \frac{\mathrm{E}\,c_N}{4}\beta_{o,2}\left(\int_0^1\int_0^{\xi_2} \Phi^2\,\Phi'^2 d\xi_2 d\xi_2 - \int_0^1 \Phi^3\Phi' d\xi_2 \right)$$
$$+ \left(1 - \beta_{o,2} - \beta_{i,2} - \beta_{i,3} + 3\chi\,\beta_{o,2}\right)\int_0^1\int_0^{\xi_2} \Phi\,\Phi'^3 d\xi_2 d\xi_2 \qquad (6.117)$$
$$- \left(1 - \beta_{o,2} - \beta_{i,2} - \beta_{i,3}\right)\int_0^1\int_0^{\xi_2}\int_{\xi_2}^1 \Phi\,\Phi'^2\Phi''d\xi_2 d\xi_2 d\xi_2 - \frac{3}{2}\chi\,\beta_{o,2}\int_0^1 \Phi^2\,\Phi'^2 d\xi_2$$

$$L_9{}^v = -\frac{\mathrm{E}\,c_N}{4u_{o,2}}\,\frac{\lambda_{ch}{}^2 - 1}{\lambda_i{}^2}\beta_{o,2}{}^{\frac{3}{2}}\int_0^1 \Phi^4 d\xi_2 \qquad (6.118)$$

$$L_{10}{}^v = \left(1 - \beta_{o,2} - \beta_{i,2} - \beta_{i,3} + \chi\,\beta_{o,2}\right)\int_0^1\int_0^{\xi_2} \Phi\,\Phi'^3 d\xi_2 d\xi_2$$
$$- \left(1 - \beta_{o,2} - \beta_{i,2} - \beta_{i,3}\right)\int_0^1\int_0^{\xi_2}\int_{\xi_2}^1 \Phi\,\Phi'^2\Phi''d\xi_2 d\xi_2 d\xi_2 - \chi\,\beta_{o,2}\int_0^1\int_{\xi_2}^1 \Phi^2\,\Phi'\Phi''d\xi_2$$

(6.119)

$$M_3{}^w = \left(1 - \beta_{o,2} - \beta_{i,2} - \beta_{i,3} + \chi\,\beta_{o,2}\right)\int_0^1 \Theta^2\,d\xi_2 + 2\beta_{i,2}\int_0^1 \Theta\Theta''\,d\xi_2 \qquad (6.120)$$

$$C_3^{\,w} = \left(2u_{i,2}\sqrt{\beta_{i,1}} - 2\chi u_{i,2}\frac{\lambda_i^{\,2}}{\lambda_{ch}^{\,2}-1}\sqrt{\beta_{o,2}} \right)\int_0^1 \Theta\Theta' d\xi_2 + \left(\frac{\mathrm{E}c_N u_{i,2}}{2}\frac{\lambda_i^{\,2}}{\lambda_{ch}^{\,2}-1}\sqrt{\beta_{o,2}} + \chi\beta_{o,2} f_t \right)\int_0^1 \Theta\Theta'' d\xi_2 + \alpha_2\beta_{m,1}\int_0^1 \Theta\Theta^{(4)} d\xi_2 \tag{6.121}$$

$$K_3^{\,w} = \left(\begin{array}{l} u_{i,2}^{\,2} - \dfrac{c_b u_{i,2}^{\,2}}{2}\left(\dfrac{\lambda_i^{\,2}}{\lambda_{ch}^{\,2}-1}\right)^2 + \chi\, u_{o,2}^{\,2} \\ -\left(\dfrac{\mathrm{E}c_T u_{i,2}^{\,2}}{2}\left(\dfrac{\lambda_i^{\,2}}{\lambda_{ch}^{\,2}-1}\right)^2 \left(1-\bar{h}\right) + \left(\gamma_1-\gamma_2\right) \right)\left(1-\xi_2\right) \end{array} \right)\int_0^1 \Theta\Theta'' d\xi_2 - \left(\frac{\mathrm{E}\left(c_N + c_T\bar{h}\right)u_{i,2}^{\,2}}{2}\left(\frac{\lambda_i^{\,2}}{\lambda_{ch}^{\,2}-1}\right)^2 - \left(\gamma_1-\gamma_2\right) \right)\int_0^1 \Theta\Theta' d\xi_2 + \beta_{m,1}\int_0^1 \Theta\Theta^{(4)} d\xi_2 \tag{6.122}$$

$$G_4^{\,w} = \frac{\chi\beta_{o,2} f_t}{2}\int_0^1 \Phi\,\Theta\,\Xi\, d\xi_2 \tag{6.123}$$

$$G_5^{\,w} = \beta_{i,3}\left(\int_0^1 \frac{\Theta\Xi\Phi'\Theta''}{\Theta'} d\xi_2 + \int_0^1 \Theta\,\Xi\Phi'' d\xi_2 + \int_0^1 \Theta\,\Xi'\Phi' d\xi_2 \right) + \chi\beta_{o,2}\int_0^1 \Phi\,\Theta\,\Xi\, d\xi_2 \tag{6.124}$$

$$G_6^{\,w} = \beta_{i,3}\left(\int_0^1 \frac{\Theta\Xi\Phi'\Theta''}{\Theta'} d\xi_2 + \int_0^1 \Theta\,\Xi\Phi'' d\xi_2 + \int_0^1 \Theta\,\Xi'\Phi' d\xi_2 \right) \tag{6.125}$$

$$H_5^{\,w} = -\frac{\mathrm{E}c_d u_{i,2}^{\,2}}{2}\left(\frac{\lambda_i^{\,2}}{\lambda_{ch}^{\,2}-1}\right)^2 \int_0^1 \Theta\,\Theta'\left|\Theta'\right| d\xi_2 \tag{6.126}$$

$$H_6^{\,w} = \frac{\mathrm{E}c_d u_{i,2}}{2}\frac{\lambda_i^{\,2}}{\lambda_{ch}^{\,2}-1}\sqrt{\beta_{o,2}}\int_0^1 \Theta\left|\Theta'\right|\Theta\, d\xi_2 \tag{6.127}$$

$$H_7^{\,w} = \frac{\mathrm{E}c_d u_{i,2}}{2}\frac{\lambda_i^{\,2}}{\lambda_{ch}^{\,2}-1}\sqrt{\beta_{o,2}}\int_0^1 \Theta\,\Theta'\left|\Theta\right| d\xi_2 \tag{6.128}$$

$$H_8^{\,w} = \frac{\mathrm{E}c_d}{2}\beta_{o,2}\int_0^1 \Theta^2\left|\Theta\right| d\xi_2 \tag{6.129}$$

$$
\begin{aligned}
I_{10}{}^{w} = {} & u_{i,2}{}^{2}\left(\int_0^1 \Theta\,\Phi'\Phi''\Theta' d\xi_2 - \int_0^1\int_{\xi_2}^1 \Theta\Theta''\Phi'\Phi'' d\xi_2 d\xi_2\right) \\
& + \left(\frac{c_b u_{i,2}{}^{2}}{2} - \chi u_{i,2}{}^{2}\right)\left(\frac{\lambda_i{}^{2}}{\lambda_{ch}{}^{2}-1}\right)^2 \int_0^1\int_{\xi_2}^1 \Phi'\Phi''\Theta\,\Theta'' d\xi_2 d\xi_2 \\
& + \beta_{m,1}\left(\int_0^1 \Theta\Theta''\Phi''^{2} d\xi_2 + \int_0^1 \Theta\Theta''\Phi'\Phi''' d\xi_2 + 3\int_0^1 \Theta\Theta'\Phi''\Phi''' d\xi_2 + \int_0^1 \Phi^{(4)}\Phi'\Theta'\Theta d\xi_2\right) \\
& + \frac{\gamma_1}{2}\left(\int_0^1 \Theta\Theta'\Phi'^{2} d\xi_2 - (1-\xi_2)\left(\int_0^1 \Theta\,\Theta''\Phi'^{2} d\xi_2 + 2\int_0^1 \Theta\,\Theta'\Phi'\Phi'' d\xi_2\right)\right) \\
& + \frac{\mathrm{E}\,u_{i,2}{}^{2}(c_T - c_N)}{2}\left(\frac{\lambda_i{}^{2}}{\lambda_{ch}{}^{2}-1}\right)^2\left(\int_0^1 \Theta\Theta'\Phi'^{2} d\xi_2 - \int_0^1\int_{\xi_2}^1 \Theta\Theta''\Phi'^{2} d\xi_2 d\xi_2\right)
\end{aligned}
\tag{6.130}
$$

$$
I_{11}{}^{w} = \beta_{m,1}\alpha_2\left(\left(\int_0^1 \Theta\Theta''\Phi''^{2} d\xi_2 + \int_0^1 \Theta\Theta''\Phi'\Phi''' d\xi_2 + 3\int_0^1 \Theta\Theta'\Phi''\Phi''' d\xi_2 + \int_0^1 \Theta\Theta'\Phi'\Phi^{(4)} d\xi_2\right)\right)
\tag{6.131}
$$

$$
I_{12}{}^{w} = \beta_{i,3}\left(2\int_0^1 \Theta\,\Phi'^{2}\Theta'' d\xi_2 + 3\int_0^1 \Theta\,\Theta'\Phi'\Phi'' d\xi_2\right)
\tag{6.132}
$$

$$
\begin{aligned}
I_{13}{}^{w} = {} & (1 - \beta_{o,2} - \beta_{i,2} - \beta_{i,3} + \chi\,\beta_{o,2})\left(\int_0^1\int_0^{\xi_2} \Theta\,\Theta'\Phi'^{2} d\xi_2\, d\xi_2\right) \\
& - \frac{\beta_{o,2}\,\mathrm{E}\,c_T}{4}\int_0^1 \Theta\,\Theta'\Phi^{2} d\xi_2 - (1 - \beta_{o,2} - \beta_{i,2} - \beta_{i,3})\left(\int_0^1\int_0^{\xi_2}\int_{\xi_2}^1 \Theta\Theta''\Phi'^{2} d\xi_2 d\xi_2 d\xi_2\right) \\
& + \frac{\beta_{o,2}\,\mathrm{E}}{4} c_T \int_0^1\int_{\xi_2}^1 \Theta\Theta''\Phi^{2} d\xi_2 d\xi_2
\end{aligned}
\tag{6.133}
$$

$$
\begin{aligned}
I_{14}{}^{w} = {} & 2\sqrt{\beta_{i,1}}\, u_{i,2}\left(\int_0^1 \Theta\,\Theta'\Phi'^2 d\xi_2 - \int_0^1\int_{\xi_2}^1 \Theta\,\Theta''\Phi'^2 d\xi_2\, d\xi_2\right) \\
& + 2\chi\sqrt{\beta_{o,2}}\, u_{i,2}\,\frac{\lambda_i^{\,2}}{\lambda_{ch}^{\,2}-1}\left(\int_0^1\int_{\xi_2}^1 \Theta\,\Theta''\Phi'^2 d\xi_2\, d\xi_2 - \int_0^1\int_0^{\xi_2} \Theta\,\Theta''\Phi'^2 d\xi_2\, d\xi_2\right) \\
& + \beta_{m,1}\,\alpha_2\left(\begin{array}{l}\displaystyle\int_0^1 \Theta\,\Theta''\Phi'\Phi''' d\xi_2 + 6\int_0^1 \Theta\Theta'\Phi''\Phi''' d\xi_2 + 2\int_0^1 \Theta\Theta'\Phi'\Phi^{(4)} d\xi_2 \\ \displaystyle + 2\int_0^1 \Theta\,\Theta''\Phi''^2 d\xi_2 + \int_0^1 \Theta\,\Theta''\Phi'\Phi^{(4)} d\xi_2\end{array}\right) \\
& + \frac{u_{i,2}\,\mathrm{E}}{4}\,\frac{\lambda_i^{\,2}}{\lambda_{ch}^{\,2}-1}\sqrt{\beta_{o,2}}\,(c_N - c_T)\left(\int_0^1 \Phi\,\Phi'\Theta\Theta' d\xi_2 - \int_0^1\int_{\xi_2}^1 \Theta\Theta''\Phi\Phi' d\xi_2\, d\xi_2\right)
\end{aligned}
\tag{6.134}
$$

$$
\begin{aligned}
I_{15}{}^{w} = {} & \left(1-\beta_{o,2}-\beta_{i,2}-\beta_{i,3}+\chi\,\beta_{o,2}\right)\left(\int_0^1\int_0^{\xi_2} \Theta\Theta'\Phi'^2 d\xi_2\, d\xi_2\right) \\
& - \left(1-\beta_{o,2}-\beta_{i,2}-\beta_{i,3}\right)\int_0^1\int_0^{\xi_2}\int_{\xi_2}^1 \Theta\Theta''\Phi'^2 d\xi_2\, d\xi_2 d\xi_2 \\
& - \chi\,\beta_{o,2}\int_0^1\int_{\xi_2}^1 \Theta\Theta''\Phi\Phi' d\xi_2\, d\xi_2
\end{aligned}
\tag{6.135}
$$

$$
\begin{aligned}
I_{16}{}^{w} = {} & 2\chi\,\beta_{o,2}\int_0^1\int_0^{\xi_2} \Theta\Theta'\Phi'^2 d\xi_2\, d\xi_2 - \frac{\mathrm{E}\,c_N}{2}\,\beta_{o,2}\int_0^1\int_0^{\xi_2} \Theta^2\Phi'^2 d\xi_2\, d\xi_2 \\
& + 2\,\beta_{i,3}\int_0^1 \Theta\Theta'\Phi'\Phi'' d\xi_2 + 2\beta_{i,3}\int_0^1 \Theta\,\Theta''\Phi'^2 d\xi_2
\end{aligned}
\tag{6.136}
$$

$$
I_{17}{}^{w} = -\frac{\chi\,\beta_{o,2}}{4}\int_0^1 \Theta^2\,\Xi^2 d\xi_2 \tag{6.137}
$$

$$
I_{18}{}^{w} = \beta_{i,3}\left(\int_0^1 \Theta\,\Xi\,\Phi' d\xi_2 + \int_0^1 \Theta\,\Xi\,\Phi'' d\xi_2 - \int_0^1 \frac{\Theta\Xi\,\Theta''\Phi'}{\Theta'} d\xi_2 + \int_0^1 \Theta\,\Xi'\,\Phi' d\xi_2\right) \tag{6.138}
$$

$$
\begin{aligned}
L_6{}^{w} = & -\frac{3}{2}\left(\begin{array}{l}\left(\frac{\mathrm{E}\, c_T\, u_{i,2}{}^2}{2}\left(\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2\left(\bar{h}+1\right)+\gamma_1+\gamma_2\right)\left(1-\xi_2\right) \\ -\frac{\mathrm{E}\, c_b\, u_{i,2}{}^2}{2}\left(\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2-\frac{2}{3}u_{i,2}{}^2+\frac{5}{3}\chi\, u_{i,2}{}^2\left(\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2\end{array}\right)\int_0^1 \Theta\,\Theta'^2\Theta''\,d\xi_2 \\
& +\frac{1}{2}\left(\frac{\mathrm{E}\, c_T\, u_{i,2}{}^2}{2}\left(\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2\left(\bar{h}+3\right)+\gamma_1+\gamma_2\right)\int_0^1 \Theta\left(\Theta'\right)^3 d\xi_2 \\
& +\left(\frac{\mathrm{E}\, u_{i,2}{}^2}{2}\left(\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2\left(c_N-c_T\right)\right)\int_0^1\int_{\xi_2}^1 \Theta\,\Theta'^2\Theta''\,d\xi_2 \\
& +\left(\frac{c_b\, u_{i,2}{}^2}{2}\left(\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2-\chi\, u_{i,2}{}^2\left(\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2-u_{i,2}{}^2\right)\int_0^1\int_{\xi_2}^1 \Theta\,\Theta''^2\Theta'\,d\xi_2 \\
& +\beta_{m,1}\left(\int_0^1 \Theta\left(\Theta''\right)^3 d\xi_2+4\int_0^1 \Theta\,\Theta'\Theta''\Theta'''\,d\xi_2+\int_0^1 \Theta\,\Theta'^2\Theta^{(4)}\,d\xi_2\right)
\end{aligned}
\tag{6.139}
$$

$$
\begin{aligned}
L_7{}^{w} = & \left(-2\sqrt{\beta_{i,2}}\,u_{i,2}+2\chi\sqrt{\beta_{o,2}}\,u_{i,2}\,\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)\int_0^1\int_{\xi_2}^1 \Theta\,\Theta'^2\Theta''\,d\xi_2\,d\xi_2 \\
& +\beta_{m,1}\alpha_2\left(3\int_0^1 \Theta\left(\Theta''\right)^3 d\xi_2+3\int_0^1 \Theta\,\Theta'^2\Theta^{(4)}\,d\xi_2+12\int_0^1 \Theta\,\Theta'\Theta''\Theta'''\,d\xi_2\right) \\
& +\left(2\sqrt{\beta_{i,2}}\,u_{i,2}+\frac{7}{2}\chi\sqrt{\beta_{o,2}}\,u_{i,2}\,\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)\int_0^1 \Theta\,\Theta'^3\,d\xi_2 \\
& -\frac{\mathrm{E}\left(c_N-c_N\right)u_{i,2}}{2}\,\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\sqrt{\beta_{o,2}}\int_0^1\int_{\xi_2}^1 \Theta^2\,\Theta'\Theta''\,d\xi_2\,d\xi_2 \\
& +\chi\sqrt{\beta_{o,2}}\,u_{i,2}\,\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\left(\frac{3}{2}\int_0^1 \Theta^2\Theta'\Theta''\,d\xi_2-2\int_0^1\int_0^{\xi_2} \Theta\,\Theta'^2\Theta''\,d\xi_2\,d\xi_2\right)
\end{aligned}
\tag{6.140}
$$

$$
\begin{aligned}
L_8{}^{w} = & \frac{\mathrm{E}\, c_T}{4}\beta_{o,2}\int_0^1\int_{\xi_2}^1 \Theta^3\,\Theta''\,d\xi_2\,d\xi_2-\frac{\mathrm{E}\, c_N}{4}\beta_{o,2}\left(\int_0^1\int_0^{\xi_2} \Theta^2\,\Theta'^2\,d\xi_2\,d\xi_2-\int_0^1 \Theta^3\Theta'\,d\xi_2\right) \\
& +\left(1-\beta_{o,2}-\beta_{i,2}-\beta_{i,3}+3\chi\,\beta_{o,2}\right)\int_0^1\int_0^{\xi_2} \Theta\,\Theta'^3\,d\xi_2\,d\xi_2 \\
& -\left(1-\beta_{o,2}-\beta_{i,2}-\beta_{i,3}\right)\int_0^1\int_0^{\xi_2}\int_{\xi_2}^1 \Theta\,\Theta'^2\Theta''\,d\xi_2\,d\xi_2\,d\xi_2-\frac{3}{2}\chi\,\beta_{o,2}\int_0^1 \Theta^2\,\Theta'^2\,d\xi_2
\end{aligned}
\tag{6.141}
$$

$$L_9{}^{w} = -\frac{\mathrm{E}\, c_N}{4u_{o,2}} \beta_{o,2}{}^{\frac{3}{2}} \int_0^1 \Theta^4 d\xi_2 \tag{6.142}$$

$$\begin{aligned} L_{10}{}^{w} &= \left(1-\beta_{o,2}-\beta_{i,2}-\beta_{i,3}+\chi\,\beta_{o,2}\right)\int_0^1\int_0^{\xi_2} \Theta\Theta'^3 d\xi_2 d\xi_2 \\ &\quad -\left(1-\beta_{o,2}-\beta_{i,2}-\beta_{i,3}\right)\int_0^1\int_0^{\xi_2}\int_{\xi_2}^1 \Theta\Theta'^2\Theta'' d\xi_2 d\xi_2 d\xi_2 - \chi\,\beta_{o,2}\int_0^1\int_{\xi_2}^1 \Theta^2\,\Theta'\Theta'' d\xi_2 \end{aligned} \tag{6.143}$$

$$M_3{}^{\varphi} = \beta_{i,3}\int_0^1 \Xi^{\,2} d\xi_2 \tag{6.144}$$

$$K_3{}^{\varphi} = -\,\beta_{m,2}\int_0^1 \Xi\,\Xi'' d\xi_2 \tag{6.145}$$

$$T_5{}^{\varphi} = 8\beta_{i,4}\, f_r \int_0^1 \Xi^2\,\Xi' d\xi_2 \tag{6.146}$$

$$T_6{}^{\varphi} = 4\beta_{i,4}\, f_r \int_0^1 \Xi^2\,\Xi' d\xi_2 \tag{6.147}$$

$$T_7{}^{\varphi} = \beta_{i,3}\int_0^1 \Xi\,\Theta'\Phi' d\xi_2 \tag{6.148}$$

$$T_8{}^{\varphi} = \beta_{i,3}\int_0^1 \Xi\,\Theta'\Phi' d\xi_2 \tag{6.149}$$

6.1.4 Extensible rotating flexible pipe conveying fluid and subjected to external axial flow

The dimensionless equations of motion (5.59-5.62), which describe the vibration of fixed-simply supported extensible pipe, that conveying fluid and subjected to external axial flow are discretized via unimodal Galerkin's method. Following the same procedure presented in the previous sections, and after simplifying and rearranging the terms, we

obtain the following four coupled ordinary nonlinear differential equations in terms of the modal coordinates p, q, r and $\breve{g}$:

$$\begin{pmatrix} \left[M_4{}^u\right]\ddot{p}+\left[C_4{}^u\right]\dot{p}+\left[K_4{}^u\right]p+\left[A_5{}^u\right]q^2+\left[A_6{}^u\right]r^2+\left[A_7{}^u\right]q\,\dot{q}+\left[A_8{}^u\right]r\,\dot{r} \\ +\left[A_9{}^u\right]\dot{q}^2+\left[A_{10}{}^u\right]\dot{r}^2+\left[A_{11}{}^u\right]q\,\ddot{q}+\left[A_{12}{}^u\right]r\,\ddot{r}+\left[B_5{}^u\right]q^2|q|+\left[B_6{}^u\right]r^2|r| \\ +\left[B_7{}^u\right]q|q|\,\dot{q}+\left[B_8{}^u\right]r|r|\,\dot{r}+\left[B_9{}^u\right]q^2|\dot{q}|+\left[B_{10}{}^u\right]r^2|\dot{r}|+\left[B_{11}{}^u\right]q\,\dot{q}|\dot{q}|+\left[B_{12}{}^u\right]r\,\dot{r}|\dot{r}| \end{pmatrix}=0$$

(6.150)

$$\begin{pmatrix} \left[M_4{}^v\right]\ddot{q}+\left[C_4{}^v\right]\dot{q}+\left[K_4{}^v\right]q+\left[D_6{}^v\right]p\,q+\left[D_7{}^v\right]\dot{p}\,q+\left[D_8{}^v\right]p\,\dot{q}+\left[D_9{}^v\right]\dot{p}\,\dot{q}+\left[D_{10}{}^v\right]\ddot{p}\,q \\ +\left[D_{11}{}^v\right]\dot{\breve{g}}\,r+\left[D_{12}{}^v\right]\dot{\breve{g}}\,\dot{r}+\left[E_6{}^v\right]q\,\dot{q}+\left[E_7{}^v\right]q|q|+\left[E_8{}^v\right]\dot{q}|q|+\left[E_9{}^v\right]q|\dot{q}|+\left[E_{10}{}^v\right]\dot{q}|\dot{q}| \\ +\left[F_6{}^v\right]q^3+\left[F_7{}^v\right]q^2\,\dot{q}+\left[F_8{}^v\right]q\,\dot{q}^2+\left[F_9{}^v\right]\dot{q}^3+\left[F_{10}{}^v\right]q^2\ddot{q}+\left[F_{11}{}^v\right]q\,\dot{r}^2+\left[F_{12}{}^v\right]\dot{\breve{g}}^2\,q \end{pmatrix}=0$$

(6.151)

$$\begin{pmatrix} \left[M_4{}^w\right]\ddot{r}+\left[C_4{}^w\right]\dot{r}+\left[K_4{}^w\right]r+\left[D_6{}^w\right]p\,r+\left[D_7{}^w\right]\dot{p}\,r+\left[D_8{}^w\right]p\,\dot{r}+\left[D_9{}^w\right]\dot{p}\,\dot{r} \\ +\left[D_{10}{}^w\right]\ddot{p}\,r+\left[D_{11}{}^w\right]\dot{\breve{g}}\,q+\left[D_{12}{}^w\right]\dot{\breve{g}}\,\dot{q}+\left[D_{13}{}^w\right]\ddot{\breve{g}}\,q+\left[E_6{}^w\right]r\,\dot{r}+\left[E_7{}^w\right]r|r|+\left[E_8{}^w\right]\dot{r}|r| \\ +\left[E_9{}^w\right]r|\dot{r}|+\left[E_{10}{}^w\right]\dot{r}|\dot{r}|+\left[F_6{}^w\right]r^3+\left[F_7{}^w\right]r^2\,\dot{r}+\left[F_8{}^w\right]r\,\dot{r}^2+\left[F_9{}^w\right]\dot{r}^3+\left[F_{10}{}^w\right]r^2\,\ddot{r} \\ +\left[F_{11}{}^w\right]r\,\dot{q}^2+\left[F_{12}{}^w\right]\dot{\breve{g}}^2\,r+\left[F_{13}{}^w\right]q^2\,\ddot{r}+\left[F_{14}{}^w\right]\dot{r}\,q\,\dot{q}+\left[F_{15}{}^w\right]\dfrac{\ddot{r}\,\dot{\breve{g}}\,q}{\dot{r}} \end{pmatrix}=0$$

(6.152)

$$\left[M_4{}^\varphi\right]\ddot{\breve{g}}+\left[K_4{}^\varphi\right]\breve{g}+\left[T_9{}^\varphi\right]\dot{\breve{g}}^2+\left[T_{10}{}^\varphi\right]\ddot{\breve{g}}\breve{g}+\left[T_{11}{}^\varphi\right]\dot{r}\,\dot{q}+\left[T_{12}{}^\varphi\right]q\,\ddot{r}=0 \qquad (6.153)$$

where the coefficients are found as follow

$$M_4^{\,u} = \left(1-\beta_{o,2}-\beta_{i,2}-\beta_{i,3}\right)\int_0^1 \Psi^2 d\xi_1 \qquad (6.154)$$

$$C_4^{\,u} = 2u_{i,2}\sqrt{\beta_{i,1}}\int_0^1 \Psi\Psi' d\xi_1 - \Pi_{o,2}\alpha_2\int_0^1 \Psi\Psi'' d\xi_1 \qquad (6.155)$$

$$K_4^{\,u} = \left(u_{i,2}^{\;2}-\Pi_{0,2}\right)\int_0^1 \Psi''\Psi .d\xi_1 + \left(\frac{\mathrm{E}\,c_T\,u_{o,2}^{\;2}}{2}\left(1-\bar{h}\right)+\left(\gamma_1-\gamma_2\right)\right)\int_0^1 \Psi\Psi' d\xi_1 \qquad (6.156)$$

$$\begin{aligned} A_5^{\,u} = & \left(\begin{array}{l}\overline{\Gamma_2}-\Pi_{iL,2}+\Pi_{oL,2}-\Pi_{0,2}-\chi\,u_{o,2}^{\;2} \\ +\left(1-2\nu\right)\overline{\Pi}_2+\left(\dfrac{\mathrm{E}\,c_T\,u_{o,2}^{\;2}}{2}\left(1-\bar{h}\right)+\left(\gamma_1-\gamma_2\right)\right)\left(\dfrac{1}{2}-\xi_1\right)\end{array}\right)\int_0^1 \Psi\Phi'\Phi'' d\xi_1 \\ & +\frac{\mathrm{E}\,u_{o,2}^{\;2}}{2}\left(c_N-c_T\left(\frac{3}{2}-\frac{\bar{h}}{2}\right)\right)\int_0^1 \Psi\Phi'^2 d\xi_1 \\ & -\beta_{m,1}\left(\int_0^1 \Psi\Phi^{(4)}\Phi' d\xi_1+\int_0^1 \Psi\Phi''\Phi''' d\xi_1\right)\end{aligned} \qquad (6.157)$$

$$\begin{aligned} A_6^{\,u} = & \left(\begin{array}{l}\overline{\Gamma_2}-\Pi_{iL,2}+\Pi_{oL,2}-\Pi_{0,2}-\chi\,u_{o,2}^{\;2} \\ +\left(1-2\nu\right)\overline{\Pi}_2+\left(\dfrac{\mathrm{E}\,c_T\,u_{o,1}^{\;2}}{2}\left(1-\bar{h}\right)+\left(\gamma_1-\gamma_2\right)\right)\left(\dfrac{1}{2}-\xi_1\right)\end{array}\right)\int_0^1 \Psi\Theta'\Theta'' d\xi_1 \\ & +\frac{\mathrm{E}\,u_{o,2}^{\;2}}{2}\left(c_N-c_T\left(\frac{3}{2}-\frac{\bar{h}}{2}\right)\right)\int_0^1 \Psi\Theta'^2 d\xi_1 \\ & -\beta_{m,1}\left(\int_0^1 \Psi\Theta^{(4)}\Theta' d\xi_1+\int_0^1 \Psi\Theta''\Theta''' d\xi_1\right)\end{aligned} \qquad (6.158)$$

$$\begin{aligned} A_7^{\,u} = & \left(\frac{\mathrm{E}\,u_{o,2}\sqrt{\beta_{o,2}}}{2}\left(c_T-c_N\right)\right)\int_0^1 \Psi\Phi\Phi' d\xi_1-\left(2\Pi_{o,2}\,\alpha_2\right)\int_0^1 \Psi\Phi'\Phi'' d\xi_1 \\ & +2\,\chi\,u_{o,2}\sqrt{\beta_{o,2}}\int_0^1 \Psi\Phi'^2 d\xi_1-2\alpha_2\left(\int_0^1 \Psi\Phi^{(4)}\Phi' dX+\int_0^1 \Psi\Phi''\Phi''' d\xi_1\right)\end{aligned} \qquad (6.159)$$

$$A_8{}^u = \left(\frac{\mathrm{E}\, u_{o,2} \sqrt{\beta_{o,2}}}{2} \left(c_T - c_N \right) \right) \int_0^1 \Psi \Theta \Theta' \, d\xi_1 - \left(2\Pi_{o,2}\, \alpha_2 \right) \int_0^1 \Psi \Theta' \Theta'' \, d\xi_1$$
$$+ 2\, \chi\, u_{o,2} \sqrt{\beta_{o,2}} \int_0^1 \Psi \Theta'^2 \, d\xi_1 - 2\alpha_2 \left(\int_0^1 \Psi \Theta^{(4)} \Theta' dX + \int_0^1 \Psi \Theta'' \Theta''' d\xi_1 \right) \tag{6.160}$$

$$A_9{}^u = \frac{\mathrm{E}\, c_T \beta_{o,2}}{4} \int_0^1 \Psi \Phi^2 d\xi_1 \tag{6.161}$$

$$A_{10}{}^u = \frac{\mathrm{E}\, c_T \beta_{o,2}}{4} \int_0^1 \Psi \Theta^2 d\xi_1 \tag{6.162}$$

$$A_{11}{}^u = -\chi\, \beta_{o,2} \int_0^1 \Psi \Phi \Phi' \, d\xi_1 \tag{6.163}$$

$$A_{12}{}^u = -\chi\, \beta_{o,2} \int_0^1 \Psi \Theta \Theta' \, d\xi_1 \tag{6.164}$$

$$B_5{}^u = \frac{\mathrm{E}\, c_d\, u_{o,2}{}^2}{2} \int_0^1 \Psi \Phi'^2 \left| \Phi' \right| d\xi_1 \tag{6.165}$$

$$B_6{}^u = \frac{\mathrm{E}\, c_d\, u_{o,2}{}^2}{2} \int_0^1 \Psi \Theta'^2 \left| \Theta' \right| d\xi_1 \tag{6.166}$$

$$B_7{}^u = -\frac{\mathrm{E}\, c_d\, u_{o,2}}{2} \sqrt{\beta_{o,2}} \int_0^1 \Psi \Phi' \left| \Phi' \right| \Phi \, d\xi_1 \tag{6.167}$$

$$B_8{}^u = -\frac{\mathrm{E}\, c_d\, u_{o,2}}{2} \sqrt{\beta_{o,2}} \int_0^1 \Psi \Theta' \left| \Theta' \right| \Theta \, d\xi_1 \tag{6.168}$$

$$B_9{}^u = -\frac{\mathrm{E}\, c_d\, u_{o,2}}{2} \sqrt{\beta_{o,2}} \int_0^1 \Psi \Phi'^2 \left| \Phi \right| d\xi_1 \tag{6.169}$$

$$B_{10}{}^u = -\frac{\mathrm{E}\, c_d\, u_{o,2}}{2} \sqrt{\beta_{o,2}} \int_0^1 \Psi \Theta'^2 \left| \Theta \right| d\xi_1 \tag{6.170}$$

$$B_{11}{}^{u} = -\frac{\mathrm{E}\,c_d}{2}\beta_{o,2}\int_0^1 \Psi\Phi'|\Phi|\Phi\, d\xi_1 \tag{6.171}$$

$$B_{12}{}^{u} = -\frac{\mathrm{E}\,c_d}{2}\beta_{o,2}\int_0^1 \Psi\Theta'|\Theta|\Theta\, d\xi_1 \tag{6.172}$$

$$M_4{}^{v} = \left(1-\beta_{o,2}-\beta_{i,2}-\beta_{i,3}+\chi\,\beta_{o,2}\right)\int_0^1 \Phi^{\,2} d\xi_1 + 2\beta_{i,2}\int_0^1 \Phi\,\Phi''\, d\xi_1 \tag{6.173}$$

$$\begin{aligned} C_4{}^{v} &= \left(2u_{i,2}\sqrt{\beta_{i,1}} - 2\chi\, u_{o,2}\sqrt{\beta_{o,2}}\right)\int_0^1 \Phi\Phi' d\xi_1 + \alpha_2\beta_{m,1}\int_0^1 \Phi\Phi^{(4)} d\xi_1 \\ &+ \left(\frac{\mathrm{E}\,c_N\, u_{o,2}}{2}\sqrt{\beta_{o,2}} + \chi\,\beta_{o,2}\, f_t\right)\int_0^1 \Phi^2\, d\xi_1 \end{aligned} \tag{6.174}$$

$$\begin{aligned} K_4{}^{v} &= \left(\begin{array}{l} u_{i,2}{}^{2} - \left(\overline{\Gamma}_2 - \Pi_{iL,2} + \Pi_{oL,2}\right) + \chi\, u_{o,2}{}^{2} - (1-2\nu)\overline{\Pi}_2 \\ -\left(\dfrac{\mathrm{E}\,c_T\, u_{o,2}{}^{2}}{2}\left(1-\overline{h}\right) + \left(\gamma_1-\gamma_2\right)\right)\left(\dfrac{1}{2}-\xi_1\right) \end{array}\right)\int_0^1 \Phi\Phi''\, d\xi_1 \\ &- \left(\frac{\mathrm{E}\left(c_N + c_T\, h\right) u_{o,2}{}^{2}}{2} - \left(\gamma_1-\gamma_2\right)\right)\int_0^1 \Phi\Phi'\, d\xi_1 + \beta_{m,1}\int_0^1 \Phi\Phi^{(4)}\, d\xi_1 \end{aligned} \tag{6.175}$$

$$\begin{aligned} D_6{}^{v} &= \left(\begin{array}{l} \left(\dfrac{\mathrm{E}\,c_T\, u_{o,2}{}^{2}}{2}\left(1-\overline{h}\right) + \left(\gamma_1-\gamma_2\right)\right)\left(\dfrac{1}{2}-\xi_1\right) \\ + \left(\overline{\Gamma}_2 - \Pi_{iL,2} + \Pi_{oL,2} - \Pi_{0,2}\right) + (1-2\nu)\overline{\Pi}_2 \end{array}\right)\int_0^1 \left(\Phi\Psi''\Phi' + \Phi\Psi'\Phi''\right) d\xi_1 \\ &+ \frac{\mathrm{E}\, u_{o,2}{}^{2}\left(c_N + c_T\right)}{2}\int_0^1 \Phi\Psi'\Phi'\, d\xi_1 + \left(\frac{\mathrm{E}\,c_T\, u_{o,2}{}^{2}\left(1-\overline{h}\right)}{2} + \left(\gamma_1-\gamma_2\right)\right)\int_0^1 \Phi\Psi\Phi''\, d\xi_1 \\ &- 3\beta_{m,1}\int_0^1 \Phi\Psi'''\Phi''\, d\xi_1 - 4\beta_{m,1}\int_0^1 \Phi\Psi''\Phi'''\, d\xi_1 - 2\beta_{m,1}\int_0^1 \Phi\Psi'\Phi^{(4)}\, d\xi_1 \\ &- \beta_{m,1}\int_0^1 \Phi\Psi^{(4)}\Phi'\, d\xi_1 - \chi\, u_{o,1}{}^{2}\int_0^1 \left(2\Phi\Psi''\Phi' + 4\Phi\Psi'\Phi''\right) d\xi_1 \end{aligned} \tag{6.176}$$

$$D_7^{\,v} = -\beta_{m,1}\alpha_2\left(3\int_0^1 \Phi\Psi'''\Phi''\,d\xi_1 + 4\int_0^1 \Phi\Psi''\Phi'''\,d\xi_1 + 2\int_0^1 \Phi\Psi'\Phi^{(4)}\,d\xi_1 + \int_0^1 \Phi\Psi^{(4)}\Phi'\,d\xi_1\right)$$
$$+ 3\chi u_{o,2}\sqrt{\beta_{o,2}}\int_0^1 \Phi\Psi'\Phi' d\xi_1 + 2\chi u_{o,2}\sqrt{\beta_{o,2}}\int_0^1 \Phi\Psi\Phi'' d\xi_1 - \Pi_{0,2}\alpha_2\int_0^1 \left(\Phi\Psi''\Phi' + \Phi\Psi'\Phi''\right) d\xi_1 \tag{6.177}$$

$$D_8^{\,v} = -\beta_{m,2}\alpha_2\left(3\int_0^1 \Phi\Psi'''\Phi''\,d\xi_1 + 4\int_0^1 \Phi\Psi''\Phi'''\,d\xi_1 + 2\int_0^1 \Phi\Psi'\Phi^{(4)}\,d\xi_1 + \int_0^1 \Phi\Psi^{(4)}\Phi'\,d\xi_1\right)$$
$$- \Pi_{0,2}\alpha_2\int_0^1 \left(\Phi\Psi''\Phi' + \Phi\Psi'\Phi''\right) d\xi_1 + \frac{\mathrm{E}\,c_N\,u_{o,2}\sqrt{\beta_{o,2}}}{2}\int_0^1 \Phi^2\Psi'\,d\xi_1$$
$$+ 4\chi\,u_{o,2}\sqrt{\beta_{o,2}}\int_0^1 \Phi\Psi'\Phi'\,d\xi_1 \tag{6.178}$$

$$D_9^{\,v} = \frac{\mathrm{E}\,c_N}{2}\beta_{o,2}\int_0^1 \Phi^2\Psi\,d\xi_1 - 2\chi\beta_{o,2}\int_0^1 \Phi\Psi\,\Phi' d\xi_1 \tag{6.179}$$

$$D_{10}^{\,v} = -\chi\,\beta_{o,2}\int_0^1 \Phi\Psi\Phi'\,d\xi_1 \tag{6.180}$$

$$D_{11}^{\,v} = \frac{\chi\,\beta_{o,2}\,f_t}{2}\int_0^1 \Phi\,\Theta\,\Xi\,d\xi_1 \tag{6.181}$$

$$D_{12}^{\,v} = \beta_{i,3}\left(\int_0^1 \Phi\,\Theta''\,\Xi\,d\xi_1 + \int_0^1 \Phi\,\Theta'\,\Xi'\,d\xi_1\right) + \chi\,\beta_{o,2}\int_0^1 \Phi\,\Theta\,\Xi\,d\xi_1 \tag{6.182}$$

$$E_6^{\,v} = -4\,\beta_{m,1}\alpha_2\int_0^1 \Phi\Phi'\Phi^{(4)}\,d\xi_1 - 3\beta_{m,1}\alpha_2\Pi_{o,2}\int_0^1 \Phi\Phi'\Phi''\,d\xi_1 \tag{6.183}$$

$$E_7^{\,v} = -\frac{\mathrm{E}\,c_d\,u_{o,2}^{\;2}}{2}\int_0^1 \Phi\Phi'\left|\Phi'\right| d\xi_1 \tag{6.184}$$

$$E_8^{\,v} = \frac{\mathrm{E}\,c_d\,u_{o,2}}{2}\sqrt{\beta_{o,2}}\int_0^1 \Phi\left|\Phi'\right|\Phi\,d\xi_1 \tag{6.185}$$

$$E_9{}^{v} = \frac{\mathrm{E}\, c_d\, u_{o,2}}{2}\sqrt{\beta_{o,2}}\int_0^1 \Phi\Phi'|\Phi|\, d\xi_1 \tag{6.186}$$

$$E_{10}{}^{v} = \frac{\mathrm{E}\, c_d}{2}\beta_{o,2}\int_0^1 \Phi^2|\Phi|\, d\xi_1 \tag{6.187}$$

$$\begin{aligned} F_6{}^{v} = \frac{3}{2}\Bigg(&\left(\frac{\mathrm{E}\, c_T\, u_{o,2}{}^2}{2}\left(1-\bar{h}\right)+\left(\gamma_1-\gamma_2\right)\right)\left(\frac{1}{2}-\xi_1\right)+\left(1-2\nu\right)\overline{\Pi}_2 \\ &+\left(\overline{\Gamma_2}-\Pi_{iL,2}+\Pi_{oL,2}-\Pi_{0,2}\right)\Bigg)\int_0^1 \Phi\Phi'^2\Phi''\, d\xi_1 \\ &-\frac{5}{2}\chi\, u_{o,2}{}^2\int_0^1 \Phi\Phi''\Phi'^2 d\xi_1+\left(\frac{\mathrm{E}\, c_T\, u_{o,2}{}^2}{4}\left(h-1\right)+\left(\gamma_1+\gamma_2\right)+\frac{\mathrm{E}\, c_N\, u_{o,2}{}^2}{2}\right)\int_0^1 \Phi\left(\Phi'\right)^3 d\xi_1 \\ &-2\beta_{m,1}\int_0^1 \Phi\Phi'^2\Phi^{(4)} d\xi_1-8\beta_{m,1}\int_0^1 \Phi\Phi'\Phi''\Phi''' d\xi_1-2\beta_{m,1}\int_0^1 \Phi\left(\Phi''\right)^3 d\xi_1 \end{aligned} \tag{6.188}$$

$$\begin{aligned} F_7{}^{v} =& -\beta_{m,1}\alpha_2\left(2\int_0^1 \Phi\Phi'^2\Phi^{(4)} d\xi_1+24\int_0^1 \Phi\Phi'\Phi''\Phi''' d\xi_1+6\int_0^1 \Phi\left(\Phi''\right)^3 d\xi_1\right) \\ &-\frac{3}{2}\Pi_{0,2}\alpha_2\int_0^1 \Phi\Phi'^2\Phi''\, d\xi_1+\frac{\mathrm{E}\, u_{o,2}}{2}\sqrt{\beta_{o,2}}\left(c_T-c_N\right)\int_0^1 \Phi^2\Phi'^2 d\xi_1 \\ &+\chi\, u_{o,2}\sqrt{\beta_{o,2}}\left(\frac{7}{2}\int_0^1 \Phi\left(\Phi'\right)^3 d\xi_1+\frac{3}{2}\int_0^1 \Phi^2\,\Phi'\Phi''\, d\xi_1\right) \end{aligned} \tag{6.189}$$

$$F_8{}^{v} = \frac{\mathrm{E}\,\beta_{o,2}}{4}\left(c_T-c_N\right)\int_0^1 \Phi^3\Phi' d\xi_1-\frac{3}{2}\chi\beta_{o,2}\int_0^1 \Phi^2\Phi'^2 d\xi_1 \tag{6.190}$$

$$F_9{}^{v} = -\frac{\mathrm{E}\, c_N}{4u_{o,2}}\beta_{o,2}{}^{\frac{3}{2}}\int_0^1 \Phi^4\, d\xi_1 \tag{6.191}$$

$$F_{10}{}^{v} = -\chi\,\beta_{o,1}\int_0^1 \Phi^2\Phi'^2\, d\xi_1 \tag{6.192}$$

$$F_{11}{}^{v} = \beta_{i,3}\left(\int_0^1 \Phi\,\Phi''\,\Theta'^2 d\xi_1+2\int_0^1 \Phi\,\Phi'\,\Theta'\Theta'' d\xi_1\right) \tag{6.193}$$

$$F_{12}{}^{v} = -\frac{\chi\,\beta_{o,2}}{4}\int_0^1 \Phi^2\,\Xi^2 d\xi_1 \tag{6.194}$$

$$M_4{}^w = \left(1-\beta_{o,2}-\beta_{i,2}-\beta_{i,3}+\chi\,\beta_{o,2}\right)\int_0^1 \Theta^2\,d\xi_1 + 2\beta_{i,2}\int_0^1 \Theta\,\Theta''\,d\xi_1 \tag{6.195}$$

$$\begin{aligned} C_4{}^w = &\left(2u_{i,2}\sqrt{\beta_{i,1}} - 2\chi\,u_{o,2}\sqrt{\beta_{o,2}}\right)\int_0^1 \Theta\,\Theta' d\xi_1 + \alpha_2\beta_{m,1}\int_0^1 \Theta\,\Theta^{(4)} d\xi_1 \\ &+\left(\frac{\mathrm{E}\,c_N\,u_{o,2}}{2}\sqrt{\beta_{o,2}} + \chi\,\beta_{o,2}\,f_t\right)\int_0^1 \Theta^2\,d\xi_1 \end{aligned} \tag{6.196}$$

$$\begin{aligned} K_4{}^w = &\left(\begin{array}{l} u_{i,2}{}^2 - \left(\overline{\Gamma}_2 - \Pi_{iL,2} + \Pi_{oL,2}\right) + \chi\,u_{o,2}{}^2 - (1-2\nu)\overline{\Pi}_2 \\ -\left(\dfrac{\mathrm{E}\,c_T\,u_{o,2}{}^2}{2}\left(1-\overline{h}\right) + \left(\gamma_1-\gamma_2\right)\right)\left(\dfrac{1}{2}-\xi_1\right) \end{array}\right)\int_0^1 \Theta\,\Theta''\,d\xi_1 \\ &-\left(\frac{\mathrm{E}\left(c_N + c_T\,\overline{h}\right)u_{o,2}{}^2}{2} - \left(\gamma_1-\gamma_2\right)\right)\int_0^1 \Theta\,\Theta'\,d\xi_1 + \beta_{m,1}\int_0^1 \Theta\Theta^{(4)}\,d\xi_1 \end{aligned} \tag{6.197}$$

$$\begin{aligned} D_6{}^w = &\left(\begin{array}{l} \left(\dfrac{\mathrm{E}\,c_T\,u_{o,2}{}^2}{2}\left(1-\overline{h}\right) + \left(\gamma_1-\gamma_2\right)\right)\left(\dfrac{1}{2}-\xi_1\right) \\ +\left(\overline{\Gamma}_2 - \Pi_{iL,2} + \Pi_{oL,2} - \Pi_{0,2}\right) + (1-2\nu)\overline{\Pi}_2 \end{array}\right)\int_0^1 \left(\Theta\Psi''\Theta' + \Theta\Psi'\Theta''\right)d\xi_1 \\ &+\frac{\mathrm{E}\,u_{o,2}{}^2\left(c_N + c_T\right)}{2}\int_0^1 \Theta\Psi'\Theta'\,d\xi_1 + \left(\frac{\mathrm{E}\,c_T\,u_{o,2}{}^2\left(1-\overline{h}\right)}{2} + \left(\gamma_1-\gamma_2\right)\right)\int_0^1 \Theta\Psi\Theta''\,d\xi_1 \\ &-3\beta_{m,1}\int_0^1 \Theta\Psi'''\Theta''\,d\xi_1 - 4\beta_{m,1}\int_0^1 \Theta\Psi''\Theta'''\,d\xi_1 - 2\beta_{m,1}\int_0^1 \Theta\Psi'\Theta^{(4)}\,d\xi_1 \\ &-\beta_{m,1}\int_0^1 \Theta\Psi^{(4)}\Theta'\,d\xi_1 - \chi\,u_{o,1}{}^2\int_0^1 \left(2\Theta\Psi''\Theta' + 4\Theta\Psi'\Theta''\right)d\xi_1 \end{aligned} \tag{6.198}$$

$$\begin{aligned} D_7{}^w = &-\beta_{m,1}\alpha_2\left(3\int_0^1 \Theta\Psi'''\Theta''\,d\xi_1 + 4\int_0^1 \Theta\Psi''\Theta'''\,d\xi_1 + 2\int_0^1 \Theta\Psi'\Theta^{(4)}\,d\xi_1 + \int_0^1 \Theta\Psi^{(4)}\Theta'\,d\xi_1\right) \\ &+3\chi u_{o,2}\sqrt{\beta_{o,2}}\int_0^1 \Theta\Psi'\Theta' d\xi_1 + 2\chi u_{o,2}\sqrt{\beta_{o,2}}\int_0^1 \Theta\Psi\Theta'' d\xi_1 - \Pi_{0,2}\alpha_2\int_0^1 \left(\Theta\Psi''\Theta' + \Theta\Psi'\Theta''\right)d\xi_1 \end{aligned} \tag{6.199}$$

$$D_8{}^{w} = -\beta_{m,2}\alpha_2\left(3\int_0^1 \Theta\Psi'''\Theta''\,d\xi_1 + 4\int_0^1 \Theta\Psi''\Theta'''\,d\xi_1 + 2\int_0^1 \Theta\Psi'\Theta^{(4)}\,d\xi_1 + \int_0^1 \Theta\Psi^{(4)}\Theta'\,d\xi_1\right)$$
$$-\Pi_{0,2}\alpha_2\int_0^1 \left(\Theta\Psi''\Theta' + \Theta\Psi'\Theta''\right)d\xi_1 + \frac{\mathrm{E}\,c_N\,u_{o,2}\sqrt{\beta_{o,2}}}{2}\int_0^1 \Theta^2\Psi'\,d\xi_1$$
$$+4\chi\,u_{o,2}\sqrt{\beta_{o,2}}\int_0^1 \Theta\Psi'\Theta'\,d\xi_1 \tag{6.200}$$

$$D_9{}^{w} = \frac{\mathrm{E}\,c_N}{2}\beta_{o,2}\int_0^1 \Theta^2\Psi\,d\xi_1 - 2\chi\beta_{o,2}\int_0^1 \Theta\Psi\,\Theta' d\xi_1 \tag{6.201}$$

$$D_{10}{}^{w} = -\chi\,\beta_{o,2}\int_0^1 \Theta\,\Psi\Theta'\,d\xi_1 \tag{6.202}$$

$$D_{11}{}^{w} = \frac{\chi\,\beta_{o,2}\,f_t}{2}\int_0^1 \Phi\,\Theta\,\Xi\,d\xi_1 \tag{6.203}$$

$$D_{12}{}^{w} = \chi\,\beta_{o,2}\int_0^1 \Phi\,\Theta\,\Xi\,d\xi_1 \tag{6.204}$$

$$D_{13}{}^{w} = \beta_{i,3}\left(\int_0^1 \frac{\Theta\Xi\Phi'\Theta''}{\Theta'}d\xi_1 + \int_0^1 \Theta\,\Xi\Phi''d\xi_1 + \int_0^1 \Theta\,\Xi'\Phi' d\xi_1\right) \tag{6.205}$$

$$E_6{}^{w} = -4\,\beta_{m,1}\alpha_2\int_0^1 \Theta\,\Theta'\Theta^{(4)}\,d\xi_1 - 3\beta_{m,1}\alpha_2\Pi_{o,2}\int_0^1 \Theta\,\Theta'\Theta''\,d\xi_1 \tag{6.206}$$

$$E_7{}^{w} = -\frac{\mathrm{E}\,c_d\,u_{o,2}{}^2}{2}\int_0^1 \Theta\,\Theta'\left|\Theta'\right|d\xi_1 \tag{6.207}$$

$$E_8{}^{w} = \frac{\mathrm{E}\,c_d\,u_{o,2}}{2}\sqrt{\beta_{o,2}}\int_0^1 \Theta\left|\Theta'\right|\Theta\,d\xi_1 \tag{6.208}$$

$$E_9{}^{w} = \frac{\mathrm{E}\,c_d\,u_{o,2}}{2}\sqrt{\beta_{o,2}}\int_0^1 \Theta\,\Theta'\left|\Theta\right|d\xi_1 \tag{6.209}$$

$$E_{10}{}^{w} = \frac{\mathrm{E}\, c_d}{2} \beta_{o,2} \int_0^1 \Theta^2 |\Theta| \, d\xi_1 \tag{6.210}$$

$$\begin{aligned} F_6{}^{w} = {} & \frac{3}{2} \left(\begin{array}{l} \left(\frac{\mathrm{E}\, c_T\, u_{o,2}{}^2}{2} \left(1 - \bar{h}\right) + \left(\gamma_1 - \gamma_2\right) \right) \left(\frac{1}{2} - \xi_1 \right) + \left(1 - 2\nu\right) \overline{\Pi}_2 \\ + \left(\overline{\Gamma_2} - \Pi_{iL,2} + \Pi_{oL,2} - \Pi_{0,2} \right) \end{array} \right) \int_0^1 \Theta\, \Theta'^2 \Theta'' \, d\xi_1 \\ & - \frac{5}{2} \chi\, u_{o,2}{}^2 \int_0^1 \Theta\, \Theta'' \Theta'^2 d\xi_1 + \left(\frac{\mathrm{E}\, c_T\, u_{o,2}{}^2}{4} \left(\bar{h} - 1\right) + \left(\gamma_1 + \gamma_2\right) + \frac{\mathrm{E}\, c_N\, u_{o,2}{}^2}{2} \right) \int_0^1 \Theta \left(\Theta'\right)^3 d\xi_1 \\ & - 2\beta_{m,1} \int_0^1 \Theta\, \Theta'^2 \Theta^{(4)} d\xi_1 - 8\beta_{m,1} \int_0^1 \Theta\, \Theta' \Theta'' \Theta''' d\xi_1 - 2\beta_{m,1} \int_0^1 \Theta \left(\Theta''\right)^3 d\xi_1 \end{aligned} \tag{6.211}$$

$$\begin{aligned} F_7{}^{w} = {} & -\beta_{m,1} \alpha_2 \left(2 \int_0^1 \Theta\, \Theta'^2 \Theta^{(4)} d\xi_1 + 24 \int_0^1 \Theta\, \Theta' \Theta'' \Theta''' d\xi_1 + 6 \int_0^1 \Theta \left(\Theta''\right)^3 d\xi_1 \right) \\ & - \frac{3}{2} \Pi_{0,2} \alpha_2 \int_0^1 \Theta\, \Theta'^2 \Theta'' \, d\xi_1 + \frac{\mathrm{E}\, u_{o,2}}{2} \sqrt{\beta_{o,2}} \left(c_T - c_N\right) \int_0^1 \Theta^2 \Theta'^2 d\xi_1 \\ & + \chi\, u_{o,2} \sqrt{\beta_{o,2}} \left(\frac{7}{2} \int_0^1 \Theta \left(\Theta'\right)^3 d\xi_1 + \frac{3}{2} \int_0^1 \Theta^2\, \Theta' \Theta'' \, d\xi_1 \right) \end{aligned} \tag{6.212}$$

$$F_8{}^{w} = \frac{\mathrm{E}\, \beta_{o,2}}{4} \left(c_T - c_N\right) \int_0^1 \Theta^3 \Theta' d\xi_1 - \frac{3}{2} \chi \beta_{o,2} \int_0^1 \Theta^2 \Theta'^2 d\xi_1 \tag{6.213}$$

$$F_9{}^{w} = -\frac{\mathrm{E}\, c_N}{4 u_{o,2}} \beta_{o,2}{}^{\frac{3}{2}} \int_0^1 \Theta^4 \, d\xi_1 \tag{6.214}$$

$$F_{10}{}^{w} = -\chi\, \beta_{o,1} \int_0^1 \Theta^2 \Theta'^2 \, d\xi_1 \tag{6.215}$$

$$F_{11}{}^{w} = 0 \tag{6.216}$$

$$F_{12}{}^{w} = -\frac{\chi\, \beta_{o,2}}{4} \int_0^1 \Theta^2\, \Xi^2 d\xi_1 \tag{6.217}$$

$$F_{13}{}^{w} = \beta_{i,3}\left(2\int_0^1 \Theta\,\Phi'^2\Theta''d\xi_1 + 3\int_0^1 \Theta\,\Theta'\Phi'\Phi''d\xi_1\right) \tag{6.218}$$

$$F_{14}{}^{w} = 2\,\beta_{i,3}\int_0^1 \Theta\Theta'\Phi'\Phi''d\xi_1 + 2\beta_{i,3}\int_0^1 \Theta\,\Theta''\Phi'^2 d\xi_1 \tag{6.219}$$

$$F_{15}{}^{w} = \beta_{i,3}\left(\int_0^1 \Theta\,\Xi\,\Phi' d\xi_1 + \int_0^1 \Theta\,\Xi\,\Phi'' d\xi_1 - \int_0^1 \frac{\Theta\Xi\,\Theta''\Phi'}{\Theta'} d\xi_1 + \int_0^1 \Theta\,\Xi'\Phi' d\xi_1\right) \tag{6.220}$$

$$M_4{}^{\varphi} = \beta_{i,3}\int_0^1 \Xi^{\,2} d\xi_2 \tag{6.221}$$

$$K_4{}^{\varphi} = -\,\beta_{m,2}\int_0^1 \Xi\,\Xi''\,d\xi_2 \tag{6.222}$$

$$T_9{}^{\varphi} = 8\,\beta_{i,4}\,f_r\int_0^1 \Xi^2\,\Xi'\,d\xi_2 \tag{6.223}$$

$$T_{10}{}^{\varphi} = 4\,\beta_{i,4}\,f_r\int_0^1 \Xi^2\,\Xi'\,d\xi_2 \tag{6.224}$$

$$T_{11}{}^{\varphi} = \beta_{i,3}\int_0^1 \Xi\,\Theta'\Phi'\,d\xi_2 \tag{6.225}$$

$$T_{12}{}^{\varphi} = \beta_{i,3}\int_0^1 \Xi\,\Theta'\Phi'\,d\xi_2 \tag{6.226}$$

6.1.5 Extensible rotating flexible pipe conveying fluid downwards, which then flows upwards as a confined annular flow

In this subsection, the dimensionless equations of motion (5.67-5.70), that describe the vibration of extensible fixed-simply supported pipe, conveying fluid downwards, which then flows upwards as a confined flow, are discretized via Galerkin's

method using single mode approach. Following the same procedure presented in section 6.1.4, we obtain the following coupled ordinary nonlinear differential equations in terms of the modal coordinates p, q, r and $\breve{g}$

$$\left(\begin{array}{l}\left[M_5{}^u\right]\ddot{p}+\left[C_5{}^u\right]\dot{p}+\left[K_5{}^u\right]p+\left[A_{13}{}^u\right]q^2+\left[A_{14}{}^u\right]r^2+\left[A_{15}{}^u\right]q\,\dot{q}+\left[A_{16}{}^u\right]r\,\dot{r}\\ +\left[A_{17}{}^u\right]\dot{q}^2+\left[A_{18}{}^u\right]\dot{r}^2+\left[A_{19}{}^u\right]q\,\ddot{q}+\left[A_{20}{}^u\right]r\,\ddot{r}+\left[B_{13}{}^u\right]q^2|q|+\left[B_{14}{}^u\right]r^2|r|\\ +\left[B_{15}{}^u\right]q|q|\,\dot{q}+\left[B_{16}{}^u\right]r|r|\,\dot{r}+\left[B_{17}{}^u\right]q^2|\dot{q}|+\left[B_{18}{}^u\right]r^2|\dot{r}|+\left[B_{19}{}^u\right]q\,\dot{q}|\dot{q}|+\left[B_{20}{}^u\right]r\,\dot{r}|\dot{r}|\end{array}\right)=0$$

(6.227)

$$\left(\begin{array}{l}\left[M_5{}^v\right]\ddot{q}+\left[C_5{}^v\right]\dot{q}+\left[K_5{}^v\right]q+\left[D_{13}{}^v\right]p\,q+\left[D_{14}{}^v\right]\dot{p}\,q+\left[D_{15}{}^v\right]p\,\dot{q}\\ +\left[D_{16}{}^v\right]\dot{p}\,\dot{q}+\left[D_{17}{}^v\right]\ddot{p}\,q+\left[D_{18}{}^v\right]\dot{\breve{g}}\,r+\left[D_{19}{}^v\right]\dot{\breve{g}}\,\dot{r}+\left[E_{11}{}^v\right]\dot{q}\,q+\left[E_{12}{}^v\right]q|q|\\ +\left[E_{13}{}^v\right]\dot{q}|q|+\left[E_{14}{}^v\right]q|\dot{q}|+\left[E_{15}{}^v\right]\dot{q}|\dot{q}|+\left[F_{13}{}^v\right]q^3+\left[F_{14}{}^v\right]q^2\,\dot{q}+\left[F_{15}{}^v\right]q\,\dot{q}^2\\ +\left[F_{16}{}^v\right]\dot{q}^3+\left[F_{17}{}^v\right]q^2\ddot{q}+\left[F_{18}{}^v\right]q\,\dot{r}^2+\left[F_{19}{}^v\right]\dot{\breve{g}}^2\,q\end{array}\right)=0 \qquad (6.228)$$

$$\left(\begin{array}{l}\left[M_5{}^w\right]\ddot{r}+\left[C_5{}^w\right]\dot{r}+\left[K_5{}^w\right]r+\left[D_{14}{}^w\right]p\,r+\left[D_{15}{}^w\right]\dot{p}\,r+\left[D_{16}{}^w\right]p\,\dot{r}+\left[D_{17}{}^w\right]\dot{p}\,\dot{r}\\ +\left[D_{18}{}^w\right]\ddot{p}\,r+\left[D_{19}{}^w\right]\dot{\breve{g}}\,q+\left[D_{20}{}^w\right]\dot{\breve{g}}\,\dot{q}+\left[D_{21}{}^w\right]\ddot{\breve{g}}\,q+\left[E_{11}{}^w\right]\dot{r}\,r+\left[E_{12}{}^w\right]r|r|+\left[E_{13}{}^w\right]\dot{r}|r|\\ +\left[E_{14}{}^w\right]r|\dot{r}|+\left[E_{15}{}^w\right]\dot{r}|\dot{r}|+\left[F_{16}{}^w\right]r^3+\left[F_{17}{}^w\right]r^2\,\dot{r}+\left[F_{18}{}^w\right]r\,\dot{r}^2+\left[F_{19}{}^w\right]\dot{r}^3+\left[F_{20}{}^w\right]r^2\,\ddot{r}\\ +\left[F_{21}{}^w\right]r\,\dot{q}^2+\left[F_{22}{}^w\right]\dot{\breve{g}}^2\,r+\left[F_{23}{}^w\right]q^2\,\ddot{r}+\left[F_{24}{}^w\right]\dot{r}\,q\,\dot{q}+\left[F_{25}{}^w\right]\dfrac{\ddot{r}\,\dot{g}\,q}{\dot{r}}\end{array}\right)=0$$

(6.229)

and,

$$\left[M_5{}^\varphi\right]\ddot{\breve{g}}+\left[K_5{}^\varphi\right]\breve{g}+\left[T_{13}{}^\varphi\right]\dot{\breve{g}}^2+\left[T_{14}{}^\varphi\right]\ddot{\breve{g}}\,\breve{g}+\left[T_{15}{}^\varphi\right]\dot{r}\,\dot{q}+\left[T_{16}{}^\varphi\right]q\,\ddot{r}=0 \qquad (6.230)$$

where the coefficients can be expressed as

$$M_5^{\,u} = \left(1-\beta_{o,2}-\beta_{i,2}-\beta_{i,3}\right)\int_0^1 \Psi^2\, d\xi_1 \tag{6.231}$$

$$C_5^{\,u} = 2u_{i,2}\sqrt{\beta_{i,1}}\int_0^1 \Psi\Psi' d\xi_1 - \Pi_{0,2}\alpha_2\int_0^1 \Psi\Psi'' d\xi_1 \tag{6.232}$$

$$K_5^{\,u} = \left(u_{i,2}^{\;2}-\Pi_{0,2}\right)\int_0^1 \Psi''\Psi .d\xi_1 + \left(\frac{\mathrm{E}\,c_T\,u_{i,2}^{\;2}}{2}\left(\frac{\lambda_i^{\;2}}{\lambda_{ch}^{\;2}-1}\right)^2\left(1-\bar{h}\right)+\left(\gamma_1-\gamma_2\right)\right)\int_0^1 \Psi\Psi' d\xi_1 \tag{6.233}$$

$$\begin{aligned} A_{13}^{\;u} = & \left(\begin{array}{l} \overline{\Gamma_2}-(1-2\nu)\overline{\Pi}_3-\aleph_1-\aleph_2\,u_{i,2}^{\;2}\left(\dfrac{\lambda_i^{\;2}}{\lambda_{ch}^{\;2}-1}\right)^2 \\ -\aleph_3\,u_{i,2}^{\;2}\left(\overline{C}\left(1-\dfrac{\lambda_i^{\;2}}{\lambda_{ch}^{\;2}-1}\right)^2-1\right)-\chi\,u_{i,2}^{\;2}\left(\dfrac{\lambda_i^{\;2}}{\lambda_{ch}^{\;2}-1}\right)^2-\Pi_{0,2} \\ +\left(\dfrac{\mathrm{E}\,c_T\,u_{i,2}^{\;2}}{2}\left(\dfrac{\lambda_i^{\;2}}{\lambda_{ch}^{\;2}-1}\right)^2\left(1-\bar{h}\right)+\left(\gamma_1-\gamma_2\right)\right)\left(\dfrac{1}{2}-\xi_1\right) \end{array}\right)\int_0^1 \Psi\Phi'\Phi'' d\xi_1 \\ & +\frac{\mathrm{E}\,u_{i,2}^{\;2}}{2}\left(\frac{\lambda_i^{\;2}}{\lambda_{ch}^{\;2}-1}\right)^2\left(c_N-c_T\left(\frac{3}{2}-\frac{\bar{h}}{2}\right)\right)\int_0^1 \Psi\Phi'^2 d\xi_1 \\ & -\beta_{m,1}\left(\int_0^1 \Psi\Phi^{(4)}\Phi'\, d\xi_1+\int_0^1 \Psi\Phi''\Phi'''\, d\xi_1\right) \end{aligned} \tag{6.234}$$

$$A_{14}{}^{u}=\left(\begin{array}{l}\overline{\Gamma_2}-(1-2\nu)\overline{\Pi}_3-\aleph_1-\aleph_2\,u_{i,2}{}^{2}\left(\dfrac{\lambda_i{}^{2}}{\lambda_{ch}{}^{2}-1}\right)^{2}\\ -\aleph_3\,u_{i,2}{}^{2}\left(\overline{C}\left(1-\dfrac{\lambda_i{}^{2}}{\lambda_{ch}{}^{2}-1}\right)^{2}-1\right)\\ -\chi\,u_{i,2}{}^{2}\left(\dfrac{\lambda_i}{\lambda_{ch}{}^{2}-1}\right)^{2}-\Pi_{0,2}\\ +\left(\dfrac{\mathrm{E}\,c_T\,u_{i,2}{}^{2}}{2}(1-\overline{h})\left(\dfrac{\lambda_i{}^{2}}{\lambda_{ch}{}^{2}-1}\right)^{2}+(\gamma_1-\gamma_2)\right)\left(\dfrac{1}{2}-\xi_1\right)\end{array}\right)\int_0^1\Psi\Theta'\Theta''d\xi_1$$

$$+\frac{\mathrm{E}\,u_{i,2}{}^{2}}{2}\left(\frac{\lambda_i{}^{2}}{\lambda_{ch}{}^{2}-1}\right)^{2}\left(c_N-c_T\left(\frac{3}{2}-\frac{\overline{h}}{2}\right)\right)\int_0^1\Psi\Theta'^{2}d\xi_1 \tag{6.235}$$

$$-\beta_{m,1}\left(\int_0^1\Psi\Theta^{(4)}\Theta'\,d\xi_1+\int_0^1\Psi\Theta''\Theta'''d\xi_1\right)$$

$$A_{15}{}^{u}=\left(\frac{\mathrm{E}\,u_{i,2}\sqrt{\beta_{o,2}}}{2}\frac{\lambda_i{}^{2}}{\lambda_{ch}{}^{2}-1}(c_T-c_N)\right)\int_0^1\Psi\Phi\Phi'\,d\xi_1-(2\Pi_{o,2}\,\alpha_2)\int_0^1\Psi\Phi'\Phi''\,d\xi_1$$

$$+2\,\chi\,u_{i,2}\sqrt{\beta_{o,2}}\,\frac{\lambda_i{}^{2}}{\lambda_{ch}{}^{2}-1}\int_0^1\Psi\Phi'^{2}\,d\xi_1-2\alpha_2\left(\int_0^1\Psi\Phi^{(4)}\Phi'dX+\int_0^1\Psi\Phi''\Phi'''d\xi_1\right) \tag{6.236}$$

$$A_{16}{}^{u}=\left(\frac{\mathrm{E}\,u_{i,2}\sqrt{\beta_{o,2}}}{2}\frac{\lambda_i{}^{2}}{\lambda_{ch}{}^{2}-1}(c_T-c_N)\right)\int_0^1\Psi\Theta\Theta'\,d\xi_1-(2\Pi_{0,2}\,\alpha_2)\int_0^1\Psi\Theta'\Theta''\,d\xi_1$$

$$+2\,\chi\,u_{i,2}\sqrt{\beta_{o,2}}\,\frac{\lambda_i{}^{2}}{\lambda_{ch}{}^{2}-1}\int_0^1\Psi\Theta'^{2}\,d\xi_1-2\alpha_2\left(\int_0^1\Psi\Theta^{(4)}\Theta'dX+\int_0^1\Psi\Theta''\Theta'''d\xi_1\right) \tag{6.237}$$

$$A_{17}{}^{u}=\frac{\mathrm{E}\,c_T\beta_{o,2}}{4}\int_0^1\Psi\Phi^{2}d\xi_1 \tag{6.238}$$

$$A_{18}{}^{u}=\frac{\mathrm{E}\,c_T\beta_{o,2}}{4}\int_0^1\Psi\Theta^{2}d\xi_1 \tag{6.239}$$

$$A_{19}{}^{u}=-\chi\,\beta_{o,2}\int_0^1\Psi\Phi\Phi'\,d\xi_1 \tag{6.240}$$

$$A_{20}{}^{u} = -\chi\,\beta_{o,2}\int_0^1 \Psi\Theta\Theta'\,d\xi_1 \tag{6.241}$$

$$B_{13}{}^{u} = \frac{\mathrm{E}\,c_d\,u_{i,2}{}^2}{2}\left(\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2\int_0^1 \Psi\Phi'^2\left|\Phi'\right|d\xi_1 \tag{6.242}$$

$$B_{14}{}^{u} = \frac{\mathrm{E}\,c_d\,u_{i,2}{}^2}{2}\left(\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2\int_0^1 \Psi\Theta'^2\left|\Theta'\right|d\xi_1 \tag{6.243}$$

$$B_{15}{}^{u} = -\frac{\mathrm{E}\,c_d\,u_{i,2}}{2}\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\sqrt{\beta_{o,2}}\int_0^1 \Psi\Phi'\left|\Phi'\right|\Phi\,d\xi_1 \tag{6.244}$$

$$B_{16}{}^{u} = -\frac{\mathrm{E}\,c_d\,u_{i,2}}{2}\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\sqrt{\beta_{o,2}}\int_0^1 \Psi\Theta'\left|\Theta'\right|\Theta\,d\xi_1 \tag{6.245}$$

$$B_{17}{}^{u} = -\frac{\mathrm{E}\,c_d\,u_{i,2}}{2}\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\sqrt{\beta_{o,2}}\int_0^1 \Psi\Phi'^2\left|\Phi\right|d\xi_1 \tag{6.246}$$

$$B_{18}{}^{u} = -\frac{\mathrm{E}\,c_d\,u_{i,2}}{2}\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\sqrt{\beta_{o,2}}\int_0^1 \Psi\Theta'^2\left|\Theta\right|d\xi_1 \tag{6.247}$$

$$B_{19}{}^{u} = -\frac{\mathrm{E}\,c_d}{2}\beta_{o,2}\int_0^1 \Psi\Phi'\left|\Phi\right|\Phi\,d\xi_1 \tag{6.248}$$

$$B_{20}{}^{u} = -\frac{\mathrm{E}\,c_d}{2}\beta_{o,2}\int_0^1 \Psi\Theta'\left|\Theta\right|\Theta\,d\xi_1 \tag{6.249}$$

$$M_5{}^{v} = \left(1-\beta_{o,2}-\beta_{i,2}-\beta_{i,3}+\chi\,\beta_{o,2}\right)\int_0^1 \Phi^2\,d\xi_1 + 2\beta_{i,2}\int_0^1 \Phi\,\Phi''\,d\xi_1 \tag{6.250}$$

$$\begin{aligned} C_5{}^{v} = &\left(2u_{i,2}\sqrt{\beta_{i,1}} - 2\chi\,u_{i,2}\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\sqrt{\beta_{o,2}}\right)\int_0^1 \Phi\Phi'\,d\xi_1 + \alpha_2\beta_{m,1}\int_0^1 \Phi\Phi^{(4)}\,d\xi_1 \\ &+\left(\frac{\mathrm{E}\,c_N\,u_{i,2}}{2}\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\sqrt{\beta_{o,2}} + \chi\,\beta_{o,2}\,f_t\right)\int_0^1 \Phi^2\,d\xi_1 \end{aligned} \tag{6.251}$$

$$K_5^{\,v} = \left(\begin{array}{l} u_{i,2}^{\;2} - \left(\begin{array}{l} \overline{\Gamma_2} - (1-2\nu)\overline{\Pi}_3 - \aleph_1 - \aleph_2\, u_{i,2}^{\;2} \left(\dfrac{\lambda_i^{\;2}}{\lambda_{ch}^{\;2} - 1} \right)^2 \\ - \aleph_3\, u_{i,2}^{\;2} \left(\overline{C} \left(1 - \dfrac{\lambda_i^{\;2}}{\lambda_{ch}^{\;2} - 1} \right)^2 - 1 \right) \end{array} \right) \\ + \chi\, u_{i,2}^{\;2} \left(\dfrac{\lambda_i^{\;2}}{\lambda_{ch}^{\;2} - 1} \right)^2 \\ - \left(\dfrac{\mathrm{E}\, c_T\, u_{i,2}^{\;2}}{2} \left(\dfrac{\lambda_i^{\;2}}{\lambda_{ch}^{\;2} - 1} \right)^2 \left(1 - \overline{h}\right) + (\gamma_1 - \gamma_2) \right) \left(\dfrac{1}{2} - \xi_1 \right) \end{array} \right) \int_0^1 \Phi\Phi''\, d\xi_1$$

$$- \left(\dfrac{\mathrm{E}\left(c_N + c_T\, \overline{h}\right) u_{i,2}^{\;2}}{2} \left(\dfrac{\lambda_i^{\;2}}{\lambda_{ch}^{\;2} - 1} \right)^2 - (\gamma_1 - \gamma_2) \right) \int_0^1 \Phi\Phi'\, d\xi_1 + \beta_{m,1} \int_0^1 \Phi\Phi^{(4)}\, d\xi_1 \tag{6.252}$$

$$D_{13}^{\;v} = \left(\begin{array}{l} \left(\dfrac{\mathrm{E}\, c_T\, u_{i,2}^{\;2}}{2} \left(\dfrac{\lambda_i^{\;2}}{\lambda_{ch}^{\;2} - 1} \right)^2 \left(1 - \overline{h}\right) + (\gamma_1 - \gamma_2) \right) \left(\dfrac{1}{2} - \xi_1 \right) \\ + \left(\begin{array}{l} \overline{\Gamma_2} - (1-2\nu)\overline{\Pi}_3 - \aleph_1 - \aleph_2\, u_{i,2}^{\;2} \left(\dfrac{\lambda_i^{\;2}}{\lambda_{ch}^{\;2} - 1} \right)^2 \\ - \aleph_3\, u_{i,2}^{\;2} \left(\overline{C} \left(1 - \dfrac{\lambda_i^{\;2}}{\lambda_{ch}^{\;2} - 1} \right)^2 - 1 \right) - \Pi_{0,2} \end{array} \right) \end{array} \right) \int_0^1 \left(\Phi\Psi''\Phi' + \Phi\Psi'\Phi'' \right) d\xi_1$$

$$+ \dfrac{\mathrm{E}\, u_{i,2}^{\;2} \left(c_N + c_T\right)}{2} \left(\dfrac{\lambda_i^{\;2}}{\lambda_{ch}^{\;2} - 1} \right)^2 \int_0^1 \Phi\Psi'\Phi'\, d\xi_1$$

$$+ \left(\dfrac{\mathrm{E}\, c_T\, u_{i,2}^{\;2} \left(1 - \overline{h}\right)}{2} \left(\dfrac{\lambda_i^{\;2}}{\lambda_{ch}^{\;2} - 1} \right)^2 + (\gamma_1 - \gamma_2) \right) \int_0^1 \Phi\Psi\Phi''\, d\xi_1$$

$$- 3\beta_{m,1} \int_0^1 \Phi\Psi'''\Phi''\, d\xi_1 - 4\beta_{m,1} \int_0^1 \Phi\Psi''\Phi'''\, d\xi_1 - 2\beta_{m,1} \int_0^1 \Phi\Psi'\Phi^{(4)}\, d\xi_1$$

$$- \beta_{m,1} \int_0^1 \Phi\Psi^{(4)}\Phi'\, d\xi_1 - \chi\, u_{i,2}^{\;2} \left(\dfrac{\lambda_i^{\;2}}{\lambda_{ch}^{\;2} - 1} \right)^2 \int_0^1 \left(2\Phi\Psi''\Phi' + 4\Phi\Psi'\Phi'' \right) d\xi_1 \tag{6.253}$$

$$D_{14}{}^{v} = -\beta_{m,1}\alpha_2\left(3\int_0^1 \Phi\Psi'''\Phi''\,d\xi_1 + 4\int_0^1 \Phi\Psi''\Phi'''\,d\xi_1 + 2\int_0^1 \Phi\Psi'\Phi^{(4)}\,d\xi_1 + \int_0^1 \Phi\Psi^{(4)}\Phi'\,d\xi_1\right)$$
$$+3\chi u_{i,2}\frac{\lambda_i^{\,2}}{\lambda_{ch}^{\,2}-1}\sqrt{\beta_{o,2}}\int_0^1 \Phi\Psi'\Phi' d\xi_1 + 2\chi u_{i,2}\frac{\lambda_i^{\,2}}{\lambda_{ch}^{\,2}-1}\sqrt{\beta_{o,2}}\int_0^1 \Phi\Psi\Phi'' d\xi_1$$
$$-\Pi_{0,2}\alpha_2\int_0^1 \left(\Phi\Psi''\Phi' + \Phi\Psi'\Phi''\right)d\xi_1 \tag{6.254}$$

$$D_{15}{}^{v} = -\beta_{m,2}\alpha_2\left(3\int_0^1 \Phi\Psi'''\Phi''\,d\xi_1 + 4\int_0^1 \Phi\Psi''\Phi'''\,d\xi_1 + 2\int_0^1 \Phi\Psi'\Phi^{(4)}\,d\xi_1 + \int_0^1 \Phi\Psi^{(4)}\Phi'\,d\xi_1\right)$$
$$-\Pi_{0,2}\alpha_2\int_0^1 \left(\Phi\Psi''\Phi' + \Phi\Psi'\Phi''\right)d\xi_1 + \frac{\mathrm{E}\,c_N\,u_{i,2}\sqrt{\beta_{o,2}}}{2}\frac{\lambda_i^{\,2}}{\lambda_{ch}^{\,2}-1}\int_0^1 \Phi^2\Psi'\,d\xi_1$$
$$+4\chi u_{i,2}\sqrt{\beta_{o,2}}\frac{\lambda_i^{\,2}}{\lambda_{ch}^{\,2}-1}\int_0^1 \Phi\Psi'\Phi'\,d\xi_1 \tag{6.255}$$

$$D_{16}{}^{v} = \frac{\mathrm{E}\,c_N}{2}\beta_{o,2}\int_0^1 \Phi^2\Psi\,d\xi_1 - 2\chi\beta_{o,2}\int_0^1 \Phi\Psi\,\Phi' d\xi_1 \tag{6.256}$$

$$D_{17}{}^{v} = -\chi\,\beta_{o,2}\int_0^1 \Phi\Psi\Phi'\,d\xi_1 \tag{6.257}$$

$$D_{18}{}^{v} = \frac{\chi\,\beta_{o,2}\,f_t}{2}\int_0^1 \Phi\,\Theta\,\Xi\,d\xi_1 \tag{6.258}$$

$$D_{19}{}^{v} = \beta_{i,3}\left(\int_0^1 \Phi\,\Theta''\,\Xi\,d\xi_1 + \int_0^1 \Phi\,\Theta'\,\Xi'\,d\xi_1\right) + \chi\,\beta_{o,2}\int_0^1 \Phi\,\Theta\,\Xi\,d\xi_1 \tag{6.259}$$

$$E_{11}{}^{v} = -4\,\beta_{m,1}\alpha_2\int_0^1 \Phi\Phi'\Phi^{(4)}\,d\xi_1 - 3\beta_{m,1}\alpha_2\Pi_{o,2}\int_0^1 \Phi\Phi'\Phi''\,d\xi_1 \tag{6.260}$$

$$E_{12}{}^{v} = -\frac{\mathrm{E}\,c_d\,u_{i,2}{}^2}{2}\left(\frac{\lambda_i^{\,2}}{\lambda_{ch}^{\,2}-1}\right)^2\int_0^1 \Phi\Phi'\left|\Phi'\right|d\xi_1 \tag{6.261}$$

$$E_{13}{}^{v} = \frac{\mathrm{E}\, c_d\, u_{i,2}}{2} \frac{\lambda_i{}^2}{\lambda_{ch}{}^2 - 1} \sqrt{\beta_{o,2}} \int_0^1 \Phi \left|\Phi'\right| \Phi \, d\xi_1 \tag{6.262}$$

$$E_{14}{}^{v} = \frac{\mathrm{E}\, c_d\, u_{i,2}}{2} \frac{\lambda_i{}^2}{\lambda_{ch}{}^2 - 1} \sqrt{\beta_{o,2}} \int_0^1 \Phi \Phi' \left|\Phi\right| d\xi_1 \tag{6.263}$$

$$E_{15}{}^{v} = \frac{\mathrm{E}\, c_d}{2} \beta_{o,2} \int_0^1 \Phi^2 \left|\Phi\right| \, d\xi_1 \tag{6.264}$$

$$\begin{aligned} F_{13}{}^{v} = \frac{3}{2} & \left(\begin{array}{l} \left(\frac{\mathrm{E}\, c_T\, u_{i,2}{}^2}{2} \left(\frac{\lambda_i{}^2}{\lambda_{ch}{}^2 - 1} \right)^2 \left(1 - \bar{h}\right) + \left(\gamma_1 - \gamma_2\right) \right) \left(\frac{1}{2} - \xi_1 \right) \\ + \left(\begin{array}{l} \overline{\Gamma_2} - \left(1 - 2\nu\right) \overline{\Pi}_3 - \aleph_1 - \aleph_2\, u_{i,2}{}^2 \left(\frac{\lambda_i{}^2}{\lambda_{ch}{}^2 - 1} \right)^2 \\ - \aleph_3\, u_{i,2}{}^2 \left(\overline{C} \left(1 - \frac{\lambda_i}{\lambda_{ch}{}^2 - 1} \right)^2 - 1 \right) - \Pi_{0,2} \end{array} \right) \end{array} \right) \int_0^1 \Phi \Phi'^2 \Phi'' \, d\xi_1 \\ & - \frac{5}{2} \chi\, u_{i,2}{}^2 \left(\frac{\lambda_i{}^2}{\lambda_{ch}{}^2 - 1} \right)^2 \int_0^1 \Phi \Phi'' \Phi'^2 d\xi_1 \\ & + \left(\frac{\mathrm{E}\, c_T\, u_{i,2}{}^2}{4} \left(\frac{\lambda_i{}^2}{\lambda_{ch}{}^2 - 1} \right)^2 \left(\bar{h} - 1\right) + \left(\gamma_1 + \gamma_2\right) + \frac{\mathrm{E}\, c_N\, u_{i,2}{}^2}{2} \left(\frac{\lambda_i{}^2}{\lambda_{ch}{}^2 - 1} \right)^2 \right) \int_0^1 \Phi \left(\Phi'\right)^3 d\xi_1 \\ & - 2\beta_{m,1} \int_0^1 \Phi \Phi'^2 \Phi^{(4)} d\xi_1 - 8\beta_{m,1} \int_0^1 \Phi \Phi' \Phi'' \Phi''' d\xi_1 - 2\beta_{m,1} \int_0^1 \Phi \left(\Phi''\right)^3 d\xi_1 \end{aligned} \tag{6.265}$$

$$\begin{aligned} F_{14}{}^{v} = & -\beta_{m,1} \alpha_2 \left(2 \int_0^1 \Phi \Phi'^2 \Phi^{(4)} d\xi_1 + 24 \int_0^1 \Phi \Phi' \Phi'' \Phi''' d\xi_1 + 6 \int_0^1 \Phi \left(\Phi''\right)^3 d\xi_1 \right) \\ & - \frac{3}{2} \Pi_{0,2} \alpha_2 \int_0^1 \Phi \Phi'^2 \Phi'' \, d\xi_1 + \frac{\mathrm{E}\, u_{i,2}}{2} \frac{\lambda_i}{\lambda_{ch}{}^2 - 1} \sqrt{\beta_{o,2}} \left(c_T - c_N\right) \int_0^1 \Phi^2 \Phi'^2 d\xi_1 \\ & + \chi\, u_{i,2} \frac{\lambda_i{}^2}{\lambda_{ch}{}^2 - 1} \sqrt{\beta_{o,2}} \left(\frac{7}{2} \int_0^1 \Phi \left(\Phi'\right)^3 d\xi_1 + \frac{3}{2} \int_0^1 \Phi^2\, \Phi' \Phi'' \, d\xi_1 \right) \end{aligned} \tag{6.266}$$

$$F_{15}{}^{v} = \frac{\mathrm{E}\, \beta_{o,2}}{4} \left(c_T - c_N\right) \int_0^1 \Phi^3 \Phi' d\xi_1 - \frac{3}{2} \chi \beta_{o,2} \int_0^1 \Phi^2 \Phi'^2 d\xi_1 \tag{6.267}$$

$$F_{16}{}^{v} = -\frac{\mathrm{E}\,c_N}{4u_{i,2}}\frac{\lambda_{ch}{}^{2}-1}{\lambda_i{}^{2}}\beta_{o,2}{}^{\frac{3}{2}}\int_0^1 \Phi^4\, d\xi_1 \tag{6.268}$$

$$F_{17}{}^{v} = -\chi\,\beta_{o,1}\int_0^1 \Phi^2\Phi'^2\, d\xi_1 \tag{6.269}$$

$$F_{18}{}^{v} = \beta_{i,3}\left(\int_0^1 \Phi\,\Phi''\Theta'^2 d\xi_1 + 2\int_0^1 \Phi\,\Phi'\,\Theta'\Theta'' d\xi_1\right) \tag{6.270}$$

$$F_{19}{}^{v} = -\frac{\chi\,\beta_{o,2}}{4}\int_0^1 \Phi^2\,\Xi^2 d\xi_1 \tag{6.271}$$

$$M_5{}^{w} = \left(1-\beta_{o,2}-\beta_{i,2}-\beta_{i,3}+\chi\,\beta_{o,2}\right)\int_0^1 \Theta^2\, d\xi_1 + 2\beta_{i,2}\int_0^1 \Theta\,\Theta''\, d\xi_1 \tag{6.272}$$

$$\begin{aligned} C_5{}^{w} = &\left(2u_{i,2}\sqrt{\beta_{i,1}} - 2\chi\,u_{i,2}\frac{\lambda_i{}^{2}}{\lambda_{ch}{}^{2}-1}\sqrt{\beta_{o,2}}\right)\int_0^1 \Theta\,\Theta' d\xi_1 + \alpha_2\beta_{m,1}\int_0^1 \Theta\,\Theta^{(4)} d\xi_1 \\ &+\left(\frac{\mathrm{E}\,c_N\,u_{i,2}}{2}\frac{\lambda_i{}^{2}}{\lambda_{ch}{}^{2}-1}\sqrt{\beta_{o,2}} + \chi\,\beta_{o,2}\,f_t\right)\int_0^1 \Theta^2\, d\xi_1 \end{aligned} \tag{6.273}$$

$$\begin{aligned} K_5{}^{w} = &\left(\begin{array}{l} u_{i,2}{}^{2} - \left(\begin{array}{l} \overline{\Gamma}_2 - (1-2\nu)\overline{\Pi}_3 - \aleph_1 - \aleph_2\,u_{i,2}{}^{2}\left(\dfrac{\lambda_i{}^{2}}{\lambda_{ch}{}^{2}-1}\right)^2 \\ -\aleph_3\,u_{i,2}{}^{2}\left(\overline{C}\left(1-\dfrac{\lambda_i{}^{2}}{\lambda_{ch}{}^{2}-1}\right)^2 - 1\right) \end{array}\right) \\ +\chi\,u_{i,2}{}^{2}\left(\dfrac{\lambda_i{}^{2}}{\lambda_{ch}{}^{2}-1}\right)^2 \\ -\left(\dfrac{\mathrm{E}\,c_T\,u_{i,2}{}^{2}}{2}\left(\dfrac{\lambda_i{}^{2}}{\lambda_{ch}{}^{2}-1}\right)^2\left(1-\overline{h}\right) + \left(\gamma_1-\gamma_2\right)\right)\left(\dfrac{1}{2}-\xi_1\right) \end{array}\right)\int_0^1 \Theta\,\Theta''\, d\xi_1 \\ &-\left(\frac{\mathrm{E}\,c_N\,u_{i,2}{}^{2}}{2}\left(\frac{\lambda_i{}^{2}}{\lambda_{ch}{}^{2}-1}\right)^2 - \left(\gamma_1-\gamma_2\right)\right)\int_0^1 \Theta\,\Theta'\, d\xi_1 + \beta_{m,1}\int_0^1 \Theta\Theta^{(4)}\, d\xi_1 \end{aligned} \tag{6.274}$$

$$
\begin{aligned}
D_{14}{}^{w} = & \left(\left(\frac{\mathrm{E}\, c_T\, u_{i,2}{}^{2}}{2} \left(\frac{\lambda_i{}^{2}}{\lambda_{ch}{}^{2}-1} \right)^{2} \left(1-\bar{h}\right) + \left(\gamma_1 - \gamma_2\right) \right) \left(\frac{1}{2} - \xi_1 \right) \right. \\
& \left. + \left(\begin{array}{l} \overline{\Gamma_2} - \left(1-2\nu\right)\overline{\Pi}_3 - \aleph_1 - \aleph_2\, u_{i,2}{}^{2} \left(\dfrac{\lambda_i}{\lambda_{ch}{}^{2}-1} \right)^{2} \\ - \aleph_3\, u_{i,2}{}^{2} \left(\overline{C} \left(1 - \dfrac{\lambda_i{}^{2}}{\lambda_{ch}{}^{2}-1} \right)^{2} - 1 \right) - \Pi_{0,2} \end{array} \right) \right) \int_0^1 \left(\Theta\Psi''\Theta' + \Theta\Psi'\Theta'' \right) d\xi_1 \\
& + \frac{\mathrm{E}\, u_{i,2}{}^{2} \left(c_N + c_T \right)}{2} \left(\frac{\lambda_i{}^{2}}{\lambda_{ch}{}^{2}-1} \right)^{2} \int_0^1 \Theta\Psi'\Theta'\, d\xi_1 \\
& + \left(\frac{\mathrm{E}\, c_T\, u_{i,2}{}^{2} \left(1-\bar{h}\right)}{2} \left(\frac{\lambda_i{}^{2}}{\lambda_{ch}{}^{2}-1} \right)^{2} + \left(\gamma_1 - \gamma_2\right) \right) \int_0^1 \Theta\Psi\Theta''\, d\xi_1 \\
& - 3\beta_{m,1} \int_0^1 \Theta\Psi'''\Theta''\, d\xi_1 - 4\beta_{m,1} \int_0^1 \Theta\Psi''\Theta'''\, d\xi_1 - 2\beta_{m,1} \int_0^1 \Theta\Psi'\Theta^{(4)}\, d\xi_1 \\
& - \beta_{m,1} \int_0^1 \Theta\Psi^{(4)}\Theta'\, d\xi_1 - \chi\, u_{o,1}{}^{2} \int_0^1 \left(2\Theta\Psi''\Theta' + 4\Theta\Psi'\Theta'' \right) d\xi_1
\end{aligned}
\tag{6.275}
$$

$$
\begin{aligned}
D_{15}{}^{w} = & -\beta_{m,1}\alpha_2 \left(3\int_0^1 \Theta\Psi'''\Theta''\, d\xi_1 + 4\int_0^1 \Theta\Psi''\Theta'''\, d\xi_1 + 2\int_0^1 \Theta\Psi'\Theta^{(4)}\, d\xi_1 + \int_0^1 \Theta\Psi^{(4)}\Theta'\, d\xi_1 \right) \\
& + 3\chi\, u_{i,2} \frac{\lambda_i{}^{2}}{\lambda_{ch}{}^{2}-1} \sqrt{\beta_{o,2}} \int_0^1 \Theta\Psi'\Theta' d\xi_1 + 2\chi\, u_{i,2} \frac{\lambda_i{}^{2}}{\lambda_{ch}{}^{2}-1} \sqrt{\beta_{o,2}} \int_0^1 \Theta\Psi\Theta'' d\xi_1 \\
& - \Pi_{0,2}\alpha_2 \int_0^1 \left(\Theta\Psi''\Theta' + \Theta\Psi'\Theta'' \right) d\xi_1
\end{aligned}
\tag{6.276}
$$

$$
\begin{aligned}
D_{16}{}^{w} = & -\beta_{m,2}\alpha_2 \left(3\int_0^1 \Theta\Psi'''\Theta''\, d\xi_1 + 4\int_0^1 \Theta\Psi''\Theta'''\, d\xi_1 + 2\int_0^1 \Theta\Psi'\Theta^{(4)}\, d\xi_1 + \int_0^1 \Theta\Psi^{(4)}\Theta'\, d\xi_1 \right) \\
& - \Pi_{0,2}\alpha_2 \int_0^1 \left(\Theta\Psi''\Theta' + \Theta\Psi'\Theta'' \right) d\xi_1 + \frac{\mathrm{E}\, c_N\, u_{i,2} \sqrt{\beta_{o,2}}}{2} \frac{\lambda_i{}^{2}}{\lambda_{ch}{}^{2}-1} \int_0^1 \Theta^2\Psi'\, d\xi_1 \\
& + 4\chi\, u_{i,2} \sqrt{\beta_{o,2}} \frac{\lambda_i{}^{2}}{\lambda_{ch}{}^{2}-1} \int_0^1 \Theta\Psi'\Theta'\, d\xi_1
\end{aligned}
\tag{6.277}
$$

$$D_{17}{}^{w} = \frac{\mathrm{E}\,c_N}{2}\beta_{o,2}\int_0^1 \Theta^2\Psi\, d\xi_1 - 2\chi\beta_{o,2}\int_0^1 \Theta\Psi\,\Theta' d\xi_1 \tag{6.278}$$

$$D_{18}{}^{w} = -\chi\,\beta_{o,2}\int_0^1 \Theta\,\Psi\Theta'\, d\xi_1 \tag{6.279}$$

$$D_{19}{}^{w} = \frac{\chi\,\beta_{o,2}\,f_t}{2}\int_0^1 \Phi\,\Theta\,\Xi\, d\xi_1 \tag{6.280}$$

$$D_{20}{}^{w} = \chi\,\beta_{o,2}\int_0^1 \Phi\,\Theta\,\Xi\, d\xi_1 \tag{6.281}$$

$$D_{21}{}^{w} = \beta_{i,3}\left(\int_0^1 \frac{\Theta\Xi\Phi'\Theta''}{\Theta'} d\xi_1 + \int_0^1 \Theta\,\Xi\Phi'' d\xi_1 + \int_0^1 \Theta\,\Xi'\Phi' d\xi_1\right) \tag{6.282}$$

$$E_{11}{}^{w} = -4\,\beta_{m,1}\alpha_2\int_0^1 \Theta\,\Theta'\Theta^{(4)}\, d\xi_1 - 3\beta_{m,1}\alpha_2\Pi_{o,2}\int_0^1 \Theta\,\Theta'\Theta''\, d\xi_1 \tag{6.283}$$

$$E_{12}{}^{w} = -\frac{\mathrm{E}\,c_d\,u_{i,2}{}^2}{2}\left(\frac{\lambda_i{}^2}{\lambda_{ch}{}^2 - 1}\right)^2\int_0^1 \Theta\,\Theta'\left|\Theta'\right| d\xi_1 \tag{6.284}$$

$$E_{13}{}^{w} = \frac{\mathrm{E}\,c_d\,u_{i,2}}{2}\frac{\lambda_i{}^2}{\lambda_{ch}{}^2 - 1}\sqrt{\beta_{o,2}}\int_0^1 \Theta\left|\Theta'\right|\Theta\, d\xi_1 \tag{6.285}$$

$$E_{14}{}^{w} = \frac{\mathrm{E}\,c_d\,u_{i,2}}{2}\frac{\lambda_i{}^2}{\lambda_{ch}{}^2 - 1}\sqrt{\beta_{o,2}}\int_0^1 \Theta\,\Theta'\left|\Theta\right| d\xi_1 \tag{6.286}$$

$$E_{15}{}^{w} = \frac{\mathrm{E}\,c_d}{2}\beta_{o,2}\int_0^1 \Theta^2\left|\Theta\right|\, d\xi_1 \tag{6.287}$$

$$F_{16}{}^{w} = \frac{3}{2}\left(\begin{array}{l}\left(\frac{\mathrm{E}\,c_T\,u_{i,2}{}^2}{2}\left(\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2\left(1-\bar{h}\right)+\left(\gamma_1-\gamma_2\right)\right)\left(\frac{1}{2}-\xi_1\right)\\ +\left(\begin{array}{l}\overline{\Gamma_2}-\left(1-2\nu\right)\overline{\Pi}_3-\aleph_1-\aleph_2\,u_{i,2}{}^2\left(\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2\\ -\aleph_3\,u_{i,2}{}^2\left(\overline{C}\left(1-\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2-1\right)-\Pi_{0,2}\end{array}\right)\end{array}\right)\int_0^1\Theta\,\Theta'^2\Theta''\,d\xi_1$$

$$-\frac{5}{2}\chi\,u_{i,2}{}^2\left(\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2\int_0^1\Theta\,\Theta''\Theta'^2 d\xi_1$$

$$+\left(\begin{array}{l}\frac{\mathrm{E}\,c_T\,u_{i,2}{}^2}{4}\left(\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2\left(\bar{h}-1\right)\\ +\left(\gamma_1+\gamma_2\right)+\frac{\mathrm{E}\,c_N\,u_{i,2}{}^2}{2}\left(\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2\end{array}\right)\int_0^1\Theta\left(\Theta'\right)^3 d\xi_1$$

$$-2\,\beta_{m,1}\int_0^1\Theta\,\Theta'^2\Theta^{(4)}d\xi_1-8\beta_{m,1}\int_0^1\Theta\,\Theta'\Theta''\Theta'''d\xi_1-2\beta_{m,1}\int_0^1\Theta\left(\Theta''\right)^3 d\xi_1 \tag{2.88}$$

$$F_{17}{}^{w} = -\beta_{m,1}\alpha_2\left(2\int_0^1\Theta\,\Theta'^2\Theta^{(4)}d\xi_1+24\int_0^1\Theta\,\Theta'\Theta''\Theta'''d\xi_1+6\int_0^1\Theta\left(\Theta''\right)^3 d\xi_1\right)$$

$$-\frac{3}{2}\Pi_{0,2}\alpha_2\int_0^1\Theta\,\Theta'^2\Theta''\,d\xi_1+\frac{\mathrm{E}\,u_{i,2}}{2}\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\sqrt{\beta_{o,2}}\left(c_T-c_N\right)\int_0^1\Theta^2\Theta'^2 d\xi_1 \tag{6.289}$$

$$+\chi\,u_{i,2}\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\sqrt{\beta_{o,2}}\left(\frac{7}{2}\int_0^1\Theta\left(\Theta'\right)^3 d\xi_1+\frac{3}{2}\int_0^1\Theta^2\,\Theta'\Theta''\,d\xi_1\right)$$

$$F_{18}{}^{w} = \frac{\mathrm{E}\,\beta_{o,2}}{4}\left(c_T-c_N\right)\int_0^1\Theta^3\Theta' d\xi_1-\frac{3}{2}\chi\beta_{o,2}\int_0^1\Theta^2\Theta'^2 d\xi_1 \tag{6.290}$$

$$F_{19}{}^{w} = -\frac{\mathrm{E}\,c_N}{4u_{i,2}}\frac{\lambda_{ch}{}^2-1}{\lambda_i{}^2}\beta_{o,2}{}^{\frac{3}{2}}\int_0^1\Theta^4\,d\xi_1 \tag{6.291}$$

$$F_{20}{}^{w} = -\chi\,\beta_{o,1}\int_0^1\Theta^2\Theta'^2\,d\xi_1 \tag{6.292}$$

$$F_{21}{}^{w} = 0 \tag{6.293}$$

$$F_{22}{}^{w} = -\frac{\chi\,\beta_{o,2}}{4}\int_0^1 \Theta^2\,\Xi^2 d\xi_1 \tag{6.294}$$

$$F_{23}{}^{w} = \beta_{i,3}\left(2\int_0^1 \Theta\,\Phi'^2\Theta'' d\xi_1 + 3\int_0^1 \Theta\,\Theta'\Phi'\Phi'' d\xi_1\right) \tag{6.295}$$

$$F_{24}{}^{w} = 2\,\beta_{i,3}\int_0^1 \Theta\Theta'\Phi'\Phi'' d\xi_1 + 2\beta_{i,3}\int_0^1 \Theta\,\Theta''\Phi'^2 d\xi_1 \tag{6.296}$$

$$F_{25}{}^{w} = \beta_{i,3}\left(\int_0^1 \Theta\,\Xi\,\Phi' d\xi_1 + \int_0^1 \Theta\,\Xi\,\Phi'' d\xi_1 - \int_0^1 \frac{\Theta\Xi\,\Theta''\Phi'}{\Theta'} d\xi_1\right) + \int_0^1 \Theta\,\Xi'\,\Phi' d\xi_1 \tag{6.297}$$

$$M_5{}^{\varphi} = \beta_{i,3}\int_0^1 \Xi^{\,2} d\xi_2 \tag{6.298}$$

$$K_5{}^{\varphi} = -\,\beta_{m,2}\int_0^1 \Xi\,\Xi''\,d\xi_2 \tag{6.299}$$

$$T_{13}{}^{\varphi} = 8\,\beta_{i,4}\,f_r\int_0^1 \Xi^2\,\Xi'\,d\xi_2 \tag{6.300}$$

$$T_{14}{}^{\varphi} = 4\,\beta_{i,4}\,f_r\int_0^1 \Xi^2\,\Xi'\,d\xi_2 \tag{6.301}$$

$$T_{15}{}^{\varphi} = \beta_{i,3}\int_0^1 \Xi\,\Theta'\Phi'\,d\xi_2 \tag{6.302}$$

$$T_{16}{}^{\varphi} = \beta_{i,3}\int_0^1 \Xi\,\Theta'\Phi'\,d\xi_2 \tag{6.303}$$

6.1.6 Extensible rotating flexible pipe with one sliding end conveying fluid downwards, which then flows upwards as a confined annular flow.

In this subsection, the dimensionless equations of motion (5.71-5.74), which describe the vibration of extensible pipe with fixed-simply supported sliding ends, conveying fluid downwards, which then flows upwards as a confined flow, are discretized via unimodal Galerkin's method. Following the same procedure described in section 6.1.5, we obtain the following coupled ordinary nonlinear differential equations in terms of the modal coordinates p, q, r and $\breve{g}$:

$$\left(\begin{array}{l}\left[M_6{}^u\right]\ddot{p}+\left[C_6{}^u\right]\dot{p}+\left[K_6{}^u\right]p+\left[A_{21}{}^u\right]q^2+\left[A_{22}{}^u\right]r^2+\left[A_{23}{}^u\right]q\,\dot{q}+\left[A_{24}{}^u\right]r\,\dot{r}\\ +\left[A_{25}{}^u\right]\dot{q}^2+\left[A_{26}{}^u\right]\dot{r}^2+\left[A_{27}{}^u\right]q\,\ddot{q}+\left[A_{28}{}^u\right]r\,\ddot{r}+\left[B_{21}{}^u\right]q^2|q|+\left[B_{22}{}^u\right]r^2|r|\\ +\left[B_{23}{}^u\right]q|q|\,\dot{q}+\left[B_{24}{}^u\right]r|r|\,\dot{r}+\left[B_{25}{}^u\right]q^2|\dot{q}|+\left[B_{26}{}^u\right]r^2|\dot{r}|+\left[B_{27}{}^u\right]q\,\dot{q}|\dot{q}|+\left[B_{28}{}^u\right]r\,\dot{r}|\dot{r}|\end{array}\right)=0$$

(6.304)

$$\left(\begin{array}{l}\left[M_6{}^v\right]\ddot{q}+\left[C_6{}^v\right]\dot{q}+\left[K_6{}^v\right]q+\left[D_{20}{}^v\right]p\,q+\left[D_{21}{}^v\right]\dot{p}\,q+\left[D_{22}{}^v\right]p\,\dot{q}\\ +\left[D_{23}{}^v\right]\dot{p}\,\dot{q}+\left[D_{24}{}^v\right]\ddot{p}\,q+\left[D_{25}{}^v\right]\dot{\breve{g}}\,r+\left[D_{26}{}^v\right]\dot{\breve{g}}\,\dot{r}+\left[E_{16}{}^v\right]q\,\dot{q}+\left[E_{17}{}^v\right]q|q|\\ +\left[E_{18}{}^v\right]\dot{q}|q|+\left[E_{19}{}^v\right]q|\dot{q}|+\left[E_{20}{}^v\right]\dot{q}|\dot{q}|+\left[F_{20}{}^v\right]q^3+\left[F_{21}{}^v\right]q^2\,\dot{q}+\left[F_{22}{}^v\right]q\,\dot{q}^2\\ +\left[F_{23}{}^v\right]\dot{q}^3+\left[F_{24}{}^v\right]q^2\ddot{q}+\left[F_{25}{}^v\right]q\,\dot{r}^2+\left[F_{26}{}^v\right]\dot{\breve{g}}^2\,q\end{array}\right)=0 \qquad (6.305)$$

$$\left(\begin{array}{l}
\left[M_6{}^w\right]\ddot{r}+\left[C_6{}^w\right]\dot{r}+\left[K_6{}^w\right]r+\left[D_{22}{}^w\right]p\,r+\left[D_{23}{}^w\right]\dot{p}\,r+\left[D_{24}{}^w\right]p\,\dot{r}+\left[D_{25}{}^w\right]\dot{p}\,\dot{r} \\
+\left[D_{26}{}^w\right]\ddot{p}\,r+\left[D_{27}{}^w\right]\dot{\breve{g}}\,q+\left[D_{28}{}^w\right]\dot{\breve{g}}\,\dot{q}+\left[D_{29}{}^w\right]\ddot{\breve{g}}\,q+\left[E_{16}{}^w\right]r\,\dot{r}+\left[E_{17}{}^w\right]r\,|r|+\left[E_{18}{}^w\right]\dot{r}\,|r| \\
+\left[E_{19}{}^w\right]r\,|\dot{r}|+\left[E_{20}{}^w\right]\dot{r}\,|\dot{r}|+\left[F_{25}{}^w\right]r^3+\left[F_{26}{}^w\right]r^2\,\dot{r}+\left[F_{27}{}^w\right]r\,\dot{r}^2+\left[F_{28}{}^w\right]\dot{r}^3+\left[F_{29}{}^w\right]r^2\,\ddot{r} \\
+\left[F_{30}{}^w\right]r\;\;\dot{q}^2+\left[F_{31}{}^w\right]\dot{\breve{g}}^2\,r+\left[F_{32}{}^w\right]q^2\,\ddot{r}+\left[F_{33}{}^w\right]\dot{r}\,q\,\dot{q}+\left[F_{34}{}^w\right]\dfrac{\ddot{r}\,\dot{\breve{g}}\,q}{\dot{r}}
\end{array}\right)=0 \tag{6.306}$$

$$\left[M_6{}^\varphi\right]\ddot{\breve{g}}+\left[K_6{}^\varphi\right]\breve{g}+\left[T_{17}{}^\varphi\right]\dot{\breve{g}}^2+\left[T_{18}{}^\varphi\right]\ddot{\breve{g}}\,g+\left[T_{19}{}^\varphi\right]\dot{r}\,\dot{q}+\left[T_{20}{}^\varphi\right]q\,\ddot{r}=0 \tag{6.307}$$

where the coefficients of the above equations are found to be as

$$M_6{}^u=\left(1-\beta_{o,2}-\beta_{i,2}-\beta_{i,3}\right)\int_0^1\Psi^2\,d\xi_1 \tag{6.308}$$

$$C_6{}^u=2u_{i,2}\sqrt{\beta_{i,1}}\int_0^1\Psi\Psi'd\xi_1-\Pi_{o,2}\alpha_2\int_0^1\Psi\Psi''d\xi_1 \tag{6.309}$$

$$K_6{}^u=\left(u_{i,2}{}^2-\Pi_{0,2}\right)\int_0^1\Psi''\Psi\,.d\xi_1+\left(\frac{\mathrm{E}\,c_T\,u_{i,2}{}^2}{2}\left(\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2\left(1-\bar{h}\right)+\left(\gamma_1-\gamma_2\right)\right)\int_0^1\Psi\Psi'd\xi_1 \tag{6.310}$$

$$A_{21}{}^{u}=\left(\begin{array}{l}-\aleph_1-\aleph_2\,u_{i,2}{}^2\left(\dfrac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2-\aleph_3\,u_{i,2}{}^2\left(\overline{C}\left(1-\dfrac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2-1\right)-\Pi_{0,2}\\ -\chi\,u_{i,2}{}^2\left(\dfrac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2+\dfrac{c_b\,u_{i,2}{}^2}{2}\left(\dfrac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2\\ +\left(\dfrac{\mathrm{E}\,c_T\,u_{i,2}{}^2}{2}\left(\dfrac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2\left(1-\overline{h}\right)+\left(\gamma_1-\gamma_2\right)\right)\left(\dfrac{1}{2}-\xi_1\right)\end{array}\right)\int_0^1\Psi\Phi'\Phi''d\xi_1$$

$$+\frac{\mathrm{E}\,u_{i,2}{}^2}{2}\left(\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2\left(c_N-c_T\left(\frac{3}{2}-\frac{\overline{h}}{2}\right)\right)\int_0^1\Psi\Phi'^2d\xi_1$$

$$-\beta_{m,1}\left(\int_0^1\Psi\Phi^{(4)}\Phi'\,d\xi_1+\int_0^1\Psi\Phi''\Phi'''d\xi_1\right)$$

(6.311)

$$A_{22}{}^{u}=\left(\begin{array}{l}-\aleph_1-\aleph_2\,u_{i,2}{}^2\left(\dfrac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2\\ -\aleph_3\,u_{i,2}{}^2\left(\overline{C}\left(1-\dfrac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2-1\right)-\chi\,u_{i,2}{}^2\left(\dfrac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2\\ -\Pi_{0,2}+\dfrac{c_b\,u_{i,2}{}^2}{2}\left(\dfrac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2\\ +\left(\dfrac{\mathrm{E}\,c_T\,u_{i,2}{}^2}{2}\left(1-\overline{h}\right)\left(\dfrac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2+\left(\gamma_1-\gamma_2\right)\right)\left(\dfrac{1}{2}-\xi_1\right)\end{array}\right)\int_0^1\Psi\Theta'\Theta''d\xi_1$$

$$+\frac{\mathrm{E}\,u_{i,2}{}^2}{2}\left(\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2\left(c_N-c_T\left(\frac{3}{2}-\frac{\overline{h}}{2}\right)\right)\int_0^1\Psi\Theta'^2d\xi_1 \qquad (6.312)$$

$$-\beta_{m,1}\left(\int_0^1\Psi\Theta^{(4)}\Theta'\,d\xi_1+\int_0^1\Psi\Theta''\Theta'''d\xi_1\right)$$

$$A_{23}{}^{u}=\left(\frac{\mathrm{E}\,u_{i,2}\sqrt{\beta_{o,2}}}{2}\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\left(c_T-c_N\right)\right)\int_0^1\Psi\Phi\Phi'\,d\xi_1-\left(2\Pi_{0,2}\,\alpha_2\right)\int_0^1\Psi\Phi'\Phi''\,d\xi_1$$

$$+2\,\chi\,u_{i,2}\sqrt{\beta_{o,2}}\,\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\int_0^1\Psi\Phi'^2\,d\xi_1-2\alpha_2\left(\int_0^1\Psi\Phi^{(4)}\Phi'dX+\int_0^1\Psi\Phi''\Phi'''d\xi_1\right) \qquad (6.313)$$

$$A_{24}{}^{u} = \left(\frac{\mathrm{E}\, u_{i,2}\sqrt{\beta_{o,2}}}{2} \frac{\lambda_i{}^2}{\lambda_{ch}{}^2 - 1} \left(c_T - c_N \right) \right) \int_0^1 \Psi\Theta\Theta'\, d\xi_1 - \left(2\Pi_{0,2}\, \alpha_2 \right) \int_0^1 \Psi\Theta'\Theta''\, d\xi_1$$
$$+ 2\,\chi\, u_{o,2}\sqrt{\beta_{o,2}} \frac{\lambda_i{}^2}{\lambda_{ch}{}^2 - 1} \int_0^1 \Psi\Theta'^2\, d\xi_1 - 2\alpha_2 \left(\int_0^1 \Psi\Theta^{(4)}\Theta' dX + \int_0^1 \Psi\Theta''\Theta''' d\xi_1 \right) \tag{6.314}$$

$$A_{25}{}^{u} = \frac{\mathrm{E}\, c_T \beta_{o,2}}{4} \int_0^1 \Psi\Phi^2 d\xi_1 \tag{6.315}$$

$$A_{26}{}^{u} = \frac{\mathrm{E}\, c_T \beta_{o,2}}{4} \int_0^1 \Psi\Theta^2 d\xi_1 \tag{6.316}$$

$$A_{27}{}^{u} = -\chi\, \beta_{o,2} \int_0^1 \Psi\Phi\Phi'\, d\xi_1 \tag{6.317}$$

$$A_{28}{}^{u} = -\chi\, \beta_{o,2} \int_0^1 \Psi\Theta\Theta'\, d\xi_1 \tag{6.318}$$

$$B_{21}{}^{u} = \frac{\mathrm{E}\, c_d\, u_{i,2}{}^2}{2} \left(\frac{\lambda_i{}^2}{\lambda_{ch}{}^2 - 1} \right)^2 \int_0^1 \Psi\Phi'^2 \left|\Phi'\right| d\xi_1 \tag{6.319}$$

$$B_{22}{}^{u} = \frac{\mathrm{E}\, c_d\, u_{i,2}{}^2}{2} \left(\frac{\lambda_i{}^2}{\lambda_{ch}{}^2 - 1} \right)^2 \int_0^1 \Psi\Theta'^2 \left|\Theta'\right| d\xi_1 \tag{6.320}$$

$$B_{23}{}^{u} = -\frac{\mathrm{E}\, c_d\, u_{i,2}}{2} \frac{\lambda_i{}^2}{\lambda_{ch}{}^2 - 1} \sqrt{\beta_{o,2}} \int_0^1 \Psi\Phi' \left|\Phi'\right| \Phi\, d\xi_1 \tag{6.321}$$

$$B_{24}{}^{u} = -\frac{\mathrm{E}\, c_d\, u_{i,2}}{2} \frac{\lambda_i{}^2}{\lambda_{ch}{}^2 - 1} \sqrt{\beta_{o,2}} \int_0^1 \Psi\Theta' \left|\Theta'\right| \Theta\, d\xi_1 \tag{6.322}$$

$$B_{25}{}^{u} = -\frac{\mathrm{E}\, c_d\, u_{i,2}}{2} \frac{\lambda_i{}^2}{\lambda_{ch}{}^2 - 1} \sqrt{\beta_{o,1}} \int_0^1 \Psi\Phi'^2 \left|\Phi\right| d\xi_1 \tag{6.323}$$

$$B_{26}{}^{u} = -\frac{\mathrm{E}\, c_d\, u_{i,2}}{2} \frac{\lambda_i{}^2}{\lambda_{ch}{}^2 - 1} \sqrt{\beta_{o,2}} \int_0^1 \Psi\Theta'^2 \left|\Theta\right| d\xi_1 \tag{6.324}$$

$$B_{27}{}^{u} = -\frac{\mathrm{E}\,c_d}{2}\beta_{o,2}\int_0^1 \Psi\Phi'\left|\Phi\right|\Phi\, d\xi_1 \tag{6.325}$$

$$B_{28}{}^{u} = -\frac{\mathrm{E}\,c_d}{2}\beta_{o,2}\int_0^1 \Psi\Theta'\left|\Theta\right|\Theta\, d\xi_1 \tag{6.326}$$

$$M_6{}^{v} = \left(1-\beta_{o,2}-\beta_{i,2}-\beta_{i,3}+\chi\,\beta_{o,2}\right)\int_0^1 \Phi^2\, d\xi_1 + 2\beta_{i,2}\int_0^1 \Phi\,\Phi''\, d\xi_1 \tag{6.327}$$

$$\begin{aligned} C_6{}^{v} = &\left(2u_{i,2}\sqrt{\beta_{i,1}} - 2\chi\, u_{i,2}\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\sqrt{\beta_{o,2}}\right)\int_0^1 \Phi\Phi' d\xi_1 + \alpha_2\beta_{m,1}\int_0^1 \Phi\Phi^{(4)} d\xi_1 \\ &+\left(\frac{\mathrm{E}\,c_N\, u_{i,2}}{2}\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\sqrt{\beta_{o,2}} + \chi\,\beta_{o,2}\, f_t\right)\int_0^1 \Phi^2\, d\xi_1 \end{aligned} \tag{6.328}$$

$$\begin{aligned} K_6{}^{v} = &\left(u_{i,2}{}^2 - \left(\begin{array}{l} -\aleph_1 - \aleph_2\, u_{i,2}{}^2\left(\dfrac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2 \\ -\aleph_3\, u_{i,2}{}^2\left(\overline{C}\left(1-\dfrac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2 - 1\right) \\ +\dfrac{c_b\, u_{i,2}{}^2}{2}\left(\dfrac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2 \end{array}\right) + \chi\, u_{i,2}{}^2\left(\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2 \right. \\ &\left. -\left(\frac{\mathrm{E}\,c_T\, u_{i,2}{}^2}{2}\left(\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2\left(1-\overline{h}\right) + \left(\gamma_1-\gamma_2\right)\right)\left(\frac{1}{2}-\xi_1\right)\right)\int_0^1 \Phi\Phi''\, d\xi_1 \\ &-\left(\frac{\mathrm{E}\left(c_N + c_T\,\overline{h}\right)u_{i,2}{}^2}{2}\left(\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2 - \left(\gamma_1-\gamma_2\right)\right)\int_0^1 \Phi\Phi'\, d\xi_1 + \beta_{m,1}\int_0^1 \Phi\Phi^{(4)}\, d\xi_1 \end{aligned}$$

(6.329)

$$
\begin{aligned}
D_{20}{}^{v} = & \left(\begin{array}{l} \left(\dfrac{\mathrm{E}\, c_T\, u_{i,2}{}^2}{2} \left(\dfrac{\lambda_i{}^2}{\lambda_{ch}{}^2 - 1} \right)^2 \left(1 - \bar{h}\right) + \left(\gamma_1 - \gamma_2\right) \right) \left(\dfrac{1}{2} - \xi_1 \right) \\ + \left(\begin{array}{l} -\aleph_1 - \aleph_2\, u_{i,2}{}^2 \left(\dfrac{\lambda_i{}^2}{\lambda_{ch}{}^2 - 1} \right)^2 + \dfrac{c_b\, u_{i,2}{}^2}{2} \left(\dfrac{\lambda_i}{\lambda_{ch}{}^2 - 1} \right)^2 \\ -\aleph_3\, u_{i,2}{}^2 \left(\overline{C} \left(1 - \dfrac{\lambda_i{}^2}{\lambda_{ch}{}^2 - 1} \right)^2 - 1 \right) - \Pi_{0,2} \end{array} \right) \end{array} \right) \int_0^1 \left(\Phi\Psi''\Phi' + \Phi\Psi'\Phi'' \right) d\xi_1 \\
& + \frac{\mathrm{E}\, u_{i,2}{}^2 \left(c_N + c_T \right)}{2} \left(\frac{\lambda_i{}^2}{\lambda_{ch}{}^2 - 1} \right)^2 \int_0^1 \Phi\Psi'\Phi'\, d\xi_1 \\
& + \left(\frac{\mathrm{E}\, c_T\, u_{i,2}{}^2 \left(1 - \bar{h}\right)}{2} \left(\frac{\lambda_i{}^2}{\lambda_{ch}{}^2 - 1} \right)^2 + \left(\gamma_1 - \gamma_2\right) \right) \int_0^1 \Phi\Psi\Phi''\, d\xi_1 \\
& - \left(\frac{\mathrm{E}\, c_T\, u_{i,2}{}^2 \left(1 - \bar{h}\right)}{2} \left(\frac{\lambda_i{}^2}{\lambda_{ch}{}^2 - 1} \right)^2 + \left(\gamma_1 - \gamma_2\right) \right) \int_0^1 \Phi\Psi(1)\Phi''\, d\xi_1 \\
& - 3\beta_{m,1} \int_0^1 \Phi\Psi'''\Phi''\, d\xi_1 - 4\beta_{m,1} \int_0^1 \Phi\Psi''\Phi'''\, d\xi_1 - 2\beta_{m,1} \int_0^1 \Phi\Psi'\Phi^{(4)}\, d\xi_1 \\
& - \beta_{m,1} \int_0^1 \Phi\Psi^{(4)}\Phi'\, d\xi_1 - \chi\, u_{i,2}{}^2 \left(\frac{\lambda_i{}^2}{\lambda_{ch}{}^2 - 1} \right)^2 \int_0^1 \left(2\Phi\Psi''\Phi' + 4\Phi\Psi'\Phi'' \right) d\xi_1
\end{aligned}
\tag{6.330}
$$

$$
\begin{aligned}
D_{21}{}^{v} = & -\beta_{m,1}\alpha_2 \left(3\int_0^1 \Phi\Psi'''\Phi''\, d\xi_1 + 4\int_0^1 \Phi\Psi''\Phi'''\, d\xi_1 + 2\int_0^1 \Phi\Psi'\Phi^{(4)}\, d\xi_1 + \int_0^1 \Phi\Psi^{(4)}\Phi'\, d\xi_1 \right) \\
& + 3\chi\, u_{i,2} \frac{\lambda_i{}^2}{\lambda_{ch}{}^2 - 1} \sqrt{\beta_{o,2}} \int_0^1 \Phi\Psi'\Phi' d\xi_1 + 2\chi\, u_{i,2} \frac{\lambda_i{}^2}{\lambda_{ch}{}^2 - 1} \sqrt{\beta_{o,2}} \int_0^1 \Phi\Psi\Phi'' d\xi_1 \\
& - \Pi_{0,2}\alpha_2 \int_0^1 \left(\Phi\Psi''\Phi' + \Phi\Psi'\Phi'' \right) d\xi_1
\end{aligned}
\tag{6.331}
$$

$$D_{22}{}^{v} = -\beta_{m,2}\alpha_2\left(3\int_0^1 \Phi\Psi'''\Phi''\,d\xi_1 + 4\int_0^1 \Phi\Psi''\Phi'''\,d\xi_1 + 2\int_0^1 \Phi\Psi'\Phi^{(4)}\,d\xi_1 + \int_0^1 \Phi\Psi^{(4)}\Phi'\,d\xi_1\right)$$
$$-\Pi_{0,2}\alpha_2\int_0^1 \left(\Phi\Psi''\Phi' + \Phi\Psi'\Phi''\right)d\xi_1 + \frac{\mathrm{E}\,c_N\,u_{i,2}\sqrt{\beta_{o,2}}}{2}\frac{\lambda_i^{\,2}}{\lambda_{ch}^{\,2}-1}\int_0^1 \Phi^2\Psi'\,d\xi_1$$
$$+4\chi\,u_{i,2}\sqrt{\beta_{o,2}}\,\frac{\lambda_i^{\,2}}{\lambda_{ch}^{\,2}-1}\int_0^1 \Phi\Psi'\Phi'\,d\xi_1$$

(6.332)

$$D_{23}{}^{v} = \frac{\mathrm{E}\,c_N}{2}\beta_{o,2}\int_0^1 \Phi^2\Psi\,d\xi_1 - 2\chi\beta_{o,2}\int_0^1 \Phi\Psi\,\Phi'd\xi_1 \tag{6.333}$$

$$D_{24}{}^{v} = -\chi\,\beta_{o,2}\int_0^1 \Phi\Psi\Phi'\,d\xi_1 \tag{6.334}$$

$$D_{25}{}^{v} = \frac{\chi\,\beta_{o,2}\,f_t}{2}\int_0^1 \Phi\,\Theta\,\Xi\,d\xi_1 \tag{6.335}$$

$$D_{26}{}^{v} = \beta_{i,3}\left(\int_0^1 \Phi\,\Theta''\Xi\,d\xi_1 + \int_0^1 \Phi\,\Theta'\,\Xi'\,d\xi_1\right) + \chi\,\beta_{o,2}\int_0^1 \Phi\,\Theta\,\Xi\,d\xi_1 \tag{6.336}$$

$$E_{16}{}^{v} = -4\,\beta_{m,1}\alpha_2\int_0^1 \Phi\Phi'\Phi^{(4)}\,d\xi_1 - 3\beta_{m,1}\alpha_2\Pi_{o,2}\int_0^1 \Phi\Phi'\Phi''\,d\xi_1 \tag{6.337}$$

$$E_{17}{}^{v} = -\frac{\mathrm{E}\,c_d\,u_{i,2}{}^{2}}{2}\left(\frac{\lambda_i^{\,2}}{\lambda_{ch}^{\,2}-1}\right)^2\int_0^1 \Phi\Phi'\left|\Phi'\right|d\xi_1 \tag{6.338}$$

$$E_{18}{}^{v} = \frac{\mathrm{E}\,c_d\,u_{i,2}}{2}\frac{\lambda_i^{\,2}}{\lambda_{ch}^{\,2}-1}\sqrt{\beta_{o,2}}\int_0^1 \Phi\left|\Phi'\right|\Phi\,d\xi_1 \tag{6.339}$$

$$E_{19}{}^{v} = \frac{\mathrm{E}\,c_d\,u_{i,2}}{2}\frac{\lambda_i^{\,2}}{\lambda_{ch}^{\,2}-1}\sqrt{\beta_{o,2}}\int_0^1 \Phi\Phi'\left|\Phi\right|d\xi_1 \tag{6.340}$$

$$E_{20}{}^{v} = \frac{\mathrm{E}\,c_d}{2}\beta_{o,2}\int_0^1 \Phi^2\left|\Phi\right|\,d\xi_1 \tag{6.341}$$

$$F_{20}{}^{v} = \frac{3}{2}\left(\begin{array}{l}\left(\frac{\mathrm{E}\,c_T\,u_{i,2}{}^2}{2}\left(\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2\left(1-\bar{h}\right)+\left(\gamma_1-\gamma_2\right)\right)\left(\frac{1}{2}-\xi_1\right)\\ +\left(\begin{array}{l}-\aleph_1-\aleph_2\,u_{i,2}{}^2\left(\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2+\frac{c_b\,u_{i,2}{}^2}{2}\left(\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2\\ -\aleph_3\,u_{i,2}{}^2\left(\overline{C}\left(1-\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2-1\right)-\Pi_{0,2}\end{array}\right)\end{array}\right)\int_0^1 \Phi\Phi'^2\Phi''\,d\xi_1$$

$$-\frac{5}{2}\chi\,u_{i,2}{}^2\left(\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2\int_0^1 \Phi\Phi''\Phi'^2 d\xi_1$$

$$+\left(\frac{\mathrm{E}\,c_T\,u_{i,2}{}^2}{4}\left(\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2\left(\bar{h}-1\right)+\left(\gamma_1+\gamma_2\right)+\frac{\mathrm{E}\,c_N\,u_{i,2}{}^2}{2}\left(\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2\right)\int_0^1 \Phi\left(\Phi'\right)^3 d\xi_1$$

$$-2\,\beta_{m,1}\int_0^1 \Phi\Phi'^2\Phi^{(4)}d\xi_1-8\beta_{m,1}\int_0^1 \Phi\Phi'\Phi''\Phi'''d\xi_1-2\beta_{m,1}\int_0^1 \Phi\left(\Phi''\right)^3 d\xi_1 \tag{6.342}$$

$$F_{21}{}^{v} = -\beta_{m,1}\alpha_2\left(2\int_0^1 \Phi\Phi'^2\Phi^{(4)}d\xi_1+24\int_0^1 \Phi\Phi'\Phi''\Phi'''d\xi_1+6\int_0^1 \Phi\left(\Phi''\right)^3 d\xi_1\right)$$

$$-\frac{3}{2}\Pi_{0,2}\alpha_2\int_0^1 \Phi\Phi'^2\Phi''\,d\xi_1+\frac{\mathrm{E}\,u_{i,2}}{2}\frac{\lambda_i}{\lambda_{ch}{}^2-1}\sqrt{\beta_{o,2}}\left(c_T-c_N\right)\int_0^1 \Phi^2\Phi'^2 d\xi_1 \tag{6.343}$$

$$+\chi\,u_{i,2}\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\sqrt{\beta_{o,2}}\left(\frac{7}{2}\int_0^1 \Phi\left(\Phi'\right)^3 d\xi_1+\frac{3}{2}\int_0^1 \Phi^2\,\Phi'\Phi''\,d\xi_1\right)$$

$$F_{22}{}^{v} = \frac{\mathrm{E}\,\beta_{o,2}}{4}\left(c_T-c_N\right)\int_0^1 \Phi^3\Phi' d\xi_1-\frac{3}{2}\chi\beta_{o,2}\int_0^1 \Phi^2\Phi'^2 d\xi_1 \tag{6.344}$$

$$F_{23}{}^{v} = -\frac{\mathrm{E}\,c_N}{4u_{i,2}}\frac{\lambda_{ch}{}^2-1}{\lambda_i{}^2}\beta_{o,2}{}^{\frac{3}{2}}\int_0^1 \Phi^4\,d\xi_1 \tag{6.345}$$

$$F_{24}{}^{v} = -\chi\,\beta_{o,1}\int_0^1 \Phi^2\Phi'^2\,d\xi_1 \tag{6.346}$$

$$F_{25}{}^{v} = \beta_{i,3}\left(\int_0^1 \Phi\,\Phi''\Theta'^2 d\xi_1+2\int_0^1 \Phi\,\Phi'\,\Theta'\Theta''d\xi_1\right) \tag{6.347}$$

$$F_{26}{}^{v} = -\frac{\chi\,\beta_{o,2}}{4}\int_0^1 \Phi^2\,\Xi^2 d\xi_1 \tag{6.348}$$

$$M_6{}^{w} = \left(1-\beta_{o,2}-\beta_{i,2}-\beta_{i,3}+\chi\,\beta_{o,2}\right)\int_0^1 \Theta^2\,d\xi_1 + 2\beta_{i,2}\int_0^1 \Theta\,\Theta''\,d\xi_1 \tag{6.349}$$

$$\begin{aligned} C_5{}^{w} = &\left(2u_{i,2}\sqrt{\beta_{i,1}} - 2\chi\,u_{i,2}\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\sqrt{\beta_{o,2}}\right)\int_0^1 \Theta\,\Theta' d\xi_1 + \alpha_2\beta_{m,1}\int_0^1 \Theta\,\Theta^{(4)} d\xi_1 \\ &+\left(\frac{\mathrm{E}\,c_N\,u_{i,2}}{2}\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\sqrt{\beta_{o,2}} + \chi\,\beta_{o,2}\,f_t\right)\int_0^1 \Theta^2\,d\xi_1 \end{aligned} \tag{6.350}$$

$$\begin{aligned} K_6{}^{w} = &\left(\begin{array}{l} u_{i,2}{}^2 - \left(\begin{array}{l} -\aleph_1 - \aleph_2\,u_{i,2}{}^2\left(\dfrac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2 \\ -\aleph_3\,u_{i,2}{}^2\left(\overline{C}\left(1-\dfrac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2 - 1\right) \\ +\dfrac{c_b\,u_{i,2}{}^2}{2}\left(\dfrac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2 \end{array}\right) \\ + \chi\,u_{i,2}{}^2\left(\dfrac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2 \\ -\left(\dfrac{\mathrm{E}\,c_T\,u_{i,2}{}^2}{2}\left(\dfrac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2\left(1-\overline{h}\right)+\left(\gamma_1-\gamma_2\right)\right)\left(\dfrac{1}{2}-\xi_1\right) \end{array}\right)\int_0^1 \Theta\,\Theta'' d\xi_1 \\ &-\left(\frac{\mathrm{E}\left(c_N + c_T\,\overline{h}\right)u_{i,2}{}^2}{2}\left(\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2 - \left(\gamma_1-\gamma_2\right)\right)\int_0^1 \Theta\,\Theta'\,d\xi_1 + \beta_{m,1}\int_0^1 \Theta\Theta^{(4)}\,d\xi_1 \end{aligned} \tag{6.351}$$

$$
\begin{aligned}
D_{22}{}^{w} = & \left(\begin{array}{l} \left(\dfrac{\mathrm{E}\, c_T\, u_{i,2}{}^{2}}{2} \left(\dfrac{\lambda_i{}^{2}}{\lambda_{ch}{}^{2}-1} \right)^{2} \left(1-\bar{h}\right) + \left(\gamma_1 - \gamma_2\right) \right) \left(\dfrac{1}{2} - \xi_1 \right) \\ + \left(\begin{array}{l} -\aleph_1 - \aleph_2\, u_{i,2}{}^{2} \left(\dfrac{\lambda_i{}^{2}}{\lambda_{ch}{}^{2}-1} \right)^{2} \\ -\aleph_3\, u_{i,2}{}^{2} \left(\bar{C} \left(1 - \dfrac{\lambda_i{}^{2}}{\lambda_{ch}{}^{2}-1} \right)^{2} - 1 \right) - \Pi_{0,2} \\ \dfrac{+c_b\, u_{i,2}{}^{2}}{2} \left(\dfrac{\lambda_i{}^{2}}{\lambda_{ch}{}^{2}-1} \right)^{2} \end{array} \right) \end{array} \right) \int_0^1 \left(\Theta\Psi''\Theta' + \Theta\Psi'\Theta'' \right) d\xi_1 \\
& + \frac{\mathrm{E}\, u_{i,2}{}^{2} \left(c_N + c_T \right)}{2} \left(\frac{\lambda_i{}^{2}}{\lambda_{ch}{}^{2}-1} \right)^{2} \int_0^1 \Theta\Psi'\Theta'\, d\xi_1 \\
& + \left(\frac{\mathrm{E}\, c_T\, u_{i,2}{}^{2} \left(1-\bar{h}\right)}{2} \left(\frac{\lambda_i{}^{2}}{\lambda_{ch}{}^{2}-1} \right)^{2} + \left(\gamma_1 - \gamma_2\right) \right) \int_0^1 \Theta\Psi\Theta''\, d\xi_1 \\
& - \left(\frac{\mathrm{E}\, c_T\, u_{i,2}{}^{2} \left(1-\bar{h}\right)}{2} \left(\frac{\lambda_i{}^{2}}{\lambda_{ch}{}^{2}-1} \right)^{2} + \left(\gamma_1 - \gamma_2\right) \right) \int_0^1 \Theta\Psi(1)\Theta''\, d\xi_1 \\
& - 3\beta_{m,1} \int_0^1 \Theta\Psi'''\Theta''\, d\xi_1 - 4\beta_{m,1} \int_0^1 \Theta\Psi''\Theta'''\, d\xi_1 - 2\beta_{m,1} \int_0^1 \Theta\Psi'\Theta^{(4)}\, d\xi_1 \\
& - \beta_{m,1} \int_0^1 \Theta\Psi^{(4)}\Theta'\, d\xi_1 - \chi\, u_{o,1}{}^{2} \int_0^1 \left(2\Theta\Psi''\Theta' + 4\Theta\Psi'\Theta'' \right) d\xi_1
\end{aligned}
\tag{6.352}
$$

$$
\begin{aligned}
D_{23}{}^{w} = & -\beta_{m,1}\alpha_2 \left(3\int_0^1 \Theta\Psi'''\Theta''\, d\xi_1 + 4\int_0^1 \Theta\Psi''\Theta'''\, d\xi_1 + 2\int_0^1 \Theta\Psi'\Theta^{(4)}\, d\xi_1 + \int_0^1 \Theta\Psi^{(4)}\Theta'\, d\xi_1 \right) \\
& + 3\chi\, u_{i,2} \frac{\lambda_i{}^{2}}{\lambda_{ch}{}^{2}-1} \sqrt{\beta_{o,2}} \int_0^1 \Theta\Psi'\Theta' d\xi_1 + 2\chi\, u_{i,2} \frac{\lambda_i{}^{2}}{\lambda_{ch}{}^{2}-1} \sqrt{\beta_{o,2}} \int_0^1 \Theta\Psi\Theta'' d\xi_1 \\
& - \Pi_{0,2}\alpha_2 \int_0^1 \left(\Theta\Psi''\Theta' + \Theta\Psi'\Theta'' \right) d\xi_1
\end{aligned}
\tag{6.353}
$$

$$D_{24}{}^{w} = -\beta_{m,2}\alpha_2\left(3\int_0^1 \Theta\Psi'''\Theta''\,d\xi_1 + 4\int_0^1 \Theta\Psi''\Theta'''\,d\xi_1 + 2\int_0^1 \Theta\Psi'\Theta^{(4)}\,d\xi_1 + \int_0^1 \Theta\Psi^{(4)}\Theta'\,d\xi_1\right)$$
$$-\Pi_{0,2}\alpha_2\int_0^1 \left(\Theta\Psi''\Theta' + \Theta\Psi'\Theta''\right)d\xi_1 + \frac{\mathrm{E}\,c_N\,u_{i,2}\sqrt{\beta_{o,2}}}{2}\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\int_0^1 \Theta^2\Psi'\,d\xi_1$$
$$+4\chi\,u_{i,2}\sqrt{\beta_{o,2}}\,\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\int_0^1 \Theta\Psi'\Theta'\,d\xi_1$$

(6.354)

$$D_{25}{}^{w} = \frac{\mathrm{E}\,c_N}{2}\beta_{o,2}\int_0^1 \Theta^2\Psi\,d\xi_1 - 2\chi\beta_{o,2}\int_0^1 \Theta\Psi\,\Theta' d\xi_1 \tag{6.355}$$

$$D_{26}{}^{w} = -\chi\,\beta_{o,2}\int_0^1 \Theta\,\Psi\Theta'\,d\xi_1 \tag{6.356}$$

$$D_{27}{}^{w} = \frac{\chi\,\beta_{o,2}\,f_t}{2}\int_0^1 \Phi\,\Theta\,\Xi\,d\xi_1 \tag{6.357}$$

$$D_{28}{}^{w} = \chi\,\beta_{o,2}\int_0^1 \Phi\,\Theta\,\Xi\,d\xi_1 \tag{6.358}$$

$$D_{29}{}^{w} = \beta_{i,3}\left(\int_0^1 \frac{\Theta\Xi\Phi'\Theta''}{\Theta'}d\xi_1 + \int_0^1 \Theta\,\Xi\Phi''d\xi_1 + \int_0^1 \Theta\,\Xi'\Phi'd\xi_1\right) \tag{6.359}$$

$$E_{16}{}^{w} = -4\,\beta_{m,1}\alpha_2\int_0^1 \Theta\,\Theta'\Theta^{(4)}\,d\xi_1 - -3\beta_{m,1}\alpha_2\Pi_{o,2}\int_0^1 \Theta\,\Theta'\Theta''\,d\xi_1 \tag{6.360}$$

$$E_{17}{}^{w} = -\frac{\mathrm{E}\,c_d\,u_{i,2}{}^2}{2}\left(\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\right)^2\int_0^1 \Theta\,\Theta'\left|\Theta'\right|d\xi_1 \tag{6.361}$$

$$E_{18}{}^{w} = \frac{\mathrm{E}\,c_d\,u_{i,2}}{2}\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\sqrt{\beta_{o,2}}\int_0^1 \Theta\left|\Theta'\right|\Theta\,d\xi_1 \tag{6.362}$$

$$E_{19}{}^{w} = \frac{\mathrm{E}\,c_d\,u_{i,2}}{2}\frac{\lambda_i{}^2}{\lambda_{ch}{}^2-1}\sqrt{\beta_{o,2}}\int_0^1 \Theta\,\Theta'\left|\Theta\right|d\xi_1 \tag{6.363}$$

$$E_{20}{}^{w} = \frac{\mathrm{E}\, c_d}{2} \beta_{o,2} \int_0^1 \Theta^2 |\Theta| \, d\xi_1 \tag{6.364}$$

$$F_{25}{}^{w} = \frac{3}{2} \left(\begin{array}{l} \left(\frac{\mathrm{E}\, c_T\, u_{i,2}{}^2}{2} \left(\frac{\lambda_i{}^2}{\lambda_{ch}{}^2 - 1} \right)^2 \left(1 - \bar{h}\right) + \left(\gamma_1 - \gamma_2\right) \right) \left(\frac{1}{2} - \xi_1 \right) \\ + \left(\begin{array}{l} -\aleph_1 - \aleph_2\, u_{i,2}{}^2 \left(\frac{\lambda_i{}^2}{\lambda_{ch}{}^2 - 1} \right)^2 \\ -\aleph_3\, u_{i,2}{}^2 \left(\overline{C} \left(1 - \frac{\lambda_i{}^2}{\lambda_{ch}{}^2 - 1} \right)^2 - 1 \right) - \Pi_{0,2} \\ + \frac{c_b\, u_{i,2}{}^2}{2} \left(\frac{\lambda_i{}^2}{\lambda_{ch}{}^2 - 1} \right)^2 \end{array} \right) \end{array} \right) \int_0^1 \Theta\, \Theta'^2 \Theta'' \, d\xi_1$$

$$- \frac{5}{2} \chi\, u_{i,2}{}^2 \left(\frac{\lambda_i{}^2}{\lambda_{ch}{}^2 - 1} \right)^2 \int_0^1 \Theta\, \Theta'' \Theta'^2 d\xi_1$$

$$+ \left(\frac{\mathrm{E}\, c_T\, u_{i,2}{}^2}{4} \left(\frac{\lambda_i{}^2}{\lambda_{ch}{}^2 - 1} \right)^2 \left(\bar{h} - 1\right) + \left(\gamma_1 + \gamma_2\right) + \frac{\mathrm{E}\, c_N\, u_{i,2}{}^2}{2} \left(\frac{\lambda_i{}^2}{\lambda_{ch}{}^2 - 1} \right)^2 \right) \int_0^1 \Theta \left(\Theta'\right)^3 d\xi_1$$

$$- 2\, \beta_{m,1} \int_0^1 \Theta\, \Theta'^2 \Theta^{(4)} d\xi_1 - 8 \beta_{m,1} \int_0^1 \Theta\, \Theta' \Theta'' \Theta''' d\xi_1 - 2 \beta_{m,1} \int_0^1 \Theta \left(\Theta''\right)^3 d\xi_1 \tag{6.365}$$

$$\begin{aligned} F_{26}{}^{w} = & - \beta_{m,1} \alpha_2 \left(2 \int_0^1 \Theta\, \Theta'^2 \Theta^{(4)} d\xi_1 + 24 \int_0^1 \Theta\, \Theta' \Theta'' \Theta''' d\xi_1 + 6 \int_0^1 \Theta \left(\Theta''\right)^3 d\xi_1 \right) \\ & - \frac{3}{2} \Pi_{0,2} \alpha_2 \int_0^1 \Theta\, \Theta'^2 \Theta'' \, d\xi_1 + \frac{\mathrm{E}\, u_{i,2}}{2} \frac{\lambda_i{}^2}{\lambda_{ch}{}^2 - 1} \sqrt{\beta_{o,2}} \left(c_T - c_N\right) \int_0^1 \Theta^2 \Theta'^2 d\xi_1 \\ & + \chi\, u_{i,2} \frac{\lambda_i{}^2}{\lambda_{ch}{}^2 - 1} \sqrt{\beta_{o,2}} \left(\frac{7}{2} \int_0^1 \Theta \left(\Theta'\right)^3 d\xi_1 + \frac{3}{2} \int_0^1 \Theta^2\, \Theta' \Theta'' \, d\xi_1 \right) \end{aligned} \tag{6.366}$$

$$F_{27}{}^{w} = \frac{\mathrm{E}\, \beta_{o,2}}{4} \left(c_T - c_N\right) \int_0^1 \Theta^3 \Theta' d\xi_1 - \frac{3}{2} \chi \beta_{o,2} \int_0^1 \Theta^2 \Theta'^2 d\xi_1 \tag{6.367}$$

$$F_{28}{}^{w} = - \frac{\mathrm{E}\, c_N}{4 u_{i,2}} \frac{\lambda_{ch}{}^2 - 1}{\lambda_i{}^2} \beta_{o,2}{}^{\frac{3}{2}} \int_0^1 \Theta^4 \, d\xi_1 \tag{6.368}$$

$$F_{29}{}^{w} = -\chi\,\beta_{o,1}\int_0^1 \Theta^2\Theta'^2\,d\xi_1 \tag{6.369}$$

$$F_{30}{}^{w} = 0 \tag{6.370}$$

$$F_{31}{}^{w} = -\frac{\chi\,\beta_{o,2}}{4}\int_0^1 \Theta^2\,\Xi^2 d\xi_1 \tag{6.371}$$

$$F_{32}{}^{w} = \beta_{i,3}\left(2\int_0^1 \Theta\,\Phi'^2\Theta'' d\xi_1 + 3\int_0^1 \Theta\,\Theta'\Phi'\Phi'' d\xi_1\right) \tag{6.372}$$

$$F_{33}{}^{w} = 2\,\beta_{i,3}\int_0^1 \Theta\Theta'\Phi'\Phi'' d\xi_1 + 2\beta_{i,3}\int_0^1 \Theta\,\Theta''\Phi'^2 d\xi_1 \tag{6.373}$$

$$F_{34}{}^{w} = \beta_{i,3}\left(\int_0^1 \Theta\,\Xi\,\Phi' d\xi_1 + \int_0^1 \Theta\,\Xi\,\Phi'' d\xi_1 - \int_0^1 \frac{\Theta\Xi\,\Theta''\Phi'}{\Theta'} d\xi_1 + \int_0^1 \Theta\,\Xi'\,\Phi' d\xi_1\right) \tag{6.374}$$

$$M_3{}^{\varphi} = \beta_{i,3}\int_0^1 \Xi^{\,2} d\xi_2 \tag{6.375}$$

$$K_6{}^{\varphi} = -\beta_{m,2}\int_0^1 \Xi\,\Xi''\,d\xi_2 \tag{6.376}$$

$$T_{17}{}^{\varphi} = 8\,\beta_{i,4}\,f_r\int_0^1 \Xi^2\,\Xi'\,d\xi_2 \tag{6.377}$$

$$T_{18}{}^{\varphi} = 4\,\beta_{i,4}\,f_r\int_0^1 \Xi^2\,\Xi'\,d\xi_2 \tag{6.378}$$

$$T_{19}{}^{\varphi} = \beta_{i,3}\int_0^1 \Xi\,\Theta'\Phi'\,d\xi_2 \tag{6.379}$$

$$T_{20}{}^{\varphi} = \beta_{i,3}\int_0^1 \Xi\,\Theta'\Phi'\,d\xi_2 \tag{6.380}$$

6.2 Assumed Modes

In order to solve the discretized ordinary nonlinear equations of motion, the approximated spatial modal functions Ψ, Φ, Θ, and Ξ should be specified. In this analysis, it is assumed that these spatial functions are the eigenfunctions of the associated linear problem, which can be obtained by solving the associated frequency equations of the corresponding linear model for each of vibration mode. Due to the fact that vibration modes are depending mainly on the assigned boundary conditions, the assumed modes for each model are treated separately in this section. It is important to recognize that the provided mode shapes are not normalized and accordingly equation (6.10) is no longer valid.

6.2.1 Extensible flexible pipe conveying fluid and subjected to external axial flow

The boundary conditions encountered in this model are fixed-fixed. By referring to [44], the assumed modes for this model can be expressed as:

- Axial motion

$$\Psi_j = \sin(j\pi\xi_1), \qquad j = 1,2,3. \tag{6.381}$$

- Flexural motion

$$\Phi_j = \left[\sin(\overline{\alpha}_j\xi_1) - \sinh(\overline{\alpha}_j\xi_1)\right] - R_j\left[\cos(\overline{\alpha}_j\xi_1) - \cosh(\overline{\alpha}_j\xi_1)\right], \quad j = 1,2,3. \tag{6.382}$$

where $\overline{\alpha}_1 = 4.730$, $\overline{\alpha}_2 = 7.853$, $\overline{\alpha}_3 = 10.996$ and R_j is given as

$$R_j = \left[\frac{\sin(\overline{\alpha}_j) - \sinh(\overline{\alpha}_j)}{\cos(\overline{\alpha}_j) - \cosh(\overline{\alpha}_j)}\right] \tag{6.383}$$

6.2.2 Inextensible rotating flexible pipe conveying fluid and subjected to external axial flow

The boundary conditions incorporated with this model are fixed-free. Referring to [44] and [45], the assumed modes for this model are found as

- Flexural motion in *X*-*Y* plane.

$$\Phi_j = \left[\sin\left(\overline{\alpha}_j \xi_2\right) - \sinh\left(\overline{\alpha}_j \xi_2\right)\right] - R_j \left[\cos\left(\overline{\alpha}_j \xi_2\right) - \cosh\left(\overline{\alpha}_j \xi_2\right)\right], \quad j = 1, 2, 3. \quad (6.384)$$

where $\overline{\alpha}_1 = 1.875$, $\overline{\alpha}_2 = 4.6943$, $\overline{\alpha}_3 = 7.855$ and R_j is given as:

$$R_j = \left[\frac{\sin\left(\overline{\alpha}_j\right) + \sinh\left(\overline{\alpha}_j\right)}{\cos\left(\overline{\alpha}_j\right) + \cosh\left(\overline{\alpha}_j\right)}\right] \quad (6.385)$$

- Flexural motion in *X*-*Z* plane.

$$\Theta_j = \left[\sin\left(\overline{\alpha}_j \xi_2\right) - \sinh\left(\overline{\alpha}_j \xi_2\right)\right] - R_j \left[\cos\left(\overline{\alpha}_j \xi_2\right) - \cosh\left(\overline{\alpha}_j \xi_2\right)\right], \quad j = 1, 2, 3. \quad (6.386)$$

where $\overline{\alpha}_1 = 1.875$, $\overline{\alpha}_2 = 4.6943$, $\overline{\alpha}_3 = 7.855$ and R_j is given as:

$$R_j = \left[\frac{\sin\left(\overline{\alpha}_j\right) + \sinh\left(\overline{\alpha}_j\right)}{\cos\left(\overline{\alpha}_j\right) + \cosh\left(\overline{\alpha}_j\right)}\right] \quad (6.387)$$

- Torsional motion.

$$\Xi_j = \sin\left(\frac{2j+1}{2}\pi \xi_2\right), \qquad j = 0, 1, 2. \quad (6.388)$$

One should recognize that the assumed modes associated with the inextensible rotating pipe conveying fluid downwards, which then flows upwards as confined axial flow are identical to the assumed modes of this model.

6.2.3 Extensible rotating flexible pipe conveying fluid and subjected to external axial flow

The boundary conditions associated with this problem are fixed-simple supported. By referring to [45], the assumed modes for this model are found as follow:

- Axial motion

$$\Psi_j = \sin(j\pi\xi_1), \qquad j = 1,2,3. \tag{6.389}$$

- Flexural motion in *X*-*Y* plane.

$$\Phi_j = \left[\sin(\overline{\alpha}_j\xi_1) - \sinh(\overline{\alpha}_j\xi_1)\right] - R_j\left[\cos(\overline{\alpha}_j\xi_1) - \cosh(\overline{\alpha}_j\xi_1)\right], \quad j = 1,2,3. \tag{6.390}$$

where $\overline{\alpha}_1 = 3.923$, $\overline{\alpha}_2 = 7.069$, $\overline{\alpha}_3 = 10.210$ and R_j is given as

$$R_j = \left[\frac{\sin(\overline{\alpha}_j) - \sinh(\overline{\alpha}_j)}{\cos(\overline{\alpha}_j) - \cosh(\overline{\alpha}_j)}\right] \tag{6.391}$$

- Flexural motion in *X*-*Z* plane.

$$\Theta_j = \left[\sin(\overline{\alpha}_j\xi_1) - \sinh(\overline{\alpha}_j\xi_1)\right] - R_j\left[\cos(\overline{\alpha}_j\xi_1) - \cosh(\overline{\alpha}_j\xi_1)\right], \quad j = 1,2,3. \tag{6.392}$$

where $\overline{\alpha}_1 = 3.923$, $\overline{\alpha}_2 = 7.069$, $\overline{\alpha}_3 = 10.210$ and R_j is given as

$$R_j = \left[\frac{\sin(\overline{\alpha}_j) - \sinh(\overline{\alpha}_j)}{\cos(\overline{\alpha}_j) - \cosh(\overline{\alpha}_j)}\right] \tag{6.393}$$

- Torsional motion.

$$\Xi_j = \sin(j\pi\xi_1), \qquad j = 1,2,3. \tag{6.394}$$

It is clear that the assumed modes associated with the extensible rotating pipe conveying fluid downwards, which then flows upwards as confined axial flow are identical to the assumed modes of this model.

6.2.4 Extensible Rotating Flexible Pipe with one sliding end Conveying Fluid downwards, which then flows upwards as a confined annular flow

The boundary conditions associated with this problem are fixed-sliding simply supported. By referring to [45], the assumed modes for this model could be found as follow:

- Axial motion

$$\Psi_j = \sin\left(\frac{2j+1}{2}\pi\,\xi_1\right), \qquad j = 0,1,2. \tag{6.395}$$

- Flexural motion in *X-Y* plane.

$$\Phi_j = \left[\sin\left(\overline{\alpha}_j\xi_1\right) - \sinh\left(\overline{\alpha}_j\xi_1\right)\right] - R_j\left[\cos\left(\overline{\alpha}_j\xi_1\right) - \cosh\left(\overline{\alpha}_j\xi_1\right)\right], \quad j = 1,2,3. \tag{6.396}$$

where $\overline{\alpha}_1 = 3.923$, $\overline{\alpha}_2 = 7.069$, $\overline{\alpha}_3 = 10.210$ and R_j is given as

$$R_j = \left[\frac{\sin\left(\overline{\alpha}_j\right) - \sinh\left(\overline{\alpha}_j\right)}{\cos\left(\overline{\alpha}_j\right) - \cosh\left(\overline{\alpha}_j\right)}\right] \tag{6.397}$$

- Flexural motion in *X-Z* plane.

$$\Theta_j = \left[\sin\left(\overline{\alpha}_j\xi_1\right) - \sinh\left(\overline{\alpha}_j\xi_1\right)\right] - R_j\left[\cos\left(\overline{\alpha}_j\xi_1\right) - \cosh\left(\overline{\alpha}_j\xi_1\right)\right], \quad j = 1,2,3. \tag{6.398}$$

where $\overline{\alpha}_1 = 3.923$, $\overline{\alpha}_2 = 7.069$, $\overline{\alpha}_3 = 10.210$ and R_j is given as

$$R_j = \left[\frac{\sin\left(\overline{\alpha}_j\right) - \sinh\left(\overline{\alpha}_j\right)}{\cos\left(\overline{\alpha}_j\right) - \cosh\left(\overline{\alpha}_j\right)}\right] \tag{6.399}$$

- Torsional motion.

$$\Xi_j = \sin\left(\frac{2j+1}{2}\pi\,\xi_1\right), \qquad j = 0,1,2. \tag{6.400}$$

CHAPTER SEVEN

NUMERICAL RESULTS AND DISCUSSION

In this chapter, the nonlinear equations of motion presented in chapter 6 are solved numerically and implemented to describe the following two important engineering applications:

- Vibrations of a tube in a double pipe heat exchanger.
- Vibrations of rotating drillstring.

The physical parameters of each case study are identified carefully to represent a realistic model. Also a set of practical design and operating conditions are assigned in order to describe similar industrial applications.

7.1 Objectives of the Numerical Analysis

Two numerical examples were considered. First, the dynamic analysis of the tube in a double pipe heat exchanger is performed in order to investigate the following aspects:

1. Estimation of the axial and lateral natural frequencies of the system for the first three modes of vibration.
2. The effect of the internal and external flows on the natural frequencies of the system.

3. The influence of several design parameters on the system's natural frequencies such as the flow velocities, material damping and annulus spacing.
4. Construction of the phase plane plots in order to check the stability of the system at the operating conditions.
5. Demonstration of the importance of nonlinear formulation in studying the dynamic behavior of such systems.

The second example is devoted to investigate the vibration of a rotating drillstring. This study is discussed from the following point of view:

1. Estimation of the lateral and torsional natural frequencies of the system for first three modes of vibration.
2. Studying the effect of different end conditions on the system, namely; fixed free and fixed-sliding simply supported ends.
3. Studying the effects of several design conditions on the system's natural frequencies such as: velocities of the drilling fluids, annulus spacing and the rotational speed.
4. Verifying the importance of considering the internal flow, external flow and rotation of the system in obtaining accurate natural frequencies.

7.2 Solution Procedure

Figure 7.1 shows the flow chart of the proposed solution algorithm. This algorithm is established to solve the governing discreatized equations of motion, which were derived in chapter 6. The main features of the solution scheme can be summarized as follow:

7.2.1 Evaluation of the coefficients of the equations of motion

The first step of performing the numerical analysis is assigning the system parameters, followed by identifying the desired mode of vibration and then evaluating the equations' coefficients, which were defined in chapter 6. Due to the fact that many differentiations and integrations are involved within this analysis, Sympolic MATLAB® is used to evaluate the coefficients of equations of motion for each case study separately. The proposed algorithm provides flexibility for the user to select, modify and solve for different system parameters and design conditions associated with each case, provided that the selected design conditions do not violate any of the model assumptions. Another advantage of this code is its ability to return the dimensionless quantities defined in chapter 5 prior to solution. This step is shown schematically in figure 7.1.

7.2.2 Solving the governing equations of motion.

The preliminary task for this step is to express the governing equations of motion in the state space representation, as most of the ODE's solvers require the system to be defined in this form. ODE45 code is utilized for solving the nonlinear system of differential equations in which all nonlinear terms are considered. ODE45 is considered as one of the initial value problems (IVPs) solvers in MATLAB® package. In this solver, the differential equations are integrated via the fourth order Runge Kutta method. It is necessary to specify the initial conditions in this step for each case study.

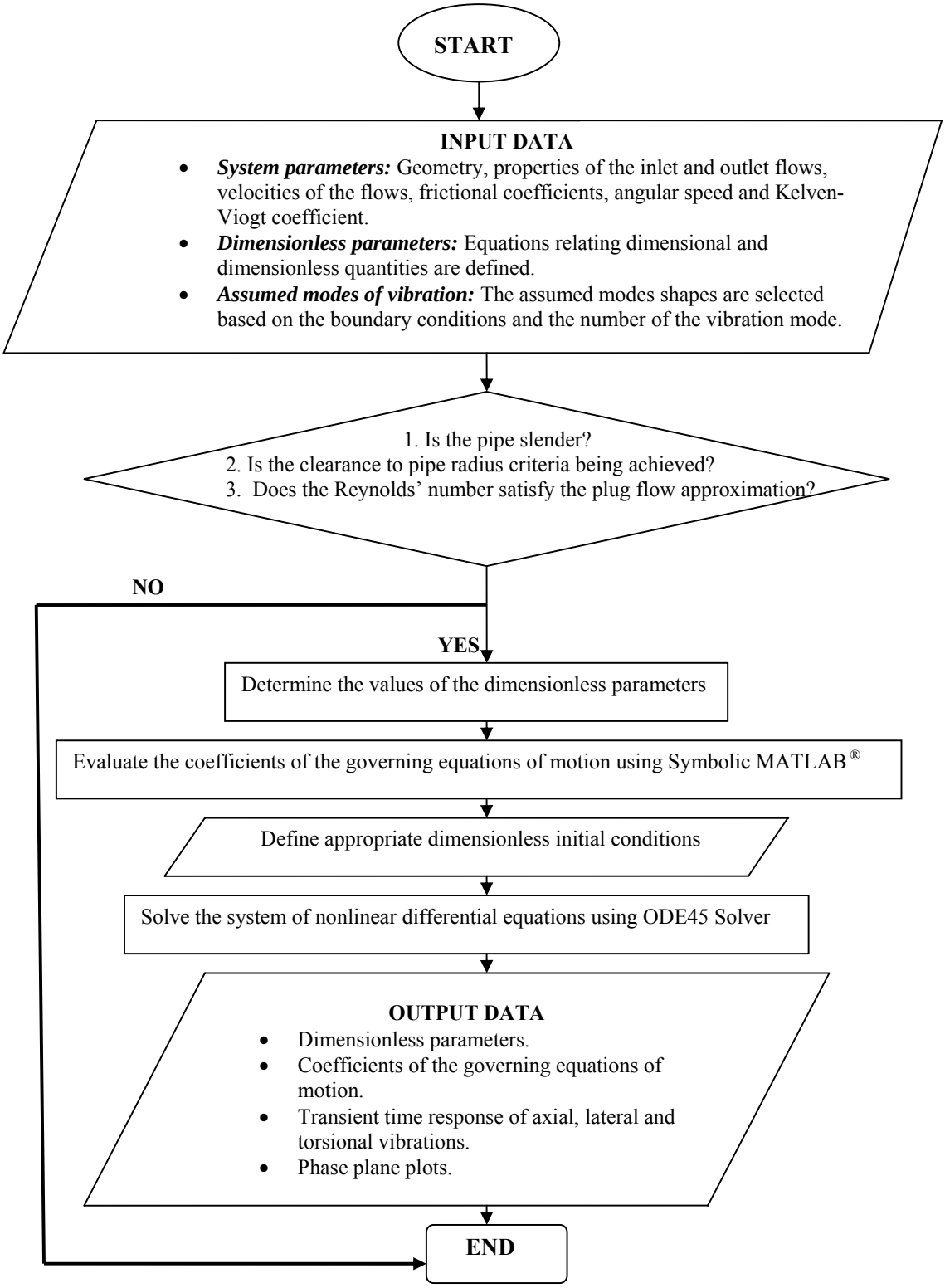


Figure 7.1: Algorithm flow chart

7.3 Flow Induced Vibration of Tube in A double-pipe Heat Exchanger

This case study represents a double pipe heat exchanger which is used to cool the lubricating oil of a large industrial gas turbine. In this type of heat exchangers, water flows through the inside pipe (tube-side flow) and the oil through the annular space between the outside and the inside pipe (shell-side flow), as illustrated schematically by figure 2.1.The flow rate of the oil through the annulus is 0.8 kg/s while the flow rate of the water inside the tube is 2 kg/s. In this model, both ends of the tube are fixed.

7.3.1 System parameters

In order to investigate the dynamic behavior of this heat exchanger, some realistic values of the system parameters are assigned as listed in table 7.1. The corresponding dimensionless quantities are calculated based on their definitions as presented in chapter 5. Table 7.2 shows the values of the dimensionless parameters associated with the current model. It is important to note that the dimensionless time equals *2.1485* times the actual time. Also the tension and internal pressurization are set to zero.

7.3.2 Transient response of the system

Based on the solution technique discussed in section 7.2, the coefficients of the equations of motion (6.11) and (6.12) were calculated. Table 7.3 shows such values for the first three modes of vibration. Based on the values listed in table 7.3, the following observations are made:

- The most dominant nonlinear terms are A_1^u and F_1^v, which are associated with q^2 and q^3,respectively, as given by equations (6.11) and (6.12). These terms emerged

Table 7.1: System parameters of a double pipe heat exchanger.

Parameters	Value
Length of elastic tube, L	4.0 m
Outside diameter of the elastic tube, D_o	0.028 m
Thickness of the elastic tube	0.0015 m
Diameter of the outer pipe, D_{ch}	0.048 m
Young's modulus of the elastic tube, E	2.1×10^{11} N/m^2
Mass of the tube per unit length, m	0.98 kg/m
Mass of the internal flow per unit length, M	0.491
Density of the lubricating oil, $\rho_{f,o}$	785 kg/m^3
Velocity of internal flow, U_i	4.07 m/s
Velocity of the external flow, U_o	0.854 m/s
Friction coefficient in normal direction, C_N	0.0125
Friction coefficient in tangential direction, C_T	0.0125
The form-drag coefficient, C_{DP}	0.0125

Table 7.2: Dimensionless parameters of a double pipe heat exchanger.

Dimensionless parameters	Value
Added mass, χ	2.032
Mass ratio of the internal fluid, β_i	0.2512
Mass ratio of the external fluid, $\beta_{o,1}$	0.2473
Velocity of the internal flow, u_i	0.2374
Velocity of the external flow, u_o	0.0494
Axial flexibility, $\Pi_{0,1}$	1.8169×10^5
Slenderness ratio, E	142.8571
Hydraulic coefficient, $\bar{h}$	1.4
Friction coefficient in normal direction, c_N	0.0159
Friction coefficient in tangential direction, c_T	0.0159
The form-drag coefficient, c_d	0.0159

Table 7.3: Coefficients of the governing equations of motion in both axial and lateral directions.

Coefficient	1st Mode	2nd Mode	3rd Mode
M_1^u	0.3763	0.3763	0.3763
C_1^u	0	0	0
K_1^u	896600	3586400	8069400
A_1^u	14573	-7.0762×10^6	-1.2609×10^7
A_2^u	1.1111	0.0026	2.2187
A_3^u	0.1309	0	0
A_4^u	-0.000983	-0.3331	-0.0540
B_1^u	0.1440	0	0
B_2^u	-0.2579	0	0
B_3^u	-0.2579	0	0
B_4^u	-0.000369	0	0
M_1^v	1.3632	1.3632	1.3632
C_1^v	0.0355	0.0829	0.0836
K_1^v	519.887	3.7994×10^3	1.3955×10^4
D_1^v	29467	-1.4153×10^7	-6.4220×10^6
D_2^v	-0.7983	-0.0033	-1.7171
D_3^v	1.251	-0.0025	2.0064
D_4^v	0.2598	-0.6661	-0.1079
D_5^v	-0.000983	-0.3331	-0.0540
E_1^v	0	0	0
E_2^v	-7.526×10^{-5}	0	0
E_3^v	-0.0965	0	0
E_4^v	1.433×10^{-8}	0	0
E_5^v	-0.3953	0	0
F_1^v	2.1515×10^7	3.2068×10^8	1.717×10^9
F_2^v	0.0209	0.0213	11.9415
F_3^v	-8.4861	-23.8852	-49.3839
F_4^v	-2.3063	0	0
F_5^v	-5.6574	-15.9234	-32.9226

from the potential energy expression as a result of elasticity of the inner tube in both axial and lateral directions, in addition to the influence of the external flow on the inner tube. Another dominant term is D_1^v, which is associated with the product $p\ q$ as shown in equation (6.12). This term reveals the importance of considering the coupling between axial and lateral displacement corresponding to the external flow and elastic coupling. Such nonlinear terms are expected to play a role in determining the dynamic behavior of the system when the tube is subjected to large deformation.

- One of the interesting findings is that the damping in the axial direction is zero, which will not be true if the boundary conditions are not fixed-fixed, or if the material dissipative damping has been considered. On the other hand, the linear damping in lateral direction is very small as the applied external flow velocity is within a small range in this case study.

Once the coefficients of the equations of motion are calculated , the system of governing equations is solved using ODE 45 solver, after applying the following initial conditions to the mid span of the pipe such that: $u(t) = 0.001,\ \dot{u}(t) = 0,\ v(t) = 0.01,\ \dot{v}(t) = 0$ .The corresponding dimensionless initial conditions are calculated based on their definitions provided in chapter 5, as: $p(t_1^*) = 0.00025,\ \dot{p}(t_1^*) = 0,\ q(t_1^*) = 0.0025,\ \dot{q}(t_1^*) = 0$. Figures 7.3-7.8 show the transient response of the system for the first three modes of vibration in addition to the corresponding phase plane plots. The natural frequencies in axial and lateral directions were calculated based on the period of oscillation, while their values are listed in table 7.4. In the following analysis, the damping factor is estimated graphically using the logarithmic decrement method.

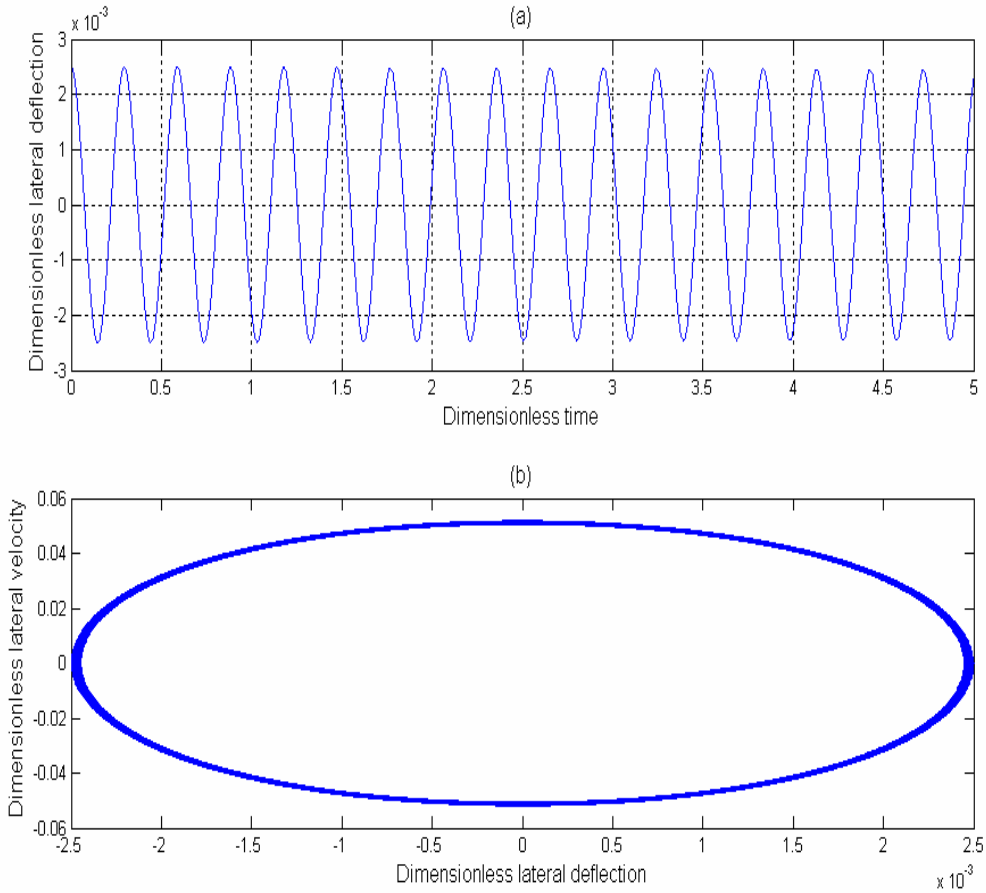


Figure 7.2: (a) Transient response of the dimensionless lateral deflection at the first mode of vibration. (b) Corresponding phase plane plot.

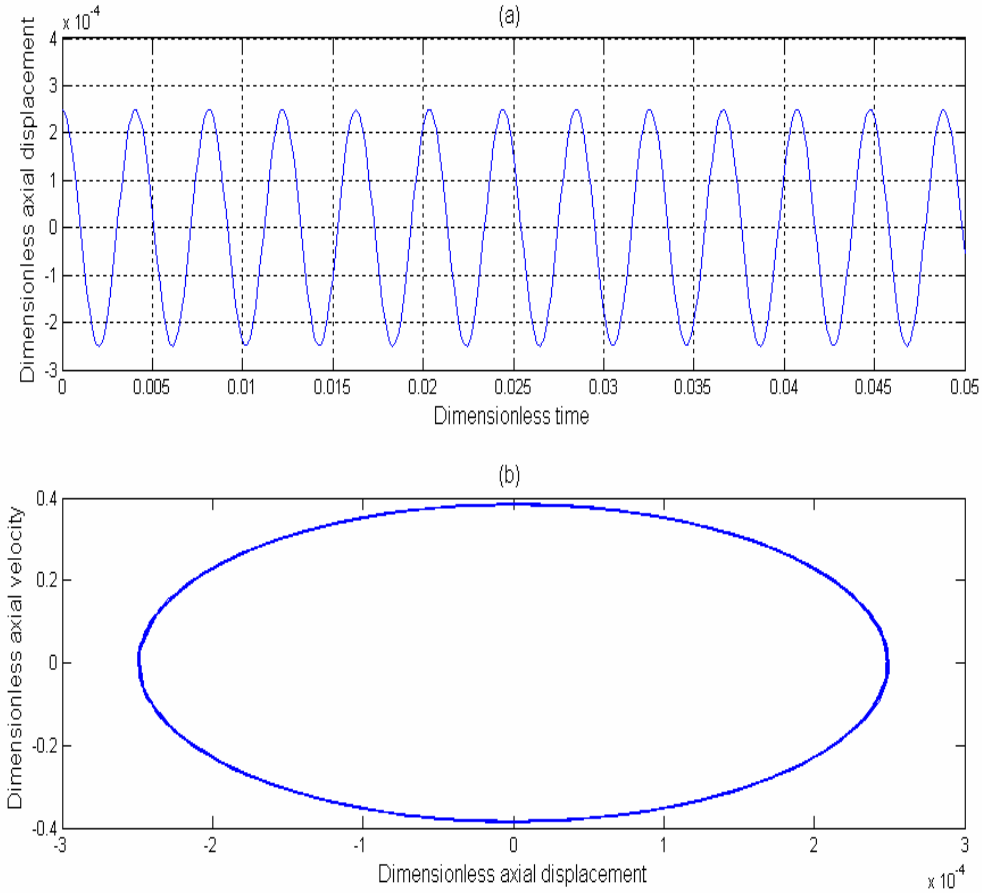


Figure 7.3: (a) Transient response of the dimensionless axial deflection at the first mode of vibration. (b) Corresponding phase plane plot.

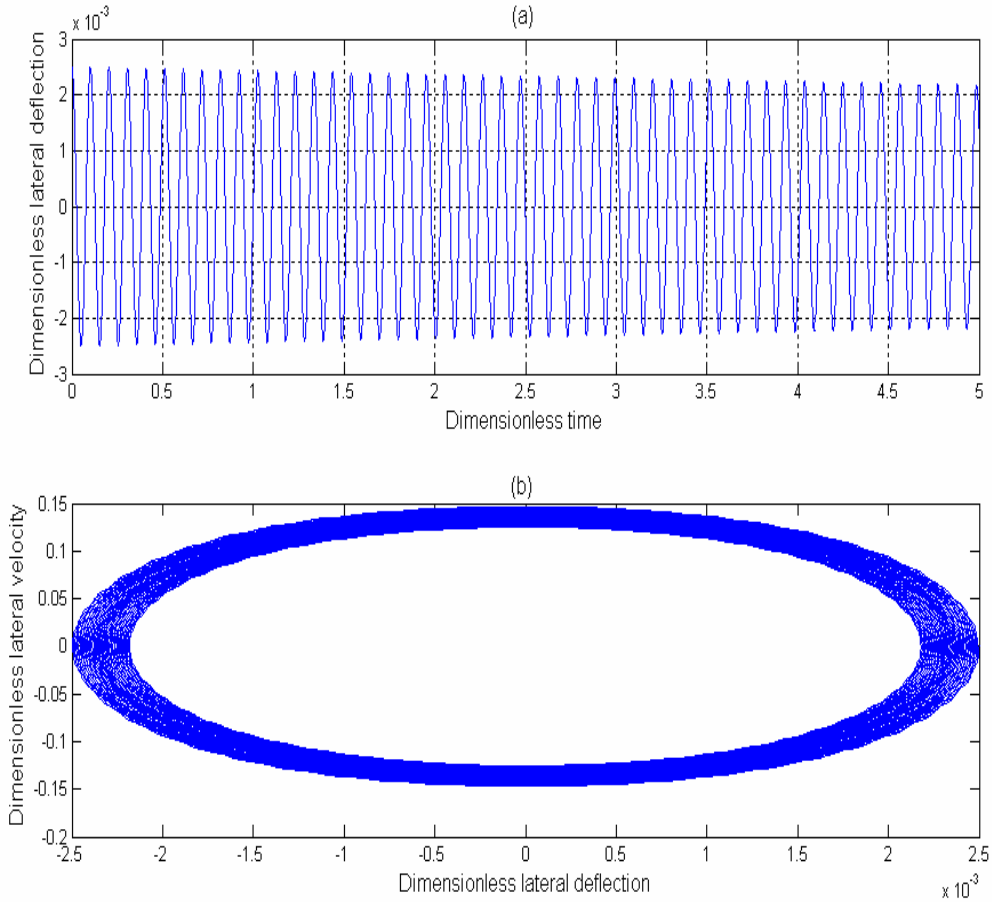


Figure 7.4: (a) Transient response of the dimensionless lateral deflection at the second mode of vibration. (b) Corresponding phase plane plot.

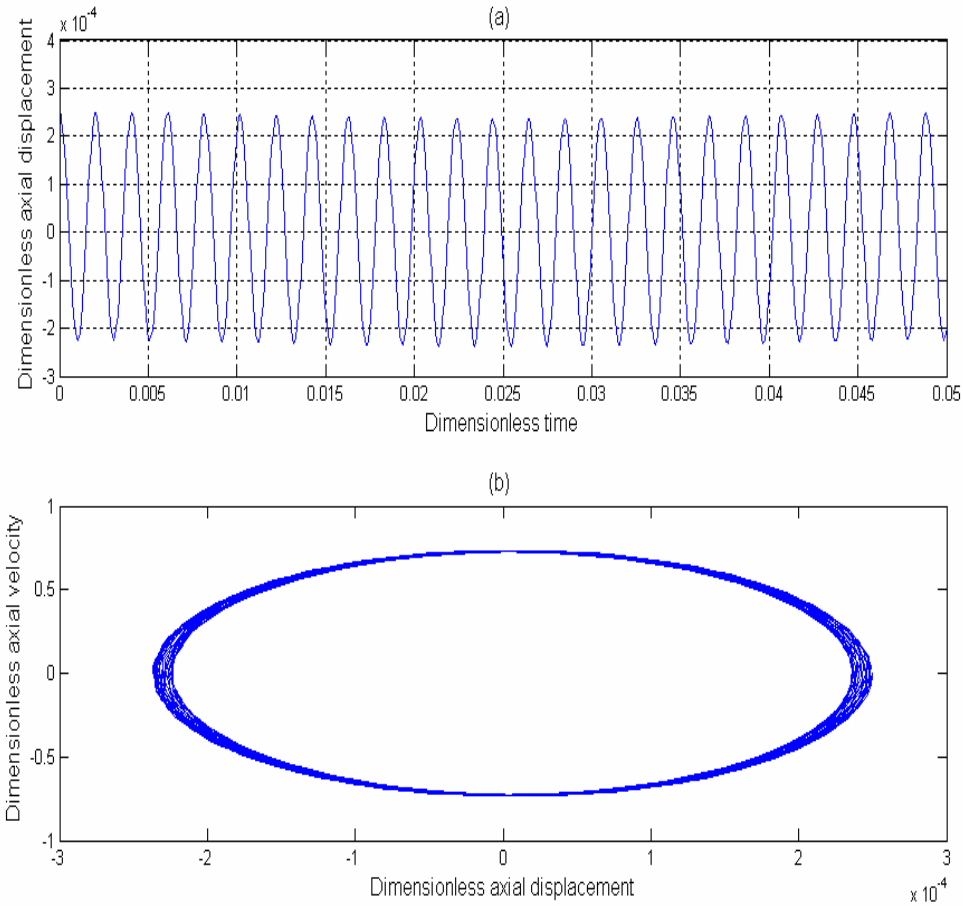


Figure 7.5: (a) Transient response of the dimensionless axial deflection at the second mode of vibration. (b) Corresponding phase plane plot.

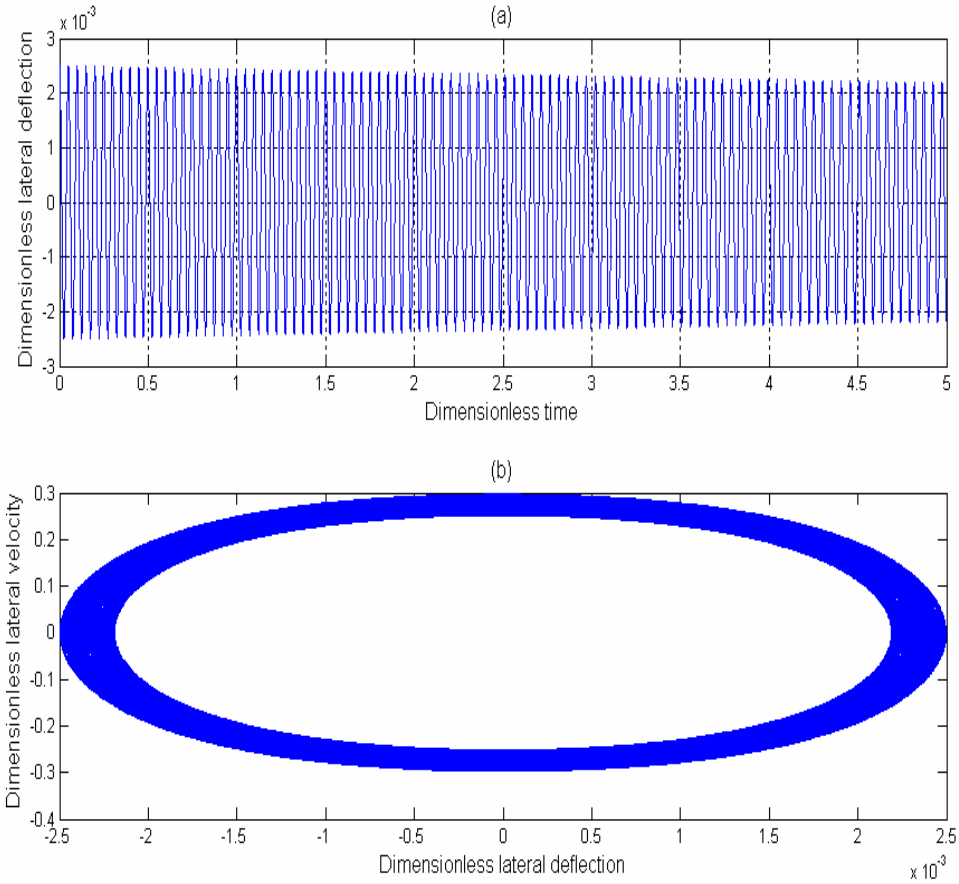


Figure 7.6: (a) Transient response of the dimensionless lateral deflection at the third mode of vibration. (b) Corresponding phase plane plot.

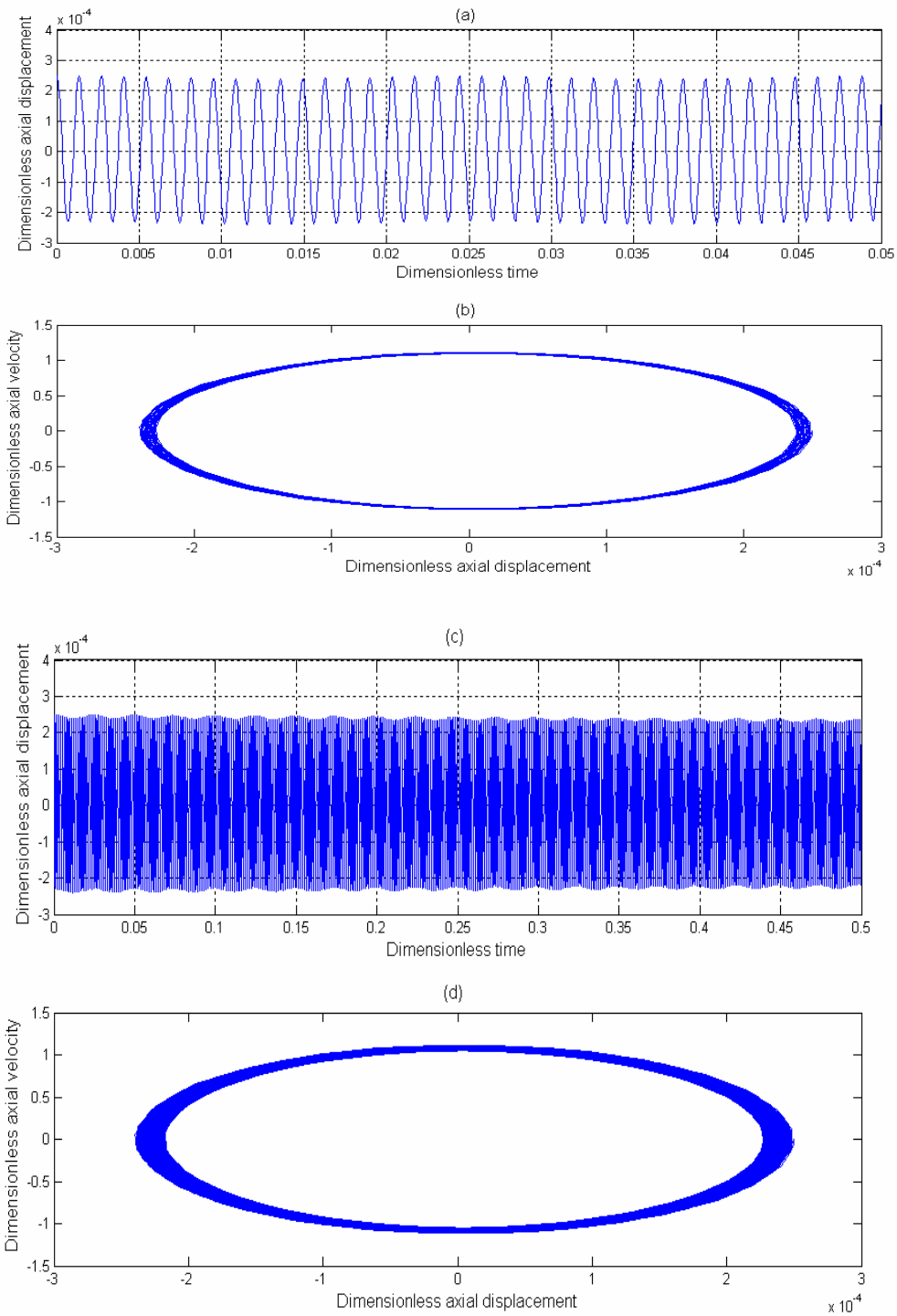


Figure 7.7: (a) Transient response of the dimensionless axial deflection at the third mode of vibration. (b) Corresponding phase plane plot. (c) Transient response of the dimensionless axial deflection at the third mode of vibration for longer time interval. (d) Corresponding phase plane plot.

Table 7.4: Lateral and axial natural frequencies of a double pipe heat exchanger.

No. of Mode	Dimensionless natural frequency of lateral vibration	Natural frequency of lateral vibration (rad/s)	Dimensionless natural frequency of axial vibration	Natural frequency of axial vibration (rad/s)
1st Mode	19.56	42.03	1541.13	3311.12
2nd mode	52.62	113.06	2990.57	6425.24
3rd mode	101.33	217.70	4695.83	10089.00

By referring to figures 7.3-7.7 and table 7.4, the following observations are made:

- The damping ratio is very small for the lateral vibrations and equals 0.004, while the system seems undamped in the axial direction.
- The phase portraits show that the system is stable for both axial and lateral vibrations at the selected operating conditions.
- The axial natural frequencies attains high values compared to the lateral ones, accordingly, one may consider that the dynamic analysis can be focused on the more significant lateral vibrations.
- At higher modes, it was observed that there are some fluctuations in the amplitude of the axial vibrations, as shown in figure 7.7(c). This beating phenomenon appears as a result of the nonlinear terms, and could not be viewed in light of linear analysis, where linear damping is zero in the axial direction.

7.3.3 Influences of the internal and external flows

In order to investigate the effects of internal and external fluids on the transient response of the system, the simulation was first performed for the dry model in which both internal and external fluids are absent. Figures 7.8-7.13 show the transient responses for both axial and lateral vibrations of the dry pipe system for the first three modes of vibration. In order to demonstrate the effect of each flow pattern separately on the system's natural frequencies, a comparative study is performed at different flow schemes; such as internal flow only, external flow only, and the presence of both external and internal flows. For brevity, lateral and axial natural frequencies for the first three modes of vibration for each case are listed in Table 7.5, without providing their transient plots.

Also in table 7.5, the natural frequencies of each flow pattern are compared to those natural frequencies associated with the dry system. It is important to consider that the dimensionless time depends on the existence of the internal and external flows and was modified for each case accordingly. The obtained natural frequencies for the dry system were found very close to those resulted from the tabulated exact formulas associated with the fixed-fixed configuration [45]. Moreover, the obtained natural frequencies are considered more accurate as they accommodates the nonlinear terms, while the exact formulas are based on the linear formulation.

By referring to figures 7.8-7.13 and table 7.5, the influence of the internal and external fluids can be summarized as follow:

- *Influence of the internal flow*

  Based on table 7.5, we can conclude that the internal flow has higher influence in lowering (softening) the lateral and axial natural frequencies of the system for the first three modes of vibration by a percentage of 17.8 % up to 19.4% compared to the fluid-free model. This finding seems logical as the internal fluid increases the inertia term of the system significantly. This finding assures that the effect of the internal flow can not be eliminated either in lateral or axial directions. Also it is noted that the internal flow provides relatively smaller amount of damping in the lateral direction when compared to the case of external flow. This result may lead us to state that the effect of the internal flow in damping this system may be negligible, particularly in lateral direction. On the other side, we note that the damping in axial direction is still

having zero value in the case of fixed-fixed ends condition and negligible effect of material damping.

❖ *Influence of the external flow.*

Upon carrying the simulation for the model containing external flow only, it was found that the external flow plays a dominant role in softening the lateral natural frequency of the system for all modes of vibration by values ranges from 29 % up to 31.7 % compared to the fluid-free model. This finding can be interpreted by recognizing that the added mass of the external flow (i.e. the hydrodynamic mass) increases the inertia of the system in the lateral direction. Also the external flow provides the system with good amount of damping compared to the internal flow. Regarding the effect of the external flow on the axial vibration, it is noted that external flow has very limited effect on the axial natural frequencies (i.e. less than 1.5 % for all modes of vibration), this is due to the fact that the hydrodynamic added mass resulted is a much smaller inertia term in the axial direction.

❖ *Influence of both external and internal flows.*

The combined softening effect of the internal and external flows is responsible for lowering the lateral natural frequencies of the system up to 38.1 %. This seems reasonable as each of the flows (i.e. either internal or external flows) has its own softening contribution. Regarding the axial natural frequency, it decreased by a percentage of 16.3 % up to 21.2% compared to the dry system for the first three modes of vibration. These results are very close to the case of internal flow only which assures the minor effect of the external flow on the axial natural frequencies.

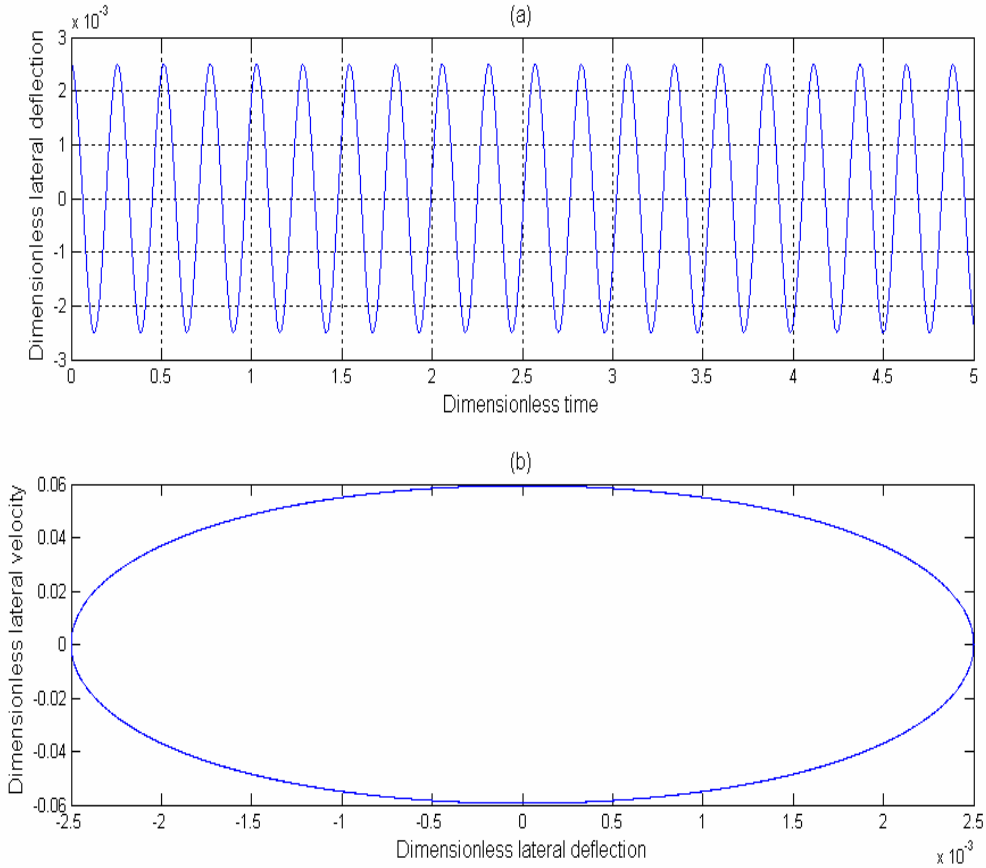


Figure 7.8: (a) Transient response of the dimensionless lateral deflection at the 1st mode of vibration for the fluid- free system.

(b) Corresponding phase plane plot.

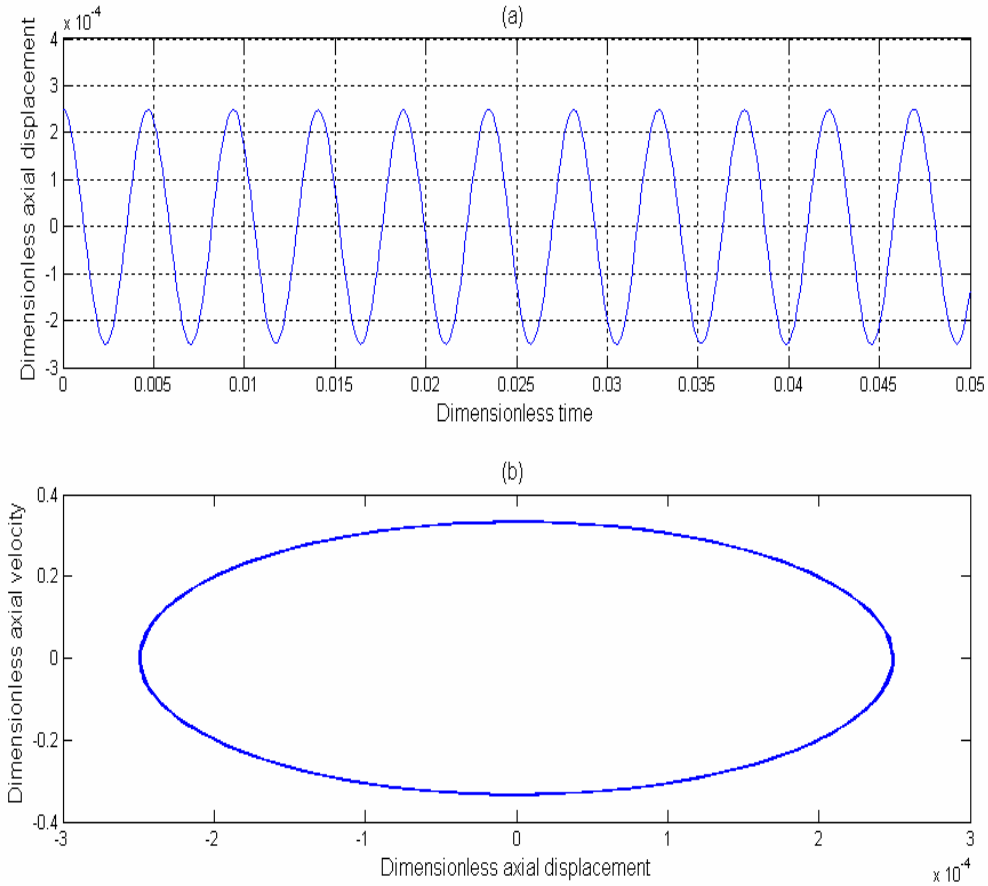


Figure 7.9: (a) Transient response of the dimensionless axial deflection at the 1st mode of vibration for the fluid- free system. (b) Corresponding phase plane plot.

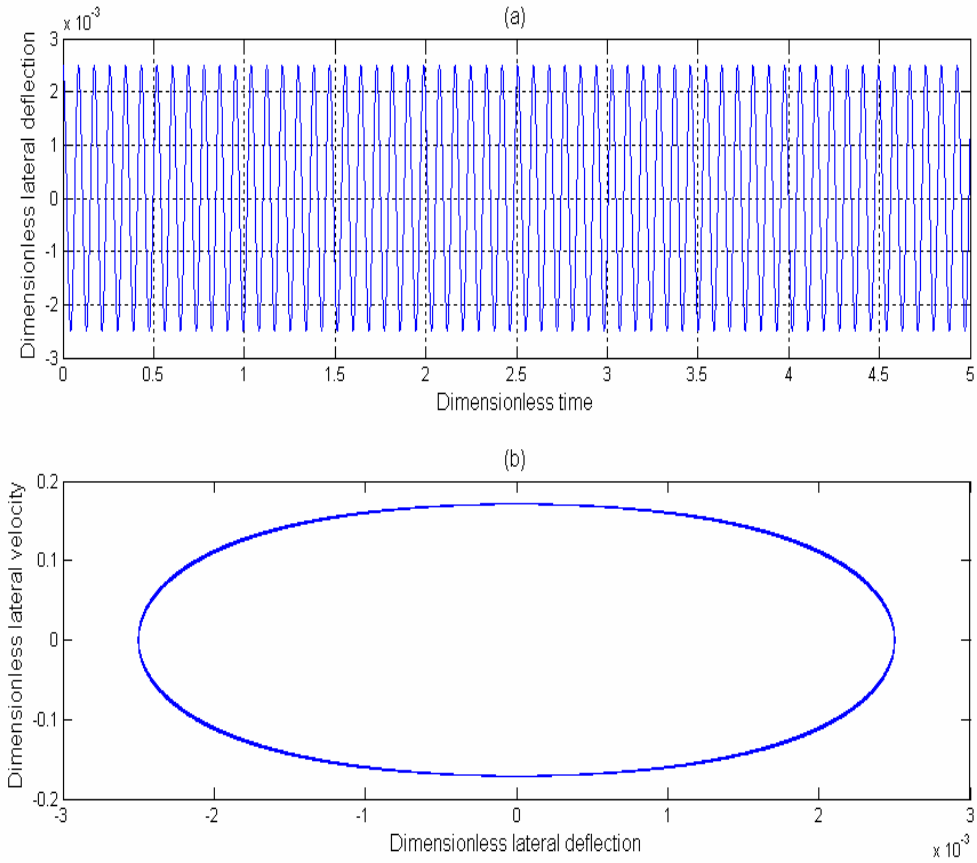


Figure 7.10: (a) Transient response of the dimensionless lateral deflection at the second mode of vibration for the fluid- free system. (b) Corresponding phase plane plot.

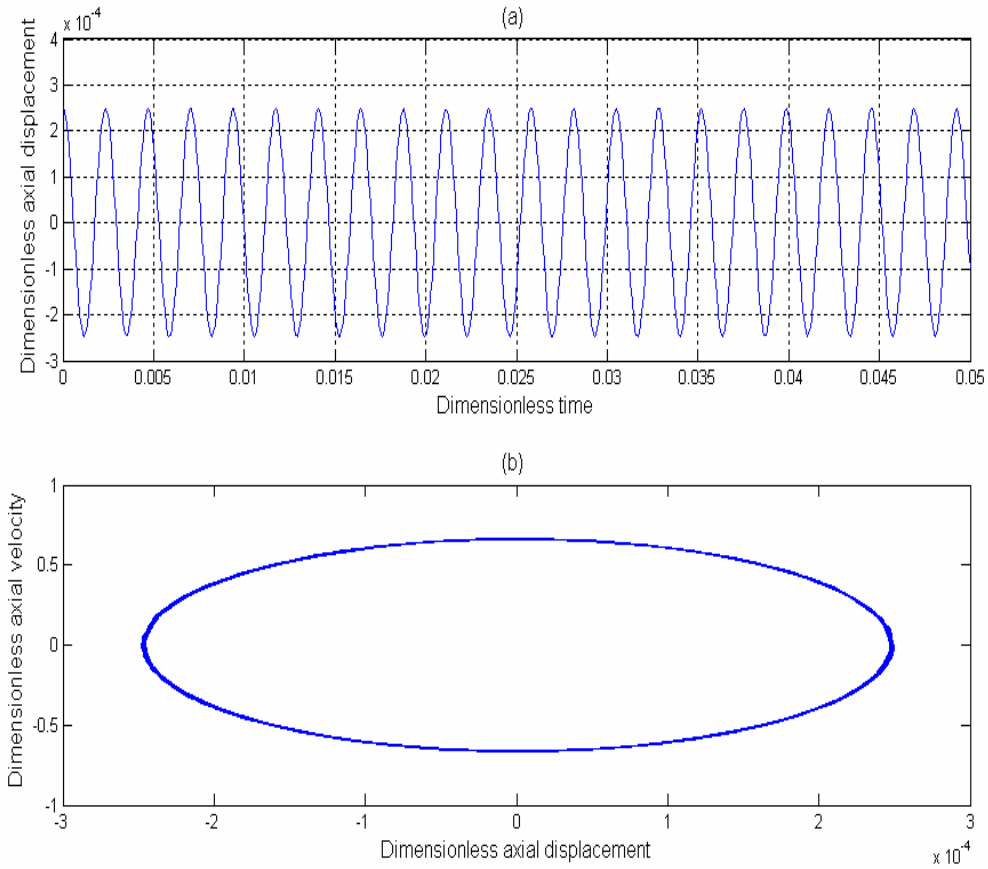


Figure 7.11: (a) Transient response of the dimensionless axial deflection at the 2nd mode of vibration for the fluid- free system. (b) Corresponding phase plane plot.

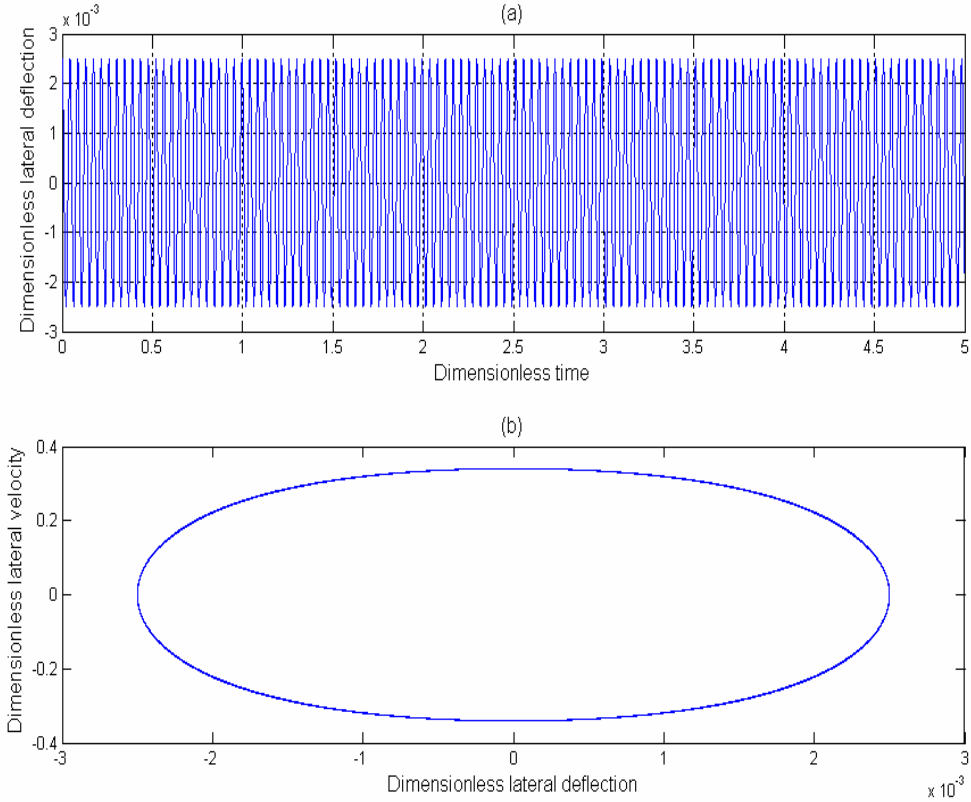


Figure 7.12: (a) Transient response of the dimensionless lateral deflection at the 3rd mode of vibration for the fluid- free system. (b) Corresponding phase plane plot.

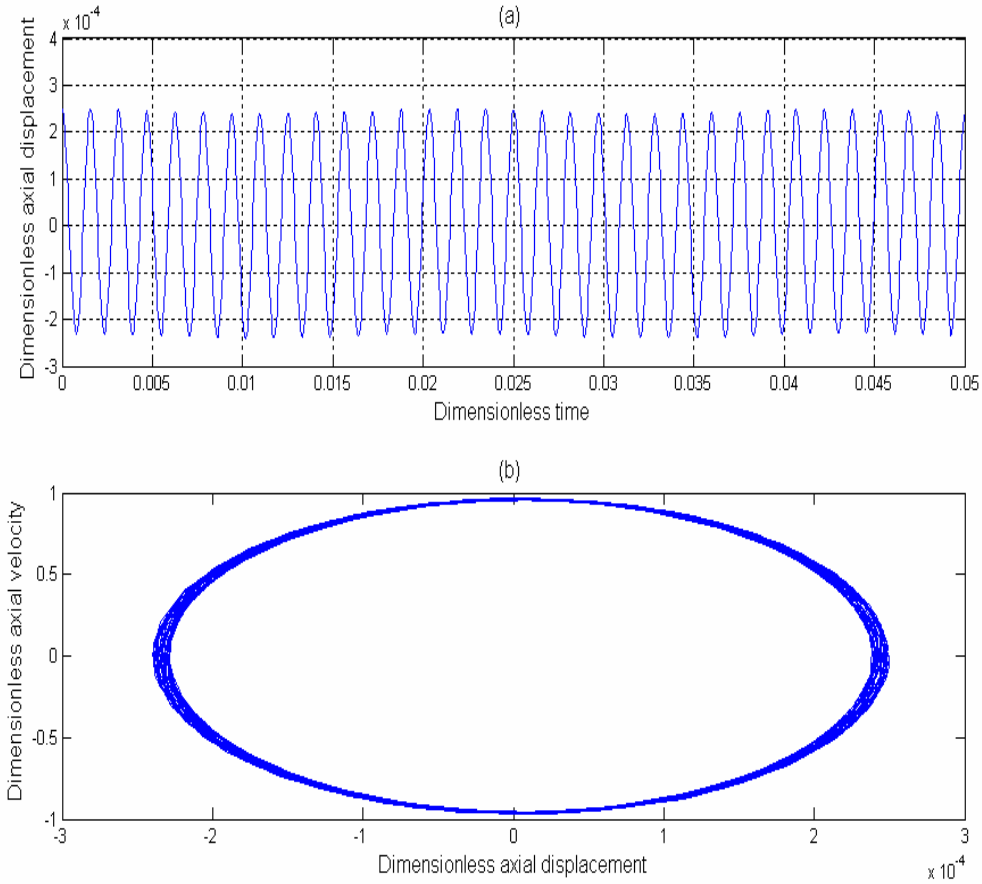


Figure 7.13: (a) Transient response of the dimensionless axial deflection at the 3rd mode of vibration for the fluid- free system.

(b) Corresponding phase plane plot.

Table 7.5: Lateral and axial natural frequencies for several flow patterns.

Model description	No. of mode of vibration	Lateral vibrations		Axial vibrations	
		Natural frequency (rad/s)	*Percentage (%)	Natural frequency (rad/s)	*Percentage (%)
The dry (fluid-free) system	1st mode	67.9	---	4032	---
	2nd mode	177	---	8150.1	---
	3rd mode	351.9	---	12050	---
Internal flow only	1st mode	55.0	-19	3291	-18.4
	2nd mode	145.4	-17.8	6566.7	-19.4
	3rd mode	284.6	-19.1	9865	-18.1
External flow only	1st mode	46.5	-31.5	4063.7	0.8
	2nd mode	125.7	-29	8128	-0.3
	3rd mode	240.4	-31.7	12195.6	1.21
Both internal and external flows	1st mode	42	-38.1	3311.1	-17.9
	2nd mode	113	-36.2	6425.2	-21.2
	3rd mode	217.7	-38.1	10089	-16.3

* The tabulated percentage represents the percentage of change of the natural frequencies between each flow pattern and the dry system.

7.3.4 Influences of the external flow velocity

Figures (7.14-7.17) show the transient response of the system for axial and lateral vibrations at external flow velocities of 5 and 10 m/s. The natural frequencies and damping factors associated with lateral and axial vibration at those external flow velocities are listed in table 7.6. Based on the tabulated results, the following remarks are made:

- The external flow velocity has a minor effect in lowering the lateral natural frequencies of the model with percentage of decrease less than 0.7 % as the external velocity was increased by ten times.
- The external flow velocity plays dominant role in increasing the damping of the system in the lateral direction, as the damping factor at highest flow velocity is about 10 times the damping coefficient at the lowest one. This phenomenon can be displayed clearly by investigating the phase plots of different cases, which show that the system is more damped at higher external velocities as it returns to the equilibrium point faster.
- The external flow velocity has no effect on the axial natural frequency of the system. Accordingly, for fixed-fixed pipe, the effect of the external flow velocity is negligible on the axial frequencies.

Finally, we must note that the above findings are obtained for the current model and are particular to the chosen boundary conditions.

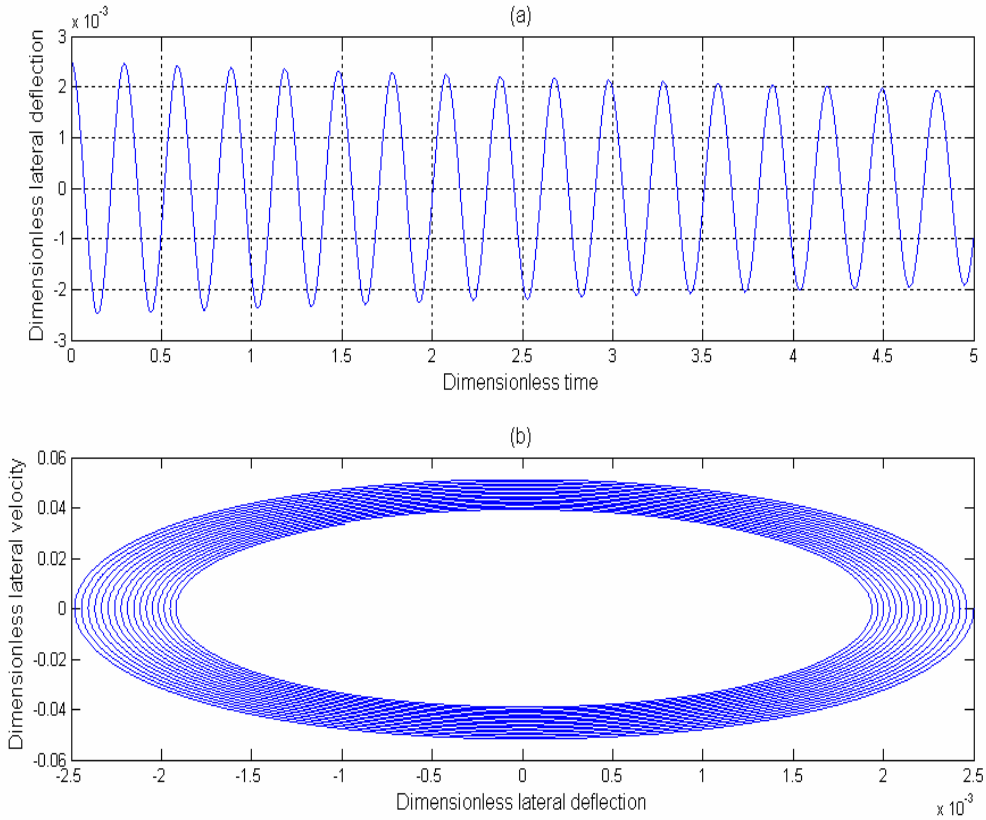


Figure 7.14: (a) Transient response of the dimensionless lateral deflection at the 1st mode of vibration at U_o= 5 m/s.

(b) Corresponding phase plane plot.

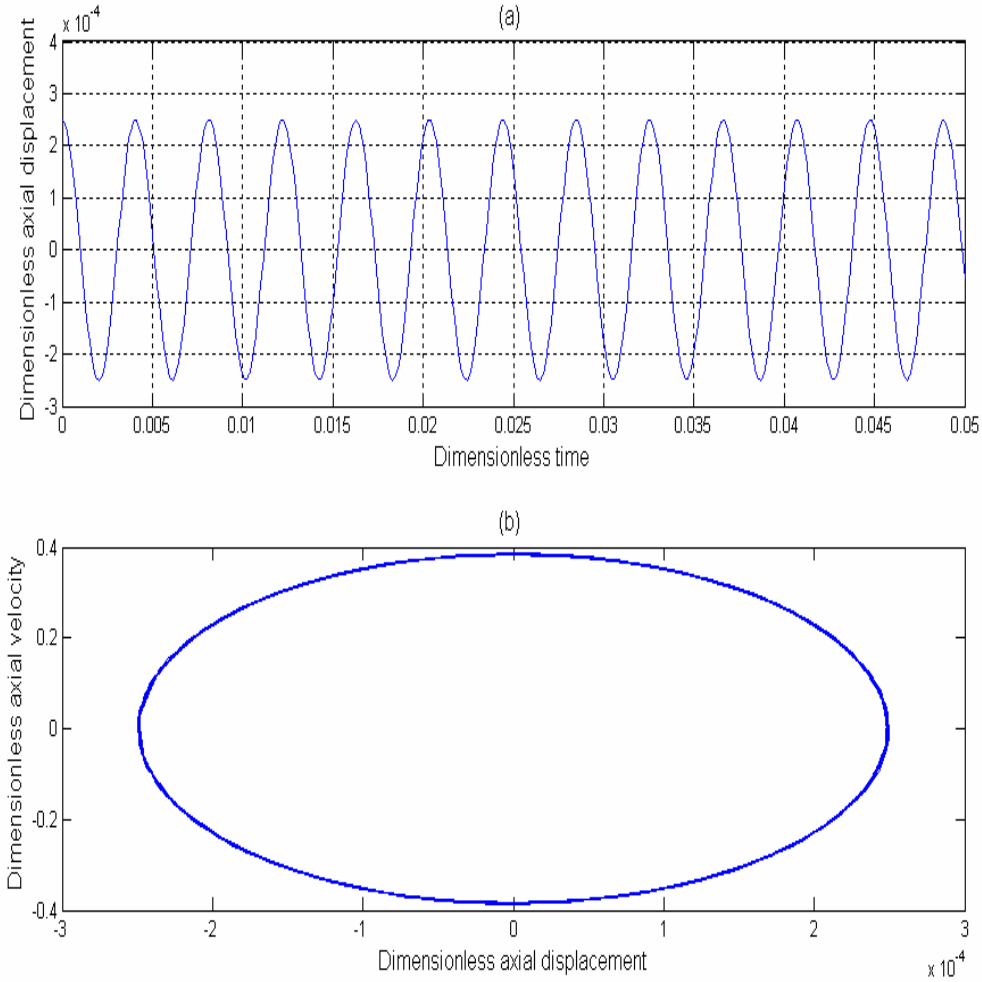


Figure 7.15: (a) Transient response of the dimensionless axial deflection at the 1st mode of vibration at U_o= 5 m/s.

(b) Corresponding phase plane plot.

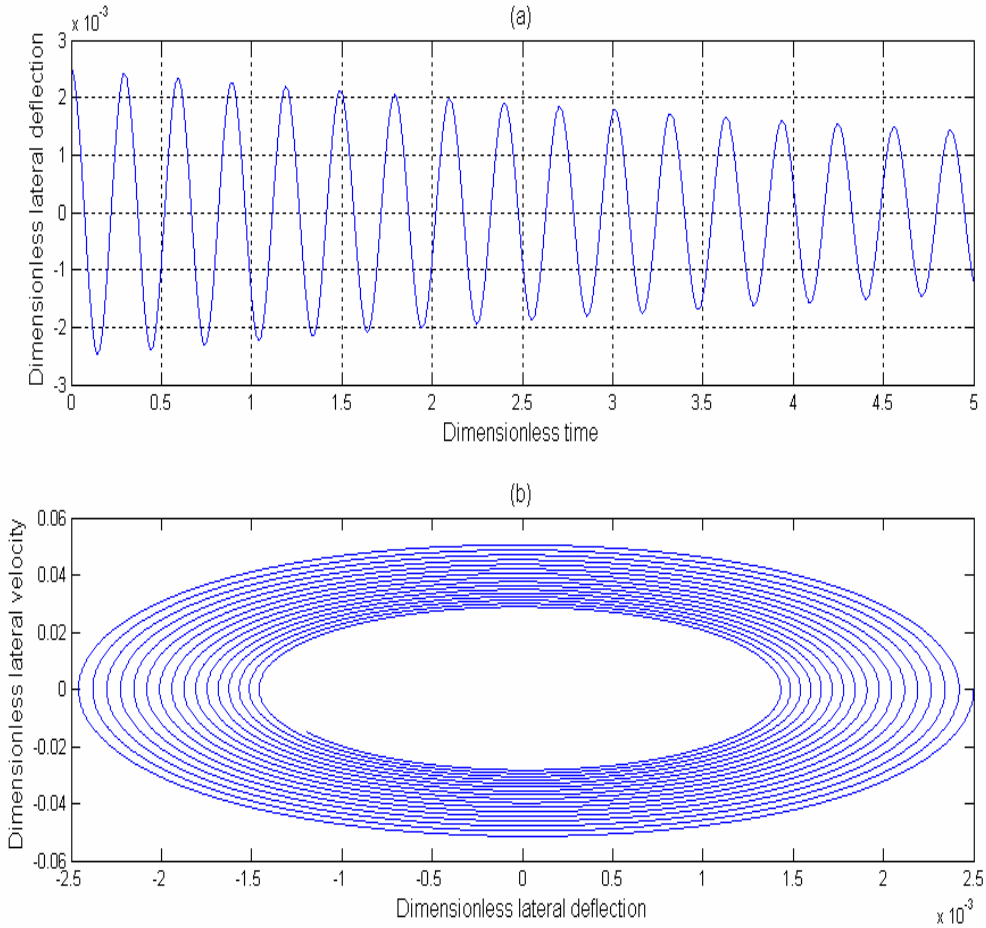


Figure 7.16: (a) Transient response of the dimensionless lateral deflection at the 1st mode of vibration at U_o= 10 m/s.

(b) Corresponding phase plane plot.

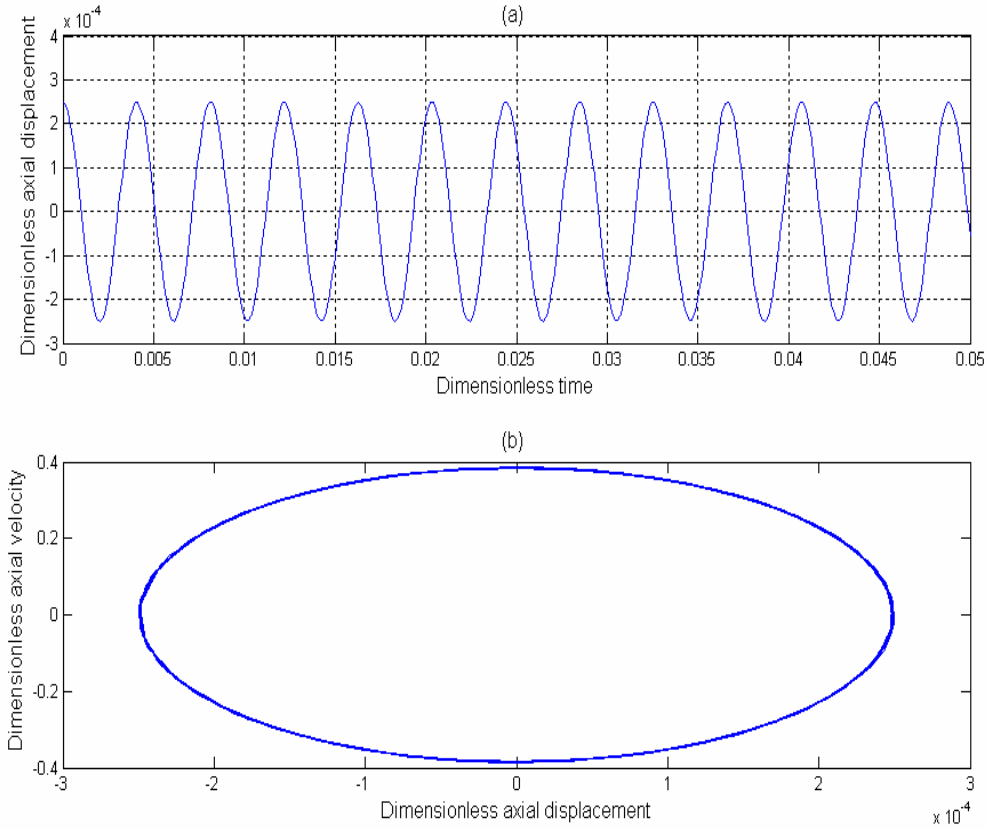


Figure 7.17: (a) Transient response of the dimensionless axial deflection at the 1st mode of vibration at U_o= 10 m/s.

(b) Corresponding phase plane plot.

Table 7.6: Lateral and axial natural frequencies and damping coefficients of a double pipe heat exchanger at various external flow velocities.

Velocity of the external flow (m/s)	**Damping ratio in lateral direction**	**Natural frequency of lateral vibration (rad/s)**	**Damping ratio in axial direction**	**Natural frequency of axial vibration (rad/s)**
0.854	0.00067	42.03	0	3311.12
5	0.0037	41.89	0	3311.12
10	0.0064	41.74	0	3311.12

Regarding the influence of the internal flow velocity, and based on several numerical runs, it was found that increasing the internal flow velocity has a minor effect on the dynamics of the system in lateral and axial directions. Therefore, the internal flow velocity effect on the natural frequencies of the system is almost negligible.

7.3.5 Influence of the annulus spacing

Figures (7.18-7.21) show the transient response of the system for the axial and lateral vibrations when the outer cylinder diameter is considered at the two values of 0.08 and 0.03 m. The natural frequencies and damping coefficients for the lateral and axial vibrations at various outer pipe diameters are listed in table 7.7. Based on the obtained results, the following remarks are made:

- The annulus spacing has a dominant softening effect on the lateral natural frequencies of the system. As the annular spacing decreased, a noticeable drop in the natural frequency occurred. It was found that, for the annular space of 1 mm, the natural frequency is lowered by a percentage of 45.1 % compared to the case when the annular spacing is 10 mm. This finding is considered justifiable since the added mass is a function of the annular spacing such that $Ma \propto \frac{(D_{ch}/D_o)^2+1}{(D_{ch}/D_o)^2-1}$. Accordingly, if the annular space decreases, the above ratio increases and leads to build up the added mass of the external fluid. Following the same philosophy, the natural frequency is increasing as the annular spacing is widening as displayed when the outer pipe diameter equals to 0.08 m.

- The annular spacing has a minor effect on the axial natural frequency of the system. It was found that axial natural frequency of the system at annular spacing of 1 mm is higher by a percentage of 3.24 % compared to the case in which the annulus spacing equals 26 mm. This minor effect is expected since the added mass is not a function of the linear coefficients of the equation of motion in the axial direction (i.e. equation 6.11), but it is associated with some nonlinear coefficients that have small impact on the dynamics of the system. One should recognize that such small variation of the axial natural frequency cannot be picked out without considering the nonlinear formulation.

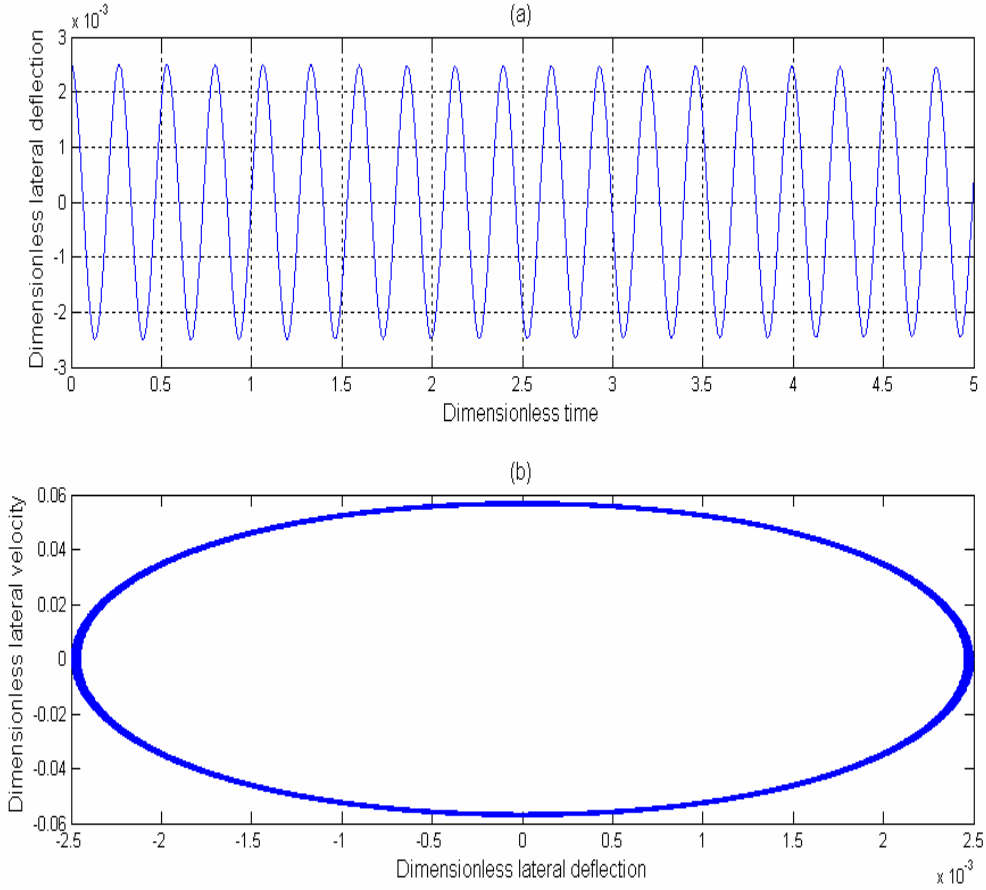


Figure 7.18: (a) Transient response of the dimensionless lateral deflection at the 1st mode of vibration for D_{ch}=0.08 m.

(b) Corresponding phase plane plot.

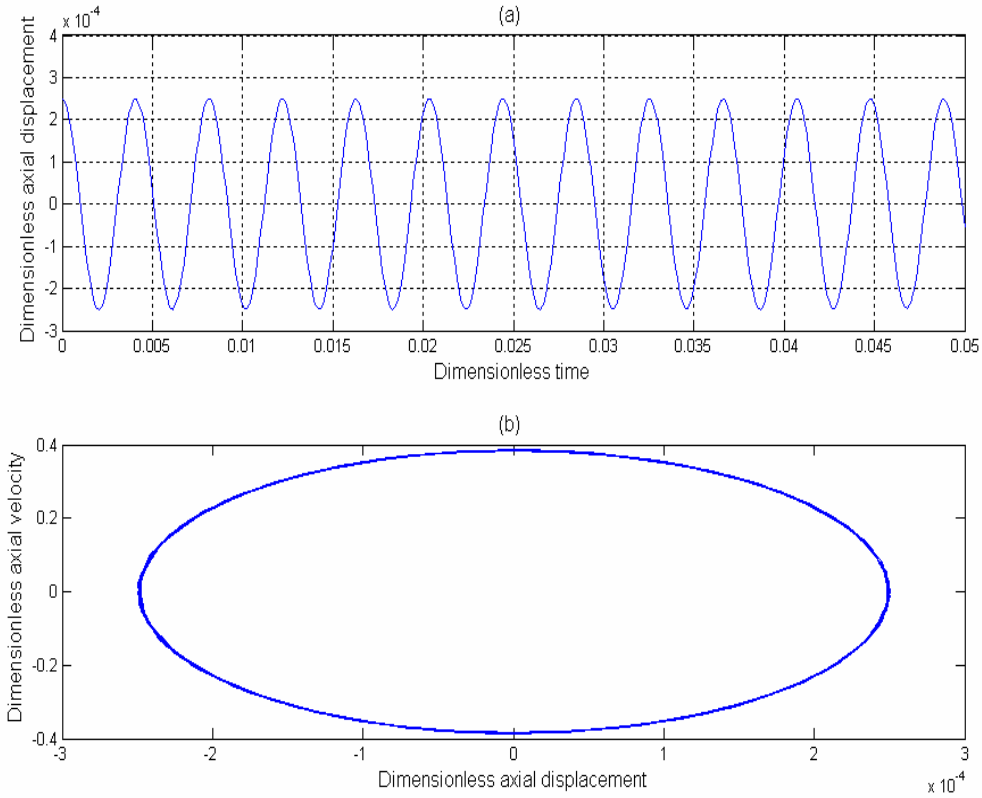


Figure 7.19: (a) Transient response of the dimensionless axial deflection at the 1st mode of vibration for D_{ch}=0.08 m.

(b) Corresponding phase plane plot.

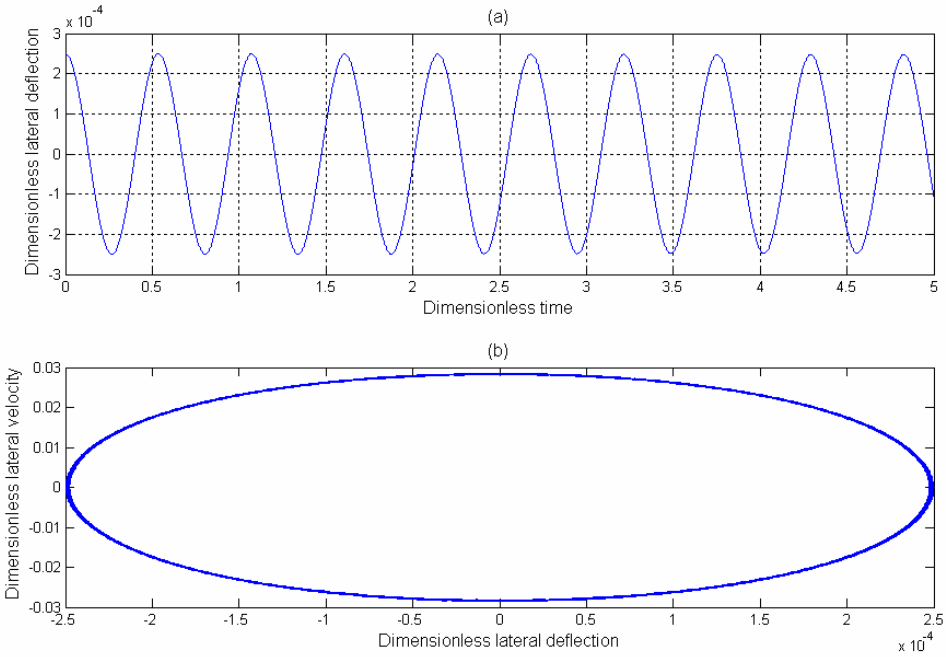


Figure 7.20: (a) Transient response of the dimensionless lateral deflection at the 1st mode of vibration for D_{ch}=0.03 m.

(b) Corresponding phase plane plot.

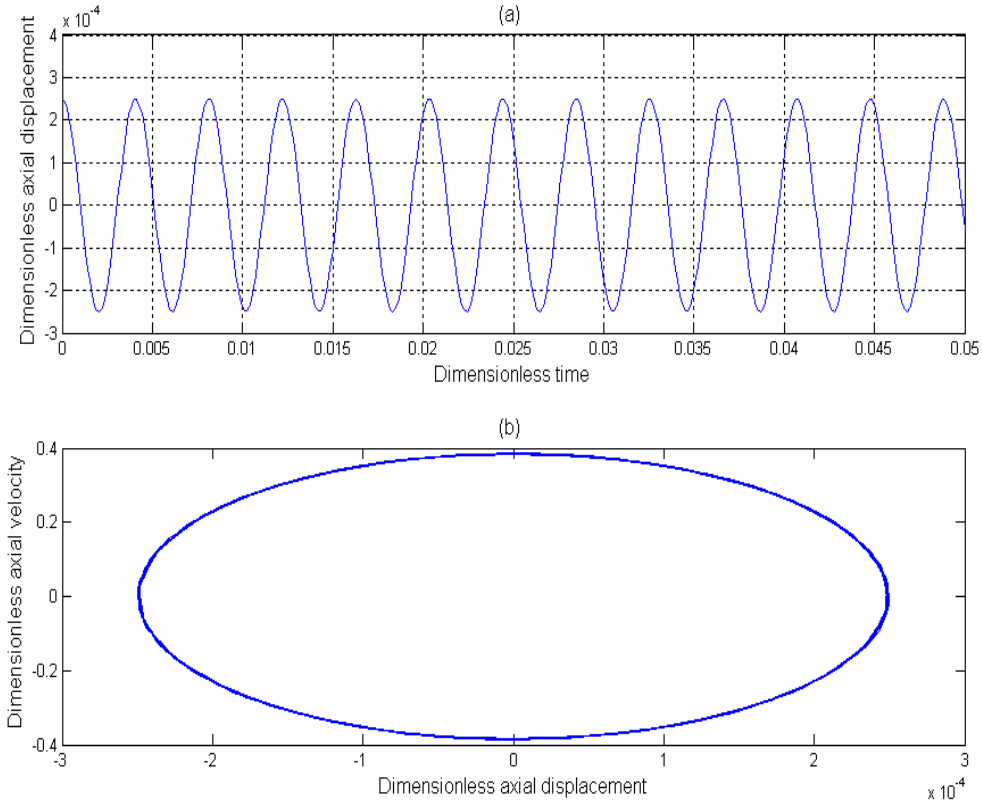


Figure 7.21: (a) Transient response of the dimensionless axial deflection at the 1st mode of vibration for D_{ch}=0.03 m.

(b) Corresponding phase plane plot.

Table 7.7: Lateral and axial natural frequencies and damping coefficients of a double pipe heat exchanger at various outer pipe diameters.

Diameter of the outer pipe (m)	Damping ratio in lateral direction	Natural frequency of lateral vibration (rad/s)	Damping ratio in axial direction	Natural frequency of axial vibration (rad/s)
0.03	0.00067	23.07	0	3204.23
0.048	0.00067	42.03	0	3311.12
0.08	0.00064	46.82	0	3307.17

7.3.6 Effect of the material damping

In the previous analysis, it was assumed that the Kelvin Voigt coefficient is zero, which implies a negligible damping of the tube material. Our task here is to investigate the effect of the material damping on the system response and the natural frequencies. Figures (7.22-7.23) show the response of the system described in subsection 7.3.2 at the 1st mode of vibration but this time when the material damping is considered and the Kelvin Voigt coefficient is assigned to be 0.0001. The obtained findings can be summarized as follow:

- The lateral natural frequency was found to be 42.2 rad/sec while the axial natural frequency is 3268.08 rad/sec. This means that the lateral natural frequency is higher by 0.4 % while the axial natural frequency is lower by 1.3 % compared to the similar system with negligible material damping. Based on the above, we can conclude that the material damping has very minor effect on the system natural frequencies. Also one should note that such small deviations could not be displayed without performing the nonlinear analysis.
- The damping coefficient in the axial direction was found to be 0.166 while it was zero in the absence of the material damping. The value of the damping coefficient in the lateral direction is 0.015 which is much higher than the case of non dissipative material. Accordingly, we can state that the material damping plays dominant role in suppressing the vibration in the axial direction while the effect is relatively smaller in the lateral direction.

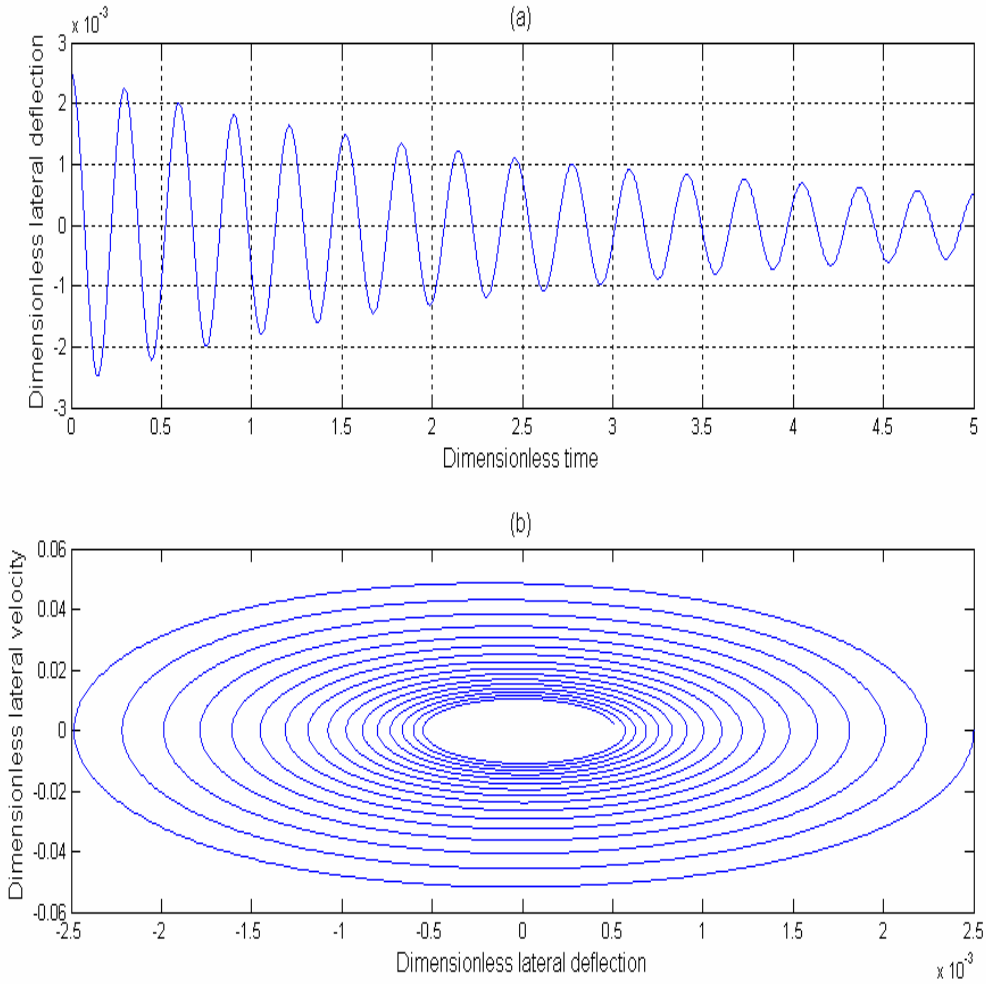


Figure 7.22: (a) Transient response of the dimensionless lateral deflection at the 1st mode of vibration for Kelvin Voigt coefficient of 0.0001. (b) Corresponding phase plane plot.

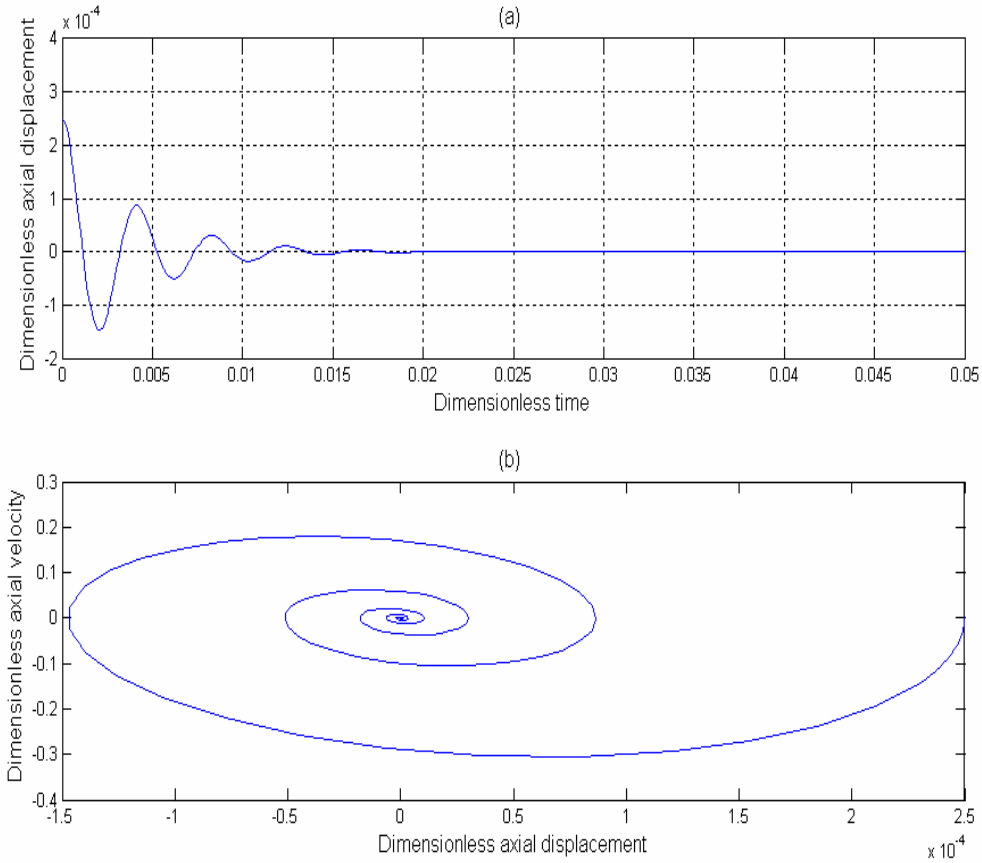


Figure 7.23: (a) Transient response of the dimensionless axial deflection at the 1st mode of vibration for Kelvin Voigt coefficient of 0.0001. (b) Corresponding phase plane plot.

7.3.7 Influence of the friction coefficients

In order to investigate the effect of the friction coefficients on the dynamic behavior of the system, a computer run has been performed for the values of C_f=C_t=C_n=0.04. Figures (7.24-7.25) show the response of the system described in subsection 7.3.2 at the 1st mode of vibration, but this time new values of friction coefficients are assigned. By referring to the figures 7.24 and 7.25, the following remarks have been established:

- The lateral natural frequency is found to be 41.6 rad/sec, which means that the lateral natural frequency is less by 1.05 % compared to the similar system with friction coefficients of 0.0125. Accordingly, we can conclude that as the friction coefficient is higher the lateral natural frequency will slightly decrease. On the other hand, it was found that the effect of the friction coefficients have negligible influence on the axial natural frequency of the system.
- The damping factor in the lateral direction was found to be 0.002 while it was 0.00067 when lower friction coefficients were used. The value of the damping factor in the axial direction remains the same, this is reasonable as the friction coefficients didn't appear in equation (6.14). Accordingly, we can state that the friction coefficients plays dominant role in increasing the damping in the lateral direction.

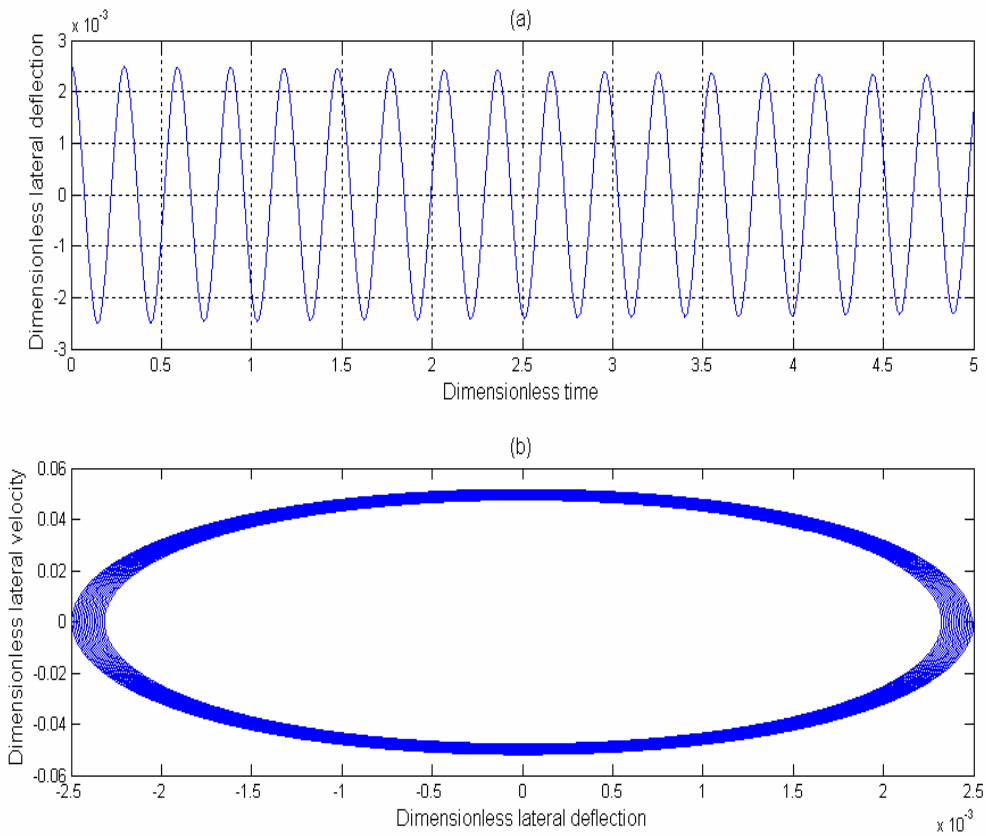


Figure 7.24: (a) Transient response of the dimensionless lateral deflection at the 1st mode of vibration at C_t=C_f=C_N=0.04.

(b) Corresponding phase plane plot.

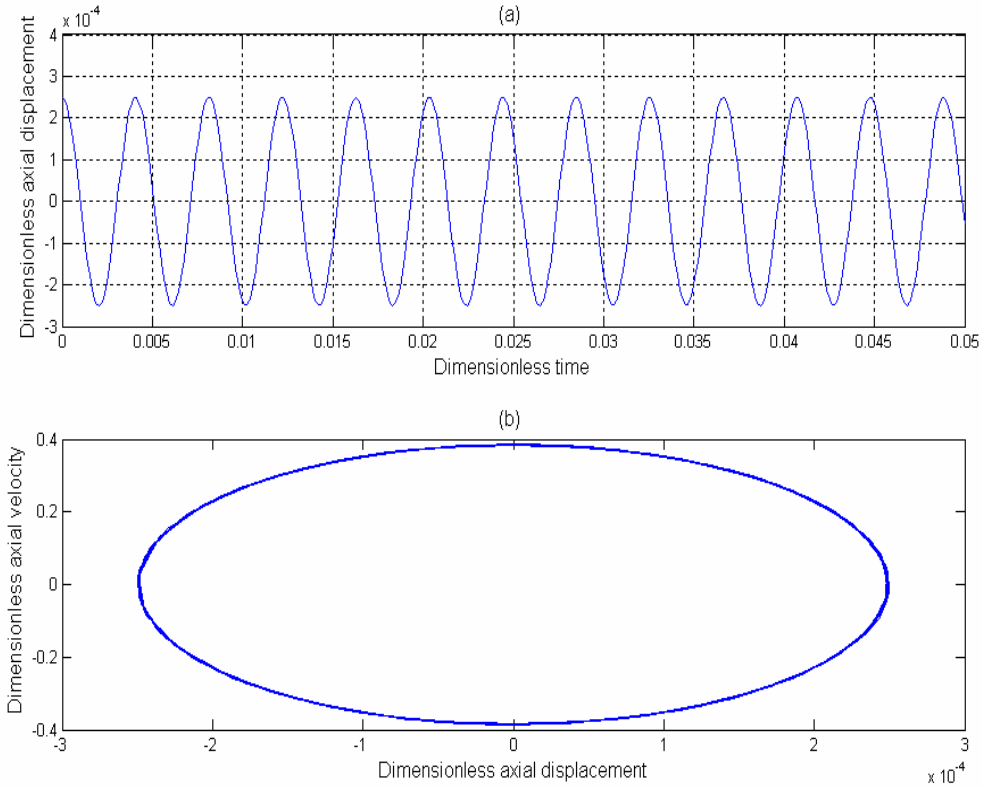


Figure 7.25: (a) Transient response of the dimensionless axial vibration at the 1st mode of vibration at C_t=C_f=C_N=0.04.

(b) Corresponding phase plane plot.

7.3.8 Comparison between linear and nonlinear responses

The main objective of this subsection is to verify the importance of the nonlinear formulation in studying the dynamic behavior of a double pipe system. Figures (7.26-7.28) show the time response for the system described in subsection 7.3.2 for both linear and nonlinear formulations. Based on the obtained results, the advantages of nonlinear formulation over the linear one are summarized as follow:

- To predict accurate natural frequencies of the system by considering the coupling between axial and lateral directions and including the quadratic and cubic nonlinearities of the system. It was found that the axial responses for both linear and nonlinear formulation are almost the same for the first three modes of vibration as shown in figures (7.26-7.28). This behavior can be explained by noting that the coefficients of the nonlinear terms that appeared in the axial equation of motion (i.e. equation 6.11) attain low values while the axial-lateral coupling and cubic nonlinear terms are absent. By referring to the same figures, it was shown that lateral response obtained from nonlinear analysis has noticeable deviation if it is compared to the linear formulation. The main terms leading to this difference are the axial-lateral coupling, quadratic and cubic nonlinearities. Although the percentage of change between the lateral natural frequencies obtained via linear and nonlinear formulations were found to be less than 0.33 % for all modes of vibration, but such deviation can be considered important when the system is operating near the critical excitation frequencies. In this case, we can conclude that linear formulation is a sufficient qualitative tool to predict the general behavior of the system.

- To estimate accurate amplitudes of vibration. In this analysis, it is observed that the amplitudes of vibration obtained from linear and nonlinear formulations have almost the same values. This is considered reasonable in the light of small deflections. However, if the system is allowed to assume large deflections, the terms of the quadratic and cubic nonlinearities will be amplified accordingly. Thus, such nonlinear formulation is very important to predict accurate amplitudes in the case of large deflections.
- Some phenomena could not be picked out without the nonlinear formulation, such as the beating phenomena shown in figure 7.7 (c), while the linear damping coefficient equals zero.

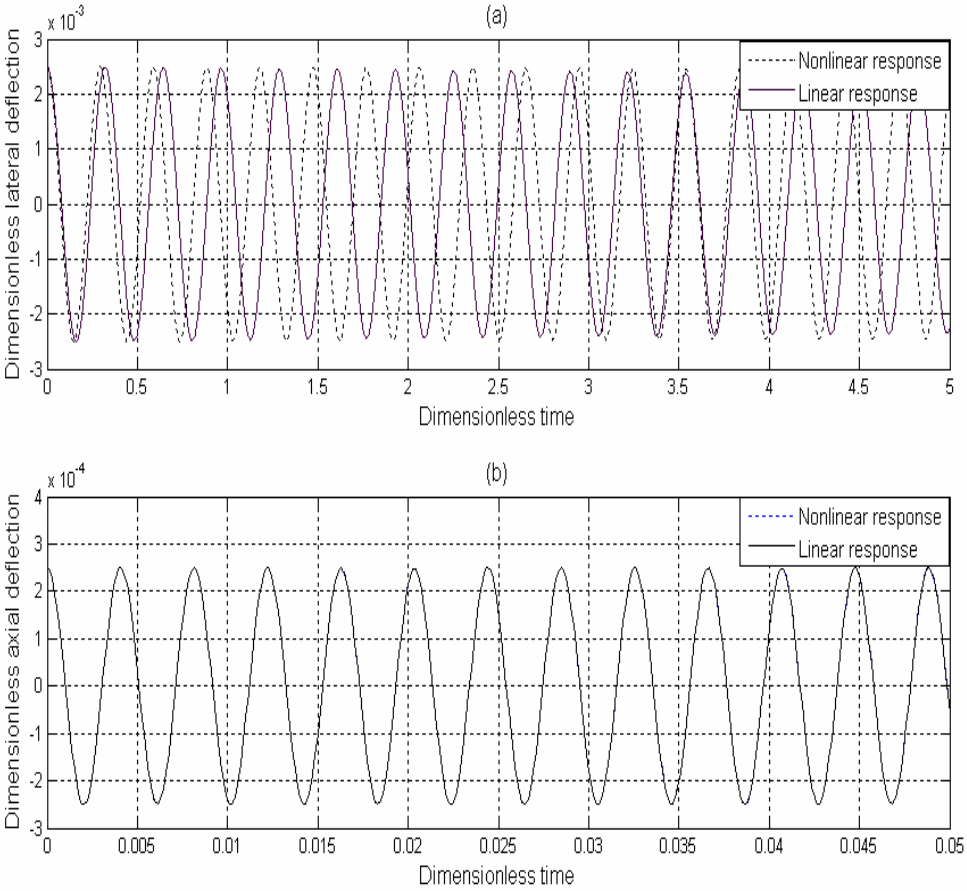


Figure 7.26: (a) Comparison between linear and nonlinear lateral responses for the 1st mode of vibration (b) Comparison between linear and nonlinear axial responses for the 1st mode of vibration.

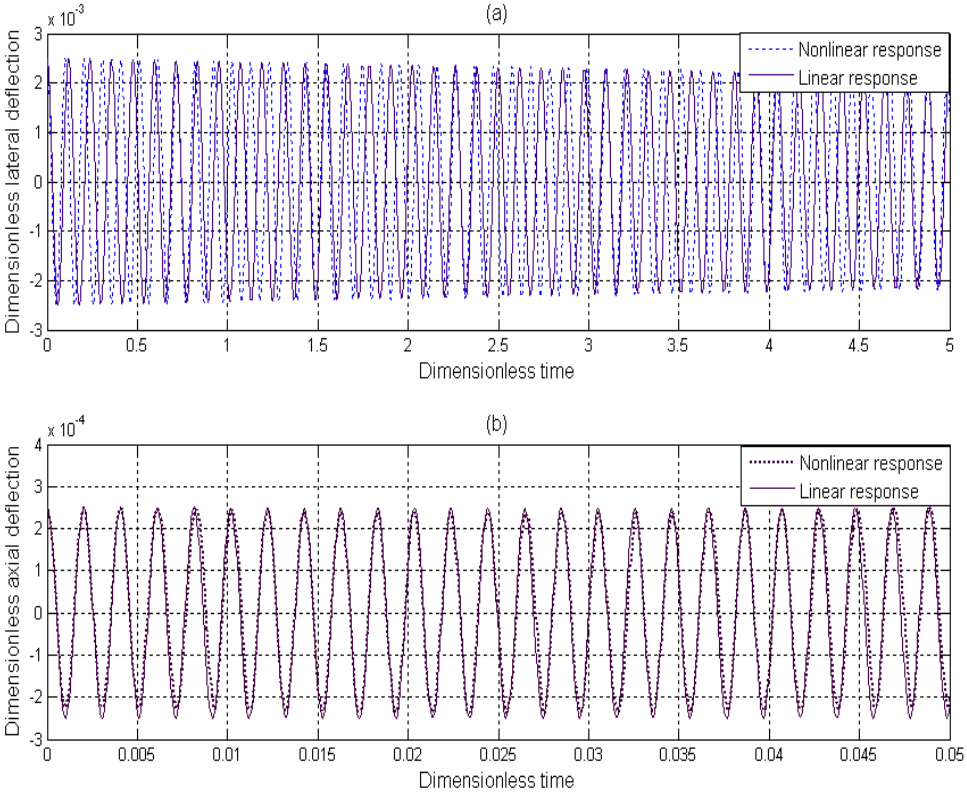


Figure 7.27: (a) Comparison between linear and nonlinear lateral responses for the 2nd mode of vibration (b) Comparison between linear and nonlinear axial responses for the 2nd mode of vibration.

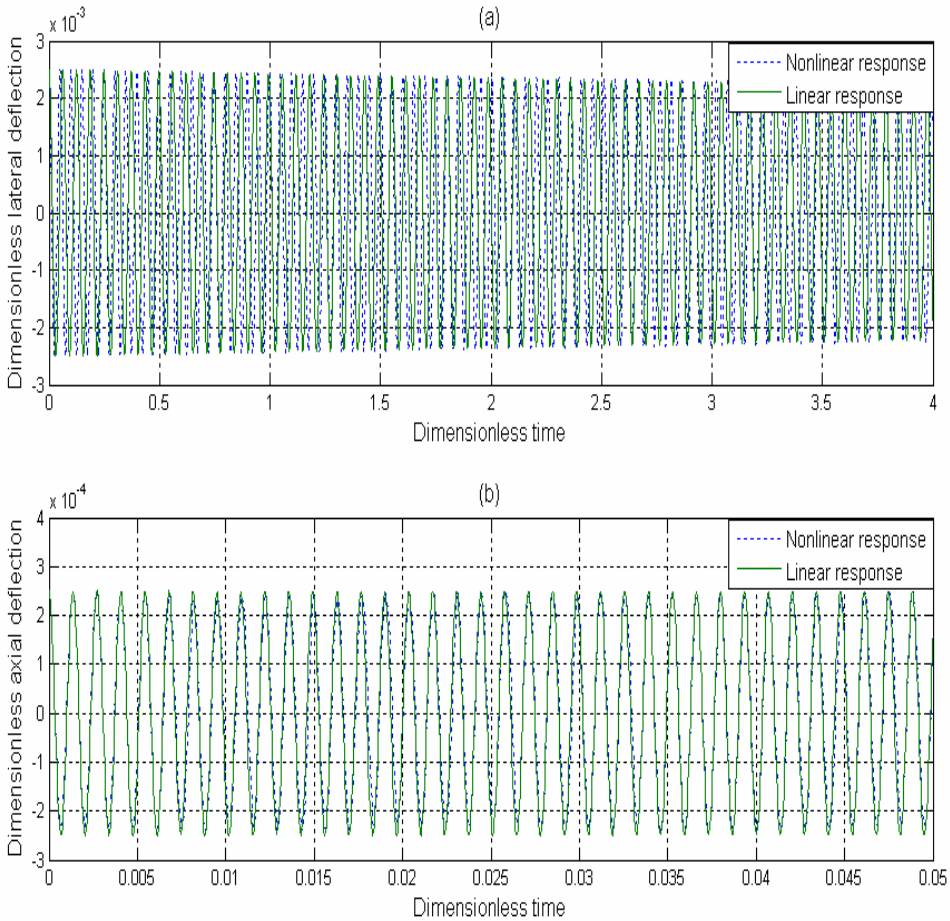


Figure 7.28: (a) Comparison between linear and nonlinear lateral responses for the 3rd mode of vibration (b) Comparison between linear and nonlinear axial responses for the 3rd mode of vibration.

7.4 Flow Induced Vibration of a Rotating Drillstring.

This case study represents a system inspired by drilling applications. The system consists of a cantilevered hollow rotating pipe conveying circulation fluid, which flows downwards through the bit, and then upwards through the borehole to the surface, as illustrated schematically in figure 3.6. The flow rate of the drilling fluid inside the drill pipe is assumed to be *9.533* kg/sec. The density of the internal fluid $\rho_{f,i}$ and the density of the fluid in the annulus $\rho_{f,o}$ are assigned with different values, as in drilling operations the internal fluid represents the circulation mud while the annular flow represents the mud together with the cutting debris, which is expected to have higher density.

7.4.1 System parameters

In order to investigate the dynamic behavior of a drill pipe, realistic values of the system parameters are assigned, which are similar to the actual practical values used in oil drilling. Table 7.8 shows the selected design parameters of the system. Table 7.9 shows the values of the corresponding dimensionless quantities which have been calculated based on their definitions provided in chapter 5. On the other hand, one should recognize that the dimensionless time equals 1.85 x 10^{-4} times the actual time. Also the tension and internal pressurization are set to be zero.

7.4.2 Transient responses of the system

By following the same procedure described in section 7.2, the coefficients of the equations of motion (6.96-6.98) have been resolved. Tables 7.10 and 7.11 show the values of such coefficients for the first three modes of vibration.

Table 7.8: System parameters of a rotating drill pipe.

Parameters	Value
Length of the drill pipe, L	1000 m
Outside diameter of the drill pipe, D_o	0.111 m
Thickness of the elastic tube	0.008 m
Diameter of the borehole , D_{ch}	0.139 m
Young's modulus of the elastic tube, E	2.1×10^{11} N/m^2
Modulus of rigidity, G	76.92×10^9 N/m^2
Mass of the drill pipe per unit length, m	20.321 kg/m
Mass of the drilling fluid inside the drill pipe per unit length, M	6.238 kg/m
Density of the internal drilling mud, $\rho_{f,i}$	880 kg/m^3
Density of the external drilling mud, $\rho_{f,o}$	1050 kg/m^3
Velocity of the drilling fluid inside the drill pipe, U_i	1.528 m/s
Friction coefficient in normal direction, C_N	0.0125
Friction coefficient in tangential direction, C_T	0.0125
The form-drag coefficient, C_{DP}	0.0125
Base drag coefficient, C_b	0.0125

Table 7.9: Dimensionless parameters of a rotating drill pipe.

Dimensionless parameters	Value
Added mass, χ	4.5556
Mass of the internal fluid, $\beta_{i,\,1}$	0.1699
Diameteral mass moment of inertia of the pipe and drilling fluid , $\beta_{i,\,2}$	8.3415×10^{-10}
Polar mass moment of inertia of the pipe, $\beta_{i,\,3}$	1.4767×10^{-9}
Polar mass moment of inertia of the fluid, $\beta_{i,\,4}$	1.9164×10^{-10}
Mass ratio of the external fluid, $\beta_{o,2}$	0.2767
Velocity of the internal flow, $u_{i,\,2}$	3.4044
Velocity of the external flow, $u_{o,\,2}$	5.4197
Flexural elasticity , $\beta_{m,\,1}$	0.5772
Flexural rigidity, $\beta_{m,\,2}$	0.4228
Slenderness ratio, E	9.0090×10^{3}
Hydraulic coefficient, $\bar{h}$	4.0000
Weight of the pipe and internal fluid, γ_1	2.0734×10^{5}
Weight of the external fluid, γ_2	7.9324×10^{4}
Friction coefficient in normal direction, c_N	0.0159
Friction coefficient in tangential direction, c_T	0.0159
The form-drag coefficient, c_d	0.0159
The form-drag coefficient, c_b	0.0159

Based on the tabulated values, the following observations are made:

- The most dominant nonlinear terms are L_6^v and L_6^w, which are associated with the cubic nonlinearity q^3 and r^3, respectively, as given by equations (6.96) and (6.97). These terms emerged from the potential energy expression, represented by elasticity of the drilling pipe and the gravitational force due to the weight of the pipe and fluid, in addition to the influence of the external hydrodynamic forces on the drill pipe. Other dominant terms are I_8^v and I_{10}^w which are associated with qr^2 and rq^2as shown by equations (6.96) and (6.97). These terms represent the elastic and hydrodynamic coupling between lateral deflections in the *X-Y* and *X-Z* planes. This observation reveals the importance of considering nonlinear formulation, since these terms play dominant role in determining the dynamic behavior of the system when the drill pipe is subjected to large deformation.
- The nonlinear terms associated with the gyroscopic effects and lateral-torsional coupling such as G_3^v, G_4^v, I_{14}^v, G_4^w, G_5^w, G_6^w, I_{17}^w, I_{18}^w which appear in equations (6.96) and (6.97) are found to have small values, and therefore a weak influence on the dynamic behavior of the current system. It is important to note that the unsymmetrical gyroscopic forces in *X-Y* and *X-Z* planes were the main reason behind 3-dimensional formulation. From another point of view, such gyroscopic effects and lateral-torsional coupling cannot be appreciated in a linear formulation, wherein these terms are ignored.

Table 7.10: Coefficients of the lateral equation of motion in *X-Y* plane.

Coefficient	1st mode	2nd mode	3rd mode
M_3^v	0.2949	1.8849	1.9457
C_3^v	26.6267	146.6750	191.3385
K_3^v	1.3653×10^4	2.2142×10^5	5.9326×10^5
G_3^v	1.1533×10^{-4}	2.2349×10^{-4}	6.1058×10^{-5}
G_4^v	0.1637	0.3173	0.0867
H_5^v	-186.9685	-2.8246×10^4	597.8553
H_6^v	16.3421	807.6646	880.7293
H_7^v	6.6604	432.9056	0.2218
H_8^v	1.4420	23.6379	24.6208
I_8^v	9.2707×10^3	1.3514×10^7	-4.3454×10^5
I_9^v	0	0	0
I_{10}^v	-0.0595	196.5096	-10.0020
I_{11}^v	8.4092	3.4783×10^3	3.0096×10^4
I_{12}^v	0.1778	258.5100	-9.9995
I_{13}^v	-0.0353	539.0960	-17.4826
I_{14}^v	-0.0366	-0.1986	-0.1557
L_6^v	1.3528×10^4	1.9598×10^7	-2.9777×10^5
L_7^v	12.6883	9.0154×10^3	2.9985×10^4
L_8^v	-0.3297	265.9202	-355.7846
L_9^v	-0.0244	-1.0460	-1.0860
L_{10}^v	0.1778	258.5100	-9.9995

Table 7.11: Coefficients of the torsional and lateral equations of motion in *X-Z* plane.

Coefficient	1st mode	2nd mode	3rd mode
M_3^w	0.2949	1.8849	1.9457
C_3^w	26.6267	146.6750	191.3385
K_3^w	1.3653×10^4	2.2142×10^5	5.9326×10^5
G_4^w	1.1533e-004	2.2349e-004	6.1058e-005
G_5^w	-2.4427×10^{-9}	-2.0229×10^{-8}	6.8628×10^{-9}
G_6^w	-2.6845×10^{-9}	-2.0697×10^{-8}	6.7348×10^{-9}
H_5^w	-186.9685	-2.8246e+004	597.8553
H_6^w	16.3421	807.6646	880.7293
H_7^w	6.6604	432.9056	0.2218
H_8^w	1.4420	23.6379	24.6208
I_{10}^w	9.2707×10^3	1.3514×10^7	-4.3454×10^5
I_{11}^w	0	0	0
I_{12}^w	-2.4262×10^{-9}	-9.2572×10^{-7}	-8.9549×10^{-6}
I_{13}^w	-0.0595	196.5096	-10.0020
I_{14}^w	8.4092	3.4783×10^3	3.0096×10^4
I_{15}^w	0.1778	258.5100	-9.9995
I_{16}^w	-0.0353	539.0960	-17.4826
I_{17}^w	-0.0366	-0.1986	-0.1557
I_{18}^w	1.835×10^{-10}	-2.5366×10^{-9}	-5.8134×10^{-10}
L_6^w	1.3528×10^4	1.9598×10^7	-2.9777×10^5
L_7^w	12.6883	9.0154×10^3	2.9985×10^4
L_8^w	-0.3297	265.9202	-355.7846
L_9^w	-0.0244	-1.0460	-1.0860
L_{10}^w	0.1778	258.5100	-9.9995
M_3^φ	7.3833×10^{-10}	7.3833×10^{-10}	7.3833×10^{-10}
K_3^φ	0.5216	4.6948	13.0412
T_5^φ	7.1997×10^{-13}	-7.1997×10^{-13}	7.1997×10^{-13}
T_6^φ	3.5999×10^{-13}	-3.5999×10^{-13}	3.5999×10^{-13}
T_7^φ	4.8904×10^{-10}	-1.6711×10^{-8}	-4.6767×10^{-9}
T_8^φ	4.8904×10^{-10}	-1.6711×10^{-8}	-4.6767×10^{-9}

After evaluating the coefficients of the equations of motion, the system of governing equations is solved using ODE45 solver, and upon applying the following actual (physical) initial conditions at the free end of the pipe such that: $v(t) = 0.002,\ \dot{v}(t) = 0.368,\ w(t) = 0.002,\ \dot{w}(t) = 0.368,\ \varphi(t) = 0.02,\ \dot{\varphi}(t) = 0.1$.

The corresponding dimensionless initial conditions are calculated based on their definitions presented in Chapter 5, while they were found to be:

$q(t_2^*) = 0.000002,\ \dot{q}(t_2^*) = 2,\ r(t_2^*) = 0.000002,\ \dot{r}(t_2^*) = 2,\ g(t_2^*) = 0.02,\ \dot{g}(t_2^*) = 500$.

Figures 7.29 -7.38 show the transient responses of the system for the first three modes of vibration in addition to the corresponding phase plane plots. The natural frequencies in the torsional and lateral directions (i.e. *X-Y* and *X-Z* planes) were calculated and their values listed in table 7.12.

By referring to figures 7.29-7.38 and table 7.4, the following observations are made:

- The drill pipe system has a considerable amount of damping in the lateral direction with damping ratio equals 0.21; which shows that the hydrodynamic forces play dominant role in suppressing the vibrations of the drill pipe during operation. The corresponding phase plane plots behave as stable spiral node and return back to the equilibrium position rapidly. On the other hand, and upon investigating the torsional vibration responses, it was found that the system has negligible amount of damping; this behavior is reasonable since the linear damping term is absent in equation (6.98), and such a little amount of damping is associated with some nonlinear terms. The phase plane plots corresponding to the torsional vibration are found to be center type.

- The phase portraits show that the system is stable for both lateral and torsional vibrations at the selected operating conditions.
- The torsional natural frequencies attain high values compared to the lateral ones. The transient response of torsional vibrations appear like a noise at higher modes of vibration, as expected. Accordingly, one may consider that lateral vibrations are the dominant ones.
- Upon investigating figure 7.38, it was observed that lateral vibrations in *X*-*Y* plane and *X*-*Z* plane have a small deviation from each other due to the gyroscopic effects. It was noted that such a deviation becomes negligible at high modes of vibration. This phenomenon can be explained by considering that the values of the nonlinear coefficients associated with gyroscopic motion are small in which, at higher modes of vibration, they appear negligible when compared to those high values of the nonlinear coefficients associated with cubic lateral nonlinearity. On the other hand, such gyroscopic effects cause slight difference between the lateral natural frequencies in *X*-*Y* and *X*-*Z* planes as shown in table 7.12.

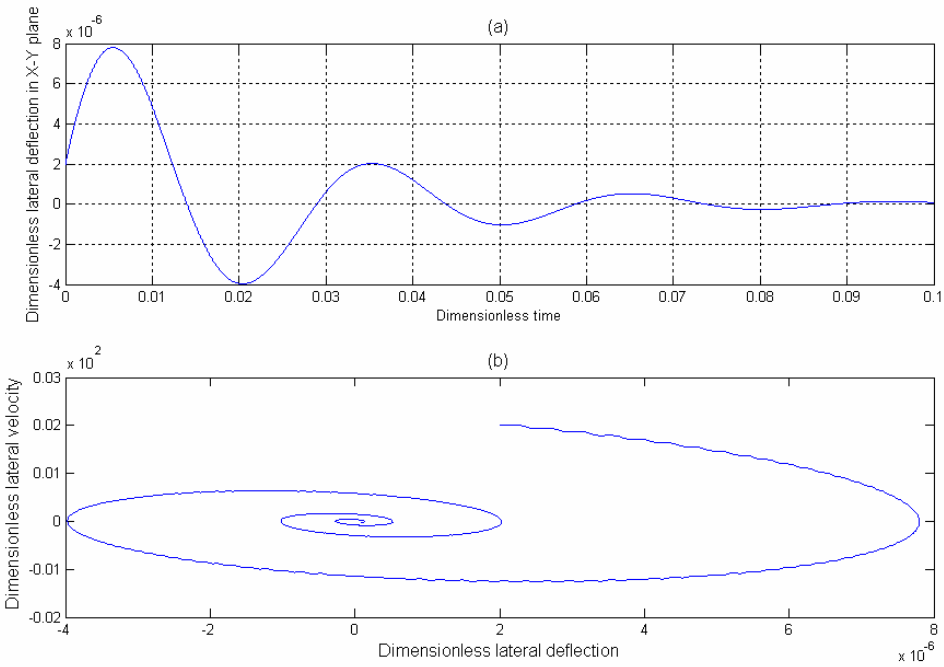


Figure 7.29: (a) Transient response of the dimensionless lateral deflection in *X-Y* plane at the 1st mode of vibration. (b) Corresponding phase plane plot.

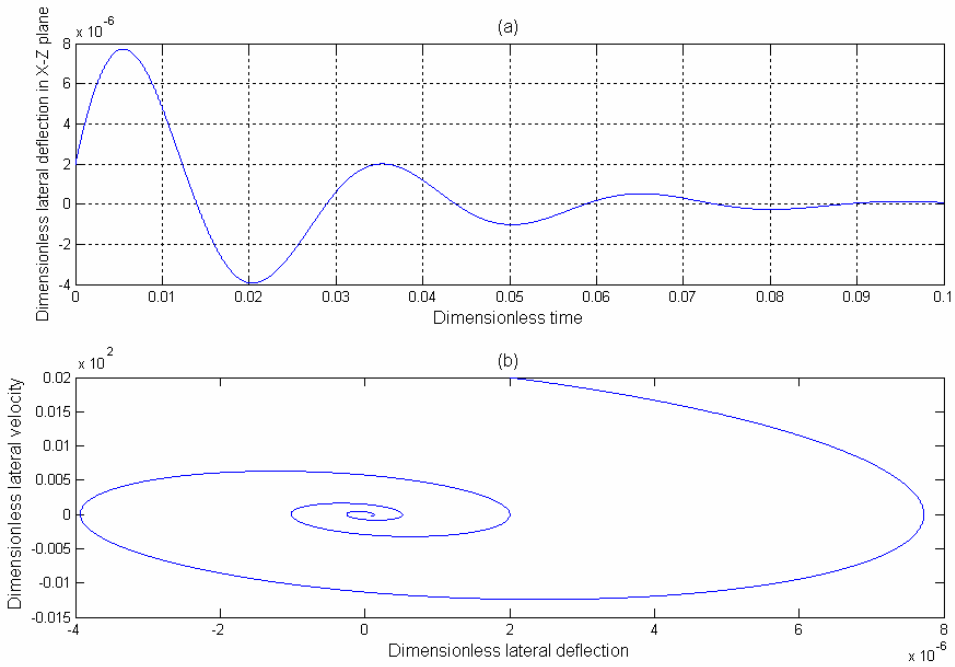


Figure 7.30: (a) Transient response of the dimensionless lateral deflection in *X-Z* plane at the 1st mode of vibration. (b) Corresponding phase plane plot.

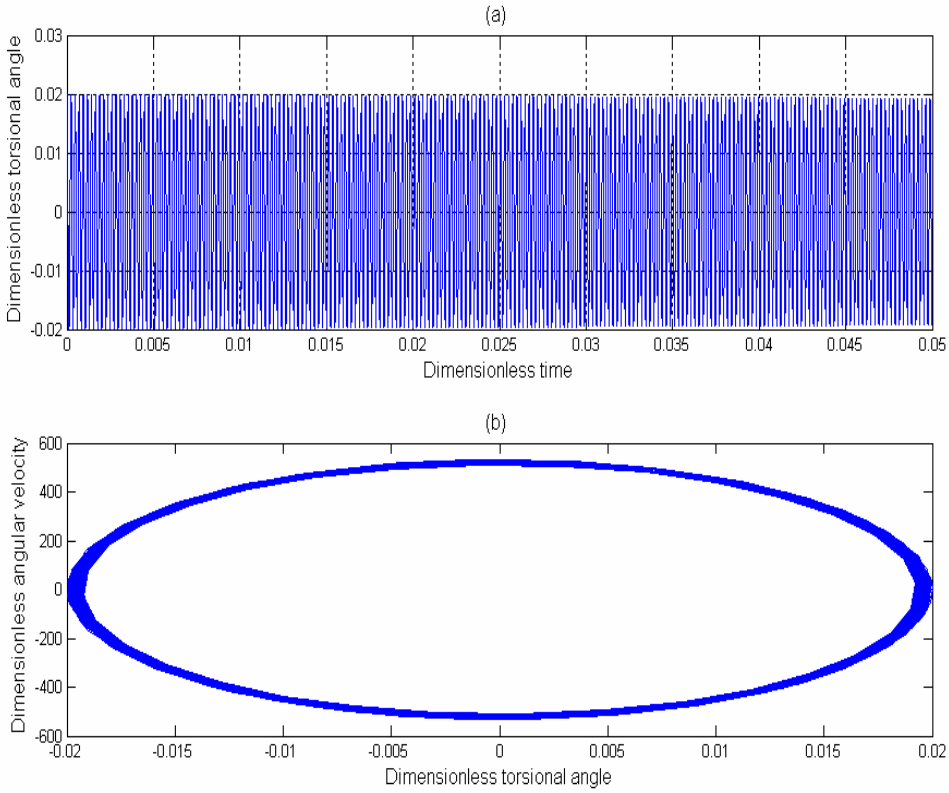


Figure 7.31: (a) Transient response of the dimensionless torsional vibration at the 1st mode of vibration. (b) Corresponding phase plane plot.

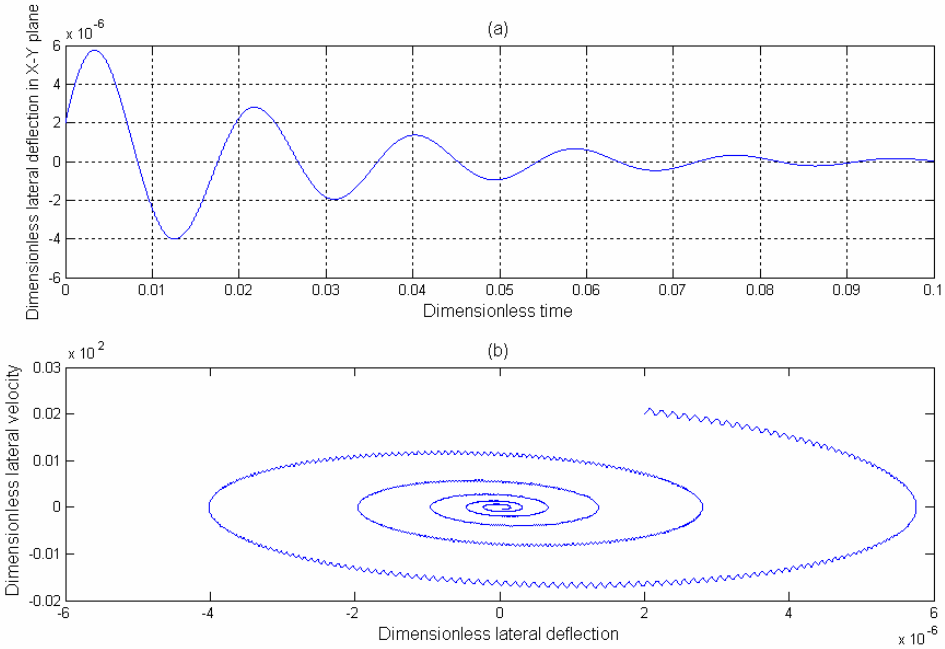


Figure 7.32: (a) Transient response of the dimensionless lateral deflection in *X*-*Y* plane at the 2nd mode of vibration. (b) Corresponding phase plane plot.

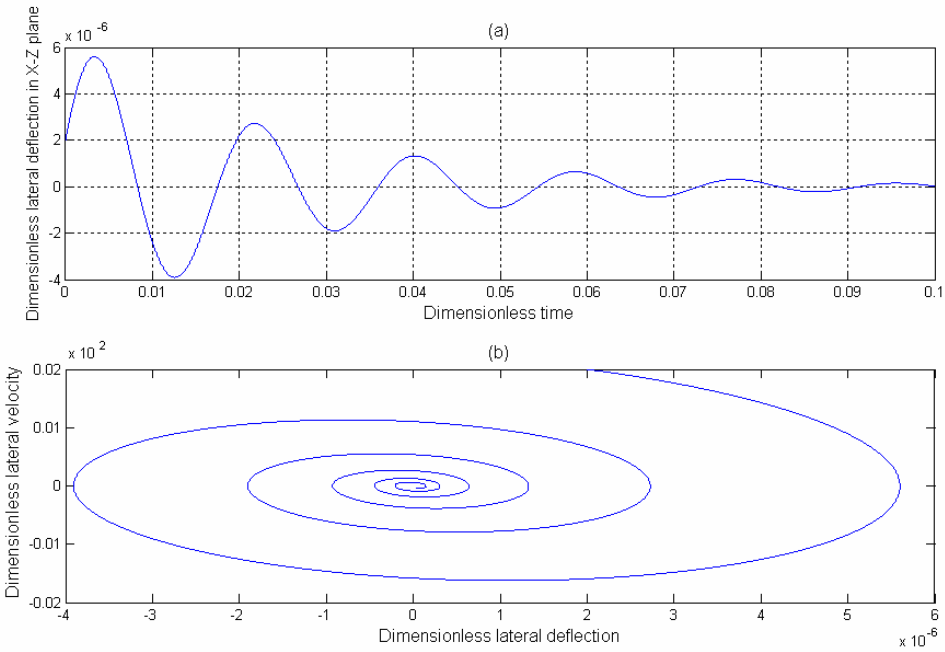
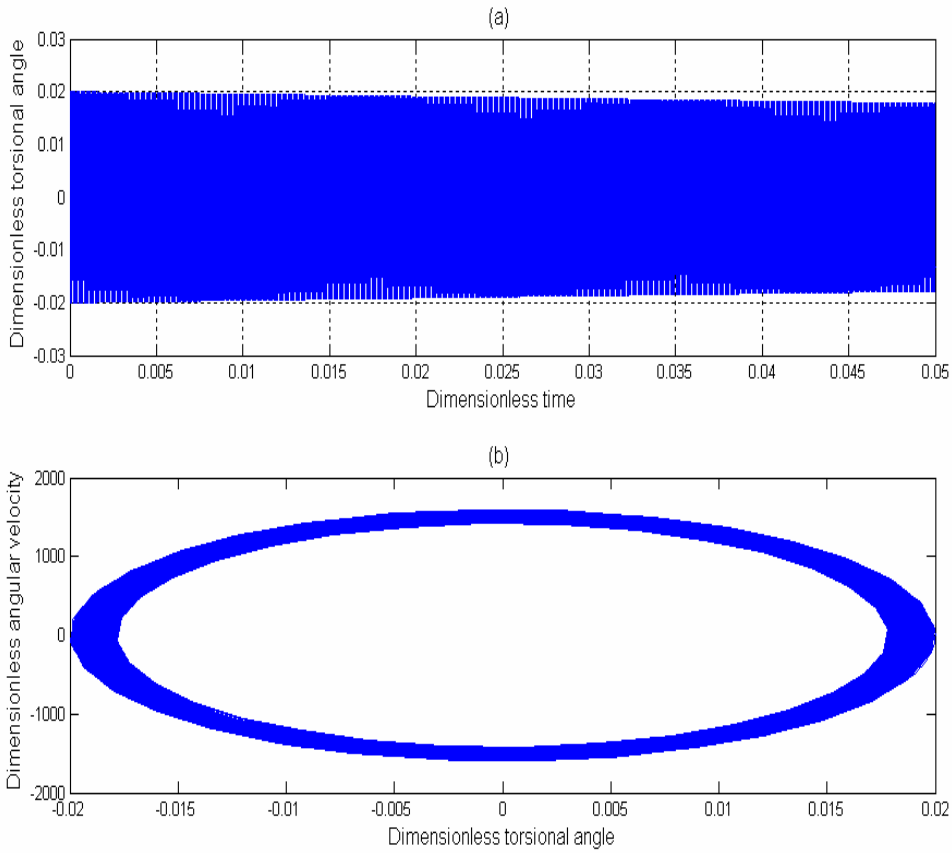


Figure 7.33: (a) Transient response of the dimensionless lateral deflection in *X-Z* plane at the 2nd mode of vibration. (b) Corresponding phase plane plot.



F**igure 7.34:** (a) Transient response of the dimensionless torsional vibration at the 2nd mode of vibration. (b) Corresponding phase plane plot.

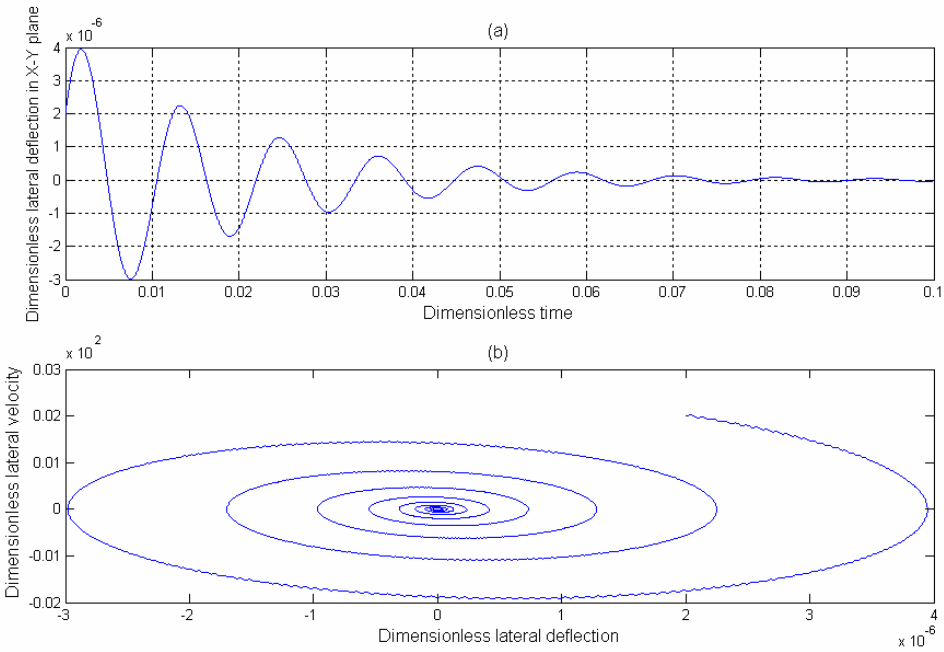


Figure 7.35: (a) Transient response of the dimensionless lateral deflection in *X-Y* plane at the 3rd mode of vibration. (b) Corresponding phase plane plot.

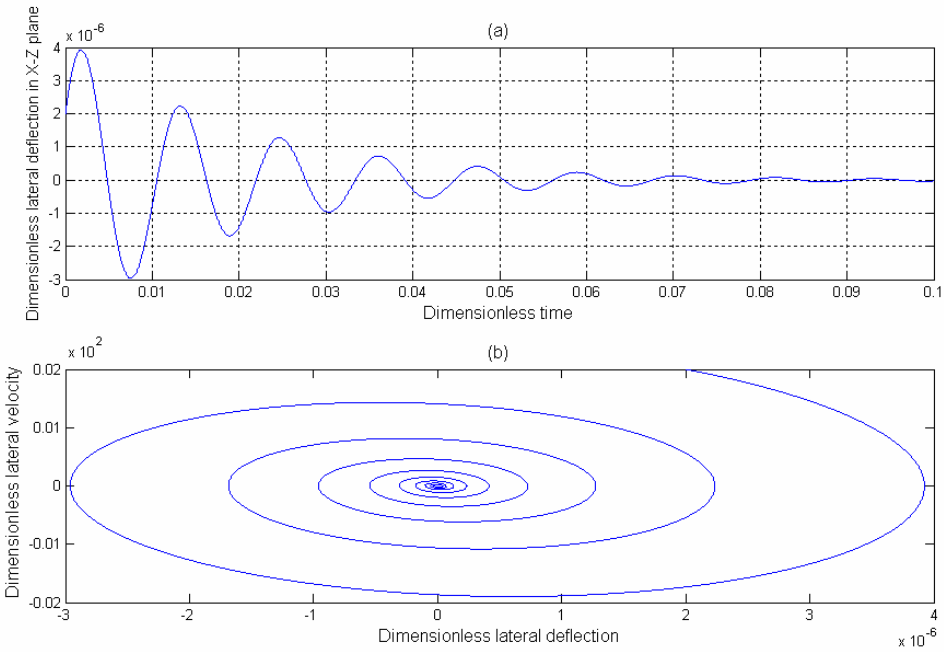


Figure 7.36: (a) Transient response of the dimensionless lateral deflection in *X-Z* plane at the 3rd mode of vibration. (b) Corresponding phase plane plot.

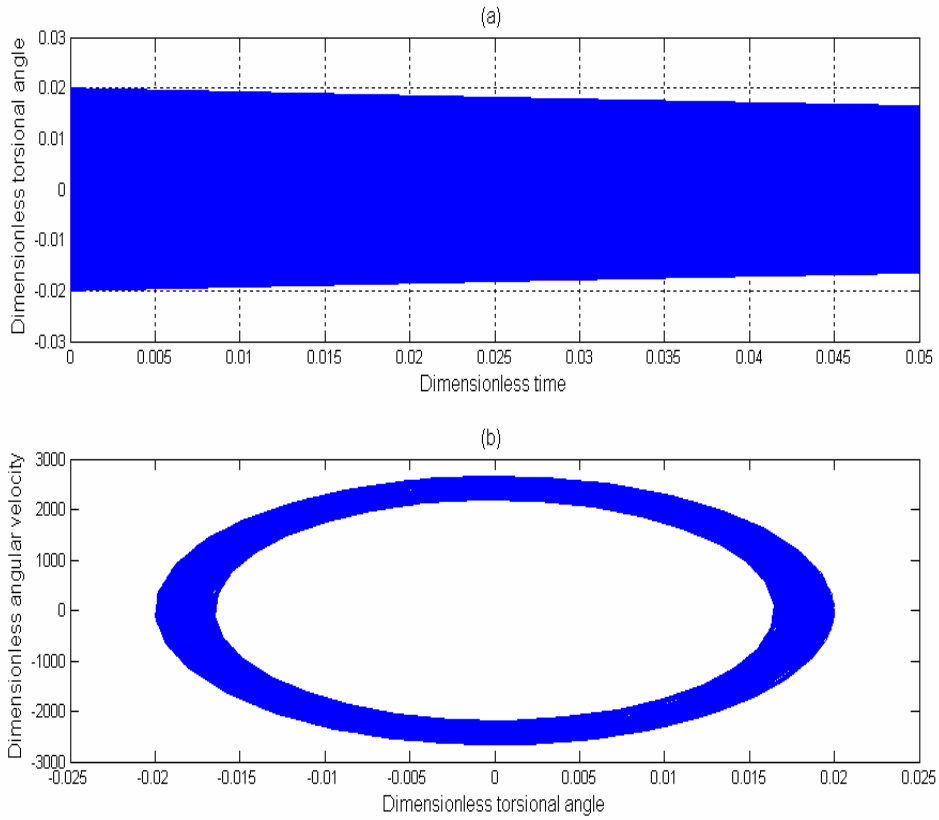


Figure 7.37: (a) Transient response of the dimensionless torsional vibration at the 3rd mode of vibration. (b) Corresponding phase plane plot.

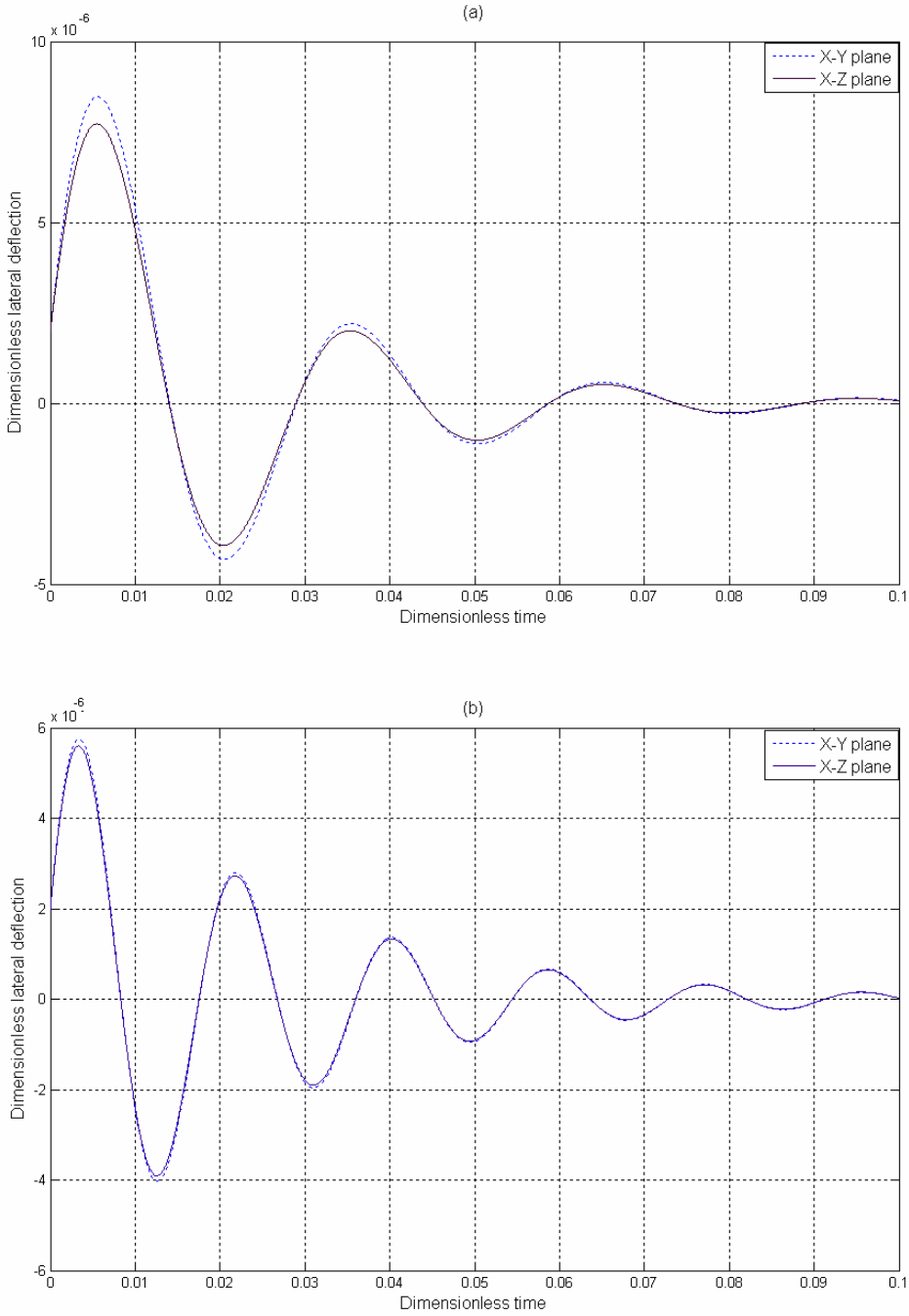


Figure 7.38: (a) Comparison between transient responses of the dimensionless lateral vibration in *X-Y* and *X-Z* planes at the 1st mode of vibration. (b) Comparison between transient responses of the dimensionless lateral vibration in *X-Y* and *X-Z* planes at the 2nd mode of vibration.

Table 7.12: Lateral and torsional natural frequencies of a rotating drill pipe.

No. of Mode	**Natural frequency of lateral vibration in *X-Y* plane (rad/s)**	**Natural frequency of lateral vibration in *X-Z* plane (rad/s)**	**Natural frequency of torsional vibration (rad/s)**
1st Mode	0.0391	0.0389	4.888
2nd mode	0.0632	0.0631	14.800
3rd mode	0.1024	0.1024	24.579

7.4.3 Influences of the internal and external flows

In order to investigate the effects of internal and external fluids on the transient response of the drill pipe, the simulation was performed for the case in which both internal and external fluids are absent. Figures 7.39-7.41 show the transient responses for both torsional and lateral vibration of the dry drill pipe at the first mode of vibration. In order to demonstrate the effect of the internal fluid separately, it is assumed that the pipe is conveying fluid with no fluid in the annulus region. One should note that such model cannot describe the drilling operation but it will be useful for the comparison purposes. Figures 7.42-7.44 show the transient responses for both torsional and lateral vibration of the system at the first mode of vibration when the external flow is discarded.

Another simulation was performed to study the effect of the external flow while the internal flow is absent. Again, it should be noted that the current system cannot describe the dynamics of the drill pipe, but it is carried out just for comparison purposes. In order to establish the required comparison study, the external flow velocity is assumed to be 1.91 m/s which equal the velocity of the external flow for the model discussed in subsection 7.4.3. Figures 7.45-7.47 show the transient responses and phase plane plots for both torsional and lateral vibrations of the system at the first mode of vibration when the internal flow is ignored.

Table 7.13 shows the lateral and torsional natural frequencies of the drill pipe at several flow patterns described above. In table 7.13, the natural frequencies of each flow pattern are compared to those natural frequencies associated with the dry system. One should recognize that the dimensionless time depends on the existence of the internal and external flows and has been modified for each case, accordingly.

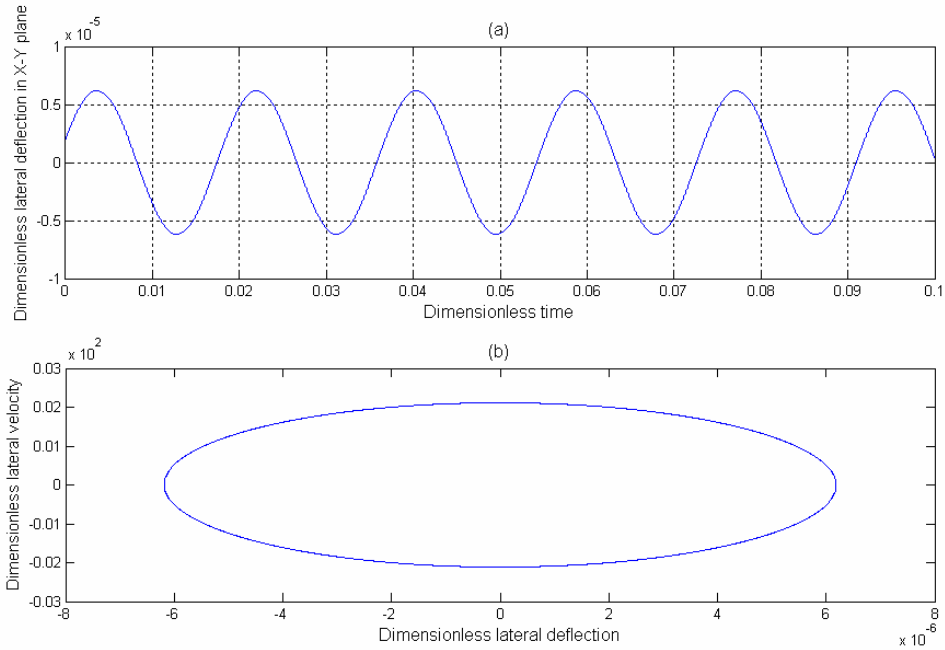


Figure 7.39: (a) Transient response of the dimensionless lateral deflection in *X*-*Y* plane at the 1st mode of vibration for the fluid-free drill pipe. (b) Corresponding phase plane plot.

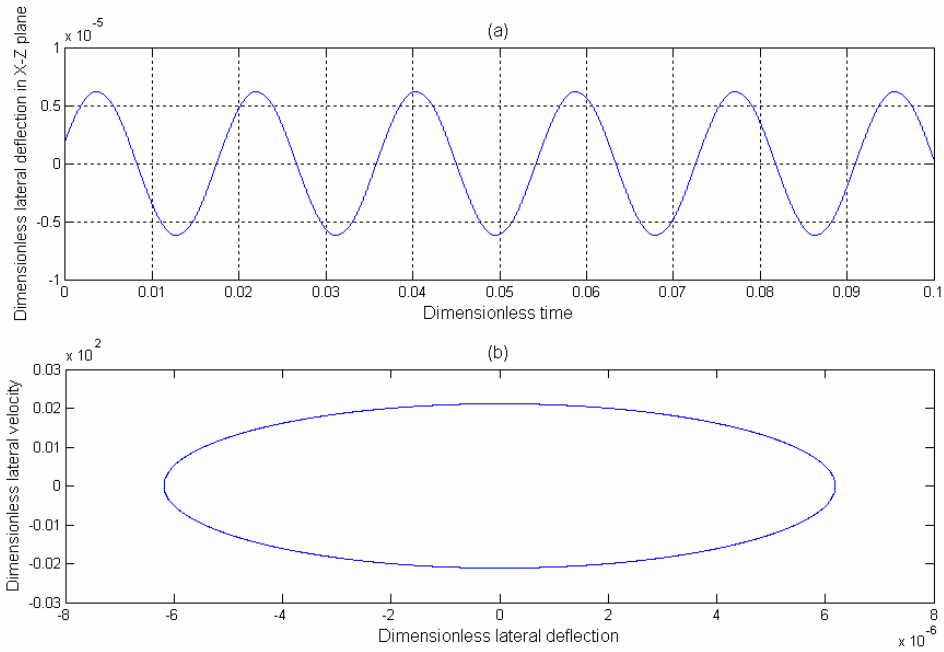


Figure 7.40: (a) Transient response of the dimensionless lateral deflection in *X-Z* plane at the 1st mode of vibration for the free of fluid drill pipe. (b) Corresponding phase plane plot.

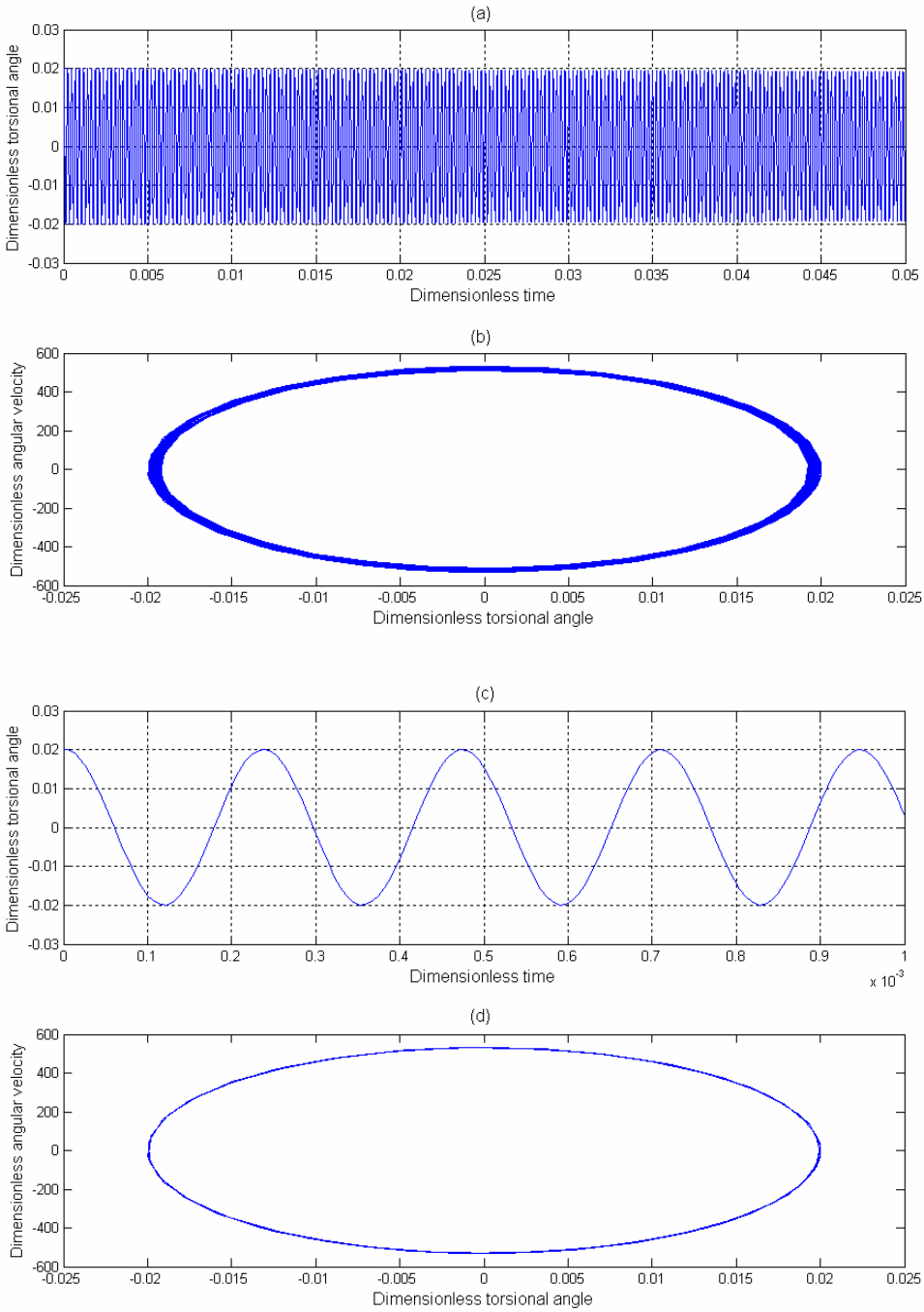


Figure 7.41: (a) Transient response of the dimensionless tosional vibration at the 1st mode of vibration for the free of fluid drill pipe. (b) Corresponding phase plane plot. (c) Transient response in a short time scale. (d) Corresponding phase plane plot.

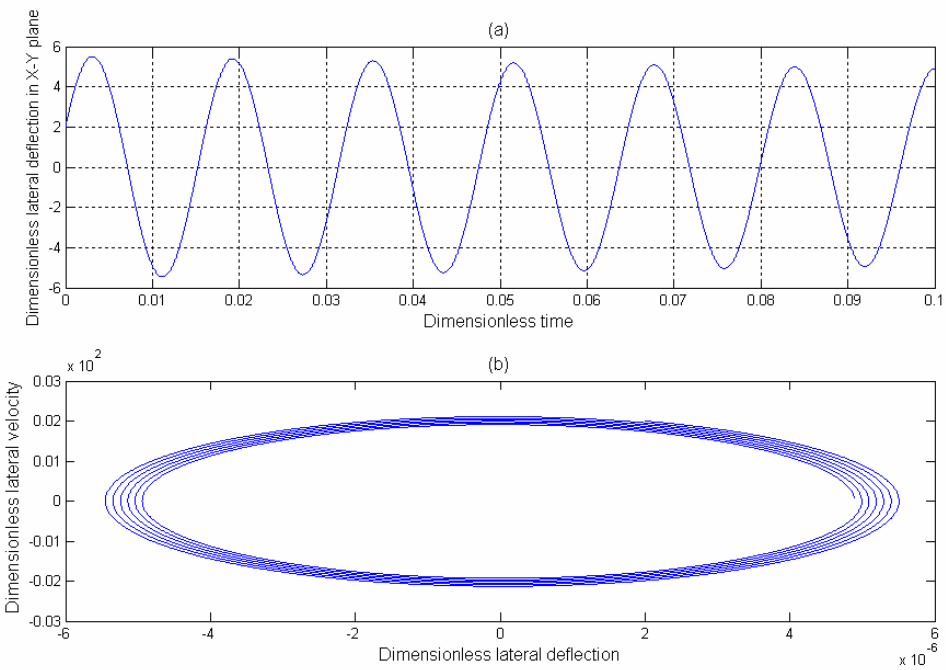


Figure 7.42: (a) Transient response of the dimensionless lateral deflection in *X*-*Y* plane at the 1st mode of vibration for the pipe system conveying fluid and free of external fluid .(b) Corresponding phase plane plot.

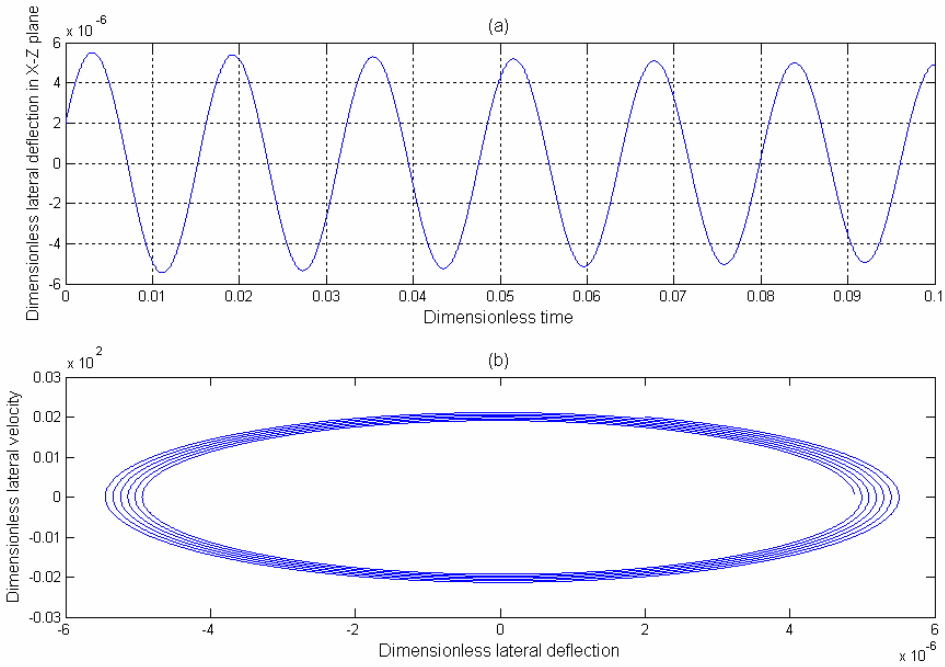


Figure 7.43: (a) Transient response of the dimensionless lateral deflection in *X*-*Z* plane at the 1st mode of vibration for the pipe system conveying fluid and free of external fluid .(b) Corresponding phase plane plot.

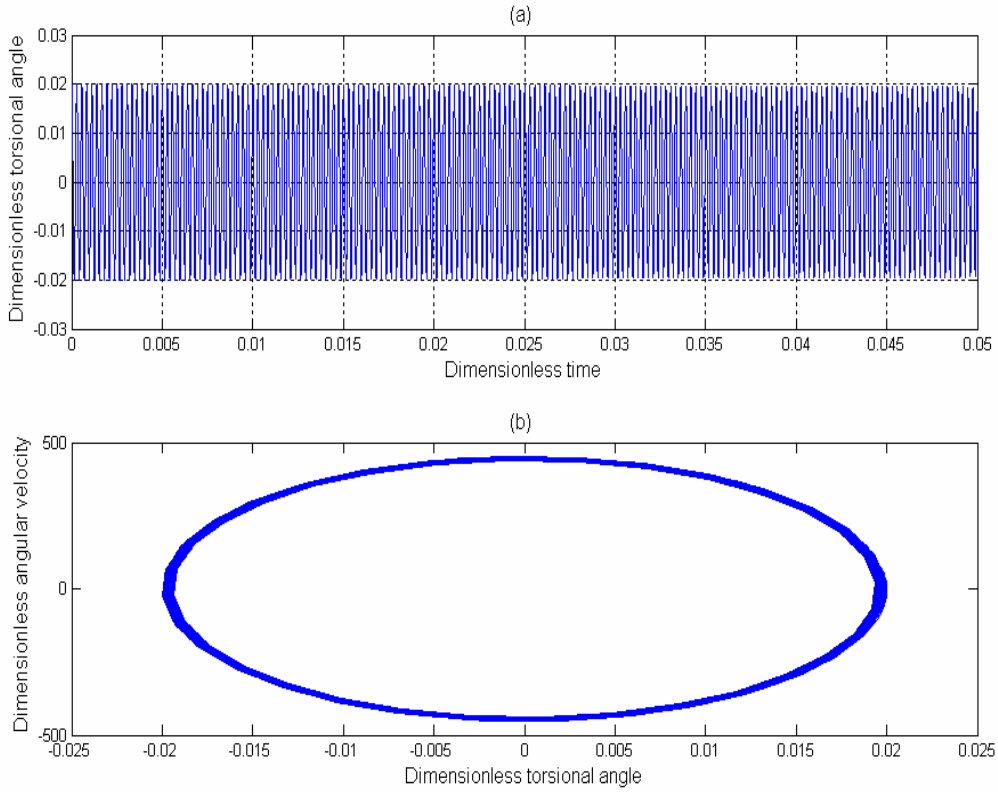


Figure 7.44: (a) Transient response of the dimensionless torsional vibration at the 1st mode of vibration for the pipe system conveying fluid and free of external fluid. (b) Corresponding phase plane plot.

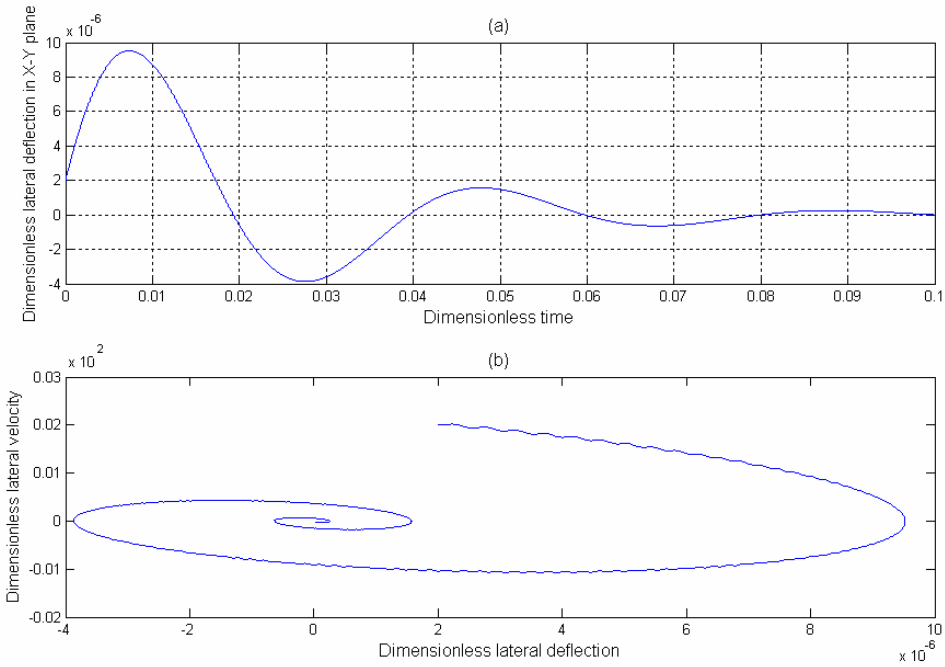


Figure 7.45: (a) Transient response of the dimensionless lateral deflection in *X-Y* plane at the 1st mode of vibration for the pipe system under the influence of external flow and free of internal fluid.

(b) Corresponding phase plane plot.

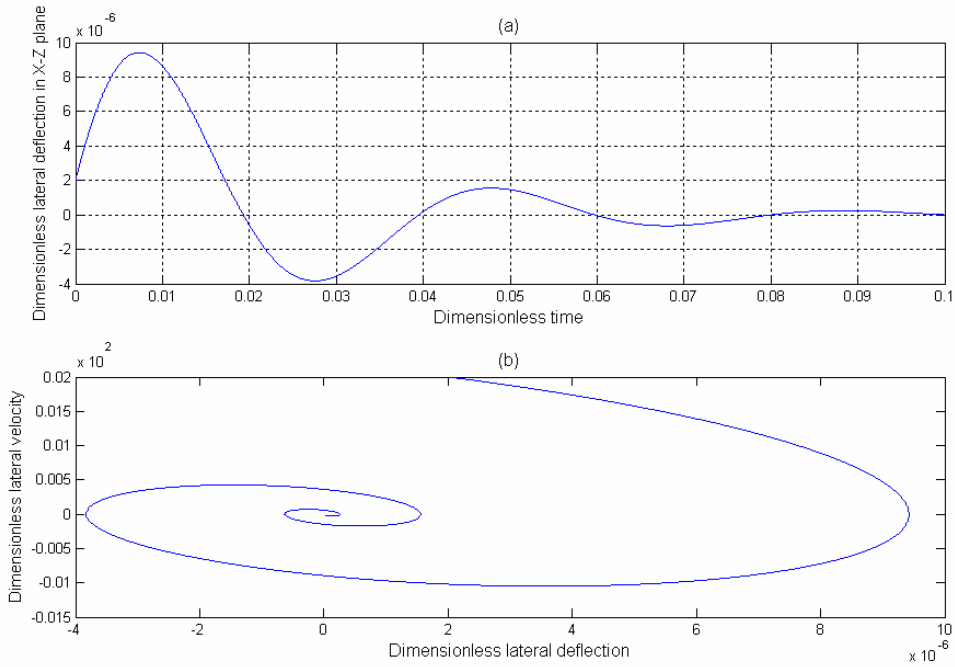
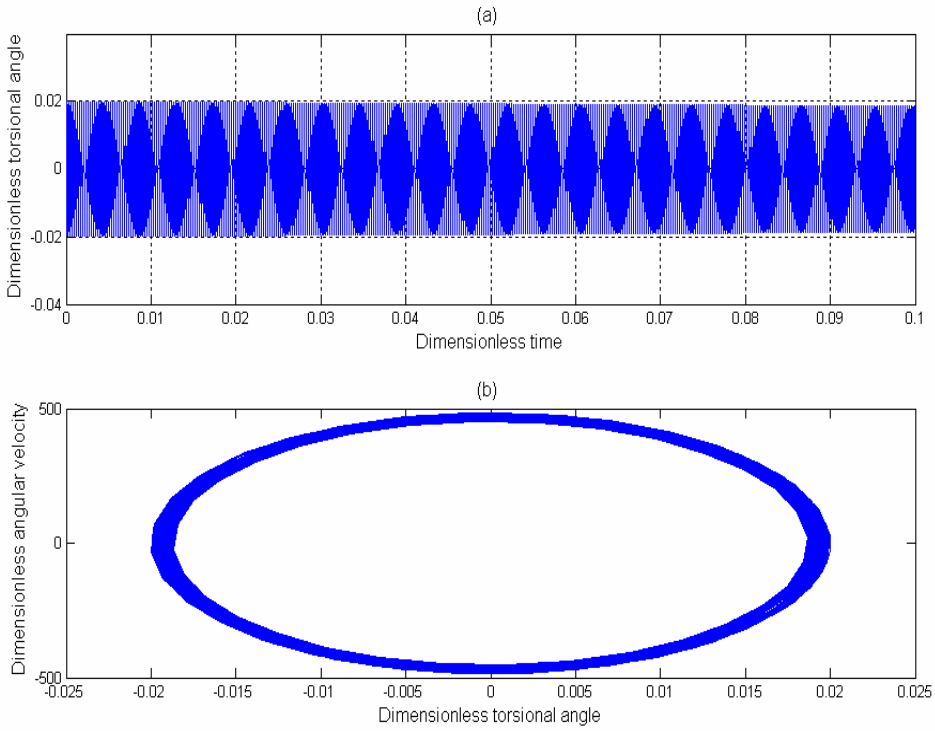


Figure 7.46: (a) Transient response of the dimensionless lateral deflection in *X-Z* plane at the 1st mode of vibration for the pipe system under the influence of external flow and free of internal fluid.

(b) Corresponding phase plane plot.



F**igure 7.47:** (a) Transient response of the dimensionless torsional vibration at the 1st mode of vibration for the pipe system under the influence of external flow and free of internal fluid.

(b) Corresponding phase plane plot.

Table 7.13: Lateral and torsional natural frequencies of rotating drill pipe at various flow patterns.

Model description	No. of mode of vibration	Lateral vibrations in *X*-*Y* and *X*-*Z* planes		Torsional vibrations	
		Natural frequency (rad/s)	*Percentage (%)	Natural frequency (rad/s)	*Percentage (%)
Fluid-free (dry) system	1st mode	0.08411	---	6.726	---
	2nd mode	0.1352	---	19.829	---
	3rd mode	0.2149	---	33.0489	---
Internal flow only	1st mode	0.073	-13.2	5.0284	-25.2
	2nd mode	0.112	-17.2	14.75	-25.6
	3rd mode	0.175	-18.6	24.59	-25.6
External flow only	1st mode	0.032	-61.95	4.851	-27.88
	2nd mode	0.052	-61.54	14.745	-25.64
	3rd mode	0.0765	-64.42	24.576	-25.64
Both internal and external flows	1st mode	0.0391	-53.1	4.888	-27.33
	2nd mode	0.0632	-53.25	14.800	-25.36
	3rd mode	0.1024	-52.35	24.579	-25.63

* The tabulated percentage represents the percentage of change of the natural frequencies between each flow pattern and the dry system.

Referring to figures 7.39-7.47 and table 7.13, the influences of the internal and external fluids can be summarized as follow:

- *Influence of the internal flow*

The internal flow has a minor effect in damping the drill pipe in the lateral direction, as shown by figures 7.42 and 7.43. The corresponding phase plots are for a stable center. Similar behavior was observed in the transient response of the torsional vibrations, as shown in figure 7.44. It was found that the internal flow has high influence in lowering (softening effect) the lateral and torsional natural frequencies of the system as shown in table 7.13. This finding can be interpreted by recognizing the dominant role of the internal flow in adding the inertia term of the system. By comparing the current case with the dry model, it was found that the lateral natural frequencies decreased by a percentage of 13.2 % up to 18.6 % for the first three modes of vibration. Also the internal flow has strong influence on the natural frequency of the torsional vibration as it is lowered up to 25.6 % compared to the dry system. These results reveal the importance of considering the internal flow in modeling the drill pipe, and show that neglecting the influence of the internal flow will lead to appreciable error in determining the natural frequencies of the system.

- *Influence of the external flow.*

The external flow has a dominant influence in suppressing the lateral vibration of the drill pipe, as shown in figures 7.45 and 7.46. The damping ratio is found to be 0.28. It was found that the external flow has dominant role in lowering the lateral natural frequencies of the system as shown in table 7.13. By comparing this case with the dry model, we

found that natural frequencies decreased by a percentage of 61.5 % up to 64.4 % for the first three modes of vibration. This finding can be interpreted by recognizing the effect of the added mass in increasing the inertia term of the system in the lateral direction. Also from table 7.13, we can observe that the external flow has stronger influence in lowering the natural frequency if it is compared with the effect of the internal flow. Now, we can conclude that the external flow should be taken into account in modeling the drill pipe and may not be neglected in most operating conditions.

- *Influence of both external and internal flows.*

The dual effect of the internal and external flows together is responsible for lowering the lateral natural frequencies of the system up to 53.35 %. This seems reasonable as each of the flows (i.e. either internal or external flows) has its own contribution in lowering the natural frequency of the system. It was observed that the lateral natural frequency of the pipe under the influence of the external flow, and free of internal flow is lower than the natural frequency of the pipe conveying fluid and subjected to external flow by 8-12 %. This phenomenon can be explained by considering that the internal flow has its own contribution in adding flexibility to the system, which means that the system without internal flow is considered stiffer, and leads to decrease in the lateral natural frequency of the system.

7.4.4 Influences of the flow velocity

Figures (7.48 -7.52) show the transient response of the system for torsional and lateral vibrations at internal flow velocities of 5 and 10 m/s. It should be considered that both internal and external velocities are dependant on the drill pipe model, whereas

such dependence based on the ratio of the cross sectional areas of the inner pipe and the outer cylinder. The natural frequencies and damping coefficients associated with lateral and torsional vibration at various external flow velocities are listed in table 7.14. Based on the tabulated results, the following observations are made:

- The external flow velocity has a considerable influence in lowering the lateral natural frequencies of the subject model with percentage of decrease reaches 11.8 % between the lowest external velocity (i.e. U_i=1.528 m/s) and the case of U_i=5 m/s. Such behavior can be explained by noting that the hydrodynamic forces have proportional relation with the external flow velocity, and such hydrodynamic forces are responsible of decreasing the stiffness of the system as noticed from equation (6.101), and leads to reduce system's natural frequency accordingly. Based on the model derivation, it is observed that the external flow velocity has direct impact on the natural frequencies of the system, while the internal flow velocity has very minor effect. On the other hand, the flow velocity has no effect on the torsional natural frequencies of the system as shown in figure 7.52 and table 7.14.
- The flow velocity plays dominant role in increasing the damping of the system in the lateral direction. It was found that the damping coefficient at flow velocity of 5 m/s is about four times the damping coefficient at the internal velocity of 1.528. This phenomenon can be displayed clearly by investigating the corresponding phase plots, which appear as stable nodes.

- The flow velocity is considered as important factor in achieving stability of the system. At high flow velocities the system is shifted to the unstable zone. Such behavior occurred at internal flow velocity equals 10 m/s, as shown in figure (7.51). This figure shows that the lateral vibration in *X*-*Y* and *X*-*Z* planes increase exponentially with time; hence the motion is unstable. The physical reason for this trend is that, at high flow velocities, the hydrodynamic forces attains large values, which tries to move the drill pipe away from its equilibrium position.

Finally, we must note that the obtained findings are corresponding to the current inextensible rotating flexible pipe conveying fluid, which then flows upwards as annular flow , in which both internal and external flow are dependent.

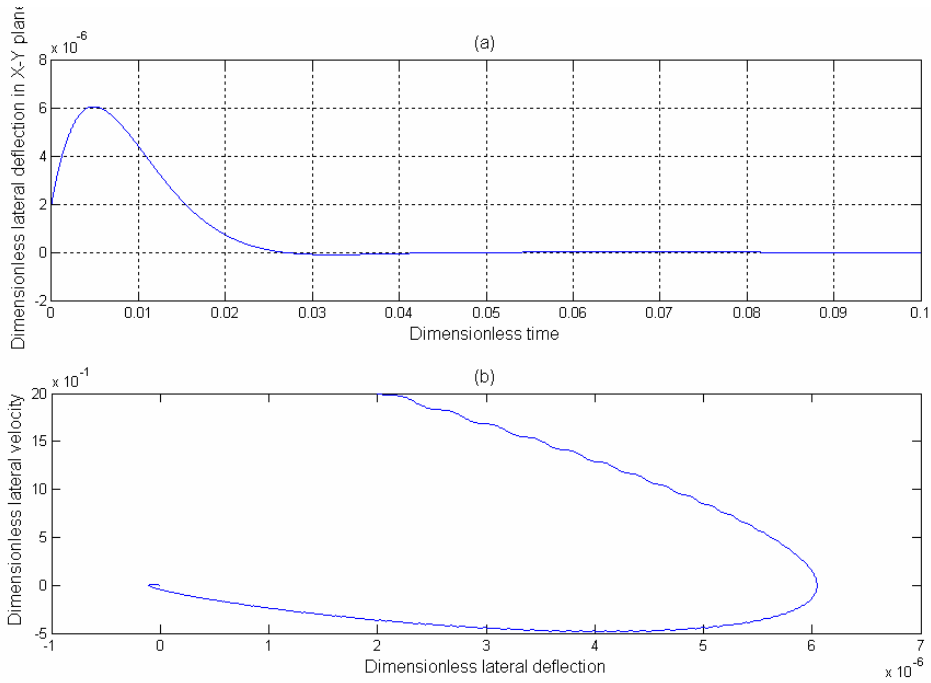


Figure 7.48: (a) Transient response of the dimensionless lateral deflection in *X-Y* plane at the 1st mode of vibration at $U_i = 5$ m/s. (b) Corresponding phase plane plot.

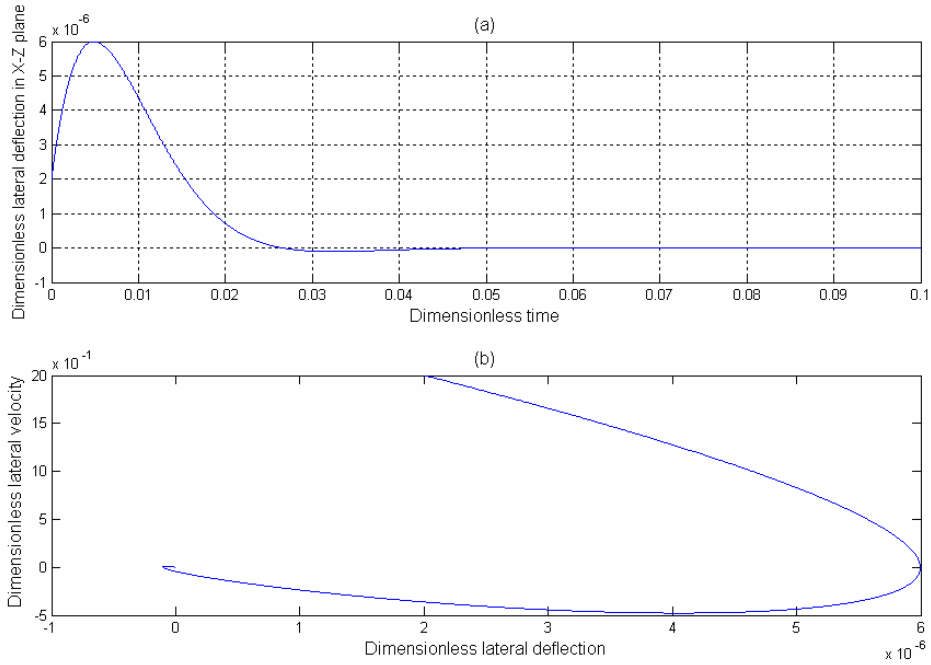


Figure 7.49: (a) Transient response of the dimensionless lateral deflection in *X-Z* plane at the 1st mode of vibration at $U_i = 5$ m/s. (b) Corresponding phase plane plot.

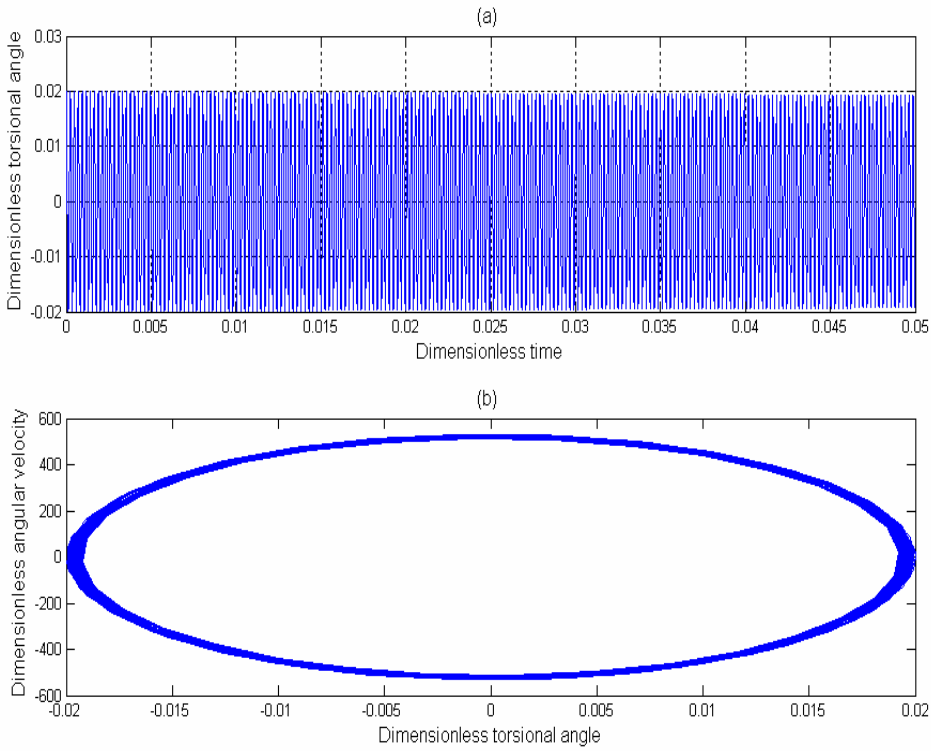


Figure 7.50: (a) Transient response of the dimensionless torsional vibration at the 1st mode of vibration at $U_i = 5$ m/s.

(b) Corresponding phase plane plot.

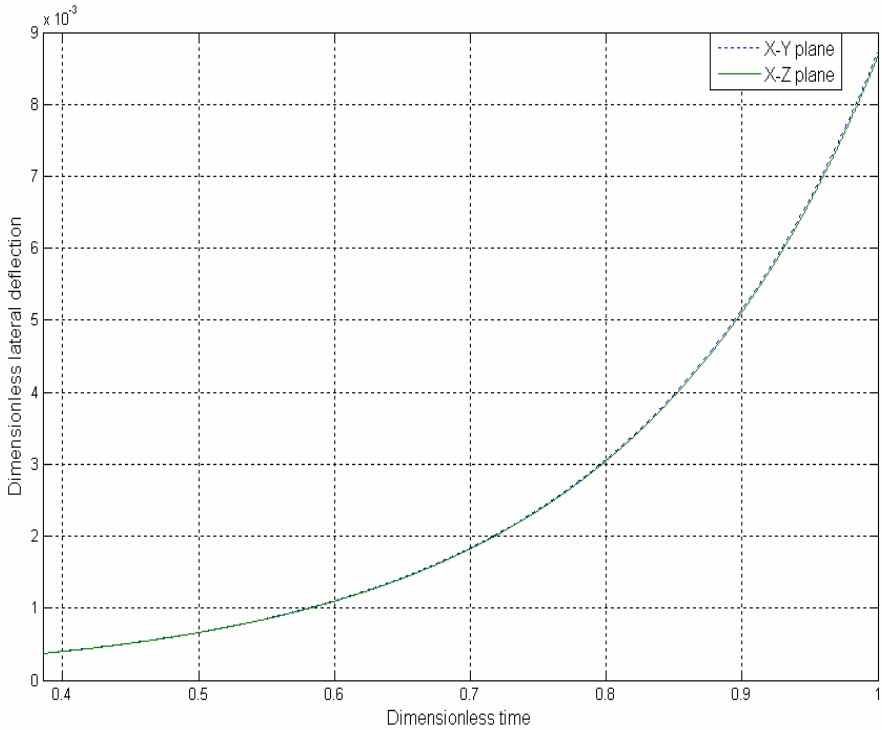


Figure 7.51: (a) Transient response of the dimensionless lateral deflection in *X-Y* and *X-Z* plane at the 1st mode of vibration at $U_i = 10$ m/s.

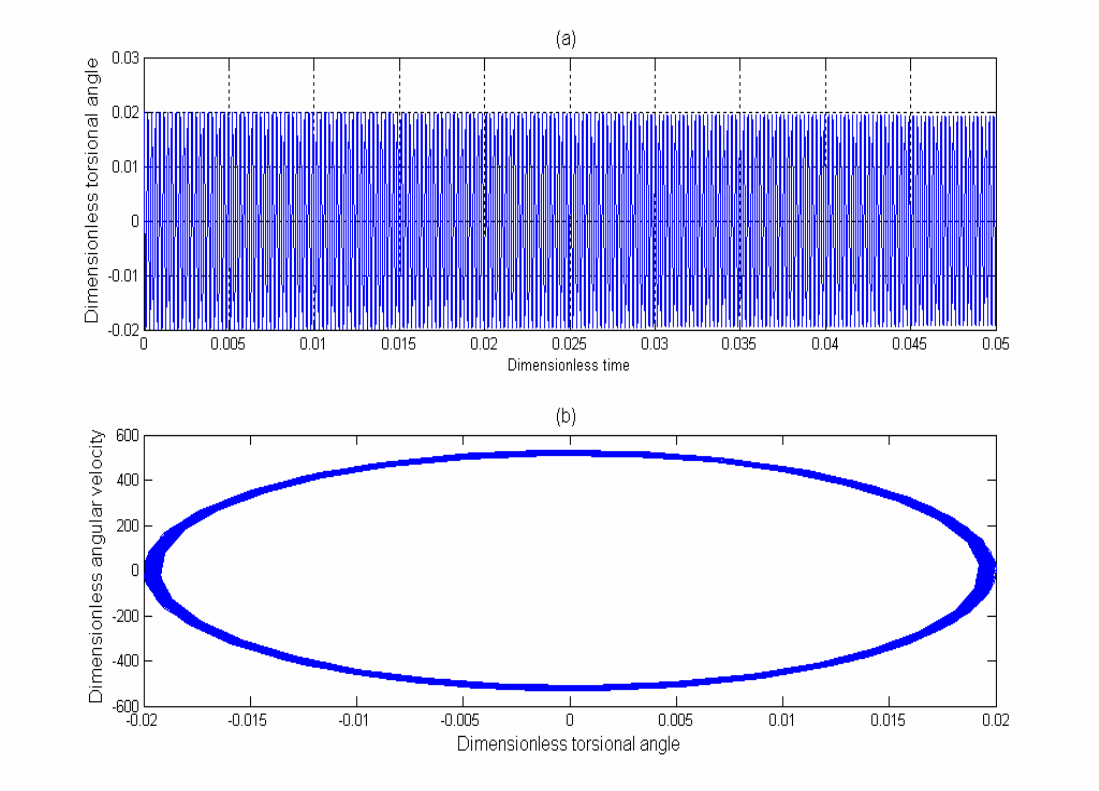


Figure 7.52: (a) Transient response of the dimensionless torsional vibration at the 1st mode of vibration at $U_i = 10$ m/s.

(b) Corresponding phase plane plot.

Table 7.14: Lateral and torsional natural frequencies and damping ratios of a drill pipe at various external flow velocities.

Velocity of the internal flow (m/s)	**Velocity of the internal flow (m/s)**	**Damping coefficient in lateral directions in *X*-*Y* and *X*-*Z* planes**	**Natural frequency of lateral vibrations (rad/s)**	**Natural frequency of torsional vibrations (rad/s)**
1.528	2.315	0.21	0.0391	4.888
5	7.608	0.7912	0.0345	4.888

7.4.5 Influence of the annulus spacing

Figures (7.53 -7.56) show the transient response of the system for the lateral and torsional vibrations when the outer cylinder diameter is considered at the two values of 0.12 and 0.2 m. The natural frequencies and damping coefficients for the lateral and torsional vibrations at those outer pipe diameters are listed in table 7.15. Based on the obtained results, the following remarks are made:

- The annulus spacing has dominant effect on the lateral natural frequencies of the system. As annular spacing decreases, a sharp drop in the natural frequency was observed. It was found that, for the annular space of 4.5 mm, the natural frequency is lowered by a percentage of 47.8 % compared to the annulus spacing of 13.9 mm, and 59.8 % compared to the annulus spacing of 45 mm. This finding is considered justifiable since the added mass is a function of the annular spacing such that $M_a \propto \frac{(D_{ch}/D_o)^2+1}{(D_{ch}/D_o)^2-1}$. Accordingly, as the annular space decreases, the above ratio increases and leads to increase the value of the added hydrodynamic mass of the external fluid accordingly. From another point of view, and by investigating equation (6.101), we can conclude that the value of stiffness is decreasing when the annular space is narrowing, which leads to a drop in the natural frequency of the system.
- The annular spacing has minor effect on the torsional natural frequency of the system. It is found that the torsional natural frequency of the system at annulus spacing of 4.5 mm is less by 5 % when it is compared to the case of 45 mm annulus spacing. This relatively small effect is expected since as the annulus

space is narrowing, the mass moment of inertia of the system is decreasing slightly, which yields a slight increase in the torsional natural frequency.

- By referring to figures 7.53 and 7.55, it was observed that the damping ratio in the lateral directions has inverse relation with the annulus spacing. This can be interpreted by examining equation (6.100), which shows that increasing the annulus spacing is responsible for minimizing the value of the linear damping of the system.

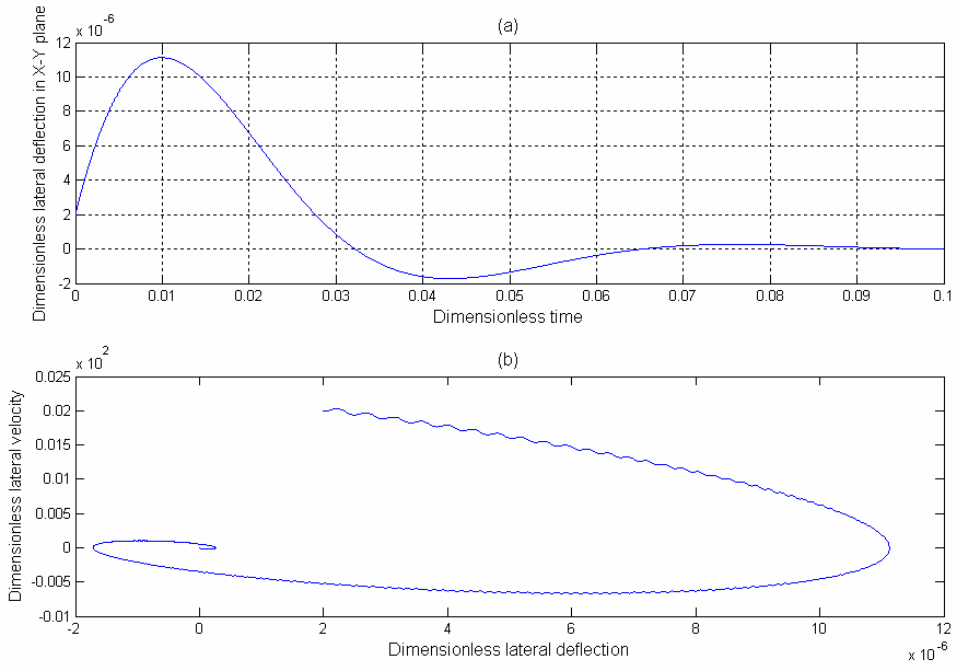


Figure 7.53: (a) Transient response of the dimensionless lateral deflection in *X-Y* plane at the 1st mode of vibration at D_{ch}=0.12 m. (b) Corresponding phase plane plot.

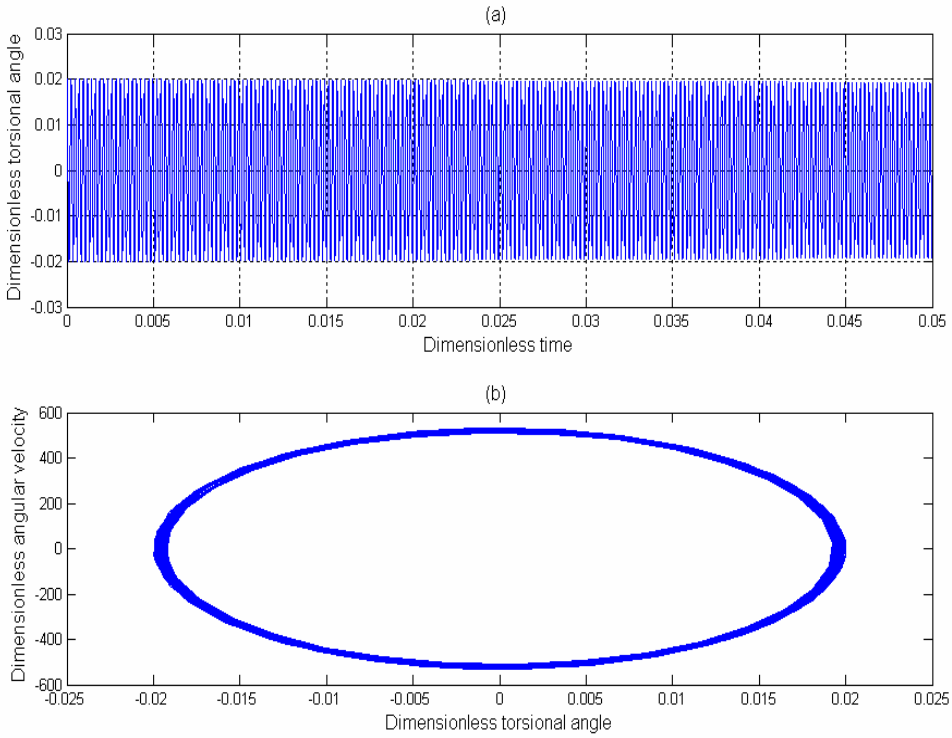


Figure 7.54: (a) Transient response of the dimensionless torsional vibration at the 1st mode of vibration at D_{ch}=0.12 m.

(b) Corresponding phase plane plot.

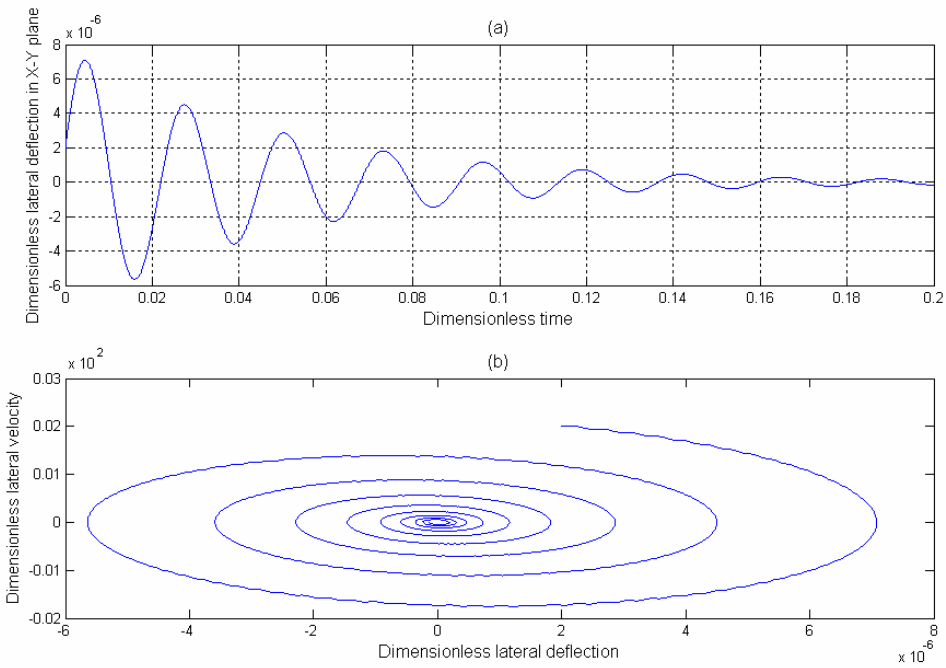


Figure 7.55: (a) Transient response of the dimensionless lateral deflection in *X-Y* plane at the 1st mode of vibration at D_{ch}=0.2 m. (b) Corresponding phase plane plot.

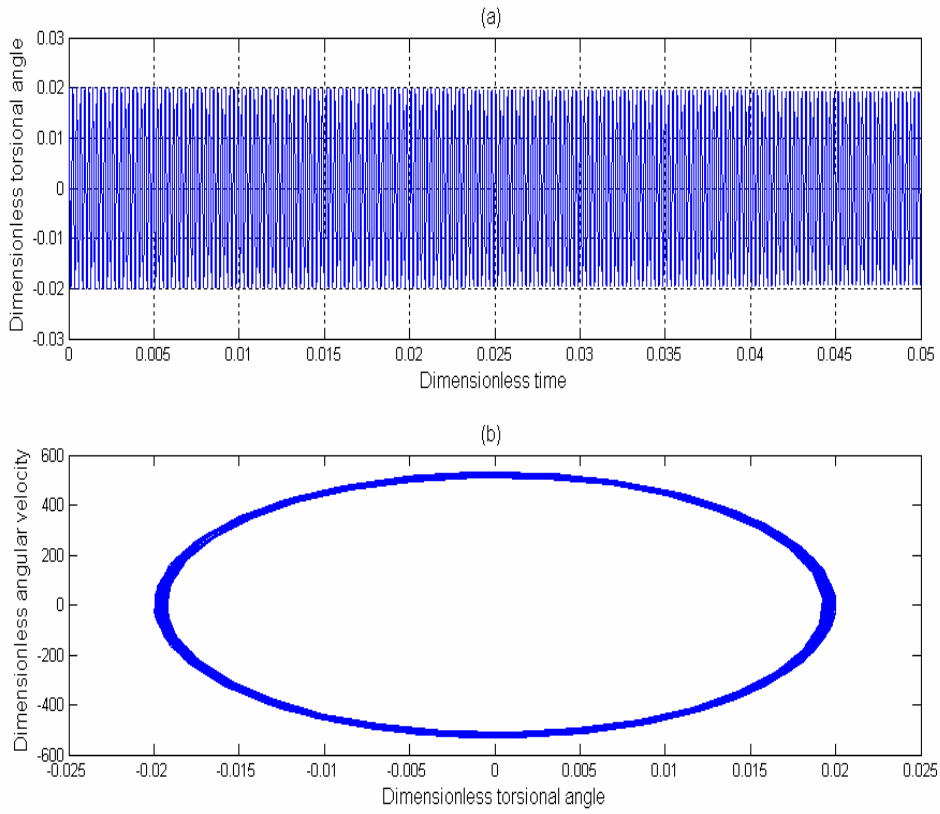


Figure 7.56: (a) Transient response of the dimensionless torsional vibration at the 1st mode of vibration at D_{ch}=0.2 m.

(b) Corresponding phase plane plot.

Table 7.15: Lateral and axial natural frequencies and damping ratios of the drill pipe system at various outer cylinder diameters.

Diameter of the outer pipe (m)	**Damping coefficient in the lateral directions in *X-Y* and *X-Z* planes**	**Natural frequency of lateral vibrations in *X-Y* and *X-Z* planes (rad/s)**	**Natural frequency of torsional vibrations (rad/s)**
0.12	0.511	0.0204	4.753
0.1388	0.21	0.0391	4.888
0.2	0.072	0.0508	5.003

7.4.6 Influence of the rotational speed

This subsection is devoted to study the effect of the rotational speed on the natural frequencies of the system. In this subsection, the drill pipe is assumed to operate at constant speed. Accordingly, the inextensible drill pipe has only two degrees of freedom represented by lateral vibrations in *X-Y* and *X-Z* planes. Figures 7.57-7.65 show the transient response of both lateral vibrations in *X-Y* and *X-Z* planes, in addition to comparison plots between them. The natural frequencies and damping ratios at various rotational speeds are listed in table 7.16. Based on figures 7.57-7.65 and table 7.16, the following remarks are concluded:

- The amplitudes of the lateral vibration in both *X-Y* and *X-Z* planes are increasing rapidly as the drill pipe rotates faster. By comparing figures 7.57 and 7.64, it was observed that the amplitude of vibrations at 1000 rpm is about 2.3 times the amplitudes of stationary pipe. This observation reveals the importance of considering the coupling between rotational speed and lateral deflections in estimating accurate amplitudes of the system.
- By examining figures 7.59, 7.61 and 7.63, we observe that the amplitudes of the lateral vibrations in *X-Y* plane is higher than the corresponding ones in *X-Z* plane, while this deviation magnifies as the rotational speed increases. This trend is due to the unsymmetrical gyroscopic forces in X-Y and X-Z planes, where their magnitudes are proportional to the rotational speed. Figure 7.65 shows that both responses of the lateral vibration in X-Y and X-Z are identical for stationary pipe since gyroscopic forces are absent. Based on this analysis, we can state that 3-

dimensional analysis is essential for predicting accurate response of the system at high rotational speeds.

- The natural frequencies of lateral vibration in X-Y and X-Z planes are almost identical. The only difference is in their amplitudes as explained above.
- The lateral natural frequency of the drill pipe is rising slightly as the rotational speed increases as shown in table 7.16. It was noticed that the lateral natural frequency of the pipe at 1000 rpm is higher by 2 % as compared to stationary pipe. This is due to the fact that increasing rotational speed is leading to stiff the drill pipe due to the frictional forces between the external fluid and the outer cylinder wall.
- The damping ratio increases slightly by increasing the rotational speed of the drill pipe as shown in table 7.16. It was found that the damping ratio at 1000 rpm is higher by 2.1 % compared to the stationary pipe. The reason behind this phenomenon is that the frictional coefficient f_t has proportional relation with the rotational speed as shown by equation (3.60), and this coefficient is responsible of adding damping to the system.

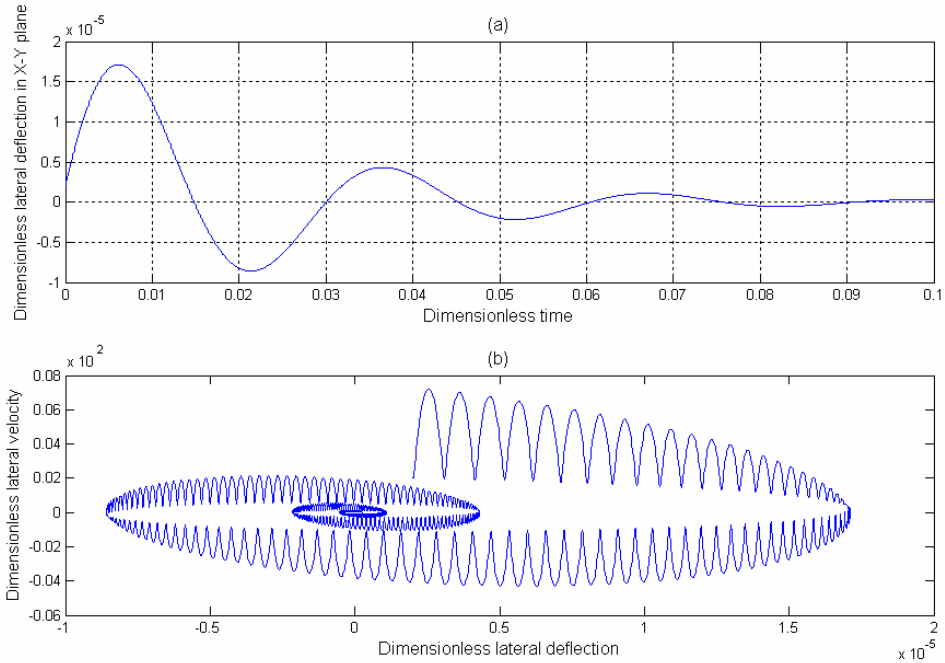


Figure 7.57: (a) Transient response of the dimensionless lateral vibration in *X*-*Y* plane for the 1st mode of vibration at Ω=1000 rpm. (b) Corresponding phase plane plot.

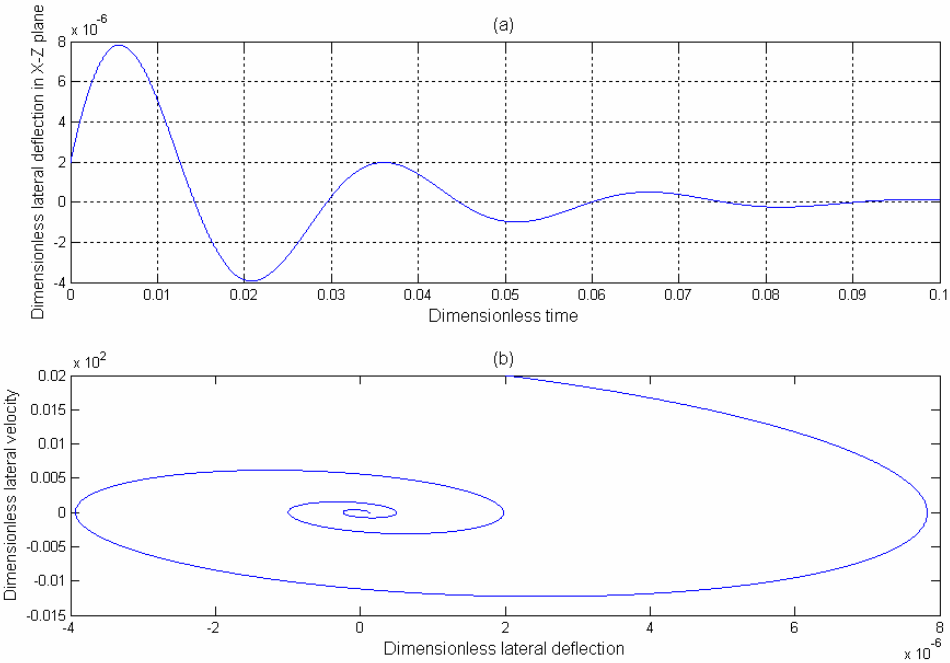


Figure 7.58: (a) Transient response of the dimensionless lateral vibration in *X-Z* plane for the 1st mode of vibration at Ω=1000 rpm. (b) Corresponding phase plane plot.

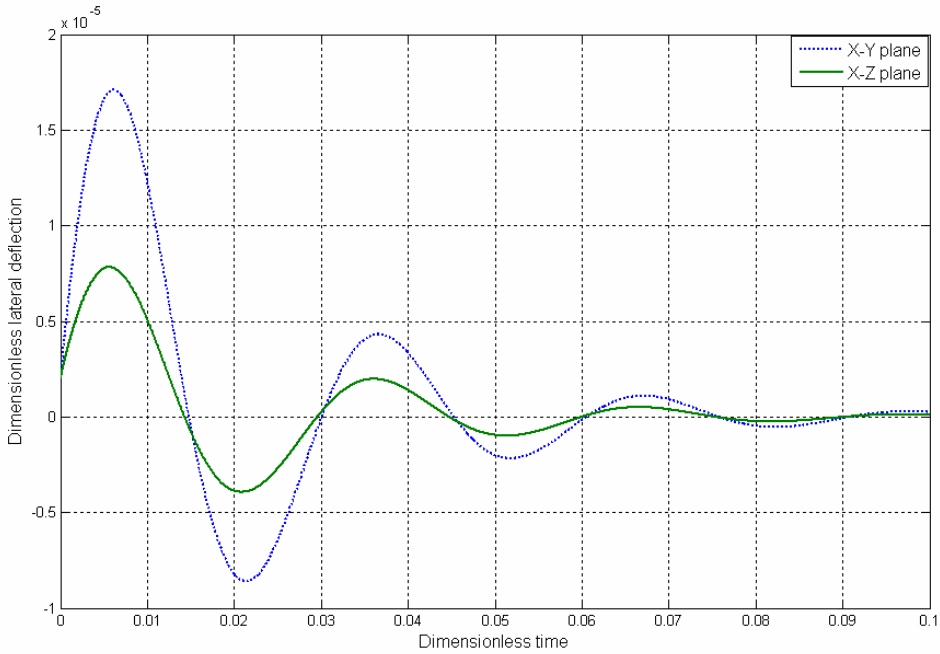


Figure 7.59: Comparison between transient responses of the dimensionless lateral vibration in *X-Y* and *X-Z* planes for the 1st mode of vibration at Ω=1000 rpm.

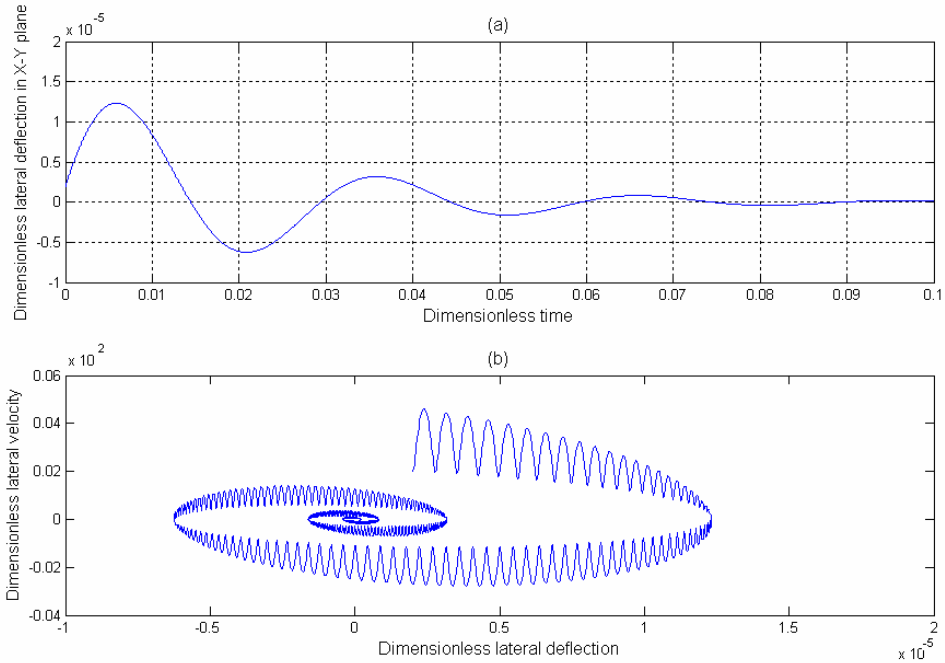


Figure 7.60: (a) Transient response of the dimensionless lateral vibration in *X-Y* plane for the 1st mode of vibration at Ω=500 rpm. (b) Corresponding phase plane plot.

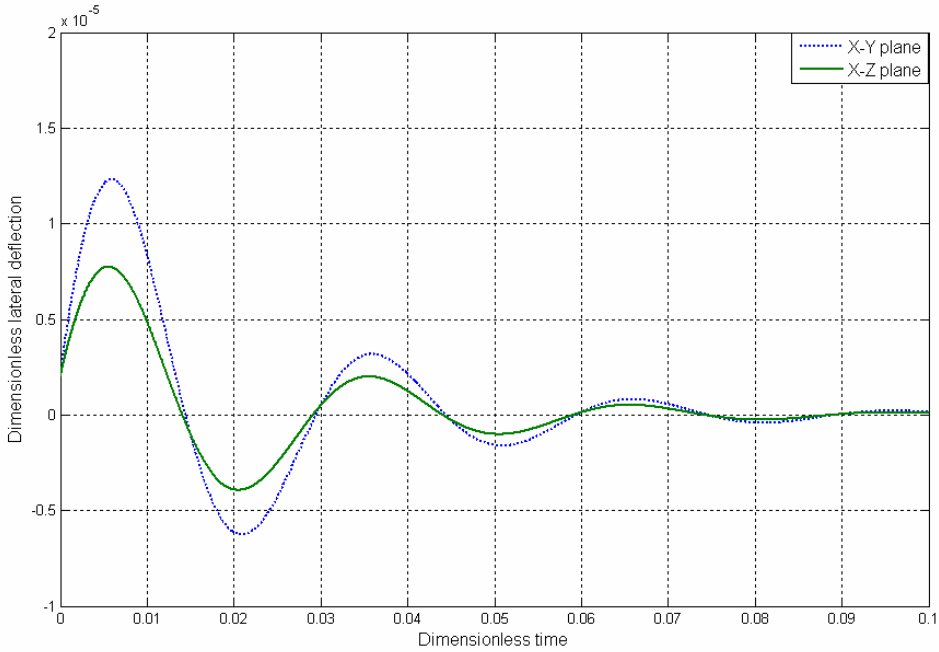


Figure 7.61: Comparison between transient responses of the dimensionless lateral vibration in *X-Y* and *X-Z* planes the for 1st mode of vibration at Ω=500 rpm.

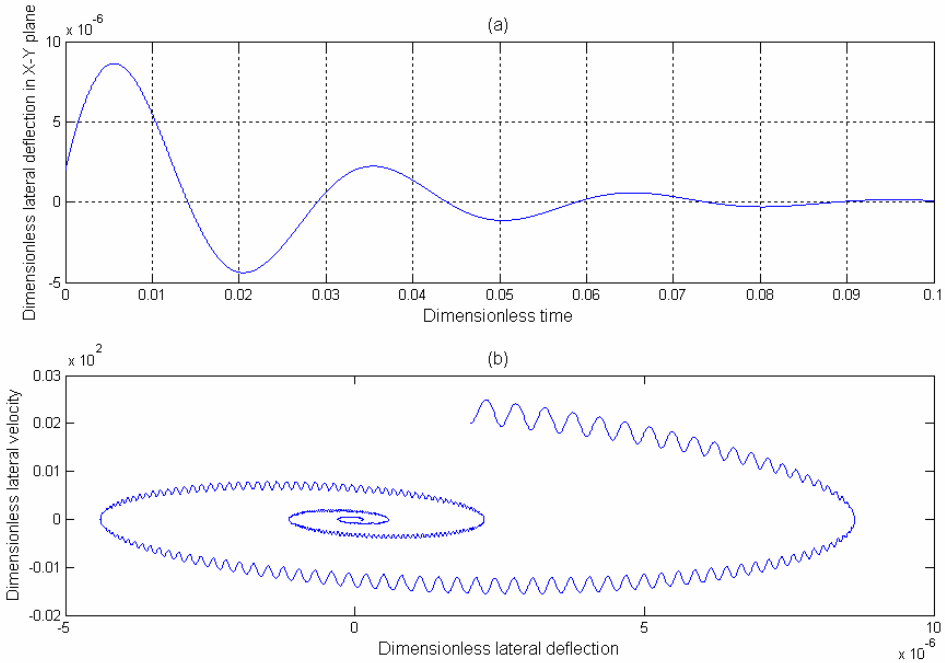


Figure 7.62: (a) Transient response of the dimensionless lateral vibration in *X*-*Y* plane for the 1st mode of vibration at Ω=100 rpm.

(b) Corresponding phase plane plot.

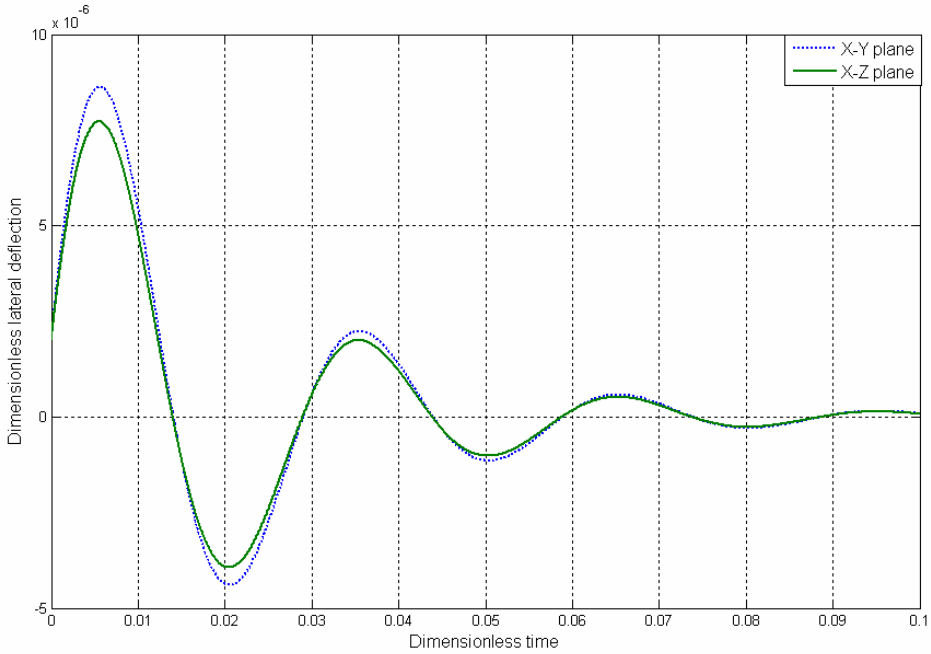


Figure 7.63: Comparison between transient responses of the dimensionless lateral vibration in *X-Y* and *X-Z* planes for the 1st mode of vibration at Ω=100 rpm.

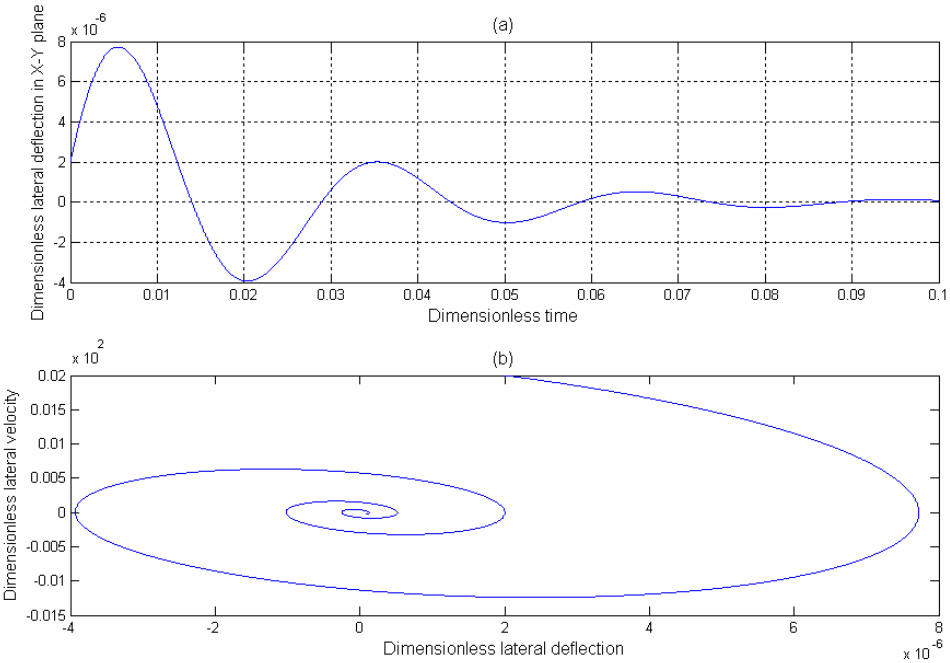


Figure 7.64: (a) Transient response of the dimensionless lateral vibration in *X*-*Y* plane for the 1st mode of vibration at Ω=0 rpm. (b) Corresponding phase plane plot.

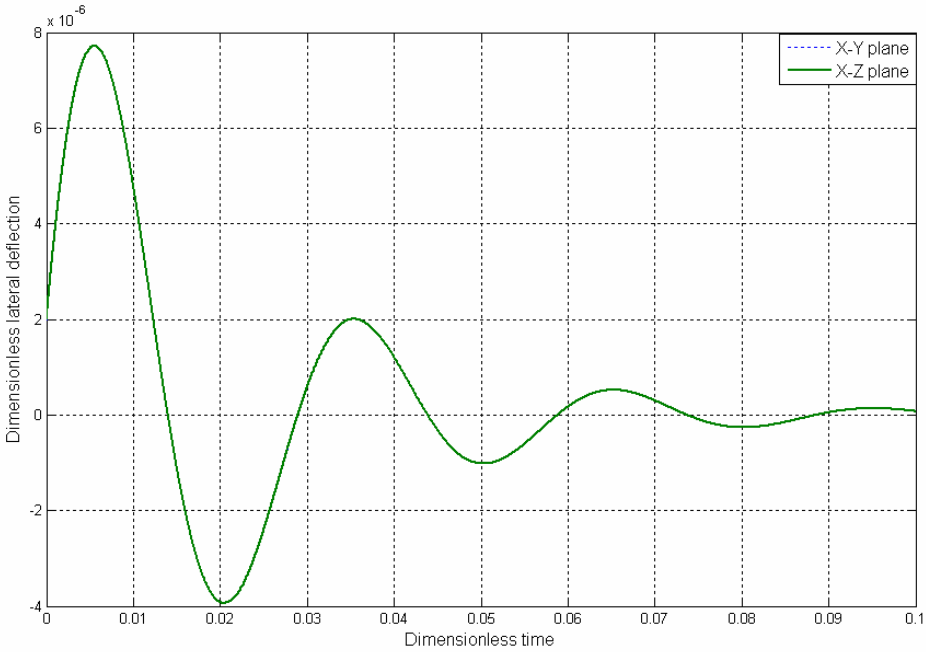


Figure 7.65: Comparison between transient responses of the dimensionless lateral vibration in *X-Y* and *X-Z* planes for the 1st mode of vibration at Ω= 0 rpm.

Table 7.16: Lateral natural frequencies and damping ratios of a rotating drillstring at various rotational speeds.

Angular speed, Ω (rpm)	**Natural frequency of lateral vibrations in** *X-Y and X-Z* **planes (rad/s)**	**Damping ratio**
1000	0.03984	0.21434
500	0.03978	0.21212
100	0.03947	0.21019
0	0.03905	0.20990

7.4.7 Influence of the end conditions

The main objective of this subsection is to investigate the flow induced vibrations of the drill pipe with fixed-simply supported sliding ends, illustrated by figure 4.8. In order to pick up the influence of the end conditions on the transient response of the drill pipe, a comparison is performed between the transient responses of cantilevered and simply supported configurations by utilizing the same system parameters presented in section 7.4.1.

Adopting the solution approach described in section 7.2, the coefficients of the equations of motion (304-307) were evaluated. It was found that the most weighing nonlinear terms are associated with quadratic and cubic nonlinearities. For brevity, the values of the coefficients of the equations of motion are not listed. After evaluating the coefficients of the equations of motion, the system of governing equations have been solved using ODE45 solver while the same dimensionless initial conditions presented in section 7.4.2 were applied to the mid span of the pipe while the dimensionless initial conditions in the axial direction are assumed to be as follow: $p(t) = 0.000002,\ \dot{p}(t) = 0.$ Figures 7.66 -7.70 show the transient axial, lateral and torsional responses of the system for the first mode of vibration in addition to the corresponding phase plane plots. Other transient responses were generated for the second and third modes of vibration, but not shown here for the purpose of brevity. Referring to these figures, the natural frequencies were calculated for the first three modes of vibrations and their values are listed in table 7.17.

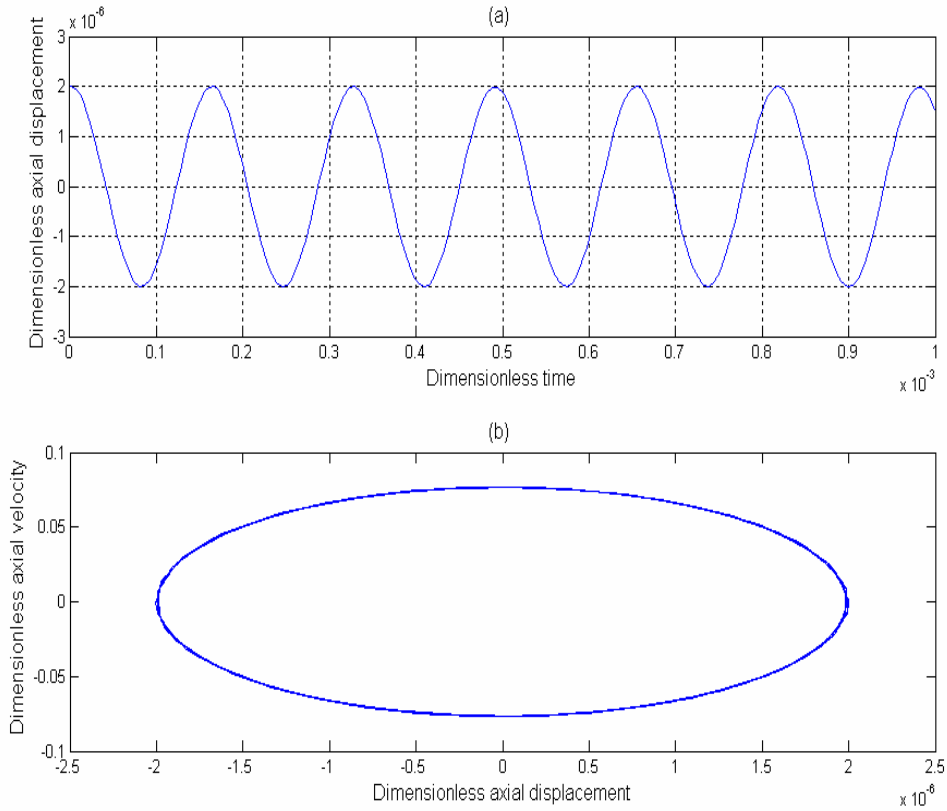


Figure 7.66: (a) Transient response of the dimensionless axial vibration of the drill pipe at the 1st mode of vibration. (b) Corresponding phase plane plot.

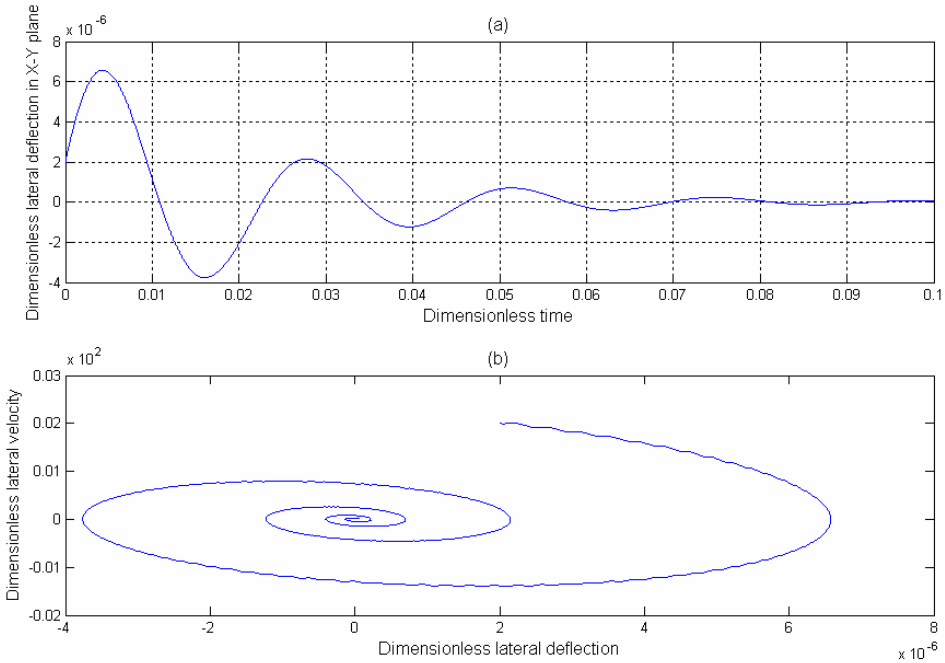


Figure 7.67: (a) Transient response of the dimensionless lateral vibrations of the extensible drill pipe in *X*-*Y* plane at the 1st mode of vibration. (b) Corresponding phase plane plot.

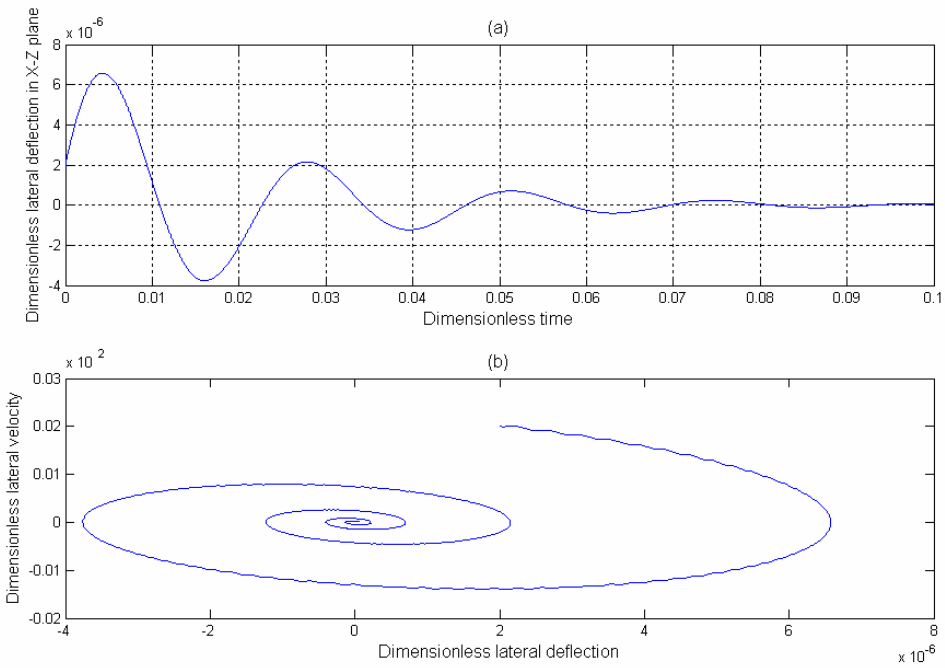


Figure 7.68: (a) Transient response of the dimensionless lateral vibrations of the extensible drill pipe in *X-Z* plane at the 1st mode of vibration. (b) Corresponding phase plane plot.

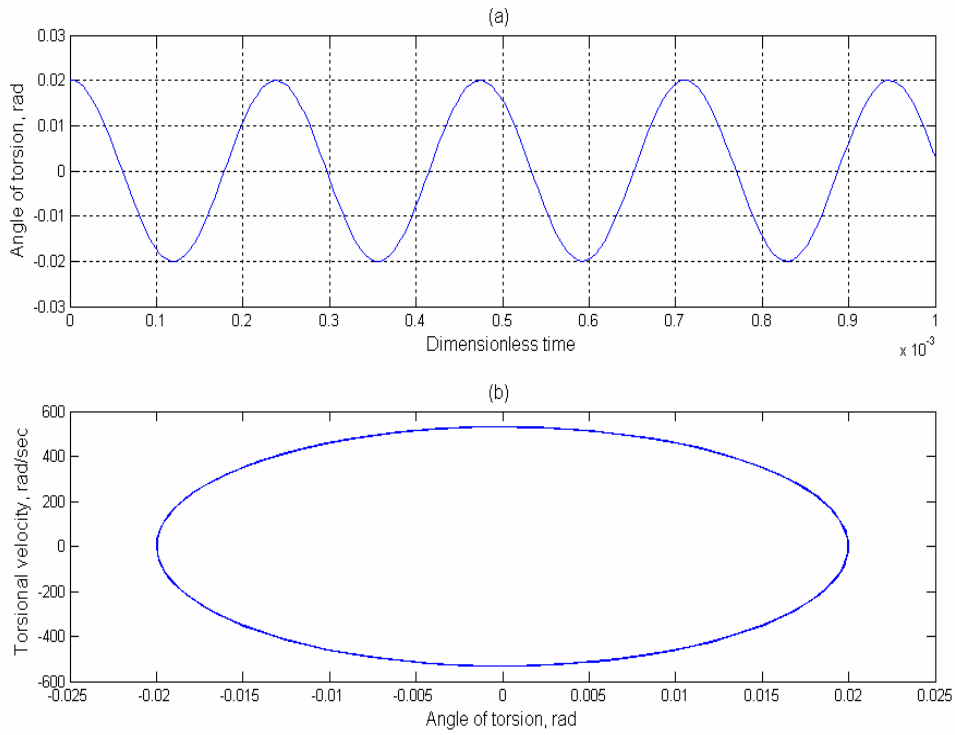


Figure 7.69: (a) Transient response of the dimensionless torsional vibrations of the extensible drill pipe at the 1st mode of vibration. (b) Corresponding phase plane plot.

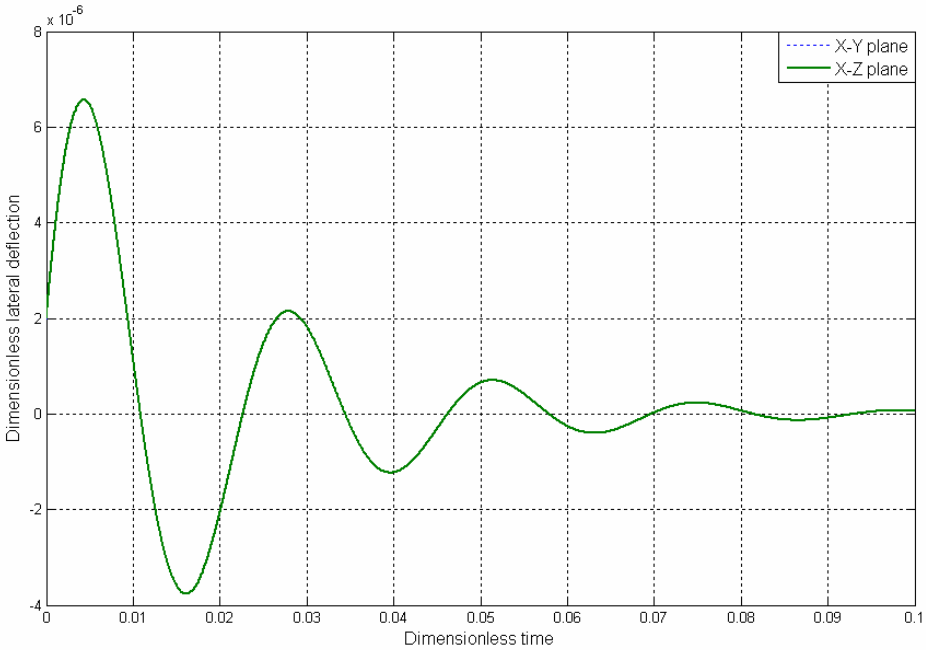


Figure 7.70: Comparison between transient responses of the dimensionless lateral vibration in *X-Y* and *X-Z* planes of the extensible drill pipe at the 1st mode of vibration.

Table 7.17: Axial**,** lateral and torsional natural frequencies of an extensible rotating drill pipe.

No. of Mode	**Natural frequency of the axial vibrations (rad/s)**	**Natural frequency of lateral vibrations in *X-Y* and *X-Z* planes (rad/s)**	**Natural frequency of torsional vibrations (rad/s)**
1st Mode	6.894	0.0494	4.917
2nd mode	21.489	0.1383	14.751
3rd mode	36.323	0.2608	24.586

Referring to figures 7.66-7.70, tables 7.12 and 7.17, the following observations are made:

- By comparing the natural frequencies of the current model and the cantilevered configuration, it was found that the lateral natural frequencies of this system are higher than the corresponding ones for the cantilevered configuration by a percentage of 27 % at the 1st mode of vibration, while it reaches up to 154.7% at the third mode of vibration. This finding can be interpreted by noting that simply supported end provides much stiffness to the drill pipe rather than cantilevered end condition. Accordingly, we can conclude that cantilevered configuration is much flexible, and can accommodate the variation in the design condition much better than the simply supported.
- The simply-supported drill pipe system has a considerable amount of damping in the lateral direction with damping ratio equals 0.175. In comparison with the cantilevered configuration, it is observed that the damping ratio for this model is less by 16.7 %. This fact implies that although both models exhibit stable spiral node, but the cantilevered configuration is much efficient in suppression the vibration of the drill pipe. This can be verified by considering the phase plane plots shown in figures 7.29 and 7.67, in which we note that the cantilevered drill pipe returns faster to its equilibrium point compared to simply-supported configuration.
- By comparing figures 7.38 (a) and 7.70, it is observed that the gyroscopic forces have higher influence on the cantilevered pipe than simply supported configuration.

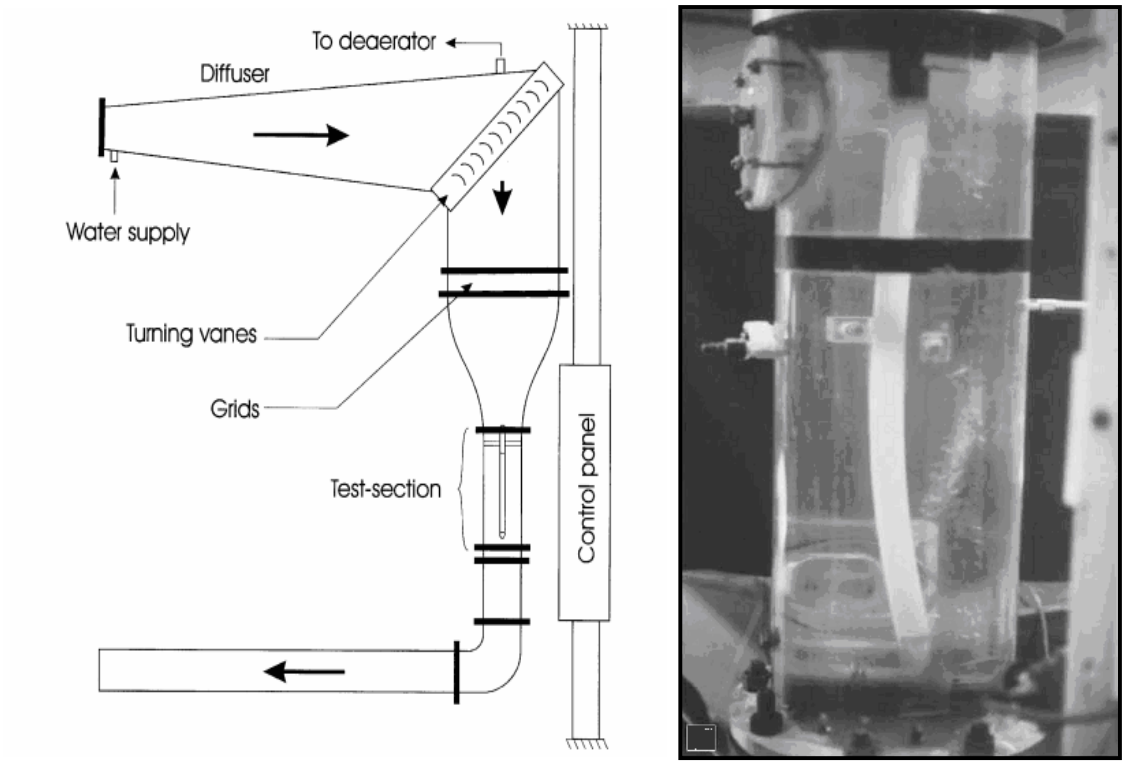
- The torsional natural frequencies of the both cantilevered and simply- supported models are approximately the same. This finding shows that torsional vibrations are the same for extensible and inextensible conditions.
- Both axial and torsional natural frequencies attain high values compared to the lateral ones. Accordingly, one may consider that lateral vibrations are dominant in achieving the stability of the drill pipe for both extensible and inextensible conditions.

Finally, and based on the above comparisons, we conclude that selecting the appropriate end conditions is very important step in achieving reliable and accurate model of the drill pipe system.

7.5 Comparison with Pertinent Experimental Work in the Available Literature

In order to validate the developed mathematical model and the associated numerical scheme, a comparison between the obtained nonlinear formulation and available experimental work reported in the literature is undertaken. It is important to mention that most of the experimental studies performed in this field were restricted to the quiescent external fluid, while those studies which involve external axial flow are very few due to the complexity of generating such uniform axial flow experimentally. Accordingly, our results are compared to that experiment performed by Paidoussis et al. [16], which was concerned with the dynamics of a cantilevered cylinder subjected to external axial flow. The test-section was placed vertically, in a water tunnel, as shown diagrammatically in figure 7.71(a). In this experiment, flow straighteners, screens and a large flow-area reduction were utilized to ensure an axial, uniform flow stream in the test-section. Figure 7.71 (b) shows a photograph of the test-section. The physical system parameters corresponding to this experiment are listed in table 7.18. The most appropriate dynamic model which may represent this system, is the case study presented in section 3.11.3 associated with a cantilevered flexible pipe conveying fluid and subjected to external axial flow. The derived mathematical model has been modified to match this experiment, by assuming the absence of the internal flow and reversing the direction of the external flow. On the other hand, since the system is symmetric in X-Y and X-Z planes due to the absence of rotation, one equation of motion in the lateral direction is found to be sufficient to describe the dynamics of this system.

(a) **(b)**



F**igure 7.71:** (a) Diagrammatic view of part of the water tunnel used in Paidoussis et al. experiments [16]. (b) Photograph of a vertical test-section, with a flexible cantilevered cylinder mounted in it, [16].

Table 7.18: System parameters of the test-section, [16].

Parameters	Value
Length of elastic tube, L	0.520 m
Outside diameter of the elastomer pipe, D_o	0.0254 m
Diameter of the outer cylinder, D_{ch}	0.203 m
Young's modulus of the elastic tube, E	2.883×10^6 N/m^2
Mass of the tube per unit length, m	0.577 kg/m
Density of the water, $\rho_{f,o}$	998 kg/m^3
Velocity of external flow, U_o	1.34 m/s
Friction coefficient in normal direction, C_N	0.0125
Friction coefficient in tangential direction, C_T	0.0125
The form-drag coefficient, C_{DP}	0.0125

Adopting the solution approach described in section 7.2, the dimensionless coefficients of the governing equation of motion in the lateral direction were evaluated. It is important to recognize that the dimensionless time equals 0.8625 times the actual time.

After evaluating the coefficients of the lateral equation of motion, the governing equation of motion is solved using ODE45 solver, and upon applying the following actual (physical) initial conditions at $s = 0.2028$ measured from the fixed end of the pipe such that: $v(t) = 0.05,\ \dot{v}(t) = 0$. The corresponding dimensionless initial conditions are calculated based on their definitions presented in Chapter 5, while their values were found to be: $q(t^*) = 0.0962,\ \dot{q}(t^*) = 0$.

Figures 7.72 -7.74 show the numerically generated transient responses of the lateral vibrations of the system for the first three modes of vibration in addition to the corresponding phase plane plots. Referring to these figures, the natural frequencies were calculated for the first three modes of vibrations, where their values are listed in table 7.18. Figure 7.75 shows the power spectral density (PSD) of the lateral vibrations resulted from the experimental work performed by Paidoussis et al. [16]. The numerically generated lateral natural frequencies were compared with the corresponding experimental values resulted from the PSD as shown in table 7.18. Based on the obtained results, it was found that the developed mathematical model and the associated numerical algorithm are able to estimate the lateral natural frequencies of this system by a percentage of error of 5 % for the first four modes of vibration. These results assure that the developed mathematical model and the associated numerical scheme are considered reliable and accurate enough to describe the actual behavior of the dynamics of such a fluid-structure interaction model.

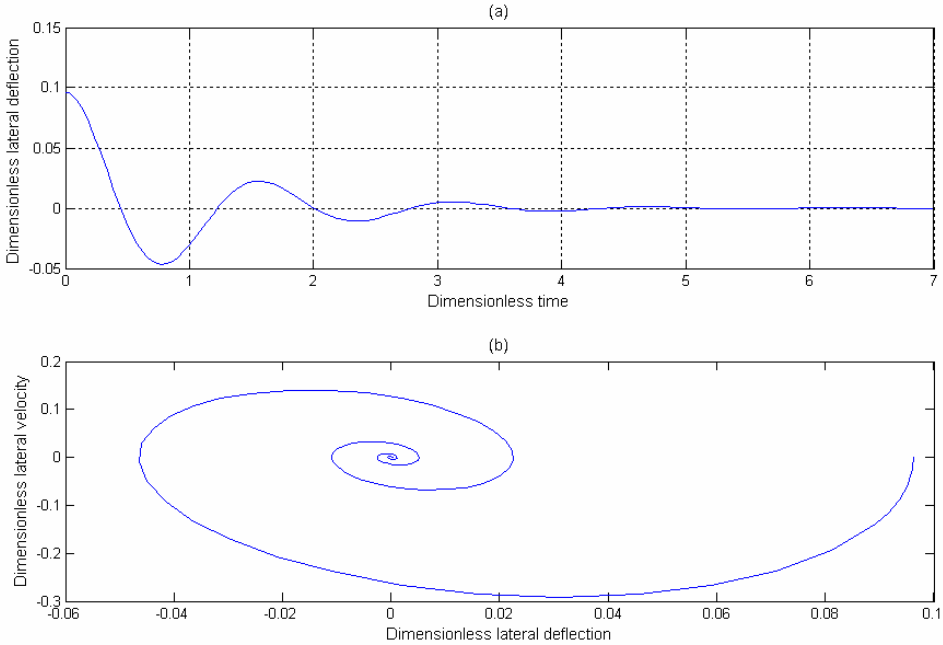


Figure 7.72: (a) Transient response of the dimensionless lateral vibrations of the cantilevered test-section at the 1st mode of vibration. (b) Corresponding phase plane plot.

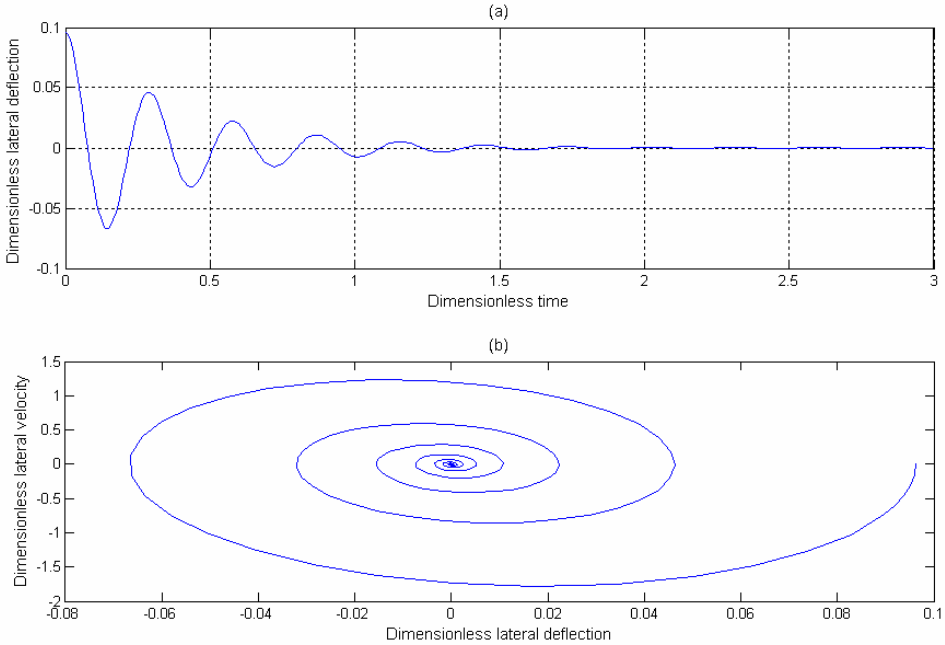


Figure 7.73: (a) Transient response of the dimensionless lateral vibrations of the cantilevered test-section at the 2nd mode of vibration. (b) Corresponding phase plane plot.

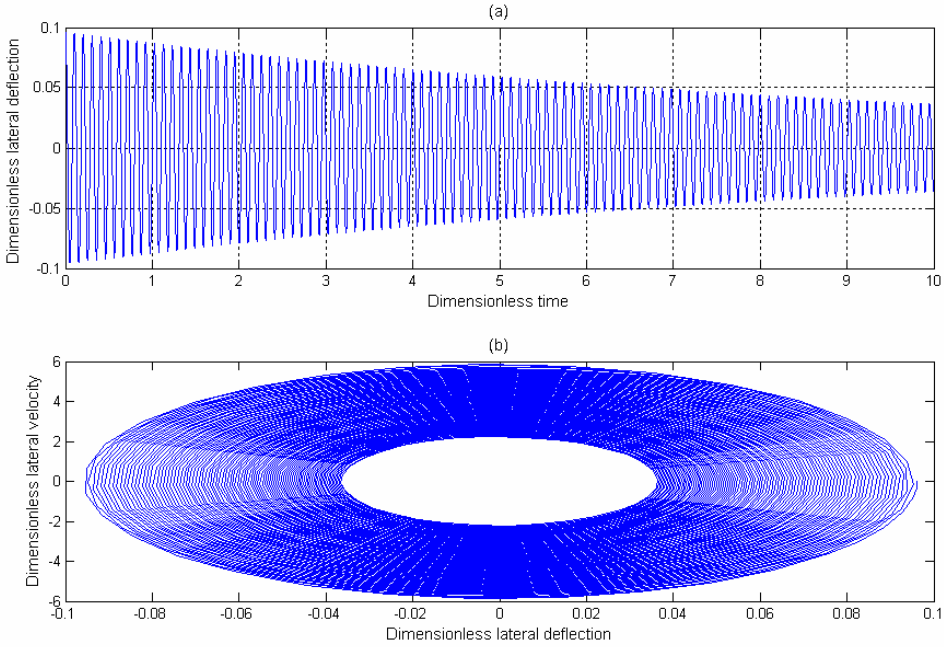
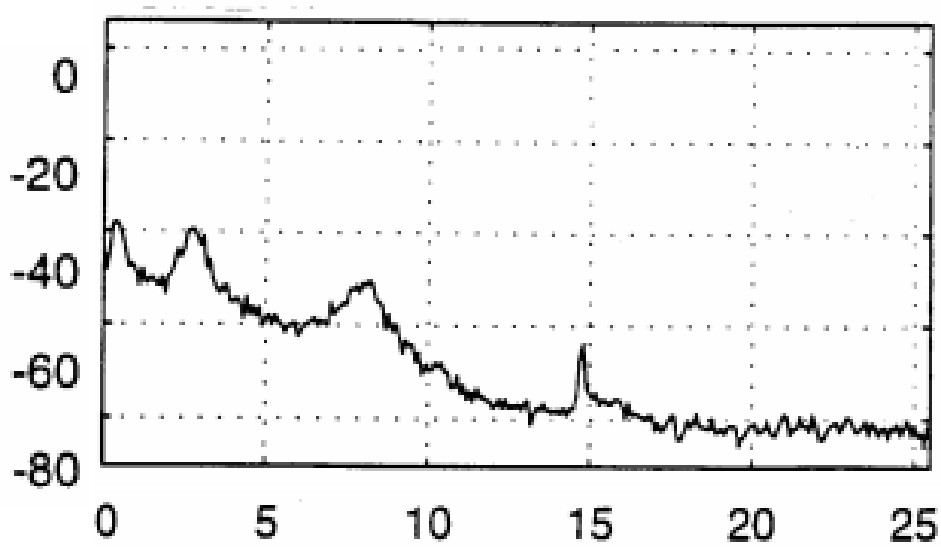


Figure 7.74: (a) Transient response of the dimensionless lateral vibrations of the cantilevered test-section at the 3rd mode of vibration. (b) Corresponding phase plane plot.



F**igure 7.75:** Power spectral density (PSD) of vibrations at s=0.2028 and U_o=1.34 m/s, obtained experimentally by Paidoussis et al. [16].

Table 7.19: Comparison between the calculated lateral natural frequencies of the cantilevered test-section and the corresponding ones estimated experimentally by Paidoussis et al. [16], for the first four modes of vibrations.

No. of Mode	**Calculated value of the lateral natural frequency based on the developed scheme (Hz)**	**Experimental value of the lateral natural frequency estimated by Paidoussis et al. [16] (Hz)**	**Percentage of error (%)**
1st Mode	0.57	0.6	5
2nd mode	3.03	2.9	4.5
3rd mode	8.41	8.0	5.1
4th mode	15.15	14.6	3.8

CHAPTER EIGHT

CONCLUSIONS AND RECOMMENDATIONS

Based on the work conducted in this dissertation, starting from problem identification, followed by detailed derivation of the governing equations of motion and ending by performing numerical simulations of some selected case studies, the following conclusions and recommendations are stated:

8.1 Conclusions

1. The Lagrangian approach together with the variational technique are considered as powerful tools to formulate the nonlinear governing equations of motion that describe the dynamics of both rotating and non rotating slender flexible pipe conveying fluid and subjected to external axial flow. This formulation is applicable to both extensible and inextensible conditions associated with different end conditions.
2. The obtained formulation is considerably comprehensive and found reliable to describe the dynamics of the following systems:
    - Extensible non rotating flexible pipe conveying fluid and subjected to external axial flow.

- Inextensible rotating flexible pipe conveying fluid and subjected to external axial flow.
- Inextensible rotating flexible pipe conveying fluid downwards, which then flows upwards as a confined flow.
- Extensible rotating flexible pipe conveying fluid and subjected to external axial flow.
- Extensible rotating flexible pipe conveying fluid downwards, which then flows upwards as a confined flow.

The derived models were employed to investigate the dynamics of selected engineering applications, represented by the vibrations of tube in a double pipe heat exchanger, and the flow induced vibrations of rotating drillstring.

2. A comprehensive analytical and numerical study of the dynamics of an extensible non rotating flexible pipe conveying fluid and subjected to external axial flow was performed. The governing equations of motion showed that the most dominant nonlinear terms of the equation of motion in the lateral direction, are associated with the coupling between axial and lateral deflection in addition to the cubic nonlinearity. The most dominant coefficients of the equation of motion in the axial direction are associated with the quadratic nonlinearity due to the elasticity of the pipe and induced hydrodynamic forces.
3. Based on the numerical simulation results of the extensible non rotating flexible pipe conveying fluid and subjected to external axial flow, which may represent the vibrations of a tube in a double pipe heat exchanger , the following findings were noted:

- The internal flow has high influence in softening the lateral and axial natural frequencies of the system for the first three modes of vibration by a percentage of 17.8 % to 19.4%, compared to the fluid-free model.
- The external flow plays a dominant role in softening the lateral natural frequency of the system for all modes of vibration by a percentage ranges from 29 % to 31.7 %, compared to the fluid-free model.
- The external flow velocity has a minor effect in lowering the lateral natural frequencies, but it has high influence in increasing damping of the system in the lateral direction.
- As the annular spacing decreased, a noticeable drop in the lateral natural frequency was observed. This finding is considered justifiable since the added mass of the external fluid has inverse relation with the annular spacing. On the other side, annular spacing has very limited effect on the axial vibration of the system.
- The friction coefficient has a dominant influence in increasing the damping in the lateral direction, while its effect on the axial vibration is negligible.

4. The derived mathematical model was extended to simulate the flow induced vibration of rotating drillstring under the inextensibility condition. In this case, the governing equations of motion showed that the most dominant nonlinear terms of the lateral equations of motion in *X*-*Y* and *X*-*Z* planes are associated with the cubic nonlinearity. These terms emerged from the potential energy expression, represented by elasticity of the drill pipe and the gravitational force due to the

weight of the pipe and fluid, in addition to the influence of the external hydrodynamic forces on the drill pipe.

5. The three-dimensional analysis was found necessary for predicting the accurate vibration behavior of a rotating flexible pipe conveying fluid and subjected to external axial flow, particularly at high rotational speeds. This behavior was due to the non symmetric gyroscopic forces in *X*-*Y* and *X*-*Z* planes.
6. The numerical simulation of the dynamics of an inextensible rotating flexible pipe conveying fluid, which then flows upwards axially in the annular region, (which may represent the vibrations of rotating drillstring), revealed the following observations:
    - The internal flow has a significant role in softening the lateral and torsional natural frequencies of the system. It was found that the external flow is responsible of lowering the lateral natural frequencies for the first three modes of vibration by a percentage of 13.2 % up to 18.6 % compared to the dry model, while the torsional natural frequency decreased by a percentage of 25.6 %, as compared to the dry model.
    - The external flow has a dominant role in lowering the lateral natural frequencies up to 64.4 %, compared to the dry model.
    - The flow velocity has significant influence in increasing the damping of the drill pipe in the lateral direction. It was found that the damping coefficient at flow velocity of 5 m/s is nearly four times the damping coefficient at the internal velocity of 1.528 m/s. It was also found that the external flow velocity is the primary cause of such a behavior, while the

internal flow has a minor effect in damping the drill pipe in the lateral direction.

- The annular spacing has a dominant effect on the lateral natural frequencies of the system. As the annular spacing decreased, a noticeable drop in the natural frequency was observed. It was found that when the annular space reached 4.5 mm, the natural frequency was lowered by a percentage of 47.8 % compared to the annular spacing of 13.9 mm, and by 59.8 % when compared to annulus spacing of 45 mm. On the other hand, the annular spacing has minor effect on the torsional natural frequency of the system.
- The lateral natural frequency of the drill pipe experiences slight increase (stiffening effect) at high rotational speeds. It was noticed that the lateral natural frequency of the pipe at 1000 rpm is higher by 2 % compared to the stationary pipe. Also, as the drill pipe rotated faster the amplitudes of the lateral vibration in both *X*-*Y* and *X*-*Z* planes increased.

7. The derived mathematical model was extended to simulate the flow induced vibrations of a rotating drillstring, by assuming fixed-simply supported sliding ends and ignoring the inextensibility condition. This study showed that the lateral natural frequencies of this system are higher than the corresponding ones for the cantilevered configuration by a percentage of 27 % at the 1st mode of vibration, and it reached up to 154.7% at the third mode of vibration. Also it was found that the damping ratio for this model is less by 16.7 % compared to the cantilevered one, which implies that a cantilevered configuration is much efficient in

suppressing the vibration of the drill pipe. This comparison reveals the importance of selecting the appropriate end conditions for achieving reliable and accurate model of the drill pipe system.

8. The nonlinear formulation of the rotating and stationary flexible pipes conveying fluid and subjected to external flow was found useful in predicting accurate natural frequencies and amplitudes by considering all types of inertia and elastic coupling between axial, lateral and torsional deflections. The nonlinear analysis becomes essential when the pipe is subjected to large deflections.
9. The developed mathematical model and associated numerical scheme have been validated by establishing a comparison with the available experimental work reported in the literature. Based on the comparison performed between the generated lateral natural frequencies resulted from the numerical simulation and the corresponding values estimated experimentally by Paidoussis et al. [16], concerning the dynamics of a cantilevered cylinder subjected to external axial flow, it was found that the developed mathematical model and the associated numerical algorithm are capable of estimating the lateral natural frequencies of this system with an approximate percentage of error of 5 % for the first three modes of vibration. These results assure that the developed mathematical model and the associated numerical scheme are considered reliable and sufficiently accurate to describe the actual behavior of the nonlinear dynamics of such fluid-structure interaction systems.

8.2 Recommendations for Future Work

1. Performing experimental investigations of the rotating pipe conveying fluid and subjected to external flow in order to validate the current mathematical model.
2. The single assumed mode Lagrangian method was used to discretize the governing nonlinear equations of motion, while multiple assumed modes solution is recommended to obtain more accurate nonlinear differential equations.
3. The current comprehensive nonlinear formulation and the associated numerical solution algorithm are found able to estimate the natural frequencies and amplitudes of the vibrations of various fluid-structure interaction systems, which involve several flow schemes, various system parameters and end conditions. Accordingly, and in light of the limited available commercial FE packages associated with the fluid-structure interaction applications, particularly under the influence of rotation, it is recommended to develop a comprehensive computer package with friendly user interface, which can estimate the vibrations of similar models and contribute to the area of fluid-structure interaction.

REFERENCES

[1] Naudascher, E. and Rockwell, D., "Flow-Induced vibrations: an Engineering Guide". Rotterdam: A. A. Balkema. 1994.

[2] Paidoussis, M. P., "Dynamics of tubular cantilevers conveying fluid". *Journal of Mechanical Engineering Science.* Vol. 12 (1970), pp. 85-103.

[3] Semler, C., Li G. and Paidoussis, M. P., "The nonlinear equations of motion of pipes conveying fluid". *Journal of Sound and Vibration.* Vol. 169, No. 5 (1994), pp. 577-599.

[4] Paidoussis, M. P. and Semler, C., "Nonlinear dynamics of fluid-conveying cantilevered pipe with a small mass attached at the free end". *International Journal of Nonlinear Mechanics.* Vol. 33, No. 1 (1998), pp. 15-32.

[5] Paidoussis, M. P. and Li, G. X., "Stability, double degeneracy and chaos in cantilevered pipes conveying fluid". *International Journal of Nonlinear Mechanics.* Vol. 29, No. 1 (1994), pp. 83-107.

[6] Szabo, ZS., "Bifurcation analysis of a pipe containing pulsate flow". *Periodica Polytechnica Ser. Mech. Eng.* Vol. 44, No. 1 (2000), pp. 149-160.

[7] Sarkar, A. and Paidoussis, M. P., "A compact Limit-Cycle oscillation model of a cantilever conveying fluid". *Journal of Fluids and Structures.* Vol. 17 (2003), pp. 525-539.

[8] Wadham-Gagnon M., Paidoussis M. and Semler C., "Dynamics of Cantilevered Pipes Conveying Fluid. Part1: Nonlinear Equations of Three-Dimensional Motion". *Journal of Fluids and Structure* Vol. 23 (2007), pp. 545 - 567.

[9] Hawthorne, W. R., "The early development of the Dracone flexible barge". *Proceedings of the Institution of Mechanical Engineers.* Vol. 175 (1961), pp. 52-83.

[10] Paidoussis, M. P., "Dynamics of flexible slender cylinders in axial flow. Part1: theory, Part2: experiments". *Journal of Fluid Mechanics.* Vol. 26(1966), pp. 717-736, 737-751.

[11] Paidoussis, M. P., “Dynamics of cylindrical structures subjected to axial flow”. *Journal of Sound and Vibrations.* Vol. 29 (1973), pp. 365-385.

[12] Lopes, J. L., Paidoussis, M. P. and Semler, C., “Linear and nonlinear dynamics of cantilevered cylinders in axial flow. Part 2: The equations of motion”. *Journal of Fluids and Structures.* Vol. 16, No. 6, (2002), pp. 715-737.

[13] Lighthill, M., “Note on the swimming of slender fish”. *Journal of Fluid Mechanics.* Vol. 9 (1960), 305-317.

[14] Semler, C., Lopes, J. L., Augu, N. and Paidoussis, M. P., “Linear and nonlinear dynamics of cantilevered cylinders in axial flow. Part 3: Nonlinear dynamics. *Journal of Fluids and Structures.* Vol. 16, No. 6 (2002), pp. 739-759.

[15] Doedel, E. and Kerneves, “AUTO: software for continuation and bifurcation problems in ordinary differential equations”. *Applied Mathmatics Report, California Institute of Technology, U.S.A.* 1986

[16] Paidoussis, M. P., Grinevich, E., Adamovic, D. and Semler, C., “Linear and nonlinear dynamics of cantilevered cylinders in axial flow. Part 1: Physical dynamics”. *Journal of Fluids and Structures.* Vol. 16, No. 6 (2002), pp. 691-713.

[17] Modarres-Sadeghi, Y., Paidoussis, M. P. and Semler, C., “A nonlinear model for extensible slender flexible cylinder subjected to axial flow”**.** *Journal of Fluids and Structures.* Vol. 21(2005), pp. 609-627.

[18] Khulief, Y., Al-Sulaiman, F. and Bashmal, S., “Vibration Analysis of drillstrings with Self-Excited Stick-Slip Oscillations”. *Journal of Sound and Vibration.* Vol. 299 (2007), pp. 540-558.

[19] Sampaio, R., Piovan, M. T. and Venero Lozano, “Coupled axial/torsional vibrations of drill-strings by means of nonlinear model”. *Mechanics Research Communications*. Vol. 4 (2007), pp. 497-502.

[20] Berlioz, Der Hogopian, J., Dufour, R. and Draoui, E., “Dynamic behavior of a drillstring: Experimental investigation of lateral instabilities”. *Trans. of ASME, Journal of Vibration and Acoustics.* Vol. 118 (1996), pp. 292-298.

[21] Lalanne, M., and Ferraris, G., “Rotordynamics Prediction In Engineering”. John Wiley. 1990.

[22] Axisa, P. and Antunes, J., “Flexural vibrations of rotors immersed in dense fluids”. *3rd Int. Symp. on Transport Phenomena and Dynamics of Rotating Machinery, ISROMAC-3, Honolulu, Hawaii,* 1990.

[23] Antunes, J., Axisa, P. and Grunenwald T., "Dynamics of rotors immersed in eccentric annular flow". *Journal of Fluids and Structures*. Vol. 10 (1996), pp. 893-918.

[24] Khulief, Y. and Al-Sulaiman, F., "Experimentally-tuned mathematical model for drillstring vibrations". *DETC2007-35057, presented at ASME IDETC/CIE, Las Vegas, Nevada.* 2007.

[25] Schmalhorst, B., Brommundt, E., Baumgart, A. and Richter, U., "Drilling Dynamics in the Presence of Mud Flow". *SPE 59236, presented at SPE/IADC Drilling Conference, New Orleans, Louisiana.* 2000.

[26] Escudier, M.P., Oliveira, P.J., Pinho, F.T. and Smith, S., "Fully developed laminar flow of non-Newtonian liquids through annuli: comparison of numerical calculations with experiments". *Experiments in Fluids*. Vol. 33 (2002), pp. 101-111.

[27] Christoforou, A. and Yigit, A., "Dynamic modeling of rotating drillstrings with borehole interactions". *Journal of Sound and Vibration.* Vol. 206, No. 5 (1997), pp. 243-260.

[28] Cull, S. and Trucker, R., "On the modeling of Coulomb friction"**.** *Journal of Physics A: Mathmatical and Theoritical.* Vol. 32 (1999), pp. 2103-2113.

[29] Zamanian, M., Khadem, S. and Ghazavi, M., "Stick-slip oscillation of drag bits by considering damping of drilling mud and active damping system". *Journal of Petroleum Science and Engineering.* Vol. 59 (2007), pp. 289-299.

[30] Hannoyer, M. J. and Paidoussis, M. P., "Instabilities of tubular beams simultaneously subjected to internal and external axial flows". *ASME Journal of Mechanical Design.* Vol. 100 (1978), pp. 328-336.

[31] Paidoussis, M. P. and Besancon, P., "Dynamics of arrays of cylinders with internal and external axial flow". *Journal of Sound and Vibration.* Vol. 76 (1981), pp. 361-380.

[32] Wang, X. and Bloom, F., "Dynamics of a submerged and inclined concentric pipe system with internal and external flow". *Journal of Fluids and Structure.* Vol. 13 (1999), pp. 443-460**.**

[33] Paidoussis, M. P., Luu, T. P. and Prabhakar, S., "Dynamics of a long tubular cantilever conveying fluid downwards, which then flows upward around the cantilever as confined annular flow". *Journal of Fluids and Structures.* Vol. 24 (2008), pp. 111-128**.**

[34] Paidoussis M. P., “Fluid-Structure Interactions: Slender Structures and Axial Flow”, Vol.1. Academic Press, UK, 1998.

[35] Stoker J. J., “Nonlinear Elasticity”. Gordon and Breach, New York, 1968.

[36] Lighthill, M., “Note on the swimming of slender fish”. *Journal of Fluid Mechanics.* Vol. 9 (1960), 305-317.

[37] Taylor G. I., “Analysis of the swimming of long and narrow animals”. *Proceedings of the Royal Society (London).* A 214 (1952), pp. 158-183.

[38] Potter M. C. and Wiggert D. C. “Mechanics of Fluids”, 3rd Edition. Brooks/Cole, USA, 2002.

[39] Lesmez, M., Wiggert,D. and Hatfield, F., “Modal Analysis of Vibrations in Liquid-Filled Piping System”. *Journal of FluidsEngineering.* Vol. 112(1990), pp. 311-318.

[40] Zavdoney L. L. and Nayfeh A. A., “The non-linear response of a slender beam carrying a lumped mass to principle parametric excitation”. *International Journal of Non-linear Mechanics.* Vol. 4 (1989), pp. 105-125.

[41] Dimarogonas, A., and Paipetis, S., “Analytical Methods in Rotor Dynamics”. *Applied Science Publishers LTD, New York,* 1983.

[42] Idel’chick, I. E., “Handbook of Hydraulic Resistance”, Hemisphere, New York, 1986.

[43] Finlayson, B. A. and Scriven L. E., “The Method of Weighted Residuals- a review”. *Applied Mechanics Reviews.* Vol. 19 (1966), pp. 735-748.

[44] Ginsberg, J. H., “Mechanical and Structural Vibrations, Theory and Applications”, 1st Edition. John Wiley & Sons Inc, 2001.

[45] Rao S. S., “Mechanical Vibrations**”**, 3rd Edition. Addison-Wesley Publishing Company, 1995.

VITA

FADI ABDELHADI GHAITH was born on June 25, 1979 in Hebron, Palestine. He holds Jordanian nationality. He has completed higher school education in Amman, Jordan. He has obtained bachelor degree in Mechanical Engineering from University of Jordan in 2002, followed by Master Degree in Mechanical Engineering from the same University in 2005. His Master thesis was concerned with the nonlinear dynamic modeling and control of cantilever beam mounted on a moving cart and carrying tip mass. Mr. Ghaith joined the industry in 2002, since he has worked as Design and Research Engineer in Jordan Petroleum Refinery for four years. In this period, he was responsible of preparing design calculations related to the stationary equipments such as pressure vessels and heat exchangers, and in accordance with the international codes.

In 2006, Mr. Ghaith enrolled the PhD program in Mechanical Engineering Department at King Fahd University of Petroleum and Minerals. His current research is focusing on the dynamic modeling of Fluid-Structure interaction systems, dynamic modeling of rotating drillstrings, Exergy analysis of inverted trickle solar still and nonlinear Finite Element modeling of Charpy impact test. He has five technical publications in the aforementioned areas of research.

Present Address: *KFUPM P.O.B 8115, Dhahran 31261, Saudi Arabia.*

Permanent Address: *Hay Nazzal, P.O.B 710497, Amman, Jordan.*
Telephone No. 00962777250778
Email: fod1979@yahoo.com