MICROWAVE FILTER DESIGN BY USING CONTINUOUSLY VARYING TRANSMISSION LINE (CVTL)

BY

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ABSTRACT

Name: YOUSEF FAWAZ FAIZ AL-SHARIF

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In this thesis, an effective method for designing planar microstrip filters by using Continuously Varying Transmission Line (CVTL) theory is presented. This method comprises of two parts. The first part is to find the mathematical expressions of the scattering parameters of CVTL. The second part is to apply an optimization technique to find the suitable values that will match the specific desired frequency response. Genetic Algorithm is the optimization technique utilized in this thesis due to its robustness and efficiency.

The aim of this thesis is to find the impedance profile and the length of CVTL that will satisfy the desired frequency response whether it is low pass, stop band, band pass or high pass filter. Accordingly, the width of the microstrip will be determined once the impedance profile is known. The advantage of this technique is that the filter impedance profile is continuous which minimizes the internal power reflection.

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خلاصة الرسالة

الاســــــــــــم : يـوســف بــن فـــواز بــن فـــائـــز الـشـــريـــف

عـنــوان الـرســالــة : تصميم مرشح اتصالات المايكروويف باستخدام خط الارسال المستمر التغيير (CVTL)

الدرجـة الممـنـوحـة : مـاجـسـتـيـر فــي الـعـلــوم

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تـاريخ مـنـح الـدرجة : يــونــيــو ٢٠٠٩

في هذه الرسالة، سيتم تقديم طريقة فعالة لتصميم المرشحات المستوية المتعلقة باتصالات المايكروويف باستخدام خط الارسال المستمر التغيير (CVTL). وتتألف هذه الطريقة من جزئين. الجزء الأول يتمثل في إيجاد العبارات الرياضية لعوامل الإستطارة الخاصة بـ CVTL. أما بالنسبة للجزء الثاني فهو عبارة عن تطبيق إحدى تقنيات التحسين لإيجاد القيم المناسبة التي تحقق وتيرة استجابة المرشح المرجوة. أما بالنسبة لتقنية التحسين المستخدمة في هذه الرسالة فهي تقنية القاعدة الوراثية نظرا للمتانة والكفاءة التي تتمتع بها هذه التقنية.

الهدف من هذه الرسالة هو إيجاد تفاصيل مقاومة و طول CVTL التي تحقق وتيرة الاستجابة المرجوة للمرشح المراد تصميمه سواء كان مرشح الترددات المنخفضة أو الترددات العالية أو مرشح تمرير النطاق أو صد النطاق. كما أن عرض CVTL سيتم تحديده عند معرفة تفاصيل المقاومة. و الجدير بالذكر، أن ميزة هذه الطريقة تتمثل في أن المرشح – بعد تصميمه – يمتلك خاصية المقاومة المستمرة التي تتميز بانعدام الحواف الحادة مما يساعد في تقليل الانعكاس الداخلي للموجة.

جـامـعــة المـلــك فـهـد لـلـبــتــرول و الـمـعــادن

الـظـهـران، الـمـمـلـكـة الـعـربـيـة الـسـعـوديـة

CHAPTER 1

BACKGROUND

1.1 Introduction

The term MICROWAVE refers to alternating signals with frequencies between 300 MHz and 300 GHz with a corresponding electrical wavelength between 1 m to 1 mm, respectively. The microwave region of the electromagnetic spectrum has certain unique properties which enable microwave signals to propagate over long distances through the atmosphere under all but the most severe weather conditions. In real life, there are many applications of microwave signals in the communications field such as satellite, radar, navigation systems. However, the microwave spectrum is a finite resource which must be efficiently utilized and this is where microwave filters come in [1].

A microwave filter is a two-port network which is used to control the frequency response at a certain stage in a microwave system. Filters enable the frequencies within the passband of the filters to be passed and attenuate the ones that within the stopband of the filter. In other word, filters discriminate between wanted and unwanted signal frequencies. The typical classifications of the filters are low pass, high pass, band pass and band stop.

In general, filters are indispensable components in wireless systems. They are used in receivers for rejecting signals outside the operating band, attenuating undesired mixer products and for setting the IF bandwidth of the receiver. In transmitters, filters are used to control the spurious responses of up-converting mixers, to select the desired sidebands and to limit the bandwidth of the radiated signal. Microwave filters are vital components in a huge variety of electronic systems including cellular radio, satellite communications, radar, test and measurement systems.

Filters are essentially frequency selective elements. The filtering behavior results frequency dependent reactances provided by inductors and capacitors. In microwave frequencies, lumped element inductors or capacitors cannot be used and thus transmission line sections, such as microstrip transmission lines are used as inductors and capacitors [2].

1.2 Planar Transmission Structures

One of the principal requirements for a transmission structure to be suitable as a circuit element in Microwave Integrated Circuits (MICs) should be “planar” in configuration. A planar configuration implies that the characteristics of the element can be determined by the dimension in a single plane. For illustration, the width of a microstrip line on a dielectric substrate can be adjusted to control its impedance. Once the impedance can be controlled by dimensions in a single plane, the circuit fabrication can be conveniently carried out by techniques of photolithography and photoetching of thin films. Utilization of these techniques at microwave and millimeter wave frequencies has led to the development of hybrid and monolithic MICs.

There are several transmission structures that satisfy the requirement of being planar such as microstrip lines, coplanar waveguides, slotlines and coplanar strips [3].

1.3 Microstrip Transmission Lines

A microstrip line is a very simple geometric structure, which consists of a metallic strip on a dielectric slab, which in turn, is backed by a conducting ground plane as shown in Figure 1. Because of this planar geometry, microstrip readily lends itself to fabrication. Microstrip lines are the most commonly used transmission structures due to the fact that the mode of propagation in a microstrip is almost transverse electromagnetic (TEM)

which allows an easy approximate analysis. Also, simple transitions to coaxial circuits are feasible.

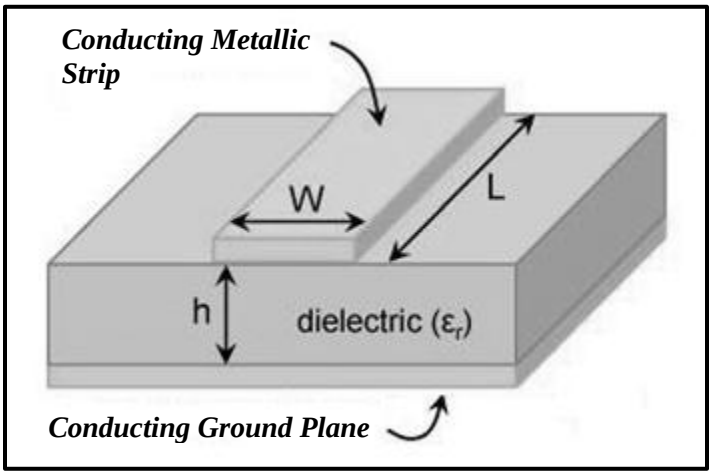


Figure 1: Basic Model of a Microstrip

The microstrip structure is open at the top. This open configuration also makes a microstrip very convenient for use in Microwave Integrated Circuits (MICs). Moreover, one of the main advantages of microstrip is that small adjustments or tuning can possibly be incorporated after the circuit has been fabricated. The most important dimensional parameters are the microstrip width (*w*) and the dielectric height (*h*). Also of great importance is the relative permittivity of the substrate (ε_r). The thickness (*t*) of the top-conducting strip has generally much less important effect and may often be neglected. There are many applications of microstrip transmission lines in MICs either passive circuits or active circuits. Passive circuits are including filters, impedance transformers, couplers, power dividers and combiners. Active circuits are including amplifiers, oscillators and mixers [3].

In this thesis, the filter design is going to be achieved by means of a specific type of microstrip which is called **C**ontinuously **V**arying **T**ransmission **L**ine (CVTL). Figure 2 shows the top view of this type of microstrip. The main advantage of the CVTL is that it has a smooth impedance variation along its length which leads to minimize the internal reflected power and maximize the output efficiency unlike the stepped impedance microstrip shown in figure 9 which suffers from the high power loss due to large impedance discontinuities [4].

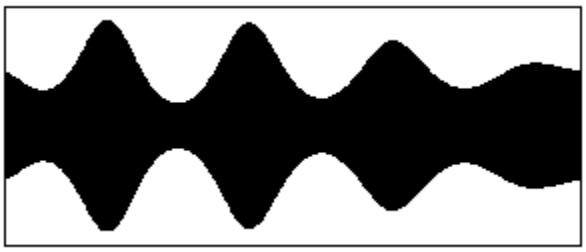


Figure 2: Top View of a CVTL Microstrip

1.4 Thesis Motivation

The subject of microwave filters is quite extensive due to the importance of these components in practical systems and the wide variety of possible implementations. Filters are one of the most widely used components for Radio Frequency (RF) and microwave communications. Filters play important roles in many RF or microwave applications. They are used to reject or pass specific frequency bands in accordance with the designer target. Emerging applications of wireless communications field continue to challenge RF

or microwave filters with ever more stringent requirements like higher performance, smaller size, lighter weight, and lower cost.

Microstrip transmission lines are widely used since they enjoy broad frequency band and they provide circuits that are compact and light in weight. Also, they are generally economical to produce since they are readily adaptable to hybrid and monolithic integrated-circuit fabrication technologies at RF and microwave frequencies. All of the aforementioned reasons make the microstrip filters to be the best filters used in microwave circuits applications specially with help of the recent advance of novel materials and fabrication technologies, including monolithic microwave integrated circuits (MMIC) and Microelectromechanical systems (MEMS) which has stimulated the rapid development of new microstrip filters. In the meantime, advances in computer-aided design (CAD) tools such as full-wave electromagnetic (EM) simulators have revolutionized filter design. Many microstrip filters with advanced filtering characteristics have been appeared.

1.5 Thesis Contribution

This thesis presents a novel method in designing and implementing microwave filters using continuously varying transmission line (CVTL) design method. Moreover, the analysis is easier to implement than a numerical electromagnetic approach and takes less computation time.

This methodology is based on obtaining the general and unified mathematical expressions for scattering parameters namely S_{11} and S_{21}. These mathematical expressions are in terms of the impedance and the length of the filter which makes them valid to be applied for any type of filter. Once they are obtained, the optimization technique is applied to get the response of any type of filter in accordance with the specified optimization criteria.

This methodology makes it possible to quickly and efficiently design any type of filter by only changing the criteria of the optimization to get the new impedance values and the new length of the filter. Therefore, one of the main advantages of this methodology is that the only thing needs to be modified is the optimization criteria in order to design any type of filter since the general mathematical expressions of S_{11} and S_{21} are unified for all filter types.

1.6 Thesis Organization

Chapter 1 gives an introduction to the microwave communications field and microwave filters in general. Moreover, it includes the thesis motivation section as well as the thesis contribution section.

Chapter 2 presents the literature review which is comprising of description of scattering parameters and the subject of filter design and filter implementation. It discusses the

most commonly used designing methods which are the Image Parameters Method and the Insertion Loss Method. Also, the most convenient and commonly used implementation methods which are Stubs Method and Stepped-Impedance Method will be discussed in this chapter. Moreover, a general review for the previous works will be presented.

Chapter 3 provides an introduction to Genetic Algorithm which is the optimization technique going to be utilized to get the required results. Also, it will highlight the important associated issues such as the operators and parameters of the genetic algorithms.

Chapter 4 concentrates on the explanation of the methodology approach to design the CVTL filters with different situations in terms of the expansions of voltage and impedance terms. The first situation is to have the voltage wave expression and the impedance expression expanded with five terms expansion. The second situation is with eight terms expansion. Also, the settings and criteria of the application of the genetic algorithm to this methodology are discussed.

Chapter 5 introduces the results found by this methodology and the effect of the variations of the values of impedance constants on the response of the designed filter. Also, it highlights a comparison between filters at 3 GHz designed by five and eight terms expansion. The results include both values and plots of the impedance profile, the physical dimensions of the filter and the scattering parameters namely S_{11} and S_{21}.

In chapter 6, conclusions and suggestions for future work are presented.

CHAPTER 2

LITERATURE REVIEW

2.1 Scattering Parameters

It is evident that for some problems it will be more convenient to formulate the transformation properties of the two-port components directly in terms of waves. The two independent quantities required for microwave components analysis are an incident and a reflected wave replacing the voltage and current at the input and output terminals [2].

2.1.1 Brief Description

As in figure 3, let the incident and reflected voltage waves on the input guide which are given in magnitude and phase V_1^+ and V_1^- respectively. Similarly, the incident and reflected waves looking toward the junction from reference plane 2 are V_2^+ and V_2^- respectively. It is common to normalize incident and reflected voltage waves as follows:

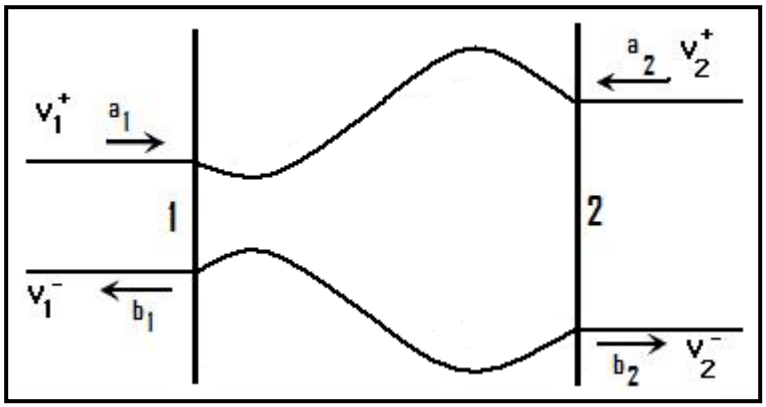


Figure 3: Incident and Reflected Waves at Ports of Microwave Network

$$a_n = \frac{V_n^+}{\sqrt{Z_{0n}}}$$

$$b_n = \frac{V_n^-}{\sqrt{Z_{0n}}}$$

where Z_{on} is the characteristic impedance at port *n*. Thus, voltage and current at reference plane *n* are related to these wave quantities as follows:

$$V_n = V_n^+ + V_n^- = \sqrt{Z_{0n}}\left(a_n + b_n\right)$$

$$I_n = \frac{1}{Z_{0n}}\left(V_n^+ - V_n^-\right) = \frac{1}{\sqrt{Z_{0n}}}\left(a_n - b_n\right)$$

Due to the linear relations between the wave quantities, the reflected waves are expressed in terms of the incident waves as follows [4]:

$$b_1 = S_{11} \times a_1 + S_{12} \times a_2$$

$$b_2 = S_{21} \times a_1 + S_{22} \times a_2$$

The coefficients S_{11}, S_{12}, S_{21} and S_{22} are known as SCATTERING Coefficients. The above equations are frequently written in the matrix form as:

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

or

$$[b] = [S]\ [a]$$

with the [S] array known as the scattering matrix. Once the S-parameters are determined, the complete characterizations of the microwave network are determined. For a physical interpretation, the signal flow graph (figure 4) shows if the output guide is matched so that a_2=0, then

$$b_1 = S_{11} \times a_1$$

and

$$b_2 = S_{21} \times a_1$$

Thus S_{11} is just the input reflection coefficient when the output is matched and S_{21} is the ratio of waves to the right at output and input under this condition [2].

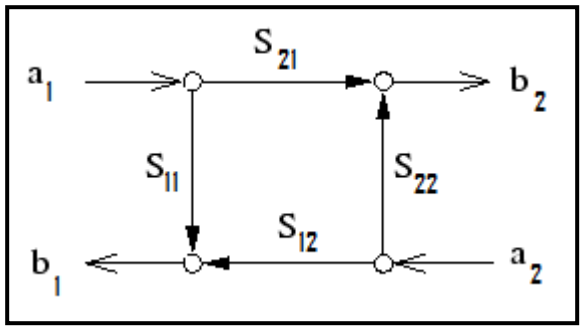


Figure 4: Signal Flow Graph of S-parameters

2.1.2 Measurement of S-parameters

The general procedure to measure or find the values of the S-parameters is to apply input voltage to one port and match the load at the other port.

2.1.2.1 Measurement of S_{11} & S_{21}

S_{11} and S_{21} are found by applying input voltage to Port 1 and terminating Port 2 with a Matched Load to have $a_2 = 0$. So,

$$S_{11} = \frac{b_1}{a_1}\Big|_{a_2=0}$$

is the reflection coefficient at Port 1 (matched load at Port 2) and

$$S_{21} = \frac{b_2}{a_1}\Big|_{a_2=0}$$

is the Voltage transfer ratio from Port 1 to Port 2 (matched load at Port 2).

2.1.2.2 Measurement of S_{22} & S_{12}

S_{22} and S_{12} are found by applying input voltage to Port 2 and terminating Port 1 with a matched load to have $a_1 = 0$. So,

$$S_{22} = \frac{b_2}{a_2}\Big|_{a_1=0}$$

is the reflection coefficient at Port 2 (matched load at Port 1) and

$$S_{12} = \frac{b_1}{a_2}\Big|_{a_1=0}$$

is the voltage transfer ratio from Port 2 to Port 1 (matched load at Port 1).

2.2 Filter Design

Filters can be designed using the image parameter method or the insertion loss method which is sometimes called network synthesis method. In the image parameter method, the design is rather simple. However, the response in the passband and the stopband cannot be precisely controlled. In the insertion loss method, design starts with a low-pass prototype based on maximally flat or Chebyshev response and the insertion loss in the passband and the stopband can be defined and controlled based on the number of sections chosen and the components used. Also, the image method for filter design does not give assurance as to how large the peak reflection loss values are in the pass-band. The network synthesis method for filter design applies tables of low-pass lumped elements filter prototypes. The use of such prototypes to determine the parameters of the RF or Microwave filter eliminates the guess work inherent in image method filter design [2].

2.2.1 Image Parameter Method

The image parameter method of filter design was developed in the late 1930s. It was useful especially for low frequency filters in radio and telephony. Filters which are designed by using the image parameters method consist of cascade of simpler two-port filter sections to provide the required cutoff frequencies and attenuation characteristics. Although, the design procedure is quite simple, the design of filters by this method often must be iterated many times to achieve the preferred results. Also, another disadvantage

is presented in this method. This disadvantage is that as arbitrary frequency response cannot be incorporated into the design unlike the insertion method [2].

The image viewpoint for the analysis of circuits is a wave viewpoint much the same as the wave viewpoint commonly used for analysis of transmission lines. Such circuits include filters. Therefore, filters can be designed by the image method. Uniform transmission line characteristic impedance can again be its image impedance, if γ is the line's propagation constant per unit length, then, γl is the image propagation function for a length l.

The relation between the image parameters and the general circuit parameters, for example open-circuit and short-circuit impedances, is that transmission properties of general circuits can be defined in terms of their image parameters. For example, the image properties of the L-section network are as follows (Figure 5):

$$Z_{11} = \sqrt{Z_a\left(Z_a + Z_c\right)}$$

$$Z_{12} = \frac{Z_a Z_c}{\sqrt{Z_a\left(Z_a + Z_c\right)}}$$

$$\gamma = \coth^{-1}\sqrt{1 + \left(\frac{Z_a}{Z_c}\right)}$$

Z_{11}, Z_{12} and γ are the image parameters. Z_a and Z_c are the transmission properties of the L-section network [1].

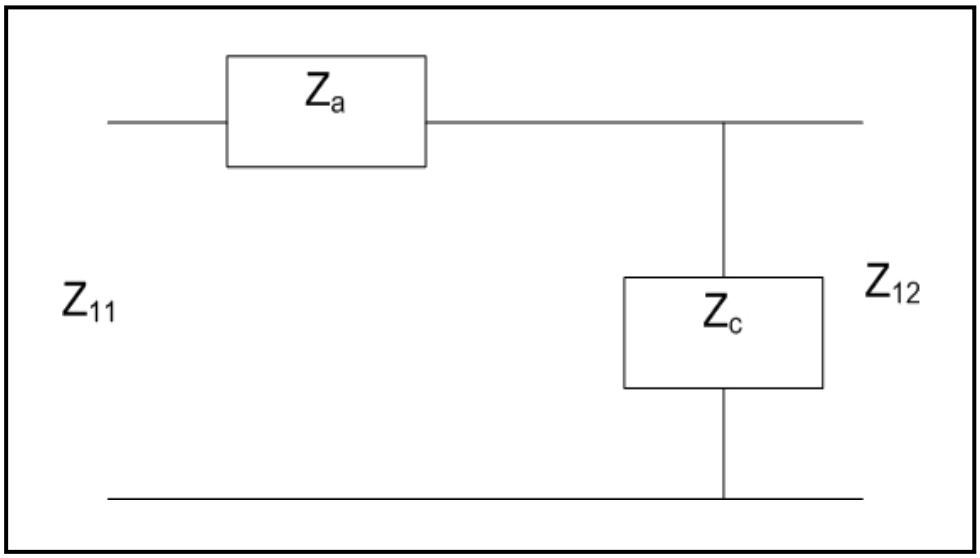


Figure 5: L-section Network [1].

2.2.1.1 Constant-*k* and Filter Sections

Constant-*k* section (figure 6) and *m*-derived section (figure 7) are designed from the image point of view. Below are the image properties of the dissipation-free sections. They are normalized so that their image impedance is R' = 1 at ω' = 0 and their cutoff frequency occurs at $\omega'_1 = 1$ radians/sec. Its image impedances are as follows:

$$Z_{11} = \sqrt{1-(\omega')^2}$$

and

$$Z_{12} = \frac{1}{Z_{11}}$$

Its propagation function is

$$\gamma = \alpha + j\beta = 0 + j\sin^{-1}(\omega')$$

for the $0 \leq \omega' \leq 1$ pass-band, and

$$\gamma = \alpha + j\beta = \cosh^{-1}(\omega') + j\frac{\pi}{2}$$

for the $1 \leq \omega \leq \infty$ stop-band, where α is in nepers and β is in radians [1].

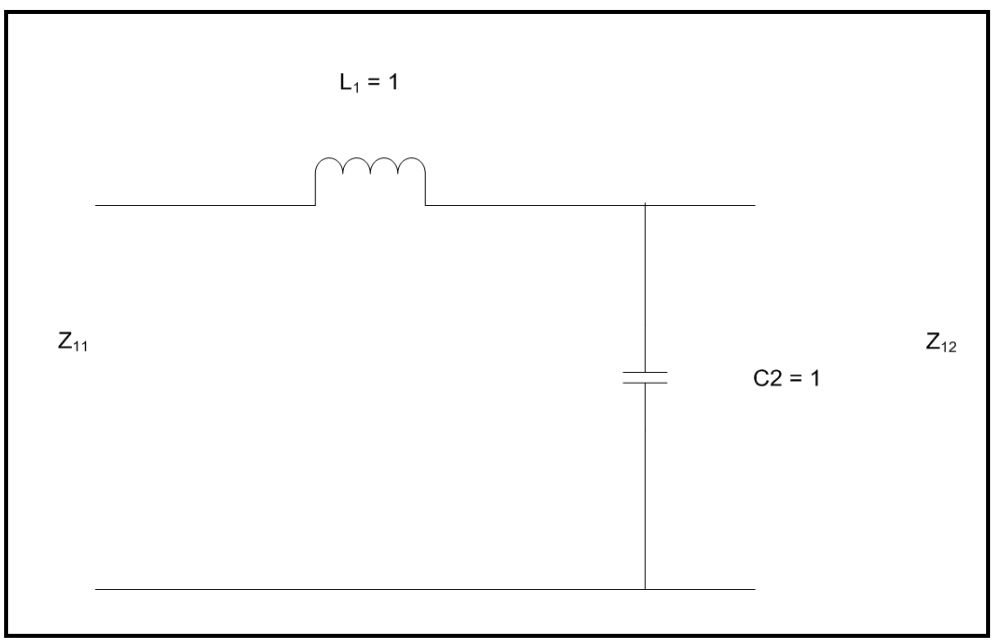


Figure 6: Constant-*k* half Section [1].

2.2.1.2 *m*-Derived Filter Sections

The *m-derived* filter section is a modification of the constant-*k* section designed to overcome the slow attenuation rate past cutoff and a non-constant image impedance found in the constant-*k* section. Its image impedances are

$$Z_{11} = \sqrt{1-(\omega')^2}$$

and

$$Z_{12} = \frac{1-\left(\dfrac{\omega'}{\omega'\infty}\right)^2}{\sqrt{1-(\omega')^2}}$$

where $\omega'\infty = \dfrac{1}{\sqrt{1-m^2}}$

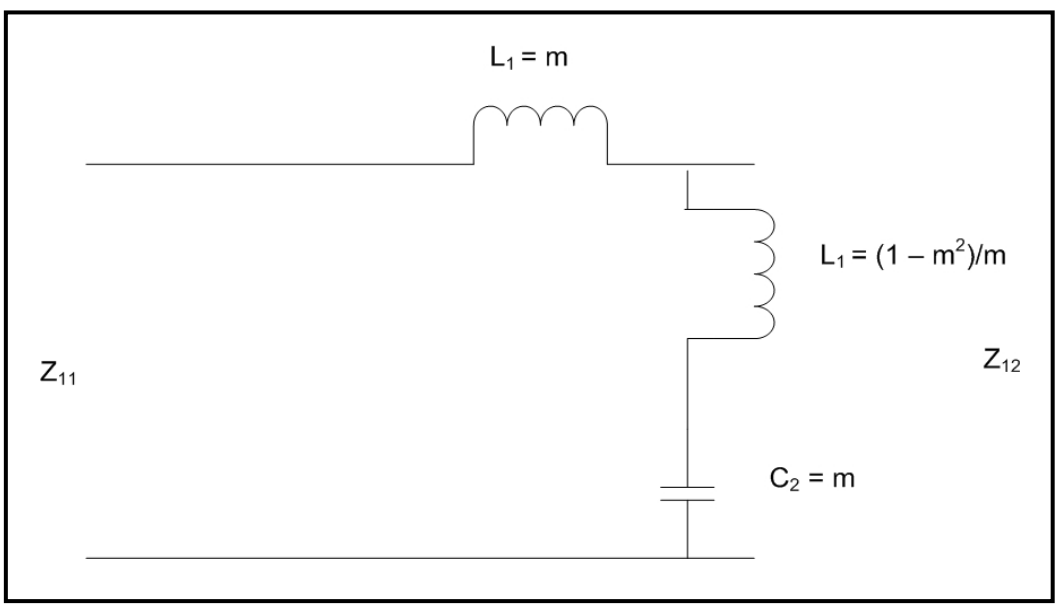


Figure 7: *m*-derived half Section [1].

The propagation constant is

$$\gamma = \alpha + j\beta = 0 + j\left(\frac{1}{2}\right)\cos^{-1}\left[1 - \frac{2m^2}{\left(\frac{1}{\omega'}\right)^2 - \left(1 - m^2\right)}\right]$$

in the $0 \leq \omega \leq 1$ pass-band, and

$$\gamma = \left(\frac{1}{2}\right)\cosh^{-1}\left[\frac{2m^2}{\left(\frac{1}{\omega'}\right)^2 - \left(1 - m^2\right)} - 1\right] + j\left(\frac{\pi}{2}\right)$$

in the $1 = \omega' \leq \omega'\infty$ stop-band, and

$$\gamma = \left(\frac{1}{2}\right)\cosh^{-1}\left[1 - \frac{2m^2}{\left(\frac{1}{\omega'}\right)^2 - \left(1 - m^2\right)}\right] + j\left(\frac{\pi}{2}\right)$$

in the $\omega'\infty \leq \omega' \leq \infty$ stop-band.

Constant-*k* and *m*-derived half sections can be connected together to form a filter. With sections chosen so that image impedances match at the junctions, the image attenuation and the image phase for the entire structure are simply the sum of the image attenuations and the phase values of the individual sections.

The resistive terminations to an image filter do not match its image impedance. Therefore, matching end sections are designed to improve the response of the filter. The *m*-derived half sections are used as the matching end sections. They improve the passband response of the filter and can further sharpen the cutoff characteristic of the filter.

The *m*-derived half sections reduce the reflections at the filter ends. On the other hand, they give no assurance as to how large the peak reflection loss values may be in the passband. Thus, though the image method is conceptually simple, it requires a great deal of trial and error if accurately defined band edges and low pass-band reflection loss are required [1].

2.2.2 Insertion Loss Method

The insertion loss method or the network synthesis method uses network synthesis techniques to design filters with a totally specified frequency response. Most of microwave filter designs performed with sophisticated computer aided design programs are based on the insertion loss method. The design is achieved by starting with low-pass filter prototypes which are normalized in terms of frequency and impedance. Then, transformations are applied to convert the prototype designs to the required impedance level and frequency range as well as the desired frequency response [2].

2.2.2.1 Characterization by Power Loss Ratio

The power loss ratio (P_{LR}) is defined as:

$$P_{LR} = \frac{\text{Power available from the source}}{\text{Power delivered to load}} = \frac{P_{inc}}{P_{load}} = \frac{1}{1-\left|\Gamma(\omega)\right|^2}$$

Where Γ is the reflection coefficient looking into the filter (assuming no loss in the filter). Design of a filter using the insertion-loss approach usually begins by designing a normalized low-pass prototype (LPP). The LPP is a low-pass filter with normalized source and load resistances of 1Ω and normalized cutoff frequency of 1 Radian/s. Figure 8 shows the characteristics of LPP. Impedance transformation and frequency scaling are then applied to denormalize the LPP and synthesize different type of filters with different cutoff frequencies.

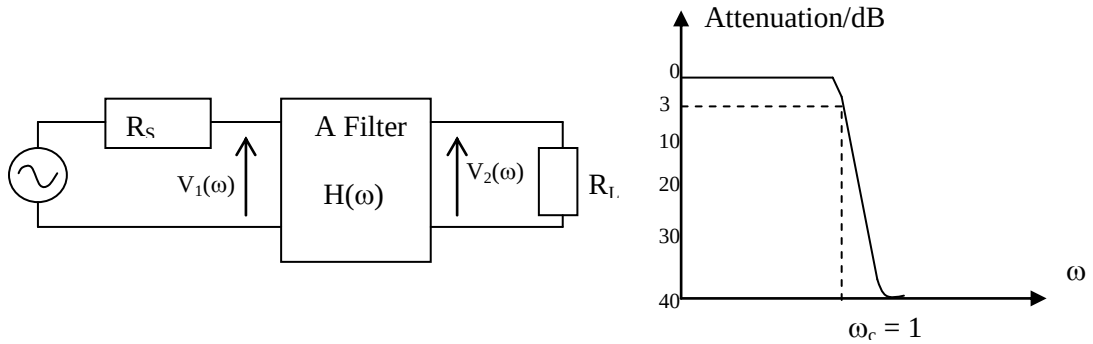


Figure 8: Normalized LPP Filter Network with Unity Cutoff Frequency (1 Radian/s).

Low-pass prototype (LPP) filters have the form shown in Figure 9 (An alternative network where the position of inductor and capacitor is interchanged). The network consists of reactive elements forming a ladder, usually known as a ladder network. There are two ladder circuits for low-pass filter prototypes. The first begins with a shunt element, and the second begins with a series element. For the ladder circuit beginning with a shunt element, the generator has series internal resistance while the generator has shunt internal conductance for a ladder circuit beginning with a series element. The two ladder circuits are used interchangeably with considerations to the final filter symmetry. Specifying the insertion loss at some frequency within the stop-band determines the size or order of the filter. The order of the network corresponds to the number of reactive elements. Impedance transformation and frequency scaling are then applied to transform the network to non-unity cutoff frequency, non-unity source/load resistance and to other types of filters such as high-pass, band-pass or band-stop.

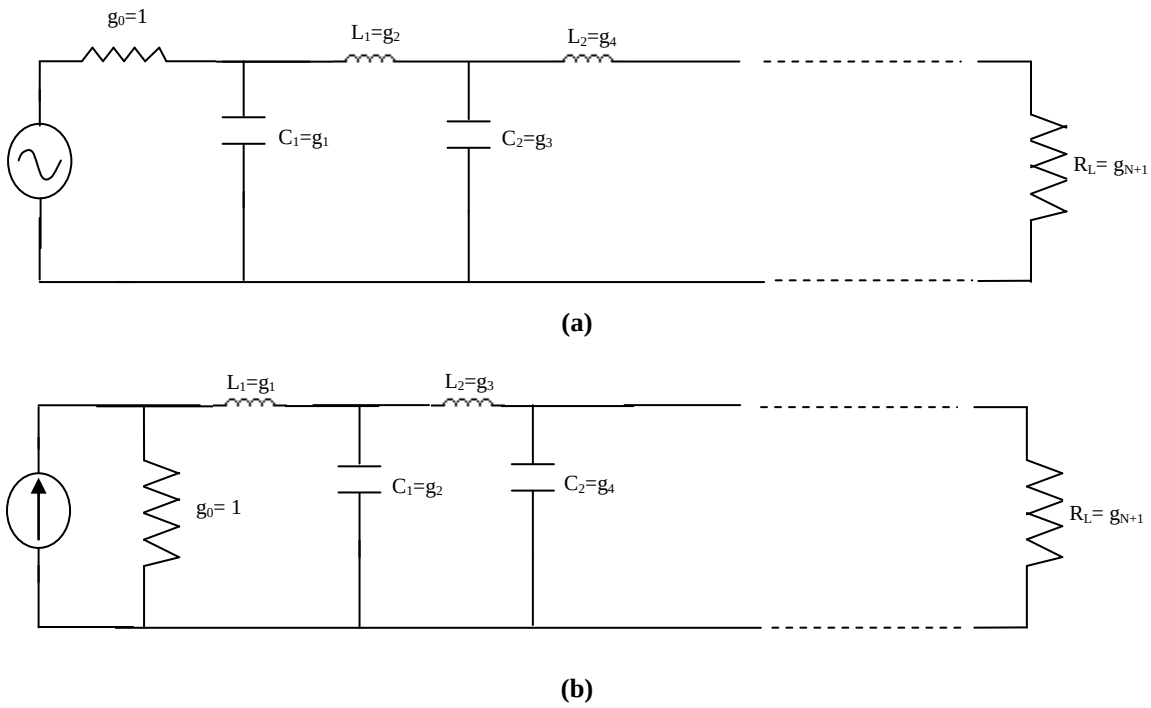


Figure 9: Ladder Circuits for Low-Pass Filter Prototypes and their Element Definitions. (a) Prototype Beginning with a Shunt Element. (b) Prototype Beginning with a Series Element

There are standard approaches to design a normalized LPP of Figure 9 that approximate an ideal low-pass filter response with cutoff frequency of unity. These standard approaches are as the following:

- Maximally flat or Butterworth Response.
- Equal ripple or Chebyshev Response.
- Linear Phase Response.

There are tables to design filters from the low-pass filter prototypes for the above mentioned responses. The tables are derived from the equations for the equivalent filter responses. These tables list the values of the reactive elements of the filter and the

generator input impedance as well as the load impedance. The element values are given for filter orders from one to ten. In these tables, the impedances and the frequency of operation are normalized. So, filter prototypes need to be converted to usable impedance level and cutoff frequency by scaling. These tables namely table 4, 5, 6 and 7 are shown in the Appendix E.

The required size or order (N) of the filter to fit a specification on the attenuation or insertion loss at some frequency in the stopband of the filter can be determined with help of graphs called "Attenuation Versus Normalized Frequency" found in the Appendix F. The definition of normalized frequency is as follows:

$$f_{norm} = \left| \frac{\omega}{\omega_c} \right| - 1$$

These graphs are for maximally flat filter prototype (Figure 40) as well as equal-ripple filter prototype (Figure 41 & 42). The graphs show the attenuation characteristics for various N up to 10, versus normalized frequency. If a filter with N > 10 is needed, such a result can usually be obtained by cascading two designs of lower order.

The insertion loss method allows a high degree of control over the passband and stopband amplitude and phase characteristics with a systematic way to synthesize a desired response. The necessary design tradeoffs can be evaluated to best meet the application requirements. For example, if a minimum insertion loss is most important, a binomial response could be used; a chebyshev response would satisfy a requirement for the sharpest cutoff. If it is possible to sacrifice the attenuation rate, a better phase

response can be obtained by using a linear phase filter design. Generally in all cases, the insertion loss method allows filter performance to be improved in a straightforward manner at the expense of a higher order filter.

2.2.2.1.1 Maximally flat or Butterworth Response

Maximally flat filter response is also called Butterworth response or Binomial response as well. It has the flattest amplitude response in the pass band. For the low-pass filter, it is defined by

$$P_{LR} = 1 + k^2 \left(\frac{\omega}{\omega_c} \right)^{2N}$$

where N is the order of the filter which is equal to the number of reactive elements and ω_c is the cutoff frequency. The passband extend from $\omega=0$ to $\omega=\omega_c$ and at the band edge the power loss ratio is $1 + k^2$. For $\omega \gg \omega_c$, insertion loss increases at the rate of 20N dB per decade increase in frequency [2].

2.2.2.1.2 Equal ripple or Chebyshev Response

Equal-Ripple or Chebyshev Filter Response is defined by

$$P_{LR} = 1 + k^2 T^2{}_N \left(\frac{\omega}{\omega_c} \right)$$

where $T^2{}_N(x)$ is a Chebyshev polynomial of order N. The pass-band response has ripples of amplitude $1 + k^2$ while the stop-band insertion loss increases at the rate of 20N dB per decade in frequency [2].

2.2.2.1.3 Linear Phase Response

The above filters specify the amplitude response, however in some applications such as multiplexing filters for communications systems it is important to have a linear phase response in the passband to avoid signal distortion. Linear phase response filter specifies the phase of the filter. A linear phase characteristic has the following response

$$\phi(\omega) = A_{\omega}\left[1 + p\left(\frac{\omega}{\omega_c}\right)^{2N}\right]$$

where $\phi(\omega)$ is the phase of the voltage transfer function and p is a constant. Group delay which is the derivative of the phase characteristic shows that the group delay for a linear phase filter is a maximally flat function [2].

2.2.2.2 Impedance and Frequency Scaling

The impedance and frequency scaling are known as Filter Transformation process because it transforms the low pass prototype to any other type of filter. The prototype supports load impedance of 1Ω and cutoff frequency of 1 radian/second.

2.2.2.2.1 Transformation to Low Pass Filter

The low-pass prototype filter can be converted into a low-pass filter which meets arbitrary cutoff frequency and impedance level specification using frequency scaling and impedance transform. For a new load impedance of R_o and cutoff frequency of ω_c, the

normalized component values for the prototype (resistance R_n , inductance L_n and capacitance C_n) are changed by the followings [5]:

$$R = R_o R_n$$

$$L = R_o \frac{L_n}{\omega_c}$$

$$C = \frac{C_n}{R_o \omega_c}$$

where *R*, *L* and *C* are the new filter component values after frequency scaling and impedance transform processes. The transformation as shown in the three aforementioned equations implies that the schematic does not need to be changed; only the element values are scaled down or up to reflect the new specifications.

2.2.2.2.2 Transformation to High Pass Filter

The frequency substitution where

$$\omega \longleftarrow -\frac{\omega_c}{\omega} \qquad (2.1)$$

can be used to convert a low pass response to a high pass response. This substitution maps ω=0 to ω= $\pm\infty$ and vice versa. The cutoff takes place when ω= $\pm\omega_c$. The negative sign is required to convert inductors and capacitors to realizable capacitors and inductors. Applying (2.1) to the series reactances, jωL_k, and the shunt susceptances, jωC_k, of the prototype filter gives

$$jX_k = -j\frac{\omega_c}{\omega}L_k = \frac{1}{j\omega C'_k}$$

$$jB_k = -j\frac{\omega_c}{\omega}C_k = \frac{1}{j\omega L'_k}$$

which shows that series inductors L_k should be replaced with capacitors C'_k and shunt capacitors C_k should be replaced with inductors L'_k. The new component values of the high pass filter without impedance scaling will be as follows:

$$C'_k = \frac{1}{\omega_C L_k}$$

$$L'_k = \frac{1}{\omega_C C_k}$$

The final values of the components of the high pass filter including impedance scaling will be as follows:

$$C'_k = \frac{1}{R_0 \omega_C L_k}$$

$$L'_k = \frac{R_0}{\omega_C C_k}$$

2.2.2.2.3 Transformation to Band Pass Filter

The band pass response can be obtained from the low pass prototype. Assume ω_1 and ω_2 denote the edges of the passband, then a bandpass response can be obtained using the following frequency substitution:

$$\omega \longleftarrow \frac{1}{\Delta}\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right) \qquad (2.2)$$

where

$$\Delta = \frac{\omega_2 - \omega_1}{\omega_0} \qquad (2.3)$$

is the fractional bandwidth of the passband. The center frequency, ω_o, is the geometric mean of ω_1 and ω_2 :

$$\omega_0 = \sqrt{\omega_2 \omega_1} \qquad (2.4)$$

then the transformation of (2.2) maps the bandpass characteristics to the low pass response as follows:

when $\omega = \omega_0$, then

$$\frac{1}{\Delta}\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right) = 0$$

when $\omega = \omega_1$, then

$$\frac{1}{\Delta}\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right) = -1$$

when $\omega = \omega_2$, then

$$\frac{1}{\Delta}\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right) = 1$$

The new filter elements are determined by using (2.2) in the expression for the series reactance and shunt susceptances. Therefore

$$jX_k = \frac{j}{\Delta}\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)L_k = j\frac{\omega}{\Delta\omega_0}L_k - j\frac{\omega_0}{\Delta\omega}L_k = j\omega L'_k - j\frac{1}{\omega C'_k}$$

which shows that a series inductor, L_k, is transformed to a series LC circuit with element values,

$$L'_k = \frac{L_k}{\Delta\omega_0}$$

$$C'_k = \frac{\Delta}{L_k\omega_0}$$

Similarly,

$$jB_k = \frac{j}{\Delta}\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)C_k = j\frac{\omega}{\Delta\omega_0}C_k - j\frac{\omega_0}{\Delta\omega}C_k = j\omega C'_k - j\frac{1}{\omega L'_k}$$

which shows that a shunt capacitor, C_k, is transformed to a shunt LC circuit with element values,

$$L'_k = \frac{\Delta}{C_k\omega_0}$$

$$C'_k = \frac{C_k}{\Delta\omega_0}$$

Therefore, the low pass filter elements are converted to series resonant circuits in the series arms and to parallel resonant circuits in the shunt arms with resonant frequency equals to ω_o.

2.2.2.2.4 Transformation to Band Stop Filter

The following transformation, which is the inverse of the transformation of the low pass filter to bandpass filter, can be used to get a band stop response:

$$\omega \longleftarrow \Delta\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)^{-1} \qquad (2.5)$$

where Δ and ω_o have the same definitions as in (2.3) and (2.4). Then series inductors of the low pass prototype are converting to parallel LC circuits having element values as follows:

$$L'_k = \frac{\Delta L_k}{\omega_0}$$

$$C'_k = \frac{1}{\Delta L_k \omega_0}$$

The shunt capacitor of the low pass prototype is converted to series LC circuits having element values obtained by the following:

$$L'_k = \frac{1}{\Delta C_k \omega_0}$$

$$C'_k = \frac{\Delta C_k}{\omega_0}$$

The element transformations from a low pass prototype to a high pass, bandpass or band stop filter are summarized in table 1 which is showing the results without impedance scaling.

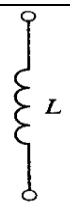
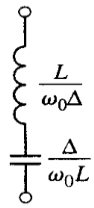
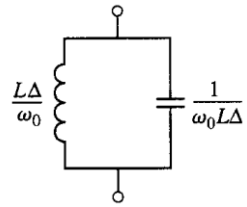
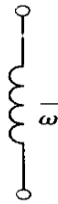
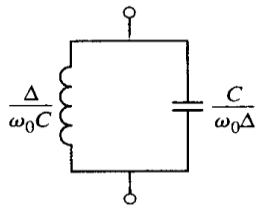
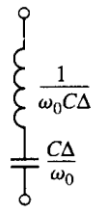
Low Pass	High Pass	Band Pass	Band Stop
			
			

Table 1: Summary of Frequency Scaling

The summary of the whole transformation process from the low pass prototype to any desired type of filter including both the impedance scaling as well as the frequency scaling is shown in table 2.

Both of the above-mentioned filter design methods provide lumped–element circuits. For microwave applications such designs usually should be modified to use distributed elements consisting of transmission line sections. This transition from lumped–element circuits to distributed elements is done by using the Richard's transformation and the Kuroda identities [2].

Low-Pass	
Simple impedance and frequency scaling:	$L_k^{'} = \frac{1}{w_c} R_0 L_k$ $C_k^{'} = \frac{1}{w_c} \frac{C_k}{R_0}$
High-Pass	
Note the swapping of capacitors and inductors:	$C_k^{'} = \frac{1}{w_c} \frac{1}{R_0 L_k}$ $L_k^{'} = \frac{1}{w_c} \frac{R_0}{C_k}$
Band Pass	
Series inductor L_k transforms to a series LC circuit with:	$L_{Lk}^{'} = \frac{1}{w_o} \frac{R_0 L_k}{\Delta}$ $C_{Lk}^{'} = \frac{1}{w_o} \frac{\Delta}{R_0 L_k}$
Shunt capacitor C_k transforms to a parallel LC circuit with:	$L_{Ck}^{'} = \frac{1}{w_o} \frac{R_0 \Delta}{C_k}$ $C_{Ck}^{'} = \frac{1}{w_o} \frac{C_k}{R_0 \Delta}$
Band Stop	
Series inductor L_k transforms to a parallel LC circuit with:	$L_{Lk}^{'} = \frac{1}{w_o} \Delta R_0 L_k$ $C_{Lk}^{'} = \frac{1}{w_o} \frac{1}{\Delta R_0 L_k}$
Shunt capacitor C_k transforms to a series LC circuit with:	$L_{Ck}^{'} = \frac{1}{w_o} \frac{R_0}{\Delta C_k}$ $C_{Ck}^{'} = \frac{1}{w_o} \frac{\Delta C_k}{R_0}$

Table 2: Filter Transformation Process

2.3 Filter Implementation

Microwave filter implementation is a filter realization process which includes the transforming of the lumped elements values produced from the design process to distributed elements values. The filter implementation is originally materialized by two methods. These two methods are Stubs method and Stepped-impedance line method.

2.3.1 Stubs Method

In this method, the Richard's transformation and the Kuroda identities will be utilized. Richard's Transformation and Kuroda's Identities focus on uses of $\lambda/8$ lines. Richard's idea is to use variable Zo (for example: width of microstrip) to create lumped elements from transmission lines. For a short circuit lossless transmission line, the input impedance is given by

$$Z_{in} = j\, Z_o \tan \beta l$$

where $\beta = 2\pi/\lambda$ and l is the length of the transmission line. If $l = \lambda/8$, then

$$Z_{in} = j\, Z_o \tan (2\pi/\lambda)\, (\lambda/8) = j\, Z_o \tan (\pi/4) = j\, Z_o$$

which is an inductive value. Further, if the line is open circuit, then

$$Z_{in} = -\, j\, Z_o \cot \beta l$$

and for $l = \lambda/8$

$$Z_{in} = -j\, Z_o \cot(2\pi/\lambda)(\lambda/8) = -j\, Z_o \cot(\pi/4) = -j\, Z_o$$

which is a capacitive value. Therefore, the inductor can be replaced with a short-circuited stub of characteristic impedance L while the capacitor can be replaced with an open-circuited stub of characteristic impedance 1/C (see figure 10).

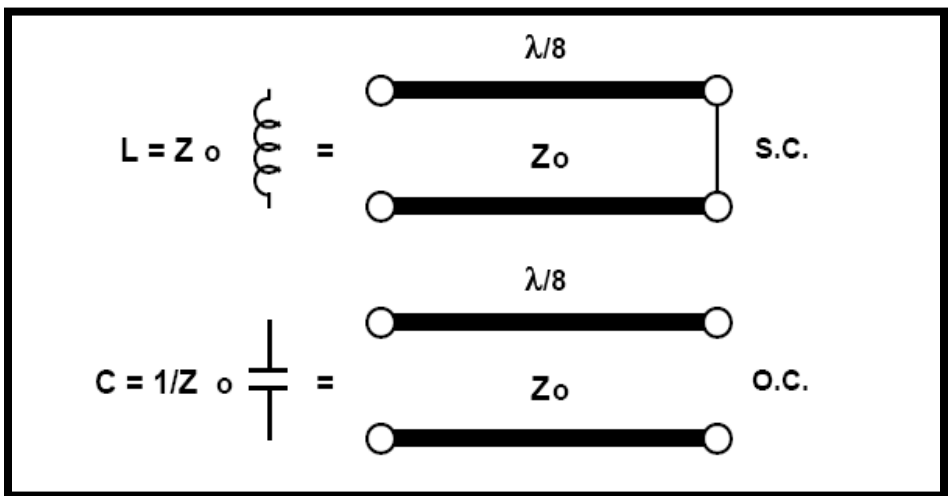


Figure 10: Richard's Transformation

Kuroda's idea is about using the λ/8 line of appropriate Zo to transform awkward or unrealizable elements to those with more tractable values and also to achieve more practical microwave filter implementation by performing physical separation of transmission line stubs, transformation of series stubs into shunt stubs or changing impractical characteristic impedances into more realizable ones. Kuroda’s Identities can be very useful in making the implementation of Richard’s transformations more

practicable. Kuroda's Identities essentially provide a list of equivalent two-port networks as shown in Table 3. By equivalent, we mean that they have precisely the same scattering, impedance, admittance and transmission parameters. In other words, we can replace one two-port network with its equivalent in a circuit and the behavior and characteristics of the circuit will not change.

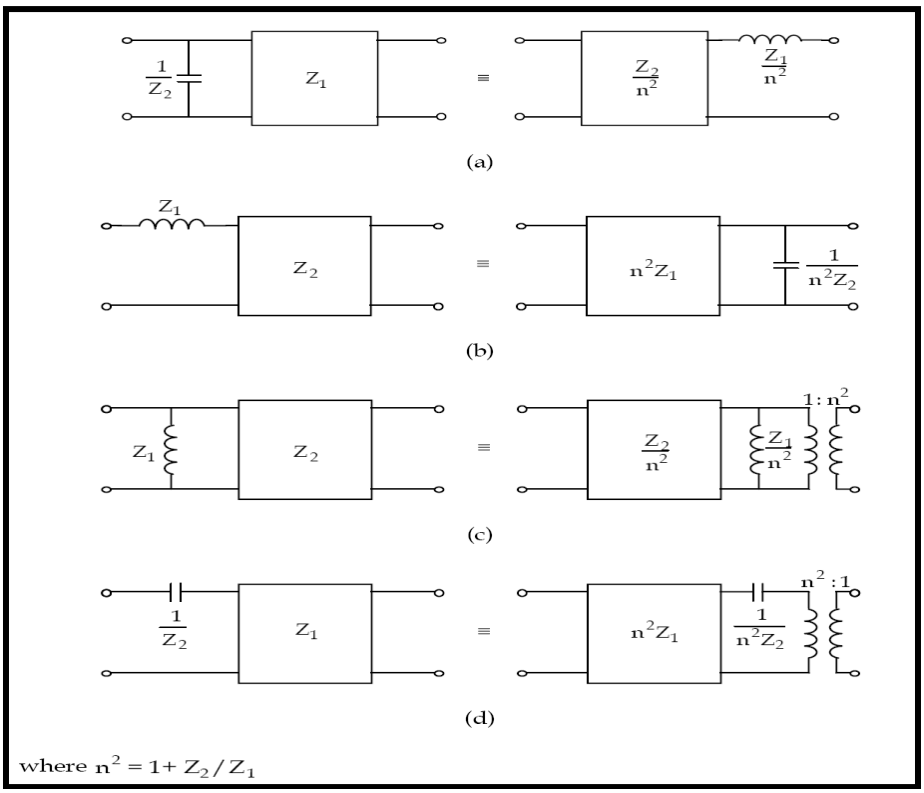


Table 3: Kuroda Identities [2]

In microwave filter implementation process, it is very difficult to implement series stubs. Therefore, Kuroda Identity no. *b* is used to convert the series stubs to shunt stubs and here comes the real importance of Kuroda Identities. So, the completed filter in microstrip form looks like the one shown in figure 11.

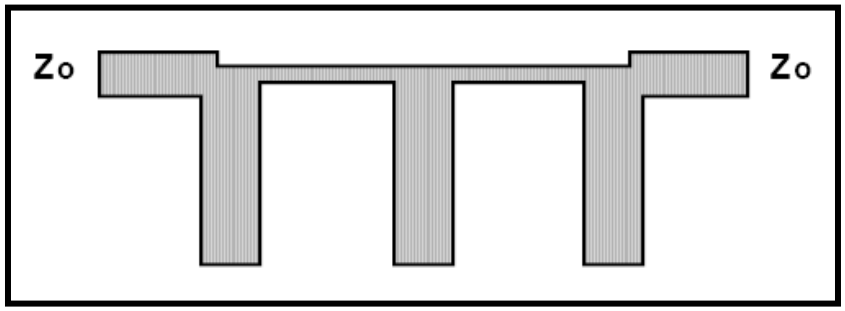


Figure 11: Microstrip Fabrication of Microwave Filter by Using Stubs Method

The advantage of this method is that it has a sharp cutoff. However, the main drawback of this method is that the filter size is quite big and takes a large space as a component of microwave system [2].

2.3.2 Stepped-Impedance Method

In this method, successive sections of transmission lines are used to represent inductor (L) or capacitor (C) by using alternating sections of very high and very low characteristic impedance lines. The line section with the high impedance (narrow width) represents a series inductor while the one with the low impedance (wide width) represents a shunt capacitor.

In order to use this method, we need to know the highest and lowest feasible transmission line impedances, Z_{max} and Z_{min}. Then the length of each transmission line element will be as follows (see figure 12):

$$\beta\ell = \frac{LR_0}{Z_{\max}}$$

for inductive elements

$$\beta\ell = \frac{CZ_{\min}}{R_0}$$

for capacitive elements

where R_o is the filter impedance and L and C are the normalized element values of the low-pass prototype.

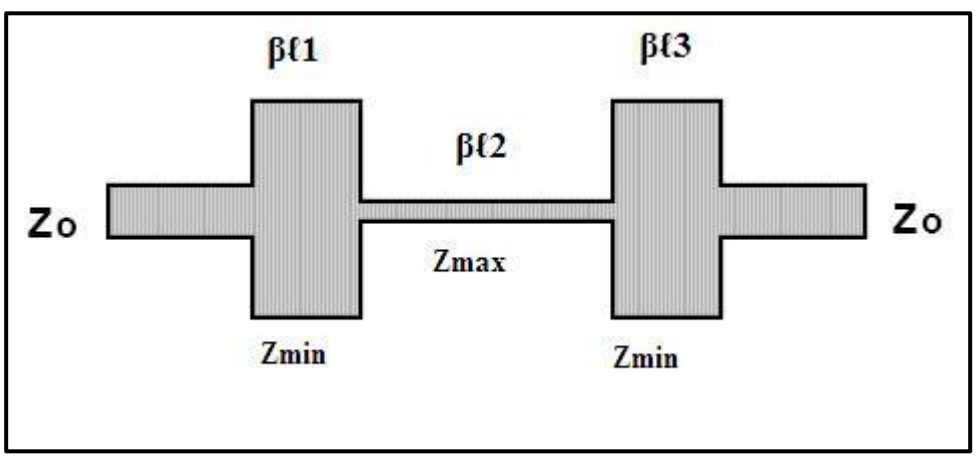


Figure 12: Microstrip Fabrication of Microwave Filter by Stepped-Impedance Method

The advantage of this method is that it is easier to implement than the stubs method and takes up less space. However, its main drawback is the high reflected wave due to large impedance discontinuities. Also, there is another drawback which is related to the cutoff

since it does not have a sharp cutoff, the use of such filters is usually limited to applications where a sharp cutoff is not required [2].

2.4 Present Approach

There are several papers which discuss the analysis of CVTL, the design of filter by scattering parameters and the application of optimization methods during the design stage. However, the ones that are related to the topic of my thesis are briefly highlighted below:

In [6], the Genetic Algorithm (GA) has been applied for designing multi-layered microwave filters that incorporates combinations of different dielectrics. The GA simultaneously chooses, optimally, the material in each layer as well as its thickness. The result is a multi-layer composite that provides a maximum absorption of both TE and TM waves for a prescribed range of frequencies and incident angles. The population size was 100 and the number of generations that has been executed was 1000. The selection of the parents was based on the tournament selection type. This selection was preferred because it converges more quickly and also because it has a faster execution time than many other competing schemes. There are many variations of the crossover operator and the uniform crossover was used. The value of the probability of the mutation was kept low and chosen to be in the range between 0.01 and 0.1.

In [7], a planar filter was implemented by CVTL microstrip and the design was achieved by using inverse scattering method. That means, the design was by means of applying the Fourier transform to the reflection coefficient. Then, a numerical method was applied to solve the design equation which contains an integral term that can be evaluated numerically.

More recently, a design procedure is suggested in [4] where a continuously varying transmission line solution for the voltage and current distribution along the line is obtained in the form of a power series. The transmission line is divided into *n*-sections and the S-parameters of each section is calculated in terms of its local characteristics impedance. The overall S-parameters are then optimized towards the required values of LPF. This technique will be adapted in this thesis. However, to avoid the excessive number of sections used in the design reported in [4] and the associated extensive numerical work, a new procedure is proposed.

In this thesis, the whole transmission line section is treated as one unit and the overall solution is assumed as a complete fourier series expansions with unknown coefficients. The coefficients are obtained in terms of the assumed impedance profile. Finally, the S-parameters are obtained in terms of the above impedances which are then optimized to achieve the required S-parameters profiles. Details of the analysis are given in chapter 4.

CHAPTER 3

INTRODUCTION TO GENETIC ALGORITHM

3.1 Introduction

Genetic Algorithms (GA) were first presented by J. H. Holland in his book “Natural and Artificial Systems” in the year 1975 and developed further by his students. GA is a problem solving strategy that uses stochastic search. GA is mainly designed to solve optimization problems. It is very robust, efficient and powerful tool that it can be applied to a wide range of problem areas such as Acoustics, Aerospace engineering, Astronomy and astrophysics, Chemistry, Electrical engineering, Financial markets, Geophysics, Mathematics and algorithmics, Military and law enforcement, Molecular biology, Pattern recognition and data mining, Robotics, Routing and scheduling and Systems engineering. It also has good performance when solving some difficult problems where no existing specialized techniques can perform well [8].

GA is a class of algorithms inspired by evolution. In GA, each possible solution is coded using a data structure known as a chromosome. The chromosome is composed of a string of genes, each gene representing a specific input variable. Collectively, the genes are used to evaluate the fitness of an individual solution. Then, proportional to their specific fitness values, chromosomes are allowed to reproduce using the genetic operators of crossover and mutation. So, the fittest chromosomes are more likely to reproduce and contribute to the next generation of chromosomes whereas less fit individuals eventually become lost to future generations. Over successive iterations, the average fitness of the whole population improves and the genetic algorithm can be expected to breed an optimal solution to the problem [9].

In such situations, the problem is viewed as a landscape upon which possible solutions may be found. As seen in Figure 13, an individual solution is located and represented by its coordinates. For optimization problems, where one is looking for the highest hill or the lowest valley, it can be seen that some individual solutions are better located than others.

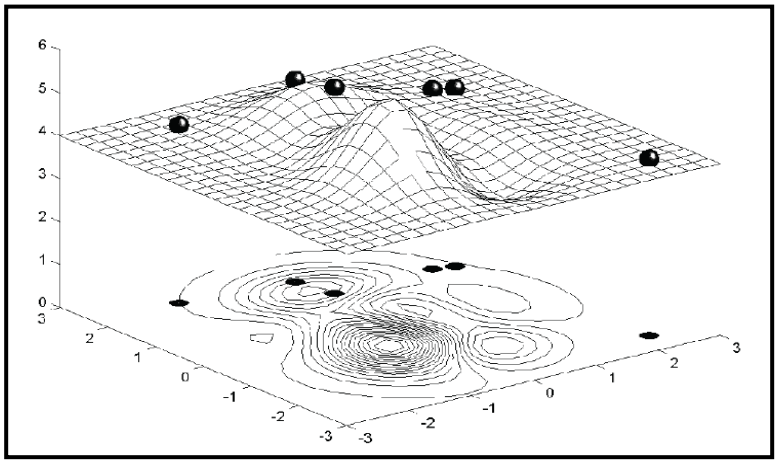


Figure 13: Landscape of the mathematical equation
$f(x, y) = (1 - x)^2 e^{-x2-(y+1)2} - (x - x^3 - y^3) e - x^2 - y^2$ **and six possible solutions [11]**

Borrowing heavily from the language of biological evolution, these solutions or individuals are said to be the fittest in a population of competing solutions. Then over successive iterations, the GA evolves the population of competing solutions until it becomes a population of consistently very fit individuals, as seen in Figure 14 [11].

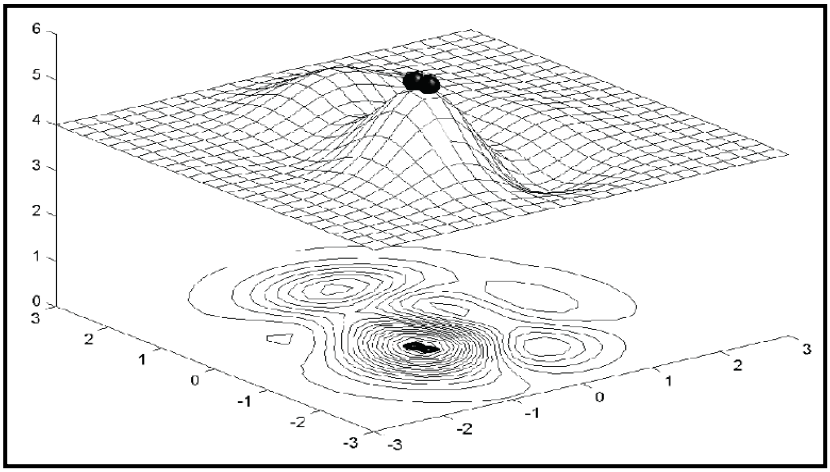


Figure 14: Landscape of the mathematical equation
$f(x, y) = (1 - x)^2 e^{-x2-(y+1)2} - (x - x^3 - y^3) e - x^2 - y^2$ **and six 'fitter' solutions [11]**

3.2 Initialization

GA operates with a set of strings instead of a single string. This set or population of strings goes through the process of evolution to produce new individual strings. To begin with, the initial population could be seeded with heuristically chosen strings or at random. In either case, the initial population should contain a wide variety of structures [12].

3.3 Evaluation Function

The evaluation function is a procedure to determine the fitness of each string in the population. Since GA proceeds in the direction of evolving better fit strings and the fitness value is the only information available to GA, the performance of the algorithm is highly sensitive to the fitness values. In case of optimization routines, the fitness is the value of the objective function to be optimized. GA is basically unconstrained search procedures in the given problem domain. Any type of constraints associated with the problem could be incorporated into the objective function as penalty function [8].

3.4 Parent Selection Techniques

When breeding new chromosomes, we need to decide which chromosomes to use as parents. The selected parents must be the fittest individuals from the population but we also want sometimes to select less fit individuals so that more of the search space is

explored and to increase the chance of producing promising offspring. There are many selection techniques such as Roulette Wheel Selection, Tournament Selection, Windowing Selection and Fitness ranking Selection [13].

3.5 Genetic Operators

3.5.1 Crossover

Crossover entails choosing two individuals to swap segments of their code, producing artificial "offspring" that are combinations of their parents. This process is intended to simulate the analogous process of recombination that occurs to chromosomes during sexual reproduction. Common forms of crossover include single-point crossover, in which a point of exchange is set at a random location in the two individuals' genes, and one individual contributes all its code from before that point and the other contributes all its code from after that point to produce an offspring, and uniform crossover, in which the value at any given location in the offspring's gene is either the value of one parent's gene at that location or the value of the other parent's gene at that location, chosen with 50/50 probability [13].

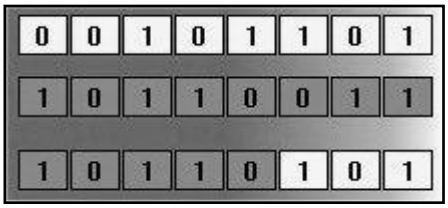


Figure 15: Single-point Crossover Operator

The above diagram (Figure 15) illustrates the effect of crossover operator on individuals in a population of 8-bit strings. It shows two individuals undergoing single-point crossover, the point of exchange is set between the fifth and sixth positions in the genome, producing a new individual that is a hybrid of its progenitors [13].

3.5.2 Mutation

Mutation is the process of random modification of the value of a string with small probability. So, mutation in a genetic algorithm causes small alterations at single points in an individual's code. It is not a primary operator but it ensures that the probability of searching any region in the problem space is never zero and prevents complete loss of genetic material through reproduction and crossover.

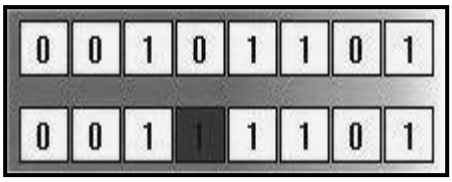


Figure 16: Mutation Operator

The above diagram (Figure 16) illustrates the effect of mutation operator on individuals in a population of 8-bit strings. It shows an individual undergoing mutation at position 4, changing the 0 at that position in its genome to a 1 [13].

3.5.3 Elitism

It is sometimes the case that a good solution is found in the early stages of the optimization process by GA but gets deleted from the population as the GA progresses. One solution is to “remember” the best solution found so far. Alternatively, a technique called elitism can be used. This technique ensures that the best members of the population are carried forward from one generation to the next [8].

3.6 Genetic Parameters

3.6.1 Population size

Population size affects the efficiency of the algorithm. If we have smaller population, it would only cover a small search space and may results in poor performance. A larger population would cover more space and prevent premature convergence to local solutions. At the same time, a large population needs more evaluation per generations and may slow down the convergence rate. As a general observation of previous works on GA, increase in complexity of the algorithm leads to a need for larger population size [14].

3.6.2 Probability of Crossover

Probability of crossover or crossover rate is the parameter that affects the rate at which the crossover operator is applied. A higher crossover rate introduces new strings more quickly into the population. For uniform crossover, a higher probability of contributing ones parents allele lowers the rate of disruption. If the crossover rate is too high, high performance strings are eliminated faster. A low crossover rate may cause stagnation due to the lower exploration rate [14].

3.6.3 Probability of Mutation

Probability of mutation or mutation rate is the probability with each bit position of each string in the new population undergoes a random change after a selection process. A low mutation rate helps to prevent any bit positions from getting stuck to a single value [14].

3.7 Genetic Algorithm in The Present Work

In this thesis, GA is going to be utilized as the optimization technique to optimize the expressions of $|S_{11}|$ & $|S_{21}|$ to satisfy the required response of the desired type of filter. Here, the expressions of $|S_{11}|$ & $|S_{21}|$ will be in terms of specific parameters pertaining to

the form of the variation of the characteristic impedance along the line. These expressions of $|S_{11}|$ & $|S_{21}|$ are forced to vary with the frequency in a prescribed manner according to the type of filter we require. The GA is utilized to minimize simultaneously the differences between the $|S_{11}|$, $|S_{21}|$ and their target functions in the least mean square sense.

CHAPTER 4

DESIGN OF CONTINUOUSLY VARYING TRANSMISSION LINE FILTER

4.1 Review of Present Practice in Filter Theory

All the design approaches including the aforementioned methods are generally composed of a sequence of two steps: first a synthesis problem is solved using techniques usually borrowed from lumped-element network synthesis, and then a suitable equivalence between the synthesized network and the actual distributed structure to be realized is established. In the past, the second step was generally carried out by means of simple approximated circuit modeling of the real filter structure, and often some experimental cut-and-try work was necessary to obtain the desired filter response. Today, the trend is to take advantage of optimization techniques and make use of the available full-wave electromagnetic simulators, which can now analyze the complete physical structure of many filters [15].

Recently, an effective method to design microwave filters is based on analyzing and optimizing the Continuously Varying Transmission Lines (CVTL) to achieve the desired frequency response. By using CVTL, there are no sharp impedance discontinuities and this is the main advantage of using CVTL in planar filter design [4].

In [4], a new continuously varying transmission line design method to design planar microwave filters is presented. The construction of the filter is by fabricating a CVTL which is a non uniform microstrip and the design was carried out by investigating and analyzing the scattering parameters of CVTL in detail in frequency domain. The transmission line is divided into *n* sections. The variation of the characteristic impedance of each section has been chosen to be a cubic polynomial. The design was accomplished by analyzing S_{11} and S_{21} because their magnitudes ($|S_{11}|$ & $|S_{21}|$) represent the return loss and insertion loss respectively.

After finding the mathematical expressions of S_{11} and S_{21} in terms of the frequency and the coefficients of the characteristic impedance, then, the optimization algorithm which is least-square criteria was applied to optimize the values of S_{11} and S_{21} to get the required frequency response of the filter. A low pass filter and stopband filter were designed and tested by comparing the simulated and theoretical results. There was a good conformity between both results which reflects the validity of the method used to analyze and optimize CVTL and filters.

4.2 Methodology Approach

The achieved work is basically designing all types of microwave filters by using CVTL microstrip with help of optimization technique namely GENETIC ALGORITHM. This is going to be achieved by finding and analyzing the scattering parameters of CVTL. Then, optimize them to get the desired frequency response of the filter. The approach used here is different from the approach of [4]. In [4], a cubic spline function is used per section along the length which generates a very lengthy and quite large numerical structure to be optimized. Our approach is based on assuming the voltage, current and impedance along the line as a finite fourier series expansion with unknown coefficients and then optimize these coefficients to obtain an acceptable solution. This is detailed in the followings:

The differential equation governing the voltage V(x) along the line with its characteristic impedance Z(x) is given by [4]:

$$\frac{d^2V}{dx^2} - \frac{1}{Z(x)} \cdot \frac{dZ}{dx} \cdot \frac{dV}{dx} + L_0 C_0 \omega^2 V(x) = 0 \qquad (1)$$

where ω is the operating frequency in radian/second. L_o and C_o are constants which their product is equal to 1 over the square of the propagation velocity c ($L_oC_o = 1/c^2$) where $c = 3 \times 10^8$ m/s.

Let V(x) & Z(x) be represented by their Fourier expansions in space along the line as follows:

$$V(x) = p\,V_0 + \sum_{l=1}^{n} \left(p\,V_l \cos l\alpha x + q\,V_{l+n} \sin l\alpha x \right)$$

where p and q are constants to be determined from the line terminations, $\alpha = \frac{2\pi b}{LF}$, LF is the physical length of the filter and b is an optimization parameter. Further, let

$$Z(x) = Z_C + \sum_{j=1}^{n} \left(Z_j \cos j\alpha x + Z_{j+n} \sin j\alpha x \right)$$

The general procedure of the methodology will be consisting of the following steps:

1. The expressions of V(x) & Z(x), their first derivatives and the second derivative of V(x) will be substituted in (1).
2. One may assume V_0 & V_{n+1} equal 1 without loss of generality.
3. Obtain the values of all $V_i(x)$ and $V_{i+n}(x)$ in terms of $Z_j(x)$ where i & j = 1, 2, ….n.
4. Then, we have a system of a certain number of unknowns (V_1,V_2, V_3 etc ..) with the same number of equations in terms of the Zs (coefficients of Z(x)) as well as LF, b, L_0C_0, ω^2, p and q. In case of five terms expansion, the matrix size is 9 X 9 while in case of eight terms expansion, the matrix size is 15 X 15.
5. Assuming the $Z_j(x)$ are known, this system will be solved by means of MATLAB to get the values of the V's.

6. After long manipulations and calculations, the expressions of S_{11} and S_{21} will be obtained in terms of Zs, LF, *b*, L_0C_0, ω^2, p and q. The ratio (**p/q**) shall be determined according to the line termination which is assumed to be matched. The values of Zs, LF and *b* are going to be determined by Genetic Algorithm.
7. If the values of S_{11} and S_{21} are acceptable, the problem is finished. If not, one assume new set for the impedance parameters Z_j and repeat the problem. The assumption of Z_j is done by means of the genetic algorithm.
8. After finding suitable values of Zs, *b*, and LF, the Z(*x*) profile as well as the length of the filters are determined. Consequently, the physical parameters of the filter will be determined based on certain formulas provided by [2] which will be mentioned later.
9. Then, the simulation of the filter will be conducted by means of professional simulator.

The main purpose of having more than five terms expansion (eight terms expansion) is to have more freedom in the impedance profile to be able to design filters at higher frequencies more than 3 GHz. It is found that the five terms expansion is adequate for filters with cut-off at 3 GHz. However, the filters of cut-off frequency at 7 GHz or more need more oscillations in the impedance profile and can be achieved by increasing the terms expansion.

4.2.1 Five Terms Expansion

In this case the expansion of V(x) & Z(x) will be up to cos5αx and sin5αx. Therefore, the compact form of the voltage V(x) will be as follows:

$$V(x) = p\, V_0 + \sum_{l=1}^{5} \left(p\, V_l \cos l\alpha x + q\, V_{l+5} \sin l\alpha x \right)$$

and the expanded form will be as follows:

$$V(x) = p\,(V_0 + V_1 \cos \alpha x + V_2 \cos 2\alpha x + V_3 \cos 3\alpha x + V_4 \cos 4\alpha x + V_5 \cos 5\alpha x)$$

$$+ q\,(V_6 \sin \alpha x + V_7 \sin 2\alpha x + V_8 \sin 3\alpha x + V_9 \sin 4\alpha x + V_{10} \sin 5\alpha x)$$

Where V_0 & V_6 are – without loss of generality – assumed to be equal to 1 ($V_0 = V_6 = 1$).

In the same manner, the expressions of Z(x) will be as the following:

$$Z(x) = Z_C + \sum_{j=1}^{5} \left(Z_j \cos j\alpha x + Z_{j+5} \sin j\alpha x \right)$$

$$Z(x) = Z_C + Z_1 \cos \alpha x + Z_2 \cos 2\alpha x + Z_3 \cos 3\alpha x + Z_4 \cos 4\alpha x + Z_5 \cos 5\alpha x$$

$$+ Z_6 \sin \alpha x + Z_7 \sin 2\alpha x + Z_8 \sin 3\alpha x + Z_9 \sin 4\alpha x + Z_{10} \sin 5\alpha x$$

Then, the first and second derivatives of V(x) are going to be

$$\frac{dv}{dx} = p\,\alpha\,(-V_1 \sin \alpha x - 2\,V_2 \sin 2\alpha x - 3\,V_3 \sin 3\alpha x - 4\,V_4 \sin 4\alpha x - 5\,V_5 \sin 5\alpha x)$$

$$+\,q\,\alpha\,(V_6 \cos \alpha x + 2\,V_7 \cos 2\alpha x + 3\,V_8 \cos 3\alpha x + 4\,V_9 \cos 4\alpha x + 5\,V_{10} \cos 5\alpha x)$$

$$\frac{dv^2}{dx^2} = -\,p\,\alpha^2\,(V_1 \cos \alpha x + 4\,V_2 \cos 2\alpha x + 9\,V_3 \cos 3\alpha x + 16\,V_4 \cos 4\alpha x + 25\,V_5 \cos 5\alpha x)$$

$$-\,q\,\alpha^2\,(V_6 \sin \alpha x + 4\,V_7 \sin 2\alpha x + 9\,V_8 \sin 3\alpha x + 16\,V_9 \sin 4\alpha x + 25\,V_{10} \sin 5\alpha x)$$

The first derivative of Z(x) is

$$\frac{dz}{dx} = -\,\alpha\,(Z_1 \sin\alpha x + 2\,Z_2 \sin 2\alpha x + 3\,Z_3 \sin 3\alpha x + 4\,Z_4 \sin 4\alpha x + 5\,Z_5 \sin 5\alpha x)$$

$$+\,\alpha\,(Z_6 \cos\alpha x + 2\,Z_7 \cos\alpha 2x + 3\,Z_8 \cos\alpha 3x + 4\,Z_9 \cos\alpha 4x + 5\,Z_{10} \cos\alpha 5x)$$

In order to facilitate the substitution in (1) and the manipulation with its terms to get the final expression, the following compact forms will be used:

$$Z(x) = Z_C + \sum_{j=1}^{5} (Z_j \cos j\alpha x + Z_{j+5} \sin j\alpha x)$$

$$\frac{dz}{dx} = \alpha \sum_{m=1}^{5} (- m\, Z_m \sin m\alpha x + m\, Z_{m+5} \cos m\alpha x)$$

$$V(x) = p\, V_0 + \sum_{l=1}^{5} (p\, V_l \cos l\alpha x + q\, V_{l+5} \sin l\alpha x)$$

$$\frac{dv(x)}{dx} = \alpha \sum_{n=1}^{5} (- n\, p\, V_n \sin n\alpha x + n\, q\, V_{n+5} \cos n\alpha x)$$

$$\frac{d^2v(x)}{dx^2} = - \alpha^2 + \sum_{u=1}^{5} (p\, u^2 V_u \cos u\alpha x + q\, u^2 V_{u+5} \sin u\alpha x)$$

$$= - \alpha^2 u^2 + \sum_{u=1}^{5} (p\, V_u \cos u\alpha x + q\, V_{u+5} \sin u\alpha x)$$

Now, equation (1) can be written as:

$$Z(x) \cdot \frac{d^2V}{dx^2} - \frac{dZ}{dx} \cdot \frac{dV}{dx} + k_0{}^2 \cdot Z(x) \cdot V(x) = 0 \qquad (2)$$

Where $k_o{}^2 = \omega^2 \cdot L_o \cdot C_o$

Now, substituting the above set of equations in the first and the last terms of equation (2) given:

$$Z(x) \,.\, \frac{d^2 v(x)}{dx^2} + k_0^2 \,.\, Z(x) \,.V(x) \;=\; Z(x)\,[\,\frac{d^2 v(x)}{dx^2} + k_0^2\, V(x)]$$

$$= \;[Z_C + \sum_{j=1}^{5} \;(Z_j \cos j\alpha x + Z_{j+5} \sin j\alpha x)]$$

$$\times\, [pV_0\, k_0^2 + (\, k_0^2 - \alpha^2\, l^{\,2}) \sum_{l=1}^{5} \;(p\, V_l \cos l\alpha x + q\, V_{l+5} \sin l\alpha x]$$

$$= \;p\, V_0\, k_0^2\, Z_C + (\, k_0^2 - \;\alpha^2\, l^{\,2})\, Z_C \sum_{l=1}^{5} \;(p\, V_l \cos l\alpha x + q\, V_{l+5} \sin l\alpha x)$$

$$+\, p\, V_0\, k_0^2 \sum_{j=1}^{5} \;(Z_j \cos j\alpha x + Z_{j+5} \sin j\alpha x\,)$$

$$+\, \{(k_0^2 - \alpha^2\, l^{\,2})\, [\sum_{j=1}^{5} \;(Z_j \cos j\alpha x + Z_{j+5} \sin j\alpha x)]\, [\sum_{l=1}^{5} \;(pV_l \cos l\alpha x + qV_{l+5} \sin l\alpha x)]\}$$

Next , we will analyze the last mentioned term which is

$$(k_0^2 - \alpha^2\, l^{\,2})\, [\sum_{j=1}^{5} \;(Z_j \cos j\alpha x + Z_{j+5} \sin j\alpha x)]\, [\sum_{l=1}^{5} \;(p\, V_l \cos l\alpha x + q\, V_{l+5} \sin l\alpha x)] =$$

$$\{(k_0^2 - \alpha^2\, l^{\,2})\, [\sum_{j=1}^{5} \;\sum_{l=1}^{5} \;(Z_j \cos j\alpha x + Z_{j+5} \sin j\alpha x)\, (p\, V_l \cos l\alpha x + q\, V_{l+5} \sin l\alpha x)]\} =$$

$$(k_0^2 - \alpha^2 l^2)\,[\sum_{j=1}^{5}\ \sum_{l=1}^{5}\ (Z_j \cos j\alpha x)\,(p\,V_l \cos l\alpha x) + (Z_j \cos j\alpha x)\,(q\,V_{l+5} \sin l\alpha x)$$

$$+ (Z_{j+5} \sin j\alpha x)\,(p\,V_l \cos l\alpha x) + (Z_{j+5} \sin j\alpha x)\,(q\,V_{l+5} \sin l\alpha x)] =$$

$$(k_0^2 - \alpha^2 l^2)\,[\sum_{j=1}^{5}\ \sum_{l=1}^{5}\ (p\,V_l\ Z_j \cos j\alpha x \cos l\alpha x) + (q\,V_{l+5}\,Z_j \cos j\alpha x \sin l\alpha x)$$

$$+ (p\,V_l\,Z_{j+5} \sin j\alpha x \cos l\alpha x) + (q\,V_{l+5}\,Z_{j+5} \sin j\alpha x \sin l\alpha x)] =$$

$$(k_0^2 - \alpha^2 l^2)\,\{\sum_{j=1}^{5}\ \sum_{l=1}^{5}\ \frac{1}{2}\,(p\,V_l\,Z_j)\,[\cos(l + j)\alpha x + \cos(l - j)\alpha\, x)]$$

$$+ \frac{1}{2}\,(q\,V_{l+5}\,Z_j)\ [\sin(l + j)\alpha x - \sin(l - j)\alpha x)]$$

$$+ \frac{1}{2}\,(pV_l\,Z_{j+5})\,[\sin(l + j)\alpha x + \sin(l - j)\alpha x)]$$

$$+ \frac{1}{2}\,(qV_{l+5}\,Z_{j+5})\,[\cos(l - j)\alpha x - \cos(l + j)\alpha x)]\} =$$

$$\sum_{j=1}^{5}\ \sum_{l=1}^{5}\ (\frac{k_0^2 - \alpha^2 l^2}{2})\,[(pV_l\,Z_j + qV_{l+5}\,Z_{j+5}) \cos(l - j)\alpha x + (pV_l\,Z_j - qV_{l+5}\,Z_{j+5}) \cos(l + j)\alpha x$$

$$+ (-p\,V_l\,Z_{j+5} + q\,V_{l+5}\,Z_j) \sin(l - j)\alpha x + (p\,V_l\,Z_{j+5} + q\,V_{l+5}\,Z_j) \sin(l + j)\alpha x]$$

Therefore,

$$Z(x)\ \frac{d^2v(x)}{dx^2} + k_0^2\ Z(x)\ V(x) = p\ V_0\, k_0^2\ Z_C$$

$$+ Z_C\ (k_0^2 - \alpha^2 l^2) \sum_{l=1}^{5} (p\ V_l \cos l\alpha x + q\ V_{l+5} \sin l\alpha x)$$

$$+ p\ V_0\, k_0^2 \sum_{j=1}^{5} (Z_j \cos j\alpha x + Z_{j+5} \sin j\alpha x)$$

$$+ \sum_{j=1}^{5} \sum_{l=1}^{5} \left(\frac{k_0^2 - \alpha^2 l^2}{2}\right) [(pV_l\ Z_j + qV_{l+5} Z_{j+5}) \cos(l - j)\alpha x + (pV_l\ Z_j - qV_{l+5}\ Z_{j+5}) \cos(l + j)\alpha x$$

$$+ (-pV_l\ Z_{j+5} + qV_{l+5} Z_j) \sin (l\text{-}j)\alpha x + (pV_l\ Z_{j+5} + qV_{l+5} Z_j) \sin(l+j)\alpha x]$$

Next, substituting of the expressions of $\frac{dv}{dx}$ and $\frac{dz}{dx}$ into the remaining term of equation (2) gives:

$$-\frac{dv}{dx} \cdot \frac{dz}{dx} = -\alpha^2 \sum_{j=1}^{5} (-j\ Z_j \sin j\alpha x + j\ Z_{j+5} \cos j\alpha x \times \sum_{l=1}^{5} (-l\ pV_l \sin l\alpha x + l\ qV_{l+5} \cos l\alpha x)$$

$$= -\alpha^2 \sum_{l=1}^{5} \sum_{j=1}^{5} (l\ j\ p\ Z_j\ V_l \sin l\alpha x \sin j\alpha x - l\ j\ p\ Z_{j+5}\ V_l \sin l\alpha x \cos j\alpha x$$

$$- j\ l\ q\ Z_j\ V_{l+5} \sin j\alpha x \cos l\alpha x + j\ l\ q\ Z_{j+5} V_{l+5} \cos l\alpha x \cos j\alpha x)$$

$$= -\alpha^2 \sum_{l=1}^{5} \sum_{j=1}^{5} \{ \frac{1}{2} l\, j\, p\, Z_j V_l [\cos(l - j)\alpha x - \cos(l + j)\alpha x]$$

$$- \frac{1}{2} l\, j\, p\, Z_{j+5} V_l [\sin(l - j)\alpha x + \sin(l + j)\alpha x]$$

$$- \frac{1}{2} l\, j\, q\, Z_j V_{l+5} [\sin(j - l)\alpha x + \sin(l + j)\alpha x]$$

$$+ \frac{1}{2} l\, j\, q\, Z_{j+5} V_{l+5} [\cos(l - j)\alpha x + \cos(l + j)\alpha x]\}$$

$$-\frac{dv}{dx} \cdot \frac{dz}{dx} = \frac{-\alpha^2 lj}{2} \{ \sum_{l=1}^{5} \sum_{j=1}^{5} [(p\, Z_j V_l + q\, Z_{j+5} V_{l+5}) \cos(l - j)\alpha x]$$

$$+ [(-p\, Z_j V_l + q\, Z_{j+5} V_{l+5}) \cos(l + j)\alpha x]$$

$$+ [(-p\, Z_{j+5} V_l + q\, Z_j V_{l+5}) \sin(l - j)\alpha x]$$

$$- [(p\, Z_{j+5} V_l + q\, Z_j V_{l+5}) \sin(l + j)\alpha x]\}$$

Considering all the above expansions, equation (1) will take the following form:

$$\sum_{l=1}^{5} \sum_{j=1}^{5} \cos(l - j)\alpha x [(\frac{k_0^2 - \alpha^2 l^2 - \alpha^2 lj}{2})(p\, Z_j V_l + q\, Z_{j+5} V_{l+5})]$$

$$+ \sum_{l=1}^{5} \sum_{j=1}^{5} \cos(l + j)\alpha x [(\frac{k_0^2 - \alpha^2 l^2 + \alpha^2 lj}{2})(p\, Z_j V_l - q\, Z_{j+5} V_{l+5})]$$

$$+\sum_{l=1}^{5}\sum_{j=1}^{5}\sin(l-j)\alpha x\left[\left(\frac{k_0^2-\alpha^2 l^2-\alpha^2 lj}{2}\right)(-p\,Z_{j+5}V_l+q\,V_{l+5}Z_j)\right]$$

$$+\sum_{l=1}^{5}\sum_{j=1}^{5}\sin(l+j)\alpha x\left[\left(\frac{k_0^2-\alpha^2 l^2+\alpha^2 lj}{2}\right)(-p\,Z_{j+5}V_l+q\,Z_jV_{l+5})\right]$$

$$+\sum_{l=1}^{5}\cos l\alpha x\left[p\,V_l Z_C(k_0^2-\alpha^2 l^2)\right]+\sum_{j=1}^{5}p\,V_0\,k_0^2\,Z_j\cos j\alpha x$$

$$+\sum_{l=1}^{5}\sin l\alpha x\left[q\,V_{l+5}Z_C(k_0^2-\alpha^2 l^2)\right]+\sum_{j=1}^{5}pV_0\,k_0^2\,Z_{j+5}\sin j\alpha x$$

$$+p\,V_0 Z_C\,k_0^2=0$$

There are terms that contain trigonometric functions higher than either cos5α*x* or sin5α*x* during solving the expressions of equation (1). However, any term contains trigonometric function that is higher than cos5α*x* or sin5α*x* will be ignored and it will not be taken into consideration. This is in the hope that high order terms are increasingly getting smaller in magnitudes. If this is not the case – as it is shown at higher frequencies – more expansion terms must be considered. Since equation (1) equals to zero, therefore, the coefficients of each term of the sine and cosine expansion must equal to Zero. Due to the length and the complexity of the expressions of the solution, the expansion of the terms, which participate in the construction of the 9 X 9 matrix system to find the values of the V's (V_1, V_2, V_3, V_4, V_5, V_7, V_8, V_9 and V_{10}) which will be part of the expressions of S_{11} and

S_{21}, is shown in Appendix A. These terms which are the constant term and the associated coefficients of the trigonometric functions can be cast in matrix form as follows:

$$\begin{bmatrix} & & \\ & M_{ij} & \\ & & \end{bmatrix} \times \begin{bmatrix} V_1 \\ V_2 \\ \cdot \\ V_5 \\ rV_7 \\ \cdot \\ \cdot \\ rV_{10} \end{bmatrix} = \begin{bmatrix} \\ A_i \\ \\ \end{bmatrix} + r \begin{bmatrix} \\ B_j \\ \\ \end{bmatrix}$$

where A_i is 9 X 1 matrix containing the coefficients of V_0 and the B_j is 9 X 1 matrix comprising of the coefficients of V_6 (both V_0 and V_6 are assumed to be 1) and r = p/q. Further, M_{ij} is 9 X 9 matrix representing the coefficients of (V_1, V_2, V_3, V_4, V_5, rV_7, rV_8, rV_9 and rV_{10}). Solution of the above matrix system will yield the values of V_1, V_2, V_3, V_4, V_5, rV_7, rV_8, rV_9 and rV_{10} all in terms of the impedance profile Z(*x*) along the transmission line path. The matrix formulation is detailed in Appendix B. The solution of the matrix formulation will be in the form of the following:

$V_1 = A_1 + r\,B_1$	$V_4 = A_4 + r\,B_4$	$rV_7 = A_6 + r\,B_6$	$rV_9 = A_8 + r\,B_8$
$V_2 = A_2 + r\,B_2$	$V_5 = A_5 + r\,B_5$	$rV_8 = A_7 + r\,B_7$	$rV_{10} = A_9 + r\,B_9$
$V_3 = A_3 + r\,B_3$			

where r – as mentioned earlier – is a constant which is equal to q divided by p (r = q/p) .

It is important to note that Z(x) is the characteristic impedance at x. Thus,

$$Z(x) = \frac{V_{in}(x)}{I_{in}(x)} = \frac{-V_{ref}(x)}{I_{ref}(x)}$$

Where I(x) is the current along the line and given by

$$I(x) = \frac{j \cdot c}{\omega \cdot Z(x)} \cdot \frac{dV(x)}{dx}$$

However, V(x) and I(x) are the total voltage and the total current along x and they are equal to the following:

$$V(x) = V_{total}(x) = V_{in}(x) + V_{ref}(x)$$

$$I(x) = I_{total}(x) = I_{in}(x) - I_{ref}(x)$$

So, when the line is matched at $x = l$, this means $Z_{load} = Z(l)$

$$\therefore\ Z_{load} = \frac{V_{total}(l)}{I_{total}(l)} = \frac{V_{in}(l) + V_{ref}(l)}{I_{in}(l) - I_{ref}(l)}$$

But, since the line is matched, then $V_{ref}(l) = 0$ and $I_{ref}(l) = 0$

$$\therefore\ Z_{load} = Z(l) = \frac{V(l)}{I\ (l)} = \frac{V_{total}(l)}{I_{total}(l)}$$

where $V(l)$ and $I(l)$ are $V(x)$ and $I(x)$ at $x = l$.

Therefore, the input impedance at x which is $Z_{in}(x)$ does not equal to $Z(x)$ which is the characteristic impedance at x ($Z_{in}(x) \neq Z(x)$).

$$\therefore Z_{in}(x) = \frac{V(x)}{I\ (x)} = \frac{V_{total}(x)}{I_{total}(x)} = \frac{V_{in}(x) + V_{ref}(x)}{I_{in}(x) - I_{ref}(x)}$$

So, $Z_{in}(x) = \dfrac{V(x)}{I\ (x)} \neq Z(x)$

Now, to find the mathematical expressions for **r**, $\mathbf{S_{11}}$ and $\mathbf{S_{21}}$, we have $\alpha = \frac{2\pi b}{l}$ and $b \neq l$

$$b_1 = S_{11}a_1 + S_{12}a_2$$

$$b_2 = S_{21}a_1 + S_{22}a_2$$

$$S_{11} = \frac{b_1}{a_1}\Big|_{a_2=0}$$

where b_1 is the reflected wave at the input port of the filter and a_1 is the incident wave at the same port. The condition $a_2 = 0$ necessitates the following:

1. $Z(x = l) = Z_{load}$.
2. $V_{ref}(x = l) = I_{ref}(x = l) = 0$.

Therefore, at $x = l$, we have $\alpha x = 2\pi b$ and

$$V(x) = V_{total}(x) = V_{in}(x)$$

$$I(x) = I_{total}(x) = I_{in}(x)$$

Therefore

$$Z_{load} = Z(l) = \frac{V(x)}{I\ (x)} = \frac{V_{in}(x)}{I_{in}(x)} = \frac{V(l)}{I\ (l)}$$

$$V(l) = \mathbf{p}\,(1 + V_1 \cos 2\pi b + V_2 \cos 4\pi b + V_3 \cos 6\pi b + V_4 \cos 8\pi b + V_5 \cos 10\pi b)$$
$$+ \mathbf{q}\,(\sin 2\pi b + V_7 \sin 4\pi b + V_8 \sin 6\pi b + V_9 \sin 8\pi b + V_{10} \sin 10\pi b)$$

$$= \mathbf{p}\,(1 + A_1 \cos 2\pi b + r\,B_1 \cos 2\pi b + A_2 \cos 4\pi b + r\,B_2 \cos 4\pi b + A_3 \cos 6\pi b + r\,B_3 \cos 6\pi b + A_4 \cos 8\pi b + r\,B_4 \cos 8\pi b + A_5 \cos 10\pi b + r\,B_5 \cos 10\pi b) + \mathbf{q}\,(\sin 2\pi b + (A_6/r) \sin 4\pi b + B_6 \sin 4\pi b + (A_7/r) \sin 6\pi b + B_7 \sin 6\pi b + (A_8/r) \sin 8\pi b + B_8 \sin 8\pi b + (A_9/r) \sin 10\pi b + B_9 \sin 10\pi b)$$

$$= \mathbf{p}\,[(1 + A_1 \cos 2\pi b + A_2 \cos 4\pi b + A_3 \cos 6\pi b + A_4 \cos 8\pi b + A_5 \cos 10\pi b) + \mathbf{r}\,(B_1 \cos 2\pi b + B_2 \cos 4\pi b + B_3 \cos 6\pi b + B_4 \cos 8\pi b + B_5 \cos 10\pi b) + \mathbf{r}\,(\sin 2\pi b + B_6 \sin 4\pi b + B_7 \sin 6\pi b + B_8 \sin 8\pi b + B_9 \sin 10\pi b) + (A_6 \sin 4\pi b + A_7 \sin 6\pi b + A_8 \sin 8\pi b + A_9 \sin 10\pi b)]$$

For the sake of simplicity and brevity, let

$$R1 = (1 + A_1 \cos 2\pi b + A_2 \cos 4\pi b + A_3 \cos 6\pi b + A_4 \cos 8\pi b + A_5 \cos 10\pi b)$$

$$R2 = (B_1 \cos 2\pi b + B_2 \cos 4\pi b + B_3 \cos 6\pi b + B_4 \cos 8\pi b + B_5 \cos 10\pi b)$$

$$R3 = (\sin 2\pi b + B_6 \sin 4\pi b + B_7 \sin 6\pi b + B_8 \sin 8\pi b + B_9 \sin 10\pi b)$$

$$R4 = (A_6 \sin 4\pi b + A_7 \sin 6\pi b + A_8 \sin 8\pi b + A_9 \sin 10\pi b)$$

Therefore, V (l) = P [R_1 + rR_2 + rR_3 + R_4] = P [(R_1 + R_4) + r (R_2 + R_3)]

Now, the expression of I(l) will be manipulated

$$I(l) = \frac{j \cdot c}{\omega \cdot Z(l)} \cdot \frac{2\pi b}{l} \cdot \frac{dV}{dl}$$

= (1/Z(l)) × j × (c/ω) × ($2\pi b/l$) × **p** [- A_1 sin $2\pi b$ – 2 A_2 sin $4\pi b$ – 3 A_3 sin $6\pi b$ – 4 A_4 sin $8\pi b$ – 5 A_5 sin $10\pi b$) + **r** (- B_1 sin $2\pi b$ – 2 B_2 sin $4\pi b$ – 3 B_3 sin $6\pi b$ – 4 B_4 sin $8\pi b$ – 5 B_5 sin $10\pi b$) + **r** (cos $2\pi b$ + 2 B_6 cos $4\pi b$ + 3 B_7 cos $6\pi b$ + 4 B_8 cos $8\pi b$ + 5 B_9 cos $10\pi b$) + (2 A_6 cos $4\pi b$ + 3 A_7 cos $6\pi b$ + 4 A_8 cos $8\pi b$ + 5 A_9 cos $10\pi b$)]

Again, for the sake of simplicity and brevity, let

S1 = (- A_1 sin $2\pi b$ – 2 A_2 sin $4\pi b$ – 3 A_3 sin $6\pi b$ – 4 A_4 sin $8\pi b$ – 5 A_5 sin $10\pi b$)

S2 = (- B_1 sin $2\pi b$ – 2 B_2 sin $4\pi b$ – 3 B_3 sin $6\pi b$ – 4 B_4 sin $8\pi b$ – 5 B_5 sin $10\pi b$)

S3 = (cos $2\pi b$ + 2 B_6 cos $4\pi b$ + 3 B_7 cos $6\pi b$ + 4 B_8 cos $8\pi b$ + 5 B_9 cos $10\pi b$)

S4 = (2 A_6 cos $4\pi b$ + 3 A_7 cos $6\pi b$ + 4 A_8 cos $8\pi b$ + 5 A_9 cos $10\pi b$)

$$\Rightarrow \text{I}(l) = \frac{j}{Z(l)}\left(\frac{\lambda b}{l}\right) \times p\left[\left(S_1 + rS_2 + rS_3 + S_4\right)\right]$$

$$= \frac{j}{Z(l)}\left(\frac{\lambda b}{l}\right) \times p\left[\left(S_1 + S_4\right) + r(S_2 + S_3)\right]$$

$$\Rightarrow \text{Z}(l) = \frac{(R_1 + R_4) + r(R_2 + R_3)}{(S_1 + S_4) + r(S_2 + S_3)} \times \text{Z}(l) \times (-j) \times \left(\frac{l}{\lambda b}\right)$$

$(S_1 + S_4) + r\,(S_2 + S_3) = (-j)\,(l / \lambda b)\,[(R_1 + R_4) + r\,(R_2 + R_3)]$

$(S_1 + S_4) + j\,(l / \lambda b)\,(R_1 + R_4) = -\,r\,[(S_2 + S_3) + j\,(l / \lambda b)\,(R_2 + R_3)]$

$$\therefore\ \text{r} = \frac{(S_1 + S_4) + j\,\dfrac{l}{\lambda b}(R_1 + R_4)}{(S_2 + S_3) + j\,\dfrac{l}{\lambda b}(R_2 + R_3)}$$

After we have finished from the analysis of the load side, we will move to the input side at which $x = 0$. Now, since $a_2 = 0$, we have $Z_{load} = Z(l)$ (matched loads) and there is no reflected voltage nor reflected current. Hence, a_1 is the only incident voltage wave and b_1 is the only reflected voltage wave. Therefore,

$$S_{11} = \frac{V_{ref}(x=0)}{V_{in}(x=0)} = \frac{\Gamma \times V_{in}(x=0)}{V_{in}(x=0)} = \Gamma$$

$$\Gamma = \frac{Z_{in} - Z_0}{Z_{in} + Z_0}$$

at $x = 0$, $Z_0 = Z(0)$

$Z(0) = Z(x = 0) = Z_C + Z_1 \cos 0 + Z_2 \cos 0 + Z_3 \cos 0 + Z_4 \cos 0 + Z_5 \cos 0$

$+ Z_6 \sin 0 + Z_7 \sin 0 + Z_8 \sin 0 + Z_9 \sin 0 + Z_{10} \sin 0$

$\Rightarrow Z(0) = (Z_C + Z_1 + Z_2 + Z_3 + Z_4 + Z_5)$

$Z_{in} = Z_{in}(x = 0) = Z_{in}(0) = \dfrac{V(0)}{I(0)}$

$V(0) = p\,[1 + V_1 + V_2 + V_3 + V_4 + V_5]$

$= p\,[(1 + A_1 + A_2 + A_3 + A_4 + A_5)] + r\,(B_1 + B_2 + B_3 + B_4 + B_5)]$

$= p\,(R_1 + r\,R_2)$

$$\text{I}(0) = \frac{1}{Z(0)} \times \left(\frac{j\lambda b}{l} \right) \times p(S_4 + rS_3)$$

$$\therefore Z_{in}(0) = \frac{(R_1 + rR_2)}{(S_4 + rS_3)} \times Z(0) \times (-j) \times \frac{(l)}{\lambda b}$$

$$\Rightarrow S_{11} = \Gamma = \frac{Z_{in}(0) - Z(0)}{Z_{in}(0) + Z(0)}$$

Next, we have to determine the mathematical expression of S_{21}

$$S_{21} = \frac{b_2}{a_1}\Big|_{a_2=0}$$

b_2 is the reflected voltage wave at the load side whereas a_1 is the incident voltage wave at the input side. However, since there is no incident wave at the load side (a_2=0) due to matching condition, b_2 is the total voltage at the load side

$$\Rightarrow b_2 = \frac{V_{total}(l)}{\sqrt{Z(l)}} = \frac{V(l)}{\sqrt{Z(l)}}$$

where Z(l) is found by replacing the variable x with l in the expression of Z(x) as follows:

$$Z(l) = Z_C + Z_1 \cos 2\pi b + Z_2 \cos 4\pi b + Z_3 \cos 6\pi b + Z_4 \cos 8\pi b + Z_5 \cos 10\pi b$$

$$+ Z_6 \sin 2\pi b + Z_7 \sin 4\pi b + Z_8 \sin 6\pi b + Z_9 \sin 8\pi b + Z_{10} \sin 10\pi b$$

$$a_1 = \frac{V_{inc}(x=0)}{\sqrt{Z(0)}}$$

$$\because \ V(x) = V_{total}(x) = V_{inc}(x) + V_{ref}(x)$$

$$\because V_{total}(x) = V_{inc}(x)\ [\ 1+\Gamma]$$

$$\Rightarrow V_{inc}(x) = \frac{V_{total}(x)}{1+\Gamma}$$

$$\because \ \Gamma = S_{11}$$

$$\Rightarrow V_{inc}(x) = \frac{V_{total}(x)}{1+S_{11}}$$

$$\Rightarrow a_1 = \frac{V_{total}(x=0)}{(1+S_{11})} \times \frac{1}{\sqrt{Z(0)}}$$

$$\because S_{21} = \frac{b_2}{a_1}$$

$$\Rightarrow S_{21} = \frac{V(l)}{\sqrt{Z(l)}} \times \frac{(1+S_{11})}{V_{total}(x=0)} \times \sqrt{Z(0)}$$

$$\therefore S_{21} = \frac{V(l)}{V\ (0)} \times \sqrt{\frac{Z(0)}{Z(l)}} \times (1+S_{11})$$

4.2.2 Eight Terms Expansion

The analysis here will be the same as the one that has been carried out in the previous section (Five Terms Expansion) except that the expansion in this case is up to cos8α*x* and sin8α*x*. Therefore, there are more terms to be included in the expressions and they will make a significant change in the analysis. The compact form of the voltage V(*x*) will be as follows:

$$V(x) = pV_0 + \sum_{l=1}^{8} (\ p\ V_l \cos l\alpha x + q\ V_{l+8} \sin l\alpha x)$$

and the expanded form will be as follows:

$$V(x) = p\ (V_0 + V_1 \cos \alpha x + V_2 \cos 2\alpha x + V_3 \cos 3\alpha x + V_4 \cos 4\alpha x + V_5 \cos 5\alpha x + V_6 \cos 6\alpha x + V_7 \cos 7\alpha x + V_8 \cos 8\alpha x) + q\ (V_9 \sin \alpha x + V_{10} \sin 2\alpha x + V_{11} \sin 3\alpha x + V_{12} \sin 4\alpha x + V_{13} \sin 5\alpha x + V_{14} \sin 6\alpha x + V_{15} \sin 7\alpha x + V_{16} \sin 8\alpha x)$$

where V_0 & V_9 are assumed to be equal to 1 ($V_0 = V_9 = 1$).

In the same manner, the expression of $Z(x)$ will be as the following:

$$Z(x) = Z_C + \sum_{j=1}^{8} (Z_j \cos j\alpha x + Z_{j+8} \sin j\alpha x)$$

$$Z(x) = Z_C + Z_1 \cos \alpha x + Z_2 \cos 2\alpha x + Z_3 \cos 3\alpha x + Z_4 \cos 4\alpha x + Z_5 \cos 5\alpha x + Z_6 \cos 6\alpha x + Z_7 \cos 7\alpha x + Z_8 \cos 8\alpha x + Z_9 \sin \alpha x + Z_{10} \sin 2\alpha x + Z_{11} \sin 3\alpha x + Z_{12} \sin 4\alpha x + Z_{13} \sin 5\alpha x + Z_{14} \sin 6\alpha x + Z_{15} \sin 7\alpha x + Z_{16} \sin 8\alpha x$$

Then, the first and second derivative of $V(x)$ are going to be

$$\frac{dv}{dx} = p\ \alpha\ (-V_1 \sin \alpha x - 2\ V_2 \sin 2\alpha x - 3\ V_3 \sin 3\alpha x - 4\ V_4 \sin 4\alpha x - 5\ V_5 \sin 5\alpha x - 6\ V_6 \sin 6\alpha x - 7\ V_7 \sin 7\alpha x - 8\ V_8 \sin 8\alpha x) + q\ \alpha\ (V_9 \cos \alpha x + 2\ V_{10} \cos 2\alpha x + 3\ V_{11} \cos 3\alpha x + 4\ V_{12} \cos 4\alpha x + 5\ V_{13} \cos 5\alpha x + 6\ V_{14} \cos 6\alpha x + 7\ V_{15} \cos 7\alpha x + 8\ V_{16} \cos 8\alpha x)$$

$$\frac{dv^2}{dx^2} = - p\,\alpha^2 (V_1 \cos \alpha x + 4\, V_2 \cos 2\alpha x + 9\, V_3 \cos 3\alpha x + 16\, V_4 \cos 4\alpha x + 25\, V_5 \cos 5\alpha x + 36\, V_6 \cos 6\alpha x + 49\, V_7 \cos 7\alpha x + 64\, V_8 \cos 8\alpha x) - q\,\alpha^2 (V_9 \sin \alpha x + 4\, V_{10} \sin 2\alpha x + 9\, V_{11} \sin 3\alpha x + 16\, V_{12} \sin 4\alpha x + 25\, V_{13} \sin 5\alpha x + 36\, V_{14} \sin 6\alpha x + 49\, V_{15} \sin 7\alpha x + 64\, V_{16} \sin 8\alpha x)$$

The first derivative of Z(x) is

$$\frac{dz}{dx} = - \alpha (Z_1 \sin \alpha x + 2\, Z_2 \sin 2\alpha x + 3\, Z_3 \sin 3\alpha x + 4\, Z_4 \sin 4\alpha x + 5\, Z_5 \sin 5\alpha x + 6\, Z_6 \sin 6\alpha x + 7\, Z_7 \sin 7\alpha x + 8\, Z_8 \sin 8\alpha x) + \alpha (Z_9 \cos \alpha x + 2\, Z_{10} \cos 2\alpha x + 3\, Z_{11} \cos 3\alpha x + 4\, Z_{12} \cos 4\alpha x + 5\, Z_{13} \cos 5\alpha x + 6\, Z_{14} \cos 6\alpha x + 7\, Z_{15} \cos 7\alpha x + 8\, Z_{16} \cos 8\alpha x)$$

As it has been done with the five terms expansion case, to simplify the substitution in (1) and the manipulations with its terms to get the final expression, the following compact forms will be used:

$$Z(x) = Z_C + \sum_{j=1}^{8} (Z_j \cos j\alpha x + Z_{j+8} \sin j\alpha x)$$

$$\frac{dz}{dx} = \alpha \sum_{m=1}^{8} (- m\, Z_m \sin m\alpha x + m\, Z_{m+8} \cos m\alpha x)$$

$$V(x) = p\, V_0 + \sum_{l=1}^{8} (p\, V_l \cos l\alpha x + q\, V_{l+8} \sin l\alpha x)$$

$$\frac{dv(x)}{dx} = \alpha \sum_{n=1}^{8} (- n\, p\, V_n \sin n\alpha x + n\, q\, V_{n+8} \cos n\alpha x)$$

$$\frac{d^2v(x)}{dx^2} = - \alpha^2 + \sum_{u=1}^{8} (p\, u^2 V_u \cos u\alpha x + q\, u^2 V_{u+8} \sin u\alpha x)$$

$$= - \alpha^2 u^2 + \sum_{u=1}^{8} (p\, V_u \cos u\alpha x + q\, V_{u+8} \sin u\alpha x)$$

After substitutions of the expressions and manipulations of the terms of (1) exactly in the same manner as it has been done with five terms expansion case, the final form of (1) will be as follows:

$$\sum_{l=1}^{8} \sum_{j=1}^{8} \cos (l - j)\alpha x \left[\left(\frac{k_0^2 - \alpha^2 l^2 - \alpha^2 lj}{2}\right) (p\, Z_j V_l + q\, Z_{j+8} V_{l+8})\right]$$

$$+ \sum_{l=1}^{8} \sum_{j=1}^{8} \cos (l + j)\alpha x \left[\left(\frac{k_0^2 - \alpha^2 l^2 + \alpha^2 lj}{2}\right) (p\, Z_j V_l - q\, Z_{j+8} V_{l+8})\right]$$

$$+\sum_{l=1}^{8}\sum_{j=1}^{8}\sin(l-j)\alpha x\left[\left(\frac{k_0^2-\alpha^2 l^2-\alpha^2 lj}{2}\right)(-p\,Z_{j+8}V_l+q\,V_{l+8}Z_j)\right]$$

$$+\sum_{l=1}^{8}\sum_{j=1}^{8}\sin(l+j)\alpha x\left[\left(\frac{k_0^2-\alpha^2 l^2+\alpha^2 lj}{2}\right)(-p\,Z_{j+8}V_l+q\,Z_jV_{l+8})\right]$$

$$+\sum_{l=1}^{8}\cos l\alpha x\,[p\,V_l Z_C(k_0^2-\alpha^2 l^2)]+\sum_{j=1}^{8}p\,V_0\,k_0^2\,Z_j\cos j\alpha x$$

$$+\sum_{l=1}^{8}\sin l\alpha x\,[q\,V_{l+8}Z_C(k_0^2-\alpha^2 l^2)]+\sum_{j=1}^{8}pV_0\,k_0^2\,Z_{j+8}\sin j\alpha x$$

$$+p\,V_0 Z_C\,k_0^2=0$$

Again, there are terms that contain trigonometric functions higher than either cos8α*x* or sin8α*x* during solving the expression of equation (1). However, any term contains trigonometric function that is higher than cos8α*x* or sin8α*x* will be ignored. Since equation (1) equals to zero, therefore, the constant term and coefficients of the trigonometric functions should be equal to zero to satisfy the solution. Due to the length and the complexity of the expressions of the solution, the important terms, that participate in the formation of the system of equations with fifteen (15) equations and fifteen (15) unknowns (V_1, V_2, V_3, V_4, V_5, V_6, V_7, V_8, V_{10}, V_{11}, V_{12}, V_{13}, V_{14}, V_{15} and V_{16}) which will be part of the expressions of S_{11} and S_{21}, are indicated in Appendix C. These terms are the constant term and the associated coefficients of the trigonometric functions. A matrix formation for these terms is exactly as done in the five terms expansion where the

voltage expansion coefficients are determined in terms of the impedance profile Z(x) along the line. Here, the matrix M_{ij} is 15 X 15 while A_i and B_j are 15 X 1 matrices as shown in Appendix D. The solution of the matrix system will be in the form of the following:

$V_1 = A_1 + r\,B_1$	$V_5 = A_5 + r\,B_5$	$rV_{10} = A_9 + r\,B_9$	$rV_{14} = A_{13} + r\,B_{13}$
$V_2 = A_2 + r\,B_2$	$V_6 = A_6 + r\,B_6$	$rV_{11} = A_{10} + r\,B_{10}$	$rV_{15} = A_{14} + r\,B_{14}$
$V_3 = A_3 + r\,B_3$	$V_7 = A_7 + r\,B_7$	$rV_{12} = A_{11} + r\,B_{11}$	$rV_{16} = A_{15} + r\,B_{15}$
$V_4 = A_4 + r\,B_4$	$V_8 = A_8 + r\,B_8$	$rV_{13} = A_{12} + r\,B_{12}$	

In order to find the mathematical expressions of r, S_{11} and S_{21}, the same analysis that has been took place in the previous section (Five Terms Expansion) will be carried out here except that it has more terms expansion. So, the analysis here will be briefly discussed and limited to the expressions that will be affected by introducing more expanded terms in the voltage wave, the current and the impedance. Hence, the expanded expressions are as follows:

$$Z_l = \frac{V(l)}{I(l)}$$

$$V(l) = \mathbf{p}\,(1 + V_1 \cos 2\pi b + V_2 \cos 4\pi b + V_3 \cos 6\pi b + V_4 \cos 8\pi b + V_5 \cos 10\pi b + V_6 \cos 12\pi b + V_7 \cos 14\pi b + V_8 \cos 16\pi b) + \mathbf{q}\,(\sin 2\pi b + V_{10} \sin 4\pi b + V_{11} \sin 6\pi b + V_{12} \sin 8\pi b + V_{13} \sin 10\pi b + V_{14} \sin 12\pi b + V_{15} \sin 14\pi b + V_{16} \sin 16\pi b)$$

$$= \mathbf{p}\ (1 + A_1 \cos 2\pi b + r\, B_1 \cos 2\pi b + A_2 \cos 4\pi b + r\, B_2 \cos 4\pi b + A_3 \cos 6\pi b + r\, B_3 \cos 6\pi b + A_4 \cos 8\pi b + r\, B_4 \cos 8\pi b + A_5 \cos 10\pi b + r\, B_5 \cos 10\pi b + A_6 \cos 12\pi b + r\, B_6 \cos 12\pi b + A_7 \cos 14\pi b + r\, B_7 \cos 14\pi b + A_8 \cos 16\pi b + r\, B_8 \cos 16\pi b) + \mathbf{q}\ (\sin 2\pi b + (A_9/r) \sin 4\pi b + B_9 \sin 4\pi b + (A_{10}/r) \sin 6\pi b + B_{10} \sin 6\pi b + (A_{11}/r) \sin 8\pi b + B_{11} \sin 8\pi b + (A_{12}/r) \sin 10\pi b + B_{12} \sin 10\pi b + (A_{13}/r) \sin 12\pi b + B_{13} \sin 12\pi b + (A_{14}/r) \sin 14\pi b + B_{14} \sin 14\pi b + (A_{15}/r) \sin 16\pi b + B_{15} \sin 16\pi b)$$

$$= \mathbf{p}\ [(1 + A_1 \cos 2\pi b + A_2 \cos 4\pi b + A_3 \cos 6\pi b + A_4 \cos 8\pi b + A_5 \cos 10\pi b + A_6 \cos 12\pi b + A_7 \cos 14\pi b + A_8 \cos 16\pi b) + \mathbf{r}\ (B_1 \cos 2\pi b + B_2 \cos 4\pi b + B_3 \cos 6\pi b + B_4 \cos 8\pi b + B_5 \cos 10\pi b + B_6 \cos 12\pi b + B_7 \cos 14\pi b + B_8 \cos 16\pi b) + \mathbf{r}\ (\sin 2\pi b + B_9 \sin 4\pi b + B_{10} \sin 6\pi b + B_{11} \sin 8\pi b + B_{12} \sin 10\pi b + B_{13} \sin 12\pi b + B_{14} \sin 14\pi b + B_{15} \sin 16\pi b) + (A_9 \sin 4\pi b + A_{10} \sin 6\pi b + A_{11} \sin 8\pi b + A_{12} \sin 10\pi b + A_{13} \sin 12\pi b + A_{14} \sin 14\pi b + A_{15} \sin 16\pi b)]$$

Again, let

$$R1 = (1 + A_1 \cos 2\pi b + A_2 \cos 4\pi b + A_3 \cos 6\pi b + A_4 \cos 8\pi b + A_5 \cos 10\pi b + A_6 \cos 12\pi b + A_7 \cos 14\pi b + A_8 \cos 16\pi b)$$

$$R2 = (B_1 \cos 2\pi b + B_2 \cos 4\pi b + B_3 \cos 6\pi b + B_4 \cos 8\pi b + B_5 \cos 10\pi b + B_6 \cos 12\pi b + B_7 \cos 14\pi b + B_8 \cos 16\pi b)$$

$$R3 = (\sin 2\pi b + B_9 \sin 4\pi b + B_{10} \sin 6\pi b + B_{11} \sin 8\pi b + B_{12} \sin 10\pi b + B_{13} \sin 12\pi b + B_{14} \sin 14\pi b + B_{15} \sin 16\pi b)$$

$$R4 = (A_9 \sin 4\pi b + A_{10} \sin 6\pi b + A_{11} \sin 8\pi b + A_{12} \sin 10\pi b + A_{13} \sin 12\pi b + A_{14} \sin 14\pi b + A_{15} \sin 16\pi b)$$

Therefore,

$$V(l) = p\,[R_1 + rR_2 + rR_3 + R_4] = p\,[(R_1 + R_4) + r\,(R_2 + R_3)]$$

Now, the expression of I (l) will be manipulated

$$I(l) = (1/Z(l)) \times j \times (c/\omega) \times (2\pi b/l) \times \mathbf{p}\,[- A_1 \sin 2\pi b - 2 A_2 \sin 4\pi b - 3 A_3 \sin 6\pi b - 4 A_4 \sin 8\pi b - 5 A_5 \sin 10\pi b - 6 A_6 \sin 12\pi b - 7 A_7 \sin 14\pi b - 8 A_8 \sin 16\pi b) + \mathbf{r}\,(- B_1 \sin 2\pi b - 2 B_2 \sin 4\pi b - 3 B_3 \sin 6\pi b - 4 B_4 \sin 8\pi b - 5 B_5 \sin 10\pi b - 6 B_6 \sin 12\pi b - 7 B_7 \sin 14\pi b - 8 B_8 \sin 16\pi b) + \mathbf{r}\,(\cos 2\pi b + 2 B_9 \cos 4\pi b + 3 B_{10} \cos 6\pi b + 4 B_{11} \cos 8\pi b + 5 B_{12} \cos 10\pi b + 6 B_{13} \cos 12\pi b + 7 B_{14} \cos 14\pi b + 8 B_{15} \cos 16\pi b) + (2 A_9 \cos 4\pi b + 3 A_{10} \cos 6\pi b + 4 A_{11} \cos 8\pi b + 5 A_{12} \cos 10\pi b + 6 A_{13} \cos 12\pi b + 7 A_{14} \cos 14\pi b + 8 A_{15} \cos 16\pi b)]$$

Now, let

S1 = (- A_1 sin $2\pi b$ – 2 A_2 sin $4\pi b$ – 3 A_3 sin $6\pi b$ – 4 A_4 sin $8\pi b$ – 5 A_5 sin $10\pi b$ – 6 A_6 sin $12\pi b$ – 7 A_7 sin $14\pi b$ – 8 A_8 sin $16\pi b$)

S2 = (- B_1 sin $2\pi b$ – 2 B_2 sin $4\pi b$ – 3 B_3 sin $6\pi b$ – 4 B_4 sin $8\pi b$ – 5 B_5 sin $10\pi b$ – 6 B_6 sin $12\pi b$ – 7 B_7 sin $14\pi b$ – 8 B_8 sin $16\pi b$)

S3 = (cos $2\pi b$ + 2 B_9 cos $4\pi b$ + 3 B_{10} cos $6\pi b$ + 4 B_{11} cos $8\pi b$ + 5 B_{12} cos $10\pi b$ + 6 B_{13} cos $12\pi b$ + 7 B_{14} cos $14\pi b$ + 8 B_{15} cos $16\pi b$)

S4 = (2 A_9 cos $4\pi b$ + 3 A_{10} cos $6\pi b$ + 4 A_{11} cos $8\pi b$ + 5 A_{12} cos $10\pi b$ + 6 A_{13} cos $12\pi b$ + 7 A_{14} cos $14\pi b$ + 8 A_{15} cos $16\pi b$)

$$\therefore\ r = \frac{(S_1 + S_4) + j\,\dfrac{l}{\lambda b}(R_1 + R_4)}{(S_2 + S_3) + j\,\dfrac{l}{\lambda b}(R_2 + R_3)}$$

The mathematical expression for S_{11} will be as the following:

$$Z_{in} = \frac{V(0)}{I(0)}$$

V (0) = p [1 + V_1 + V_2 + V_3 + V_4 + V_5 + V_6 + V_7 + V_8]

= p [(1 + A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7 + A_8)]
+ r (B_1 + B_2 + B_3 + B_4 + B_5 + B_6 + B_7 + B_8)]

= p (R_1 + r R_2)

$$\because \text{ I (0) } = \frac{1}{Z(0)} \times \left(\frac{j\lambda b}{l} \right) \times p(S_4 + rS_3)$$

$$Z(0) = Z(x = 0) = Z_C + Z_1 \cos 0 + Z_2 \cos 0 + Z_3 \cos 0 + Z_4 \cos 0 + Z_5 \cos 0 + Z_6 \cos 0$$

$$+ Z_7 \cos 0 + Z_8 \cos 0 + Z_9 \sin 0 + Z_{10} \sin 0 + Z_{11} \sin 0 + Z_{12} \sin 0$$

$$+ Z_{13} \sin 0 + Z_{14} \sin 0 + Z_{15} \sin 0 + Z_{16} \sin 0$$

$$\Rightarrow Z(0) = (Z_C + Z_1 + Z_2 + Z_3 + Z_4 + Z_5 + Z_6 + Z_7 + Z_8)$$

$$\therefore Z_{in} = \frac{(R_1 + rR_2)}{(S_4 + rS_3)} \times Z(0) \times (-j) \times \frac{(l)}{\lambda b}$$

$$\Rightarrow S_{11} = \frac{Z_{in} - Z(0)}{Z_{in} + Z(0)}$$

Now, the mathematical expression of S_{21} will be as follows:

$$Z(l) = Z_C + Z_1 \cos 2\pi b + Z_2 \cos 4\pi b + Z_3 \cos 6\pi b + Z_4 \cos 8\pi b + Z_5 \cos 10\pi b + Z_6 \cos 12\pi b + Z_7 \cos 14\pi b + Z_8 \cos 16\pi b + Z_9 \sin 2\pi b + Z_{10} \sin 4\pi b + Z_{11} \sin 6\pi b + Z_{12} \sin 8\pi b + Z_{13} \sin 10\pi b + Z_{14} \sin 12\pi b + Z_{15} \sin 14\pi b + Z_{16} \sin 16\pi b$$

$$\therefore S_{21} = \frac{V(l)}{V\ (0)} \times \sqrt{\frac{Z(0)}{Z(l)}} \times (1 + S_{11})$$

4.3 Application of Genetic Algorithm to the Methodology

The genetic algorithm (GA) is going to be applied to this methodology by means of MATLAB®. There is an optimization toolbox in MATLAB® which includes GA and many other optimization techniques. In order to get this toolbox, one should type the word ”*optimtool*” in the main window.

Here, each possible solution (chromosome) is consisting of the values of Zs, *b* and LF respectively. These values are initially created by either the GA as random values or by the user as deterministic values. The population size of the possible solutions was 250 and the selection function which chooses the parents that will participate in producing the next generation was based on stochastic uniform function. The crossover rate was

0.7 whereas the mutation rate was very low. The number of generations was chosen to be 3000.

The objective function to be optimized is the Error Function. The optimization of the error function means minimizing it to have the minimum error between the required values and the optimized values. The general form of the error function is as follows:

$$\text{Error} = \sum_{j=0}^{n} (\left|S_{11}(j)_{optimized}\right| - \left|S_{11\,required}\right|)^2 + \sum_{j=0}^{n} (\left|S_{21}(j)_{optimized}\right| - \left|S_{21\,required}\right|)^2$$

where [j = 0, 0.1, 0.2, 0.3, ….. n] is the portion of the frequency spectrum in GHz that we want to optimize the filter response at. $|S_{11\ required}|$ and $|S_{21\ required}|$ are known and their values are set by the user. For example, if the desired filter to be designed is low pass filter with cutoff frequency at 3 GHz. Then, $|S_{11\ required}|$ has the values of zero below 3 GHz, values of 1 above 3 GHz and 0.5 at 3 GHz. Similarly, $|S_{21\ required}|$ has the values of 1 below 3 GHz, values of zero above 3 GHz and 0.5 at 3 GHz.

In the optimization procedure, the quantity l/λ (length over wavelength) is considered variable so that the filter length is optimized as well. In order to have the characteristic impedances at the input and at the output of the filter equal to 50 ohm to match them with the characteristic impedance of the transmission line, there are two constraints need to be incorporated with GA. The first one is to have the characteristic impedance at the input equal to 50 ohm. This is a linear constraint as indicated below:

$Z(0) = Z_C + Z_1 + Z_2 + Z_3 + Z_4 + Z_5 = 50$ (In case of 5 Terms Expansion)

$Z(0) = Z_C + Z_1 + Z_2 + Z_3 + Z_4 + Z_5 + Z_6 + Z_7 + Z_8 = 50$ (In case of 8 Terms Expansion)

On the other hand, the second one is nonlinear constraint which matches the characteristic impedance at the load side with 50 ohm transmission line. In case of 5 Terms Expansion, the constraint is

$$Z(l) = Z_C + Z_1 \cos 2\pi b + Z_2 \cos 4\pi b + Z_3 \cos 6\pi b + Z_4 \cos 8\pi b + Z_5 \cos 10\pi b$$
$$+ Z_6 \sin 2\pi b + Z_7 \sin 4\pi b + Z_8 \sin 6\pi b + Z_9 \sin 8\pi b + Z_{10} \sin 10\pi b = 50$$

while in case of 8 Terms Expansion, the constraint is

$$Z(l) = Z_C + Z_1 \cos 2\pi b + Z_2 \cos 4\pi b + Z_3 \cos 6\pi b + Z_4 \cos 8\pi b + Z_5 \cos 10\pi b$$
$$+ Z_6 \cos 12\pi b + Z_7 \cos 14\pi b + Z_8 \cos 16\pi b + Z_9 \sin 2\pi b + Z_{10} \sin 4\pi b + Z_{11} \sin 6\pi b$$
$$+ Z_{12} \sin 8\pi b + Z_{13} \sin 10\pi b + Z_{14} \sin 12\pi b + Z_{15} \sin 14\pi b + Z_{16} \sin 16\pi b = 50.$$

4.4 Width of the Filter

Once the design results are obtained from the GA, they include the values of the Zs, the optimization constant (*b*) and the physical length of the Filter (*LF*). The design starts by assigning the dielectric thickness (d) and the dielectric constant (ϵ_r). We take d = 1.575 mm and ϵ_r = 6 based on the datasheet on microwave laminates from "ARLON Company". So, the only thing left and not found yet is the Width of the filter. However, for a given characteristic impedance Z_0 and dielectric constant ϵ_r, the width (W) over the thickness (d) ratio can be calculated as [2]:

For W/d < 2

$$\frac{W}{d} = \frac{8e^{A}}{e^{2A} - 2}$$

For W/d >2

$$\frac{W}{d} = \frac{2}{\pi}\left[B - 1 - \ln(2B - 1) + \frac{\varepsilon_r - 1}{2\varepsilon_r}\left\{\ln(B - 1) + 0.39 - \frac{0.61}{\varepsilon_r}\right\}\right]$$

where

$$A = \frac{Z_0}{60}\sqrt{\frac{\varepsilon_r + 1}{2}} + \frac{\varepsilon_r - 1}{\varepsilon_r + 1}\left(0.23 + \frac{0.11}{\varepsilon_r}\right)$$

$$B = \frac{377\pi}{2Z_0\sqrt{\varepsilon_r}}$$

Now, all the required values of the parameters and the variables to implement the filter are obtained. The next step is to confirm our findings through either building and testing the element or through simulation. The building process is attempted, but due to the lack of the proper device, we had to resort to simulation using professional simulator software.

CHAPTER 5

RESULTS

In this chapter, the obtained results from applying either the five terms expansion or the eight terms expansion analysis in obtaining the filter parameters (length and all coefficients of the impedance expansion) as outlined in the following results and graphs.

5.1 Results of Five Terms Expansion

5.1.1 Filters with Cut-off at 3 GHz

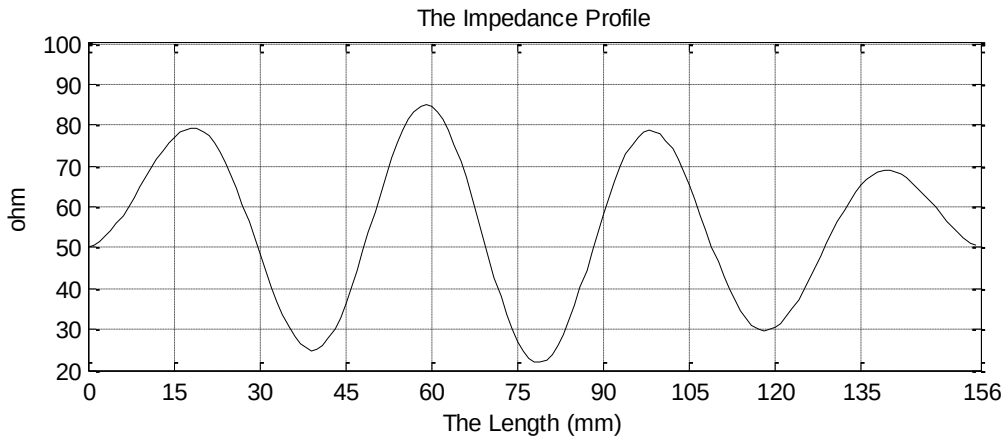
In this section, the obtained filter designs at 3 GHz including low pass filter, band stop filter, high pass filter and band pass filter are presented.

5.1.1.1 Low Pass Filter

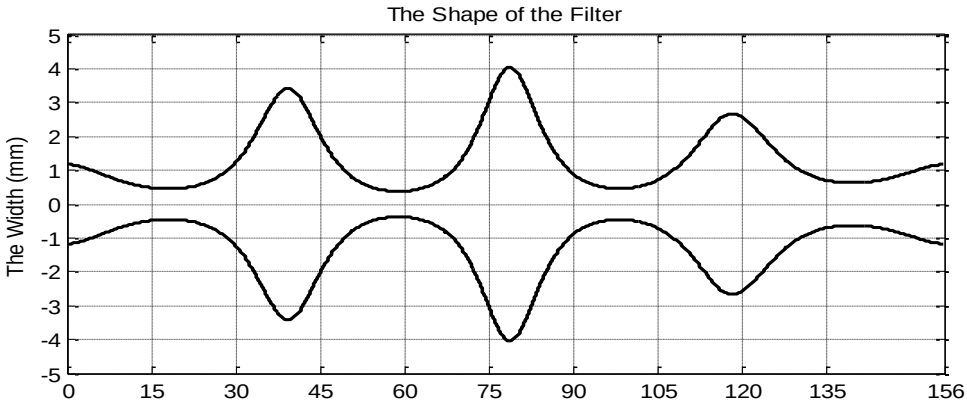
A design of low pass filter has been achieved with a length of 155.98 mm and a value of 0.9855 for the optimization constant (*b*). The values of Zs are as follows:

Zc = 54.69, Z_1 = 4.17, Z_2 = 4.31, Z_3 = 6.74, Z_4 = -23.02, Z_5 = 3.11, Z_6 = 1.65, Z_7 = 0.95, Z_8 = 2.44, Z_9 = 1.58 and Z_{10} = -1.64.

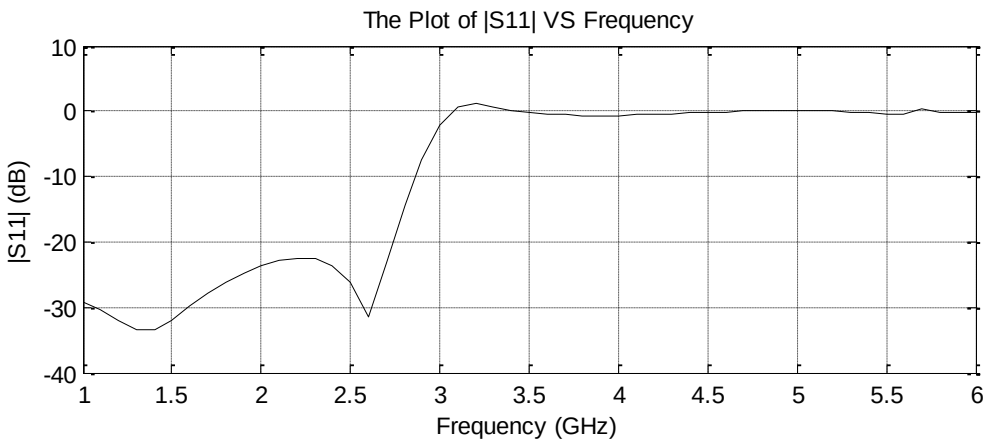
Figure 17 (a, b, c & d) shows the plot of the impedance profile, the filter shape, S_{11} and S_{21} respectively.



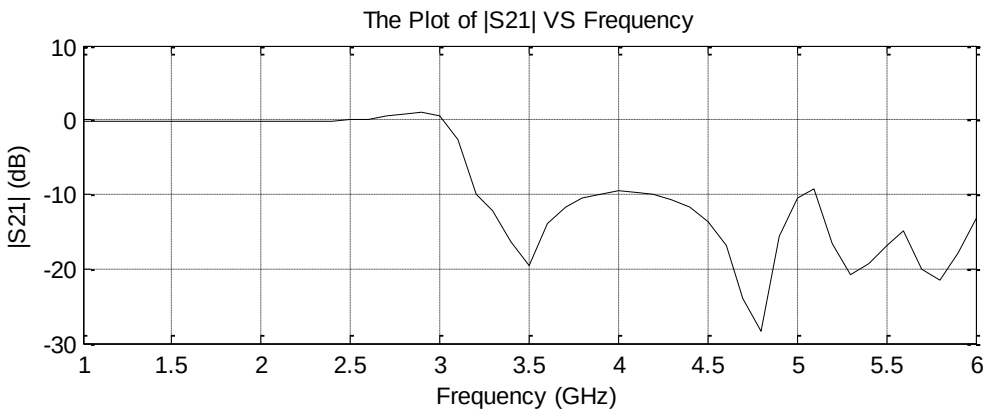
(a)



(b)



(c)



(d)

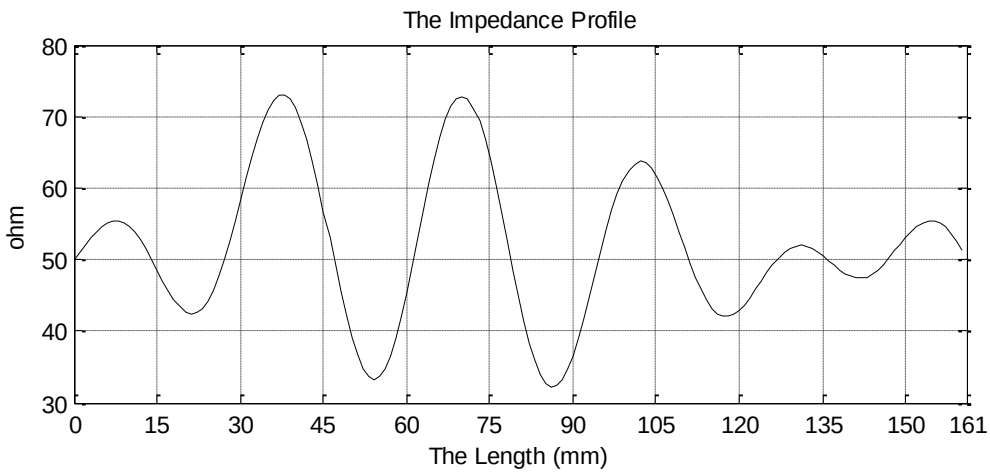
Figure 17: Results of Low Pass Filter with Cut-off at 3 GHz (5 Terms Expansion)

5.1.1.2 Band Stop Filter

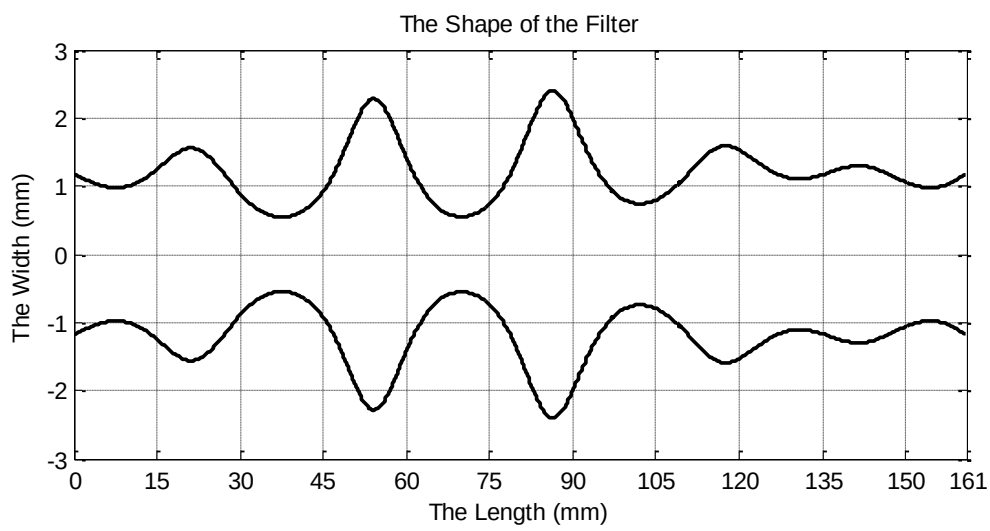
The values of the Zs that are satisfying the response of band stop filter at 3 GHz are as the following:

Zc = 51.61, Z_1 = -0.9536, Z_2 = -1.48, Z_3 = -1.18, Z_4 = 9.13, Z_5 = -7.13, Z_6 = 2.74, Z_7 = 0.0628, Z_8 = -0.3503, Z_9 = -4.52 and Z_{10} = 7.51.

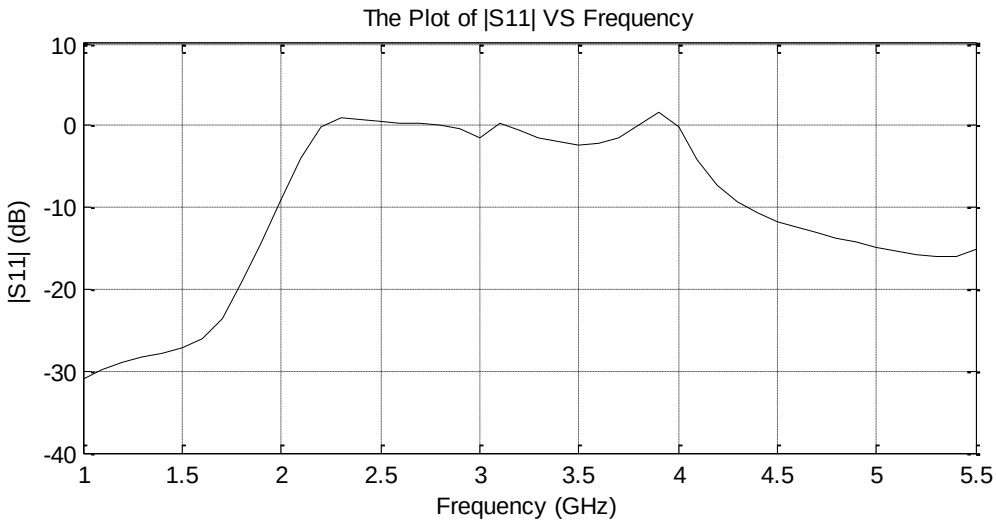
The filter length is 160.76 mm and the optimization constant (*b*) is 1.0962. The plot of the impedance profile, the filter shape and the profile plots of S_{11} and S_{21} are shown below in Figure 18 (a, b, c & d) respectively.



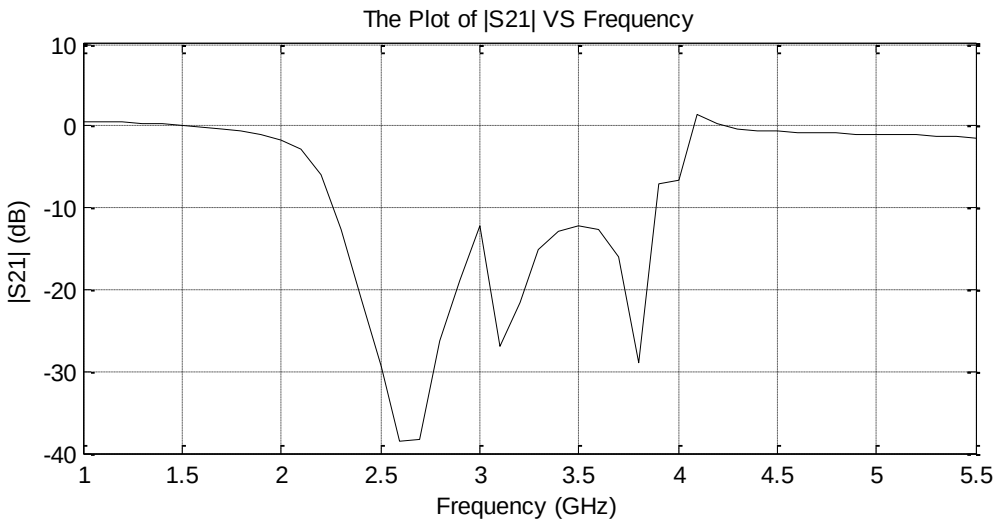
(a)



(b)



(c)



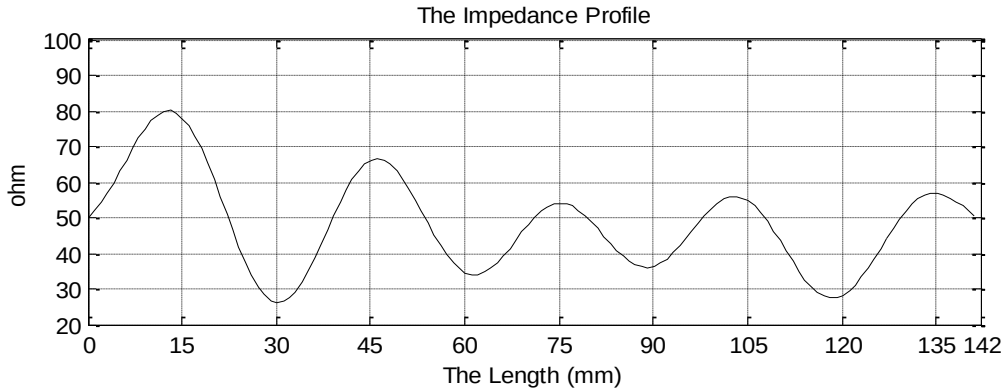
(d)

Figure 18: Results of Band Stop Filter at 3 GHz (5 Terms Expansion)

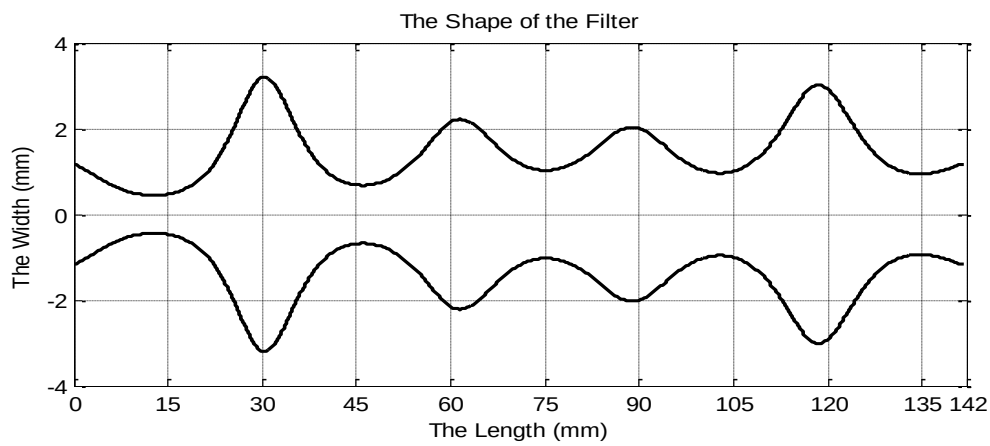
5.1.1.3 Band Pass Filter

The designed band pass filter at 3 GHz has 141.58 mm as a length and 0.9574 as a value of the optimization constant (*b*). The plot of the impedance profile of the designed filter, the filter shape and the plots of S_{11} and S_{21} are shown below in Figure 19 (a, b, c & d) respectively. The values of Zs are as follows:

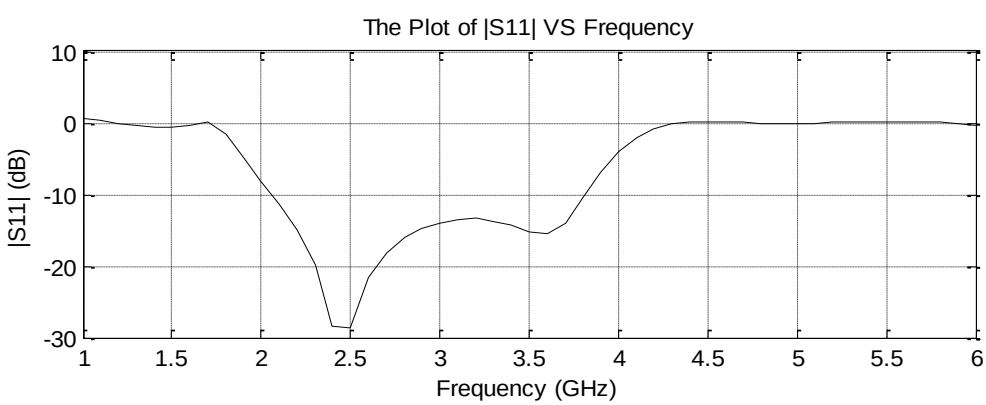
$Zc = 48.49$, $Z_1 = 5.30$, $Z_2 = 5.14$, $Z_3 = 6.33$, $Z_4 = -1.81$, $Z_5 = -13.45$, $Z_6 = 4.25$, $Z_7 = 3.12$, $Z_8 = 3.17$, $Z_9 = 4.69$ and $Z_{10} = -0.7052$.



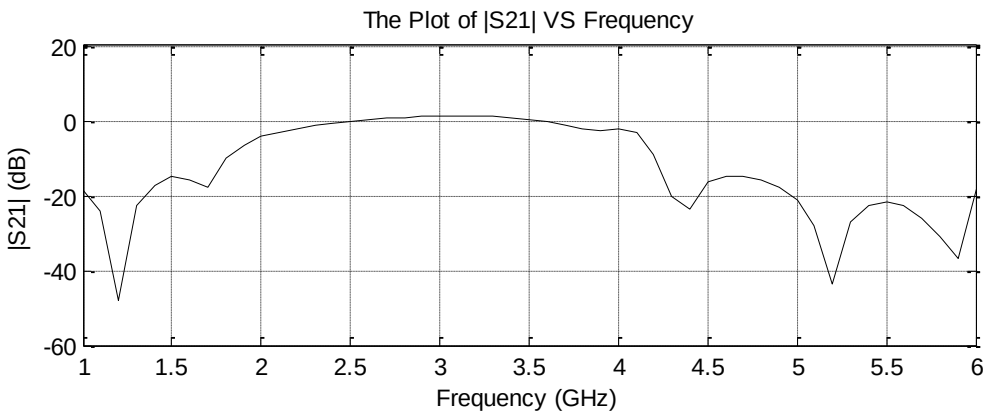
(a)



(b)



(c)



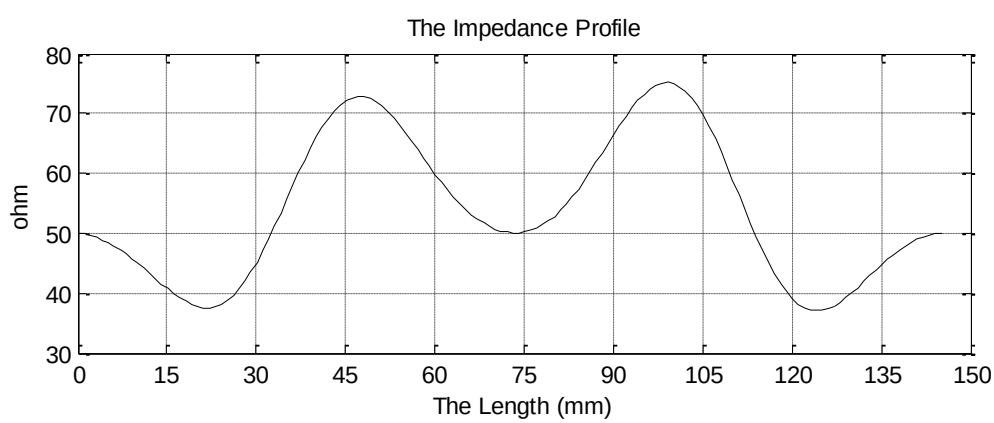
(d)

Figure 19: Results of Band Pass Filter at 3 GHz (5 Terms Expansion)

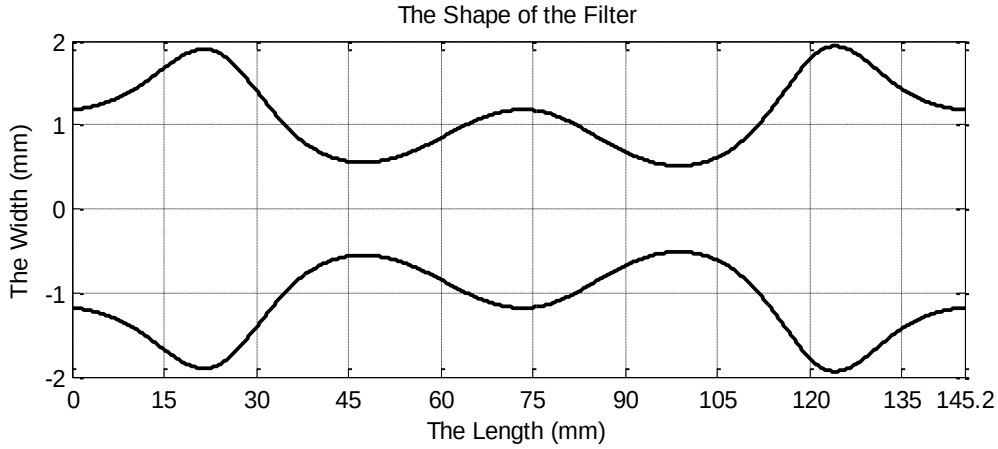
5.1.1.4 High Pass Filter

The designed high pass filter at 3 GHz has 145.21 mm as a length and 0.98784 as a value of the optimization constant (*b*). The plot of the impedance profile, the filter shape, the plot of S_{11} and the plot of S_{21} of the designed filter are shown in below Figure respectively (Figure 20: a, b, c & d). The values of Zs are as follows:

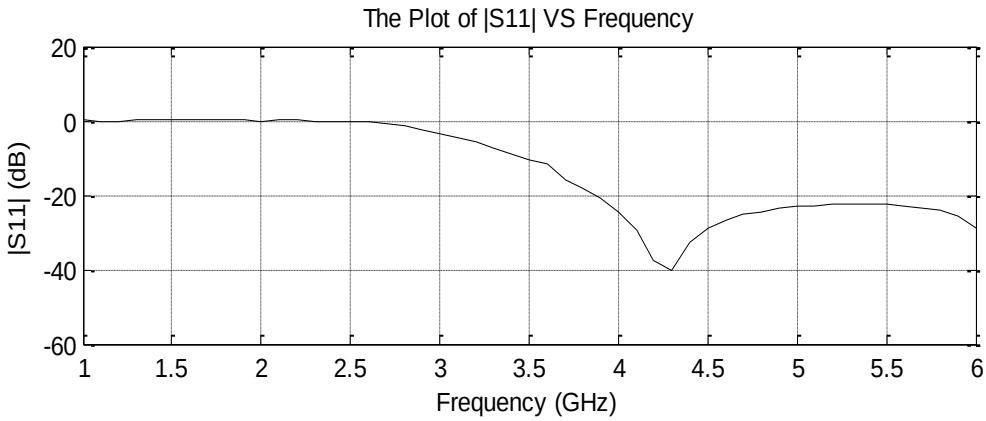
Zc = 53.89, Z_1 = -10.38, Z_2 = -4.448, Z_3 = 11.96, Z_4 = 0.568833, Z_5 = -1.5877, Z_6 = -0.305975, Z_7 = 0.515413, Z_8 = -0.584391, Z_9 = -0.501818 and Z_{10} = 0.161258.



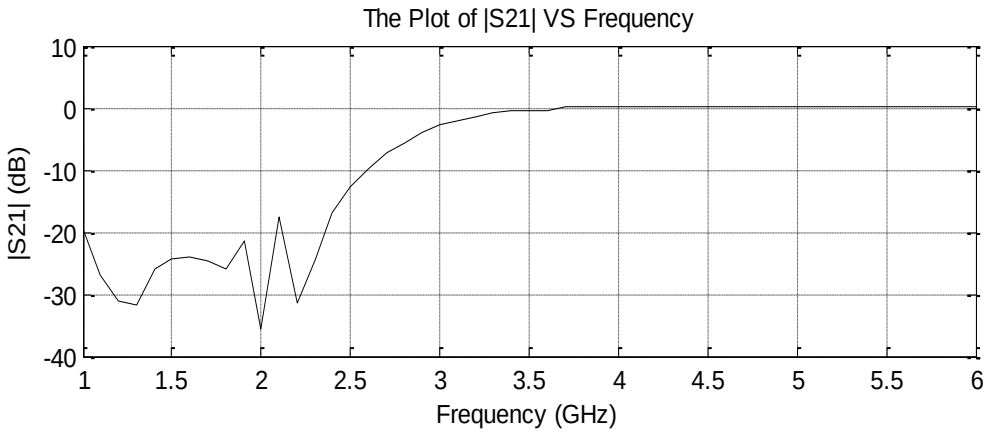
(a)



(b)



(c)



(d)

Figure 20: Results of High Pass Filter with Cut-off at 3 GHz (5 Terms Expansion)

5.1.2 Filters at Frequencies Higher than 3 GHz

The design of filters with cutoff frequencies higher than 3 GHz by means of five terms expansion was not possible through a large number of trials. This is due to the need for more oscillations in the impedance profile to satisfy the desired response. In order to have more oscillations in the impedance profile, the expansion should be performed on number of terms higher than the current expansion.

5.2 Results of Eight Terms Expansion

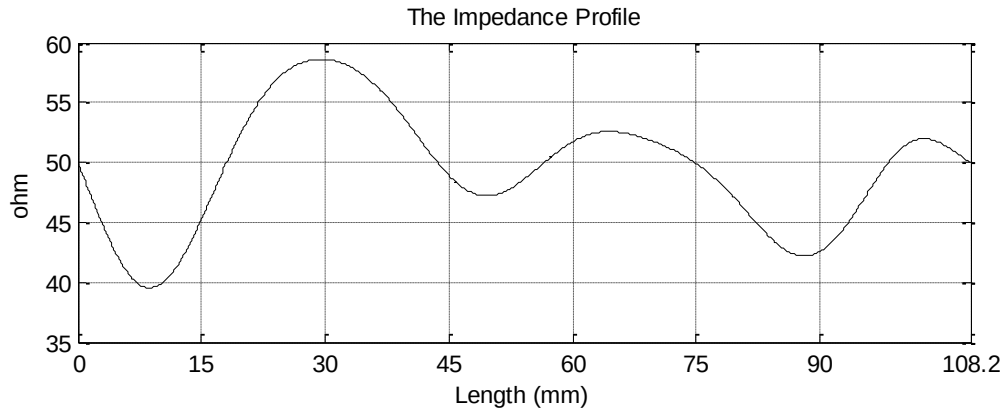
5.2.1 Filters with Cut-off at 3 GHz

5.2.1.1 Low Pass Filter

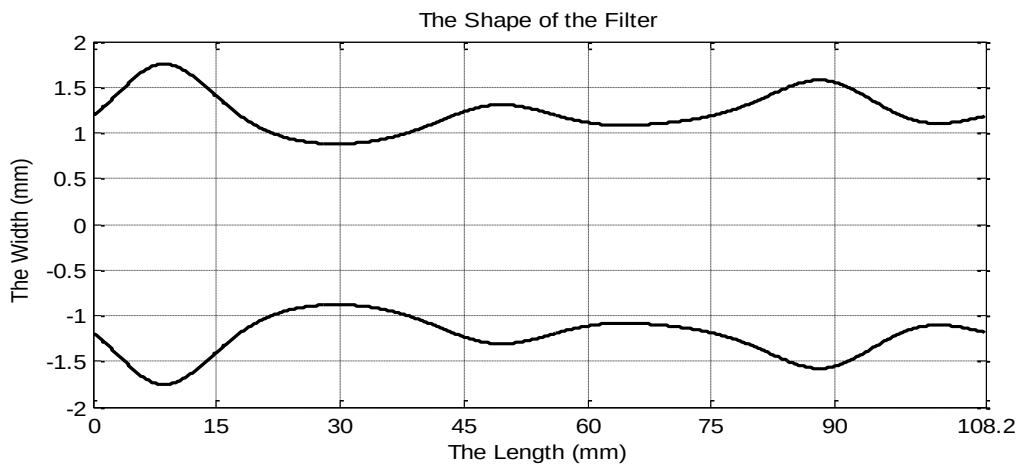
The values of the Zs that are satisfying the response of low pass filter at 3 GHz are as the following:

Zc = 50.311, Z_1 = 0.85533, Z_2 = -2.42054, Z_3 = 0.3662567, Z_4 = -1.96868, Z_5 = 0.806049, Z_6 = 0.35959, Z_7 = 1.8608, Z_8 = -0.16935, Z_9 = 0.248715, Z_{10} = 0.527828, Z_{11} = -1.27853, Z_{12} = -3.9273, Z_{13} = -2.60142, Z_{14} = 0.39674, Z_{15} = -0.73389 and Z_{16} = -1.64497.

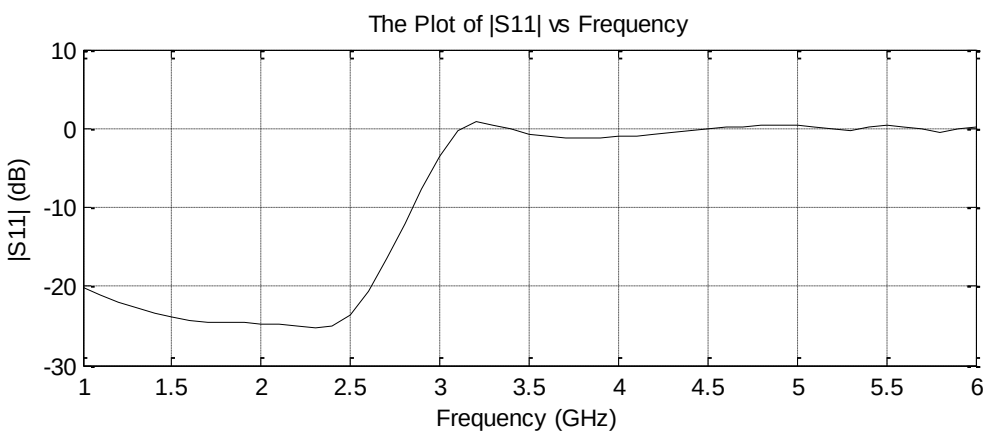
The filter length is 108.18 mm and the optimization constant (*b*) is 0.61844. The plot of the impedance profile, the filter shape, the plot of S_{11} and the plot of S_{21} are shown below in Figure 21 (a, b, c & d) respectively.


The Impedance Profile
ohm
60
55
50
45
40
35
0
15
30
45
60
75
90
108.2
Length (mm)

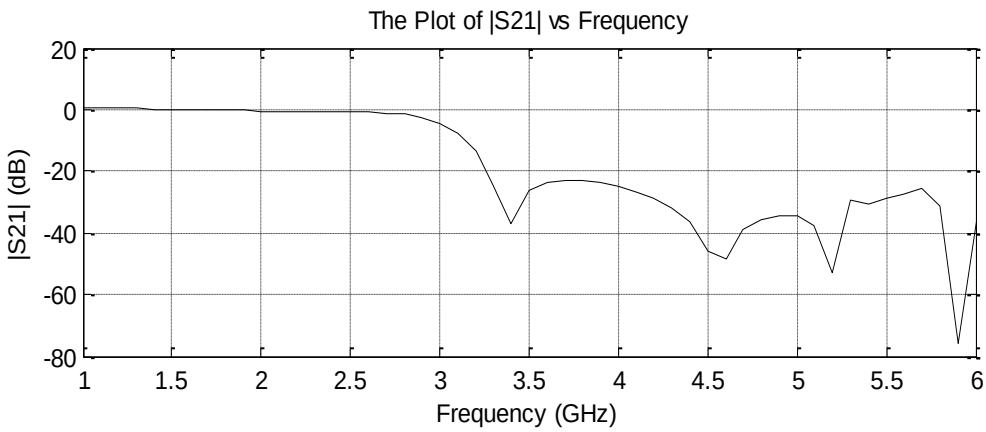
(a)


The Shape of the Filter
The Width (mm)
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
0
15
30
45
60
75
90
108.2
The Length (mm)

(b)


The Plot of |S11| vs Frequency
|S11| (dB)
10
0
-10
-20
-30
1
1.5
2
2.5
3
3.5
4
4.5
5
5.5
6
Frequency (GHz)

(c)



(d)

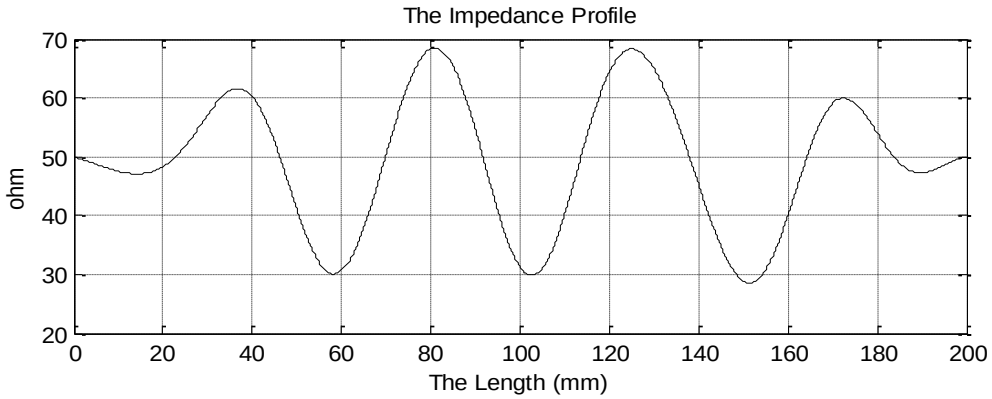
Figure 21: Results of Low Pass Filter with Cut-off at 3 GHz (8 Terms Expansion)

5.2.1.2 Band Stop Filter

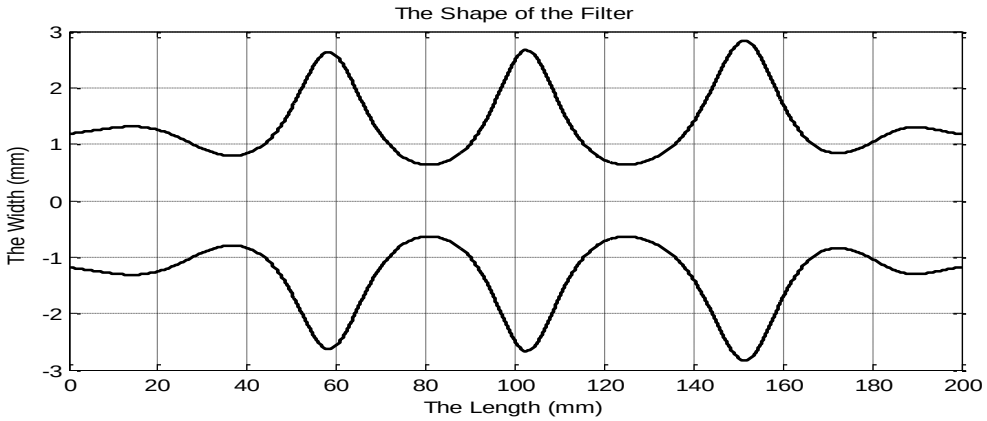
The designed band stop filter at 3 GHz has 200 mm as a length and 0.96978 as a value of the constant *b*. The values of Zs are as follows:

Zc = 49.20, Z_1 = 0.3234, Z_2 = 2.33, Z_3 = -0.4236, Z_4 = -11.31, Z_5 = 10.11, Z_6 = 0.6104, Z_7 = 0.0471, Z_8 = -0.8852, Z_9 = 0.7214, Z_{10} = 0.8499, Z_{11} = -1.37, Z_{12} = -0.9782, Z_{13} = 0.6426, Z_{14} = 1.06, Z_{15} = -0.9138 and Z_{16} = -0.4270.

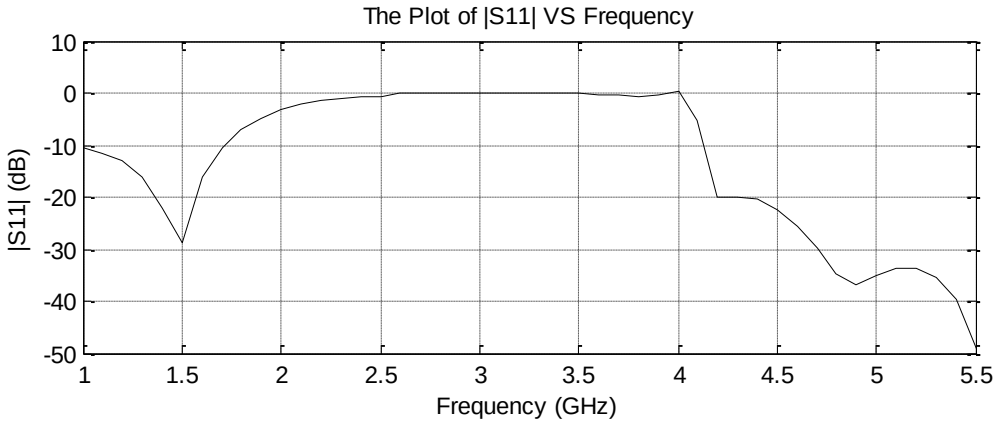
The plot of the impedance profile, the filter shape and the profile plots of S_{11} and S_{21} of the designed filter are shown in below Figure respectively (Figure 22: a, b, c & d).


The Impedance Profile
ohm
70
60
50
40
30
20
0
20
40
60
80
100
120
140
160
180
200
The Length (mm)

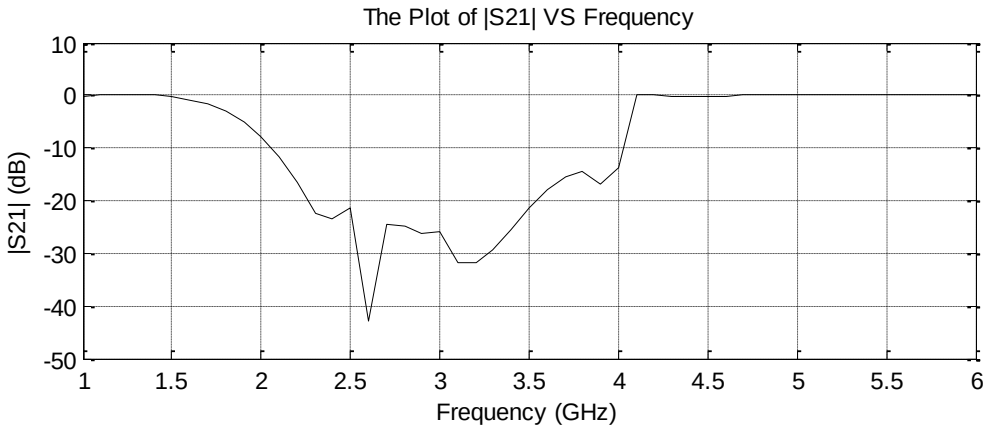
(a)


The Shape of the Filter
The Width (mm)
3
2
1
0
-1
-2
-3
0
20
40
60
80
100
120
140
160
180
200
The Length (mm)

(b)


The Plot of |S11| VS Frequency
|S11| (dB)
10
0
-10
-20
-30
-40
-50
1
1.5
2
2.5
3
3.5
4
4.5
5
5.5
Frequency (GHz)

(c)



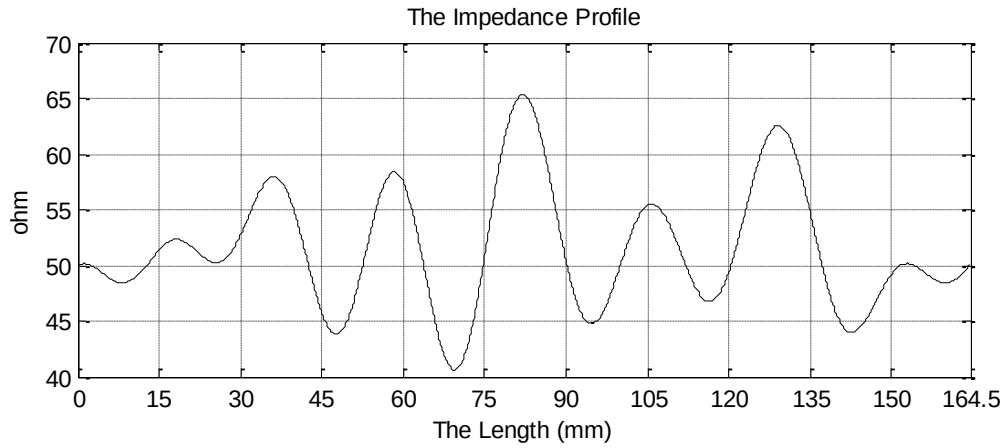
(d)

Figure 22: Results of Band Stop Filter at 3 GHz (8 Terms Expansion)

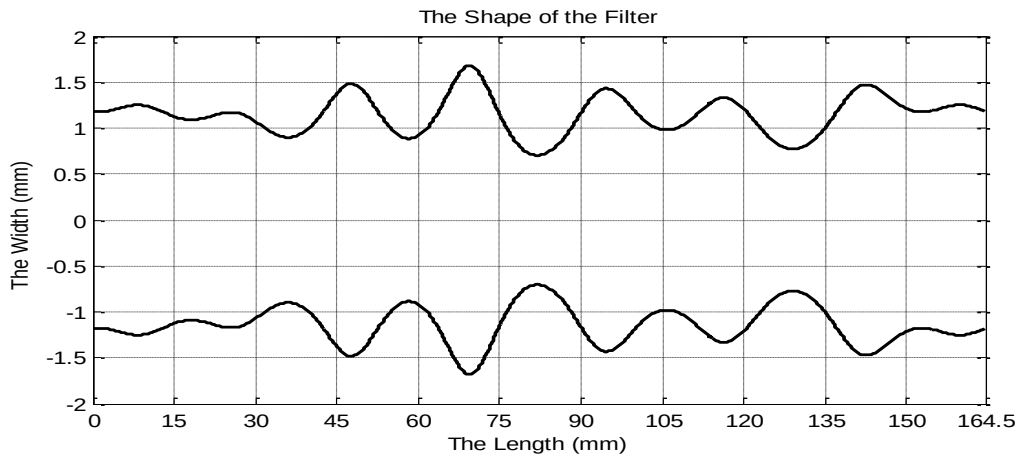
5.2.1.3 Band Pass Filter

The designed band pass filter at 3 GHz has 164.46 mm as a length and 1.08463 as a value of the constant *b*. The plot of the impedance profile, the filter shape and the plots of S_{11} and S_{21} of the designed filter are shown in below Figure (Figure 23: a, b, c & d) respectively. The values of Zs are as follows:

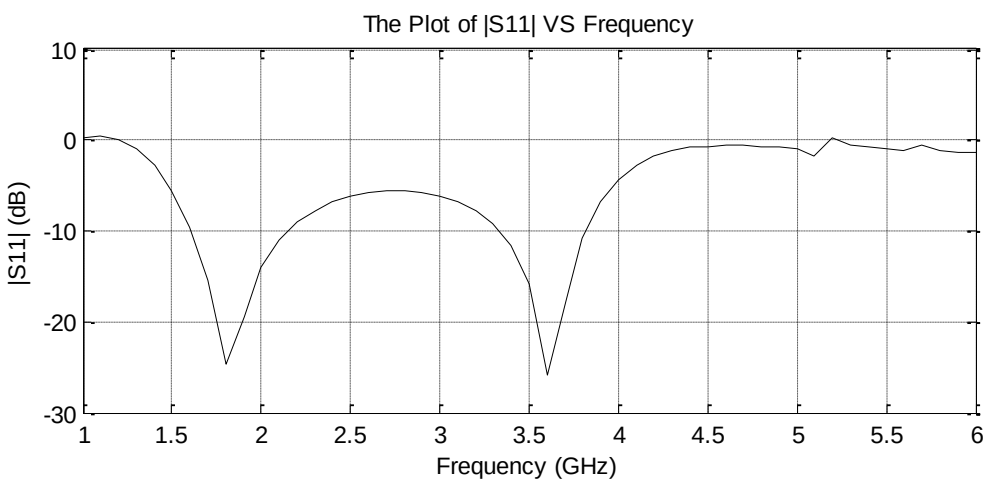
Zc = 51.66, Z_1 = -0.7686, Z_2 = -0.8779, Z_3 = -2.7489, Z_4 = -0.6512, Z_5 = 0.26828, Z_6 = 0.40352, Z_7 = 1.51085, Z_8 = 1.2024, Z_9 = -0.71457, Z_{10} = 0.12074, Z_{11} = -1.423, Z_{12} = 1.8772, Z_{13} = -0.19283, Z_{14} = 5.1157, Z_{15} = -4.0456 and Z_{16} = 0.300332786.


The Impedance Profile
ohm
70
65
60
55
50
45
40
0
15
30
45
60
75
90
105
120
135
150
164.5
The Length (mm)

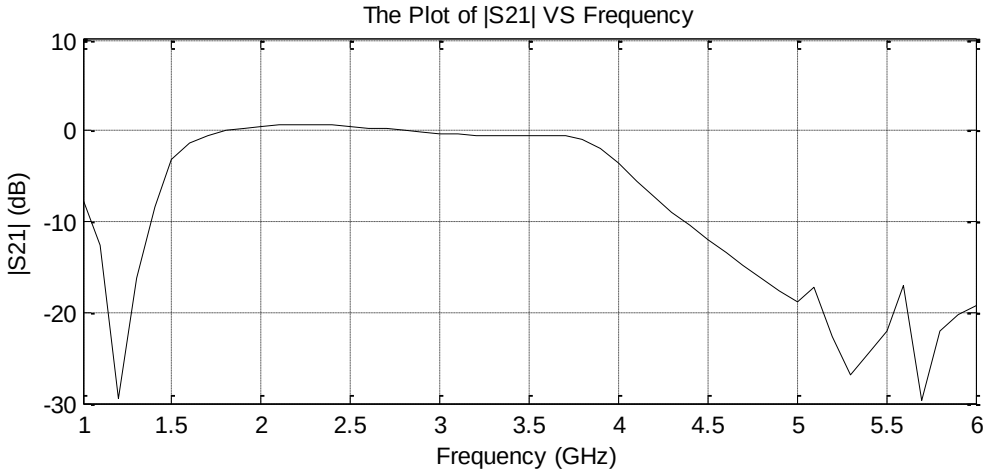
(a)


The Shape of the Filter
The Width (mm)
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
0
15
30
45
60
75
90
105
120
135
150
164.5
The Length (mm)

(b)


The Plot of |S11| VS Frequency
|S11| (dB)
10
0
-10
-20
-30
1
1.5
2
2.5
3
3.5
4
4.5
5
5.5
6
Frequency (GHz)

(c)



(d)

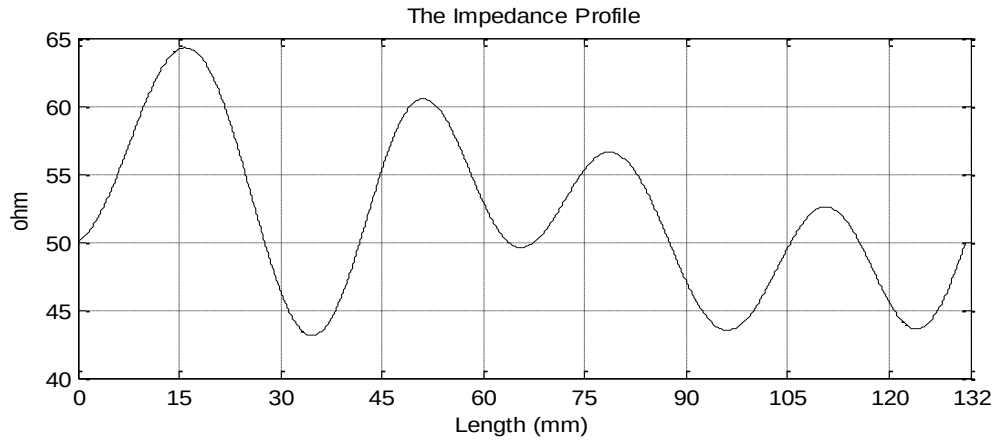
Figure 23: Results of Band Pass Filter at 3 GHz (8 Terms Expansion)

5.2.1.4 High Pass Filter

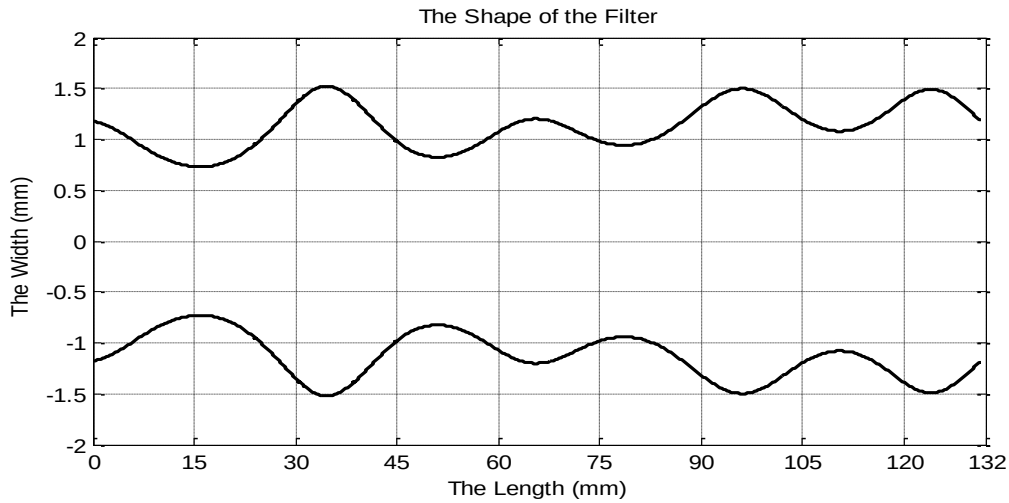
The designed high pass filter at 3 GHz has 131.294 mm as a length and 0.742195 as a value of the constant *b*. The values of Zs are as the following:

Zc = 51.9, Z_1 = 1.87827, Z_2 = 0.383658, Z_3 = 0.87141, Z_4 = -0.556578, Z_5 = -1.43934, Z_6 = -2.21986, Z_7 = -0.7112, Z_8 = -0.10586, Z_9 = 2.34224, Z_{10} = -0.65619, Z_{11} = 2.7229, Z_{12} = 2.7388, Z_{13} = 1.73304, Z_{14} = -4.3139, Z_{15} = 0.2533 and Z_{16} = 0.539473.

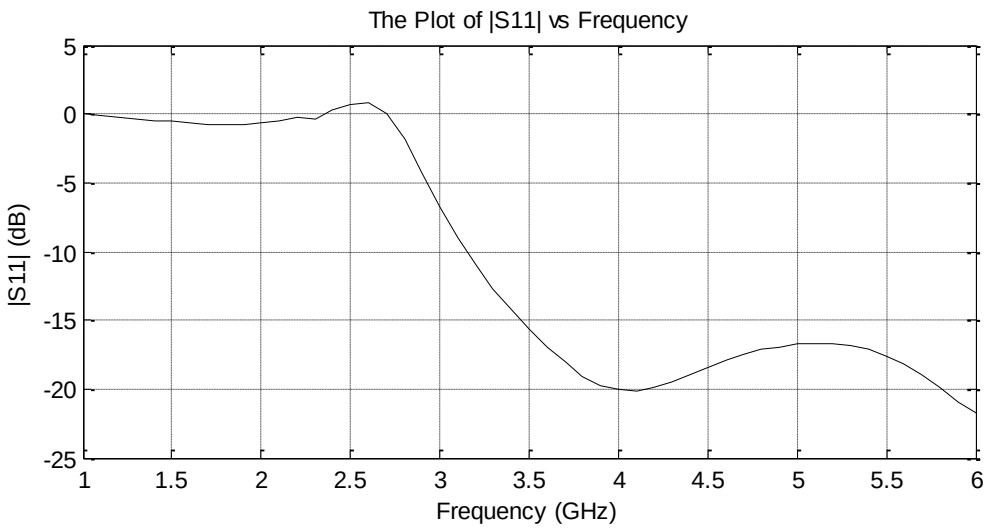
The plot of the impedance profile of the designed filter, the shape of the filter and the plots of S_{11} and S_{21} are shown in Figure 24 respectively.


The Impedance Profile
ohm
65
60
55
50
45
40
0
15
30
45
60
75
90
105
120
132
Length (mm)

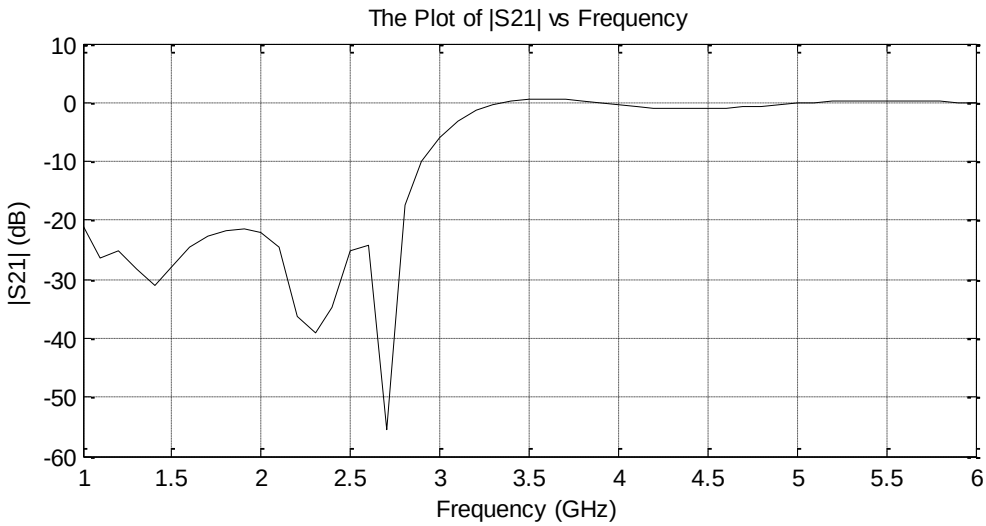
(a)


The Shape of the Filter
The Width (mm)
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
0
15
30
45
60
75
90
105
120
132
The Length (mm)

(b)


The Plot of |S11| vs Frequency
|S11| (dB)
5
0
-5
-10
-15
-20
-25
1
1.5
2
2.5
3
3.5
4
4.5
5
5.5
6
Frequency (GHz)

(c)



(d)

Figure 24: Results of High Pass Filter with Cut-off at 3 GHz (8 Terms Expansion)

5.2.1.5 Comparisons Between Filters at 3 GHz (5 & 8 Terms Expansions)

Up to now, filters at 3 GHz have been designed based on this method with five terms expansion and eight terms expansion. In this section, the responses of low pass filters and high pass filters with both five and eight terms expansion are going to be compared. However, before we do that, there is a worth mentioning point related to the length of the filters needs to be highlighted. During the design of filters at 3 GHz with eight terms expansion, after finishing the designs with five terms expansion, the length was chosen to be shorter than five terms expansion case when designing low and high pass filters and chosen to be longer when designing band pass and band stop filters.

In both cases, the designs have been achieved. This shows one of the main advantages of this methodology. This advantage is about having a great deal of flexibility in the design process which leads to variety of filters shapes, sizes and lengths satisfying exactly the same objectives and criteria.

Figure 25 and 26 show the plots of the return loss ($|S_{11}|$) and the insertion loss ($|S_{21}|$) of LPF designed with five and eight terms expansion respectively. Both are having very sharp cutoffs especially the one designed with five terms expansion. However, the values of the return and insertion losses in the case of the eight terms expansion are less than the five terms expansion case especially the values of the insertion loss.

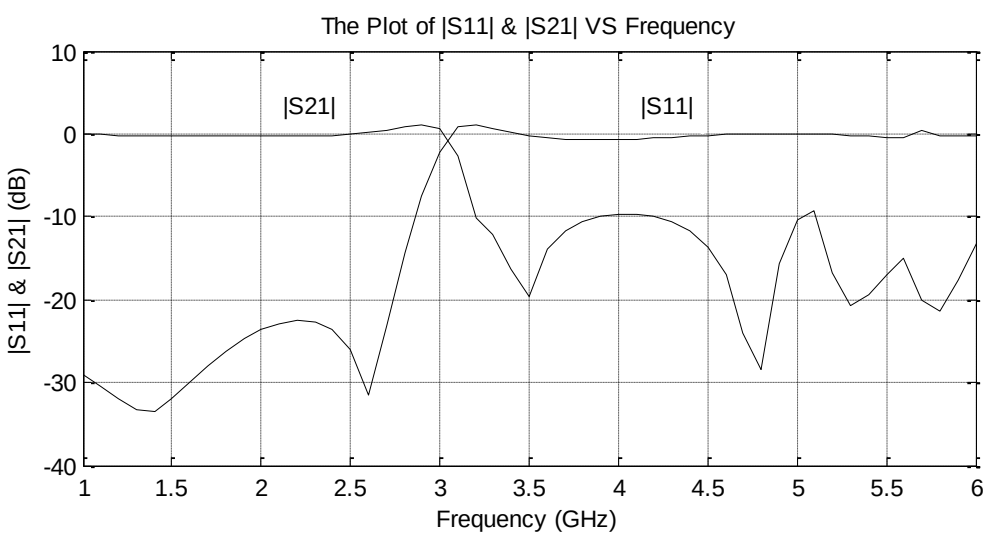


Figure 25: Return & Insertion Losses of LPF with Cut-off at 3 GHz (5 Terms Expansion)

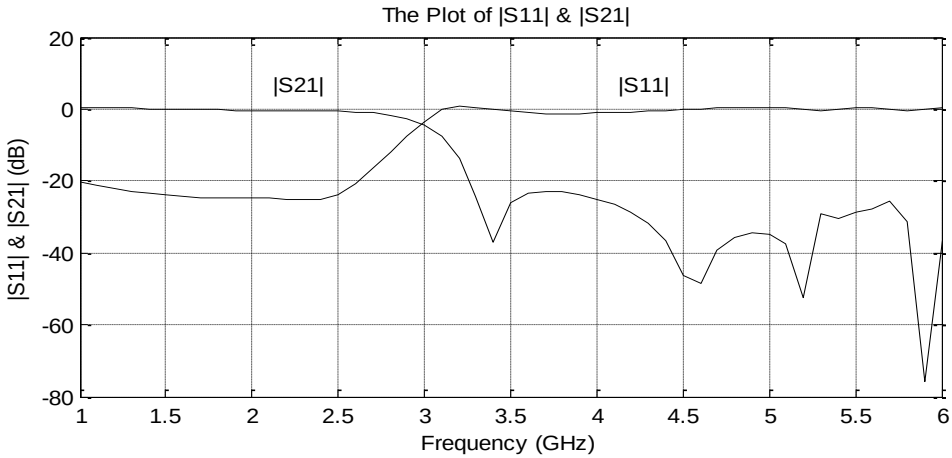


Figure 26: Return & Insertion Losses of LPF with Cut-off at 3 GHz (8 Terms Expansion)

Figure 27 and 28 show the plots of the return loss ($|S_{11}|$) and the insertion loss ($|S_{21}|$) of HPF designed with five and eight terms expansion respectively. It is so obvious that the filter designed by eight terms expansion has sharper cutoff. Also, it has less insertion loss compared to the five terms expansion case. However, the filter designed by five terms expansion has less return loss compared with eight terms expansion case. In general, in order to have filter with both sharp cutoff and minimum insertion loss, it should be designed with eight terms expansion.

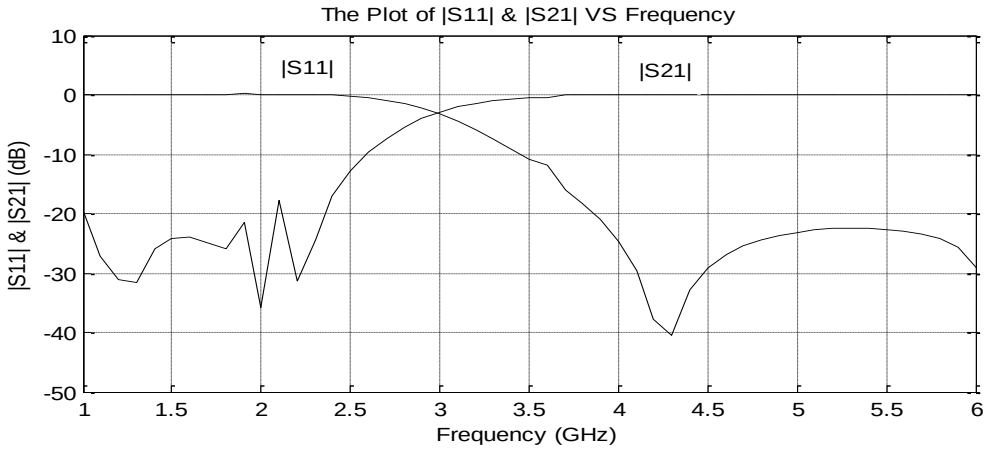


Figure 27: Return & Insertion Losses of HPF with Cut-off at 3 GHz (5 Terms Expansion)

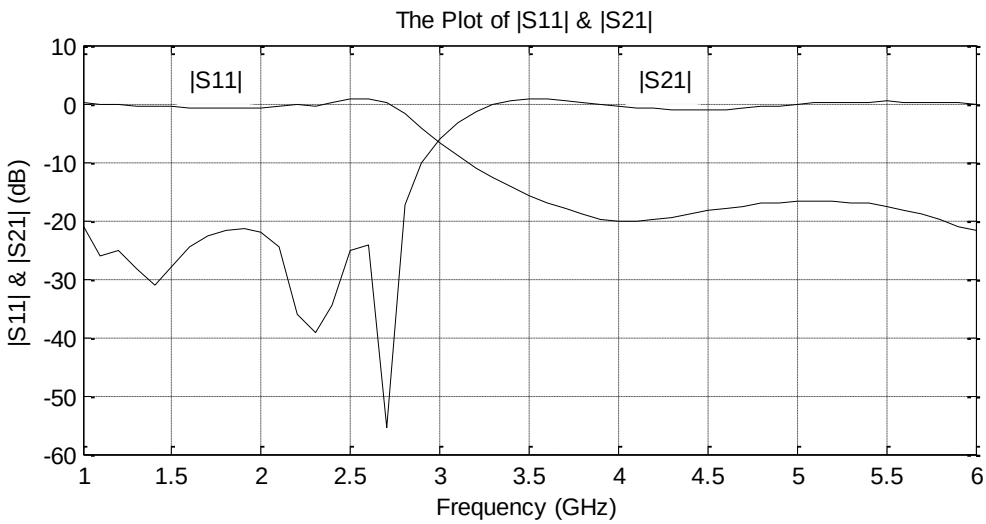


Figure 28: Return & Insertion Losses of HPF with Cut-off at 3 GHz (8 Terms Expansion)

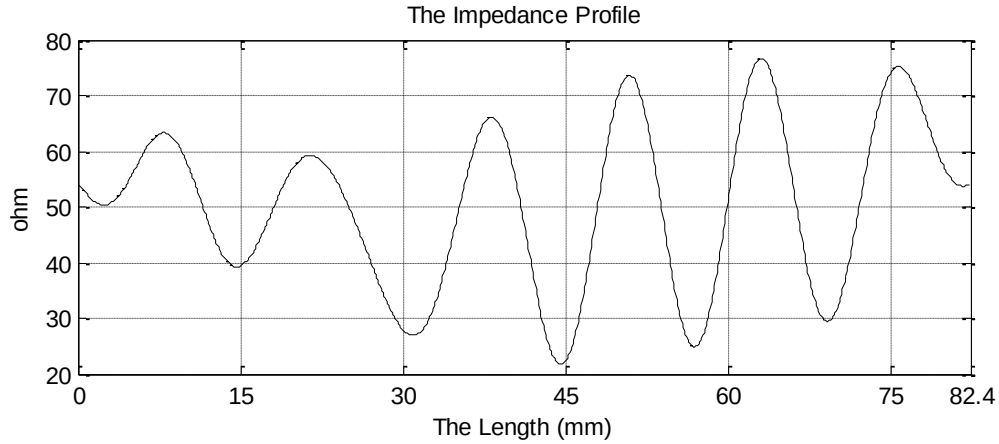
5.2.2 Filters with Cut-off at 7 GHz

5.2.2.1 Low Pass Filter

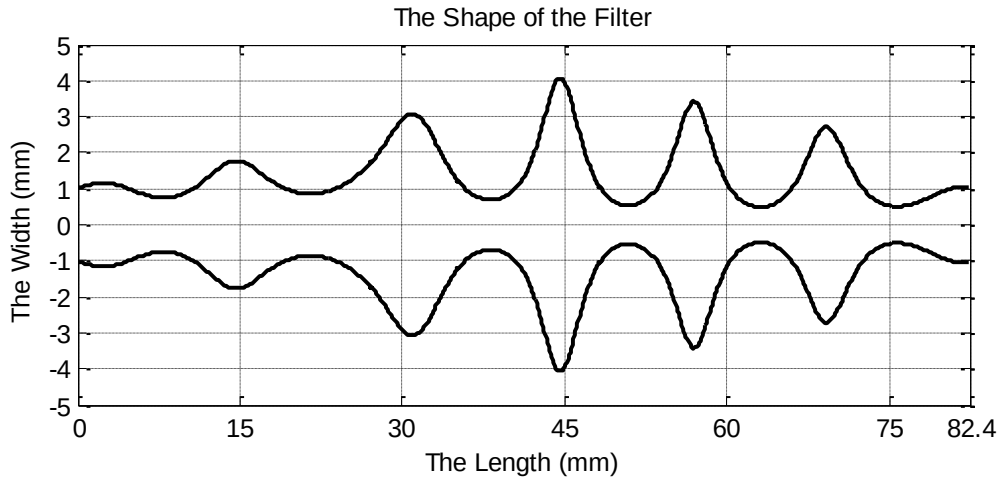
The values of the Zs that are satisfying the response of low pass filter at 7 GHz are as the following:

Zc = 50.71, Z_1 = 6.04, Z_2 = 1.11, Z_3 = -1.03, Z_4 = -0.5306, Z_5 = -1.11, Z_6 = -10.30, Z_7 = 11.79, Z_8 = -2.37, Z_9 = -3.59, Z_{10} = 0.1585, Z_{11} = -1.45, Z_{12} = -1.82, Z_{13} = 3.78, Z_{14} = 4.00, Z_{15} = -0.3882 and Z_{16} = -7.44.

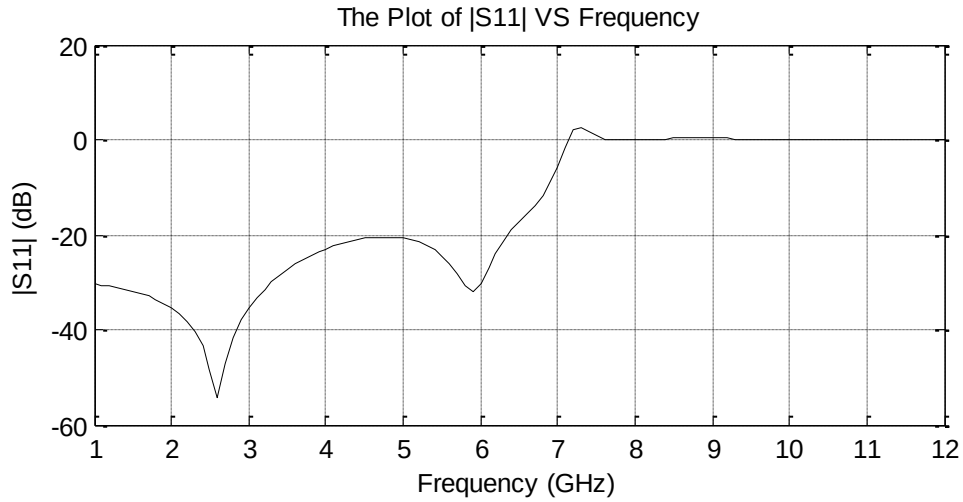
The designed filter has a length of 82.35 mm and optimization constant value of 0.9368. Figure 29 (a, b, c and d) shows the plot of the impedance profile, the filter shape, the plot of S_{11} and the plot of S_{21} respectively.


The Impedance Profile
ohm
80
70
60
50
40
30
20
0
15
30
45
60
75
82.4
The Length (mm)

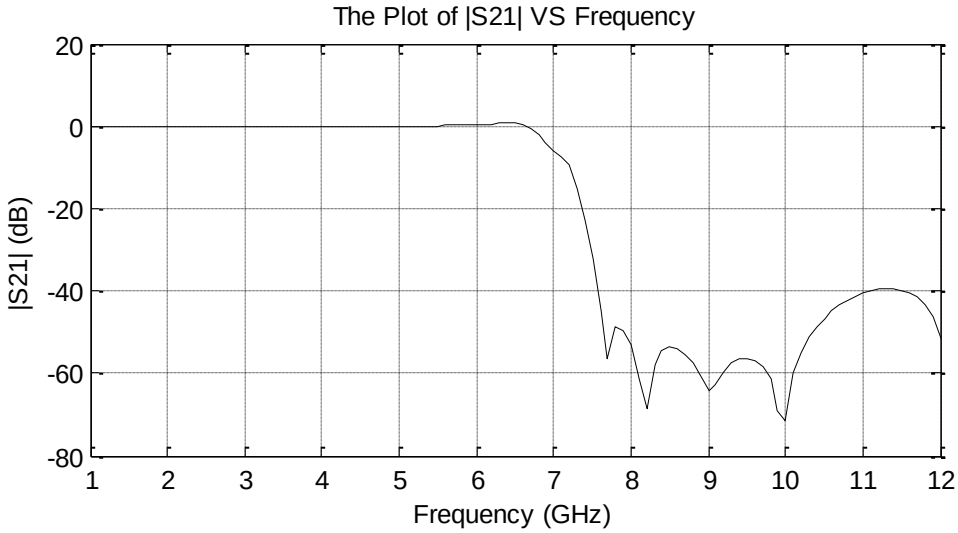
(a)


The Shape of the Filter
The Width (mm)
5
4
3
2
1
0
-1
-2
-3
-4
-5
0
15
30
45
60
75
82.4
The Length (mm)

(b)


The Plot of |S11| VS Frequency
|S11| (dB)
20
0
-20
-40
-60
1
2
3
4
5
6
7
8
9
10
11
12
Frequency (GHz)

(c)



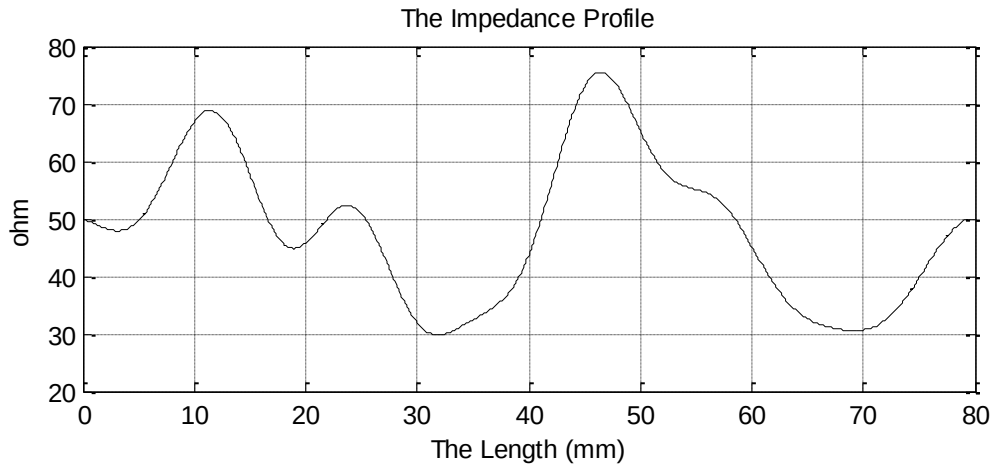
(d)

Figure 29: Results of Low Pass Filter with Cut-off at 7 GHz (8 Terms Expansion)

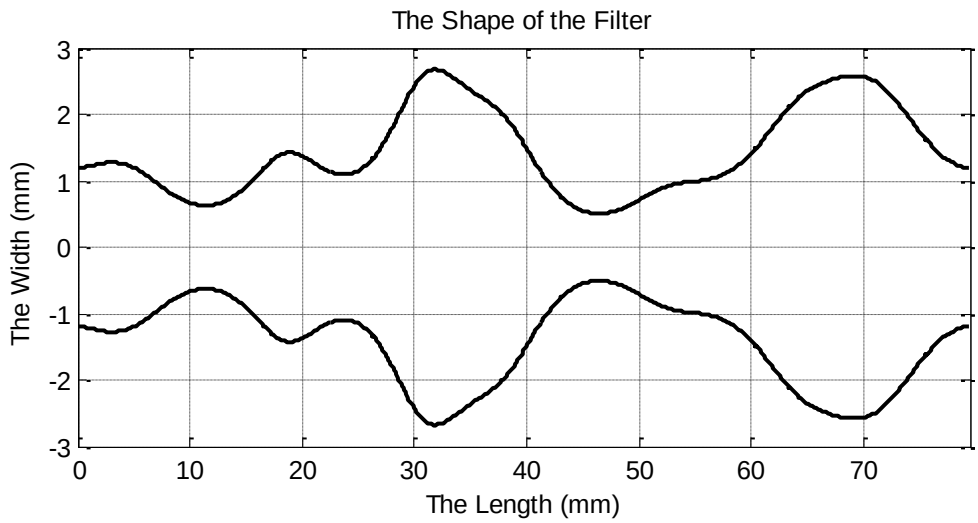
5.2.2.2 High Pass Filter

The designed high pass filter at 7 GHz has 79.49 mm as a length and 0.97613 as a value of the constant *b*. The plot of the impedance profile of the designed filter, the shape of the filter and the plots of S_{11} and S_{21} are shown in Figure 30 respectively. The values of Zs are as the following:

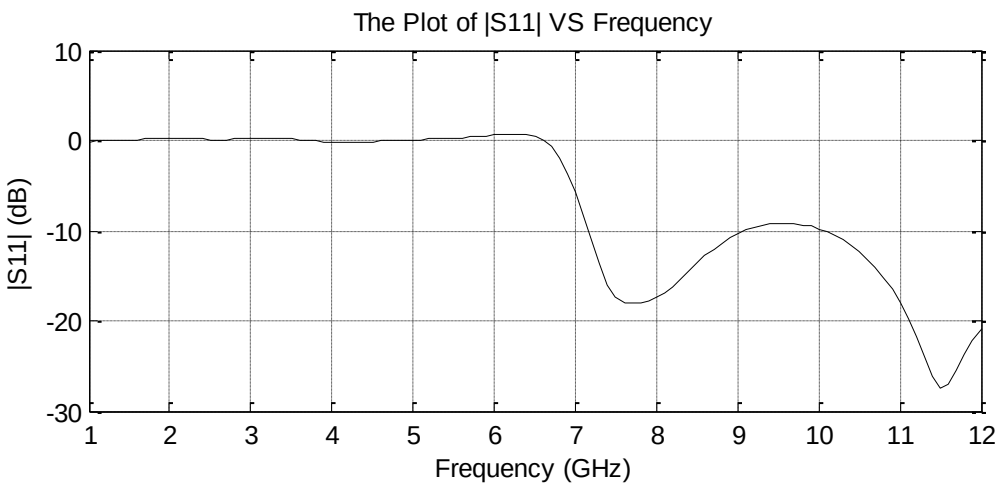
Zc = 48.1572, Z_1 = -1.60038, Z_2 = 2.07573, Z_3 = 0.939436, Z_4 = -0.730876, Z_5 = -1.42879, Z_6 = 0.362858, Z_7 = 3.283191, Z_8 = -1.057364, Z_9 = 1.61567, Z_{10} = 15.20248, Z_{11} = -5.273709, Z_{12} = 2.30311, Z_{13} = -4.911012, Z_{14} = -1.529156, Z_{15} = -0.4651368 and Z_{16} = 0.417186.


The Impedance Profile
ohm
80
70
60
50
40
30
20
0
10
20
30
40
50
60
70
80
The Length (mm)

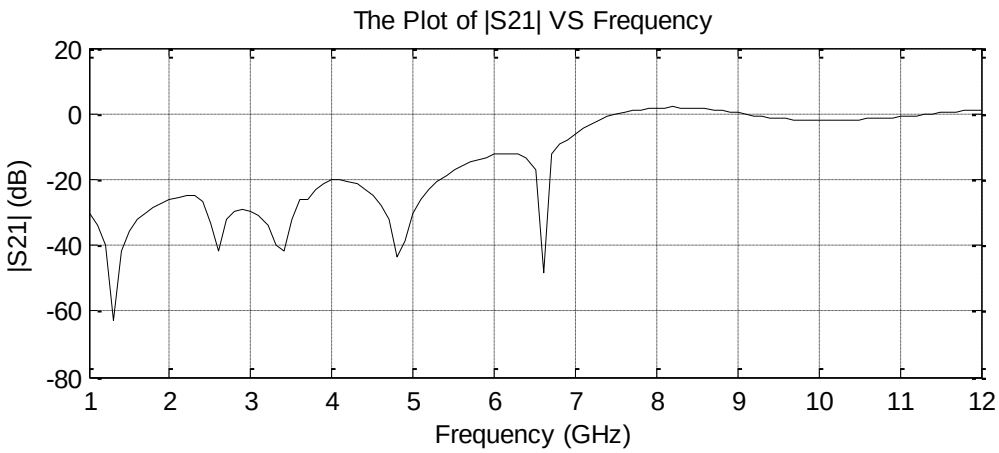
(a)


The Shape of the Filter
The Width (mm)
3
2
1
0
-1
-2
-3
0
10
20
30
40
50
60
70
The Length (mm)

(b)


The Plot of |S11| VS Frequency
|S11| (dB)
10
0
-10
-20
-30
1
2
3
4
5
6
7
8
9
10
11
12
Frequency (GHz)

(c)



(d)

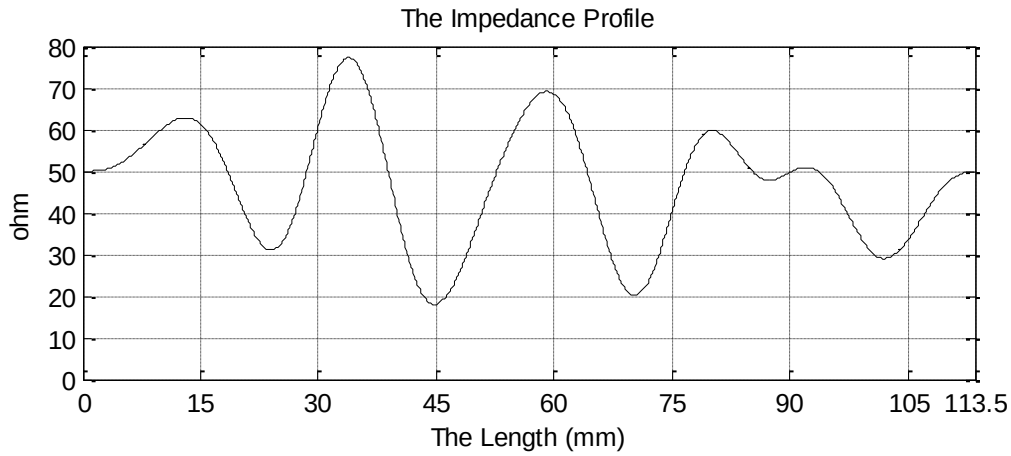
Figure 30: Results of High Pass Filter with Cut-off at 7 GHz (8 Terms Expansion)

5.2.2.3 Band Stop Filter

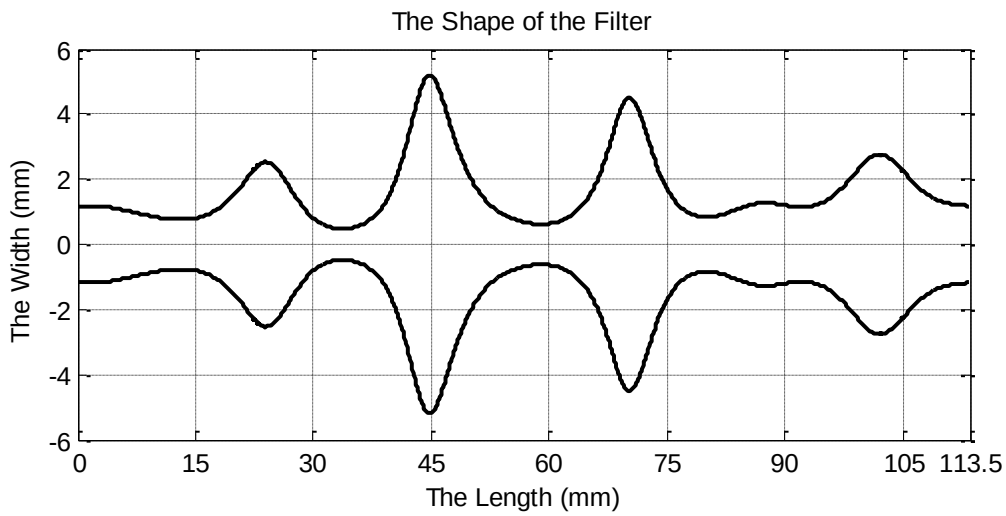
The designed band stop filter at 7 GHz has 113.44 mm as a length and 0. 9867 as a value of the optimization constant. The values of Zs are as follows:

Zc = 47.00, Z_1 = 1.44, Z_2 = -0.9598, Z_3 = -0.7358, Z_4 = 10.17, Z_5 = -12.22, Z_6 = 5.34, Z_7 = 2.75, Z_8 = -2.77, Z_9 = 3.04, Z_{10} = 3.86, Z_{11} = 2.81, Z_{12} = 6.19, Z_{13} = -1.67, Z_{14} = -4.71, Z_{15} = -2.98 and Z_{16} = 1.59.

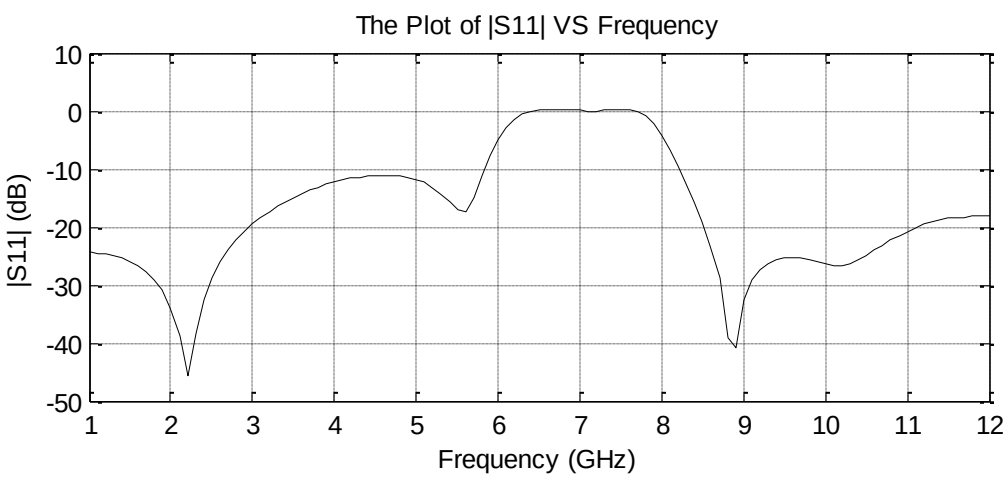
The plot of the impedance profile, the filter shape, the plot of S_{11} and the plot of S_{21} of the designed filter are shown in below Figure respectively (Figure 31: a, b, c & d).


The Impedance Profile
ohm
80
70
60
50
40
30
20
10
0
0
15
30
45
60
75
90
105
113.5
The Length (mm)

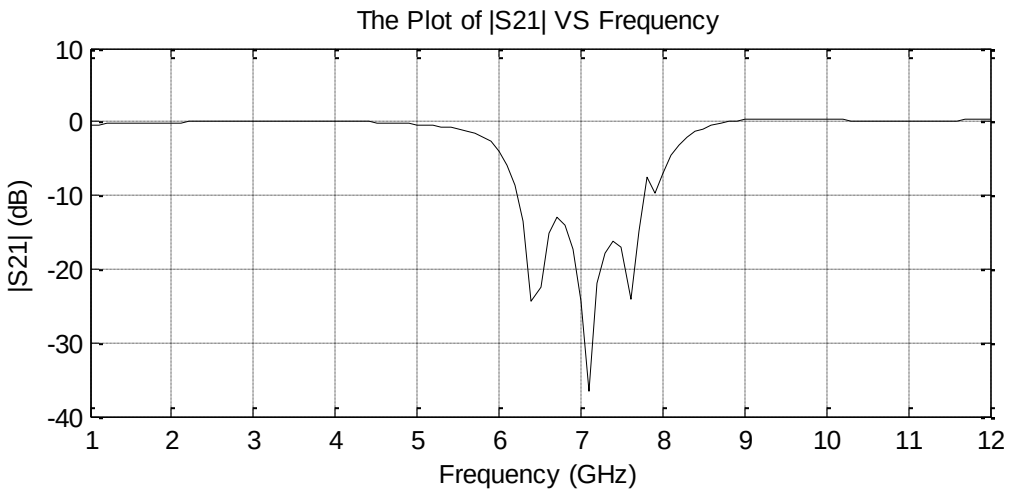
(a)


The Shape of the Filter
The Width (mm)
6
4
2
0
-2
-4
-6
0
15
30
45
60
75
90
105
113.5
The Length (mm)

(b)


The Plot of |S11| VS Frequency
|S11| (dB)
10
0
-10
-20
-30
-40
-50
1
2
3
4
5
6
7
8
9
10
11
12
Frequency (GHz)

(c)



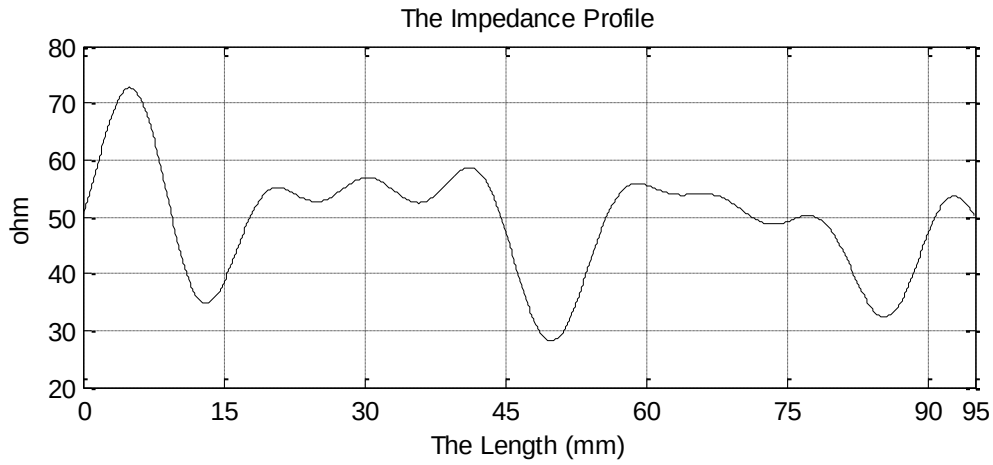
(d)

Figure 31: Results of Band Stop Filter at 7 GHz (8 Terms Expansion)

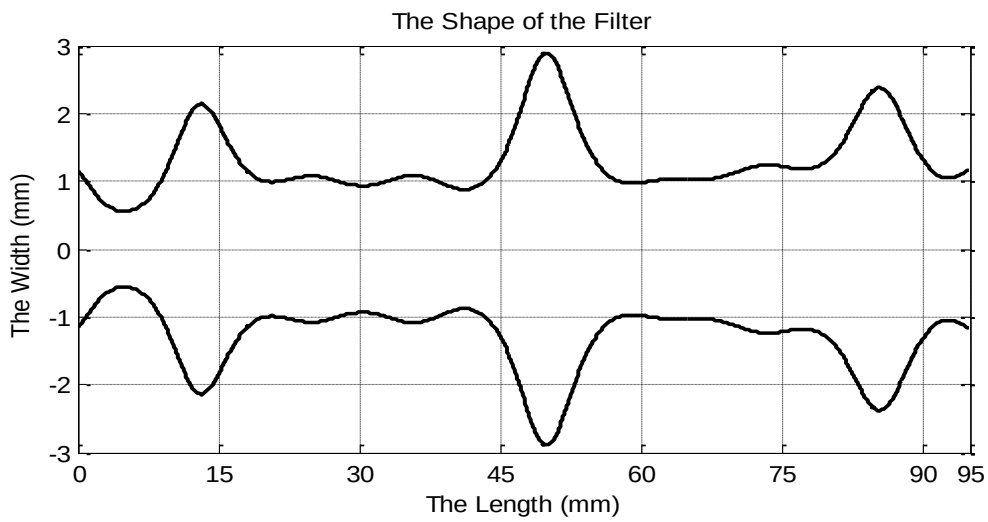
5.2.2.4 Band Pass Filter

The designed band pass filter at 7 GHz has a length of 94.98 mm and an optimization constant (*b*) value of 0.944454. Figure 32 shows the plot of the impedance profile, the plot of the filter shape and the plots of S_{11} and S_{21} of the designed filter respectively. The values of Zs are as follows:

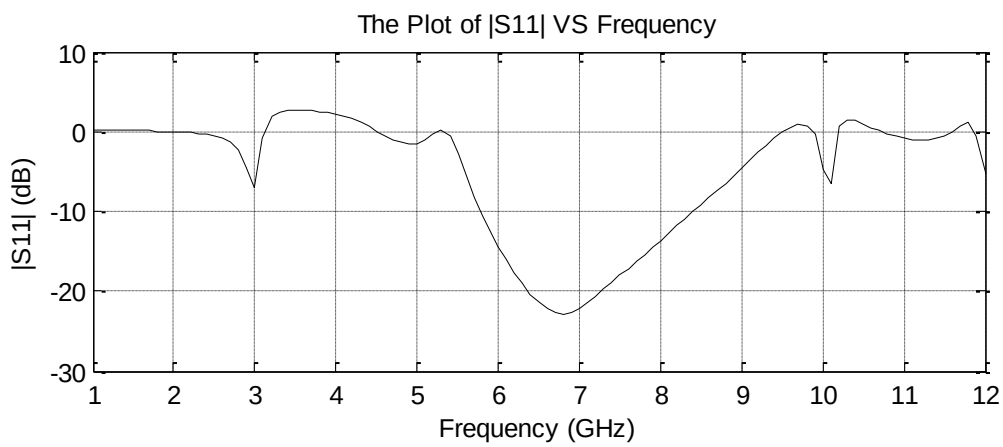
Zc = 49.4952, Z_1 = 0.918223, Z_2 = -0.848646, Z_3 = 8.292301, Z_4 = -0.1031266, Z_5 = 3.262291, Z_6 = -5.087931, Z_7 = -1.6194802, Z_8 = -4.3088298, Z_9 = 2.8433077, Z_{10} = 1.9570063, Z_{11} = 0.6908, Z_{12} = 1.3779977, Z_{13} = 2.0585, Z_{14} = 2.713939, Z_{15} = 2.556518 and Z_{16} = 2.6138707.


The Impedance Profile
ohm
80
70
60
50
40
30
20
0
15
30
45
60
75
90
95
The Length (mm)

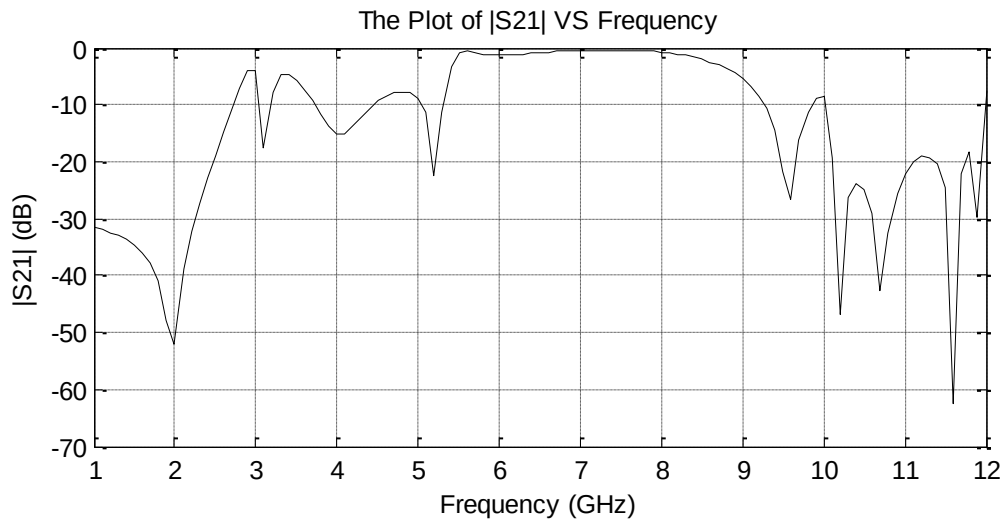
(a)


The Shape of the Filter
The Width (mm)
3
2
1
0
-1
-2
-3
0
15
30
45
60
75
90
95
The Length (mm)

(b)


The Plot of |S11| VS Frequency
|S11| (dB)
10
0
-10
-20
-30
1
2
3
4
5
6
7
8
9
10
11
12
Frequency (GHz)

(c)



(d)

Figure 32: Results of Band Pass Filter at 7 GHz (8 Terms Expansion)

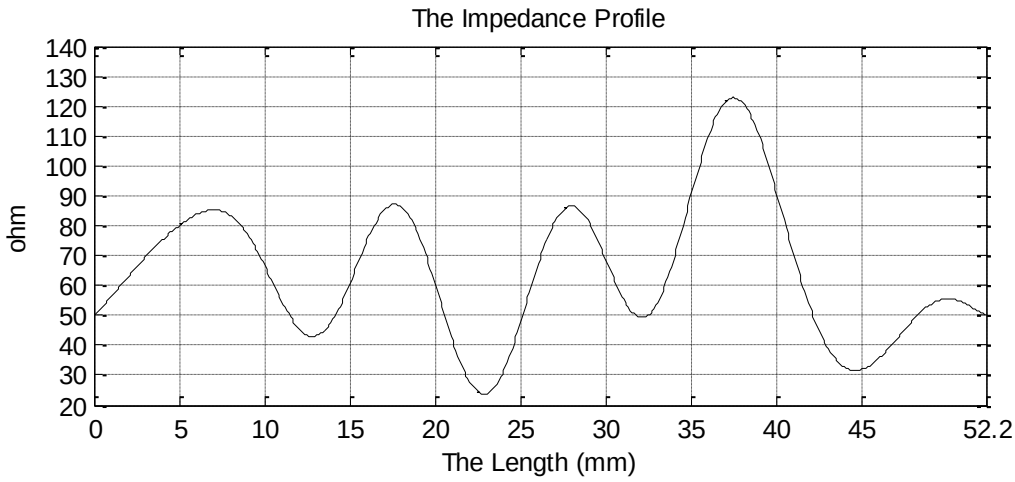
5.2.3 Filters with Cut-off at 10 GHz

5.2.3.1 Low Pass Filter

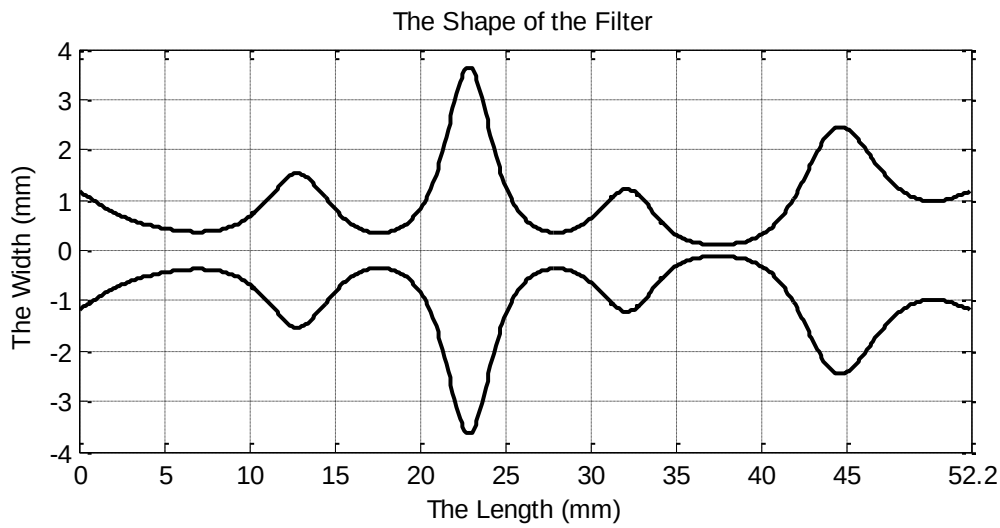
The filter length is 52.13 mm and the constant *b* is 0.94287. Figure 33 (a, b, c and d) shows the plot of the impedance profile of the designed filter, the filter shape, the plot of S_{11} and the plot of S_{21} respectively.

The values of the Zs that are satisfying the response of low pass filter at 10 GHz are as the following:

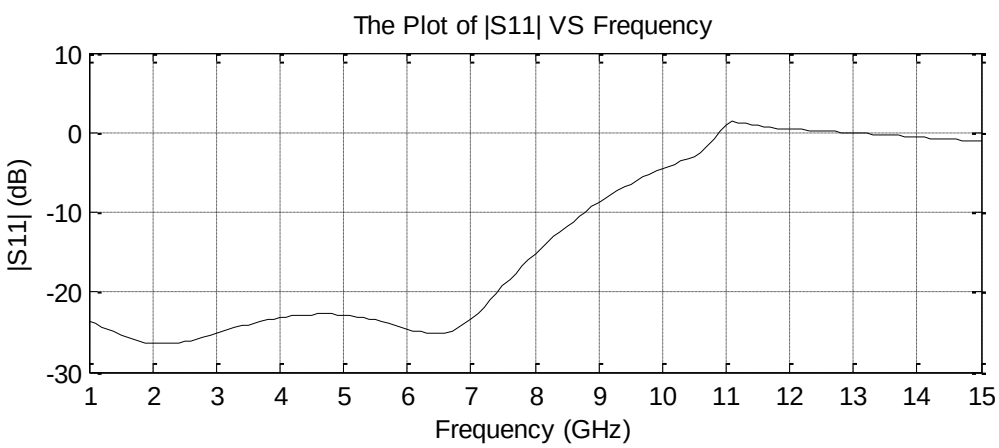
Zc = 63.79, Z_1 = -5.15, Z_2 = -4.33, Z_3 = 9.47, Z_4 = -2.59, Z_5 = -19.97, Z_6 = 11.48, Z_7 = -2.33, Z_8 = -0.3664, Z_9 = -2.15, Z_{10} = 18.03, Z_{11} = 1.85, Z_{12} = -2.95, Z_{13} = -1.26, Z_{14} = 0.6633, Z_{15} = 1.33 and Z_{16} = 0.9357.


The Impedance Profile
ohm
140
130
120
110
100
90
80
70
60
50
40
30
20
0
5
10
15
20
25
30
35
40
45
52.2
The Length (mm)

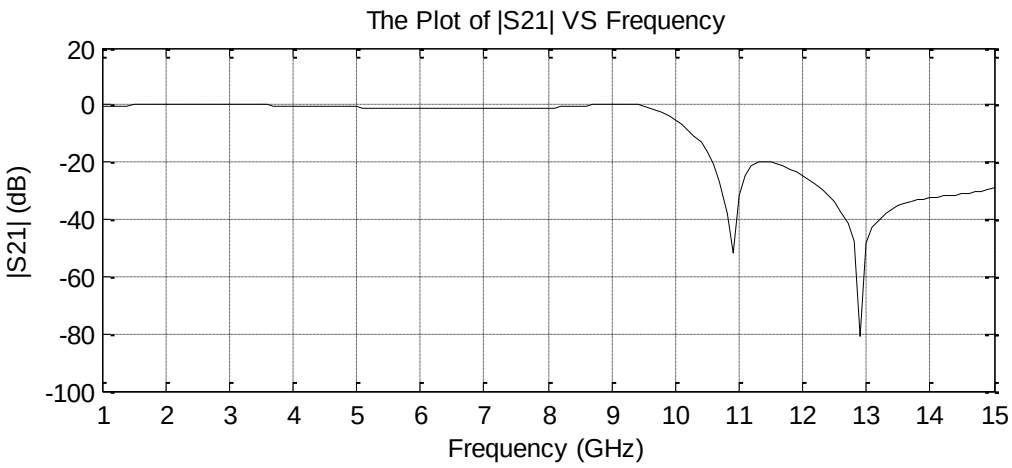
(a)


The Shape of the Filter
The Width (mm)
4
3
2
1
0
-1
-2
-3
-4
0
5
10
15
20
25
30
35
40
45
52.2
The Length (mm)

(b)


The Plot of |S11| VS Frequency
|S11| (dB)
10
0
-10
-20
-30
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
Frequency (GHz)

(c)



(d)

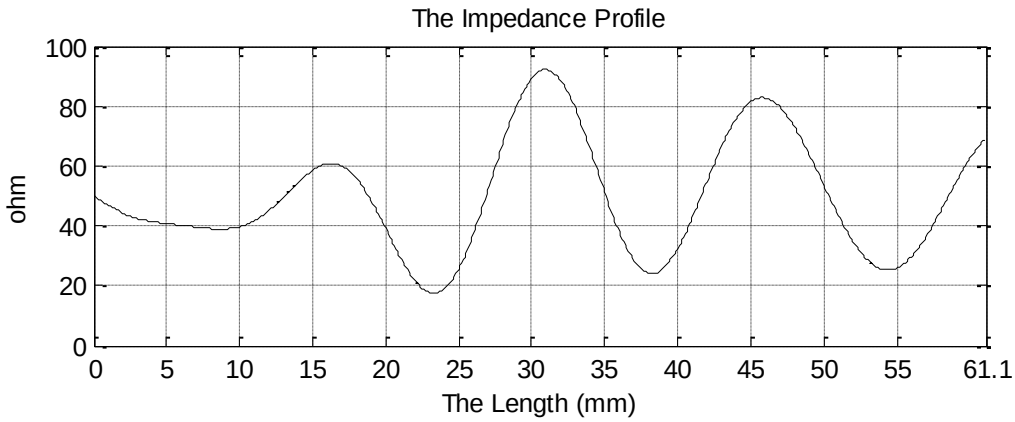
Figure 33: Results of Low Pass Filter with Cut-off at 10 GHz (8 Terms Expansion)

5.2.3.2 High Pass Filter

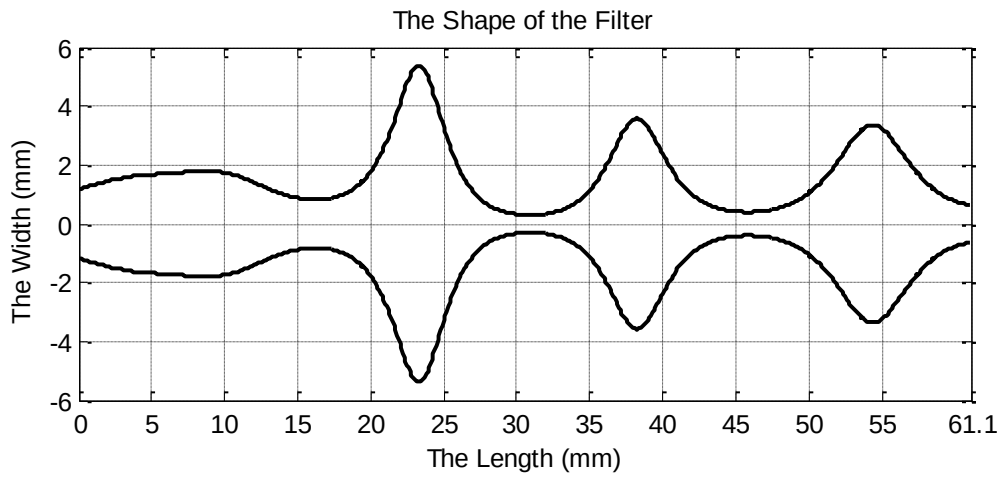
The designed high pass filter at 10 GHz has 61.1 mm as a length and 0.65245 as a value of the optimization constant. The values of Zs are as follows:

Zc = 50.105, Z_1 = -4.0561, Z_2 = -0.959964, Z_3 = -2.00367, Z_4 = -1.18517, Z_5 = -2.8188, Z_6 = 23.01252, Z_7 = -6.43557, Z_8 = -5.6581, Z_9 = -0.96491, Z_{10} = -5.185789, Z_{11} = 2.8444, Z_{12} = 4.6227, Z_{13} = 0.17206, Z_{14} = -6.3204, Z_{15} = 0.03057 and Z_{16} = -3.122.

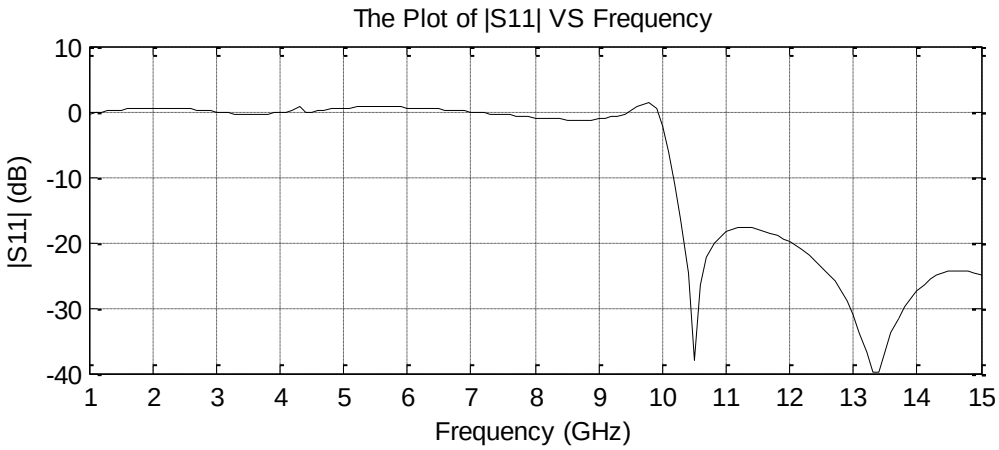
The plot of the impedance profile, the filter shape, the plot of S_{11} and the plot of S_{21} of the designed filter are shown in below Figure respectively (Figure 34: a, b, c & d).


The Impedance Profile
ohm
100
80
60
40
20
0
0 5 10 15 20 25 30 35 40 45 50 55 61.1
The Length (mm)

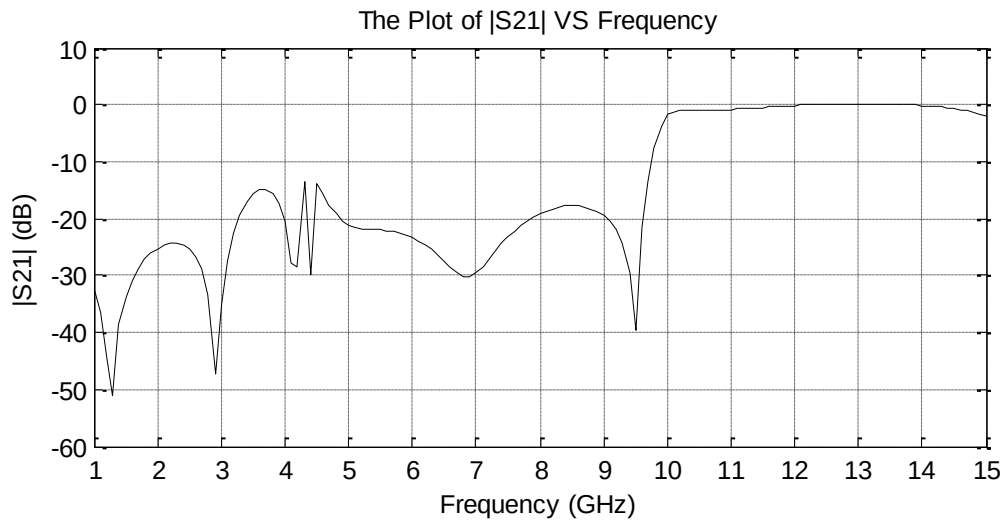
(a)


The Shape of the Filter
The Width (mm)
6
4
2
0
-2
-4
-6
0 5 10 15 20 25 30 35 40 45 50 55 61.1
The Length (mm)

(b)


The Plot of |S11| VS Frequency
|S11| (dB)
10
0
-10
-20
-30
-40
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15
Frequency (GHz)

(c)



(d)

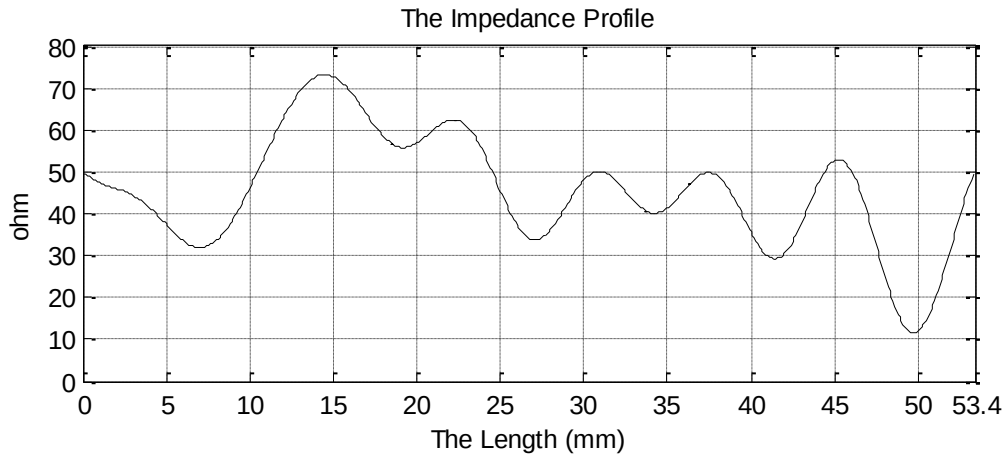
Figure 34: Results of High Pass Filter with Cut-off at 10 GHz (8 Terms Expansion)

5.2.3.3 Band Stop Filter

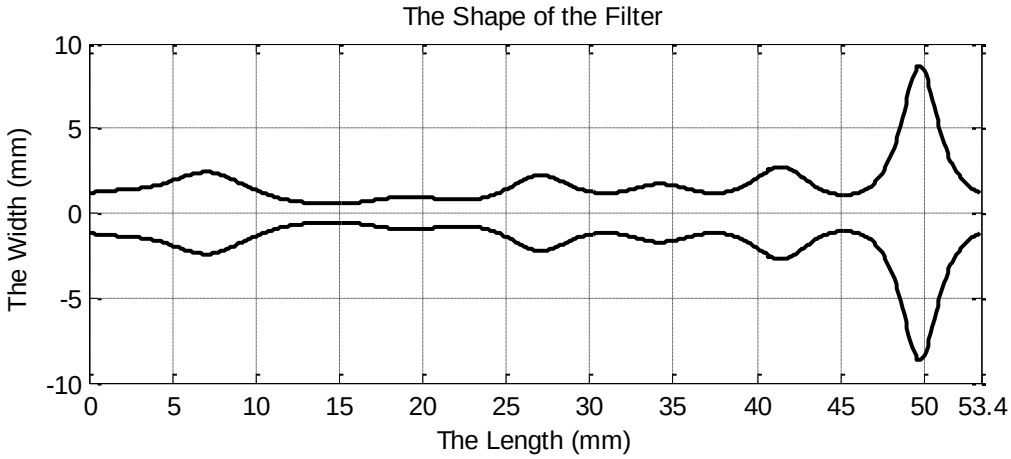
The length of the designed band stop filter is 53.38 mm and the constant *b* is 0.93148. Figure 35 (a, b, c and d) shows the plot of the impedance profile of the designed filter, the filter shape and the profile plots of S_{11} and S_{21} respectively.

The values of the Zs that are satisfying the response of low pass filter at 10 GHz are as the following:

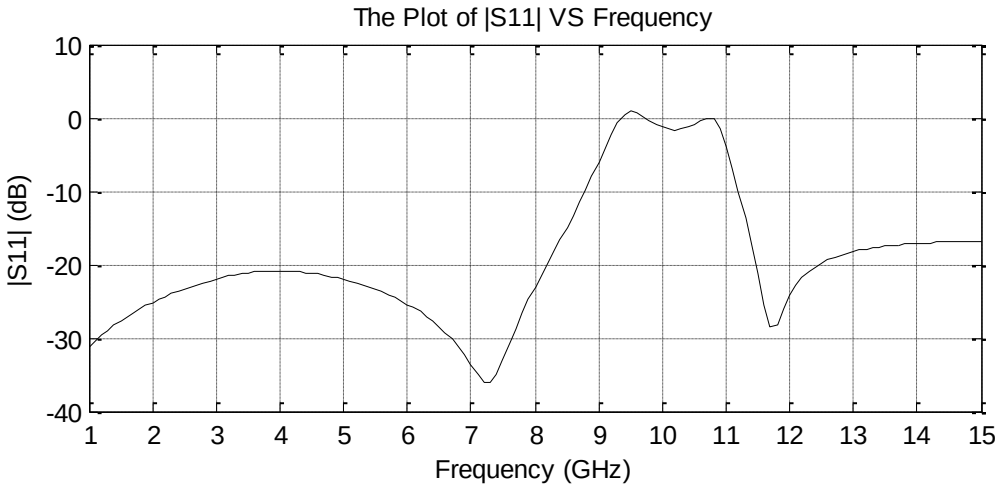
Zc = 45.955, Z_1 = -3.7516, Z_2 = -3.4197, Z_3 = 7.1091, Z_4 = 6.214, Z_5 = 6.2891, Z_6 = -1.6928, Z_7 = -4.6309, Z_8 = -2.0712, Z_9 = 10.1093, Z_{10} = -1.380, Z_{11} = -3.8658, Z_{12} = 0.34976, Z_{13} = -1.3310, Z_{14} = -2.0357, Z_{15} = -5.2882 and Z_{16} = 3.5271.


The Impedance Profile
ohm
80
70
60
50
40
30
20
10
0
0
5
10
15
20
25
30
35
40
45
50
53.4
The Length (mm)

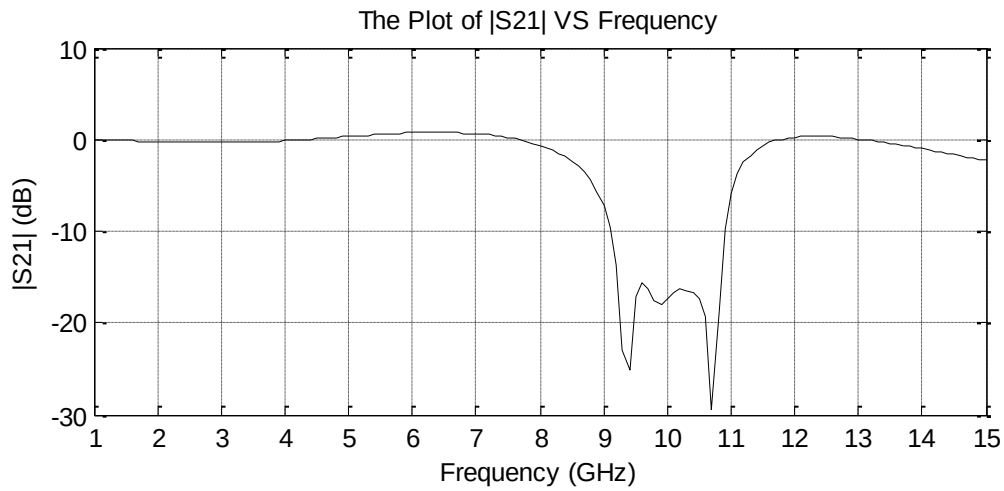
(a)


The Shape of the Filter
The Width (mm)
10
5
0
-5
-10
0
5
10
15
20
25
30
35
40
45
50
53.4
The Length (mm)

(b)


The Plot of |S11| VS Frequency
|S11| (dB)
10
0
-10
-20
-30
-40
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
Frequency (GHz)

(c)



(d)

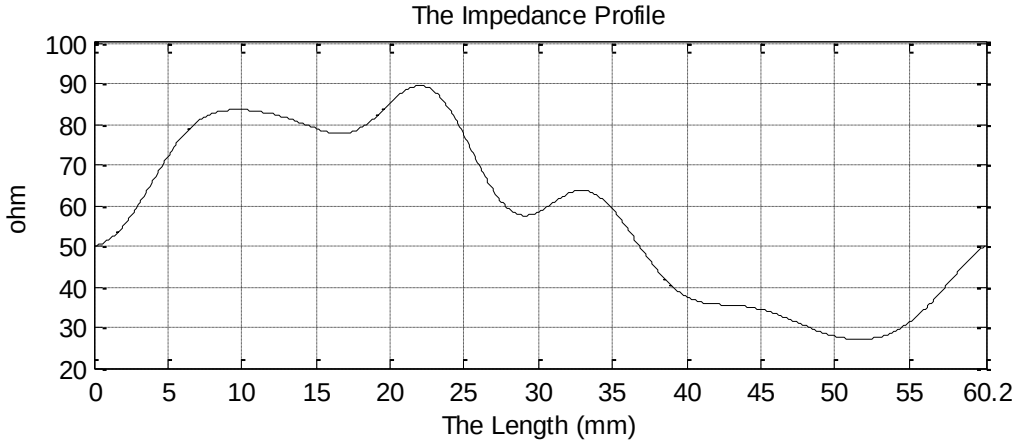
Figure 35: Results of Band Stop Filter at 10 GHz (8 Terms Expansion)

5.2.3.4 Band Pass Filter

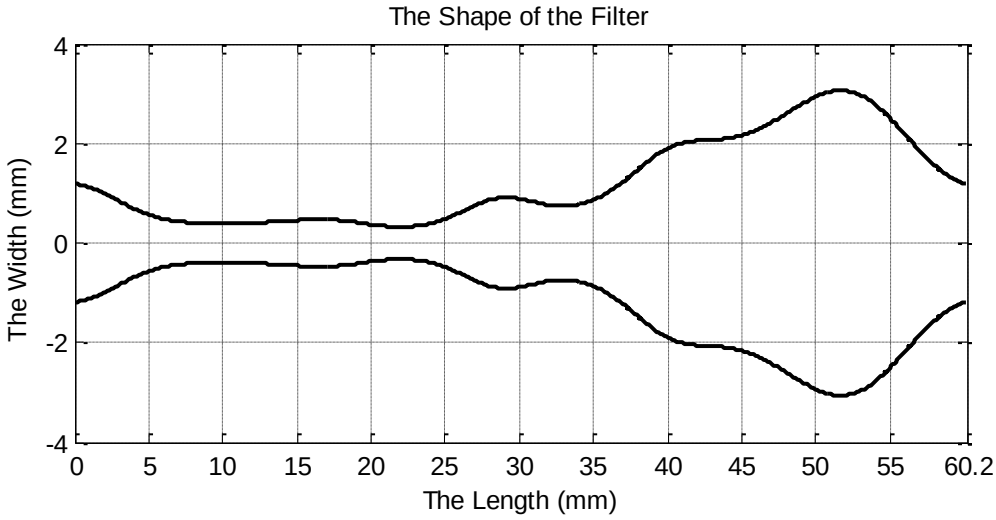
The values of the Zs that are satisfying the response of band pass filter at 10 GHz are as the following:

Zc = 57.297, Z_1 = -0.61412, Z_2 = 0.4731, Z_3 = -0.32035, Z_4 = -2.3116, Z_5 = -3.8735, Z_6 = 1.2636, Z_7 = -1.9845, Z_8 = 0.0704512, Z_9 = 27.2699, Z_{10} = 0.80697, Z_{11} = -0.840165, Z_{12} = 0.545752, Z_{13} = -4.263, Z_{14} = 0.90241, Z_{15} = -0.50575 and Z_{16} = -0.44424.

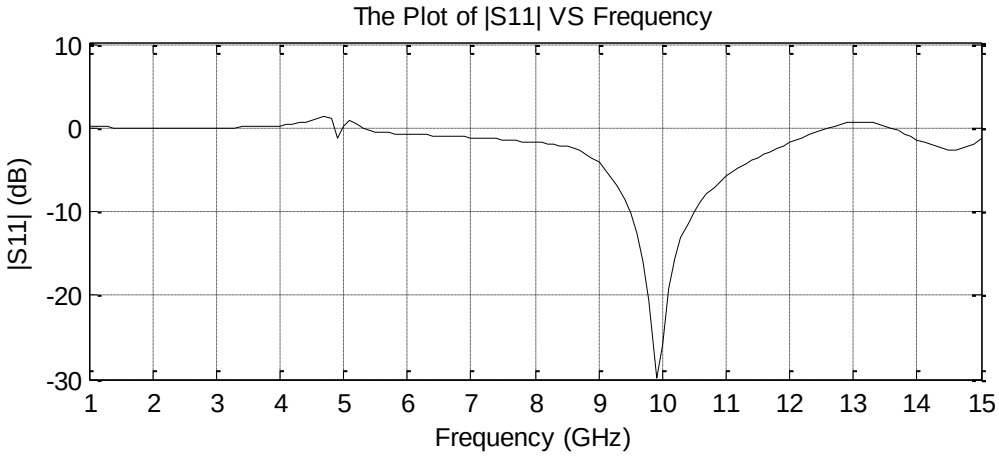
The filter length is 60.11 mm and the value of the optimization constant (*b*) is 0.91988. Figure 36 (a, b, c and d) shows the plot of the impedance profile of the designed filter, the filter shape, the plot of S_{11} and the plot of S_{21} respectively.


The Impedance Profile
ohm
100
90
80
70
60
50
40
30
20
0
5
10
15
20
25
30
35
40
45
50
55
60.2
The Length (mm)

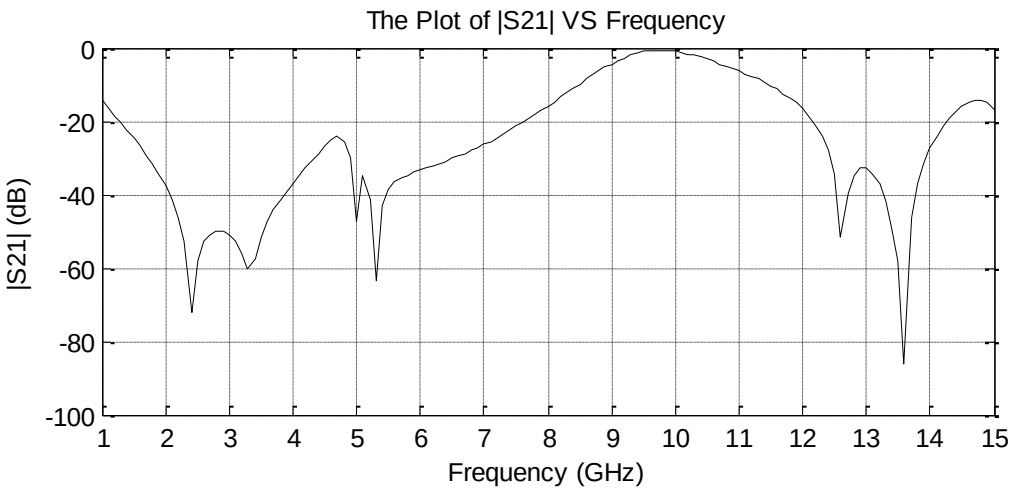
(a)


The Shape of the Filter
The Width (mm)
4
2
0
-2
-4
0
5
10
15
20
25
30
35
40
45
50
55
60.2
The Length (mm)

(b)


The Plot of |S11| VS Frequency
|S11| (dB)
10
0
-10
-20
-30
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
Frequency (GHz)

(c)



(d)

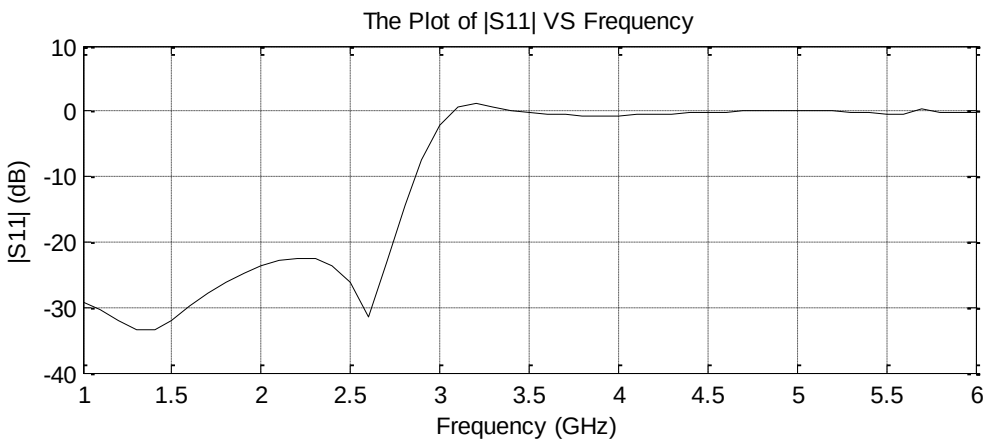
Figure 36: Results of Band Pass Filter at 10 GHz (8 Terms Expansion)

5.3 Sensitivity and Validation of this Methodology

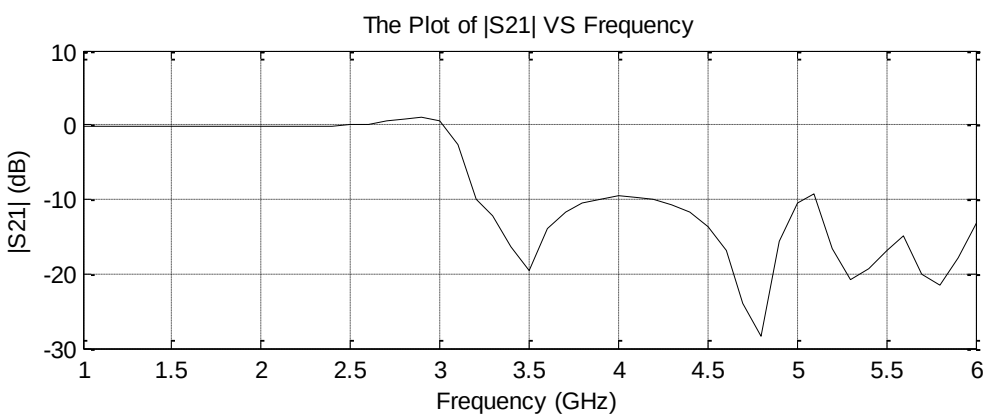
In this section, the sensitivity of this methodology with the expansions (five and eight terms) is going to be discussed to assess its efficiency with respect to the number of terms in the expansions. Here, the sensitivity means that the changing of the filter response (changing the values of S_{11} and S_{21}) by a small changing in the values of Zs. Also, the validation of this methodology with the current expansions (five and eight terms) is going to be investigated by means of simulating the designed filter with a professional simulator software. The assessment is by no means complete since it is basically an observation for few results. It is very important to investigate and validate the sensitivity of these current expansions to determine if they are adequate for this methodology so that they may be efficiently applied.

5.3.1 Five Terms Expansion

In order to examine the response sensitivity of the filters designed by five terms expansion, the low pass filter designed in section 5.1.1.1 will be our example here. It has the following plots of S_{11} and S_{21} respectively (Figure 37):



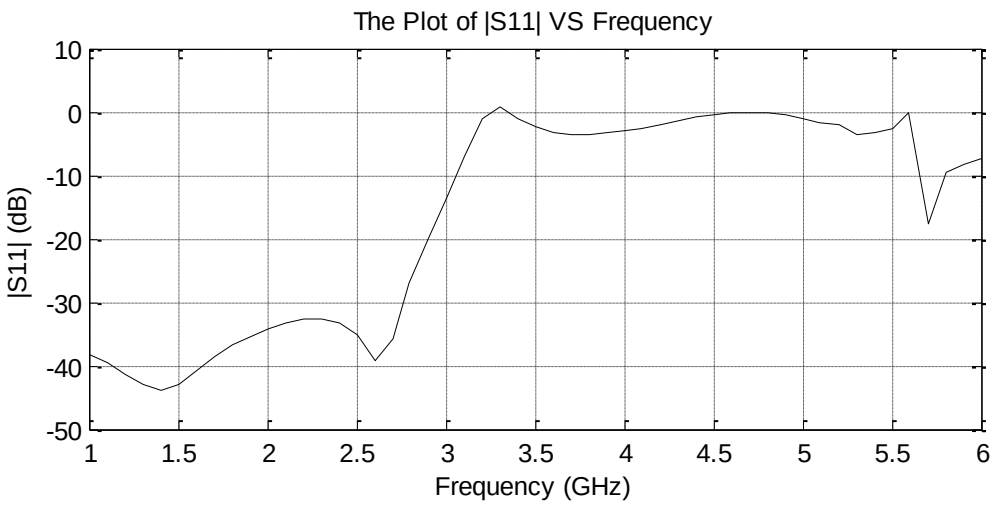
(a)



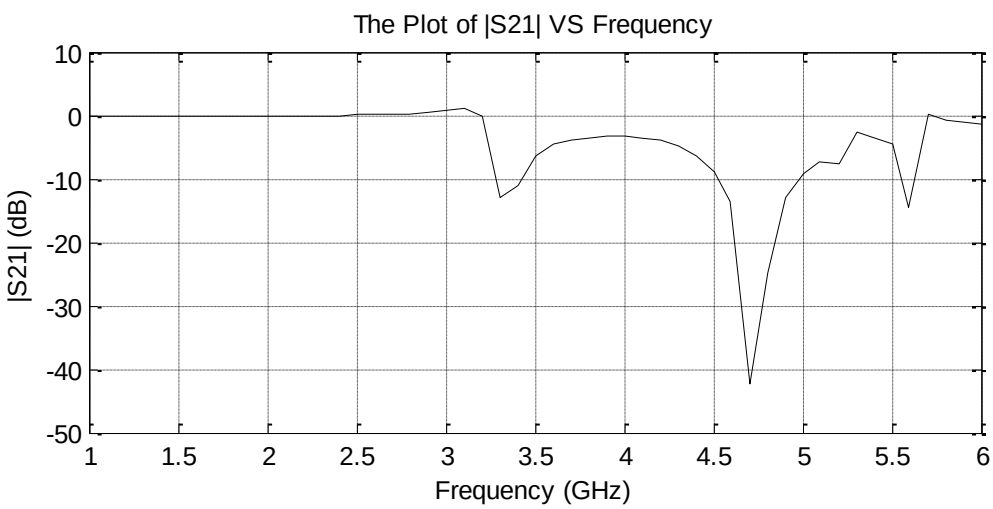
(b)

Figure 37: Plots of S_{11} and S_{21} for the Low Pass Filter with Cut-off at 3 GHz (5 Terms Expansion)

When only one value of the Zs is changed by only ± 2%, the filter response (S_{11} & S_{21}) will be dramatically changed to a limit that it is almost not a low pass filter anymore as it can be shown in figure 38. (Notice the drastic change in $|S_{21}|$)



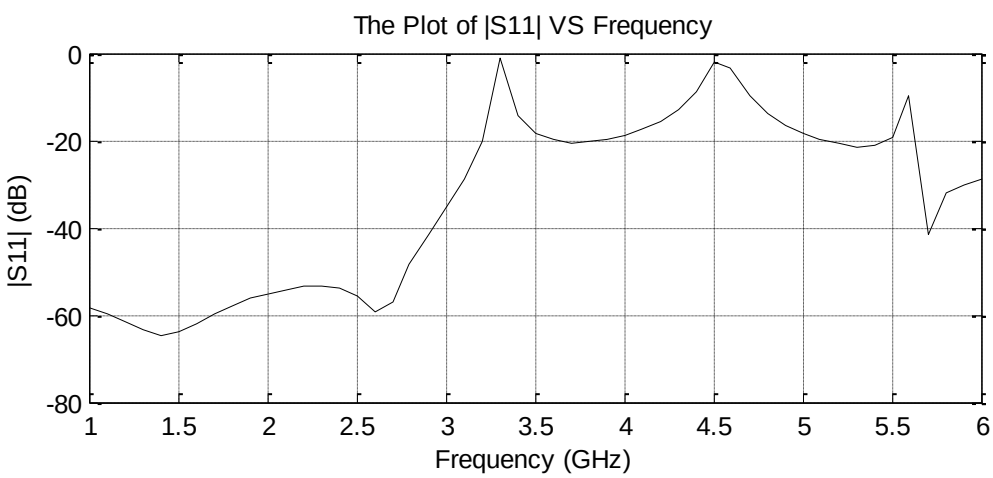
(a)



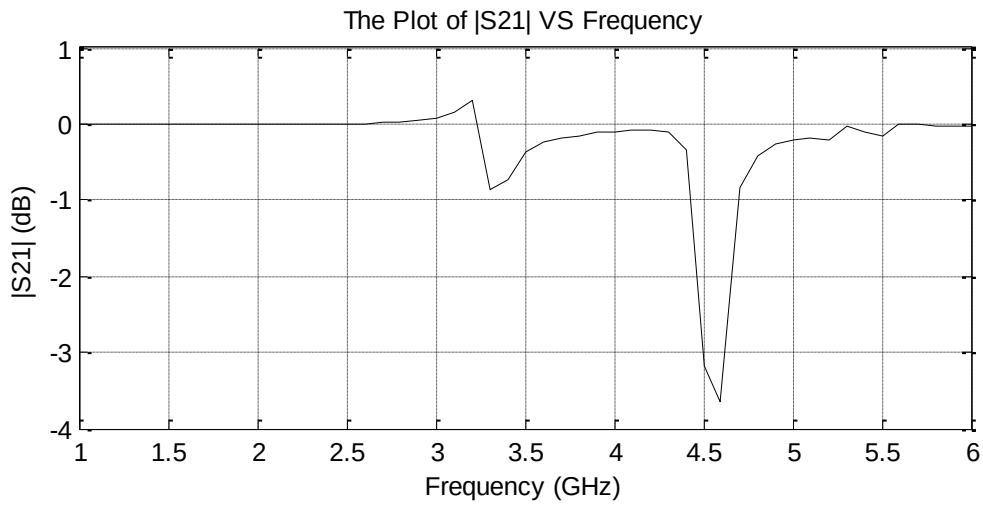
(b)

Figure 38: Plots of S_{11} and S_{21} for the LPF after Changing one Value of Zs (5 Terms Expansion)

When all the values of Zs are changed by only ±0.5%, the filter response represented by the below plots of S_{11} and S_{21} will be completely not related to a low pass filter response (figure 39). This shows that the response is very sensitive to any minor change in the values of the Zs. That means, any small change in any value of the Zs cause a huge change in the filter response leading to have an unstable system in the real life. Therefore, any tiny change in the width of the filter will affect the value of the characteristic impedance which leads to unpredictable changes in the values of Zs resulting in a considerable change in the filter response. This obviously explains why there was some differences between the theoretical results and the drawn filters in the simulator. Also, the equations used to find the width of the microstrip are not very accurate because they provide a very good approximation of the width by using quasi-static approach analysis not by using a full wave analysis which is more accurate.



(a)

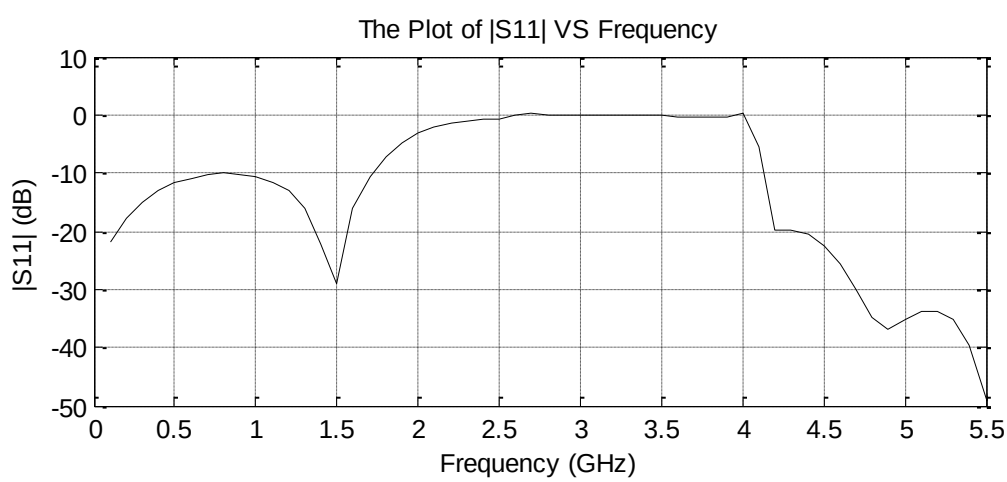


(b)

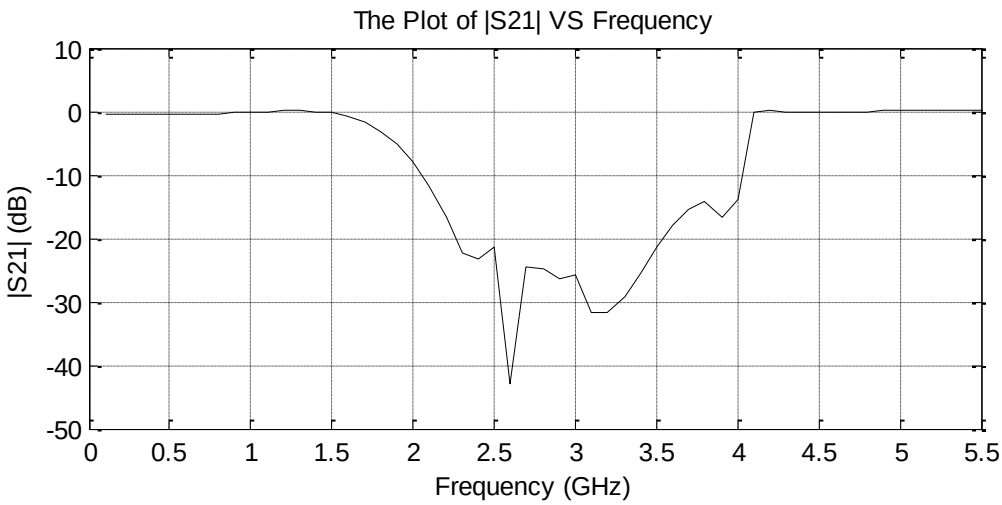
Figure 39: Plots of S_{11} and S_{21} for the LPF after Changing all Value of Zs (5 Terms Expansion)

5.3.2 Eight Terms Expansion

Again, in order to evaluate the sensitivity of this methodology with the eight terms expansion, the designed band stop filter which has been introduced in section 5.2.1.2 will be investigated here. It has the following plots of S_{11} and S_{21} respectively (Figure 40):



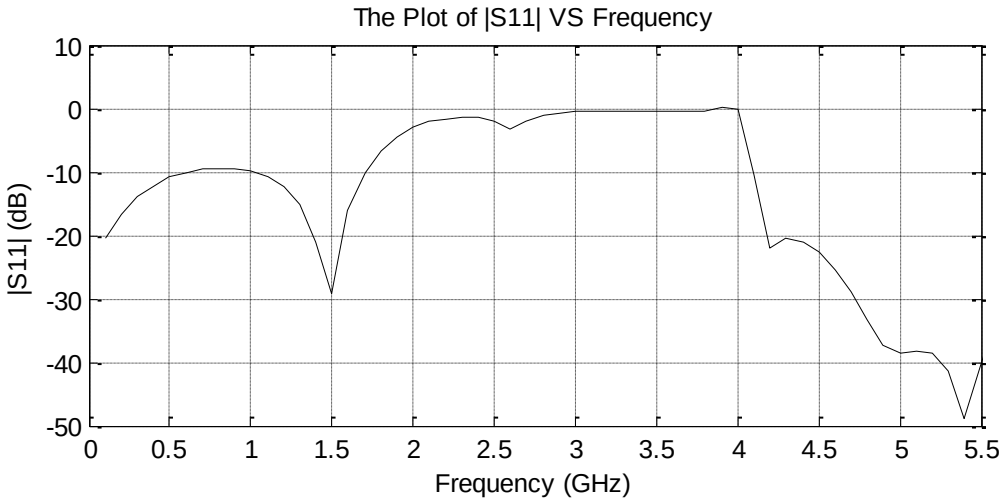
(a)



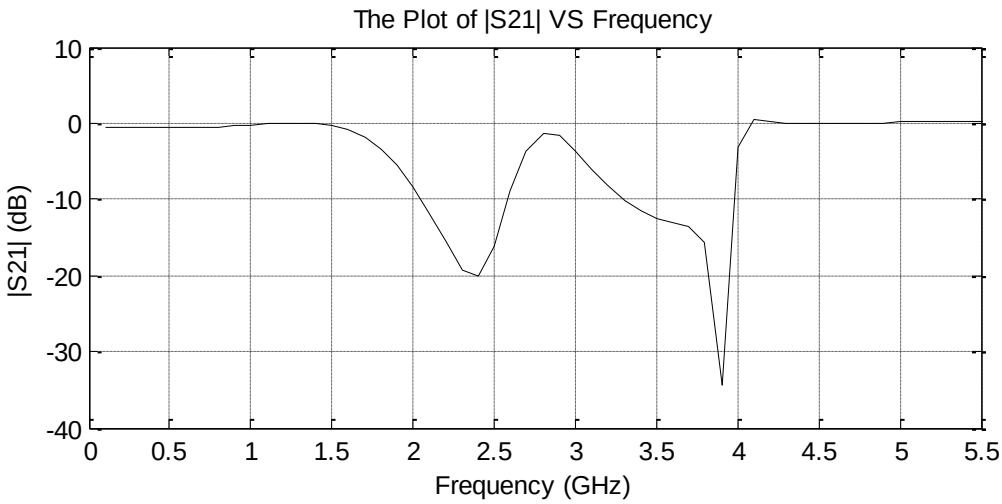
(b)

Figure 40: Plots of S_{11} and S_{21} for the Band Stop Filter at 3 GHz (8 Terms Expansion)

When only one value of the Zs is changed by less than ±5%, the response is slightly distorted. However, when one value is changed by ±5%, the filter response (S_{11} & S_{21}) will be noticeably distorted but the general response is still reflecting a response of a band stop filter as seen in figure 41. In some cases, the distortions are so high that it could destroy the whole desired response.



(a)

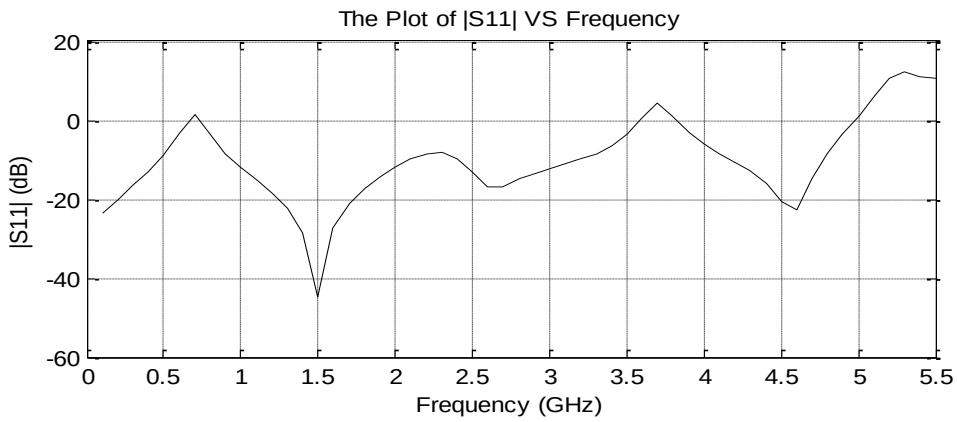


(b)

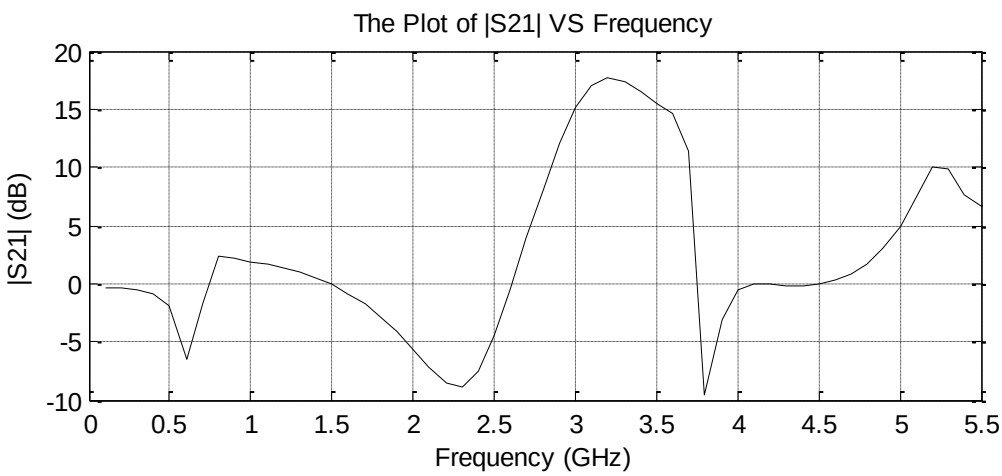
Figure 41: Plots of S_{11} and S_{21} for the BSF after Changing one Value of Zs (8 Terms Expansion)

When all the values of Zs are changed by either ±0.5%, ±1% or ±1.5%, the response of the filter is highly distorted, but, it still respond as a band stop filter. However, when all the values of Zs are changed by ±2%, the filter response represented by the below plots of S_{11} and S_{21} will be completely distorted as shown in figure 42. This is because we have a non-linear system producing a non-linear error which could be very high at some cases. The best way to reduce this non-linear error is by expanding the system with more terms so that when the characteristic impedance changes, this change will be distributed among all the values of Zs. This leads to very small changes in the values of Zs so that they will not harmfully affect the response.

It is so obvious that the methodology with eight terms expansion is much less sensitive than five terms expansion which enables obtaining a simulation result as it can be seen in figure 43.



(a)



(b)

Figure 42: Plots of S_{11} and S_{21} for the BSF after Changing all Value of Zs (8 Terms Expansion)

In figure 43, the blue line represents S_{11} and the red one represents S_{21}. Comparing the theoretical results (figure 40) with the simulated results (figure 43), one can see that both figures show a response of band sop filter. However, the bandwidth of the simulated

result is less than the theoretical one. Also, there is a poor conformity between the starting and ending points of the stopped band since the theoretical result starts the stopped band at around 2 GHz whereas it is the end point of the stopped band in the simulated result. Again, this is because of the high sensitivity of the methodology with eight terms expansion. In order to have a stable response and consequently an excellent conformity between the theoretical and simulated results, there should be more terms expansion.

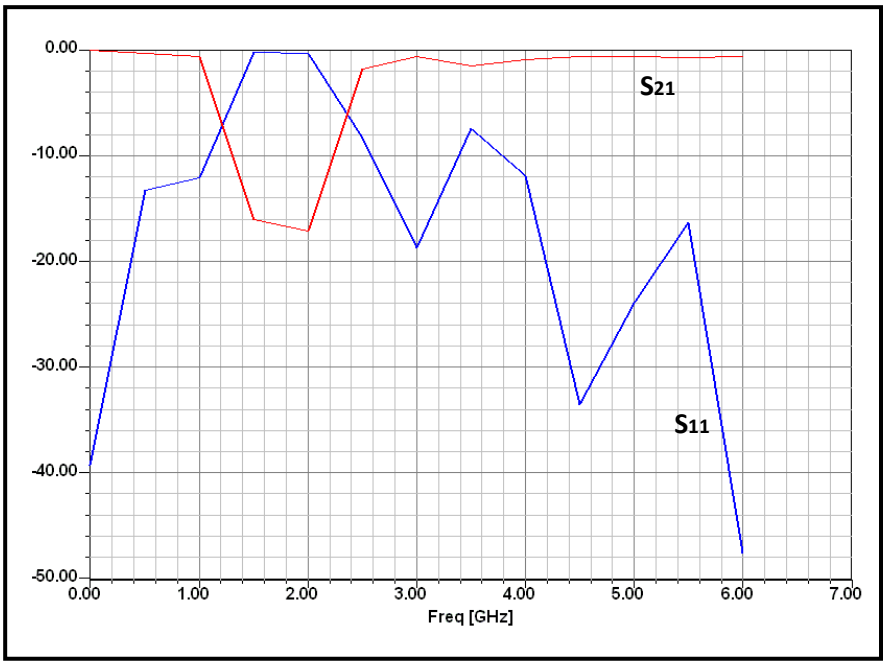


Figure 43: Plots of S_{11} and S_{21} for the BSF at 3 GHz (8 Terms Expansion)

CHAPTER 6

CONCLUSIONS AND RECOMMENDATIONS FOR FUTURE WORK

6.1 Conclusions

In this thesis, the design of the filters is based on a global assumption of voltage, current and impedance along the entire length of the transmission line and then using genetic algorithm to optimize the impedance parameters and the length of the filter to fit the required values of S_{11} and S_{21}. The method is quite successful and several types of filters are designed and presented. Low pass, band pass, band stop and high pass filters were designed at different frequency ranges.

There are several advantages of using this technique such as the following:

- The designed filters have the characteristic impedance at their inputs and their outputs equals to 50 ohm which matches the characteristic impedance of the

transmission line within the microwave integrated circuit. Therefore, impedance matching techniques are not required anymore. This is achieved by using genetic algorithm because all constraints whether they are linear or nonlinear can be incorporated in the optimization procedure.

- There is no sharp impedance discontinuity influence which minimizes the internal reflected power.

- The design process by this technique is very flexible which enables designing filters with different shapes, sizes and lengths satisfying exactly the same objectives and having exactly the same criteria.

- In order to design any type of filter, the only thing needs to be modified is the optimization criteria, since the general mathematical expressions of S_{11} and S_{21} may be altered at will.

In fact, the technique avoids the standard methods of filter design and tackles the filter characteristics directly through its S-parameters (S_{11} & S_{21}). Also, this technique enjoys minimal numerical work – compared to the work in [4] – on the expense of long and tedious mathematical analysis, particularly for a large number of expansion terms. On the other hand, in [4], the line is divided into *n* sections and a power series is assumed to describe discrete points of the characteristic impedance per section and then using cubic spline interpolation to describe the characteristic impedance variation between the discrete points. After that, the global scattering parameters of all sections are determined by finding the transmission matrix of every section of the transmission line, transform it

by using chain cascaded matrices and then converted back into scattering parameters. This generates a very lengthy and quite large numerical structure to be optimized.

A comparison between the two techniques show that it is a tradeoff between extensive mathematical work with limited numerical work – which is the present work – as opposed to extensive numerical work with limited mathematical analysis as presented in [4].

Although, Optimization is a very powerful tool, it must be applied judiciously. In fact, without a good starting point, the most elegant and robust optimization procedures may not be able to find an acceptable solution and this is considered as a disadvantage of this technique since it is capitalizing on an optimization tool which is largely dependent on the initial assumptions.

Finally, a comparison between filters at 3 GHz cut-off design with five terms expansion and eight terms expansion shows that the filters designed with eight terms expansion have both sharp cutoff and minimum values of insertion loss simultaneously which is an advantage for this design. One may conclude that increasing the number of terms would lead to a better and more refined design.

6.2 Recommendations for Future Work

This work has focused on the filter designs at cutoff frequencies of 3 GHz, 7 GHz and 10 GHz. A recommendation for the future work will be to consider filters at the same frequencies but with larger expansion terms to reduce the sensitivity to small variation in the filter characteristic impedance. Also, another recommendation will be to consider filters at higher frequencies such as 20 GHz, 26 GHz or 30 GHz by increasing the number of the expansion terms. Therefore, the increase in the number of the expansion of the terms is highly needed for both cases either for having filter designs that are non-sensitive to the change of impedance parameters and for filters at higher frequencies.

Another idea for the higher frequency filters is to reduce the number of expansion terms by starting at higher spatial frequency terms directly assuming that the low spatial frequency terms are of minimal effect. This has been tested and found to be possible.

Finally, other optimization techniques should be attempted and compared to this work.

APPENDIX

7.1 Appendix A: Five Terms Expansion Constant Term and Coefficients of Trigonometric Functions

In this section, the terms that should be equal to zero to satisfy the solution of the equation (1) and form the system of equations with 9 equations and 9 unknowns (V_1, V_2, V_3, V_4, V_5, V_7, V_8, V_9 and V_{10}) will be listed below:

$$\text{The Constant Term} = p\, V_0\, Z_C\, k^2{}_0 + \left(\frac{k^2{}_0 - 2\alpha^2}{2}\right)(p\, Z_1\, V_1 + q\, Z_6\, V_6) +$$

$$\left(\frac{k^2{}_0 - 8\alpha^2}{2}\right)(p\, Z_2\, V_2 + q\, Z_7\, V_7) + \left(\frac{k^2{}_0 - 18\alpha^2}{2}\right)(p\, Z_3\, V_3 + q\, Z_8\, V_8) +$$

$$\left(\frac{k^2{}_0 - 32\alpha^2}{2}\right)(p\, Z_4\, V_4 + qZ_9V_9) + \left(\frac{k^2{}_0 - 50\alpha^2}{2}\right)(pZ_5\, V_5 + q\, Z_{10}\, V_{10}) = 0$$

$$\text{Cos}\alpha x \text{ Coefficient} = p\,V_1 Z_C (k^2_0 - \alpha^2) + p\,V_0 Z_1 k^2_0 + \left(\frac{k^2_0 - 3\alpha^2}{2}\right)(p\,Z_2 V_1 + q\,Z_7 V_6)$$

$$+\left(\frac{k^2_0 - 6\alpha^2}{2}\right)(p\,Z_1 V_2 + q\,Z_6 V_7) + \left(\frac{k^2_0 - 10\alpha^2}{2}\right)(p\,Z_3 V_2 + q\,Z_8 V_7)$$

$$+\left(\frac{k^2_0 - 15\alpha^2}{2}\right)(p\,Z_2 V_3 + q\,Z_7 V_8) + \left(\frac{k^2_0 - 21\alpha^2}{2}\right)(p\,Z_4 V_3 + q\,Z_9 V_8)$$

$$+\left(\frac{k^2_0 - 28\alpha^2}{2}\right)(p\,Z_3 V_4 + q\,Z_8 V_9) + \left(\frac{k^2_0 - 36\alpha^2}{2}\right)(p\,Z_5 V_4 + q\,Z_{10} V_9)$$

$$+\left(\frac{k^2_0 - 45\alpha^2}{2}\right)(p\,Z_4 V_5 + q\,Z_9 V_{10}) = 0$$

$$\text{Cos}2\alpha x \text{ Coefficient} = p\,V_2 Z_C (k^2_0 - 4\alpha^2) + p\,V_0 Z_2 k^2_0 + \left(\frac{k^2_0 - 4\alpha^2}{2}\right)(p\,Z_3 V_1 + q\,Z_8 V_6)$$

$$+ \frac{k^2_0}{2}(p\,Z_1 V_1 - q\,Z_6 V_6) + \left(\frac{k^2_0 - 12\alpha^2}{2}\right)(p\,Z_4 V_2 + q\,Z_9 V_7)$$

$$+\left(\frac{k^2_0 - 12\alpha^2}{2}\right)(p\,Z_1 V_3 + q\,Z_6 V_8) + \left(\frac{k^2_0 - 24\alpha^2}{2}\right)(p\,Z_5 V_3 + q\,Z_{10} V_8)$$

$$+\left(\frac{k^2_0 - 24\alpha^2}{2}\right)(p\,Z_2 V_4 + q\,Z_7 V_9) + \left(\frac{k^2_0 - 40\alpha^2}{2}\right)(p\,Z_3 V_5 + q\,Z_8 V_{10})$$

$$= 0$$

$$\text{Cos}3\alpha x \text{ Coefficient} = p\,V_3\,Z_C\,(k^2_0 - 9\alpha^2) + pV_0\,Z_3\,k^2_0 + \left(\frac{k^2_0 - 5\alpha^2}{2}\right)(p\,Z_4\,V_1 + q\,Z_9\,V_6)$$

$$+\left(\frac{k^2_0 + \alpha^2}{2}\right)(p\,Z_2\,V_1 - q\,Z_7\,V_6) + \left(\frac{k^2_0 - 14\alpha^2}{2}\right)(p\,Z_5\,V_2 + q\,Z_{10}\,V_7)$$

$$+\left(\frac{k^2_0 - 2\alpha^2}{2}\right)(p\,Z_1\,V_2 - q\,Z_6\,V_7) + \left(\frac{k^2_0 - 20\alpha^2}{2}\right)(p\,Z_1\,V_4 + q\,Z_6\,V_9)$$

$$+\left(\frac{k^2_0 - 35\alpha^2}{2}\right)(p\,Z_2\,V_5 + q\,Z_7\,V_{10}) = 0$$

$$\text{Cos}4\alpha x \text{ Coefficient} = pV_4\,Z_C\,(k^2_0 - 16\alpha^2) + p\,V_0\,Z_4\,k^2_0 + \left(\frac{k^2_0 - 6\alpha^2}{2}\right)(pZ_5V_1 + qZ_{10}V_6)$$

$$+\left(\frac{k^2_0 + 2\alpha^2}{2}\right)(p\,Z_3\,V_1 - q\,Z_8\,V_6) + \left(\frac{k^2_0 - 6\alpha^2}{2}\right)(p\,Z_1\,V_3 - q\,Z_6\,V_8)$$

$$+\frac{k^2_0}{2}(p\,Z_2\,V_2 - q\,Z_7\,V_7) + \left(\frac{k^2_0 - 30\alpha^2}{2}\right)(p\,Z_1\,V_5 + q\,Z_6\,V_{10}) = 0$$

$$\text{Cos}5\alpha x \text{ Coefficient} = p\,V_5\,Z_C\,(k^2_0 - 25\alpha^2) + p\,V_0\,Z_5\,k^2_0 + \left(\frac{k^2_0 + 3\alpha^2}{2}\right)(pZ_4V_1 - qZ_9V_6)$$

$$+\left(\frac{k^2_0 - 3\alpha^2}{2}\right)(p\,Z_2\,V_3 - q\,Z_7\,V_8) + \left(\frac{k^2_0 - 12\alpha^2}{2}\right)(p\,Z_1\,V_4 - q\,Z_6\,V_9)$$

$$+\left(\frac{k^2_0 + 2\alpha^2}{2}\right)(p\,Z_3\,V_2 - q\,Z_8\,V_7) = 0$$

$$\text{Sin}\alpha x \text{ Coefficient} = q\, V_6\, Z_C\, (k^2_0 - \alpha^2) + p\, V_0\, Z_6\, k^2_0 - \left(\frac{k^2_0 - 3\alpha^2}{2}\right)(-p\, Z_7\, V_1 + q\, Z_2\, V_6)$$

$$+\left(\frac{k^2_0 - 6\alpha^2}{2}\right)(-p\, Z_6\, V_2 + q\, Z_1\, V_7) - \left(\frac{k^2_0 - 10\alpha^2}{2}\right)(-p\, Z_8\, V_2 + q\, Z_3\, V_7)$$

$$-\left(\frac{k^2_0 - 15\alpha^2}{2}\right)(-p\, Z_7\, V_3 + q\, Z_2\, V_8) - \left(\frac{k^2_0 - 21\alpha^2}{2}\right)(-p\, Z_9\, V_3 + q\, Z_4\, V_8)$$

$$+\left(\frac{k^2_0 - 28\alpha^2}{2}\right)(-p\, Z_8 V_4 + q\, Z_3 V_9) - \left(\frac{k^2_0 - 36\alpha^2}{2}\right)(-p\, Z_{10} V_4 + q\, Z_5 V_9)$$

$$+\left(\frac{k^2_0 - 45\alpha^2}{2}\right)(-p\, Z_9\, V_5 + q\, Z_4\, V_{10}) = 0$$

$$\text{Sin}2\alpha x \text{ Coefficient} = q\, V_7\, Z_C\, (k^2_0 - 4\alpha^2) + pV_0\, Z_7\, k^2_0 - \left(\frac{k^2_0 - 4\alpha^2}{2}\right)(-p\, Z_8 V_1 + q\, Z_3 V_6)$$

$$+\frac{k^2_0}{2}(p\, Z_6\, V_1 + q\, Z_1\, V_6) - \left(\frac{k^2_0 - 12\alpha^2}{2}\right)(-p\, Z_9\, V_2 + q\, Z_4\, V_7)$$

$$+\left(\frac{k^2_0 - 12\alpha^2}{2}\right)(-p\, Z_6\, V_3 + q\, Z_1\, V_8) - \left(\frac{k^2_0 - 24\alpha^2}{2}\right)(-p\, Z_{10} V_3 + q\, Z_5 V_8)$$

$$-\left(\frac{k^2_0 - 24\alpha^2}{2}\right)(-pZ_7\, V_4 + q\, Z_2\, V_9) + \left(\frac{k^2_0 - 40\alpha^2}{2}\right)(-p\, Z_8 V_5 + qZ_3 V_{10})$$

$$= 0$$

$$\text{Sin}3\alpha x \text{ Coefficient} = qV_8 Z_C (k^2_0 - 9\alpha^2) + pV_0 Z_8 k^2_0 - \left(\frac{k^2_0 - 5\alpha^2}{2}\right)(-p\ Z_9V_1 + q\ Z_4V_6)$$

$$+\left(\frac{k^2_0 + \alpha^2}{2}\right)(p\ Z_7 V_1 + q\ Z_2\ V_6) - \left(\frac{k^2_0 - 14\alpha^2}{2}\right)(-p\ Z_{10} V_2 + q\ Z_5 V_7)$$

$$+\left(\frac{k^2_0 - 2\alpha^2}{2}\right)(p\ Z_6 V_2 + q\ Z_1 V_7) + \left(\frac{k^2_0 - 20\alpha^2}{2}\right)(-p\ Z_6 V_4 + q\ Z_1 V_9)$$

$$+\left(\frac{k^2_0 - 35\alpha^2}{2}\right)(-p\ Z_7 V_5 + q\ Z_2 V_{10}) = 0$$

$$\text{Sin}4\alpha x \text{ Coefficient} = qV_9 Z_C (k^2_0 - 16\alpha^2) + pV_0Z_9 k^2_0 - \left(\frac{k^2_0 - 6\alpha^2}{2}\right)(-p\ Z_{10}V_1 + qZ_5V_6)$$

$$+\left(\frac{k^2_0 + 2\alpha^2}{2}\right)(p\ Z_8\ V_1 + q\ Z_3\ V_6) + \frac{k^2_0}{2}(p\ Z_7\ V_2 + q\ Z_2\ V_7)$$

$$+\left(\frac{k^2_0 - 6\alpha^2}{2}\right)(p\ Z_6 V_3 + q\ Z_1 V_8) + \left(\frac{k^2_0 - 30\alpha^2}{2}\right)(-p\ Z_6 V_5 + q\ Z_1 V_{10})$$

$$= 0$$

$$\text{Sin}5\alpha x \text{ Coefficient} = qV_{10} Z_C (k^2_0 - 25\alpha^2) + p\ V_0 Z_{10} k^2_0 + \left(\frac{k^2_0 + 3\alpha^2}{2}\right)(p\ Z_9 V_1 + q\ Z_4 V_6)$$

$$+\left(\frac{k^2_0 + 2\alpha^2}{2}\right)(p\ Z_8 V_2 + q\ Z_3 V_7) + \left(\frac{k^2_0 - 3\alpha^2}{2}\right)(p\ Z_7 V_3 + q\ Z_2 V_8)$$

$$+\left(\frac{k^2_0 - 12\alpha^2}{2}\right)(p\ Z_6 V_4 + q\ Z_1 V_9) = 0$$

7.2 Appendix B: Equations Matrix of Five Terms Expansion

	V_1	V_2	V_3	V_4
Const.	$\left(\frac{k_0^2-2\alpha^2}{2}\right)Z_1$	$\left(\frac{k_0^2-8\alpha^2}{2}\right)Z_2$	$\left(\frac{k_0^2-18\alpha^2}{2}\right)Z_3$	$\left(\frac{k_0^2-32\alpha^2}{2}\right)Z_4$
Cosαx	$(k_0^2-\alpha^2)\,Z_C+\left(\frac{k_0^2-3\alpha^2}{2}\right)Z_2$	$\left(\frac{k_0^2-6\alpha^2}{2}\right)Z_1+\left(\frac{k_0^2-10\alpha^2}{2}\right)Z_3$	$\left(\frac{k_0^2-15\alpha^2}{2}\right)Z_2+\left(\frac{k_0^2-21\alpha^2}{2}\right)Z_4$	$\left(\frac{k_0^2-28\alpha^2}{2}\right)Z_3+\left(\frac{k_0^2-36\alpha^2}{2}\right)Z_5$
Cos2αx	$\left(\frac{k_0^2-4\alpha^2}{2}\right)Z_3+\left(\frac{k_0^2}{2}\right)Z_1$	$(k_0^2-4\alpha^2)\,Z_C+\left(\frac{k_0^2-12\alpha^2}{2}\right)Z_4$	$\left(\frac{k_0^2-12\alpha^2}{2}\right)Z_1+\left(\frac{k_0^2-24\alpha^2}{2}\right)Z_5$	$\left(\frac{k_0^2-24\alpha^2}{2}\right)Z_2$
Cos3αx	$\left(\frac{k_0^2-5\alpha^2}{2}\right)Z_4+\left(\frac{k_0^2+\alpha^2}{2}\right)Z_2$	$\left(\frac{k_0^2-14\alpha^2}{2}\right)Z_5+\left(\frac{k_0^2-2\alpha^2}{2}\right)Z_1$	$(k_0^2-9\alpha^2)\,Z_C$	$\left(\frac{k_0^2-20\alpha^2}{2}\right)Z_1$
Cos4αx	$\left(\frac{k_0^2-6\alpha^2}{2}\right)Z_5+\left(\frac{k_0^2+2\alpha^2}{2}\right)Z_3$	$\left(\frac{k_0^2}{2}\right)Z_2$	$\left(\frac{k_0^2-6\alpha^2}{2}\right)Z_1$	$(k_0^2-16\alpha^2)\,Z_C$
Sinαx	$\left(\frac{k_0^2-3\alpha^2}{2}\right)Z_7$	$-\left(\frac{k_0^2-6\alpha^2}{2}\right)Z_6+\left(\frac{k_0^2-10\alpha^2}{2}\right)Z_8$	$-\left(\frac{k_0^2-15\alpha^2}{2}\right)Z_7+\left(\frac{k_0^2-21\alpha^2}{2}\right)Z_9$	$-\left(\frac{k_0^2-28\alpha^2}{2}\right)Z_8+\left(\frac{k_0^2-36\alpha^2}{2}\right)Z_{10}$
Sin2αx	$\left(\frac{k_0^2-4\alpha^2}{2}\right)Z_8+\left(\frac{k_0^2}{2}\right)Z_6$	$\left(\frac{k_0^2-12\alpha^2}{2}\right)Z_9$	$\left(\frac{k_0^2-12\alpha^2}{2}\right)Z_6+\left(\frac{k_0^2-24\alpha^2}{2}\right)Z_{10}$	$\left(\frac{k_0^2-24\alpha^2}{2}\right)Z_7$
Sin3αx	$\left(\frac{k_0^2-5\alpha^2}{2}\right)Z_9+\left(\frac{k_0^2+\alpha^2}{2}\right)Z_7$	$\left(\frac{k_0^2-14\alpha^2}{2}\right)Z_{10}+\left(\frac{k_0^2-2\alpha^2}{2}\right)Z_6$	0	$-\left(\frac{k_0^2-20\alpha^2}{2}\right)Z_6$
Sin4αx	$\left(\frac{k_0^2-6\alpha^2}{2}\right)Z_{10}+\left(\frac{k_0^2+2\alpha^2}{2}\right)Z_8$	$\left(\frac{k_0^2}{2}\right)Z_7$	$\left(\frac{k_0^2-6\alpha^2}{2}\right)Z_6$	0

	V_5	rV_7	rV_8	rV_9
Const.	$\left(\frac{k_0^2 - 50\alpha^2}{2}\right)Z_5$	$\left(\frac{k_0^2 - 8\alpha^2}{2}\right)Z_7$	$\left(\frac{k_0^2 - 18\alpha^2}{2}\right)Z_8$	$\left(\frac{k_0^2 - 32\alpha^2}{2}\right)Z_9$
Cosαx	$\left(\frac{k_0^2 - 45\alpha^2}{2}\right)Z_4$	$\left(\frac{k_0^2 - 6\alpha^2}{2}\right)Z_1 + \left(\frac{k_0^2 - 10\alpha^2}{2}\right)Z_3$	$\left(\frac{k_0^2 - 15\alpha^2}{2}\right)Z_7 + \left(\frac{k_0^2 - 21\alpha^2}{2}\right)Z_9$	$\left(\frac{k_0^2 - 28\alpha^2}{2}\right)Z_8 + \left(\frac{k_0^2 - 36\alpha^2}{2}\right)Z_{10}$
Cos2αx	$\left(\frac{k_0^2 - 40\alpha^2}{2}\right)Z_3$	$\left(\frac{k_0^2 - 12\alpha^2}{2}\right)Z_4$	$\left(\frac{k_0^2 - 12\alpha^2}{2}\right)Z_6 + \left(\frac{k_0^2 - 24\alpha^2}{2}\right)Z_{10}$	$\left(\frac{k_0^2 - 24\alpha^2}{2}\right)Z_7$
Cos3αx	$\left(\frac{k_0^2 - 35\alpha^2}{2}\right)Z_2$	$\left(\frac{k_0^2 - 14\alpha^2}{2}\right)Z_5 + \left(\frac{k_0^2 - 2\alpha^2}{2}\right)Z_1$	0	$\left(\frac{k_0^2 - 20\alpha^2}{2}\right)Z_6$
Cos4αx	$\left(\frac{k_0^2 - 30\alpha^2}{2}\right)Z_1$	$-\left(\frac{k_0^2}{2}\right)Z_7$	$-\left(\frac{k_0^2 - 6\alpha^2}{2}\right)Z_6$	0
Sinαx	$-\left(\frac{k_0^2 - 45\alpha^2}{2}\right)Z_9$	$\left(\frac{k_0^2 - 6\alpha^2}{2}\right)Z_1 - \left(\frac{k_0^2 - 10\alpha^2}{2}\right)Z_3$	$\left(\frac{k_0^2 - 15\alpha^2}{2}\right)Z_2 - \left(\frac{k_0^2 - 21\alpha^2}{2}\right)Z_4$	$\left(\frac{k_0^2 - 28\alpha^2}{2}\right)Z_3 - \left(\frac{k_0^2 - 36\alpha^2}{2}\right)Z_5$
Sin2αx	$-\left(\frac{k_0^2 - 40\alpha^2}{2}\right)Z_8$	$(k_0^2-4\alpha^2)\, Z_C - \left(\frac{k_0^2 - 12\alpha^2}{2}\right)Z_4$	$\left(\frac{k_0^2 - 12\alpha^2}{2}\right)Z_1 - \left(\frac{k_0^2 - 24\alpha^2}{2}\right)Z_5$	$-\left(\frac{k_0^2 - 24\alpha^2}{2}\right)Z_2$
Sin3αx	$-\left(\frac{k_0^2 - 35\alpha^2}{2}\right)Z_7$	$-\left(\frac{k_0^2 - 14\alpha^2}{2}\right)Z_5 + \left(\frac{k_0^2 - 2\alpha^2}{2}\right)Z_1$	$(k_0^2 - 9\alpha^2)\, Z_C$	$\left(\frac{k_0^2 - 20\alpha^2}{2}\right)Z_1$
Sin4αx	$-\left(\frac{k_0^2 - 30\alpha^2}{2}\right)Z_6$	$\left(\frac{k_0^2}{2}\right)Z_2$	$\left(\frac{k_0^2 - 6\alpha^2}{2}\right)Z_1$	$(k_0^2 - 16\alpha^2)\, Z_C$

	rV_{10}		Constant		r *Constant
Const.	$\left(\frac{k_0^2 - 50\alpha^2}{2}\right)Z_{10}$		$-Z_C k_0^2$		$-\left(\frac{k_0^2 - 2\alpha^2}{2}\right)Z_6$
Cosαx	$\left(\frac{k_0^2 - 45\alpha^2}{2}\right)Z_9$		$-Z_1 k_0^2$		$-\left(\frac{k_0^2 - 3\alpha^2}{2}\right)Z_7$
Cos2αx	$\left(\frac{k_0^2 - 40\alpha^2}{2}\right)Z_8$		$-Z_2 k_0^2$		$-\left(\frac{k_0^2 - 4\alpha^2}{2}\right)Z_8 + \left(\frac{k_0^2}{2}\right)Z_6$
Cos3αx	$\left(\frac{k_0^2 - 35\alpha^2}{2}\right)Z_7$		$-Z_3 k_0^2$		$-\left(\frac{k_0^2 - 5\alpha^2}{2}\right)Z_9 + \left(\frac{k_0^2 + \alpha^2}{2}\right)Z_7$
Cos4αx	$\left(\frac{k_0^2 - 30\alpha^2}{2}\right)Z_6$	=	$-Z_4 k_0^2$	+ **r**	$-\left(\frac{k_0^2 - 6\alpha^2}{2}\right)Z_{10} + \left(\frac{k_0^2 + 2\alpha^2}{2}\right)Z_8$
Sinαx	$\left(\frac{k_0^2 - 45\alpha^2}{2}\right)Z_4$		$-Z_6 k_0^2$		$-(k_0^2 - \alpha^2)\, Z_C - \left(\frac{k_0^2 - 3\alpha^2}{2}\right)Z_2$
Sin2αx	$\left(\frac{k_0^2 - 40\alpha^2}{2}\right)Z_3$		$-Z_7 k_0^2$		$\left(\frac{k_0^2 - 4\alpha^2}{2}\right)Z_3 - \left(\frac{k_0^2}{2}\right)Z_1$
Sin3αx	$\left(\frac{k_0^2 - 35\alpha^2}{2}\right)Z_2$		$-Z_8 k_0^2$		$\left(\frac{k_0^2 - 5\alpha^2}{2}\right)Z_4 - \left(\frac{k_0^2 + \alpha^2}{2}\right)Z_2$
Sin4αx	$\left(\frac{k_0^2 - 30\alpha^2}{2}\right)Z_1$		$-Z_9 k_0^2$		$\left(\frac{k_0^2 - 6\alpha^2}{2}\right)Z_5 - \left(\frac{k_0^2 + 2\alpha^2}{2}\right)Z_3$

7.3 *Appendix C : Eight Terms Expansion Constant Term and Coefficients of Trigonometric Functions*

In this section, the terms that should be equal to zero to satisfy the solution of the equation (1) and form the system of equations with 15 equations and 15 unknowns (V_1, V_2, V_3, V_4, V_5, V_6, V_7, V_8, V_{10}, V_{11}, V_{12}, V_{13}, V_{14}, V_{15} and V_{16}) will be listed below:

$$\text{The Constant Term} = \left(\frac{k^2{}_0 - 2\alpha^2}{2}\right)(p\, Z_1 V_1 + q\, Z_9 V_9) + \left(\frac{k^2{}_0 - 8\alpha^2}{2}\right)(p\, Z_2 V_2 + q\, Z_{10} V_{10})$$

$$+\left(\frac{k^2{}_0 - 18\alpha^2}{2}\right)(p\, Z_3 V_3 + q\, Z_{11} V_{11}) + \left(\frac{k^2{}_0 - 32\alpha^2}{2}\right)(p\, Z_4 V_4 + q\, Z_{12} V_{12})$$

$$+\left(\frac{k^2{}_0 - 50\alpha^2}{2}\right)(pZ_5 V_5 + qZ_{13} V_{13}) + \left(\frac{k^2{}_0 - 72\alpha^2}{2}\right)(pZ_6 V_6 + q\, Z_{14} V_{14})$$

$$+\left(\frac{k^2{}_0 - 98\alpha^2}{2}\right)(pZ_7 V_7 + q\, Z_{15} V_{15}) + \left(\frac{k^2{}_0 - 128\alpha^2}{2}\right)(pZ_8 V_8 + q\, Z_{16} V_{16})$$

$$+ p\, V_0 Z_C k^2{}_0 = 0$$

$$\text{Cos}\alpha x \text{ Coefficient} = \left(\frac{k^2{}_0 - 3\alpha^2}{2}\right)(p\, Z_2 V_1 + q\, Z_{10} V_9) + \left(\frac{k^2{}_0 - 6\alpha^2}{2}\right)(p\, Z_1 V_2 + q\, Z_9 V_{10})$$

$$+\left(\frac{k^2{}_0 - 10\alpha^2}{2}\right)(p\, Z_3 V_2 + qZ_{11} V_{10}) + \left(\frac{k^2{}_0 - 15\alpha^2}{2}\right)(pZ_2 V_3 + q\, Z_{10} V_{11})$$

$$+\left(\frac{k^2{}_0 - 21\alpha^2}{2}\right)(p\, Z_4 V_3 + qZ_{12} V_{11}) + \left(\frac{k^2{}_0 - 28\alpha^2}{2}\right)(p\, Z_3 V_4 + q\, Z_{11} V_{12})$$

$$+\left(\frac{k^2{}_0-36\alpha^2}{2}\right)(p\ Z_5V_4+qZ_{13}V_{12})+\left(\frac{k^2{}_0-45\alpha^2}{2}\right)(pZ_4\ V_5+q\ Z_{12}\ V_{13})$$

$$+\left(\frac{k^2{}_0-55\alpha^2}{2}\right)(pZ_6V_5+qZ_{14}V_{13})+\left(\frac{k^2{}_0-66\alpha^2}{2}\right)(p\ Z_5\ V_6+q\ Z_{13}\ V_{14})$$

$$+\left(\frac{k^2{}_0-78\alpha^2}{2}\right)(pZ_7V_6+qZ_{15}V_{14})+\left(\frac{k^2{}_0-91\alpha^2}{2}\right)(p\ Z_6\ V_7+q\ Z_{14}\ V_{15})$$

$$+\left(\frac{k^2{}_0-105\alpha^2}{2}\right)(p\ Z_8V_7+qZ_{16}V_{15})+\left(\frac{k^2{}_0-120\alpha^2}{2}\right)(pZ_7V_8+q\ Z_{15}\ V_{16})$$

$$+\ p\ V_1\ Z_C\ (k_0^2-\alpha^2)+p\ V_0\ k^2{}_0\ Z_1=0$$

$$\text{Cos}2\alpha x\ \text{Coefficient} = \left(\frac{k^2{}_0-4\alpha^2}{2}\right)(p\ Z_3\ V_1+qZ_{11}V_9)+\left(\frac{k^2{}_0-12\alpha^2}{2}\right)(p\ Z_4\ V_2+q\ Z_{12}\ V_{10})$$

$$+\left(\frac{k^2{}_0-12\alpha^2}{2}\right)(pZ_1V_3+qZ_9V_{11})+\left(\frac{k^2{}_0-24\alpha^2}{2}\right)(p\ Z_5\ V_3+q\ Z_{13}\ V_{11})$$

$$+\left(\frac{k^2{}_0-24\alpha^2}{2}\right)(p\ Z_2V_4+q\ Z_{10}\ V_{12})+\left(\frac{k^2{}_0-40\alpha^2}{2}\right)(pZ_6V_4+q\ Z_{14}\ V_{12})$$

$$+\left(\frac{k^2{}_0-40\alpha^2}{2}\right)(p\ Z_3\ V_5+qZ_{11}V_{13})+\left(\frac{k^2{}_0-60\alpha^2}{2}\right)(pZ_7V_5+qZ_{15}\ V_{13})$$

$$+\left(\frac{k^2{}_0-60\alpha^2}{2}\right)(pZ_4V_6+qZ_{12}V_{14})+\left(\frac{k^2{}_0-84\alpha^2}{2}\right)(pZ_8\ V_6+q\ Z_{16}\ V_{14})$$

$$+\left(\frac{k^2{}_0-84\alpha^2}{2}\right)(pZ_5V_7+qZ_{13}V_{15})+\left(\frac{k^2{}_0-112\alpha^2}{2}\right)(p\ Z_6\ V_8+q\ Z_{14}\ V_{16})$$

$$+\left(\frac{k^2{}_0}{2}\right)(p\ Z_1\ V_1-q\ Z_9\ V_9)+p\ V_2\ Z_C\ (k^2{}_0-4\alpha^2)+p\ V_0\ k^2{}_0\ Z_2=0$$

$$\text{Cos}3\alpha x \text{ Coefficient} = \left(\frac{k^2{}_0 - 5\alpha^2}{2}\right)(p\ Z_4 V_1 + q\ Z_{12} V_9) + \left(\frac{k^2{}_0 - 14\alpha^2}{2}\right)(p\ Z_5 V_2 + q\ Z_{13} V_1)$$

$$+ \left(\frac{k^2{}_0 - 27\alpha^2}{2}\right)(p\ Z_6 V_3 + q\ Z_{14} V_{11}) + \left(\frac{k^2{}_0 - 20\alpha^2}{2}\right)(p\ Z_1 V_4 + q\ Z_9 V_{12})$$

$$+ \left(\frac{k^2{}_0 - 44\alpha^2}{2}\right)(pZ_7 V_4 + q\ Z_{15} V_{12}) + \left(\frac{k^2{}_0 - 35\alpha^2}{2}\right)(pZ_2 V_5 + q\ Z_{10} V_{13})$$

$$+ \left(\frac{k^2{}_0 - 65\alpha^2}{2}\right)(pZ_8 V_5 + q\ Z_{16} V_{13}) + \left(\frac{k^2{}_0 - 54\alpha^2}{2}\right)(pZ_3 V_6 + q\ Z_{11} V_{14})$$

$$+ \left(\frac{k^2{}_0 - 77\alpha^2}{2}\right)(pZ_4 V_7 + q\ Z_{12} V_{15}) + \left(\frac{k^2{}_0 - 104\alpha^2}{2}\right)(pZ_5 V_8 + qZ_{13} V_{16})$$

$$+ \left(\frac{k^2{}_0 + \alpha^2}{2}\right)(p\ Z_2 V_1 + q\ Z_{10} V_9) + \left(\frac{k^2{}_0 - 2\alpha^2}{2}\right)(p\ Z_1 V_2 + q\ Z_9 V_{10})$$

$$+ p\ V_3 Z_C\ (k^2{}_0 - 9\alpha^2) + p\ V_0 k^2{}_0 Z_3 = 0$$

$$\text{Cos}4\alpha x \text{ Coefficient} = \left(\frac{k^2{}_0 - 6\alpha^2}{2}\right)(pZ_5 V_1 + q\ Z_{13} V_9) + \left(\frac{k^2{}_0 - 16\alpha^2}{2}\right)(p\ Z_6 V_2 + q\ Z_{14} V_{10})$$

$$+ \left(\frac{k^2{}_0 - 30\alpha^2}{2}\right)(P\ Z_7 V_3 + q\ Z_{15} V_{11}) + \left(\frac{k^2{}_0 - 48\alpha^2}{2}\right)(pZ_8 V_4 + qZ_{16} V_{12})$$

$$+ \left(\frac{k^2{}_0 - 30\alpha^2}{2}\right)(pZ_1 V_5 + qZ_9 V_{13}) + \left(\frac{k^2{}_0 - 48\alpha^2}{2}\right)(p\ Z_2 V_6 + qZ_{10} V_{14})$$

$$+ \left(\frac{k^2{}_0 - 70\alpha^2}{2}\right)(pZ_3 V_7 + qZ_{11} V_{15}) + \left(\frac{k^2{}_0 - 96\alpha^2}{2}\right)(pZ_4 V_8 + q\ Z_{12} V_{16})$$

$$+\left(\frac{k^2{}_0+2\alpha^2}{2}\right)(p\,Z_3\,V_1-q\,Z_{11}\,V_9)+\left(\frac{k^2{}_0}{2}\right)(p\,Z_2\,V_2-q\,Z_{10}\,V_{10})$$

$$+\left(\frac{k^2{}_0-6\alpha^2}{2}\right)(p\,Z_1\,V_3-q\,Z_9\,V_{11})+p\,V_4\,Z_C\,(k^2{}_0-16\alpha^2)+p\,V_0\,k^2{}_0\,Z_4$$

$$=0$$

$$\text{Cos}5\alpha x\text{ Coefficient}=\left(\frac{k^2{}_0-7\alpha^2}{2}\right)(pZ_6V_1+q\,Z_{14}\,V_9)+\left(\frac{k^2{}_0-18\alpha^2}{2}\right)(p\,Z_7\,V_2+q\,Z_{15}\,V_{10})$$

$$+\left(\frac{k^2{}_0-33\alpha^2}{2}\right)(p\,Z_8V_3+q\,Z_{16}\,V_{11})+\left(\frac{k^2{}_0-42\alpha^2}{2}\right)(pZ_1\,V_6+q\,Z_9\,V_{14})$$

$$+\left(\frac{k^2{}_0-63\alpha^2}{2}\right)(pZ_2V_7+q\,Z_{10}\,V_{15})+\left(\frac{k^2{}_0-88\alpha^2}{2}\right)(pZ_3\,V_8+q\,Z_{11}\,V_{16})$$

$$+\left(\frac{k^2{}_0+3\alpha^2}{2}\right)(pZ_4V_1-qZ_{12}V_9)+\left(\frac{k^2{}_0+2\alpha^2}{2}\right)(p\,Z_3\,V_2-q\,Z_{11}\,V_{10})$$

$$+\left(\frac{k^2{}_0-3\alpha^2}{2}\right)(pZ_2V_3-qZ_{10}V_{11})+\left(\frac{k^2{}_0-12\alpha^2}{2}\right)(p\,Z_1\,V_4-q\,Z_9\,V_{12})$$

$$+p\,V_5\,Z_C\,(k^2{}_0-25\alpha^2)+p\,V_0\,k^2{}_0\,Z_5=0$$

$$\text{Cos}6\alpha x\text{ Coefficient}=\left(\frac{k^2{}_0-8\alpha^2}{2}\right)(p\,Z_7\,V_1+qZ_{15}\,V_9)+\left(\frac{k^2{}_0-20\alpha^2}{2}\right)(p\,Z_8V_2+q\,Z_{16}\,V_{10})$$

$$+\left(\frac{k^2{}_0-56\alpha^2}{2}\right)(p\,Z_1V_7+qZ_{\,9}V_{15})+\left(\frac{k^2{}_0-80\alpha^2}{2}\right)(p\,Z_2\,V_8+q\,Z_{10}\,V_{16})$$

$$+\left(\frac{k^2{}_0+4\alpha^2}{2}\right)(p\ Z_5\ V_1-q\ Z_{13}\ V_9)+\left(\frac{k^2{}_0+4\alpha^2}{2}\right)(p\ Z_4\ V_2-q\ Z_{12}\ V_{10})$$

$$+\left(\frac{k^2{}_0}{2}\right)(p\ Z_3\ V_3-q\ Z_{11}\ V_{11})+\left(\frac{k^2{}_0-8\alpha^2}{2}\right)(p\ Z_2\ V_4-q\ Z_{10}\ V_{12})$$

$$+\left(\frac{k^2{}_0-20\alpha^2}{2}\right)(p\ Z_1\ V_5-q\ Z_9\ V_{13})+p\ V_6\ Z_C\ (k^2{}_0-36\alpha^2)+p\ V_0\ k^2{}_0\ Z_6$$

$$=0$$

$$\text{Cos}7\alpha x\ \text{Coefficient}=\left(\frac{k^2{}_0-9\alpha^2}{2}\right)(pZ_8\ V_1+q\ Z_{16}\ V_9)+\left(\frac{k^2{}_0-72\alpha^2}{2}\right)(pZ_1\ V_8+q\ Z_9\ V_{16})$$

$$+\left(\frac{k^2{}_0+5\alpha^2}{2}\right)(pZ_6\ V_1-q\ Z_{14}\ V_9)+\left(\frac{k^2{}_0+6\alpha^2}{2}\right)(p\ Z_5\ V_2-q\ Z_{13}\ V_{10})$$

$$+\left(\frac{k^2{}_0+3\alpha^2}{2}\right)(p\ Z_4\ V_3-q\ Z_{12}\ V_{11})+\left(\frac{k^2{}_0-4\alpha^2}{2}\right)(p\ Z_3\ V_4-qZ_{11}\ V_{12})$$

$$+\left(\frac{k^2{}_0-15\alpha^2}{2}\right)(p\ Z_2V_5-q\ Z_{10}V_{13})+\left(\frac{k^2{}_0-30\alpha^2}{2}\right)(p\ Z_1V_6-qZ_9\ V_{14})$$

$$+p\ V_7\ Z_C\ (k^2{}_0-49\alpha^2)+p\ V_0\ k^2{}_0\ Z_7=0$$

$$\text{Cos}8\alpha x\ \text{Coefficient}=\left(\frac{k^2{}_0+6\alpha^2}{2}\right)(pZ_7\ V_1-q\ Z_{15}\ V_9)+\left(\frac{k^2{}_0+8\alpha^2}{2}\right)(p\ Z_6\ V_2-q\ Z_{14}\ V_{10})$$

$$+\left(\frac{k^2{}_0+6\alpha^2}{2}\right)(p\ Z_5\ V_3-q\ Z_{13}\ V_{11})+\left(\frac{k^2{}_0}{2}\right)(p\ Z_4\ V_4-q\ Z_{12}\ V_{12})$$

$$+\left(\frac{k^2{}_0-10\alpha^2}{2}\right)(pZ_3V_5-q\,Z_{11}V_{13})+\left(\frac{k^2{}_0-24\alpha^2}{2}\right)(pZ_2V_6-q\,Z_{10}\,V_{14})$$

$$+\left(\frac{k^2{}_0-42\alpha^2}{2}\right)(p\,Z_1V_7-q\,Z_9\,V_{15})+p\,V_8\,Z_C\,(k^2{}_0-64\alpha^2)+P\,V_0\,k^2{}_0\,Z_8$$

$$=0$$

$$\text{Sin}\alpha x\text{ Coefficient} = -\left(\frac{k^2{}_0-3\alpha^2}{2}\right)(-\,pZ_{10}V_1+q\,V_9\,Z_2)+\left(\frac{k^2{}_0-6\alpha^2}{2}\right)(-\,pZ_9\,V_2+q\,V_{10}\,Z_1)$$

$$-\left(\frac{k^2{}_0-10\alpha^2}{2}\right)(-pZ_{11}V_2+qV_{10}\,Z_3)+\left(\frac{k^2{}_0-15\alpha^2}{2}\right)(-pZ_{10}V_3+qV_{11}\,Z_2)$$

$$-\left(\frac{k^2{}_0-21\alpha^2}{2}\right)(-pZ_{12}V_3+qV_{11}\,Z_4)+\left(\frac{k^2{}_0-28\alpha^2}{2}\right)(-pZ_{11}V_4+q\,V_{12}\,Z_3)$$

$$-\left(\frac{k^2{}_0-36\alpha^2}{2}\right)(-pZ_{13}V_4+qV_{12}\,Z_5)+\left(\frac{k^2{}_0-45\alpha^2}{2}\right)(-pZ_{12}V_5+qV_{13}\,Z_4)$$

$$-\left(\frac{k^2{}_0-55\alpha^2}{2}\right)(-pZ_{14}V_5+qV_{13}\,Z_6)+\left(\frac{k^2{}_0-66\alpha^2}{2}\right)(-pZ_{13}V_6+qV_{14}\,Z_5)$$

$$-\left(\frac{k^2{}_0-78\alpha^2}{2}\right)(-pZ_{15}V_6+qV_{14}\,Z_7)+\left(\frac{k^2{}_0-91\alpha^2}{2}\right)(-pZ_{14}V_7+qV_{15}\,Z_6)$$

$$-\left(\frac{k^2{}_0-105\alpha^2}{2}\right)(-pZ_{16}V_7+qV_{15}\,Z_8)+\left(\frac{k^2{}_0-120\alpha^2}{2}\right)(-pZ_{15}V_8+qV_{16}\,Z_7)$$

$$+\,q\,V_9\,Z_C\,(k^2{}_0-\alpha^2)+p\,V_0\,k^2{}_0\,Z_9=0$$

$$\text{Sin}2\alpha x \text{ Coefficient} = -\left(\frac{k^2{}_0 - 4\alpha^2}{2}\right)(-pZ_{11}V_1+qV_9 Z_3) - \left(\frac{k^2{}_0 - 12\alpha^2}{2}\right)(-pZ_{12}V_2 + q\, V_{10} Z_4)$$

$$+\left(\frac{k^2{}_0 - 12\alpha^2}{2}\right)(-pZ_9V_3+qV_{11}Z_1) - \left(\frac{k^2{}_0 - 24\alpha^2}{2}\right)(-pZ_{13}V_3 + qV_{11} Z_5)$$

$$+\left(\frac{k^2{}_0 - 24\alpha^2}{2}\right)(-pZ_{10}V_4+qV_{12} Z_2) - \left(\frac{k^2{}_0 - 40\alpha^2}{2}\right)(-pZ_{14}V_4+qV_{12} Z_6)$$

$$+\left(\frac{k^2{}_0 - 40\alpha^2}{2}\right)(-pZ_{11}V_5+qV_{13} Z_3) - \left(\frac{k^2{}_0 - 60\alpha^2}{2}\right)(-pZ_{15}V_5+qV_{13} Z_7)$$

$$+\left(\frac{k^2{}_0 - 60\alpha^2}{2}\right)(-pZ_{12}V_6+qV_{14} Z_4) - \left(\frac{k^2{}_0 - 84\alpha^2}{2}\right)(-pZ_{16}V_6+qV_{14} Z_8)$$

$$+\left(\frac{k^2{}_0 - 84\alpha^2}{2}\right)(-pZ_{13}V_7+qV_{15} Z_5) + \left(\frac{k^2{}_0 - 112\alpha^2}{2}\right)(-pZ_{14}V_8+qV_{16} Z_6)$$

$$+\left(\frac{k^2{}_0}{2}\right)(p\, Z_9 V_1 + q\, Z_1 V_9) + q\, V_{10} Z_C\, (k^2{}_0 - 4\alpha^2) + p\, V_0\, k^2{}_0\, Z_{10} = 0$$

$$\text{Sin}3\alpha x \text{ Coefficient} = -\left(\frac{k^2{}_0 - 5\alpha^2}{2}\right)(-pZ_{12}V_1+qV_9 Z_4) - \left(\frac{k^2{}_0 - 14\alpha^2}{2}\right)(-pZ_{13}V_2 + q\, V_{10} Z_5)$$

$$-\left(\frac{k^2{}_0 - 27\alpha^2}{2}\right)(-pZ_{14}V_3+q\, V_{11} Z_6) + \left(\frac{k^2{}_0 - 20\alpha^2}{2}\right)(-pZ_9V_4+qV_{12} Z_1)$$

$$-\left(\frac{k^2{}_0 - 44\alpha^2}{2}\right)(-pZ_{15}V_4+qV_{12} Z_7) + \left(\frac{k^2{}_0 - 35\alpha^2}{2}\right)(-pZ_{10}V_5 +qV_{13} Z_2)$$

$$-\left(\frac{k^2{}_0 - 65\alpha^2}{2}\right)(-pZ_{16}V_5+qV_{13} Z_8) + \left(\frac{k^2{}_0 - 54\alpha^2}{2}\right)(-pZ_{11}V_6 +qV_{14} Z_3)$$

$$+\left(\frac{k^2{}_0-77\alpha^2}{2}\right)(-pZ_{12}V_7+qV_{15}Z_4)+\left(\frac{k^2{}_0-104\alpha^2}{2}\right)(-pZ_{13}V_8+qV_{16}Z_5)$$

$$+\left(\frac{k^2{}_0+\alpha^2}{2}\right)(pZ_{10}V_1+q\,Z_2V_9)+\left(\frac{k^2{}_0-2\alpha^2}{2}\right)(pZ_9V_2+q\,Z_1V_{10})$$

$$+\,q\,V_{11}Z_C(k^2{}_0-9\alpha^2)+p\,V_0k^2{}_0Z_{11}=0$$

$$\text{Sin}4\alpha x\text{ Coefficient} = -\left(\frac{k^2{}_0-6\alpha^2}{2}\right)(-pZ_{13}V_1+qV_9Z_5)-\left(\frac{k^2{}_0-16\alpha^2}{2}\right)(-pZ_{14}V_2+q\,V_{10}Z_6)$$

$$-\left(\frac{k^2{}_0-30\alpha^2}{2}\right)(-pZ_{15}V_3+qV_{11}Z_7)-\left(\frac{k^2{}_0-48\alpha^2}{2}\right)(-pZ_{16}V_4+qV_{12}Z_8)$$

$$+\left(\frac{k^2{}_0-30\alpha^2}{2}\right)(-pZ_9V_5+qV_{12}Z_7)+\left(\frac{k^2{}_0-48\alpha^2}{2}\right)(-pZ_{10}V_6+qV_{14}Z_2)$$

$$+\left(\frac{k^2{}_0-70\alpha^2}{2}\right)(-pZ_{11}V_7+qV_{15}Z_3)+\left(\frac{k^2{}_0-96\alpha^2}{2}\right)(-pZ_{12}V_8+qV_{16}Z_4)$$

$$+\left(\frac{k^2{}_0+2\alpha^2}{2}\right)(p\,Z_{11}V_1+q\,V_3Z_9)+\left(\frac{k^2{}_0}{2}\right)(p\,Z_{10}V_2+q\,V_2Z_{10})$$

$$+\left(\frac{k^2{}_0-6\alpha^2}{2}\right)(pZ_9V_3+q\,V_1Z_{11})+q\,V_{12}Z_C(k^2{}_0-16\alpha^2)+p\,V_0k^2{}_0Z_{12}$$

$$=0$$

$$\text{Sin}5\alpha x \text{ Coefficient} = -\left(\frac{k^2{}_0 - 7\alpha^2}{2}\right)(-p\,Z_{14}\,V_1 + qV_9\,Z_6) - \left(\frac{k^2{}_0 - 18\alpha^2}{2}\right)(-p\,Z_{15}V_2 + qV_{10}\,Z_7)$$

$$-\left(\frac{k^2{}_0 - 33\alpha^2}{2}\right)(-pZ_{16}V_3 + qV_{11}\,Z_8) + \left(\frac{k^2{}_0 - 42\alpha^2}{2}\right)(-pZ_9V_6 + qV_{14}\,Z_1)$$

$$+\left(\frac{k^2{}_0 - 63\alpha^2}{2}\right)(-pZ_{10}V_7 + qV_{15}\,Z_2) + \left(\frac{k^2{}_0 - 88\alpha^2}{2}\right)(-pZ_{11}V_8 + qV_{16}\,Z_3)$$

$$+\left(\frac{k^2{}_0 + 3\alpha^2}{2}\right)(pZ_{12}V_1 + qV_4\,Z_9) + \left(\frac{k^2{}_0 + 2\alpha^2}{2}\right)(pZ_{11}V_2 + q\,V_3\,Z_{10})$$

$$+\left(\frac{k^2{}_0 - 3\alpha^2}{2}\right)(pZ_{10}V_3 + qV_2\,Z_{11}) + \left(\frac{k^2{}_0 - 12\alpha^2}{2}\right)(p\,Z_9\,V_4 + q\,V_1\,Z_{12})$$

$$+ q\,V_{13}\,Z_C\,(k^2{}_0 - 25\alpha^2) + p\,V_0\,k^2{}_0\,Z_{13} = 0$$

$$\text{Sin}6\alpha x \text{ Coefficient} = -\left(\frac{k^2{}_0 - 8\alpha^2}{2}\right)(-pZ_{15}V_1 + qV_9\,Z_7) - \left(\frac{k^2{}_0 - 20\alpha^2}{2}\right)(-pZ_{16}V_2 + q\,V_{10}\,Z_8)$$

$$+\left(\frac{k^2{}_0 - 56\alpha^2}{2}\right)(-pZ_9V_7 + qV_{15}\,Z_1) + \left(\frac{k^2{}_0 - 80\alpha^2}{2}\right)(-p\,Z_{10}V_8 + qV_{16}\,Z_2)$$

$$+\left(\frac{k^2{}_0 + 4\alpha^2}{2}\right)(pZ_{13}V_1 + q\,V_5\,Z_9) + \left(\frac{k^2{}_0 + 4\alpha^2}{2}\right)(p\,Z_{12}\,V_2 + q\,V_4\,Z_{10})$$

$$+\left(\frac{k^2{}_0}{2}\right)(p\,Z_{11}\,V_3 + q\,V_3\,Z_{11}) + \left(\frac{k^2{}_0 - 8\alpha^2}{2}\right)(p\,Z_{10}\,V_4 + q\,V_2\,Z_{12})$$

$$+\left(\frac{k^2{}_0 - 20\alpha^2}{2}\right)(pZ_9V_5 + qV_1Z_{13}) + qV_{14}\,Z_C\,(k^2{}_0 - 36\alpha^2) + p\,V_0\,k^2{}_0\,Z_{14}$$

$$= 0$$

$$\text{Sin}7\alpha x \text{ Coefficient} = -\left(\frac{k^2{}_0 - 9\alpha^2}{2}\right)(-pZ_{16}V_1 + qV_9 Z_8) + \left(\frac{k^2{}_0 - 72\alpha^2}{2}\right)(-pZ_9V_8 + q\,V_{16}\,Z_1)$$

$$+\left(\frac{k^2{}_0 + 5\alpha^2}{2}\right)(pZ_{14}V_1 + q\,Z_6\,V_9) + \left(\frac{k^2{}_0 + 6\alpha^2}{2}\right)(p\,Z_{13}V_2 + q\,Z_5\,V_{10})$$

$$+\left(\frac{k^2{}_0 + 3\alpha^2}{2}\right)(pZ_{12}V_3 + q\,Z_4V_{11}) + \left(\frac{k^2{}_0 - 4\alpha^2}{2}\right)(p\,Z_{11}\,V_4 + q\,Z_3\,V_{12})$$

$$+\left(\frac{k^2{}_0 - 15\alpha^2}{2}\right)(pZ_{10}V_5 + qZ_2V_{13}) + \left(\frac{k^2{}_0 - 30\alpha^2}{2}\right)(pZ_9V_6 + q\,Z_1\,V_{14})$$

$$+ q\,V_{15}\,Z_C\,(k^2{}_0 - 49\alpha^2) + p\,V_0\,k^2{}_0\,Z_{15} = 0$$

$$\text{Sin } 8\alpha x \text{ Coefficient} = \left(\frac{k^2{}_0 + 6\alpha^2}{2}\right)(pZ_{15}V_1 + qZ_7\,V_9) + \left(\frac{k^2{}_0 + 8\alpha^2}{2}\right)(p\,Z_{14}\,V_2 + q\,Z_6\,V_{10})$$

$$+\left(\frac{k^2{}_0 + 6\alpha^2}{2}\right)(p\,Z_{13}\,V_3 + q\,Z_5\,V_{11}) + \left(\frac{k^2{}_0}{2}\right)(p\,Z_{12}\,V_4 + q\,Z_4\,V_{12})$$

$$+\left(\frac{k^2{}_0 - 10\alpha^2}{2}\right)(pZ_{11}V_5 + qZ_3V_{13}) + \left(\frac{k^2{}_0 - 24\alpha^2}{2}\right)(pZ_{10}V_6 + q\,Z_2\,V_{14})$$

$$+\left(\frac{k^2{}_0 - 42\alpha^2}{2}\right)(pZ_9V_7 + qZ_1V_{15}) + qV_{16}\,Z_C\,(k^2{}_0 - 64\alpha^2) + pV_0\,k^2{}_0\,Z_{16}$$

$$= 0$$

7.4 Appendix D: Equations Matrix of Eight Terms Expansion

	V_1	V_2	V_3
Const.	$\left(\frac{k_0^2-2\alpha^2}{2}\right)Z_1$	$\left(k_0^2-8\alpha^2\right)Z_2$	$\left(\frac{k_0^2-18\alpha^2}{2}\right)Z_3$
Cos*α*x	$\left(\frac{k_0^2-3\alpha^2}{2}\right)Z_2+(k_0^2-\alpha^2)Z_C$	$\left(\frac{k_0^2-6\alpha^2}{2}\right)Z_1+\left(\frac{k_0^2-10\alpha^2}{2}\right)Z_3$	$\left(\frac{k_0^2-15\alpha^2}{2}\right)Z_2+\left(\frac{k_0^2-21\alpha^2}{2}\right)Z_4$
Cos2*α*x	$\left(\frac{k_0^2-4\alpha^2}{2}\right)Z_3+\frac{k_0^2}{2}Z_1$	$\left(\frac{k_0^2-12\alpha^2}{2}\right)Z_4+Z_C(k_0^2-4\alpha^2)$	$\left(\frac{k_0^2-12\alpha^2}{2}\right)Z_1+\left(\frac{k_0^2-24\alpha^2}{2}\right)Z_5$
Cos3*α*x	$\left(\frac{k_0^2-5\alpha^2}{2}\right)Z_4+\left(\frac{k_0^2+\alpha^2}{2}\right)Z_2$	$\left(\frac{k_0^2-14\alpha^2}{2}\right)Z_5+\left(\frac{k_0^2-2\alpha^2}{2}\right)Z_1$	$\left(\frac{k_0^2-27\alpha^2}{2}\right)Z_6+Z_C\left(k_0^2-9\alpha^2\right)$
Cos4*α*x	$\left(\frac{k_0^2-6\alpha^2}{2}\right)Z_5+\left(\frac{k_0^2+2\alpha^2}{2}\right)Z_3$	$\left(\frac{k_0^2-16\alpha^2}{2}\right)Z_6+\left(\frac{k_0^2}{2}\right)Z_2$	$\left(\frac{k_0^2-30\alpha^2}{2}\right)Z_7+\left(\frac{k_0^2-6\alpha^2}{2}\right)Z_1$
Cos5*α*x	$\left(\frac{k_0^2-7\alpha^2}{2}\right)Z_6+\left(\frac{k_0^2+3\alpha^2}{2}\right)Z_4$	$\left(\frac{k_0^2-18\alpha^2}{2}\right)Z_7+\left(\frac{k_0^2+2\alpha^2}{2}\right)Z_3$	$\left(\frac{k_0^2-33\alpha^2}{2}\right)Z_8+\left(\frac{k_0^2-3\alpha^2}{2}\right)Z_2$
Cos6*α*x	$\left(\frac{k_0^2-8\alpha^2}{2}\right)Z_7+\left(\frac{k_0^2+4\alpha^2}{2}\right)Z_5$	$\left(\frac{k_0^2-20\alpha^2}{2}\right)Z_8+\left(\frac{k_0^2+4\alpha^2}{2}\right)Z_4$	$\left(\frac{k_0^2}{2}\right)Z_3$
Cos7*α*x	$\left(\frac{k_0^2-9\alpha^2}{2}\right)Z_8+\left(\frac{k_0^2+5\alpha^2}{2}\right)Z_6$	$\left(\frac{k_0^2+6\alpha^2}{2}\right)Z_5$	$\left(\frac{k_0^2+3\alpha^2}{2}\right)Z_4$
Sin*α*x	$\left(\frac{k_0^2-3\alpha^2}{2}\right)Z_{10}$	$-\left(\frac{k_0^2-6\alpha^2}{2}\right)Z_9+\left(\frac{k_0^2-10\alpha^2}{2}\right)Z_{11}$	$-\left(\frac{k_0^2-15\alpha^2}{2}\right)Z_{10}+\left(\frac{k_0^2-21\alpha^2}{2}\right)Z_{12}$
Sin2*α*x	$\left(\frac{k_0^2-4\alpha^2}{2}\right)Z_{11}+\left(\frac{k_0^2}{2}\right)Z_9$	$\left(\frac{k_0^2-12\alpha^2}{2}\right)Z_{12}$	$-\left(\frac{k_0^2-12\alpha^2}{2}\right)Z_9+\left(\frac{k_0^2-24\alpha^2}{2}\right)Z_{13}$
Sin3*α*x	$\left(\frac{k_0^2-5\alpha^2}{2}\right)Z_{12}+\left(\frac{k_0^2+\alpha^2}{2}\right)Z_{10}$	$\left(\frac{k_0^2-14\alpha^2}{2}\right)Z_{13}+\left(\frac{k_0^2-2\alpha^2}{2}\right)Z_9$	$\left(\frac{k_0^2-27\alpha^2}{2}\right)Z_{14}$
Sin4*α*x	$\left(\frac{k_0^2-6\alpha^2}{2}\right)Z_{13}+\left(\frac{k_0^2+2\alpha^2}{2}\right)Z_{11}$	$\left(\frac{k_0^2-16\alpha^2}{2}\right)Z_{14}+\left(\frac{k_0^2}{2}\right)Z_{10}$	$\left(\frac{k_0^2-30\alpha^2}{2}\right)Z_{15}+\left(\frac{k_0^2-6\alpha^2}{2}\right)Z_9$
Sin5*α*x	$\left(\frac{k_0^2-7\alpha^2}{2}\right)Z_{14}+\left(\frac{k_0^2+3\alpha^2}{2}\right)Z_{12}$	$\left(\frac{k_0^2-18\alpha^2}{2}\right)Z_{15}+\left(\frac{k_0^2+2\alpha^2}{2}\right)Z_{11}$	$\left(\frac{k_0^2-33\alpha^2}{2}\right)Z_{16}+\left(\frac{k_0^2-3\alpha^2}{2}\right)Z_{10}$
Sin6*α*x	$\left(\frac{k_0^2-8\alpha^2}{2}\right)Z_{15}+\left(\frac{k_0^2+4\alpha^2}{2}\right)Z_{13}$	$\left(\frac{k_0^2-20\alpha^2}{2}\right)Z_{16}+\left(\frac{k_0^2+4\alpha^2}{2}\right)Z_{12}$	$\left(\frac{k_0^2}{2}\right)Z_{11}$
Sin7*α*x	$\left(\frac{k_0^2-9\alpha^2}{2}\right)Z_{16}+\left(\frac{k_0^2+5\alpha^2}{2}\right)Z_{14}$	$\left(\frac{k_0^2+6\alpha^2}{2}\right)Z_{13}$	$\left(\frac{k_0^2+3\alpha^2}{2}\right)Z_{12}$

	V_4	V_5	V_6
Const.	$\left(\frac{k_0^2 - 32\alpha^2}{2}\right)Z_4$	$\left(\frac{k_0^2 - 50\alpha^2}{2}\right)Z_5$	$\left(\frac{k_0^2 - 72\alpha^2}{2}\right)Z_6$
Cos*α*x	$\left(\frac{k_0^2 - 28\alpha^2}{2}\right)Z_3 + \left(\frac{k_0^2 - 36\alpha^2}{2}\right)Z_5$	$\left(\frac{k_0^2 - 45\alpha^2}{2}\right)Z_4 + \left(\frac{k_0^2 - 55\alpha^2}{2}\right)Z_6$	$\left(\frac{k_0^2 - 66\alpha^2}{2}\right)Z_5 + \left(\frac{k_0^2 - 78\alpha^2}{2}\right)Z_7$
Cos2*α*x	$\left(\frac{k_0^2 - 24\alpha^2}{2}\right)Z_2 + \left(\frac{k_0^2 - 40\alpha^2}{2}\right)Z_6$	$\left(\frac{k_0^2 - 40\alpha^2}{2}\right)Z_3 + \left(\frac{k_0^2 - 60\alpha^2}{2}\right)Z_7$	$\left(\frac{k_0^2 - 60\alpha^2}{2}\right)Z_4 + \left(\frac{k_0^2 - 84\alpha^2}{2}\right)Z_8$
Cos3*α*x	$\left(\frac{k_0^2 - 20\alpha^2}{2}\right)Z_1 + \left(\frac{k_0^2 - 44\alpha^2}{2}\right)Z_7$	$\left(\frac{k_0^2 - 35\alpha^2}{2}\right)Z_2 + \left(\frac{k_0^2 - 65\alpha^2}{2}\right)Z_8$	$\left(\frac{k_0^2 - 54\alpha^2}{2}\right)Z_3$
Cos4*α*x	$\left(\frac{k_0^2 - 48\alpha^2}{2}\right)Z_8 + Z_C\left(k_0^2 - 16\alpha^2\right)$	$\left(\frac{k_0^2 - 30\alpha^2}{2}\right)Z_1$	$\left(\frac{k_0^2 - 48\alpha^2}{2}\right)Z_2$
Cos5*α*x	$\left(\frac{k_0^2 - 12\alpha^2}{2}\right)Z_1$	$Z_C\,(k_0^2 - 25\alpha^2)$	$\left(\frac{k_0^2 - 42\alpha^2}{2}\right)Z_1$
Cos6*α*x	$\left(\frac{k_0^2 - 8\alpha^2}{2}\right)Z_2$	$\left(\frac{k_0^2 - 20\alpha^2}{2}\right)Z_1$	$Z_C\,(k_0^2 - 36\alpha^2)$
Cos7*α*x	$\left(\frac{k_0^2 - 4\alpha^2}{2}\right)Z_3$	$\left(\frac{k_0^2 - 15\alpha^2}{2}\right)Z_2$	$\left(\frac{k_0^2 - 30\alpha^2}{2}\right)Z_1$
Sin*α*x	$-\left(\frac{k_0^2 - 28\alpha^2}{2}\right)Z_{11} + \left(\frac{k_0^2 - 36\alpha^2}{2}\right)Z_{13}$	$-\left(\frac{k_0^2 - 45\alpha^2}{2}\right)Z_{12} + \left(\frac{k_0^2 - 55\alpha^2}{2}\right)Z_{14}$	$-\left(\frac{k_0^2 - 66\alpha^2}{2}\right)Z_{13} + \left(\frac{k_0^2 - 78\alpha^2}{2}\right)Z_{15}$
Sin2*α*x	$-\left(\frac{k_0^2 - 24\alpha^2}{2}\right)Z_{10} + \left(\frac{k_0^2 - 40\alpha^2}{2}\right)Z_{14}$	$-\left(\frac{k_0^2 - 40\alpha^2}{2}\right)Z_{11} + \left(\frac{k_0^2 - 60\alpha^2}{2}\right)Z_{15}$	$-\left(\frac{k_0^2 - 60\alpha^2}{2}\right)Z_{12} + \left(\frac{k_0^2 - 84\alpha^2}{2}\right)Z_{16}$
Sin3*α*x	$-\left(\frac{k_0^2 - 20\alpha^2}{2}\right)Z_9 + \left(\frac{k_0^2 - 44\alpha^2}{2}\right)Z_{15}$	$-\left(\frac{k_0^2 - 35\alpha^2}{2}\right)Z_{10} + \left(\frac{k_0^2 - 65\alpha^2}{2}\right)Z_{16}$	$-\left(\frac{k_0^2 - 54\alpha^2}{2}\right)Z_{11}$
Sin4*α*x	$\left(\frac{k_0^2 - 48\alpha^2}{2}\right)Z_{16}$	$-\left(\frac{k_0^2 - 30\alpha^2}{2}\right)Z_9$	$-\left(\frac{k_0^2 - 48\alpha^2}{2}\right)Z_{10}$
Sin5*α*x	$\left(\frac{k_0^2 - 12\alpha^2}{2}\right)Z_9$	0	$-\left(\frac{k_0^2 - 42\alpha^2}{2}\right)Z_9$
Sin6*α*x	$\left(\frac{k_0^2 - 8\alpha^2}{2}\right)Z_{10}$	$\left(\frac{k_0^2 - 20\alpha^2}{2}\right)Z_9$	0
Sin7*α*x	$\left(\frac{k_0^2 - 4\alpha^2}{2}\right)Z_{11}$	$\left(\frac{k_0^2 - 15\alpha^2}{2}\right)Z_{10}$	$\left(\frac{k_0^2 - 30\alpha^2}{2}\right)Z_9$

	V_7	V_8	rV_{10}
Const.	$\left(\frac{k^2{}_0 - 98\alpha^2}{2}\right)Z_7$	$\left(\frac{k^2{}_0 - 128\alpha^2}{2}\right)Z_8$	$\left(\frac{k^2{}_0 - 8\alpha^2}{2}\right)Z_{10}$
Cos*α x*	$\left(\frac{k^2{}_0 - 91\alpha^2}{2}\right)Z_6 + \left(\frac{k^2{}_0 - 105\alpha^2}{2}\right)Z_8$	$\left(\frac{k^2{}_0 - 120\alpha^2}{2}\right)Z_7$	$\left(\frac{k^2{}_0 - 6\alpha^2}{2}\right)Z_9 + \left(\frac{k^2{}_0 - 10\alpha^2}{2}\right)Z_{11}$
Cos2*α x*	$\left(\frac{k^2{}_0 - 84\alpha^2}{2}\right)Z_5$	$\left(\frac{k^2{}_0 - 112\alpha^2}{2}\right)Z_6$	$\left(\frac{k^2{}_0 - 12\alpha^2}{2}\right)Z_{12}$
Cos3*α x*	$\left(\frac{k^2{}_0 - 77\alpha^2}{2}\right)Z_4$	$\left(\frac{k^2{}_0 - 104\alpha^2}{2}\right)Z_5$	$\left(\frac{k^2{}_0 - 14\alpha^2}{2}\right)Z_{13} - \left(\frac{k^2{}_0 - 2\alpha^2}{2}\right)Z_9$
Cos4*α x*	$\left(\frac{k^2{}_0 - 70\alpha^2}{2}\right)Z_3$	$\left(\frac{k^2{}_0 - 96\alpha^2}{2}\right)Z_4$	$\left(\frac{k^2{}_0 - 16\alpha^2}{2}\right)Z_{14} - \left(\frac{k^2{}_0}{2}\right)Z_{10}$
Cos5*α x*	$\left(\frac{k^2{}_0 - 63\alpha^2}{2}\right)Z_2$	$\left(\frac{k^2{}_0 - 88\alpha^2}{2}\right)Z_3$	$\left(\frac{k^2{}_0 - 18\alpha^2}{2}\right)Z_{15} - \left(\frac{k^2{}_0 + 2\alpha^2}{2}\right)Z_{11}$
Cos6*α x*	$\left(\frac{k^2{}_0 - 56\alpha^2}{2}\right)Z_1$	$\left(\frac{k^2{}_0 - 80\alpha^2}{2}\right)Z_2$	$\left(\frac{k^2{}_0 - 20\alpha^2}{2}\right)Z_{16} - \left(\frac{k^2{}_0 + 4\alpha^2}{2}\right)Z_{12}$
Cos7*α x*	$Z_C\,(k_0^2 - 49\alpha^2)$	$\left(\frac{k^2{}_0 - 72\alpha^2}{2}\right)Z_1$	$-\left(\frac{k^2{}_0 + 6\alpha^2}{2}\right)Z_{13}$
Sin*α x*	$-\left(\frac{k^2{}_0 - 91\alpha^2}{2}\right)Z_{14} + \left(\frac{k^2{}_0 - 105\alpha^2}{2}\right)Z_{16}$	$-\left(\frac{k^2{}_0 - 120\alpha^2}{2}\right)Z_{15}$	$\left(\frac{k^2{}_0 - 6\alpha^2}{2}\right)Z_1 - \left(\frac{k^2{}_0 - 10\alpha^2}{2}\right)Z_3$
Sin2*α x*	$-\left(\frac{k^2{}_0 - 84\alpha^2}{2}\right)Z_{13}$	$-\left(\frac{k^2{}_0 - 112\alpha^2}{2}\right)Z_{14}$	$-\left(\frac{k^2{}_0 - 12\alpha^2}{2}\right)Z_4 + Z_C\left(k^2{}_0 - 4\alpha^2\right)$
Sin3*α x*	$-\left(\frac{k^2{}_0 - 77\alpha^2}{2}\right)Z_{12}$	$-\left(\frac{k^2{}_0 - 104\alpha^2}{2}\right)Z_{13}$	$-\left(\frac{k^2{}_0 - 14\alpha^2}{2}\right)Z_5 + \left(\frac{k^2{}_0 - 2\alpha^2}{2}\right)Z_1$
Sin4*α x*	$-\left(\frac{k^2{}_0 - 70\alpha^2}{2}\right)Z_{11}$	$-\left(\frac{k^2{}_0 - 96\alpha^2}{2}\right)Z_{12}$	$-\left(\frac{k^2{}_0 - 16\alpha^2}{2}\right)Z_6 + \left(\frac{k^2{}_0}{2}\right)Z_2$
Sin5*α x*	$-\left(\frac{k^2{}_0 - 63\alpha^2}{2}\right)Z_{10}$	$-\left(\frac{k^2{}_0 - 88\alpha^2}{2}\right)Z_{11}$	$-\left(\frac{k^2{}_0 - 18\alpha^2}{2}\right)Z_7 + \left(\frac{k^2{}_0 + 2\alpha^2}{2}\right)Z_3$
Sin6*α x*	$-\left(\frac{k^2{}_0 - 56\alpha^2}{2}\right)Z_9$	$-\left(\frac{k^2{}_0 - 80\alpha^2}{2}\right)Z_{10}$	$-\left(\frac{k^2{}_0 - 20\alpha^2}{2}\right)Z_8 + \left(\frac{k^2{}_0 + 4\alpha^2}{2}\right)Z_4$
Sin7*α x*	0	$-\left(\frac{k^2{}_0 - 72\alpha^2}{2}\right)Z_9$	$\left(\frac{k^2{}_0 + 6\alpha^2}{2}\right)Z_5$

	rV_{11}	rV_{12}	rV_{13}
Const.	$\left(\frac{k^2{}_0 - 18\alpha^2}{2}\right)Z_{11}$	$\left(\frac{k^2{}_0 - 32\alpha^2}{2}\right)Z_{12}$	$\left(\frac{k^2{}_0 - 50\alpha^2}{2}\right)Z_{13}$
Cos*α x*	$\left(\frac{k^2{}_0 - 15\alpha^2}{2}\right)Z_{10} + \left(\frac{k^2{}_0 - 21\alpha^2}{2}\right)Z_{12}$	$\left(\frac{k^2{}_0 - 28\alpha^2}{2}\right)Z_{11} + \left(\frac{k^2{}_0 - 36\alpha^2}{2}\right)Z_{13}$	$\left(\frac{k^2{}_0 - 45\alpha^2}{2}\right)Z_{12} + \left(\frac{k^2{}_0 - 55\alpha^2}{2}\right)Z_{14}$
Cos2*α x*	$\left(\frac{k^2{}_0 - 12\alpha^2}{2}\right)Z_{9} + \left(\frac{k^2{}_0 - 24\alpha^2}{2}\right)Z_{13}$	$\left(\frac{k^2{}_0 - 24\alpha^2}{2}\right)Z_{10} + \left(\frac{k^2{}_0 - 40\alpha^2}{2}\right)Z_{14}$	$\left(\frac{k^2{}_0 - 40\alpha^2}{2}\right)Z_{11} + \left(\frac{k^2{}_0 - 60\alpha^2}{2}\right)Z_{15}$
Cos3*α x*	$\left(\frac{k^2{}_0 - 27\alpha^2}{2}\right)Z_{14}$	$\left(\frac{k^2{}_0 - 20\alpha^2}{2}\right)Z_{9} + \left(\frac{k^2{}_0 - 44\alpha^2}{2}\right)Z_{15}$	$\left(\frac{k^2{}_0 - 35\alpha^2}{2}\right)Z_{10} + \left(\frac{k^2{}_0 - 65\alpha^2}{2}\right)Z_{16}$
Cos4*α x*	$\left(\frac{k^2{}_0 - 30\alpha^2}{2}\right)Z_{15} - \left(\frac{k^2{}_0 - 6\alpha^2}{2}\right)Z_{9}$	$\left(\frac{k^2{}_0 - 48\alpha^2}{2}\right)Z_{16}$	$\left(\frac{k^2{}_0 - 30\alpha^2}{2}\right)Z_{9}$
Cos5*α x*	$\left(\frac{k^2{}_0 - 33\alpha^2}{2}\right)Z_{16} - \left(\frac{k^2{}_0 - 3\alpha^2}{2}\right)Z_{10}$	$-\left(\frac{k^2{}_0 - 12\alpha^2}{2}\right)Z_{9}$	0
Cos6*α x*	$-\left(\frac{k^2{}_0}{2}\right)Z_{11}$	$-\left(\frac{k^2{}_0 - 8\alpha^2}{2}\right)Z_{10}$	$-\left(\frac{k^2{}_0 - 20\alpha^2}{2}\right)Z_{9}$
Cos7*α x*	$-\left(\frac{k^2{}_0 + 3\alpha^2}{2}\right)Z_{12}$	$-\left(\frac{k^2{}_0 - 4\alpha^2}{2}\right)Z_{11}$	$-\left(\frac{k^2{}_0 - 15\alpha^2}{2}\right)Z_{10}$
Sin*α x*	$\left(\frac{k^2{}_0 - 15\alpha^2}{2}\right)Z_{2} - \left(\frac{k^2{}_0 - 21\alpha^2}{2}\right)Z_{4}$	$\left(\frac{k^2{}_0 - 28\alpha^2}{2}\right)Z_{3} - \left(\frac{k^2{}_0 - 36\alpha^2}{2}\right)Z_{5}$	$\left(\frac{k^2{}_0 - 45\alpha^2}{2}\right)Z_{4} - \left(\frac{k^2{}_0 - 55\alpha^2}{2}\right)Z_{6}$
Sin2*α x*	$\left(\frac{k^2{}_0 - 12\alpha^2}{2}\right)Z_{1} - \left(\frac{k^2{}_0 - 24\alpha^2}{2}\right)Z_{5}$	$\left(\frac{k^2{}_0 - 24\alpha^2}{2}\right)Z_{2} - \left(\frac{k^2{}_0 - 40\alpha^2}{2}\right)Z_{6}$	$\left(\frac{k^2{}_0 - 40\alpha^2}{2}\right)Z_{3} - \left(\frac{k^2{}_0 - 60\alpha^2}{2}\right)Z_{7}$
Sin3*α x*	$-\left(\frac{k^2{}_0 - 27\alpha^2}{2}\right)Z_{6} + Z_C\left(k^2{}_0 - 9\alpha^2\right)$	$\left(\frac{k^2{}_0 - 20\alpha^2}{2}\right)Z_{1} - \left(\frac{k^2{}_0 - 44\alpha^2}{2}\right)Z_{7}$	$\left(\frac{k^2{}_0 - 35\alpha^2}{2}\right)Z_{2} - \left(\frac{k^2{}_0 - 65\alpha^2}{2}\right)Z_{8}$
Sin4*α x*	$-\left(\frac{k^2{}_0 - 30\alpha^2}{2}\right)Z_{7} + \left(\frac{k^2{}_0 - 6\alpha^2}{2}\right)Z_{1}$	$-\left(\frac{k^2{}_0 - 48\alpha^2}{2}\right)Z_{8} + Z_C\left(k^2{}_0 - 16\alpha^2\right)$	$\left(\frac{k^2{}_0 - 30\alpha^2}{2}\right)Z_{1}$
Sin5*α x*	$-\left(\frac{k^2{}_0 - 33\alpha^2}{2}\right)Z_{8} + \left(\frac{k^2{}_0 - 3\alpha^2}{2}\right)Z_{2}$	$\left(\frac{k^2{}_0 - 12\alpha^2}{2}\right)Z_{1}$	$Z_C\,(k^2{}_0 - 25\alpha^2)$
Sin6*α x*	$\left(\frac{k^2{}_0}{2}\right)Z_{3}$	$\left(\frac{k^2{}_0 - 8\alpha^2}{2}\right)Z_{2}$	$\left(\frac{k^2{}_0 - 20\alpha^2}{2}\right)Z_{1}$
Sin7*α x*	$\left(\frac{k^2{}_0 + 3\alpha^2}{2}\right)Z_{4}$	$\left(\frac{k^2{}_0 - 4\alpha^2}{2}\right)Z_{3}$	$\left(\frac{k^2{}_0 - 15\alpha^2}{2}\right)Z_{2}$

	rV_{14}	rV_{15}	rV_{16}
Const.	$\left(\frac{k^2{}_0-72\alpha^2}{2}\right)Z_{14}$	$\left(\frac{k^2{}_0-98\alpha^2}{2}\right)Z_{15}$	$\left(\frac{k^2{}_0-128\alpha^2}{2}\right)Z_{16}$
Cos*α x*	$\left(\frac{k^2{}_0-66\alpha^2}{2}\right)Z_{13}+\left(\frac{k^2{}_0-78\alpha^2}{2}\right)Z_{15}$	$\left(\frac{k^2{}_0-91\alpha^2}{2}\right)Z_{14}+\left(\frac{k^2{}_0-105\alpha^2}{2}\right)Z_{16}$	$\left(\frac{k^2{}_0-120\alpha^2}{2}\right)Z_{15}$
Cos2*α x*	$\left(\frac{k^2{}_0-60\alpha^2}{2}\right)Z_{12}+\left(\frac{k^2{}_0-84\alpha^2}{2}\right)Z_{16}$	$\left(\frac{k^2{}_0-84\alpha^2}{2}\right)Z_{13}$	$\left(\frac{k^2{}_0-112\alpha^2}{2}\right)Z_{14}$
Cos3*α x*	$\left(\frac{k^2{}_0-54\alpha^2}{2}\right)Z_{11}$	$\left(\frac{k^2{}_0-77\alpha^2}{2}\right)Z_{12}$	$\left(\frac{k^2{}_0-104\alpha^2}{2}\right)Z_{13}$
Cos4*α x*	$\left(\frac{k^2{}_0-48\alpha^2}{2}\right)Z_{10}$	$\left(\frac{k^2{}_0-70\alpha^2}{2}\right)Z_{11}$	$\left(\frac{k^2{}_0-96\alpha^2}{2}\right)Z_{12}$
Cos5*α x*	$\left(\frac{k^2{}_0-42\alpha^2}{2}\right)Z_{9}$	$\left(\frac{k^2{}_0-63\alpha^2}{2}\right)Z_{10}$	$\left(\frac{k^2{}_0-88\alpha^2}{2}\right)Z_{11}$
Cos6*α x*	0	$\left(\frac{k^2{}_0-56\alpha^2}{2}\right)Z_{9}$	$\left(\frac{k^2{}_0-80\alpha^2}{2}\right)Z_{10}$
Cos7*α x*	$-\left(\frac{k^2{}_0-30\alpha^2}{2}\right)Z_{9}$	0	$\left(\frac{k^2{}_0-72\alpha^2}{2}\right)Z_{9}$
Sin*α x*	$\left(\frac{k^2{}_0-66\alpha^2}{2}\right)Z_{5}-\left(\frac{k^2{}_0-78\alpha^2}{2}\right)Z_{7}$	$\left(\frac{k^2{}_0-91\alpha^2}{2}\right)Z_{6}-\left(\frac{k^2{}_0-105\alpha^2}{2}\right)Z_{8}$	$\left(\frac{k^2{}_0-120\alpha^2}{2}\right)Z_{7}$
Sin2*α x*	$\left(\frac{k^2{}_0-60\alpha^2}{2}\right)Z_{4}-\left(\frac{k^2{}_0-84\alpha^2}{2}\right)Z_{8}$	$\left(\frac{k^2{}_0-84\alpha^2}{2}\right)Z_{5}$	$\left(\frac{k^2{}_0-112\alpha^2}{2}\right)Z_{6}$
Sin3*α x*	$\left(\frac{k^2{}_0-54\alpha^2}{2}\right)Z_{3}$	$\left(\frac{k^2{}_0-77\alpha^2}{2}\right)Z_{4}$	$\left(\frac{k^2{}_0-104\alpha^2}{2}\right)Z_{5}$
Sin4*α x*	$\left(\frac{k^2{}_0-48\alpha^2}{2}\right)Z_{2}$	$\left(\frac{k^2{}_0-70\alpha^2}{2}\right)Z_{3}$	$\left(\frac{k^2{}_0-96\alpha^2}{2}\right)Z_{4}$
Sin5*α x*	$\left(\frac{k^2{}_0-42\alpha^2}{2}\right)Z_{1}$	$\left(\frac{k^2{}_0-63\alpha^2}{2}\right)Z_{2}$	$\left(\frac{k^2{}_0-88\alpha^2}{2}\right)Z_{3}$
Sin6*α x*	$Z_C\,(k^2{}_0-36\,\alpha^2)$	$\left(\frac{k^2{}_0-56\alpha^2}{2}\right)Z_{1}$	$\left(\frac{k^2{}_0-80\alpha^2}{2}\right)Z_{2}$
Sin7*α x*	$\left(\frac{k^2{}_0-30\alpha^2}{2}\right)Z_{1}$	$Z_C\,(k^2{}_0-49\,\alpha^2)$	$\left(\frac{k^2{}_0-72\alpha^2}{2}\right)Z_{1}$

		Constant		r * Constant
Const.		$-k_0^2 Z_C$		$-\left(\frac{k^2{}_0 - 2\alpha^2}{2}\right)Z_9$
Cosαx		$-k_0^2 Z_1$		$-\left(\frac{k^2{}_0 - 3\alpha^2}{2}\right)Z_{10}$
Cos2αx		$-k_0^2 Z_2$		$-\left(\frac{k^2{}_0 - 4\alpha^2}{2}\right)Z_{11} + \left(\frac{k^2{}_0}{2}\right)Z_9$
Cos3αx		$-k_0^2 Z_3$		$-\left(\frac{k^2{}_0 - 5\alpha^2}{2}\right)Z_{12} + \left(\frac{k^2{}_0 + \alpha^2}{2}\right)Z_{10}$
Cos4αx		$-k_0^2 Z_4$		$-\left(\frac{k^2{}_0 - 6\alpha^2}{2}\right)Z_{13} + \left(\frac{k^2{}_0 + 2\alpha^2}{2}\right)Z_{11}$
Cos5αx		$-k_0^2 Z_5$		$-\left(\frac{k^2{}_0 - 7\alpha^2}{2}\right)Z_{14} + \left(\frac{k^2{}_0 + 3\alpha^2}{2}\right)Z_{12}$
Cos6αx		$-k_0^2 Z_6$		$-\left(\frac{k^2{}_0 - 8\alpha^2}{2}\right)Z_{15} + \left(\frac{k^2{}_0 + 4\alpha^2}{2}\right)Z_{13}$
Cos7αx	=	$-k_0^2 Z_7$	+ **r**	$-\left(\frac{k^2{}_0 - 9\alpha^2}{2}\right)Z_{16} + \left(\frac{k^2{}_0 + 5\alpha^2}{2}\right)Z_{14}$
Sinαx		$-k_0^2 Z_9$		$\left(\frac{k^2{}_0 - 3\alpha^2}{2}\right)Z_2 - Z_C\left(k^2{}_0 - \alpha^2\right)$
Sin2αx		$-k_0^2 Z_{10}$		$\left(\frac{k^2{}_0 - 4\alpha^2}{2}\right)Z_3 - \left(\frac{k^2{}_0}{2}\right)Z_1$
Sin3αx		$-k_0^2 Z_{11}$		$\left(\frac{k^2{}_0 - 5\alpha^2}{2}\right)Z_4 - \left(\frac{k^2{}_0 + \alpha^2}{2}\right)Z_2$
Sin4αx		$-k_0^2 Z_{12}$		$\left(\frac{k^2{}_0 - 6\alpha^2}{2}\right)Z_5 - \left(\frac{k^2{}_0 + 2\alpha^2}{2}\right)Z_3$
Sin5αx		$-k_0^2 Z_{13}$		$\left(\frac{k^2{}_0 - 7\alpha^2}{2}\right)Z_6 - \left(\frac{k^2{}_0 + 3\alpha^2}{2}\right)Z_4$
Sin6αx		$-k_0^2 Z_{14}$		$\left(\frac{k^2{}_0 - 8\alpha^2}{2}\right)Z_7 - \left(\frac{k^2{}_0 + 4\alpha^2}{2}\right)Z_5$
Sin7αx		$-k_0^2 Z_{15}$		$\left(\frac{k^2{}_0 - 9\alpha^2}{2}\right)Z_8 - \left(\frac{k^2{}_0 + 5\alpha^2}{2}\right)Z_6$

7.5 Appendix E: *Tables of Element Values for Low Pass Filter Prototypes*

N	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9	g_{10}	g_{11}
1	2.0000	1.0000									
2	1.4142	1.4142	1.0000								
3	1.0000	2.0000	1.0000	1.0000							
4	0.7654	1.8478	1.8478	0.7654	1.0000						
5	0.6180	1.6180	2.0000	1.6180	0.6180	1.0000					
6	0.5176	1.4142	1.9318	1.9318	1.4142	0.5176	1.0000				
7	0.4450	1.2470	1.8019	2.0000	1.8019	1.2470	0.4450	1.0000			
8	0.3902	1.1111	1.6629	1.9615	1.9615	1.6629	1.1111	0.3902	1.0000		
9	0.3473	1.0000	1.5321	1.8794	2.0000	1.8794	1.5321	1.0000	0.3473	1.0000	
10	0.3129	0.9080	1.4142	1.7820	1.9754	1.9754	1.7820	1.4142	0.9080	0.3129	1.0000

Table 4: Element Values for Butterworth (Maximally Flat) Low-Pass Filter Prototypes (g_o=1,w_c=1)

N	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9	g_{10}	g_{11}
1	2.0000	1.0000									
2	1.5774	0.4226	1.0000								
3	1.2550	0.5528	0.1922	1.0000							
4	1.0598	0.5116	0.3181	0.1104	1.0000						
5	0.9303	0.4577	0.3312	0.2090	0.0718	1.0000					
6	0.8377	0.4116	0.3158	0.2364	0.1480	0.0505	1.0000				
7	0.7677	0.3744	0.2944	0.2378	0.1778	0.1104	0.0375	1.0000			
8	0.7125	0.3446	0.2735	0.2297	0.1867	0.1387	0.0855	0.0289	1.0000		
9	0.6678	0.3203	0.2547	0.2184	0.1859	0.1506	0.1111	0.0682	0.0230	1.0000	
10	0.6305	0.3002	0.2384	0.2066	0.1808	0.1539	0.1240	0.0911	0.0557	0.0187	1.0000

Table 5: Element Values for Bessel (Maximally Flat Time Delay) Low-Pass Filter Prototypes (g_o=1,w_c=1)

0.5dB Ripple Band

N	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9	g_{10}	g_{11}
1	0.6986	1.0000									
2	1.4029	0.7071	1.9841								
3	1.5963	1.0967	1.5963	1.0000							
4	1.6703	1.1926	2.3661	0.8419	1.9841						
5	1.7058	1.2296	2.5408	1.2296	1.7058	1.0000					
6	1.7254	1.2479	2.6064	1.3137	2.4758	0.8696	1.9841				
7	1.7372	1.2583	2.6381	1.3444	2.6381	1.2583	1.7372	1.0000			
8	1.7451	1.2647	2.6564	1.3590	2.6964	1.3389	2.5093	0.8796	1.9841		
9	1.7504	1.2690	2.6678	1.3673	2.7239	1.3673	2.6678	1.2690	1.7504	1.0000	
10	1.7543	1.2721	2.6754	1.3725	2.7392	1.3806	2.7231	1.3485	2.5239	0.8842	1.9841

Table 6: Element Values for Chebyshev (*0.5dB* Equal-Ripple) Low-Pass Filter Prototypes (g_o=1,w_c=1)

3.0dB Ripple Band

N	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9	g_{10}	g_{11}
1	1.9953	1.0000									
2	3.1013	0.5339	5.8095								
3	3.3487	0.7117	3.3487	1.0000							
4	3.4289	0.7483	4.3471	0.5920	5.8095						
5	3.4817	0.7618	4.5381	0.7618	3.4817	1.0000					
6	3.5045	0.7685	4.6061	0.7929	4.4641	0.6033	5.8095				
7	3.5182	0.7723	4.6386	0.8039	4.6386	0.7723	3.5182	1.0000			
8	3.5277	0.7745	4.6575	0.8089	4.6990	0.8018	4.4990	0.6073	5.8095		
9	3.5340	0.7760	4.6692	0.8118	4.7272	0.8118	4.6692	0.7760	3.5340	1.0000	
10	3.5384	0.7771	4.6768	0.8136	4.7425	0.8164	4.7260	0.8051	4.5142	0.6091	5.8095

Table 7: Element Values for Chebyshev (*3.0dB* Equal-Ripple) Low-Pass Filter Prototypes (g_o=1,w_c=1)

7.6 Appendix F: Graphs of Attenuation Versus Normalized Frequency

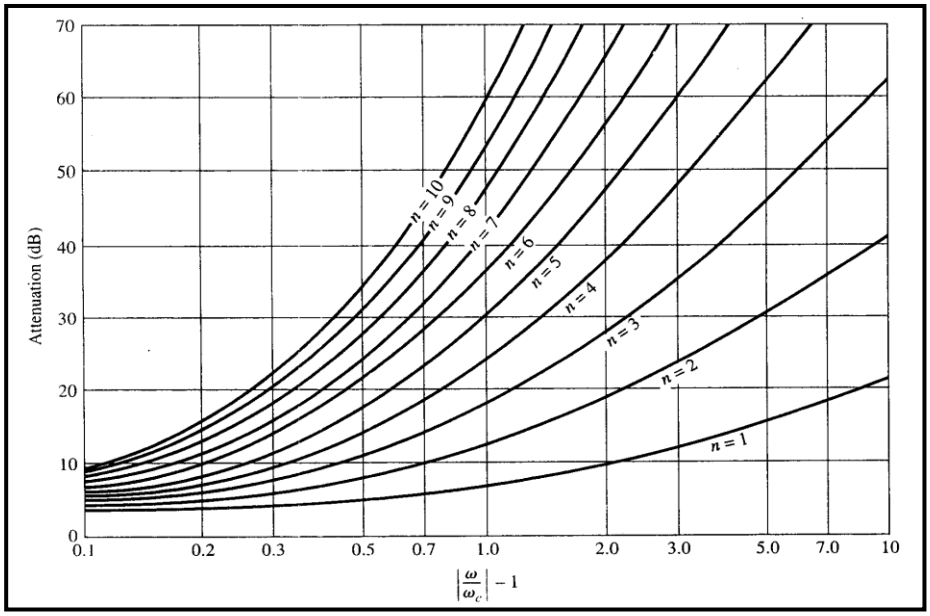


Figure 44: Attenuation versus Normalized Frequency for Maximally Flat Filter Prototypes [2]

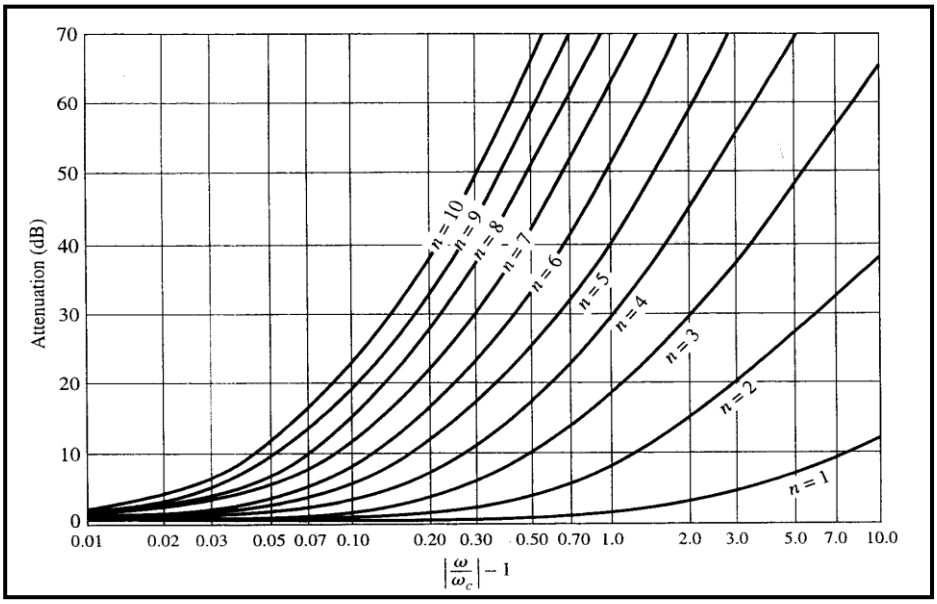


Figure 45: Attenuation versus Normalized Frequency for Equal-Ripple Filter Prototypes (0.5 dB Ripple Level) [2]

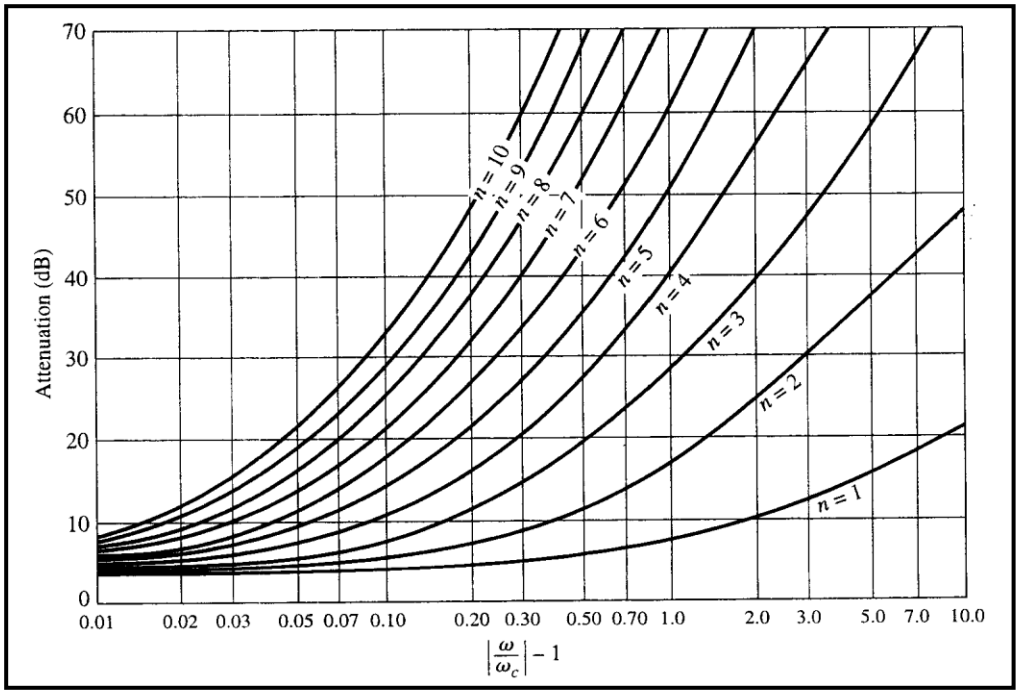


Figure 46: Attenuation versus Normalized Frequency for Equal-Ripple Filter Prototypes (3.0 dB Ripple Level) [2]

Bibliography

[1] G. L. Matthaei, L. Young, and E. M. T. Jones, *"Microwave Filters, Impedance-Matching Networks, and Coupling Structures*", 6th edition, Artech House 1980.

[2] D. Pozar, "*Microwave Engineering"*, 4th edition, 2005.

[3] K. C. Gupta, R. Garg, I. Bahl, and P. Bhartia, "*Microstrip Lines and Slotlines*", 2nd edition. Norwood, MA: Artech House, 1996.

[4] M. L. Roy, A. Perennec, S. Toutain and L. C. Calvez, "*The Continuously Varying Transmission-Line Technique—Application to Filter Design"*, IEEE Transactions on Microwave Theory and Techniques, Vol. 47, No. 9, SEPTEMBER 1999.

[5] F. F. Kuo, "*Network Analysis and Synthesis*", 2nd edition, 1966 John-Wiley & Sons.

[6] S. Chakravarty and R. Mittra, "*Design of Microwave Filters Using A Binary Coded Genetic Algorithm*", IEEE 2000.

[7] P. P. Roberts and G. E. Town, "*Design of Microwave Filters by Inverse Scattering*", IEEE Transactions on Microwave Theory and Techniques, Vol. 43, No. 4, APRIL 1995.

[8] B. P. Buckles and F. E. Petry, "*Genetic Algorithms*", IEEE Computer Society Press, 1992.

[9] M. Mitchell, "*An Introduction to Genetic Algorithms"*, 1996.

[10] M. Negnevitsky, "*Artificial Intelligence: A Guide to Intelligent Systems"*, 2nd Edition, 2005.

[11] A. Venables and G. Tan, "*A 'Hands on' Strategy for Teaching Genetic Algorithms*", Journal of Information Technology Education, Volume 6, 2007.

[12] G. Winter, J. Periaux, M. Galan and P. Cuesta, *"Genetic Algorithms in Engineering and Computer Science"*, 1995.

[13] R. S. Zebulum, M. A. Pacheco and M. M. Vellasco, *"Automatic Design of Electronic Circuits and Systems by Genetic Algorithms"*, 2002.

[14] R. K. Belew and M. D. Vose, *"Foundations of Genetic Algorithms*", 1997.

[15] D. Swanson and G. Macchiarella, "*Microwave Filter Design by Synthesis and Optimization*", IEEE Microwave Magazine, APRIL 2007.

[16] M. L. Roy, A. Perennec, S. Toutain, L. C. Calvez, "*A New Design of Microwave Filters by Using Continuously Varying Transmission lines*", IEEE 1997.

[17] R. Tascone, P. Savi, D. Trinchero and R. Orta, *" Scattering Matrix Approach for the Design of Microwave Filters*", IEEE Transactions on Microwave Theory and Techniques, Vol. 48, No. 3, MARCH 2000.

[18] C. W. Hsue, "*Time-Domain Scattering Parameters of an Exponential Transmission Line*", IEEE 1991.

[19] K. Lu, "*An Efficient Method for Analysis of Arbitrary Nonuniform Transmission Lines*", IEEE Transactions on Microwave Theory and Techniques, Vol. 45, No. 1, JANUARY 1997.

[20] C. W. Hsue and C. D. Hechtman, "*Transient Analysis of Nonuniform High-Pass Transmission Lines*", IEEE Transactions on Microwave Theory and Techniques, vol. 38, pp. 1023–1030, Aug. 1990.

[21] M. J. Ahmed, "*Impedance Transformation for Exponential, Cosine Squared, and Parabolic Tapered Transmission Lines"*, IEEE Transactions on Microwave Theory and Techniques, vol. MTT-29, pp. 67–68, January 1981.

[22] P. Bouchard and R. J. Gagne, "*Transient Analysis of Lossy Parabolic Transmission Lines with Nonlinear Loads*", IEEE Transactions on Microwave Theory and Techniques, vol. 43, pp. 1330–1333, June 1995.

[23] J. A. G. Malherbe, "*Microwave Transmission Line Filters*", Artech House 1979.

[24] G. V. Reklaitis, A. Ravindran and K. M. Ragsdell, "*Engineering Optimization: Methods and Applications*", 1983.

[25] J. N. Carvalho, J. C. Silve and A. G. Neto, *"Analysis of Continuously Varying Microstrip Line by the Step Segmentation Technique*", IEEE Transactions on Microwave Theory and Techniques, Vol. 1, No. 720, 2001.