ABSOLUTELY PURE MODULES AND RELATED CONCEPTS

BY

Hussain Yahya Al-Hammali

A Thesis Presented to the
DEANSHIP OF GRADUATE STUDIES

KING FAHD UNIVERSITY OF PETROLEUM & MINERALS
DHAHRAN, SAUDI ARABIA

In Partial Fulfillment of the
Requirements for the Degree of

MASTER OF SCIENCE

In

Mathematical Sciences

May, 2005

KING FAHD UNIVERSITY OF PETROLEUM & MINERALS
DHAHRAN 31261, SAUDI ARABIA

DEANSHIP OF GRADUATE STUDIES

This thesis, written by Hussain Al Hammali under the direction of his thesis advisor and approved by his thesis committee, has been presented to and accepted by the Dean of Graduate Studies, in partial fulfillment of the requirements for the degree of MASTER OF SCIENCE IN MATHEMATICAL SCIENCES.

Thesis Committee

Dr. Abdallah Laradji (Chairman)

Dr. Hassan Azad (Member)

Dr. Salah Eddine Kabbaj (Member)

Dr. Khaled M. Furati
Department Chairman

Dr. Mohammad Al-Ohali
Dean of Graduate studies

May, 2005 ١٤٢٦/٤/٢٩هـ

ACKNOWLEDGMENTS

Thanks to my God Allah Who takes the care of all works I do through His mercy and benignity. May his blessings be upon his messenger Mohammad SAW.

I wish to express my humble thanks and respect to my advisor Dr. Abdallah Laradji who was careful with all my reading and research in this field of Mathematics. Without his patience and knowledge I would not be able to think effectively.

My thanks also go to all other members of my thesis committee Dr. Hassan Azad and Dr. Salah Eddine Kabbaj. Also, I wish to thank Mr. Ibrahim Al-Salem who provided me with all the needs of necessary softwares as well as computer instructions.

My thanks go to my parents for their prayers and screen that were my main resource of success.

Hussain Y. Al-Hammali

May, 2005

CONTENTS

THESIS ABSTRACT

Full Name of Student:	Hussain Yahya Ahmed Al Hammali
Title of Study:	Absolutely Pure Modules and Related Concepts
Major field:	Mathematics
Date of Degree:	May, 2005

The notion of pure subgroups was first introduced by Prüfer in 1923. A subgroup G of a group A is said to be pure in A if the equation $nx = g$, where $n \in Z$ and $g \in G$, is solvable in G whenever it is solvable in A. The notion of purity was extended to submodules by several authors (e.g. Cohn, Fuchs, and Walker), and has since become an extensively studied topic in module theory. In 1967, B. H. Maddox introduced absolutely pure modules. This notion was studied and developed by other authors, for example C. Megibben and B. Stenström. In this thesis we will present various properties of absolute purity, as well as some ring characterizations that use this notion. A generalization of absolute purity, due to Lee, will also be discussed.

MASTER OF SCIENCE DEGREE
KING FAHAD UNIVERSITY OF PETROLEUM & MINERALS
Dhahran, Saudi Arabia
May, 2005

0 INTRODUCTION

The notion of pure subgroups was first introduced by Prüfer in 1923. A subgroup G of a group A is said to be pure in A if the equation $nx = g$, where $n \in \mathbb{Z}$ and $g \in G$, is solvable in G whenever it is solvable in A. In 1959, the notion of purity was extended by P. M. Cohn [5] to modules, and has since become an extensively studied topic in module theory (see for example J. Dauns [7], D. J. Fieldhouse [9], L. Fuchs [10], B. Stenström [22], R. B. Warfield [24] and R. Wisbauer [25]). Pure submodules are defined as follows.

An exact sequence of left R-modules $0 \longrightarrow A' \xrightarrow{\lambda} A \longrightarrow A'' \longrightarrow 0$ is said to be *pure-exact* if, for every right R-module B, we have exactness of $0 \longrightarrow B \otimes A' \xrightarrow{1\otimes\lambda} B \otimes A \longrightarrow B \otimes A'' \longrightarrow 0$. In this case, we say $\lambda(A')$ is *a pure submodule* of A. The following equational characterization will often be useful.

Theorem 1. *A submodule N of an R-module M is pure in M if and only if every finite system of equations over N which is solvable in M is also solvable in N, that is, if and only if, every system of equations*

$$\sum_{k=1}^{n} r_{jk}x_k = n_j \in N \;\; (1 \leq j \leq m)$$

that has a solution in M, has necessarily a solution in N, i.e. there exist $a_1, a_{2,...}, a_n$ in N such that

$$\sum_{k=1}^{n} r_{jk}a_k = n_j \; (1 \leq j \leq m).$$

In 1967, B. H. Maddox [16] introduced absolutely pure modules: A module is *absolutely pure* (a.p.) if it is pure in every module containing it as a submodule. This notion was studied and developed by many authors (see for example, B. H. Maddox [16], C. Megibben

[18], and B. Stenström [21]). Since an injective module is a direct summand of every module containing it, it is easy to show that injective modules are necessarily absolutely pure. This means that absolute purity is a weak form of injectivity, and the question as to whether there are absolutely pure modules that are not injective naturally arises. As we shall see in this thesis, these two notions coincide precisely when the ring is noetherian.

Recall that *a ring R is semisimple if and only if every left R-module is injective*. Another natural question therefore is the following: What are the rings over which all modules are absolutely pure? A third interesting question is to characterize the rings such that quotients of absolutely pure modules are absolutely pure. These and other related questions will be discussed in the thesis.

We will show how several results on injective modules can be extended, with appropriate modifications, to absolutely pure modules. For example, an analogue of Baer's Criterion established by C. Megibben [18] is:

Theorem 2. *An R-module A is absolutely pure if and only if every R-homomorphism $\psi : K \longrightarrow A$ in the diagram*

$$\begin{array}{ccccc} 0 & \longrightarrow & K & \xrightarrow{\sigma} & F \\ & & \psi \downarrow & & \\ & & A & & \end{array}$$

where F is free and both F and *K are f.g. and where σ is the inclusion map, can be extended by an R-homomorphism $\beta : F \longrightarrow A$, i.e. $\psi = \beta\sigma$.*

A useful, closely related concept to absolute purity is flatness. A right R-module B is *flat* if whenever $f : X \longrightarrow Y$ is a monomorphism then so too is $1_B \otimes f : B \otimes X \longrightarrow B \otimes Y$.

As we shall see later, flat modules are dual in a certain sense to absolutely pure modules. A characterization of flatness that we will repeatedly use in the thesis is the following classical result:

Theorem 3 (J. Lambek [14]). *An R-module A is flat if and only if A^* is an injective R-module, where $A^* = Hom_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$.*

The work is organized as follows. In Chapter 1, we give basic terminology and preliminaries that will be used in the sequel. In Chapter 2, we introduce purity of modules (in the sense of P. M. Cohn [5]) and absolute purity, and prove various results about these two notions. The role of absolute purity in characterizing a number of important classes of rings is explained in Chapter 3. Finally, in Chapter 4, we discuss generalizations of absolute purity, flatness and coherence due to S. B. Lee [15].

Chapter 1
BASIC TERMINOLOGY AND PRELIMINARIES

In this section we summarize some definitions and basic results in module theory and homological algebra that will be used in the thesis. Proofs of these results can be found in any standard algebra book, see for example [12] and [20]. For more basic results that are not included here, we refer to [4, 12, 19, 20]. Throughout the thesis, all rings (denoted R or S) are with 1, all modules are unitary and, unless otherwise stated, left modules. Let I be a non-empty set and M be an R-module, then $\mid I \mid$ denotes the cardinality of I and M^I (respectively $M^{(I)}$) denotes the direct product (respectively sum) of $\mid I \mid$ copies of M.

Definition. A diagram $A \xrightarrow{f} B \xrightarrow{g} C$ where A, B and C are R-modules, f and g are R-homomorphisms is called an exact sequence if im f =ker g . In general, a sequence ... $\longrightarrow A_{-2} \xrightarrow{f_{-2}} A_{-1} \xrightarrow{f_{-1}} A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \longrightarrow$ ... of R-modules and R-homomorphisms is exact if im $f_n =$ ker f_{n+1} for all $n \in \mathbb{Z}$.

The following useful theorem can be proved using standard diagram chasing.

Theorem 1.1 (Five Lemma). *Consider the commutative diagram with exact rows*

$$\begin{array}{ccccccccc} A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & A_4 & \xrightarrow{f_4} & A_5 \\ \downarrow t_1 & & \downarrow t_2 & & \downarrow t_3 & & \downarrow t_4 & & \downarrow t_5 \\ B_1 & \xrightarrow{h_1} & B_2 & \xrightarrow{h_2} & B_3 & \xrightarrow{h_3} & B_4 & \xrightarrow{h_4} & B_5 \end{array}$$

(i) If t_2 and t_4 are epic and t_5 is monic, then t_3 is epic.

(ii) If t_2 and t_4 are monic and t_1 epic, then t_3 is monic.

In particular if t_1, t_2, t_4, t_5 are isomorphisms, then t_3 is an isomorphism.

Theorem 1.2. *Let* $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ $(*)$ *be an exact sequence of* R*-modules and* R*-homomorphisms. Then, the following conditions are equivalent:*

(i) There is a homomorphism $k : B \longrightarrow A$ *such that* $kf = 1_A$.

(ii) There is a homomorphism $h : C \longrightarrow B$ *such that* $gh = 1_C$.

(iii) There is an isomorphism ϕ *such that*

$$\begin{array}{ccccccc} 0 \longrightarrow A & \longrightarrow & A \oplus C & \longrightarrow & C \longrightarrow 0 \\ 1_A \downarrow & & \downarrow \phi & & \downarrow 1_C \\ 0 \longrightarrow A & \xrightarrow{f} & B & \xrightarrow{g} & C \longrightarrow 0 \end{array}$$

is commutative.

The exact sequence $(*)$ is called *split exact* if any one of the above equivalent statements is satisfied.

A module Y is called *a direct summand* of X if there is a diagram $Y \xrightarrow{f} X \xrightarrow{g} Y$ such that $gf = 1_Y$. Hence, in (i) of Theorem 1.2, A is a direct summand of B.

Definition. A category $\mathfrak{C}$ consists of a class of objects, obj $\mathfrak{C}$, pairwise disjoint sets of morphisms, $Hom_C(A, B)$, for every ordered pair of objects, and compositions $Hom_C(A, B) \times Hom_C(B, C) \longrightarrow Hom_C(A, C)$, denoted $(f, g) \mapsto gf$, satisfying the following axioms:

(i) for each object A, there exists an identity morphism $1_A \in Hom_C(A, A)$ such that $f1_A = f$ for all $f \in Hom_C(A, B)$ and $1_A g = g$ for all $g \in Hom_C(C, A)$;

(ii) associativity of composition holds whenever possible: if $f \in Hom_C(A, B)$, $g \in Hom_C(B, C)$ and $h \in Hom_C(C, D)$, then $h(gf) = (hg)f$.

Definition. Let $\mathfrak{C}$ and $\mathfrak{D}$ be categories. A *covariant functor* $F : \mathfrak{C} \longrightarrow \mathfrak{D}$ is a function satisfying:

(i) If $A \in obj$ $\mathfrak{C}$, then $FA \in obj$ $\mathfrak{D}$;

(ii) if $f : A \longrightarrow B$ is a morphism in $\mathfrak{C}$, then $Ff : FA \longrightarrow FB$ is a morphism in $\mathfrak{D}$;

(iii) if $A \xrightarrow{f} B \xrightarrow{g} C$ are morphisms in $\mathfrak{C}$, then $F(gf) = FgFf$;

(iv) for every $A \in obj\ \mathfrak{C}$, we have $F(1_A) = 1_{FA}$.

Definition. A covariant functor F is *left exact* if exactness of

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C$$

implies exactness of

$$0 \longrightarrow FA \xrightarrow{F\alpha} FB \xrightarrow{F\beta} FC;$$

a covariant functor F is *right exact* if exactness of

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

implies exactness of

$$FA \xrightarrow{F\alpha} FB \xrightarrow{F\beta} FC \longrightarrow 0.$$

Definition. Let $\mathfrak{C}$ and $\mathfrak{D}$ be categories. A *contravariant functor* $F : \mathfrak{C} \longrightarrow \mathfrak{D}$ is a function satisfying:

(i) If $A \in obj\ \mathfrak{C}$, then $FA \in obj\ \mathfrak{D}$;

(ii) if $f : A \longrightarrow B$ is a morphism in $\mathfrak{C}$, then $Ff : FB \longrightarrow FA$ is a morphism in $\mathfrak{D}$;

(iii) if $A \xrightarrow{f} B \xrightarrow{g} C$ are morphisms in $\mathfrak{C}$, then

$$F(gf) = FfFg;$$

(iv) for every $A \in obj\ \mathfrak{C}$, we have $F(1_A) = 1_{FA}$.

Definition. A contravariant functor F is *left exact* if exactness of

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

implies exactness of

$$0 \longrightarrow FC \xrightarrow{F\beta} FB \xrightarrow{F\alpha} FA;$$

a contravariant functor F is *right exact* if exactness of

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C$$

implies exactness of

$$FC \xrightarrow{F\beta} FB \xrightarrow{F\alpha} FA \longrightarrow 0.$$

A functor is *exact* if it is both left and right exact.

Definition. A morphism $f : A \longrightarrow B$ in a category $\mathfrak{C}$ is called an *equivalence* if there is a morphism $g : B \longrightarrow A$ in $\mathfrak{C}$ such that $gf = 1_A$ and $fg = 1_B$.

Definition. Let E and F be covariant functors, $E : \mathfrak{U} \longrightarrow \mathfrak{B}$ and $F : \mathfrak{U} \longrightarrow \mathfrak{B}$. A *natural transformation* $t : E \longrightarrow F$ is a class of morphisms $t_A : EA \longrightarrow FA$, one for each $A \in obj\ \mathfrak{U}$, giving commutativity of

$$\begin{array}{ccc} EA & \xrightarrow{Ef} & EA' \\ \downarrow t_A & & \downarrow t_{A'} \\ FA & \xrightarrow{Ff} & FA' \end{array}$$

for every $f : A \longrightarrow A'$ in $\mathfrak{U}$.There is a similar definition if both E and F are contravariant.

Definition. If F, G: $\mathfrak{U} \longrightarrow \mathfrak{B}$ are functors of the same variance, then F and G are *naturally equivalent*, denoted $F \cong G$, if there is a natural transformation $t : F \longrightarrow G$ with each $t_A : FA \longrightarrow GA$ an equivalence.

Projective and injective modules form important classes in module theory and homological algebra. We will present the definition of each and some of their basic features.

Definition. An R-module P is said to be *projective* if for every exact sequence $A \xrightarrow{f} B \longrightarrow 0$ of R-modules and for every R-homomorphism $g : P \longrightarrow B$, there exists an R-homomorphism $h : P \longrightarrow A$ such that $fh = g$.

Theorem 1.3. *Let F be a left R-module and let X be a set. Then, the following statements are equivalent:*

(i) F has a basis indexed by X.

(ii) F is a direct sum of $\mid X \mid$ cyclic R-modules each of which is isomorphic to R (as a left R-module).

(iii) $F \cong R^{(X)}$.

A module F over a ring R, that satifies any of the above equivalent statements, is called a *free* R-module on the set X.

Theorem 1.4. *Every free R-module is projective.*

Theorem 1.5. *For any R-module P, the following conditions are equivalent:*

(i) P is projective;

(ii) every exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow P \longrightarrow 0$ of R-modules is split exact;

(iii) there is a free R-module F and an R-module K such that $F \cong K \oplus P$.

Theorem 1.6. *Let $\{P_i \mid i \in I\}$ be a family of R-modules. Then, $\oplus_{i \in I} P_i$ is projective if and only if each P_i is projective.*

Definition. A module M is *finitely generated* (f.g.) if there is a finite subset $\{x_1, x_2, ..., x_n\}$ of M with $M = Rx_1 + Rx_2 + ... + Rx_n$.

Definition. An R-module P is *finitely presented* (f.p.) if there is an exact sequence $0 \longrightarrow K \longrightarrow F \longrightarrow P \longrightarrow 0$ of R-modules where F is free and both F and K are finitely generated (f.g.).

Definition. An R-module A is said to be *injective* if for every exact sequence $0 \longrightarrow X \xrightarrow{f} Y$ of R-modules and for every R-homomorphism $g : X \longrightarrow A$, there exists an R-homomorphism $h : Y \longrightarrow A$ such that $hf = g$, i.e. $g : X \longrightarrow A$ can be extended to $h : Y \longrightarrow A$.

Theorem 1.7. *Let $\{A_i \mid i \in I\}$ be a family of R-modules. Then, $\prod_{i \in I} A_i$ is injective if and only if each A_i is injective.*

The following result is a well-known characterization of injective modules.

Theorem 1.8 *(Baer's Criterion). An R-module M is injective if and only if for any left ideal I of R and any R-homomorphism $f : I \longrightarrow M$ there exists an R-homomorphism $g : R \longrightarrow M$ that extends f.*

Theorem 1.9. *For any R-module M, The following conditions are equivalent:*

(i) M is injective;

(ii) every exact sequence $0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0$ of R-modules is split exact;

(iii) M is a direct summand of every module that contains it.

Definition. A *free resolution* of a module M is an exact sequence

$$... \longrightarrow F_n \xrightarrow{d_n} F_{n-1} \longrightarrow ... \longrightarrow F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \longrightarrow 0$$

in which each F_n is a free module.

The following results will be used repeatedly later.

Theorem 1.10. *For any R-module M, there exists a free R-module F and an epimorphism $f : F \longrightarrow M$. If M is f.g., F can be chosen to be f.g.*

One direct consequence is that a finitely generated projective module is finitely presented. To see this, let $0 \longrightarrow K \longrightarrow F \longrightarrow P \longrightarrow 0$ be an exact sequence where P is a f.g. projective module and F is f.g. free (see Theorem 1.10). The sequence is split since P is projective by Theorem 1.5, so $F \cong K \oplus P$ which leads to have K f.g. (being an epic image of the f.g. module F) and hence P is f.p.

Theorem 1.11. *Every R-module can be embedded in an injective R-module. (That is, for every R-module M, there exists a monomorphism $g : M \longrightarrow E$ where E is an injective R-module)*

Theorem 1.12. *Let M and N be R-modules. Then,*

(i) $Hom(M, \_)$ is a left exact covariant functor and $Hom(\_, M)$ is a left exact contravariant

functor.

(ii) The functors $M \otimes_R \_$ and $\_ \otimes_R N$ are right exact covariant functors.

We note that a module M is projective if and only if $Hom(M, \_)$ is exact; and M is injective if and only if $Hom(\_, M)$ is exact.

Theorem 1.13. *Let A be a right R-module and B be a left R-module. Then, there exist isomorphisms of abelian groups $A \otimes_R R \cong A$ and $R \otimes_R B \cong B$.*

The following two identities will be needed later.

Theorem 1.14. *If* $\{A_i \mid i \in I\}$ *is a family of* R*-modules, then*

(i) $Hom\ (\oplus_i A_i, B) \cong \prod_i Hom(A_i, B)$ *for all* R*-modules* B.

(ii) $Hom\ (B, \prod_i A_i) \cong \prod_i Hom(B, A_i)$ *for all* R*-modules* B.

Theorem 1.15. *Let* A *be a left* R*-module and let* $\{B_i \mid i \in I\}$ *be a family of right* R*-modules. Then, there is an isomorphism*

$$(\oplus_i B_i) \otimes_R A \cong \oplus_i (B_i \otimes_R A).$$

Theorem 1.16 (Adjoint Isomorphism). *For rings* R *and* S, *consider the situation (*${}_RA$, ${}_SB_R$, ${}_SC$*), i.e. where* A *is a left* R*-module,* B *an* (S, R)*-bimodule and* C *a left* S*-module. Then,*

$$Hom_S(B \otimes_R A, C) \cong Hom_R(A, Hom_S(B, C)).$$

And in the situation $(A_R, {}_RB_S, C_S)$ *we have*

$$Hom_S(A \otimes_R B, C) \cong Hom_R(A, Hom_S(B, C)).$$

Definition. A set I is *quasi-ordered* if it has a relation $\leq$ that is reflexive and transitive. A quasi-ordered set I is *directed* if, for each $i, j \in I$, there exists $k \in I$ with $i \leq k$ and $j \leq k$.

Now, let us define direct systems and direct limits.

Definition. Let I be a quasi-ordered set and $\mathfrak{C}$ be a category. A *direct system* in $\mathfrak{C}$ with index set I is a function $F : I \longrightarrow \mathfrak{C}$ such that for each $i \in I$, there is an object F_i and, whenever $i, j \in I$ satisfy $i \leq j$, there is a morphism $\phi_i^j : F_i \longrightarrow F_j$ such that:

(i) $\phi_i^i : F_i \longrightarrow F_i$ is the identity for every $i \in I$

(ii) if $i \leq j \leq k$, then $\phi_j^k \phi_i^j = \phi_i^k$.

Definition. Let $F = \{F_i, \phi_i^j\}$ be a direct system in a category $\mathfrak{C}$. The *direct limit* of this system, denoted as $\varinjlim F_i$, is an object and a family of morphisms $\alpha_i : F_i \longrightarrow \varinjlim F_i$ with $\alpha_i = \alpha_j \phi_i^j$ whenever $i \leq j$ satisfying the following universal mapping problem:

$$\begin{tikzcd}
\varinjlim F_i \arrow[rr, "\beta"] & & X \\
& F_i \arrow[ul, "\alpha_i"'] \arrow[ur, "f_i"] \arrow[d, "\phi^j{}_i"] & \\
& F_j \arrow[uul, "\alpha_j"] \arrow[uur, "f_j"'] &
\end{tikzcd}$$

for every object X and every family of morphisms $f_i : F_i \longrightarrow X$ with $f_i = f_j \phi_i^j$ whenever $i \leq j$, there is a unique morphism $\beta : \varinjlim F_i \longrightarrow X$ making the above diagram commute.

It can be shown that the class of direct systems with index set I is a category $\mathrm{Dir}(I)$, where a morphism $t : \{F_i, \phi_i^j\} \longrightarrow \{G_i, \psi_i^j\}$ is a family of maps $t_i : F_i \longrightarrow G_i$ making all the following diagrams commute (when $i \leqslant j$):

$$\begin{array}{ccc}
F_i & \xrightarrow{t_i} & G_i \\
\phi_i^j \downarrow & & \psi_i^j \downarrow \\
F_j & \xrightarrow{t_j} & G_j.
\end{array}$$

For future reference, the following results are included.

Theorem 1.17. (N. Bourbaki [1]). *Every R-module is a direct limit of finitely presented R-modules.*

Theorem 1.18. *Let I be a directed quasi-ordered set. Suppose there are morphisms of direct systems over I*

$$\{A_i, \phi_i^j\} \longrightarrow \{B_i, \psi_i^j\} \longrightarrow \{C_i, \theta_i^j\}$$

such that

$$0 \longrightarrow A_i \xrightarrow{t_i} B_i \xrightarrow{s_i} C_i \longrightarrow 0$$

is exact for each $i \in I$. Then there is an exact sequence of modules

$$0 \longrightarrow \varinjlim A_i \xrightarrow{t} \varinjlim B_i \xrightarrow{s} \varinjlim C_i \longrightarrow 0.$$

Theorem 1.19. *For any right R-module B, the functor $B \otimes_R$ _ preserves direct limits.*

Theorem 1.20. *Let $\{A_i, \phi_i^j\}$ be a direct system of R-modules and homomorphisms with directed index set I, let λ_j be the j^{th} injection $\lambda_j : A_j \longrightarrow \oplus A_i$, and let $\varinjlim A_i = \oplus A_i \diagup S$ where S is the submodule of $\oplus F_i$ generated by $\lambda_j \phi_i^j(a_i) - \lambda_i(a_i)$ ($a_i \in F_i$ and $i \leq j$). Then,*

(i) $\varinjlim A_i$ consists of all $\lambda_i\,(a_i) + S$;

(ii) $\lambda_i\,(a_i) + S$ is 0 if and only if $\phi_i^t(a_i) = 0$ for some $t \geqslant i$.

Chapter 2
ABSOLUTE PURITY

In this chapter we will discuss the notions of purity and absolute purity of modules. As we shall see, their basic properties generalize splitness and injectivity. We first start with purity.

Definition. An exact sequence of left R-modules

$$0 \longrightarrow A' \xrightarrow{\lambda} A \longrightarrow A'' \longrightarrow 0$$

is said to be *pure-exact* if, for every right R-module B, we have exactness of

$$0 \longrightarrow B \otimes A' \xrightarrow{1\otimes\lambda} B \otimes A \longrightarrow B \otimes A'' \longrightarrow 0.$$

In this case, we say $\lambda(A')$ is *a pure submodule* of A.

This is the definition of pure exact sequences introduced by P. M. Cohn [5]. It is clear that every R-module M has 0 and M as pure submodules.

For simplicity, we will use an equational characterization of purity of modules to deduce some results. To prove this characterization, we need the following lemma (see B. Stenström [23, p.38]).

Lemma 2.1. *Let the diagram*

$$\begin{array}{ccccccc} M' & \xrightarrow{\mu'} & M & \xrightarrow{\mu} & M'' & \longrightarrow 0 \\ \downarrow \alpha' & & \downarrow \alpha & & \downarrow \alpha'' & \\ N' & \xrightarrow{\eta'} & N & \xrightarrow{\eta} & N'' & \longrightarrow 0 \end{array}$$

be commutative with exact rows. Suppose that α is monic. Then, α'' is monic if and only if im $\eta'\cap$ im $\alpha =$ im $\alpha\mu'$.

Proof. Let α'' be monic and let $x =$ im $\eta'\cap$ im α. Then, $x = \eta'(t) = \alpha(m)$ for some $t \in N'$ and $m \in M$. $\eta'(t) \in$ im $\eta' =$ ker η implies $\eta\eta'(t) = 0$. Then, $\eta\alpha(m) = \eta\eta'(t) = 0$ which leads to have $\alpha''\mu(m) = 0$ and this implies $\mu(m) = 0$ since α'' is monic. Then, $m \in$ ker $\mu =$ im μ' which means that $m = \mu'(c)$ for some $c \in M'$. Thus, $x = \eta'(t) = \alpha(m) = \alpha\mu'(c) \in$ im $\alpha\mu'$. The other inclusion is clear since im $\alpha\mu' \subseteq$ im α and im $\alpha\mu' =$ im $\eta'\alpha' \subseteq$ im η'.

For the converse, Let $\alpha''(m'') = 0$ for some $m'' \in M''$. Then, $m'' = \mu(m)$ for some $m \in M$ since μ is epic. Then, $\alpha''\mu(m) = \alpha''(m'') = 0$ implies $\eta\alpha(m) = 0$ and hence $\alpha(m) \in$ ker $\eta =$ im η'. But $\alpha(m) \in$ im α, so $\alpha(m) \in$ im $\alpha\mu'$ by the hypothesis, which means that $\alpha(m) = \alpha\mu'(m')$ for some $m' \in M$. This leads to have $m = \mu'(m') \in$ im $\mu' =$ ker μ. Therefore, $m'' = \mu(m) = 0$ which means that α'' is monic. □

Remark. It is clear from the proof of Lemma 2.1 that α'' is monic if

im $\eta'\cap$ im $\alpha \subseteq$ im $\alpha\mu'$.

Proposition 2.2. *For a submodule N of an R-module M, the following statements are equivalent:*

(i) N is pure submodule of M.

(ii) Every finite system of equations over N which is solvable in M is also solvable in N. That is, if the system of equations

$$\sum_{k=1}^{n} r_{jk}x_k = b_j \in N \ \ (1 \leq j \leq m)$$

has a solution in M, say $(a_1, a_2, ..., a_n)$ then the system has a solution $(t_1, t_{2,}..., t_n)$ in N, i.e.

$$\sum_{k=1}^{n} r_{jk}t_k = b_j \ (1 \leq j \leq m).$$

Proof. (ii)$\Longrightarrow$(i) We want to prove that for every right R-module B, $1_B \otimes f : B \otimes N \longrightarrow B \otimes M$ is monic whenever $f : N \longrightarrow M$ is the inclusion map. Without loss of generality, we may assume that B is a finitely presented R-module by Theorem 1.17 and Theorem 1.19. Then, there is an exact sequence

$$R^n \xrightarrow{\alpha} R^m \longrightarrow B \longrightarrow 0,$$

where $\alpha : (x_k)_{1\leq k\leq n} \mapsto (\sum\limits_{k=1}^{n} r_{jk} \ x_k)_{1\leq j\leq m}$ for some $r_{jk} \in R$. Let $f : N \longrightarrow M$ be the inclusion map. Then, the diagram

$$\begin{array}{ccccccc} R^n \otimes N & \longrightarrow & R^m \otimes N & \longrightarrow & B \otimes N & \longrightarrow 0 \\ \downarrow 1_{R^n} \otimes f & & \downarrow 1_{R^m} \otimes f & & \downarrow 1_B \otimes f & \\ R^n \otimes M & \longrightarrow & R^m \otimes M & \longrightarrow & B \otimes M & \longrightarrow 0 \end{array}$$

is commutative of exact rows since $\otimes M$ is a right exact functor for every left R-module M by Theorem 1.12. The maps $1_{R^n} \otimes f$ and $1_{R^m} \otimes f$ are monomorphisms since each free module is flat. For simplicity, we will use the natural isomorphisms in Theorem 1.13 and Theorem 1.15 and hence, we have the following diagram

$$\begin{array}{ccccccc} N^n & \xrightarrow{\alpha_1} & N^m & \longrightarrow & B \otimes N & \longrightarrow 0 \\ \downarrow \psi & & \downarrow \phi & & \downarrow 1_B \otimes f & \\ M^n & \xrightarrow{\alpha_2} & M^m & \longrightarrow & B \otimes M & \longrightarrow 0 \end{array}$$

where α_1 and α_2 are defined in the same way as α on N^n and M^n respectively and $\phi : N^m \longrightarrow M^m$ defined as $\phi((c_j)_j) = (f(c_j))_j = (c_j)_j$. Now, we want to prove that im $\alpha_2 \cap$ im $\phi \subseteq$ im $\phi\alpha_1$, so let $b \in$ im $\alpha_2 \cap$ im ϕ. Then, $b = \alpha_2((a_k)_k) = (\sum_{k=1}^{n} r_{jk}a_k)_j$ for some $a_k \in M$ and $b = \phi((c_j)_j)$ for some $c_j \in N$. Thus, $(\sum_{k=1}^{n} r_{jk}a_k)_j = \phi((c_j)_j) = (c_j)_j$. Thus, $\sum_{k=1}^{n} r_{jk}x_k = c_j \ (1 \leq j \leq m)$ be a system over N which is solvable in M and hence there are t_k 's $\in N$ such that $\sum_{k=1}^{n} r_{jk}t_k = c_j \, (1 \leq j \leq m)$ by hypothesis. Then, $(c_j)_j = (\sum_{k=1}^{n} r_{jk}t_k)_j = \alpha_1((t_k)_k)$. Now, $b = \phi((c_j)_j) = \phi\alpha_1((t_k)_k) \in$ im $\phi\alpha_1$. Thus, im $\alpha_2 \cap$ im $\phi \ \subseteq$ im $\phi\alpha_1$. Therefore, $1_B \otimes f$ is monic by Lemma 2.2.

(i)$\Longrightarrow$(ii) Let N be pure in M and let $\sum_{k=1}^{n} r_{jk}x_k = b_j \in N \ (1 \leq j \leq m)$ be a system over N which is solvable in M say by $(a_1, a_2, ..., a_n)$. Let $B = R^m \diagup$ im α where $\alpha : R^n \longrightarrow R^m$ defined as $\alpha((x_k)_{1 \leq k \leq n}) = (\sum\limits_{k=1}^{n} r_{jk} \ x_k)_{1 \leq j \leq m}$. Now, if f be the inclusion map in the above diagram, then $1_B \otimes f$ is monic since N is pure in M and hence im $\alpha_2 \cap$ im $\phi =$ im $\phi\alpha_1$ by Lemma 2.2. Now, $(b_j)_j = (\ \sum_{k=1}^{n} r_{jk}a_k)_j = \alpha_2((a_k)_k) \in$ im α_2 but $b_j \in N$ then $(b_j)_j = (f(b_j)_j) = \phi((b_j)_j) \in$ im ϕ implies $(b_j)_j \in$ im $\alpha_2 \cap$ im ϕ. Thus, $(b_j)_j \in$ im $\phi\alpha_1$ and hence $(b_j)_j = \phi\alpha_1((t_k)_k)$ for some $t_k \in N$. Then, $(\sum_{k=1}^{n} r_{jk}t_k)_j = (\sum_{k=1}^{n} r_{jk}f(t_k))_j = (f(\sum_{k=1}^{n} r_{jk}t_k)_j) = \phi(((\sum_{k=1}^{n} r_{jk}t_k)_j)) = \phi\alpha_1((t_k)_k) = (b_j)_j \ (1 \leq j \leq m)$ where $t_k \in N$. Hence, the system $\sum_{k=1}^{n} r_{jk}x_k = b_j \in N \ (1 \leq j \leq m)$ is solvable in N. $\square$

Proposition 2.3. *If N is pure in M and M is pure in P then N is pure in P.*

Proof. Let N be pure in M and M be pure in P. Let $\sum\limits_{k=1}^{n} r_{jk}x_j = n_j \in N \ (1 \leq j \leq m)$ be a finite system over N which is solvable in P. Then, the system is solvable in M since M is pure in P. Hence, since N is pure in M, the system is solvable in N, and so N is pure in P. $\square$

Proposition 2.4. *The direct sum of a family of R-modules is pure in their direct product.*

Proof. Let $\{E_i \mid i \in I\}$ be a family of R-modules and let $\sum_{k=1}^{n} r_{jk}(x_{ki})_{i \in I} = (q_{ji})_{i \in I}$ $(1 \leq j \leq m)$ be a system of linear equations over $\oplus_{i \in I} E_i$ which is solvable in $\prod_{i \in I} E_i$, say by $(t_{ki})_{i \in I}$, i.e. $\sum_{k=1}^{n} r_{jk} \ (t_{ki})_{i \in I} = (q_{ji})_{i \in I} \ (1 \leq j \leq m)$. Now almost all q_{ji} are zero since $(q_{ji})_{i\in I} \in \oplus_{i\in I} E_i$. Let $T = \bigcup_{j=1}^{m}$ support(q_{ji}) and for each k, define s_{ki} as follows

$$s_{ki} = \begin{cases} t_{ki} & \text{if } i \in \quad T \\ 0 & \text{if } i \in \quad I \diagdown T \end{cases} .$$

It is clear that $(s_{ki})_{i\in I} \in \oplus_{i\in I} E_i$ because T is finite, and that $\sum_{k=1}^{n} r_{jk}\,(s_{ki})_{i\in I} = (q_{ji})_{i\in I}$ $(1 \leq j \leq m)$. This proves that $\oplus_{i\in I} E_i$ is pure in $\prod_{i\in I} E_i$. □

The following result was proved by D. J. Fieldhouse [9] in the context of pure theories. We prove it here using the equational characterization of purity of modules.

Proposition 2.5. *If P and Q are two submodules of M, then:*

(i) $P \cap Q$ pure in Q implies P pure in $P + Q$.

(ii) $P + Q$ pure in M and $P \cap Q$ pure in Q implies P pure in M.

(iii) $P + Q$ pure in M and $P \cap Q$ pure in M implies P pure in M and Q pure in M.

(iv) $P \cap Q$ pure in $P + Q$ implies P and Q are both pure in $P + Q$.

Proof. (i) Let $\sum\limits_{k=1}^{n} r_{jk}x_k = p_j$ $(1 \leq j \leq m)$ be a system over P which is solvable in $P + Q$ say by $t_k + s_k$ $(1 \leq k \leq n)$ where $t_k \in P$ and $s_k \in Q$ i.e. $\sum\limits_{k=1}^{n} r_{jk}(t_k + s_k) = p_j$. Then $\sum\limits_{k=1}^{n} r_{jk}s_k = p_j - \sum\limits_{k=1}^{n} r_{jk}t_k \in P \cap Q$. Now, the system $\sum\limits_{k=1}^{n} r_{jk}x_k = p_j - \sum\limits_{k=1}^{n} r_{jk}t_k$ has the solution s_k $(1 \leq k \leq n)$ in Q and so it has a solution in $P \cap Q$ by hypothesis, say q_k $(1 \leq k \leq n)$, i.e. $\sum\limits_{k=1}^{n} r_{jk}q_k = p_j - \sum\limits_{k=1}^{n} r_{jk}t_k$. Thus, we have $\sum\limits_{k=1}^{n} r_{jk}(q_k + t_k) = p_j$ $(1 \leq j \leq m)$ where $q_k \in P \cap Q \subseteq P$ and $t_k \in P$. Hence, P is pure in $P + Q$.

(ii) Let $P + Q$ be pure in M and $P \cap Q$ be pure in Q. By (i), $P \cap Q$ pure in Q implies P pure in $P + Q$. Thus, P is pure in $P + Q$ and $P + Q$ is pure in M, so that P pure in M.

(iii) Suppose $P \cap Q$ is pure in M, then $P \cap Q$ is pure in Q, since Q is a submodule of M. So, if $P + Q$ is pure in M, then P is pure in M by (ii). Similarly, Q is pure in M.

(iv) Let $P \cap Q$ be pure in $P + Q$. Then $P \cap Q$ is pure in Q, as $Q \subseteq P + Q$. Hence, P is pure in $P + Q$ by (i). Similarly, Q is pure in $P + Q$. $\square$

2.1 Absolutely Pure Modules

In this section we present the notion of absolutely pure modules and some of its fundamental properties. B. H. Maddox [16] introduced the following

Definition. An R-module is *absolutely pure* (a.p.) if it is a pure submodule in every module containing it.

Recall that an exact sequence $0 \longrightarrow N \xrightarrow{\alpha} M \xrightarrow{\beta} P \longrightarrow 0$ of R-modules and R-homomorphisms is *pure-exact* if $\alpha(N)$ is pure in M.

Proposition 2.6. *An R-module N is absolutely pure if and only if every short exact sequence with first term N is pure-exact.*

Proof. Suppose first that N is an absolutely pure module and assume that

$$0 \longrightarrow N \xrightarrow{\alpha} M \xrightarrow{\beta} P \longrightarrow 0$$

is a short exact sequence. Now let $\sum\limits_{k=1}^{n} r_{jk}\, x_k = \alpha(s_j)\ (1 \leq j \leq m)$ be a system over $\alpha(N)$ which is solvable in M, say by $(a_1, a_2, ..., a_n)$, i.e. $\sum\limits_{k=1}^{n} r_{jk}\, a_k = \alpha(s_j)\ (1 \leq j \leq m)$. Now, $\alpha(N) \cong N$ and so $\alpha(s_j) = b_j$ for some $b_j \in N$. Then, $\sum\limits_{k=1}^{n} r_{jk}\, a_k = b_j \in N$. Since N is absolutely pure, there are t_k's $\in N$ such that $\sum\limits_{k=1}^{n} r_{jk}\, t_k = b_j$. Now, $t_k \in N \cong \alpha(N)$ and so $t_k = \alpha(u_k)$ for some $u_k \in N$. Therefore, $\sum\limits_{k=1}^{n} r_{jk}\, \alpha(u_k) = \alpha(s_j)$. Thus, the system is solvable in $\alpha(N)$ and hence $\alpha(N)$ is pure in M.

For the converse, simply take α to be the inclusion map. □

Proposition 2.7. *An R-module N is absolutely pure if and only if it is pure in some injective R-module.*

Proof. One implication follows by taking an injective envelope of N. For the converse, let M be a module containing N and suppose that N is pure in some injective R-module E. Let $\sum_{k=1}^{n} r_{jk}x_k = a_j$ $(1 \leq j \leq m)$ be a finite system of equations over N, solvable in M by $(b_1, b_2, ..., b_n)$, i.e. $\sum_{k=1}^{n} r_{jk}b_k = a_j$. We show that the system is solvable in N. Consider the following diagram

$$\begin{array}{ccc} 0 \longrightarrow & N & \xrightarrow{\alpha} M \\ & i \downarrow & \\ & E & \end{array}$$

where α and i are the inclusion maps. Then, there is a map $\mu : M \longrightarrow E$ such that $\mu\alpha = i$ since E is injective. Then,

$$\sum_{k=1}^{n} r_{jk}\mu(b_k) = \mu(\sum_{k=1}^{n} r_{jk}b_k) = \mu\alpha(a_j) = a_j \ (1 \leq j \leq m)$$

and so the finite system $\sum_{k=1}^{n} r_{jk}x_k = a_j$ $(1 \leq j \leq m)$ is solvable in E, and so it is solvable in N. Hence, there exist $t_k \in N$ $(1 \leq k \leq n)$ such that $\sum_{k=1}^{n} r_{jk}t_k = a_j$ $(1 \leq j \leq m)$, as required. □

Proposition 2.8. *Every split exact sequence is pure-exact. In particular, every direct summand of an R-module M is isomorphic to a pure submodule of M.*

Proof. Let $0 \longrightarrow N \xrightarrow{\lambda} M \xrightarrow{\sigma} P \longrightarrow 0$ be a split exact sequence and let $\sum_{k=1}^{n} r_{jk}x_k = \lambda(n_j)$ $(1 \leq j \leq m)$ be a finite system over $\lambda(N)$ which is solvable in M. Then, there are $b_k \in M$ $(1 \leq k \leq n)$ such that $\sum_{k=1}^{n} r_{jk}b_k = \lambda(n_j)$. Since the sequence is split, there is $\beta : M \longrightarrow N$ with $\beta\lambda = 1_N$. So we get $\sum_{k=1}^{n} r_{jk}\beta(b_k) = \beta\lambda(n_j) = n_j$, which implies that

$\sum_{k=1}^{n} r_{jk}\lambda\beta(b_k) = \lambda(n_j)$. Since $\lambda\beta(b_k) \in \lambda(N)$, the system is solvable in $\lambda(N)$, as required. $\square$

Corollary 2.9. *Injective modules are absolutely pure.*

Proof. Let N be an injective submodule of an R-module M. Then, consider the exact sequence

$$0 \longrightarrow N \xrightarrow{\subseteq} M \longrightarrow M/N \longrightarrow 0.$$

This sequence is split exact, since N is injective by Theorem 1.9, and hence it is pure exact by Proposition 2.8. Therefore, N is pure in M. $\square$

Because of the above results, we can consider absolute purity as a weak form of injectivity, and purity as a weak form of splitness. We shall see later that there are absolutely pure modules that are not injective. We next prove some additional results on absolute purity.

The following proposition was proved by B. H. Maddox [16].

Proposition 2.10. *(i) Pure submodules of absolutely pure modules are absolutely pure. In particular, direct summands of absolutely pure modules are absolutely pure.*

(ii) Direct products and direct sums of absolutely pure modules are absolutely pure.

Proof. (i) Let $N \subseteq M$ where N is a pure submodule of M and M is absolutely pure. Then, M is pure in some injective module say E by Proposition 2.7, so by Proposition 2.3, N is pure in E. Hence, N is absolutely pure again by Proposition 2.7.

(ii) Let $\{N_i \mid i \in I\}$ be a family of absolutely pure modules. Then N_i is pure in some injective module, say E_i. Now, let $\sum_{k=1}^{n} r_{jk}(x_{ki})_{i\in I} = (a_{ji})_{i\in I}$ $(1 \leq j \leq m)$ be a system of equations over $\prod_{i\in I} N_i$ which is solvable in $\prod_{i\in I} E_i$. Consider each component separately, i.e. consider the system $\sum\limits_{k=1}^{n} r_{jk}(x_{kt}) = (a_{jt})$ for each $t \in I$. Then, it will be a system over N_t which is solvable in E_t. By the hypothesis, the system will be solvable in N_t. This is true for all $t \in I$. So, $\sum\limits_{k=1}^{n} r_{jk}(x_{ki})_{i\in I} = (a_{ji})_{i\in I}$ is solvable in $\prod\limits_{i\in I} N_i$, i.e. $\prod\limits_{i\in I} N_i$ is pure in $\prod\limits_{i\in I} E_i$. Since is $\prod\limits_{i\in I} E_i$ is injective (Theorem 1.7), $\prod\limits_{i\in I} N_i$ is absolutely pure by Proposition 2.7. Also, the direct sum $\oplus_{i\in I} N_i$ of the family $\{N_i \mid i \in I\}$ is a pure submodule of their direct product by Proposition 2.4. So, by (i), $\oplus_{i\in I} N_i$ is absolutely pure. □

Recall the definition of finitely presented modules.

Definition. An R-module P is *finitely presented* (f.p.) if there is an exact sequence $0 \longrightarrow K \longrightarrow F \longrightarrow P \longrightarrow 0$ of R-modules where F is free and both F and K are finitely generated (f.g.).

Definition. A system of equations over a module A is said to be *consistent* if it is solvable in some extension B of A.

Proposition 2.11. *A system* $\sum_{k=1}^{n} r_{jk}x_k = a_j$ $(1 \leq j \leq m)$ *over a module* A *is consistent if and only if* $\sum_{j=1}^{m} \lambda_j a_j = 0$ *for all scalars* λ_j *such that* $\sum_{j=1}^{m} \lambda_j r_{jk} = 0$ *for all* k.

Proof. Let $\sum_{k=1}^{n} r_{jk}\ x_k = a_j$ be a consistent system over A. Then, the system is solvable in some extension B of A, i.e.$\sum_{k=1}^{n} r_{jk}\ t_k = a_j$ $(1 \leq j \leq m)$ for some $t_k \in B$. Now, assume that $\sum_{j=1}^{m} \lambda_j r_{jk} = 0$ for all k, then $\sum_{j=1}^{m} \lambda_j a_j = \sum_{j=1}^{m} \lambda_j (\sum_{k=1}^{n} r_{jk}t_k) = \sum_{k=1}^{n} (\sum_{j=1}^{m} \lambda_j r_{jk}) t_k = 0$.

Conversely, let $\sum_{k=1}^{n} r_{jk}x_k = a_j$ $(1 \leq j \leq m)$ be a system over A. Then, consider the following diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & K & \xrightarrow{\subseteq} & R^n \\ & & \psi \downarrow & & \\ 0 & \longrightarrow & A & \xrightarrow{\sigma} & E \end{array}$$

where K is a f.g submodule of R^n generated by $y_j = \sum_{k=1}^{n} r_{jk} e_k$ $(1 \leq j \leq m)$, where $\{e_k\}_{1 \leq k \leq n}$ is the standard basis of R^n, E is an injective envelope of A and σ is the inclusion map. Now, let $\psi : K \longrightarrow A$ defined as $y_j \mapsto a_j$ (ψ is well-defined because if $\sum_{j=1}^{m} \lambda_j y_j = 0$ then $0 = \sum_{j=1}^{m} \lambda_j (\sum_{k=1}^{n} r_{jk} e_k) = \sum_{k=1}^{n} (\sum_{j=1}^{m} \lambda_j r_{jk}) e_k$, i.e. $\sum_{j=1}^{m} \lambda_j r_{jk} = 0$ which, by hypothesis leads to have $0 = \sum_{j=1}^{m} \lambda_j a_j = \sum_{j}^{m} \lambda_j \psi(y_j) = \psi(\sum_{j=1}^{m} \lambda_j y_j))$. Now, E is injective and so there is an extension $\phi : R^n \longrightarrow E$ of $\sigma\psi$. We claim that $\{\phi(e_k)\}_{1 \leq k \leq n}$ solves the system $\sum_{k=1}^{n} r_{jk} \phi(e_k) = \phi(\sum_{k=1}^{n} r_{jk} e_k) = \phi(y_j) = \sigma\psi(y_j) = \psi(y_j) = a_j$ $(1 \leq j \leq m)$. Thus, the system is solvable in an extension E of A and hence the system is consistent. □

Proposition 2.12 (A. Kertész [13]). *An R-module A is injective if and only if every consistent system of equations over A is solvable in A.*

Proof. Let A be an injective R-module. Let $\sum\limits_{k \in K} r_{jk} x_k = a_j \in A$ $(j \in J)$ be a consistent system of equations over A and let G be the submodule of the free module $F = R^{(K)}$ generated by $\{\sum_{k \in K} r_{jk} e_k\}_{j \in J}$, where $\{e_k\}_{k \in K}$ is the standard basis of F. Consider the following diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & G & \xrightarrow{\subseteq} & F \\ & & f \downarrow & & \\ & & A & & \end{array}$$

where $f : G \longrightarrow A$ is defined as $\sum\limits_{k \in K} r_{jk} e_k \mapsto a_j$ (f is well-defined). Then, there exists an extension $\psi : F \longrightarrow A$. Let $t_k = \psi(e_k)$ then $\sum\limits_{k \in K} r_{jk} t_k = \sum\limits_{k \in K} r_{jk} \psi(e_k) = \psi(\sum\limits_{k \in K} r_{jk} e_k) = f(\sum\limits_{k \in K} r_{jk} e_k) = a_{j.}$ Thus, the system is solvable in A.

Conversely, consider the following diagram

$$\begin{array}{ccccc} 0 \longrightarrow & I & \xrightarrow{\subseteq} R \\ & f \downarrow & \\ & A & \end{array}$$

where $\{a_j \mid j \in J\}$ is a set of generators of the ideal I. Consider the system of equations $a_j x = f(a_j)$. This system is consistent by Proposition 2.11. So, by the hypothesis the system is solvable in A, i.e. $a_j t = f(a_j)$ for some t in A and all $j \in J$. Let $\psi : R \longrightarrow A$ be defined as $\psi(r) = rt$. Then, $\psi(a_j) = a_j t = f(a_j)$. Hence, ψ extends f, and A is an injective R-module by Baer's Criterion. $\square$

Proposition 2.13. *A module A is absolutely pure if and only if every consistent finite system over A is solvable in A.*

Proof. Let A be an absolutely pure R-module. Then, A is pure in some injective R-module E by Proposition 2.7. Let $\sum\limits_{k=1}^{n} r_{jk}x_k = a_j \in A$ $(1 \leq j \leq m)$ be a finite consistent system over A. Then, this system is over E and the system is consistent. Thus, the system is solvable in E by Proposition 2.12. and so the system is solvabe in A since A is pure in E.

For the converse, let A be a submodule of an R-module B and let $\sum\limits_{k=1}^{n} r_{jk}x_k = a_j \in A$ $(1 \leq j \leq m)$ be a system over A which is solvable in B, say $\sum\limits_{k=1}^{n} r_{jk}b_k = a_j \in A$ $(1 \leq j \leq m)$, where $b_k \in B$. This finite system is consistent by Proposition 2.11. Hence, there is $\{t_k\}_{1 \leq k \leq n}$ in A such that $\sum\limits_{k=1}^{n} r_{jk}t_k = a_j$. Therefore, A is pure in B, and so A is absolutely pure. $\square$

2.2 An Analogue of Baer's Criterion

One powerful test of injectivity is Baer's Criterion. One may therefore want to have an analogue of this for absolutely pure modules. The following proposition gives the desired criterion.

Proposition 2.14. *For an R-module A, the following conditions are equivalent.*

(i) *A is absolutely pure;*

(ii) *every homomorphism $\psi : K \longrightarrow A$ in the diagram*

$$\begin{array}{ccc} 0 \longrightarrow & K & \xrightarrow{\sigma} F \\ & \psi \downarrow & \\ & A & \end{array}$$

where F is free and both F and *K are f.g.*, *and σ is the inclusion map, can be extended by $\beta : F \longrightarrow A$ i.e. $\psi = \beta\sigma$.*

Proof. (i)⇒(ii) (C. Megibben [18]). Let $0 \longrightarrow K \longrightarrow F \longrightarrow P \longrightarrow 0$ be an exact sequence where F is free and both F and K are f.g. Let $\psi : K \longrightarrow A$ be an R-homomorphism. Let K be generated by $\{a_{1,}a_{2,}..., a_m\}$, F have basis $\{e_k\}_{1\leq k\leq n}$. Then, there are r_{jk}'s in R such that $a_j = \sum\limits_{k=1}^{n} r_{jk}e_k$ $(1 \leq j \leq m)$. Consider the system $\sum\limits_{k=1}^{n} r_{jk}x_k = \psi(a_j)$ $(1 \leq j \leq m)$. Now, if there exist scalars λ_j such that $\sum\limits_{j=1}^{m} \lambda_j r_{jk} = 0$ for all k, then $0 = \sum\limits_{k=1}^{n}(\sum\limits_{j=1}^{m} \lambda_j r_{jk})e_k = \sum\limits_{j=1}^{m} \lambda_j(\sum\limits_{k=1}^{n} r_{jk}e_k) = \sum\limits_{j=1}^{m} \lambda_j a_j$, which implies that $\sum\limits_{j=1}^{m} \lambda_j\psi(a_j) = 0$ and hence the system is consistent by Proposition 2.11. Then, by Proposition 2.13, the system is solvable in A say by $(t_{1,}t_2, ..., t_n)$, i.e $\sum\limits_{k=1}^{n} r_{jk}t_k = \psi(a_j)$. Let $\beta : F \longrightarrow A$ be defined by $\beta(e_k) = t_k$ $(1 \leq k \leq n)$. Then, $\beta(a_j) = \beta(\sum\limits_{k=1}^{n} r_{jk}e_k) = \sum\limits_{k=1}^{n} r_{jk}\beta(e_k) = \sum\limits_{k=1}^{n} r_{jk}t_k = \psi(a_j)$, so β extends ψ, as required.

(ii)$\Longrightarrow$(i) Let $\sum\limits_{k=1}^{n} r_{jk}x_k = a_j \in A$ $(1 \leq j \leq m)$ be a system of equations over A, solvable by $(c_1, c_2, ..., c_n)$ in an injective envelope E of A. Now, let $F = R^n$ with standard basis $\{e_k\}_{1 \leq k \leq n}$ and K be the submodule of F generated by $b_j = \sum\limits_{k=1}^{n} r_{jk}e_k$ $(1 \leq j \leq m)$. Define an R-homomorphism $\psi : K \longrightarrow A$ by $\psi(b_j) = a_j$ (ψ is well-defined because if $\sum\limits_{j=1}^{m} \lambda_j b_j = 0$ then $0 = \sum\limits_{j=1}^{m} \lambda_j (\sum\limits_{k=1}^{n} r_{jk}e_k) = \sum\limits_{k=1}^{n} (\sum\limits_{j=1}^{m} \lambda_j r_{jk})e_k$, and so $\sum\limits_{j=1}^{m} \lambda_j r_{jk} = 0$, i.e. $0 = \sum\limits_{k=1}^{n} (\sum\limits_{j=1}^{m} \lambda_j r_{jk})c_k = \sum\limits_{j=1}^{m} \lambda_j (\sum\limits_{k=1}^{n} r_{jk}c_k) = \sum\limits_{j=1}^{m} \lambda_j a_j$). Then, by the hypothesis, the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & K & \longrightarrow & F \\ & & \psi \downarrow & & \\ & & A & & \end{array}$$

has an extension $g : F \longrightarrow A$ of ψ. Let $g(e_k) = d_k \in A$. Then, $\sum\limits_{k=1}^{n} r_{jk}d_k = \sum\limits_{k=1}^{n} r_{jk}g(e_k) = g(\sum\limits_{k=1}^{n} r_{jk}e_k) = g(b_j) = \psi(b_j) = a_j$. So, the system is solvable in A and hence A is absolutely pure by Proposition 2.13. $\square$

Proposition 2.15. *Consider the short exact sequence*

$$0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0$$

where both M' and M'' are absolutely pure. Then, M is absolutely pure.

Proof. Consider the following diagram of exact sequences

$$\begin{array}{ccccccccc} & & 0 & \longrightarrow & K & \xrightarrow{\sigma} & F & & \\ & & & & f \downarrow & & & & \\ 0 & \longrightarrow & M' & \xrightarrow{\alpha} & M & \xrightarrow{\beta} & M'' & \longrightarrow & 0 \end{array}$$

where F is free and both F and K are f.g., and σ is the inclusion map. Then, $\phi : F \longrightarrow M''$ exists such that $\phi\sigma = \beta f$. So, there is a map $\psi : F \longrightarrow M$ such that $\beta\psi = \phi$ since F is projective. Now, $\beta\psi\sigma = \beta f$ implies that $\beta(\psi\sigma - f) = 0$ which means that $\psi\sigma - f \in$ ker $\beta =$ im $\alpha = \alpha(M')$ and hence $\psi\sigma - f$ maps K into $\alpha(M')$. Since α is monic, there is an isomorphism γ that maps $\alpha(M')$ onto M'. Thus, $\gamma(\psi\sigma - f)$ is a homomorphism that maps K into M'. Therefore, there is a homomorphism $\tau : F \longrightarrow M'$ such that $\tau\sigma = \gamma(\psi\sigma - f)$ since M' is an absolutely pure module by Proposition 2.14. But $\alpha\tau\sigma = \alpha\gamma(\psi\sigma - f) = \psi\sigma - f$ since $\alpha\gamma = 1_{\alpha(M')}$ which leads to have $f = \psi\sigma - \alpha\tau\sigma = (\psi - \alpha\tau)\sigma$ and hence $\psi - \alpha\tau$ is an extension of f. Therefore, M is absolutely pure by Proposition 2.14. $\square$

Chapter 3
RING CHARACTERIZATIONS

In this chapter we will study some types of rings using such concepts as absolute purity, flatness and direct limit. For example, it is known that a direct sum of injective modules is not necessarly injective, hence the question as to which rings have the property that injectivity is closed under direct sums naturally arises. This and other similar questions are the object of this chapter.

3.1 Noetherian and Coherent Rings

We begin with the following important class of rings.

Definition. A ring R is *left noetherian* if every left ideal is finitely generated.

In the following classical result, we see that the noetherianess of a ring ensures that injectivity is closed under direct sums. This result is needed later.

Proposition 3.1. *A ring R is left noetherian if and only if every direct sum of injective modules is injective.*

Proof. Suppose R is noetherian and let $\{E_i\}_{i\in J}$ be a family of injective modules. Consider the following diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & I & \overset{\subseteq}{\longrightarrow} & R \\ & & \downarrow f & & \\ & & \oplus E_i & & \end{array}$$

where I is a left ideal and f is an R-homomorphism. Then $I = \langle a_1, a_2, ..., a_n\rangle$ since R is noetherian. Now, $f(\sum\limits_{i=1}^{n} r_i a_i) = \sum\limits_{i=1}^{n} r_i f(a_i)$ where each $f(a_i) \in \oplus E_i$ has only finitely many non-zero coordinates. Hence, im $f \subseteq E_{i_1} \oplus E_{i_2} \oplus ... \oplus E_{i_m}$ for some $m \in \mathbb{N}$ and $E_{i_1} \oplus E_{i_2} \oplus ... \oplus E_{i_m}$ is injective by Theorem 1.7. Thus, there is a map $g : R \longrightarrow E_{i_1} \oplus E_{i_2} \oplus ... \oplus E_{i_m} \subseteq \oplus E_i$ extending f.

Assume now that R is not left noetherian, then there is strictly ascending sequence of ideals

$$I_1 \subset I_2 \subset I_3 \subset ...$$

Let $I = \bigcup_{n=1}^{\infty} I_n$ and note that $I \diagup I_n \neq 0$ for all n. Imbed $I \diagup I_n$ in an injective module E_n. We claim that $\oplus E_n$ is not injective. Let $\pi_n : I \longrightarrow I \diagup I_n$ be the natural map. Then for each $a \in I$, $\pi_n(a) = 0$ for sufficiently large n, so the map $f : I \longrightarrow \oplus E_n$ given by $a \longmapsto (\pi_1(a), \pi_2(a), ..., \pi_n(a), ...)$ has its image in $\oplus E_n$. Suppose $\oplus E_n$ is injective, then there is a map $g : R \longrightarrow \oplus E_n$ extending f. Write $g(1) = (x_n)_{n\in\mathbb{N}}$. Choose $m \in \mathbb{N}$ and $a \in I \backslash I_m$. Now, $\pi_m(a) \neq 0$, so $g(a) = f(a)$ has a non-zero m^{th} coordinate $\pi_m(a)$. But $g(a) = ag(1) = a(x_n)_{n\in\mathbb{N}} = (ax_n)_{n\in\mathbb{N}}$, thus $\pi_m(a) = ax_m$, so that $x_m \neq 0$. Since m is arbitrary, we contradicted the fact that almost all the coordinates $g(a) \in \oplus E_n$ are zeros. $\square$

Remark. We proved that every injective module is absolutely pure module. A natural question now arises. Is there an absolutely pure module which is not injective? We know that a ring R is noetherian if and only if every direct sum of injective modules is injective. So, if R is not left noetherian then not every direct sum of injective R-modules is injective. Let $\{E_i\}_{i\in J}$ be a family of injective modules such that $\oplus_i E_i$ is not injective. Using Corollary 2.9, each E_i is absolutely pure, hence $\oplus_i E_i$ is also absolutely pure by Proposition 2.10. We may now ask the following question. What are the rings over which all absolutely pure modules are injective? The following result gives the desired answer.

Proposition 3.2. *The following statements are equivalent for any ring R.*

(i) R is noetherian.

(ii) An R-module A is injective if and only if it is an absolutely pure R-module.

Proof. (i) $\Longrightarrow$(ii) It is known that if A is injective then it is absolutely pure by Corollary 2.9. Conversely, let M be an absolutely pure module, and consider the following diagram

$$\begin{array}{ccc} 0 \longrightarrow & I & \xrightarrow{\sigma} R \\ & f \downarrow & \\ & M & \end{array}$$

where I is an ideal in R, f is an R-homomorphism, and σ is the inclusion map. As R is noetherian, I is f.g., but R is f.g. free, and so there is a map $\beta : R \longrightarrow M$ such that $\beta\sigma = f$, by Proposition 2.14. Thus, M is an injective R-module.

(ii) $\Longrightarrow$ (i) By way of contradiction, assume that R is not noetherian and choose a family $\{B_i\}_{i \in J}$ of injective R-modules such that $A = \oplus\ B_i$ is not injective (this can be done by Proposition 3.1). Now, $A = \oplus\ B_i$ is an absolutely pure module by Corollary 2.9 and Proposition 2.10, and this contradicts the hypothesis (ii). Therefore, R is noetherian. $\Box$

J. M. Maranda [17] undertook a study of modules M for which the contravariant functor $Hom(\_, M)$ takes maps from a restricted class of monomorphisms of R-modules to epimorphisms in the category of abelian groups. These monomorphisms were labelled "pure monomorphisms". Accordingly, an R-module M is said to be *pure-injective* if $Hom_R(\_, M)$ preserves exactness in pure-exact sequences or, equivalently, if M has the injective property with respect to pure-exact sequences. It is clear that every injective module is pure-injective.

We have seen above that a ring R is noetherian if and only if every absolutely pure R-module is an injective module. Now, over arbitrary rings if we impose pure-injectivity on absolutey pure modules, then they become injective modules. To see this, let N be an absolutely pure and pure injective R-module. Then, every short exact sequence $0 \longrightarrow N \xrightarrow{\alpha} M \xrightarrow{\beta} P \longrightarrow 0$ is pure-exact by Proposition 2.6. Now, let 1_N be the identity map on N. Then, there is $g : M \longrightarrow N$ such that $g\alpha = 1_N$, since N is pure-injective. Hence, the sequence is split exact. It follows that N is injective by Theorem 1.9. It is now clear that a module is injective if and only if it is both absolutely pure and pure-injective. □

Now, let us consider another important class of rings.

Definition. A ring R is *left coherent* if every f.g. left ideal is f.p.

Note that every left noetherian is left coherent. To see this, let R be a left noetherian ring and let I be a f.g. left ideal. Then, we have the exact sequence
$0 \longrightarrow \ker\phi \xrightarrow{\subseteq} F \xrightarrow{\phi} I \longrightarrow 0$ where F is chosen to be f.g. free. Since R is noetherian, ker ϕ is f.g. being a submodule of the f.g. R-module F. Therefore, I is a f.p. left ideal and hence R is left coherent.

Definition. If B is a right R-module, its *character module* B^* is the left R-module $Hom_{\mathbb{Z}}(B, \mathbb{Q}/\mathbb{Z})$.

Proposition 3.3. *A sequence* $0 \longrightarrow N \longrightarrow M \longrightarrow P \longrightarrow 0$ *of* R*-modules is pure-exact if and only if* $0 \longrightarrow P^* \longrightarrow M^* \longrightarrow N^* \longrightarrow 0$ *is split exact.*

Proof. Suppose that

$$0 \longrightarrow N \longrightarrow M \longrightarrow P \longrightarrow 0$$

is a pure-exact sequence. Then, for all right R-modules X, the exact sequence of abelian groups

$$0 \longrightarrow X \otimes_R N \longrightarrow X \otimes_R M \longrightarrow X \otimes_R P \longrightarrow 0$$

is exact. Now $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module, so we have

$$0 \rightarrow Hom_{\mathbb{Z}}(X \otimes_R P, \mathbb{Q}/\mathbb{Z}) \rightarrow Hom_{\mathbb{Z}}(X \otimes_R M, \mathbb{Q}/\mathbb{Z}) \rightarrow Hom_{\mathbb{Z}}(X \otimes_R N, \mathbb{Q}/\mathbb{Z}) \rightarrow 0$$

is exact. By Theorem 1.16 (Adjoint Isomorphism)

$$0 \longrightarrow Hom_R(X, P^*) \longrightarrow Hom_R(X, M^*) \longrightarrow Hom_R(X, N^*) \longrightarrow 0$$

is exact. In particular, if we set $X = N^*$, then

$$0 \longrightarrow P^* \longrightarrow M^* \longrightarrow N^* \longrightarrow 0$$

is a split exact sequence.

Conversely, suppose that

$$0 \longrightarrow P^* \longrightarrow M^* \longrightarrow N^* \longrightarrow 0$$

is split exact. Then,

$$0 \longrightarrow Hom_R(X, P^*) \longrightarrow Hom_R(X, M^*) \longrightarrow Hom_R(X, N^*) \longrightarrow 0$$

is exact and again by Theorem 1.16 (Adjoint Isomorphism), we will have the exact sequence

$$0 \to Hom_{\mathbb{Z}}(X \otimes_R P, \mathbb{Q}/\mathbb{Z}) \to Hom_{\mathbb{Z}}(X \otimes_R M, \mathbb{Q}/\mathbb{Z}) \to Hom_{\mathbb{Z}}(X \otimes_R N, \mathbb{Q}/\mathbb{Z}) \to 0.$$

Then,

$$0 \longrightarrow X \otimes_R N \longrightarrow X \otimes_R M \longrightarrow X \otimes_R P \longrightarrow 0$$

is exact for all X since $\mathbb{Q}/\mathbb{Z}$ is a cogenerator for abelian groups. Hence, $0 \longrightarrow N \longrightarrow M \longrightarrow P \longrightarrow 0$ is pure-exact. □

Lemma 3.4. *Let X be a f.p. left R-module, Y an (R, S)-bimodule and Z an injective right S-module. Then, $Hom_S(Hom_R(X, Y), Z) \cong Hom_S(Y, Z) \otimes_R X$.*

Proof. Let X be a f.p. left R-module i.e. there is an exact sequence

$$F_1 \xrightarrow{\alpha} F_0 \longrightarrow X \longrightarrow 0$$

where F_0 and F_1 are f.g. free. Since $Hom_R(\_, Y)$ is left exact, the sequence

$$0 \longrightarrow Hom_R(X, Y) \longrightarrow Hom_R(F_0, Y) \longrightarrow Hom_R(F_1, Y)$$

is exact. $Hom_S(\_, Z)$ is exact since Z is injective, therefore we have a diagram

$$\begin{array}{ccccccc} Hom_S(Y,Z) \otimes_R F_1 & \to & Hom_S(Y,Z) \otimes_R F_0 & \to & Hom_S(Y,Z) \otimes_R X & \to & 0 \\ \downarrow \sigma & & \downarrow \beta & & \downarrow \psi & & \\ Hom_S(Hom_R(F_1,Y),Z) & \to & Hom_S(Hom_R(F_0,Y),Z) & \to & Hom_S(Hom_R(X,Y),Z) & \to & 0 \end{array}$$

where $\sigma(f \otimes a) : g \mapsto f(g(a))$ whenever $g \in Hom_R(F_1, Y)$ and $a \in F_1$, and where β and ψ are defined in the same way. The top row above is exact since the tensor functor is right exact by Theorem 1.12. The first two vertical maps are isomorphisms. Thus, by the Five Lemma, the last vertical map is an isomorphism. □

Proposition 3.5. *Let* $0 \longrightarrow N \longrightarrow M \longrightarrow P \longrightarrow 0$ *be a pure-exact sequence of* R-*modules and* B *a f.p.* R-*module. Then, the sequence*

$0 \longrightarrow Hom_R(B,N) \longrightarrow Hom_R(B,M) \longrightarrow Hom_R(B,P) \longrightarrow 0$ *is pure-exact.*

Proof. The sequence

$$0 \longrightarrow P^* \longrightarrow M^* \longrightarrow N^* \longrightarrow 0$$

is split exact by Proposition 3.3. Thus, the sequence

$$0 \longrightarrow P^* \otimes_R B \longrightarrow M^* \otimes_R B \longrightarrow N^* \otimes_R B \longrightarrow 0$$

is also split exact. By Lemma 3.4, we have a split exact sequence

$$0 \longrightarrow (Hom_R(B,P))^* \longrightarrow (Hom_R(B,M))^* \longrightarrow (Hom_R(B,N))^* \longrightarrow 0.$$

Then,

$$0 \longrightarrow Hom_R(B,N) \longrightarrow Hom_R(B,M) \longrightarrow Hom_R(B,P) \longrightarrow 0$$

is pure-exact by Proposition 3.3. □

Definition. A right R-module B is *flat* if whenever $f : X \longrightarrow Y$ is a monomorphism then so too is $1_B \otimes f : B \otimes X \longrightarrow B \otimes Y$.

A fundamental characterization of flatness is

Lemma 3.6 (J. Lambek [14]). *A right* R-*module* A *is flat if and only if* A^* *is an injective left* R-*module.*

Proof. We give a sketch of the proof by showing that $Hom_R(\_, Hom_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z}))$ is an exact functor. We have $Hom_R(\_, Hom_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z}))$ naturally equivalent to

$Hom_{\mathbb{Z}}(A \otimes_R \_, \mathbb{Q}/\mathbb{Z})$, the latter being the composition $Hom_{\mathbb{Z}}(\_, \mathbb{Q}/\mathbb{Z}) \circ (A \otimes_R \_)$, where each of these two composed functors is exact, since $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module and A is flat. Hence, A^* is an injective R-module.

Conversely, if A^* is an injective R-module and if $f : B' \longrightarrow B$ is monic, then, by Theorem 1.16 (Adjoint Isomorphism), we have

$$Hom_{\mathbb{Z}}(A \otimes_R B, \mathbb{Q}/\mathbb{Z}) \longrightarrow Hom_{\mathbb{Z}}(A \otimes_R B', \mathbb{Q}/\mathbb{Z}) \longrightarrow 0$$

is exact. Hence, $0 \longrightarrow A \otimes_R B' \longrightarrow A \otimes_R B$ is exact since $\mathbb{Q}/\mathbb{Z}$ is a cogenerator for abelian groups, i.e. the right R-module A is flat. $\Box$

An example of a flat R-module is R itself. To see this, consider the following diagram.

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \downarrow\cong & & \downarrow\cong \\ R \otimes M & \xrightarrow{1 \otimes f} & R \otimes N \end{array}$$

where M and N are left R-modules, f is an R-monomorphism and the vertical maps are the natural isomorphisms (Theorem 1.13). Then, $1 \otimes f$ is an R-monomorphism.

In fact, free R-modules, or more generally, projective R-modules are always flat. To see this, consider the following commutative diagram

$$\begin{array}{ccc} (\oplus B_i) \otimes M & \xrightarrow{1_{\oplus B_i} \otimes f} & (\oplus B_i) \otimes N \\ \downarrow\cong & & \downarrow\cong \\ \oplus(B_i \otimes M) & \xrightarrow{\oplus(1_{B_i} \otimes f)} & \oplus(B_i \otimes N) \end{array}$$

where $\{B_i \mid i \in I\}$ is a family of right R-modules, N and M are left R-modules, f is an R-monomorphism and the vertical maps are natural isomorphisms (Theorem 1.15). Now, we can conclude that $\oplus B_i$ is flat if and only if each B_i is flat. Hence, a free R-module is flat being a direct sum of copies of R. Also, a projective R-module is flat since it is a direct summand of some free R-module.

Lemma 3.7. *If B is a right R-module such that $0 \longrightarrow B \otimes_R I \stackrel{1_B \otimes \sigma}{\longrightarrow} B \otimes_R R$ is an exact sequence whenever I is a f.g. left ideal of R and σ is the inclusion $I \longrightarrow R$, then B is flat.*

Proof. Every left ideal I is equal to $\varinjlim I_k$ for some f.g. left ideals I_k's by Theorem 1.17 (this is simply the directed union of the finitely generated subideals of I). Now,

$$0 \longrightarrow B \otimes_R I_k \longrightarrow B \otimes_R R$$

is exact by the hypothesis, and so,

$$0 \longrightarrow \varinjlim(B \otimes_R I_k) \longrightarrow \varinjlim(B \otimes_R R)$$

is an exact sequence by Theorem 1.18. Then,

$$0 \longrightarrow \varinjlim(B \otimes_R I_k) \longrightarrow B \otimes_R R$$

is exact since $\varinjlim(B \otimes_R R) = B \otimes_R R$. Then,

$$0 \longrightarrow B \otimes_R \varinjlim I_k \longrightarrow B \otimes_R R$$

is exact since the functor $B \otimes_R \_$ preserves direct limits (Theorem 1.19). It follows that

$$(B \otimes_R R)^* \longrightarrow (B \otimes_R I)^* \longrightarrow 0$$

is exact which gives the exactness of

$$Hom_R(R, B^*) \longrightarrow Hom_R(I, B^*) \longrightarrow 0$$

by the Adjoint Isomorphism. Hence, B^* is injective by Baer's criterion, and this shows that B is flat by Lemma 3.6. □

An analogue of Lemma 3.6 is

Proposition 3.8 *A right R-module A is flat if and only if $A^*$is an absolutely pure left R-module.*

Proof. If A is a flat right R-module, then $A^*$is an injective left R-module by Lemma 3.6 and hence absolutely pure.

For the converse, let A^* be an absolutely pure left R-module and let I be a f.g. left ideal. Then, consider the exact sequence

$$0 \longrightarrow I \xrightarrow{\sigma} R$$

where σ is the inclusion map. The sequence $Hom_R(R, A^*) \longrightarrow Hom_R(I, A^*) \longrightarrow 0$ is exact by Proposition 2.14 since A^* is an absolutely pure R-module. Now, by Theorem 1.16 (Adjoint Isomorphism), we will have the exact sequence

$$Hom_{\mathbb{Z}}(A \otimes R, \mathbb{Q}/\mathbb{Z}) \longrightarrow Hom_{\mathbb{Z}}(A \otimes I, \mathbb{Q}/\mathbb{Z}) \longrightarrow 0.$$

Then, the sequence

$$0 \longrightarrow A \otimes I \longrightarrow A \otimes R$$

is exact since $\mathbb{Q}/\mathbb{Z}$ is a cogenerator for abelian groups. Therefore, A is a flat right R-module by Lemma 3.7. □

Definition. An R-module is *coflat* if it has the injective property relative to any exact sequence $0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0$ where I is a f.g. ideal of R.

One can consider coflatness as a weak form of absolute purity, in view of Proposition 2.14. Consequently, for each module M we have the following implications

M is injective $\Longrightarrow$ M is absolutely pure $\Longrightarrow$ M is coflat.

Lemma 3.9. *If R is coherent and $0 \longrightarrow N \longrightarrow M \xrightarrow{\alpha} P \longrightarrow 0$ is pure-exact and M is coflat, then P is coflat.*

Proof. Recall that

$$0 \longrightarrow Hom_R(I, N) \longrightarrow Hom_R(I, M) \xrightarrow{\alpha_*} Hom_R(I, P) \longrightarrow 0 \; (*)$$

is exact for all f.p. ideals of R (Proposition 3.5). Thus, the sequence $(*)$ is exact for all f.g. ideals I since R is coherent. Now, let $f : I \longrightarrow P$ be an R-homomorphism, then there is a map $\psi : I \longrightarrow M$ such that $\alpha_*(\psi) = f$ since $(*)$ is an exact sequence. But M is coflat, so there is a map $\lambda : R \longrightarrow M$ which extends ψ , i.e. $\lambda i = \psi$ where i is the inclusion map $I \longrightarrow R$. Let $\phi : R \longrightarrow P$ be defined as $\phi = \alpha\lambda$. Then $\phi i = \alpha\lambda i = \alpha\psi = \alpha_*(\psi) = f$, i.e. ϕ extends f. Hence, P is coflat. □

Proposition 3.10. *Let R be a left coherent ring and let M be a coflat R-module. Then M is absolutely pure.*

Proof. Let I be a f.g. ideal. Now, consider the exact sequence $0 \longrightarrow I \longrightarrow R$, then $Hom_R(R, M) \longrightarrow Hom_R(I, M) \longrightarrow 0$ is exact, since M is coflat. Then,

$0 \longrightarrow (Hom_R(I, M))^* \longrightarrow (Hom_R(R, M))^*$ is exact, and hence by Lemma 3.4, we have the exact sequence $0 \longrightarrow M^* \otimes_R I \longrightarrow M^* \otimes_R R$ since I is f.p. being a f.g. ideal in the coherent ring R. Thus, M^* is flat by Lemma 3.7, and this leads M^{**} to be injective by Lemma 3.6. Now, define a homomorphism $\mu_M : M \longrightarrow M^{**}$ by $(\mu_M(m))(f) = f(m)$ for each $m \in M$ and $f \in M^*$. μ_M is monic since $\mathbb{Q}/\mathbb{Z}$ is a cogenerator. Consider the homomorphisms $(\mu_M)^* : M^{***} \longrightarrow M^*$ defined as $(\mu_M)^*(h) = h\mu_M$ for all $h \in M^{***}$, and $\mu_{M^*} : M^* \longrightarrow M^{***}$ defined in the same way as μ_M. Then, $\mu_{M^*}(g) : M^{**} \longrightarrow \mathbb{Q}/\mathbb{Z}$

where $g \in M^*$ given by $(\mu_{M^*}(g))(f) = f(g)$ for all $f \in M^{**}$. Hence, $((\mu_M)^* \circ \mu_{M^*})(g) = (\mu_M)^*(\mu_{M^*}(g)) = \mu_{M^*}(g) \circ \mu_M$ and, for each $m \in M$, $\{((\mu_M)^* \circ \mu_{M^*})(g)\}(m) = \mu_{M^*}(g)(\mu_M(m)) = \mu_M(m)(g) = g(m)$. Hence, $((\mu_M)^* \circ \mu_{M^*})(g) = g$ and so $(\mu_M)^* \circ \mu_{M^*} = 1_{M^*}$. Thus, M^* is a direct summand of M^{***}, i.e. $0 \longrightarrow \ker(\mu_M)^* \longrightarrow M^{***} \xrightarrow{(\mu_M)^*} M^* \longrightarrow 0$ is split exact. It follows that $0 \longrightarrow M \longrightarrow M^{**} \longrightarrow K \longrightarrow 0$ is pure-exact for some R-module K by Proposition 3.3. Thus, M is pure in M^{**} (an injective R-module). Therefore, M is absolutely pure. □

Definition. If $0 \longrightarrow N \longrightarrow M \longrightarrow P \longrightarrow 0$ is a pure-exact sequence of R-modules then P is said to be *a pure quotient* of M.

Proposition 3.11. *If R is a left coherent ring, then pure quotients of absolutely pure modules are absolutely pure.*

Proof. Let P be a pure quotient of an absolutely pure R-module M i.e. there is a pure-exact sequence

$$0 \longrightarrow N \longrightarrow M \longrightarrow P \longrightarrow 0.$$

M is absolutely pure implies that M is coflat, and so P is coflat by Lemma 3.9. Hence, P is absolutely pure by Proposition 3.10. □

Corollary 3.12. *If R is a left coherent ring and A is a pure submodule of an injective module Q, then the quotient Q/A is an absolutely pure module.*

Proof. Let A be a pure submodule of an injective R-module Q. Then, the sequence

$$0 \longrightarrow A \xrightarrow{\subseteq} Q \longrightarrow Q/A \longrightarrow 0$$

is pure-exact. Hence, by Proposition 3.11, Q/A is absolutely pure being a pure quotient of Q. □

Recall that if $\{F_i, \phi_i^j\}$ is a direct system of R-modules and if $\lambda_j : F_j \longrightarrow \oplus F_i$ is the canonical injection map, then $\oplus F_i/S$, where S is the submodule of $\oplus F_i$ generated by $\lambda_j \phi_i^j(a_i) - \lambda_i(a_i)$ ($a_i \in F_i$ and $i \leq j$), is a direct limit $\varinjlim F_i$.

Lemma 3.13. *With above notation, let $x \in \oplus_{i\in I} F_i$ with $x = (x_i)_{i\in I}$. Then $x \in S$ if and only if there is $k \in I$ with $k \geq i$ for each $i \in$ support(x) and $\sum_i \phi_i^k(x_i) = 0$. Moreover, S is pure in $\oplus_{i\in I} F_i$.*

Proof. Let $x \in S$, then x is a finite R-linear combination of elements $\sigma_1, \sigma_2, ..., \sigma_n$ in S of form $\sigma_t = (..., 0, -a_i^t, 0, ..., 0, \phi_i^j(a_i^t), 0, ...)$ in $\oplus_{i\in I} F_i$. Let $k \in I$ such that $k \geq j$ for all $j \in \bigcup_{t=1}^n$ support(σ_t). Put $\sigma_t = (\sigma_i^t)_{i\in I}$, then $\sum_i \phi_i^k(\sigma_i^t) = \phi_i^k(-a_i^t) + \phi_j^k \phi_i^j(a_i^t) = \phi_i^k(-a_i^t) + \phi_i^k(a_i^t) = 0$. There are $r_1, ..., r_n \in R$ such that $x = r_1\sigma_1 + ... + r_n\sigma_n$, so $\sum_i \phi_i^k(x_i) = \sum_i \phi_i^k(\sum_t r_t \sigma_i^t) = \sum_t r_t(\sum_i \phi_i^k(\sigma_i^t)) = 0$ as required.

For the converse, write $x = (..., x_{i_1}, 0, ..., x_{i_n}, 0, ...)$ (so that support(x) $\subseteq \{i_1, ..., i_n\}$) and let $k \in I$ such that $k \geq i_1, ..., i_n$ with $\sum_{t=1}^n \phi_{i_t}^k(x_{i_t}) = 0$. Then,

$x = (..., 0, x_{i_1}, 0, ..., 0, -\phi_{i_1}^k(x_{i_1}), 0, ...) + ... + (..., 0, x_{i_n}, 0, ..., 0, -\phi_{i_n}^k(x_{i_n}), 0, ...) \in S$.

We next prove that S is a pure submodule of $\underset{i\in I}{\oplus} F_i$. Let $\sum_{k\in K} r_{hk} x_k = a_h$ ($h \in H$) where $a_h \in S$, be a finite system of equations over S, solvable in $\underset{i\in I}{\oplus} F_i$ by $\{m_k\}_{k\in K}$, say. By the first part, there exists for each $h \in H$, $N_h \in I$ such that $N_h \geq j$ for all $j \in$ support(a_h) and $\sum_i \phi_i^{N_h}(a_h(i)) = 0$. Let $T = (\bigcup_{k\in K}$ support(m_k)$) \bigcup \{N_h\}_{h\in H}$ and let $N \geq j$ for all $j \in T$. Then, if we define $\nu_k \in \underset{i\in I}{\oplus} F_i$ by $\nu_k = m_k - \sum_i \lambda_N \phi_i^N(m_k(i))$, we obtain $\nu_k = \sum_i(\lambda_i(m_k(i)) - \lambda_N \phi_i^N(m_k(i)))$, so that $\nu_k \in S$ ($k \in K$). Also, for each

$h \in H$, $\sum\limits_k r_{hk}\nu_k = \sum\limits_k r_{hk}m_k - \sum\limits_k \sum\limits_i \lambda_N \phi_i^N(r_{hk}m_k(i)) = a_h - \sum\limits_i \lambda_N \phi_i^N(\sum\limits_k r_{hk}m_k(i)) = a_h - \sum\limits_i \lambda_N \phi_i^N(a_h(i)) = a_h - \lambda_N \phi_{N_h}^N(\sum\limits_i \phi_i^{N_h}(a_h(i))) = a_h$. This proves that the finite system of equations is solvable in S, and hence S is pure in $\bigoplus\limits_{i\in I} F_i$. $\square$

Lemma 3.14. *For an R-module P, the following statements are equivalent.*

(i) P is f.p.

(ii) For every direct system $\{X_i\}_{i\in I}$ of R-modules and for every R-homomorphism $\alpha : P \longrightarrow \varinjlim X_i$, there exists an R-homomorphism $\beta : P \longrightarrow X_j$ for some $j \in I$ such that $\alpha_j\beta = \alpha$,where $\alpha_j : X_j \longrightarrow \varinjlim X_i$ is the canonical homomorphism.

Proof. (i)$\Longrightarrow$(ii) Using Lemma 3.13 and its notation, we obtain a pure-exact sequence

$$0 \longrightarrow S \longrightarrow \oplus_{i\in I}X_i \xrightarrow{\pi} \varinjlim X_i \longrightarrow 0$$

where $\pi((x_i)_{i\in I}) = (x_i)_{i\in I} + S$. For each $j \in I$, let $\lambda_j : X_i \longrightarrow \oplus_{i\in I}X_i$ be the injection map, so that $\alpha_j = \pi\lambda_j$ is the canonical homomorphism $X_j \longrightarrow \varinjlim X_i$. Let P be f.p. with generating set $\{p_1, p_2, ..., p_n\}$ and let $g : P \longrightarrow \varinjlim X_i$ be an R-homomorphism. By Proposition 3.5, there is an R-homomorphism $h : P \longrightarrow \oplus_{i\in I}X_i$ such that $\pi h = g$. Let $T = \bigcup\limits_{k=1}^{n}$support$(h(p_k))$, $j \geq i$ for all $i \in T$ (this is possible as I is directed), and let $\delta_j : \oplus_{i\in I}X_i \longrightarrow X_j$ be the homomorphism $\delta_j((x_i)_{i\in I}) = \sum\limits_{i\in T} \phi_i^j(x_i)$. We have for all $x \in h(P)$, $\pi\lambda_j\delta_j(x) = \alpha_j\delta_j(x) = \alpha_j(\sum\limits_{i\in T}\phi_i^j(x_i)) = \sum\limits_{i\in I}\alpha_j(x_i) = \pi(\sum\limits_{i\in I}\lambda_i(x_i)) = \pi(x)$, i.e. for all $p \in P$, $\pi\lambda_j\delta_j h(p) = \pi h(p) = g(p)$. This implies $g = \pi\lambda_j\delta_j h$, i.e. the map g can be factored through X_j as the following diagram shows

$$\begin{tikzcd} P \arrow[rr, "g"] \arrow[dr, "\delta_j h"'] & & \varinjlim X_i \\ & X_j \arrow[ur, "\alpha_j"'] & \end{tikzcd}$$

(ii)$\Longrightarrow$(i) By Theorem 1.17, $X = \varinjlim X_i$ for some direct system $\{X_i\}_{i\in I}$ of f.p. R-modules. Let $\alpha : X \longrightarrow X$ be the identity map, then there is an R-homomorphism $\tau : X \longrightarrow X_j$ for some $j \in I$ such that $\alpha_j\tau = \alpha$ where $\alpha_j : X_j \longrightarrow \varinjlim X_i$ is the canonical map. This means that X is a direct summand of X_j, and there exists an R-module B such that $X \oplus B \cong X_j$. Both X and B are f.g. since X_j is f.g. The exact sequence of R-modules $0 \longrightarrow H \xrightarrow{\sigma} F \xrightarrow{\phi} X \longrightarrow 0$ where F is f.g. free, σ is the inclusion and ϕ is the canonical epimorphism, induces the sequence $0 \longrightarrow H \xrightarrow{h} F \oplus B \xrightarrow{f} X \oplus B \longrightarrow 0$, where $h(a) = (a, 0)$ for all $a \in H$ and $f(a, b) = (\phi(a), b)$ for all $(a, b) \in F \oplus B$. It is easy to check that this is an exact sequence. Since $X \oplus B$ is f.p. and $F \oplus B$ is f.g., it follows that H is f.g. This shows that X is f.p. $\square$

Lemma 3.15. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{\pi} Z/T$ be the zero map, where f, g, π are homomorphisms and X, Y, Z are modules, T a submodule of Z, and π the natural epimorphism. Then, the composition gf factors over T.*

Proof. We show that we have a commutative diagram

$$\begin{array}{ccccccc} & & X & \xrightarrow{f} & Y & & \\ & & \tau\downarrow & & \downarrow g & & \\ 0 \longrightarrow & & T & \xrightarrow{\sigma} & Z & \xrightarrow{\pi} & Z/T \longrightarrow 0 \end{array}$$

for some τ, where σ is the inclusion map. Let $\tau : X \longrightarrow T$ be the map defined by $\tau(x) = gf(x)$ for each $x \in X$. We need only check that $gf(x) \in T$. But $\pi gf(x) = 0$, i.e. $gf(x) \in \ker\pi = T$, as required. $\square$

Proposition 3.16 (B. Stenström [21]**).** *A ring R is a left coherent ring if and only if every direct limit of absolutely pure R-modules is absolutely pure.*

Proof. Let R be left coherent and let $\{F_i\}_{i\in I}$ be a direct system of absolutely pure R-modules. By Lemma 3.13 $\varinjlim F_i$ is a pure quotient of the absolutely pure module $\underset{i\in I}{\oplus} F_i$. Therefore, $\varinjlim F_i$ is an absolutely pure R-module by Proposition 3.11.

For the converse, let L be a f.g. submodule of a f.g. free R-module F and let $\alpha : L \longrightarrow \varinjlim M_i$ be a homomorphism where $\left\{M_i, \psi_i^j\right\}$ is a direct system of R-modules. Choose a direct family of injective modules $E(M_i) \supseteq M_i$. Since $\varinjlim E(M_i)$ is absolutely pure, α extends to a homomorphism $\beta : F \longrightarrow \varinjlim E(M_i)$ as the following diagram shows

$$\begin{array}{ccccc} 0 \longrightarrow & L & & \xrightarrow{\sigma} F \\ & \alpha \downarrow & & \beta \downarrow \\ 0 \longrightarrow & \varinjlim M_i & \xrightarrow{\varepsilon} & \varinjlim E(M_i) \end{array}$$

where σ is the inclusion and ε is the map induced by $\varinjlim$ on the inclusions $0 \longrightarrow M_i \longrightarrow E(M_i)$. By Lemma 3.14 β factors over some $E(M_j)$, i.e. there is a map $h : F \longrightarrow E(M_j)$ such that $\alpha_j h = \beta$ (where $\alpha_j : E(M_j) \longrightarrow \varinjlim E(M_i)$ is the canonical map). Consider the diagram $(*)$

$$\begin{array}{ccccccc} 0 \longrightarrow & L \xrightarrow{\sigma} & & F \xrightarrow{h} E(M_j) \xrightarrow{\pi_j} & & E(M_j)/M_j & \\ & \downarrow \alpha & & \searrow \beta \quad \downarrow \alpha_j & & \downarrow f_j & \\ 0 \longrightarrow & \varinjlim M_i \xrightarrow{\varepsilon} & & \varinjlim E(M_i) \xrightarrow{\pi} & & \varinjlim (E(M_i)/M_i) \longrightarrow 0 & \end{array}$$

where the bottom row is exact, and where f_j is induced by the commutative diagram of exact rows

$$\begin{array}{ccccc} 0 \longrightarrow & M_j \xrightarrow{\subseteq} & E(M_j) \xrightarrow{\pi_j} & E(M_j)/M_j \longrightarrow 0 \\ & \downarrow \psi_j & \downarrow \alpha_j & \downarrow f_j \\ 0 \longrightarrow & \varinjlim M_i \xrightarrow{\varepsilon} & \varinjlim E(M_i) \xrightarrow{\pi} & \varinjlim (E(M_i)/M_i) \longrightarrow 0. \end{array}$$

We claim that $f_j \pi_j h\sigma = 0$.This is so because $f_j \pi_j h\sigma = \pi\alpha_j h\sigma = \pi\beta\sigma = \pi\varepsilon\alpha = 0$. This shows that the composition $L \longrightarrow E(M_j) \longrightarrow E(M_j)/M_j$ goes to zero in the direct limit. Since L is f.g. there exists $j' \geqslant j$ in I such that $L \longrightarrow E(M_{j'}) \longrightarrow E(M_{j'})/M_{j'}$ is the zero map by Theorem 1.20. Then, $L \longrightarrow E(M_{j'})$ factors over $M_{j'}$ by Lemma 3.15. Without loss of generality, we may assume that $j = j'$, so that the above diagram $(*)$ together with this factoring property becomes

$$\begin{tikzcd}
 & & M_j \arrow[drr, "\subseteq"] & & \\
0 \arrow[r] & L \arrow[ur, "\tau"] \arrow[r, "\sigma"] \arrow[d, "\alpha"'] & F \arrow[rr, "h"] \arrow[d, "\beta"] & & E(M_j) \arrow[dll, "\alpha_j"] \\
0 \arrow[r] & \varinjlim M_i \arrow[r, "\varepsilon"] & \varinjlim E(M_i) & &
\end{tikzcd}$$

for some map $\tau : L \longrightarrow M_j$. It is clear that $\varepsilon\alpha = \beta\sigma = \alpha_j h\sigma = \alpha_j\tau = \varepsilon\psi_j\tau$ and so $\alpha = \psi_j\tau$ since ε is monic, which shows that $\alpha : L \longrightarrow \varinjlim M_i$ factors over M_j. Hence, L is f.p. by Lemma 3.14. $\square$

Remark. If B is a right R-module and I is a left ideal, then

$BI = \{\sum b_j i_j \mid b_j \in B,\ i_j \in I\}$ is a subgroup of B. If B is flat then the natural map $f : B \otimes I \longrightarrow B \otimes R$ is monic. So, the composition $B \otimes I \xrightarrow{f} B \otimes R \xrightarrow{g} B$, where g is the natural isomorphism, is monic. Clearly im$gf = BI$, so $b \otimes i \longmapsto bi$ is an isomorphism.

Lemma 3.17. *A module V is flat if and only if for every relation*

$$\sum_{j=1}^{n} v_j a_j = 0 \ (v_j \in V, a_j \in R),$$

there exist elements $u_1, u_2, ..., u_m \in V$ and elements $c_{ij} \in R$ $(i = 1, ..., m,\ j = 1, ..., n)$ such that

$$\sum_{j=1}^{n} c_{ij} a_j = 0 \ (i = 1, ..., m)$$

and

$$\sum_{i=1}^{m} u_i c_{ij} = v_j \ (j = 1, ..., n)$$

Proof. Suppose that V is flat and $\sum_{j=1}^{n} v_j a_j = 0$. Let $I = \sum_{j=1}^{n} Ra_j$ (f.g. left ideal) and consider the free left R-module $F = \oplus_{j=1}^{n} Rx_j$ with basis $\{x_1, ..., x_n\}$ and the short exact sequence

$$0 \longrightarrow K \xrightarrow{\sigma} F \xrightarrow{f} I \longrightarrow 0$$

where $f(x_j) = a_j$ $(j = 1, ..., n)$, $K = \text{ker } f$, and σ is the inclusion map. Then, $(1 \otimes f)(\sum_{j=1}^{n} v_j \otimes x_j) = \sum_{j=1}^{n} v_j \otimes f(x_j) = \sum_{j=1}^{n} v_j \otimes a_j = 0$ since $\sum_{j=1}^{n} v_j a_j = 0$ by hypothesis and the isomorphism given in the above remark. Now, the sequence

$$0 \longrightarrow V \otimes K \xrightarrow{1\otimes\sigma} V \otimes F \xrightarrow{1\otimes f} V \otimes I \longrightarrow 0$$

is exact since V is flat. But $\sum_j v_j \otimes x_j \in \text{ker } (1 \otimes f) = \text{im } (1 \otimes \sigma)$. Thus, there exist $u_i \in V$ and $k_i \in K$ $(1 \leq i \leq m)$ such that $\sum_{i=1}^{m} u_i \otimes k_i$ =$\sum_{j=1}^{n} v_j \otimes x_j$. For each $k_i \in F$, $k_i = \sum_{j=1}^{n} c_{ij} x_j$ $(i = 1, ..., m)$ for some $c_{ij} \in R$, so that

$$\sum_{j=1}^{n} c_{ij} a_j = \sum_{j=1}^{n} c_{ij} f(x_j) = f(\sum_{j=1}^{n} c_{ij} x_j) = f(k_i) = 0 \ (i = 1, ..., m).$$

Moreover, this also gives

$$\sum_{j=1}^{n} (v_j \otimes x_j) = \sum_{i=1}^{m} (u_i \otimes k_i) = \sum_{i=1}^{m} (u_i \otimes (\sum_{j=1}^{n} c_{ij} x_j)) = \sum_{j=1}^{n} ((\sum_{i=1}^{m} u_i c_{ij}) \otimes x_j).$$

Since the x_j 's form a basis of F, and $\sum_{j=1}^{n} v_j \otimes x_j \in V \otimes F$, this implies $\sum_{i=1}^{m} u_i c_{ij} = v_j$ $(j = 1, ..., n)$.

For the converse, consider the following commutative diagram

$$\begin{array}{ccc} V\otimes I & \stackrel{1\otimes\sigma}{\longrightarrow} & V\otimes R \\ \downarrow\phi & & \downarrow\psi \\ VI & \stackrel{\alpha}{\longrightarrow} & V \end{array}$$

where I is a f.g. left ideal of R, σ is the inclusion $I \longrightarrow R$, ψ is the natural isomorphism, α is the inclusion and the map ϕ is defined as $\phi(\sum_j v_j \otimes a_j) = \sum_j v_j a_j$ $(j = 1, ..., n)$. Now, let $\phi(\sum_j v_j \otimes a_j) = 0$ then $\sum_j v_j a_j = 0$, and so, by hypothesis, there exist $u_i \in V$ and $c_{ij} \in R$ such that $\sum_{j=1}^n c_{ij} a_j = 0$ $(i = 1, ..., m)$, and $\sum_{i=1}^m u_i c_{ij} = v_j$ $(j = 1, ..., n)$. Thus, $\sum_{j=1}^n v_j \otimes a_j = \sum_{j=1}^n (\sum_{i=1}^m u_i c_{ij}) \otimes a_j = \sum_{i=1}^m u_i \otimes (\sum_{j=1}^n c_{ij} a_j) = 0$ and hence ϕ is monic. Therefore, $\psi^{-1}\alpha\phi$ is monic and so V is flat by Lemma 3.7. $\square$

Lemma 3.18. *Let F be flat and $0 \longrightarrow K \longrightarrow F \stackrel{f}{\longrightarrow} B \longrightarrow 0$ be an exact sequence of right R-modules where $K = \ker f$. Then, B is flat if and only if $K \cap FI = KI$ for every f.g. left ideal I.*

Proof. Consider the following commutative diagram

$$\begin{array}{ccccccc} & K\otimes I & \longrightarrow & F\otimes I & \stackrel{f\otimes 1_I}{\longrightarrow} & B\otimes I & \longrightarrow 0 \\ & \mu\downarrow & & \phi\downarrow & & \psi\downarrow & \\ 0\longrightarrow & K\cap FI & \stackrel{\subseteq}{\longrightarrow} & FI & \stackrel{f|_{FI}}{\longrightarrow} & BI & \end{array}$$

where ψ is defined by $\sum_{j=1}^n b_j \otimes a_j \mapsto \sum_{j=1}^n b_j a_j$, for $b_j \in B$ and $a_j \in I$ and where ϕ and μ are defined similarly (note that $KI \subseteq K \cap FI$). The two rows are easily verified to be exact. Using diagram chasing one can prove that ψ is monic iff μ is epic iff $KI = K \cap FI$. Suppose that B is flat, then ψ is an isomorphism (this can be shown in the same way as ϕ in the proof of Lemma 3.17). By the above part $KI = K \cap FI$.

For the converse, let $KI = K \cap FI$ for all f.g. left ideals of R. Then, consider the following commutative diagram

$$\begin{array}{ccc} B \otimes I & \stackrel{1_B \otimes i}{\longrightarrow} & B \otimes R \\ \downarrow \psi & & \downarrow \sigma \\ BI & \stackrel{\subseteq}{\longrightarrow} & B \end{array}$$

where i is the inclusion map, σ is the natural isomorphism, and ψ is defined as above. μ is epic by the first part and therefore ψ is monic. Hence, $1_B \otimes i : B \otimes I \longrightarrow B \otimes R$ is monic and this leads B to be flat by Lemma 3.7. □

In Proposition 2.6, an R-module N is absolutely pure module if and only if every short exact sequence with first term N is pure-exact. There is a dual result of this for flat modules in the following sense.

An R-module F is flat if and only if every short exact sequence with third term F is pure.

To see this, let F be a flat R-module and $0 \longrightarrow A \longrightarrow B \longrightarrow F \longrightarrow 0$ $(*)$ be an exact sequence of R-modules. Then, F^* is injective by Lemma 3.6. Thus, the dual exact sequence $0 \longrightarrow F^* \longrightarrow B^* \longrightarrow A^* \longrightarrow 0$ is split by Theorem 1.9 and hence the sequence $(*)$ is pure-exact by Proposition 3.3.

For the converse, let $0 \longrightarrow A \longrightarrow B \longrightarrow F \longrightarrow 0$ be an exact sequence where B is a free R-module. Then, by hypothesis, this sequence is pure and so the dual sequence $0 \longrightarrow F^* \longrightarrow B^* \longrightarrow A^* \longrightarrow 0$ splits by Proposition 3.3. By Lemma 3.6, B^* is injective and hence F^* is injective, and so F is flat by Lemma 3.6 again.

Proposition 3.19 (S. U. Chase [2]). *The following statements are equivalent:*

(i) The ring R is left coherent.

(ii) Every direct product of flat right R-modules is flat.

(iii) R^A is flat (as a right R-module) for any set A.

Proof. (i)$\Longrightarrow$(ii) First let us prove that $(R^{(B)})^A$ is flat for all sets A and B. Let $v_j \in (R^{(B)})^A$ and $a_j \in R$ with $\sum_{j=1}^n v_j a_j = 0$. Let F be the free left R-module with basis $x_1, x_2, ..., x_n$. Then,

$$0 \longrightarrow \text{ker}f \longrightarrow F \xrightarrow{f} \sum_{j=1}^{n} Ra_j \longrightarrow 0$$

is an exact sequence where $f(x_j) = a_j$ $(j = 1, ..., n\ )$ and $\sum_{j=1}^n Ra_j$ is a left ideal with generators $a_1, ..., a_n$. Since R is left coherent we can write $K = \text{ker}f = \sum_{i=1}^m Rk_i$ where $k_i = \sum_{j=1}^n c_{ij}x_j$ and $\sum_{j=1}^n c_{ij}a_j = \sum_{j=1}^n c_{ij}f(x_j) = f(\sum_{j=1}^n c_{ij}x_j) = f(k_i) = 0$. Now, observe that for $\alpha \in A$ and $\beta \in B$ we have $\sum_{j=1}^n [v_j(\alpha)](\beta)x_j \in K$ (since $f(\sum_{j=1}^n [v_j(\alpha)](\beta)x_j) = \sum_{j=1}^n [v_j(\alpha)](\beta)f(x_j) = \sum_{j=1}^n [v_j(\alpha)](\beta)a_j =$ $\sum_{j=1}^n [v_j a_j(\alpha)](\beta) = [(\sum_{j=1}^n v_j a_j)(\alpha)](\beta) = 0$, where $v_j(\alpha)$ is the α coordinate of ν_j and $[v_j(\alpha)](\beta)$ is the β coordinate of $v_j(\alpha)$). Thus, we may choose $b_{i\alpha\beta} \in R$ such that $b_{i\alpha\beta} = 0$ whenever $[v_j(\alpha)](\beta) = 0$ $(j = 1, ..., n)$ and $\sum_{j=1}^n [v_j(\alpha)](\beta)x_j = \sum_{i=1}^m b_{i\alpha\beta}k_i$ since the $k_i's$ are the generators of K. Now, define $u_1, ..., u_m \in (R^{(B)})^A$ by $[u_i(\alpha)](\beta) = b_{i\alpha\beta}$ $(\alpha \in A, \beta \in B,\ i = 1, ..., m)$ so that $\sum_{j=1}^n [v_j(\alpha)](\beta)x_j = \sum_{i=1}^m [u_i(\alpha)](\beta)k_i = \sum_{i=1}^m [u_i(\alpha)](\beta)(\sum_{j=1}^n c_{ij}x_j) = \sum_{j=1}^n (\sum_{i=1}^m [u_i(\alpha)](\beta)c_{ij})x_j$ or, equivalently,

$$\sum_{i=1}^{m} u_i c_{ij} = v_j \ (j = 1, ..., n)$$

Thus, $(R^{(B)})^A$ is flat by Lemma 3.17. Now, suppose that $\{V_\alpha\}_{\alpha \in A}$ is a family of flat modules and choose B to be a set such that the free right modules $F_\alpha = R^{(B)}$ map onto V_α $(\alpha \in A)$. Then,

$$0 \longrightarrow K_\alpha \xrightarrow{\subseteq} F_\alpha \xrightarrow{g_\alpha} V_\alpha \longrightarrow 0$$

is exact for all $\alpha \in A$. Then, we have an exact sequence

$$0 \longrightarrow \prod K_\alpha \longrightarrow \prod F_\alpha \longrightarrow \prod V_\alpha \longrightarrow 0$$

where $\prod_A F_\alpha = (R^{(B)})^A$ is flat. Now, let I be a f.g. left ideal in R. Then,

$$(\prod_A K_\alpha)I = \prod_A (K_\alpha I) = \prod_A (K_\alpha \cap F_\alpha I) = (\prod_A K_\alpha) \cap (\prod_A (F_\alpha I)) = (\prod_A K_\alpha) \cap (\prod_A F_\alpha)I$$

since $K_\alpha I = K_\alpha \cap F_\alpha I$ by Lemma 3.18. Then, $\prod_A V_\alpha$ is flat again by Lemma 3.18.

(ii)$\Longrightarrow$(iii) is obvious.

(iii)$\Longrightarrow$(i) Suppose I is a f.g. left ideal of R. Then, we have an exact sequence $0 \longrightarrow \ker f \longrightarrow F \xrightarrow{f} I \longrightarrow 0$ where F is a free module with a basis $\{x_1, ..., x_n\}$, say. Let $a_j = f(x_j)$ $(j = 1, ..., n)$ and let $K = \ker f$. Now, define $v_1, v_2, ..., v_n \in R^K$ by $k = \pi_k(v_1)x_1 + \pi_k(v_2)x_2 + ... + \pi_k(v_n)x_n$ $(k \in K)$, where π_k is the canonical projection. Then, $0 = f(k) = \sum_{j=1}^n \pi_k(v_j)f(x_j) = \sum_{j=1}^n \pi_k(v_j)a_j = \pi_k(\sum_{j=1}^n v_j a_j)$ for all $k \in K$, and so $\sum_{j=1}^n v_j a_j = 0$. By the hypothesis R^K is flat since R is flat and hence by Lemma 3.17 there exist $u_1, ..., u_m \in R^K$ and $c_{ij} \in R$ with $\sum_{j=1}^n c_{ij}a_j = 0$ $(i = 1, ..., m)$ and $\sum_{i=1}^m u_i c_{ij} = v_j$ $(j = 1, ..., n)$. Define $k_i = \sum_{j=1}^n c_{ij}x_j$ ($i = 1, ..., m)$, then $f(k_i) = f(\sum_{j=1}^n c_{ij}x_j) = \sum_{j=1}^n c_{ij}f(x_j) = \sum_{j=1}^n c_{ij}a_j = 0$ and hence $k_i \in K = \ker f$. Now, $k = \sum_{j=1}^n \pi_k(v_j)x_j = \sum_{j=1}^n \pi_k(\sum_{i=1}^m u_i c_{ij})x_j = \sum_{i=1}^m \sum_{j=1}^n \pi_k(u_i)c_{ij}x_j = \sum_{i=1}^m \pi_k(u_i)(\sum_{j=1}^n c_{ij}x_j) = \sum_{i=1}^m \pi_k(u_i)k_i$. Thus, K is f.g. and hence R is left coherent. $\square$

We say that a module A is *simple* if $A \neq 0$ and A has no proper submodules and that A is *semisimple* if it is a direct sum of simple modules. A ring R is *left semisimple* if, as a left R-module, it is semisimple.

For a semisimple ring, we have the following characterization:

A ring R is semisimple if and only if every left R-module is injective (see J. Rotman [20]). Since absolute purity is weaker than injectivity, a natural question arises: What are the rings all of whose modules are absolutely pure? To answer this, we state the following

Definition. A ring R is *von Neumann regular* if, for each $a \in R$, there is an element $a' \in R$ with $aa^{'}a = a$.

Lemma 3.20. *If R is von Neumann regular then every f.g. left ideal is principal, generated by an idempotent (i.e. an element e with $e^2 = e$)*

Proof. First we show that every principal left ideal is generated by an idempotent. Let $a \in R$, then there is an element $a' \in R$ such that $aa'a = a$, and so $a'aa'a = a'a$. It follows that $e = a'a$ is an idempotent. Moreover, $Ra = Re$ since $a = aa'a \in Re$ and $e = a'a \in Ra$. Now, without loss of generality, to show that every f.g. ideal is principal, it suffices to show that $Ra + Rb$ is principal for any a, $b \in R$. Let $e = a'a$, where $aa'a = a$. Then, $Ra = Re$ and $b = be+b(1-e) \in Re+Rb(1-e)$, so $Ra+Rb \subseteq Re+Rb(1-e)$. For the other inclusion, let $x \in Re+Rb(1-e)$ then, $x = re+sb(1-e)$ for some $r, s \in R$. Thus, $x = ta+sb(1-e)$ for some $t \in R$ since $Re = Ra$ and so $x = ta+sb-sbe = ta+sb+\nu a$ for some $\nu \in R$ since $Re = Ra$. Therefore, $x = (t + \nu)a + sb \in Ra + Rb$. Now, there is an idempotent f such that $Rb(1 - e) = Rf$ since $Rb(1 - e)$ is principal; moreover, $f = r_1b(1 - e)$ for some $r_1 \in R$. It follows that $fe = r_1b(1 - e)e = 0$. Now define $g = (1 - e)f$. Then, $ge = 0 = eg$, and $g^2 = (1 - e)f(1 - e)f = (1 - e)(f - fe)f = (1 - e)f^2 = (1 - e)f = g$ so that g is an idempotent. Also, $Rg \subseteq Rf$ and $Rf \subseteq Rg$ (since $f = r_1b(1-e)$ implies $f^2 = r_1b(1-e)f$, i.e $f = f^2 = r_1bg \in Rg$). Therefore, $Ra+Rb = Re + Rb(1 - e) = Re + Rf = Re + Rg$. It is clear that $Re + Rg \supseteq R(e + g)$ and that for all $v, u \in R$, $ve + ug = ve^2 + ug^2 = (ve + ug)(e + g) \in R(e + g)$ as $ge = 0 = eg$. Consequently, $Ra + Rb = R(e + g)$. □

We note that every division ring and, more generally, every semisimple ring is von Neumann regular.

Proposition 3.21. *A ring R is von Neumann regular if and only if every right R-module is flat.*

Proof. Let B be a right R-module and $0 \longrightarrow K \xrightarrow{\subseteq} F \longrightarrow B \longrightarrow 0$ be an exact sequence, where F is free. Then it is enough to show that $Ka \supseteq K \cap Fa$ since the other inclusion is obvious. Let $x \in K \cap Fa$ then $x = k = fa = faa^{'}a = ka^{'}a \in Ka$ for some $k \in K$, $a' \in R$, and $f \in F$. Therefore, B is flat by Lemma 3.18.

Conversely, let $0 \longrightarrow aR \xrightarrow{\subseteq} R \longrightarrow R/aR \longrightarrow 0$ be an exact sequence. Then, by hypothesis R/aR is a flat right R-module, and for any f.g left ideal I, $(aR)I = aR \cap RI = aR \cap I$. Thus, $aI = aR \cap I$. In particular, let $I = Ra$. Then $aRa = aR \cap Ra$, but since $a \in aR \cap Ra$, we get $a \in aRa$, i.e. there is $a' \in R$ such that $aa'a = a$. Hence, R is von Neumann regular. □

Proposition 3.22. *A ring R is von Neumann regular if and only if every right R-module is absolutely pure.*

Proof. Let A be an R-module. Then, by Proposition 3.21, A^* is flat and so A^{**} is an injective R-module by Lemma 3.6. In Proposition 3.10, we showed that A is pure in $A^{**}$and hence A is absolutely pure by Proposition 2.7.

Conversely, let $f : A' \longrightarrow A$ be a monic map. Then, the short exact sequence

$$0 \longrightarrow A' \xrightarrow{f} A \longrightarrow A \diagup \mathrm{im}\, f \longrightarrow 0,$$

is pure-exact by Proposition 2.6 since A' is absolutely pure by hypothesis. Thus, the sequence

$$0 \longrightarrow B \otimes_R A' \xrightarrow{1_B \otimes f} B \otimes_R A \longrightarrow B \otimes_R (A \diagup \mathrm{im}\, f) \longrightarrow 0,$$

is exact for all right R-modules B . In particular, the map $1_B \otimes f$ is monic, i.e. B is a flat R-module. Therefore, R is von Neumann regular by Proposition 3.21. □

The following proposition gives an important relationship between flatness and absolute purity.

Proposition 3.23. *Let A be a module over a coherent ring R. Then, the character module A^* is flat if and only if A is absolutely pure.*

Proof. Let A^* be a flat R-module. Then, A^{**} is injective by Proposition 3.6. In Proposition 3.10, we have A pure in A^{**} and hence A is absolutely pure.

Conversely, let I be a f.g. left ideal and consider the exact sequence $0 \longrightarrow I \xrightarrow{\subseteq} R$. Then,

$$Hom_R(R, A) \longrightarrow Hom_R(I, A) \longrightarrow 0$$

is an exact sequence with A absolutely pure by Proposition 2.14, and so

$$0 \longrightarrow (Hom_R(I, A))^* \longrightarrow (Hom_R(R, A))^*$$

is exact. Now, I is f.p. since R is a left coherent ring and so we have an exact sequence $0 \longrightarrow A^* \otimes_R I \longrightarrow A^* \otimes_R R$, by Lemma 3.4. Thus, A^* is flat by Lemma 3.7. □

One implication of Proposition 3.23 can be generalized as follows for commutative rings.

Proposition 3.24. *Let A be an absolutely pure module over a commutative coherent ring R. Then, $Hom_R(A, H)$ is flat for every injective R-module H.*

Proof. First note that $Hom_R(A, H)$ has a structure of an R-module. Let I be a f.g. ideal of R and let A be an absolutely pure R-module. Then, the exact sequence

$$0 \longrightarrow I \xrightarrow{\subseteq} R$$

induces an exact sequence

$$Hom_R(R, A) \longrightarrow Hom_R(I, A) \longrightarrow 0$$

by Proposition 3.17. Now, $Hom_R(\_, H)$ is a left exact functor and so, the sequence

$$0 \longrightarrow Hom_R(Hom_R(I, A), H) \longrightarrow Hom_R(Hom_R(R, A), H)$$

is exact. I is f.p. (as R is coherent), so we have the exact sequence

$$0 \longrightarrow Hom_R(A, H) \otimes_R I \longrightarrow Hom_R(A, H) \otimes_R R$$

by Lemma 3.4. Hence, $Hom_R(A, H)$ is flat by Lemma 3.7. □

Proposition 3.25. *If R is a commutative coherent ring, then $E \otimes_R A$ is absolutely pure whenever E is flat and A is absolutely pure.*

Proof. Observe that $E \otimes_R A$ has the structure of an R-module. By Theorem 1.16 (Adjoint Isomorphism), we have the following identity:

$$(E \otimes_R A)^* = Hom_{\mathbb{Z}}(E \otimes_R A, \mathbb{Q}/\mathbb{Z}) \cong Hom_R(A, Hom_{\mathbb{Z}}(E, \mathbb{Q}/\mathbb{Z})) = Hom_R(A, E^*).$$

Now, E is flat implies that E^* is injective by Lemma 3.6. Hence, $Hom_R(A, E^*)$ is flat by Proposition 3.24 since R is coherent. So, $(E \otimes_R A)^*$ is flat by using the above identity. Therefore, $E \otimes_R A$ is absolutely pure by Proposition 3.23. □

3.2 Semihereditary Rings

It is well-known that a ring is hereditary if and only if quotients of injective modules are injective. The question as to which rings have quotients of absolutely pure modules absolutely pure is therefore worthwhile. To discuss this, we begin with the following

Definition. A ring R is *left semihereditary* if every f.g. left ideal is projective. A commutative semihereditary domain is called a *Prüfer ring.*

We remark that every von Neumann regular is left and right semihereditary. Also, left semihereditary rings are left coherent since f.g. projectives are always f.p.

Lemma 3.26. *If R is left semihereditary, then every f.g. submodule A of a free R-module F is isomorphic to a direct sum of finitely many f.g. left ideals.*

Proof. Let F have basis $\{x_k \mid k \in K\}$ and let $A \subseteq F = \underset{k \in K}{\oplus} Rx_k$. Now, if $A = \sum_{i=1}^{m} Ra_i$, then each a_i is a linear combination of finitely many x_k's. We may therefore assume that F is free with basis $\{x_1, x_2, ..., x_n\}$. We use an induction on n. If $n = 1$, F is isomorphic to R and hence A is isomorphic to a f.g. ideal of R. Now, for $n > 1$, let us assume that every module contained in $Rx_1 \oplus ... \oplus Rx_{n-1}$ is isomorphic to a direct sum of finitely many f.g. left ideals. Let A be a f.g. submodule of $Rx_1 \oplus ... \oplus Rx_n$ and let $B = A \cap (Rx_1 \oplus ... \oplus Rx_{n-1})$. For each $a \in A$ we have unique $b \in Rx_1 \oplus ... \oplus Rx_{n-1}$ and $r \in R$ such that $a = b + rx_n$. Let $\phi : A \longrightarrow R$ be a map given by $a \longmapsto r$, then ϕ is a well-defined R-homomorphism (by uniqueness of $b \in Rx_1 \oplus ... \oplus Rx_{n-1}$ and $r \in R$ for each a in A). Then, we have the following exact sequence

$$0 \longrightarrow B \xrightarrow{\subseteq} A \xrightarrow{\phi} I \longrightarrow 0,$$

where $I = \mathrm{im}\phi$ is a f.g. ideal of R, since I is a homomorphic image of a f.g. R-module. I is projective since R is semihereditary. Thus, the sequence splits, and $A \cong B \oplus I$. Now, $B \subseteq Rx_1 \oplus ... \oplus Rx_{n-1}$ and so by the inductive hypothesis, B is isomorphic to a direct sum of finitely many f.g. ideals. Hence, A is also isomorphic to a direct sum of finitely many f.g. ideals. □

Lemma 3.27. *A ring R is left semihereditary if and only if every f.g. submodule of a projective module is projective.*

Proof. Let A be a f.g. submodule of a projective module P. Then, $F \cong P \oplus K$ for some free R-module F since P is projective (using Theorem 1.5). Thus, we may assume that A is a f.g. submodule of a free module. By Lemma 3.26, A is a finite direct sum of f.g. ideals, each of which is projective since R is semihereditary. Hence, A is projective.

Conversely, if I is f.g. left ideal, then I is projective since R itself is projective. □

Lemma 3.28. *A module P is projective if and only if every diagram*

$$\begin{array}{ccccc} & & P & & \\ & & \downarrow & & \\ Q & \longrightarrow & Q'' & \longrightarrow & 0 \end{array}$$

with Q injective can be completed to a commutative diagram.

Proof. If P is projective, then the diagram can clearly be completed.

For the converse, consider the diagram

$$\begin{array}{ccccc} & & P & & \\ & & \gamma \downarrow & & \\ A & \stackrel{\tau}{\longrightarrow} & A'' & \longrightarrow & 0 \end{array}$$

where the row is an exact sequence. Our desire is to show that there is a homomorphism $P \longrightarrow A$ making the diagram commute. Let Q be an injective envelope of A. Then, we can extend the diagram into

$$\begin{array}{ccccccccc} & & & & & & P & & \\ & & & & & & \gamma \downarrow & & \\ 0 \longrightarrow & & A' & \stackrel{i}{\longrightarrow} & A & \stackrel{\tau}{\longrightarrow} & A'' & \longrightarrow & 0 \\ & & \downarrow 1 & & \downarrow \sigma & & \downarrow \rho & & \\ 0 \longrightarrow & & A' & \stackrel{\sigma i}{\longrightarrow} & Q & \stackrel{\pi}{\longrightarrow} & Q'' & \longrightarrow & 0 \end{array}$$

where i and σ are inclusion maps, $Q'' = \text{coker } \sigma i$, π is the natural map, and ρ exists by diagram chasing as follows: let $a'' \in A''$ then there is $a \in A$ such that $\tau(a) = a''$ and so $\pi\sigma(a) \in Q''$. Define $\rho : A'' \longrightarrow Q''$ by $a'' \mapsto \pi\sigma(a)$, where $\tau(a) = a''$. It can be shown that ρ is a well-defined homomorphism. Consider the homomorphism $\rho\gamma : P \longrightarrow Q''$. Then, by hypothesis there is a homomorphism $\psi : P \longrightarrow Q$. Now, it can be shown that im$\psi \subseteq$ imσ, which leads to have a homomorphism $P \longrightarrow A$. Therefore, P is projective. □

Recall that a ring R is left *hereditary* if every left ideal is projective. A characterization of these rings (due to Cartan and Eilenberg) is the following

A ring R is left hereditary if and only if every submodule of a projective module is projective if and only if every quotient of an injective module is injective.

An analogue of this is

Proposition 3.29 (C. Megibben [18]). *The following statements are equivalent:*

(i) R is left semihereditary.

(ii) Every f.g. submodule of a projective module is projective.

(iii) Every quotient of an absolutely pure module is absolutely pure.

Proof. (i)$\Longleftrightarrow$(ii) follows by Lemma 3.27.

(i)$\Longrightarrow$ (iii) Suppose that R is left semihereditary. Then, consider the diagram

$$\begin{array}{ccccc} 0 \longrightarrow & I & \stackrel{\sigma}{\longrightarrow} R \\ & \downarrow f & \\ M \stackrel{h}{\longrightarrow} & M'' & \longrightarrow 0 \end{array}$$

where the rows are exact sequences of left R-modules, σ is the inclusion map, and M is an absolutely pure module. Let I be a f.g. left ideal and let $f : I \longrightarrow M''$ be an R-homomorphism. Then, I is projective since R is semihereditary and so there is a homomorphism $g : I \longrightarrow M$ such that $hg = f$. But M is absolutely pure and so there is an extension $\phi : R \longrightarrow M$ such that $\phi\sigma = g$. Thus, $h\phi : R \longrightarrow M''$ is a homomorphism such that $h\phi\sigma = hg = f$. Therefore, M'' is coflat and hence it is absolutely pure by Proposition 3.10 since every left semihereditary ring is left coherent.

(iii)$\Longrightarrow$(i) Let I be a f.g. left ideal. Then, consider the diagram

$$\begin{array}{ccccc} 0 \longrightarrow & I & \stackrel{\sigma}{\longrightarrow} R \\ & \downarrow f & \\ Q \stackrel{h}{\longrightarrow} & Q'' & \longrightarrow 0 \end{array}$$

where the rows are exact sequences and σ is the inclusion map. If Q is injective then it is absolutely pure by Corollary 2.9, and so Q'' is absolutely pure by hypothesis. Hence, there is an extension $\psi : R \longrightarrow Q''$ with $\psi\sigma = f$ by Proposition 2.14. But R is projective and hence there is a homomorphism $\phi : R \longrightarrow Q$ such that $h\phi = \psi$. Thus, there is a homomorphism $\phi\sigma : I \longrightarrow Q$ satisfying $h\phi\sigma = \psi\sigma = f$. Therefore, I is projective by Lemma 3.28. Hence, R is left semihereditary. □

We now end this section by proving further characterizations of semihereditary rings using absolute purity. (See L. Fuchs and L. Salce [11].)

Proposition 3.30. *For any ring R, the following statements are equivalent:*

(i) Every module contains a unique maximal absolutely pure submodule.

(ii) The sum of any family of absolutely pure submodules of an R-module is absolutely pure.

(iii) The quotient of an absolutely pure module is absolutely pure.

(iv) R is a left semihereditary ring.

Proof. (i)$\Longrightarrow$ (ii) Let $\{M_i\}$ be the family of all a. p. submodules of an R-module M. Then, $\sum M_i$ has a unique maximal absolutely pure submodule M_0 by hypothesis. Fix $j \in I$ and let $\mathcal{F}$ be the family of all a.p. submodules of $\sum M_i$ containing M_j and partial order it by inclusion. $\mathcal{F} \neq \phi$ and if $\mathcal{C}$ is a chain $\{X_t\}_{t\in T}$ of modules in $\mathcal{F}$, then $\bigcup\limits_{t\in T} X_t$ is a submodule of $\sum M_i$ containing M_j and also a.p., so $\bigcup\limits_{t\in T} X_t$ is an upper bound for $\mathcal{C}$ in $\mathcal{F}$. By Zorn's Lemma, $\mathcal{F}$ has a maximal member M'. So by uniqueness $M' = M_0$. This means $M_j \subseteq M_0$. Similarly all $M_i \subseteq M_0$, so $\sum M_i \subseteq M_0$ i.e. $M_0 = \sum M_i$.

(ii)$\Longrightarrow$ (i) Let M be an R-module and let $\{M_i\}$ be the family of all absolutely pure submodules of M then ($\{M_i\}, \subseteq$) is a partially ordered set. Let $S = \{M_j\}$ be a chain of absolutely pure submodules of M. Then this chain has $\bigcup\limits_{j} M_j$ as an upper bound in $\{M_i\}$. Thus, $\{M_i\}$ contains a maximal element by Zorn's Lemma. Now let us show the uniqueness, let M_1 and M_2 be two maximal absolutely pure submodules of M. Then, $M_1 + M_2$ is an absolutely pure submodule contained in M and containing M_1. By maximality of M_1, we have $M_1 + M_2 = M_1$. Similarly, $M_1 + M_2 = M_2$. Thus, $M_1 = M_2$.

(ii)$\Longrightarrow$ (iii) Let M be an absolutely pure module, N a submodule of M. Define $H = M \oplus M$, $K = \{(n, n) \mid n \in N\}$. Then, K is a submodule of H. Let $M_1 = \{(x, 0) + K \mid x \in M\}$, $M_2 = \{(0, y) + K \mid y \in M\}$, then M_1, M_2 are submodules of H/K. Now, it can be shown that $H/K = M_1 + M_2$ and $M_1 \cap M_2 = \{(x, 0) + K \mid x \in N\}$. Now, define $f : M \longrightarrow M_1$ by $f(x) = (x, 0) + K$. Then, f is an isomorphism: it is clear that f is epic, f is monic because if $f(x) = 0$ then $(x, 0) + K = 0_{M_1}$ implies $(x, 0) \in K$ leads to have $x = 0$. Therefore, $M \cong M_1$ and similarly $M \cong M_2$. Hence, both M_1 and M_2 are absolutely pure modules. Also, $f(N) = \{(x, 0) + K \mid x \in N\} = M_1 \cap M_2 \cong N$. H/K is absolutely pure being the sum of the two absolutely pure modules M_1 and M_2. Also, $M \cong M_2$ and $N \cong M_1 \cap M_2$. We have therefore $M/N \cong M_2/(M_1 \cap M_2) \cong (M_1 + M_2)/M_1$ by using the Second Isomorphism Theorem. But $M_1 + M_2 = H/K$, so that $M/N \cong (H/K)/M_1$. Define a map $\pi : (M \oplus M)/K \longrightarrow M_1$ as $\pi((x, y) + K) = (x - y, 0) + K$. Let us prove that π is well-defined. If $(x_1, y_1) + K = (x_2, y_2) + K$, then $(x_1 - x_2, y_1 - y_2) \in K$ and so $x_1 - x_2 = y_1 - y_2 \in M$, i.e. $x_1 - y_1 = x_2 - y_2$. So, $(x_1 - y_1, 0) + K = (x_2 - y_2, 0) + K$, as required. Let $\sigma : M_1 \longrightarrow (M \oplus M)/K$ be given by $\sigma((x, 0) + K) = (x, 0) + K$. Then, $\pi\sigma$ is easily shown to be the identity map on M_1, and so M_1 is a direct summand of $(M \oplus M)/K$ which is absolutely pure. Therefore, $M/N \cong (H/K)/M_1$ is absolutely pure.

(iii)$\Longrightarrow$ (ii) Let $\{M_i\}_{i \in I}$ be a family of absolutely pure submodules of an R-module M. Then, $\oplus_{i \in I} M_i$ is absolutely pure. Define $\psi : \oplus_{i \in I} M_i \longrightarrow \sum M_i$ as $\psi((m_i)_{i \in I}) = \sum\limits_{i \in I} m_i$. Then ψ is an epimorphism, and so, $\sum M_i$ is a quotient of the absolutely pure module $\oplus M_i$. Hence, $\sum M_i$ is absolutely pure by the hypothesis.

(iii)$\Longleftrightarrow$ (iv) Follows directly from Proposition 3.29. □

Chapter 4
n-ABSOLUTELY PURE MODULES AND RELATED CONCEPTS

In this chapter, we discuss generalizations, introduced by S. B. Lee [15], of absolute purity, flatness, as well as coherence of rings. We should mention that coherence has been generalized, in various other ways, by several authors, for example D. L. Costa [6], J. Chen and N. Ding [3], and J. Dauns [7,8].

A concept used frequently in this chapter is the long exact sequence of Ext. For the sake of completeness, we first recall the following definitions and results (see J. Rotman [20]).

Definition. A *complex* (or *a chain complex*) **A** is a sequence of modules and homomorphisms

$$\mathbf{A} = ... \longrightarrow A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \xrightarrow{d_{n-1}} ..., n \in \mathbb{Z}.$$

with $d_n d_{n+1} = 0$ for all n.

It is clear that every short exact sequence is a complex.

Definition. Let **X** be a complex of the form

$$\mathbf{X} = ... \longrightarrow X_1 \longrightarrow X_0 \longrightarrow M \longrightarrow 0$$

the complex obtained by suppressing M is

$$\mathbf{X}_M = ... \longrightarrow X_1 \longrightarrow X_0 \longrightarrow 0$$

and is called the *deleted complex* of $\mathbf{X}$. Similarly, we define the deleted complex $\mathbf{Y}_N$ of the complex

$$\mathbf{Y} = 0 \longrightarrow N \longrightarrow Y^0 \longrightarrow Y^1 \longrightarrow ...$$

by suppressing N.

Definition. If $(\mathbf{A}, d)$ is a complex, its n^{th} *homology module* is

$$H_n(\mathbf{A}) = \ker d_n \diagup \text{im } d_{n+1}.$$

Notation. We shall denote $\ker d_n$ by $Z_n(\mathbf{A})$ and im d_{n+1} by $B_n(\mathbf{A})$. Thus, $H_n(\mathbf{A}) = Z_n(\mathbf{A}) \diagup B_n(\mathbf{A})$.

Definition. If $\mathbf{A}$ and $\mathbf{A}'$ are complexes, a *chain map* $f : \mathbf{A} \longrightarrow \mathbf{A}'$ is a sequence of maps $f_n : A \longrightarrow A'$, $(n \in \mathbb{Z})$, such that the following diagram commutes

$$\begin{array}{ccccccc}
... \longrightarrow & A_{n+1} & \stackrel{d_{n+1}}{\longrightarrow} & A_n & \stackrel{d_n}{\longrightarrow} & A_{n-1} & \longrightarrow ... \\
& \downarrow f_{n+1} & & \downarrow f_n & & \downarrow f_{n-1} & \\
... \longrightarrow & A'_{n+1} & \stackrel{d'_{n+1}}{\longrightarrow} & A'_n & \stackrel{d'_n}{\longrightarrow} & A'_{n-1} & \longrightarrow ... .
\end{array}$$

Definition. If $f : \mathbf{A} \longrightarrow \mathbf{A}'$ is a chain map, define

$$H_n(f) : H_n(\mathbf{A}) \longrightarrow H_n(\mathbf{A}')$$

by

$$z_n + B_n(\mathbf{A}) \mapsto f_n(z_n) + B_n(\mathbf{A}').$$

$H_n(f)$ is a well-defined homomorphism and is called the map induced by f and it is usually denoted as f_*.

The notion of short exact sequence of R-modules and homomorphism can be extended to complexes and chain maps. We then obtain the notion of a short exact sequence of compexes.

Theorem (Connecting Homomorphism). *Let* $\mathbf{0} \longrightarrow \mathbf{A}' \xrightarrow{i} \mathbf{A} \xrightarrow{p} \mathbf{A}'' \longrightarrow \mathbf{0}$ *be an exact sequence of complexes. For each* n, *there is a homomorphism, called the natural connecting homomorphism*

$$\partial_n : H_n(\mathbf{A}'') \longrightarrow H_{n-1}(\mathbf{A}')$$

defined by

$$z'' + B_n(\mathbf{A}'') \mapsto i_{n-1}^{-1} d_n p_n^{-1}(z'') + B_{n-1}(\mathbf{A}').$$

Definition. If T is a covariant functor, its *right derived functors* R^nT are defined on a module A by $(R^nT)A = H^n(T\mathbf{E}_A) = \ker(Td^n)/\mathrm{im}(Td^{n-1})$, where $\mathbf{E}_A$ denotes a deleted injective resolution of A.

Definition (I). If $T = Hom_R(C, \_)$ is a covariant functor, then R^nT is defined by Ext_R^n $(C, \_)$. In particular, $Ext_R^n(C, A) = \ker d_*^n / \mathrm{im} d_*^{n-1}$, where $0 \longrightarrow A \longrightarrow E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1}$ $E^2 \xrightarrow{d^2}$ ... is an injective resolution of A.

Remark. One can see that $Ext_R^n(C, A) = H^n(\ Hom_R(C, \mathbf{E}_A))$.

Theorem. *The definition of* $Ext_R^n(C, A)$ *is independent of the choice of injective resolution of* A.

Definition. If T is a contravariant functor, then $(R^nT)C = \ker Td_{n+1} / \ \mathrm{im} Td_n$, where... $P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \longrightarrow C \longrightarrow 0$ is a projective resolution of C.

Definition (II). If $T = Hom_R(\_, A)$ is a contravariant functor, then $R^nT = Ext^n(\_, A)$. In particular, $Ext^n_R(C, A) = \ker d^*_{n+1} \diagup \operatorname{im} d^*_n$, where $... \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \longrightarrow C \longrightarrow 0$ is a projective resolution of C.

Remark. One can see that $Ext^n_R(C, A) = H^n(Hom_R(\mathbf{P}_C, A))$ where $\mathbf{P}_C$ is a deleted projective resolution.

Theorem. *The definition of $Ext^n_R(C, A)$ is independent of the choice of projective resolution of C.*

Theorem (I). *If $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ is an exact sequence of modules and T is a covariant functor, then there is an exact sequence $0 \longrightarrow R^0TA' \longrightarrow R^0TA \longrightarrow R^0TA'' \longrightarrow ... \longrightarrow R^nTA' \longrightarrow R^nTA \longrightarrow R^nTA'' \xrightarrow{\partial} R^{n+1}TA' \longrightarrow ...$ with natural connecting homomorphism ∂.*

Theorem. *$Ext^0_R(A, \_)$ is naturally equivalent to $Hom_R(A, \_)$ and $Ext^0_R(\_, B)$ is naturally equivalent to $Hom_R(\_, B)$.*

Theorem (II). *If $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ is an exact sequence of modules and T is contravariant, then there is an exact sequence $0 \longrightarrow R^0TA'' \longrightarrow R^0TA \longrightarrow R^0TA' \longrightarrow ... \longrightarrow R^nTA'' \longrightarrow R^nTA \longrightarrow R^nTA' \xrightarrow{\partial} R^{n+1}TA'' \longrightarrow ...$ with natural connecting homomorphism ∂.*

Now, if we apply Definition (I) in Theorem (I) and Definition (II) in Theorem (II), then we get the following result

Theorem *(Long Exact Sequence for Ext)* (i) *If* $0 \longrightarrow B' \longrightarrow B \longrightarrow B'' \longrightarrow 0$ *is an exact sequence, then there is a long exact sequence with natural connecting homomorphisms*

$$0 \longrightarrow Hom_R(A, B') \longrightarrow Hom_R(A, B) \longrightarrow Hom_R(A, B'') \xrightarrow{\partial} Ext^1_R(A, B') \longrightarrow ...,$$

(ii) if $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ *is an exact sequence, then there is a long exact sequence with natural connecting homomorphisms*

$$0 \longrightarrow Hom_R(A'', B) \longrightarrow Hom_R(A, B) \longrightarrow Hom_R(A', B) \xrightarrow{\partial} Ext^1_R(A'', B) \longrightarrow ....$$

In a similar way, one may apply the tensor product functor instead of the Hom_R functor to define a dual notion of Ext_R, denoted as Tor^R. We have (see J. Rotman [20, p. 221])

Theorem *(Long Exact Sequence for Tor). If* $0 \longrightarrow B' \longrightarrow B \longrightarrow B'' \longrightarrow 0$ *is an exact sequence, then there is a long exact sequence*

$$... \longrightarrow Tor^R_1(A, B'') \longrightarrow A \otimes B' \longrightarrow A \otimes B \longrightarrow A \otimes B'' \longrightarrow 0$$

with natural connecting homomorphisms; similarly in the other variable.

4.1 n-Absolutely Pure and n-Flat Modules

In this section we will discuss a generalization of the notions of absolute purity and flatness. To introduce this generalization, we need the following basic results. We will drop the subscript R from Ext^n_R, Hom_R, and the superscript R from Tor^R_n, when there is no risk of ambiguity.

Lemma 4.1. *If* B *is injective, then* $Ext^n(A, B) = 0$ *for all modules* A *and* $n \geqslant 1$.

Proof. If B is injective, then $0 \longrightarrow B \xrightarrow{\theta} E^0 \longrightarrow 0$ is an injective resolution, where $E^0 = B$ and $\theta = 1_B$. With respect to this choice of injective resolution, we can see that $Ext^n(A, B) = H^n(Hom(A, \mathbf{E}_B)) = 0$ where $n \geqslant 1$. $\square$

Remark. It can be shown that if $Ext^1(A, B) = 0$ for all R-modules A, then B is injective. Hence, a left R-module B is injective if and only if $Ext^1(A, B) = 0$ for all R-modules A.

Lemma 4.2. *If A is projective, then $Ext^n(A, B) = 0$ for all modules B and all $n \geqslant 1$.*

Proof. If P is projective, then

$$0 \xrightarrow{d_1} P_0 \xrightarrow{d_0} A \longrightarrow 0$$

is projective resolution where $P_0 = A$ and $d_0 = 1_A$. With respect to this choice of projective resolution, we can see that $Ext^n(A, B) = H^n(Hom(\mathbf{P}_A, B)) = 0$ where $n \geq 1$. $\square$

As an analogue to the remark above, we have

Proposition 4.3 (C. Megibben [18]). *A left R-module A is absolutely pure if and only if, for all finitely presented R-modules N, it satisfies $Ext^1_R(N, A) = 0$.*

Proof. Let A be an absolutely pure left R-module and let N be a finitely presented left R-module, i.e. there is a short exact sequence $0 \longrightarrow K \xrightarrow{\sigma} F \longrightarrow N \longrightarrow 0$ where F is free and both F and K are f.g.. Then, every homomorphism $g : K \longrightarrow A$ can be extended by $\beta : F \longrightarrow A$ i.e. $g = \beta\sigma$ by Proposition 2.14. Using the long exact sequence for Ext, we have the exact sequence

$$Hom(F, A) \xrightarrow{\sigma^*} Hom(K, A) \xrightarrow{\phi} Ext^1_R(N, A) \longrightarrow Ext^1_R(F, A)\ (*),$$

where $Ext^1_R(F, A) = 0$ by Lemma 4.7. Now, let $f \in Ext^1_R(N, A)$, then there is $g \in Hom(K, A)$ such that $\phi(g) = f$. By the hypothesis, there is $\beta \in Hom(F, A)$ with $\beta\sigma = g$, i.e. $\sigma^*(\beta) = g$ and so $f = \phi(g) = \phi(\sigma^*(\beta)) = 0$ since the sequence $(*)$ is exact. Thus, $Ext^1_R(N, A) = 0$ for all finitely presented modules N.

For the converse, let N be a f.p. R-module. Then, we have an exact sequence $0 \longrightarrow K \longrightarrow F \longrightarrow N \longrightarrow 0$, where F is free and both F and K are f.g. By using the long exact sequence for Ext, we have the exact sequence $(*)$. Now, if $Ext^1_R(N, A) = 0$ then σ^* is epic. Thus, if $g : K \longrightarrow A$, then there is $\beta \in Hom(F, A)$ such that $\sigma^*(\beta) = g$ i.e. $\beta\sigma = g$. Hence, A is absolutely pure by Proposition 2.14. $\square$

Definition. Let A be a left R-module. The projective dimension of A (abbreviated pd(A)) is defined as the smallest integer $n \geq 0$ for which there is a projective resoultion

$$0 \longrightarrow P_n \longrightarrow ... \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0.$$

If no such n exists, we write pd(A) $= \infty$.

Note that if A is an R-module, then pd(A) $= 0$ iff A is projective.

Following S. B. Lee [15], we define n-absolute purity as follows.

Definition. A left R-module A is *n-absolutely pure* for $n \geq 0$ if, for all finitely presented R-modules N with pd(N) $\leqslant n$, we have $Ext^1_R(N, A) = 0$.

It is clear that all modules are 0-absolutely pure since all modules with projective dimension zero are projective, and that absolutely pure modules are n-absolutely pure modules for all $n \geq 0$. Also, every $(n+1)$-absolutely pure is n-absolutely pure. Consequently, for each module A we have the following implications.

$$A \text{ is a.p. } \Longrightarrow ... \Longrightarrow A \text{ is 2-a.p.} \Longrightarrow A \text{ is 1-a.p.} \Longrightarrow A \text{ is 0-a.p.}$$

Before giving examples of R-modules that are n-a.p. (for some $n < \infty$) but not a.p., we give the following

Definition. A ring R is *quasi-Frobenius* (or simply qF) if it is left and right noetherian and R is an injective left R-module.

A well-known charaterization of qF rings is

A ring R is qF if and only if every projective R-module is injective. (See J. Rotman [20, p. 131].)

It can be shown that if the ring is qF, then every module N with pd$(N) < \infty$ is projective. Now, let N be a f.p. R-module with pd$(N) = n < \infty$ where R is qF. Then, N is projective and so $Ext^1(N, A) = 0$ for all R-modules A, i.e. all R-modules are n-a.p. for each $n < \infty$. So, if R is qF but not von Neumann regular (for example $\mathbb{Z}/4\mathbb{Z}$), then there is an R-module that is n-a.p. for all $n < \infty$ but not a.p.

Before stating a generalization of flatness dual to n-absolute purity, we first prove

Proposition 4.4. *If M is a flat right R-module, then $Tor_n(M, N) = 0$ for all $n \geqslant 1$ and all R-modules N.*

Proof. Let $\mathbf{Q}_N = ... \longrightarrow N_1 \longrightarrow N_0 \longrightarrow 0$ be an exact sequence. Since M is flat, the functor $M \otimes \_$ is exact. It follows that the complex $M \otimes \mathbf{Q}_N$ has all homology groups 0 except the zeroth. $\square$

Proposition 4.5. *A right R-module A is flat if $Tor_1(A, B) = 0$ for all finitely presented R-modules B.*

Proof. R/I, where I is a f.g. ideal, is finitely presented. Then, the exact sequence

$$0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0$$

induces the exact sequence

$$Tor_1(A, R/I) \longrightarrow A \otimes_R I \longrightarrow A \otimes_R R$$

Suppose $Tor_1(A, R/I) = 0$. Then, we obtain the exact sequence

$$0 \longrightarrow A \otimes_R I \longrightarrow A \otimes_R R$$

and hence A is flat by Lemma 3.7. □

From Propositions 4.4 and 4.5, we obtain that, A is a flat right R-module if and only if $Tor_1(A, B) = 0$ for all finitely presented R-modules B. This motivates the following generalization of flatness (see S. B. Lee [15]).

Definition. A right R-module M is *n-flat* for $n \geq 0$ if, for all finitely presented left R-modules N with pd(N) $\leqslant n$, we have $Tor_1^R(M, N) = 0$.

It is clear that all modules are 0-flat since all modules with projective dimension zero are projective, and that flat modules are n-flat modules for all $n \geq 0$. Also, every $(n+1)$-flat module is n-flat. Consequently, for each module M we have the following implications

$$M \text{ is flat} \Longrightarrow \ldots \Longrightarrow M \text{ is 2-flat} \Longrightarrow M \text{ is 1-flat} \Longrightarrow M \text{ is 0-flat.}$$

Recall that if the ring is qF, then every module N with pd$(N) < \infty$ is projective. Now, let N be a f.p. left R-module with pd$(N) = n < \infty$ where R is qF. Then, N is projective and so $Tor_1(A, N) = 0$ for all right R-modules A, i.e. all right R-modules are n-flat for each $n < \infty$. So, if R is qF but not von Neumann regular, then there is an R-module that is n-flat for all $n < \infty$ but not flat.

Lemma 4.6. Let $\{A_k \mid k \in K\}$ be a family of right R-modules and B be a left R-module. Then, $Tor_n(\oplus A_k, B) \cong \oplus Tor_n(A_k, B)$ *for all* $n \geqslant 0$.

Proof. By induction on n. For each R-module A_k there is an exact sequence

$$0 \longrightarrow S_k \longrightarrow F_k \longrightarrow A_k \longrightarrow 0 \ (*),$$

where F_k is free. This yields the exact sequence

$$0 \longrightarrow \oplus S_k \longrightarrow \oplus F_k \longrightarrow \oplus A_k \longrightarrow 0 \ (**).$$

By applying the long exact sequence of Tor on $(*)$, we have the exact sequence

$$Tor_1(F_k, B) \longrightarrow Tor_1(A_k, B) \longrightarrow S_k \otimes B \longrightarrow F_k \otimes B,$$

which implies the exactness of

$$\oplus Tor_1(F_k, B) \longrightarrow \oplus Tor_1(A_k, B) \longrightarrow \oplus(S_k \otimes B) \longrightarrow \oplus(F_k \otimes B).$$

Now, by applying the long exact sequence of Tor on $(**)$, we obtain the exact sequence

$$Tor_1(\oplus F_k, B) \longrightarrow Tor_1(\oplus A_k, B) \longrightarrow (\oplus S_k) \otimes B \longrightarrow (\oplus F_k) \otimes B$$

But each F_k is free so it is flat, and hence $\oplus Tor_1(F_k, B) = 0 = Tor_1(\oplus F_k, B)$ by Proposition 4.4 and the fact that a direct sum of flat modules is flat (see the examples below the definition of flat modules in Chapter 3). Thus, we have the following diagram.

$$\begin{array}{ccccccc}
\oplus Tor_1(F_k,B) & \longrightarrow & \oplus Tor_1(A_k,B) & \longrightarrow & \oplus(S_k\otimes B) & \longrightarrow & \oplus(F_k\otimes B)\\
 & & \downarrow \sigma & & \downarrow\cong & & \downarrow\cong\\
Tor_1(\oplus F_k,B) & \longrightarrow & Tor_1(\oplus A_k,B) & \longrightarrow & (\oplus S_k)\otimes B & \longrightarrow & (\oplus F_k)\otimes B
\end{array}$$

where the last two vertical maps are natural isomorphisms (see Theorem 1.15) and σ is defined in a natural way by standard diagram chasing. Then, σ is an isomorphism by the Five Lemma and hence $Tor_1(\oplus A_k, B) \cong \oplus Tor_1(A_k, B)$. Assume the conclusion of the Lemma is true for $n-1$, then by applying the long exact sequence of Tor on $(**)$ and $(*)$ we have

$$\begin{array}{ccccccc}
0{=}Tor_n(\oplus F_k,B) & \to & Tor_n(\oplus A_k,B) & \to & Tor_{n-1}(\oplus S_k,B) & \to & Tor_{n-1}(\oplus F_k,B){=}0\\
 & & \downarrow \sigma_n & & \downarrow\cong & & \\
0{=}\oplus Tor_n(F_k,B) & \to & \oplus Tor_n(A_k,B) & \to & \oplus Tor_{n-1}(S_k,B) & \to & \oplus Tor_{n-1}(F_k,B){=}0
\end{array}$$

Then, by the inductive hypothesis and the Five Lemma again we have $Tor_n(\oplus A_k, B) \cong \oplus Tor_n(A_k, B)$. $\square$

Lemma 4.7. *Let $\{B_k \mid k \in K\}$ be a family of left R-modules and A be a left R-module. Then, $Ext^n(A, \prod B_k) \cong \prod Ext^n(A, B_k)$ for all $n \geq 0$.*

Proof. We do an induction on n. Consider the following exact sequence

$$0 \longrightarrow B_k \longrightarrow E_k \longrightarrow Q_k \longrightarrow 0$$

where E_k is an injective envelope for B_k. Then, we have the following exact sequence

$$0 \longrightarrow \prod B_k \longrightarrow \prod E_k \longrightarrow \prod Q_k \longrightarrow 0$$

where $\prod E_k$ is injective (see Theorem 1.7). We apply the long exact sequence for Ext to have

$$\begin{array}{ccccccc}
Hom(A,\prod E_k) & \longrightarrow & Hom(A,\prod Q_k) & \longrightarrow & Ext^1(A,\prod B_k) & \longrightarrow & Ext^1(A,\prod E_k)=0 \\
\downarrow\cong & & \downarrow\cong & & \downarrow \sigma & & \\
\prod Hom(A,E_k) & \longrightarrow & \prod Hom(A,Q_k) & \longrightarrow & \prod Ext^1(A,B_k) & \longrightarrow & \prod Ext^1(A,E_k)=0
\end{array}$$

where the first two vertical maps are natural isomorphisms (see Theorem 1.14) and σ is defined in a natural way by standard diagram chasing. Then, by the Five Lemma σ is an isomorphism. Now, assume that the conclusion of the Lemma is true for $n-1$. Then,

$$\begin{array}{ccccccc}
Ext^{n-1}(A,\prod E_k) & \longrightarrow & Ext^{n-1}(A,\prod Q_k) & \longrightarrow & Ext^n(A,\prod B_k) & \longrightarrow & Ext^n(A,\prod E_k) \\
\downarrow\cong & & \downarrow\cong & & \downarrow \sigma_n & & \\
\prod Ext^{n-1}(A,E_k) & \longrightarrow & \prod Ext^{n-1}(A,Q_k) & \longrightarrow & \prod Ext^n(A,B_k) & \longrightarrow & \prod Ext^n(A,E_k)
\end{array}$$

where $Ext^n(A,\prod E_k) = 0 = \prod Ext^n(A,E_k)$ by Lemma 4.1. Hence, σ_n is an isomorphism by the inductive hypothesis and the Five Lemma. $\square$

Proposition 4.8. *Let $\{M_i\}_{i\in I}$ be a family of right R-modules and n a positive integer.*

(i) $\oplus M_i$ is n-flat if and only if each M_i is n-flat.

(ii) $\prod M_i$ is n-absolutely pure if and only if each M_i is n-absolutely pure.

Proof. (i) Let $\oplus M_i$ be n-flat. Then, $\oplus Tor_1(M_i,B) \cong Tor_1(\oplus M_i,B) = 0$ for all finitely presented left R-modules B with pd(B) $\leqslant n$. Hence, each M_i is n-flat. Conversely, let each M_i be n-flat. Then, $Tor_1(M_i,B) = 0$ for all finitely presented left R-modules B with pd$(B) \leqslant n$. Thus, $Tor_1(\oplus M_i,B) \cong \oplus Tor_1(M_i,B) = 0$ for all finitely presented left R -modules B with pd$(B) \leqslant n$ and hence $\oplus M_i$ is n-flat.

(ii) Let $\prod M_i$ be n-absolutely pure. Then, $\prod Ext^1(N,M_i) \cong Ext^1(N,\prod M_i) = 0$ for all finitely presented R-modules N with pd(N) $\leqslant n$. Thus, each $Ext^1(N,M_i) = 0$ for all finitely presented R-modules with pd$(N) \leqslant n$ and hence each M_i is n -absolutely pure. Conversely, let each M_i be n-absolutely pure, i.e. $Ext^1(N,M_i) = 0$ for all finitely presented R-modules N with pd$(N) \leqslant n$. Then, $Ext^1(N,\prod M_i) \cong \prod Ext^1(N,M_i) = 0$

for all finitely presented R-modules N with pd$(N) \leqslant n$ and hence $\prod M_i$ is n-absolutely pure. $\square$

Lemma 4.9. *For rings R and S, consider the situation $(A_R, {}_R B_S, C_S)$. Then, $Ext_R^n(A, Hom_S(B,C)) \cong Hom_S(Tor_n^R(A,B), C)$ whenever C is an injective right S-module.*

Proof. By induction on n. For any R-module B, we have the following exact sequence

$$0 \longrightarrow K \longrightarrow F \longrightarrow B \longrightarrow 0$$

where F is a free R-module. Now, we apply the long exact sequence for Tor to have the exact sequence

$$Tor_1(A,F) \longrightarrow Tor_1(A,B) \longrightarrow A \otimes_R K \longrightarrow A \otimes_R F.$$

F is free and hence is flat, so $Tor_1(A,F) = 0$ by Proposition 4.4. Then,

$$Hom_S(A \otimes_R F, C) \longrightarrow Hom_S(A \otimes_R K, C) \longrightarrow Hom_S(Tor_1(A,B), C) \longrightarrow 0 \; (*)$$

is exact since C an injective S-module and so $Hom_S(\_, C)$ is an exact functor. But

$$0 \longrightarrow Hom_S(B,C) \longrightarrow Hom_S(F,C) \longrightarrow Hom_S(K,C) \longrightarrow 0$$

is an exact sequence, as C is an injective S-module, so we use the long exact sequence of Ext to obtain following exact sequence

$$\begin{aligned} Hom_R(A, Hom_S(F,C)) &\longrightarrow Hom_R(A, Hom_S(K,C)) \longrightarrow \\ Ext_R^1(A, Hom_S(B,C)) &\longrightarrow Ext_R^1(A, Hom_S(F,C)) \; (**). \end{aligned}$$

The last term is zero since $F \cong R^{(I)}$ for some index set I, and so $Hom_S(F,C) \cong Hom_S(R^{(I)},C) \cong (Hom_S(R,C))^I \cong C^I$ is an injective R-module. From $(*)$ and $(**)$, we have the following diagram

$$\begin{array}{ccccc} Hom_S(A\otimes_R F,C)\rightarrow & & Hom_S(A\otimes_R K,C)\rightarrow & & Hom_S(Tor_1(A,B),C)\rightarrow 0 \\ \downarrow f & & \downarrow g & & \downarrow \sigma \\ Hom_R(A,Hom_S(F,C))\rightarrow & & Hom_R(A,Hom_S(K,C))\rightarrow & & Ext^1_R(A,Hom_S(B,C))\rightarrow 0 \end{array}$$

The maps f and g are natural isomorphisms (see Theorem 1.16) and σ is defined in a natural way by standard diagram chasing. Thus, σ is an isomorphism by the Five Lemma and hence $Ext^1_R(A,Hom_S(B,C)) \cong Hom_S(Tor^R_1(A,B),C)$.

Now, assume that the conclusion of the Lemma is true for $n-1$

i.e. $Ext^{n-1}_R(A,Hom_S(B,C)) \cong Hom_S(Tor^R_{n-1}(A,B),C)$. Then, we have

$$\begin{array}{ccccc} Hom_S(Tor^R_{n-1}(A,F),C)\rightarrow & & Hom_S(Tor^R_{n-1}(A,K),C)\rightarrow & & Hom_S(Tor^R_n(A,B),C)\rightarrow 0 \\ \downarrow\cong & & \downarrow\cong & & \downarrow \sigma_n \\ Ext^{n-1}_R(A,Hom_S(F,C))\rightarrow & & Ext^{n-1}_R(A,Hom_S(K,C))\rightarrow & & Ext^n_R(A,Hom_S(B,C))\rightarrow 0 \end{array}$$

where σ_n is defined in a natural way by standard diagram chasing. Thus, σ_n is an isomorphism by the Five Lemma and hence $Ext^n_R(A,Hom_S(B,C)) \cong Hom_S(Tor^R_n(A,B),C)$.

□

An analogue of Proposition 3.8 is

Proposition 4.10. *Let M be a right R-module and n a positive integer. Then, M is n-flat if and only if M^* is n-absolutely pure.*

Proof. From Lemma 4.9, we have the following identity

$$Ext^1_R(N,Hom_{\mathbb{Z}}(M,\mathbb{Q}/\mathbb{Z})) \cong Hom_{\mathbb{Z}}(Tor^R_1(N,M),\mathbb{Q}/\mathbb{Z}).$$

Now, let M be n-flat i.e. $Tor^R_1(N,M) = 0$ for all finitely presented R-modules N with pd$(N) \leqslant n$. Then, $(Tor_1(N,M))^* = 0$ implies $Ext^1_R(N,Hom_{\mathbb{Z}}(M,\mathbb{Q}/\mathbb{Z})) = 0$ for all finitely presented R-modules N with pd$(N) \leqslant n$ and hence $M^* = Hom_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is n-absolutely pure.

For the converse, let M^* be n-absolutely pure i.e. $Ext^1_R(N, M^*) = 0$ for all finitely presented R-modules N with pd$(N) \leqslant n$. Then, $Hom_{\mathbb{Z}}(Tor^R_1(N, M), \mathbb{Q}/\mathbb{Z}) = 0$ implies $Tor^R_1(N, M) = 0$ for all finitely presented R-modules N with pd$(N) \leqslant n$, since $\mathbb{Q}/\mathbb{Z}$ is an injective cogenerator, and hence M is n-flat. □

4.2 n-Coherent Rings

The following generalization of coherence is due to Lee [15]

Definition. A ring R is left n*-coherent* for $n \geq 1$ (*or* $n = \infty$), if every finitely generated submodule K of a free left R-module with pd$(K) \leqslant n - 1$ is finitely presented.

Clearly, all rings are 1-coherent since every f.g. projective module is finitely presented, and all coherent rings are n-coherent for all $n \geq 1$. Also, every $(n+1)$-coherent ring is n-coherent. Consequently, we have the following implications for a ring R.

R is coherent $\Longrightarrow$ ... $\Longrightarrow$ R is 3-coherent $\Longrightarrow$ R is 2-coherent $\Longrightarrow$ R is 1-coherent.

Now, if ${}_R\mathfrak{M}$ is the class of all left R-modules and $d = lpD(R) = sup\,\{pd(A) \mid A \in_R \mathfrak{M}\}$ is the left projective global dimension of R, then from the following diagram

$$\begin{tikzcd}
0 \arrow[r] & F_d \arrow[r] & \dots \arrow[r] & F_2 \arrow[rr, dashed] \arrow[dr] & & F_1 \arrow[rr, dashed] \arrow[dr] & & R \arrow[r] & R/I \arrow[r] & 0 \\
& & & & \dots\ K \arrow[ur] \arrow[dr] & & I \arrow[ur] \arrow[dr] & & & \\
& & & \dots 0 \arrow[ur] & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

where the row is a projective resolution of R/I, we have pd $(I)\ \leq d-1$. This means that the left coherent rings are exactly those which are d-coherent. In particular, left ∞-coherent rings are precisely the left coherent ones

Proposition 4.11. *If R is left n-coherent, then for every finitely presented left R-module N with pd $(N) \leqslant n$, every n-absolutely pure left R-module A satisfies $Ext^2_R(N, A) = 0$.*

Proof. Let R be left n-coherent ring, A be an n-absolutely pure module, and let N be finitely presented with pd(N) $\leqslant n$. Then, there is an exact sequence

$$0 \longrightarrow K \longrightarrow F \longrightarrow N \longrightarrow 0,$$

where F is free and both F and K are f.g. Thus, pd$(K) \leqslant n-1$ since pd$(N) \leqslant n$. So, K is a f.g. submodule of a free R-module F with pd$(K) \leqslant n-1$ and hence K is finitely presented. The long exact sequence of Ext induces the following exact sequence

$$Ext^1_R(K, A) \longrightarrow Ext^2_R(N, A) \longrightarrow Ext^2_R(F, A),$$

where $Ext^2_R(F, A) = 0$ since F is projective, and $Ext^1_R(K, A) = 0$ since A is n-absolutely pure. Therefore, $Ext^2_R(N, A) = 0$. $\square$

Lemma 4.12. *Let R be an n-coherent ring. Consider the situation $({}_RA, {}_RN_S, C_S)$ where N is f.p. with pd$(N) \leq n$, C is an injective S-module. Then,*

$$Tor^R_n(Hom_S(A, C), N) \cong Hom_S(Ext^n_R(N, A), C).$$

Proof. Since N is f.p., we have an exact sequence

$$0 \longrightarrow K \longrightarrow F \longrightarrow N \longrightarrow 0,$$

where F is free and both F and K are f.g. Then, by using the long exact sequence of Ext, we obtain the exact sequence

$$Hom_R(F,A) \longrightarrow Hom_R(K,A) \longrightarrow Ext^1_R(N,A) \longrightarrow Ext^1_R(F,A).$$

$Ext^1_R(F,A) = 0$ since F is free. So, we obtain an exact sequence

$$\begin{aligned} 0 \quad \longrightarrow \quad & Hom_S(Ext^1_R(N,A),C) \longrightarrow Hom_S(Hom_R(K,A),C) \longrightarrow \\ & Hom_S(Hom_R(F,A),C) \ (*). \end{aligned}$$

On the other hand, we use the long exact sequence of Tor on the sequence $0 \longrightarrow K \longrightarrow F \longrightarrow N \longrightarrow 0$ to obtain the exact sequence

$$\begin{aligned} Tor^R_1(Hom_S(A,C),F) \quad &\longrightarrow \quad Tor^R_1(Hom_S(A,C),N) \longrightarrow \\ Hom_S(A,C) \otimes_R K \quad &\longrightarrow \quad Hom_S(A,C) \otimes_R F \ (**) \end{aligned}$$

By Proposition 4.4 we have $Tor^R_1(Hom_S(A,C),F) = 0$ since F is free. Now, F is f.p. since it is f.g. projective, and K, being a f.g. submodule of a free module with pd$(K) \leqslant n-1$, is also f.p. as R is n-coherent. By Lemma 3.4, there are isomorphisms $f : Hom_S(Hom_R(K,A),C) \longrightarrow Hom_S(A,C) \otimes_R K$ and $g : Hom_S(Hom_R(F,A),C) \longrightarrow Hom_S(A,C) \otimes_R F$. Consider now the diagram

$$\begin{array}{ccccc} 0 \to Hom_S(Ext^1_R(N,A),C) & \to & Hom_S(Hom_R(K,A),C) & \to & Hom_S(Hom_R(F,A),C) \\ \downarrow \sigma & & \downarrow f & & \downarrow g \\ 0 \to Tor^R_1(Hom_S(A,C),N) & \to & Hom_S(A,C) \otimes_R K & \to & Hom_S(A,C) \otimes_R F \end{array}$$

where the rows are exact by $(*)$ and $(**)$, and σ is defined in a natural way by standard diagram chasing. Then, by the Five Lemma, σ is an isomorphism. Now, assume that the identity is true for $n-1$, i.e. there is an isomorphism $f_n : Hom_S(Ext^{n-1}(K,A),C) \longrightarrow Tor^R_{n-1}(Hom_S(A,C),K)$. Then, we have the following diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & Hom_S(Ext^n(N,A),C) & \longrightarrow & Hom_S(Ext^{n-1}(K,A),C) & \longrightarrow 0 \\ & \downarrow \sigma_n & & \downarrow f_n & \\ 0 \longrightarrow & Tor^R_n(Hom(A,C),N) & \longrightarrow & Tor^R_{n-1}(Hom_S(A,C),K) & \longrightarrow 0 \end{array}$$

where the rows are exact and where σ_n is an isomorphism by the Five Lemma. $\square$

An analogue of Proposition 3.23 is the following

Proposition 4.13. *Let R be a left n-coherent ring. Then, a right R-module A^* is n-flat if and only if A is n-absolutely pure left R-module.*

Proof. By Lemma 4.8 with $n=1$, $C=\mathbb{Q}/\mathbb{Z}$ and $S=\mathbb{Z}$, we have the identity

$$Tor^R_1(Hom_{\mathbb{Z}}(A,\mathbb{Q}/\mathbb{Z}),N) \cong Hom_{\mathbb{Z}}(Ext^1_R(N,A),\mathbb{Q}/\mathbb{Z}).$$

Let A^* be n-flat i.e. $Tor_1(A^*,N)=0$ for all f.p. R-modules N with pd$(N) \leqslant n$. Then, $Hom_{\mathbb{Z}}(Ext^1_R(N,A),\mathbb{Q}/\mathbb{Z}) \cong Tor^R_1(Hom_{\mathbb{Z}}(A,\mathbb{Q}/\mathbb{Z}),N)=0$ for all f.p. R-modules N with pd$(N) \leqslant n$. Thus, $Ext^1_R(N,A)=0$ for all f.p. R-modules N with pd$(N) \leqslant n$, i.e. A is n-absolutely pure.

For the converse, let A be n-absolutely pure i.e. $Ext^1_R(N,A)=0$ for all f.p. R-modules N with pd$(N) \leqslant n$. Then

$Tor^R_1(Hom_{\mathbb{Z}}(A,\mathbb{Q}/\mathbb{Z}),N) \cong Hom_{\mathbb{Z}}(Ext^1_R(N,A),\mathbb{Q}/\mathbb{Z})=0$ for all f.p. R-modules N with pd$(N) \leqslant n$ and hence $A^*=Hom_{\mathbb{Z}}(A,\mathbb{Q}/\mathbb{Z})$ is n-flat. $\square$

Proposition 4.14. *If R is left n-coherent, then a left R-module A is n-absolutely pure if and only if A^{**} is n-absolutely pure.*

Proof. A is n-absolutely pure iff A^* is n-flat by Proposition 4.13, and A^* is n-flat iff A^{**} is n-absolutely pure by Proposition 4.10. □

In Proposition 2.10, we had an important result that says a pure submodule of an absolutely pure module is again absolutely pure. This proposition was used to deduce various results. An analogue of this for n-coherent rings is given in

Proposition 4.15. *If R is a left n-coherent ring, then pure submodules of n-absolutely pure modules are n-absolutely pure.*

Proof. If B is pure in A and A is n-absolutely pure, then the exact sequence $0 \longrightarrow B \xrightarrow{\subseteq} A \longrightarrow A/B \longrightarrow 0$ leads to have $0 \longrightarrow (A/B)^* \longrightarrow A^* \longrightarrow B^* \longrightarrow 0$ split exact by Proposition 3.3. Hence $0 \longrightarrow B^{**} \longrightarrow A^{**} \longrightarrow (A/B)^{**} \longrightarrow 0$ is split by Proposition 2.8 and Proposition 3.3. So $B^{**} \oplus (A/B)^{**} \cong A^{**}$. By Proposition 4.14 A^{**} is n-absolutely pure since A is n -absolutely pure. Then, by Lemma 4.7 we have $Ext^1(N, B^{**}) \oplus Ext^1(N, (A/B)^{**}) \cong Ext^1(N, B^{**} \oplus (A/B)^{**}) \cong Ext^1(N, A^{**}) = 0$ for all f.p. R-modules N with pd$(N) \leqslant n$. Thus, B^{**} is n-absolutely pure and hence B is so by Proposition 4.14. □

Proposition 4.16. *Pure submodules of n-flat modules are n-flat.*

Proof. Let A be a pure submodule of an n-flat module B (B^* is n-absolutely pure by Proposition 4.10). Then, the pure exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow B/A \longrightarrow 0$ induces the split exact sequence $0 \longrightarrow (B/A)^* \longrightarrow B^* \longrightarrow A^* \longrightarrow 0$ by Proposition 3.3, i.e. $A^* \oplus (B/A)^* \cong B^*$. Then, $Ext^1_R(N, A^*) \oplus Ext^1_R(N, (B/A)^*) \cong Ext^1_R(N, A^* \oplus (B/A)^*) \cong Ext^1_R(N, B^*) = 0$ for all f.p. R-modules N with pd$(N) \leq n$, since B^* is n-absolutely pure. Hence A^* is n-absolutely pure and thus A is n-flat by Proposition 4.10. □

An analogue of Proposition 3.11 is

Proposition 4.17. *Let R be a left n-coherent ring. Then, pure quotients of n-absolutely pure left R-modules are n-absolutely pure.*

Proof. Assume that R is n-coherent and that $0 \longrightarrow B \longrightarrow A \longrightarrow A/B \longrightarrow 0$ is a pure exact sequence where A is an n-absolutely pure left R-module. Let N be f.p. with pd$(N) \leqslant n$. Then, we have an exact sequence

$$Ext^1_R(N, A) \longrightarrow Ext^1_R(N, A/B) \longrightarrow Ext^2_R(N, B).$$

$Ext^1_R(N, A) = 0$ since A is$n$-absolutely pure and $Ext^2_R(N, B) = 0$ by Proposition 4.15 and Proposition 4.11. Hence, $Ext^1_R(N, A/B) = 0$ and A/B is n-absolutely pure. □

In Proposition 4.17, we can omit the purity hypothesis for the case $n = 1$ as the following proposition shows.

Proposition 4.18. *Every quotient of a 1-absolutely pure R-module is 1-absolutely pure.*

Proof. First let us show that if pd$(N) \leq 1$, then $Ext^2_R(N, A) = 0$ for all left R-modules A. To see this, let $0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow N \longrightarrow 0$ be a projective resolution of N. Then, we obtain the exact sequence $Ext^1_R(P_1, A) \longrightarrow Ext^2_R(N, A) \longrightarrow Ext^2_R(P_0, A)$ by the long exact sequence for Ext. The first and the last terms are zero since both P_1 and P_0 are projective modules and hence $Ext^2_R(N, A) = 0$.

Now, let N be a f.p. R-module with pd$(N) \leq 1$ and let A be a submodule of a 1-a.p. R-module B.Then, the exact sequence $0 \longrightarrow A \xrightarrow{\subseteq} B \longrightarrow B/A \longrightarrow 0$ induces the exact sequence

$$Ext^1_R(N, B) \longrightarrow Ext^1_R(N, B/A) \longrightarrow Ext^2_R(N, A),$$

by the long exact sequence for Ext. $Ext^1_R(N, B) = 0$ since B is 1-a.p. and $Ext^2_R(N, A) = 0$ by the first part above. Thus, $Ext^1_R(N, B/A) = 0$ for all f.p R-modules N with pd$(N) \leq 1$, i.e. B/A is 1-a.p. □

Combining Propositions 4.10 and 4.18, one can drop the purity condition from Proposition 4.16:

Proposition 4.19. *Submodules of* 1*-flat modules are* 1*-flat.*

Proof. Let A be a submodule of a 1-flat R-module B. Then, B^* is 1-a.p. by Proposition 4.10. The short exact sequence

$$0 \longrightarrow A \xrightarrow{\subseteq} B \longrightarrow B/A \longrightarrow 0,$$

induces the dual exact sequence

$$0 \longrightarrow (B/A)^* \longrightarrow B^* \longrightarrow A^* \longrightarrow 0,$$

and so A^* is a quotient of the 1-a.p. module B^*. Therefore, A^* is 1-a.p. by Proposition 4.18 and hence A is 1-flat by Proposition 4.10. □

The following two lemmas will be needed later. First we fix the notation. Let A be a left R-module and $\{Bj \mid j \in J\}$ be a family of a right R-modules. The map $\psi : (\prod_{j\in J} B_j) \times A \longrightarrow \prod_{j\in J}(B_j \otimes A)$ defined as $\psi((b_j)_{j\in J}, a) = (b_j \otimes a)_{j\in J}$ is a middle linear map and so there is a homomorphism $\phi_A : (\prod_{j\in J} B_j) \otimes A \longrightarrow \prod_{j\in J}(B_j \otimes A)$ given by $\phi_A((b_j)_{j\in J} \otimes a) = (b_j \otimes a)_{j\in J}$. Let ρ_A be the composition $R^J \otimes A \xrightarrow{\phi_A} (R \otimes A)^J \xrightarrow{\mu} A^J$ where $\mu((r_j \otimes a_j)_{j\in J}) = (r_j a_j)_{j\in J}$. (See B. Stenström [23, pp. 41-42].)

Lemma 4.20. *With the above notation the following conditions are equivalent for any left R-module A.*

(i) A is f.g.

(ii) For every family $\{B_j \mid j \in J\}$ of right R-modules $\phi_A : (\prod_{j\in J} B_j) \otimes A \longrightarrow \prod_{j\in J}(B_j \otimes A)$ is an epimorphism.

(iii) For every set J, the map $\rho_A : R^J \otimes A \longrightarrow A^J$ is an epimorphism.

Proof. (i)$\Longrightarrow$(ii) Let A be a f.g. left R-module with generating set $\{a_1, a_2, ..., a_n\}$ and consider the map $\phi_A : (\prod_{j\in J} B_j) \otimes A \longrightarrow \prod_{j\in J}(B_j \otimes A)$ given by $(b_j)_{j\in J} \otimes a \mapsto (b_j \otimes a)_{j\in J}$. If $(u_j)_{j\in J} \in \prod_{j\in J}(B_j \otimes A)_{j\in J}$, then each $u_j = \sum\limits_{k=1}^{n} b_{jk} \otimes a_k$ for some $b_{jk} \in B_j$. Thus, $(u_j)_{j\in J}$ is an image under ϕ and so ϕ is an epimorphism.

(ii)$\Longrightarrow$(iii) is obvious.

(iii)$\Longrightarrow$(i) Choose $J = A$ and consider the element $u \in A^A$ whose x^{th} component is x. Since $\rho_A : R^A \otimes A \longrightarrow A^A$ is an epimorphism, we have $u = \rho_A(\sum\limits_{j=1}^{n}((r_{jx})_{x\in A} \otimes x_j))$ for some $(r_{jx})_{x\in A} \in R^A$ and $x_j \in A$. Then, $u = \sum\limits_{j=1}^{n} \rho_A((r_{jx})_{x\in A} \otimes x_j) = \sum\limits_{j=1}^{n}(r_{jx}x_j)_{x\in A} = (\sum\limits_{j=1}^{n} r_{jx}x_j)_{x\in A}$ and hence $x = \sum\limits_{j=1}^{n} r_{jx}x_j$ for all $x \in A$. Therefore, A is f.g. □

Lemma 4.21. *With the above notation the following conditions are equivalent for any left R-module A.*

(i) A is f.p.

(ii) For every family $\{B_j \mid j \in J\}$ of right R-modules $\phi_A : (\prod_{j\in J} B_j) \otimes A \longrightarrow \prod_{j\in J}(B_j \otimes A)$ is an isomorphism.

(iii) For every set J, the map $\rho_A : R^J \otimes A \longrightarrow A^J$ is an isomorphism.

Proof. (i)$\Longrightarrow$(ii) If A is a f.p. R-module, then there is an exact sequence $0 \longrightarrow K \longrightarrow F \longrightarrow A \longrightarrow 0$, where F is free and both F and K are f.g. R-modules. Consider the following commutative diagram

$$\begin{array}{ccccccc} (\prod_{j\in J} B_j)\otimes K & \longrightarrow & (\prod_{j\in J} B_j)\otimes F & \longrightarrow & (\prod_{j\in J} B_j)\otimes A & \longrightarrow 0 \\ \downarrow \phi_K & & \downarrow \phi_F & & \downarrow \phi_A & \\ \prod_{j\in J}(B_j\otimes K) & \longrightarrow & \prod_{j\in J}(B_j\otimes F) & \longrightarrow & \prod_{j\in J}(B_j\otimes A) & \longrightarrow 0 \end{array}$$

with exact rows. Then, ϕ_K and ϕ_A are epimorphisms by Lemma 4.20 and ϕ_F is an isomorphism since F is free. Thus, ϕ_A is a monomorphism by the Five Lemma and so ϕ_A is an isomorphism.

(ii)$\Longrightarrow$(iii) is obvious.

(iii)$\Longrightarrow$(i) By Lemma 4.20, we know that A is f.g. Choose an exact sequence $0 \longrightarrow K \longrightarrow F \longrightarrow A \longrightarrow 0$ where F is f.g free. We obtain a commutative diagram

$$\begin{array}{cccccccc} & R^I\otimes K & \longrightarrow & R^I\otimes F & \longrightarrow & R^I\otimes A & \longrightarrow 0 \\ & \downarrow \rho_K & & \downarrow \rho_F & & \downarrow \rho_K & \\ 0\longrightarrow & F^I & \longrightarrow & K^I & \longrightarrow & A^I & \longrightarrow 0 \end{array}$$

with exact rows, where ρ_F and ρ_A are isomorphisms by hypothesis. It follows by a diagram chase that ρ_K is an epimorphism, so K is f.g. by Lemma 4.20. and hence A is f.p. $\square$

Proposition 4.22. *The following statements are equivalent for any ring R.*

(i) R is left semihereditary.

(ii) 1-absolutely pure left R-modules are absolutely pure.

(iii) 1-flat right R-modules are flat.

Proof. (i)$\Longrightarrow$(ii) Let M be a 1-a.p. R-module and N be a f.p. right R-modules. Then we have an exact sequence $0 \longrightarrow K \longrightarrow F \longrightarrow N \longrightarrow 0$ where F is free and both F and K are f.g. Now, K is projective by Proposition 3.29. Thus, pd$(N) \leq 1$ and so $Ext^1_R(N, M) = 0$, i.e. M is a.p.

(ii)$\Longrightarrow$(iii) Let M be a 1-flat right R-module. Then, by Proposition 4.10, M^* is 1-a.p., and so it is a.p. This means that M is flat by Proposition 3.8.

(iii)$\Longrightarrow$(i) Let R be a ring such that all 1-flat right R-modules are flat. Since all submodules of 1-flat are 1-flat (Proposition 4.19), we obtain that every submodule of a flat module is flat. Moreover, R is left coherent. To see this, let J be any set. We prove that R^J is 1-flat as a right R-module. Let A be f.p. with pd$(A) \leq 1$. Then, there exists an exact sequence $0 \longrightarrow K \longrightarrow F \longrightarrow A \longrightarrow 0$ where both F and K are f.g. projective and thus, by the long exact sequence for Tor, we obtain the exact sequence

$$Tor_1^R(R^J, A) \longrightarrow R^J \otimes K \longrightarrow R^J \otimes F.$$

Now, to show that $Tor_1^R(R^J, A) = 0$ it suffices to show that the sequence $0 \to R^J \otimes K \to R^J \otimes F$ is exact. It is clear that K and F are f.p. since they are f.g. projective. By Lemma 4.21, we obtain

$$R^J \otimes K \cong K^J \text{and } R^J \otimes F \cong F^J.$$

Thus, the homomorphism $R^J \otimes K \longrightarrow R^J \otimes F$ is clearly monic since $K^J \longrightarrow F^J$ is so. This shows that R^J is 1-flat, and hence it is flat and R is left coherent by Proposition 3.19. Therefore, every f.g. left ideal is f.p. and also flat by the above part, and hence it is projective. So, R is left semihereditary. $\square$

This characterization can be used to give an example of an R-module that is not 1-a.p. (i.e. a 0-a.p. module that is not 1-a.p.). To see this, let R be a ring that is semihereditary but not von Neumann regular (for example $\mathbb{Z}$). Then, there is an R-module which is not a.p. and so not 1-a.p. (as R is semihereditary, see Proposition 4.22). In a similar way, there is an R-module that is 0-flat but not 1-flat since not all right R-modules are flat when the ring is not von Neumann regular by Proposition 3.21, and so they are not all 1-flat as R is semihereditary.

In Proposition 3.19, we have a characterization of coherent rings via flatness. We have an analogous result using n-coherence and n-flat modules.

Proposition 4.23. (S. B. Lee [15]). *A ring R is left n-coherent if and only if every direct product of n-flat right R-modules is n-flat.*

Proof. Let $\{M_i\}_{i\in I}$ be a set of n-flat modules. Then, $\prod_i M_i^*$ is n -absolutely pure by Proposition 4.10 and Proposition 4.8. But $\oplus_i M_i^*$ is a pure submodule of $\prod_i M_i^*$ and so, $\oplus_i M_i^*$ is n-absolutely pure by Proposition 4.15 since R is n-coherent. Then, $\prod_i M_i^{**} = \prod_i Hom_{\mathbb{Z}}(M_i^*, \mathbb{Q}/\mathbb{Z}) \cong Hom_{\mathbb{Z}}(\oplus_i M_i^*, \mathbb{Q}/\mathbb{Z}) = (\oplus_i M_i^*)^*$ is n-flat by Proposition 4.13. Now, by the last part of Proposition 3.10, M_i is a pure submodule of M_i^{**}. Thus, $\prod_i M_i$ is pure in $\prod_i M_i^{**}$ and hence, $\prod_i M_i$ is n-flat by Proposition 4.16.

For the converse, let $n \geq 1$ and suppose that every product of n-flat right R-modules is n-flat. Let H be a f.g. submodule of a f.g. free left R-module F, with pd$(H) \leq n-1$. We need to show that H is f.p. Let I be any index set and consider the commutative diagram

$$\begin{array}{ccccc} & & R^I \otimes H & \stackrel{1\otimes f}{\longrightarrow} & R^I \otimes F \\ & & \downarrow \sigma & & \downarrow \tau \\ 0 & \longrightarrow & H^I & \stackrel{g}{\longrightarrow} & F^I \end{array}$$

where $f : H \longrightarrow F$ and g are the inclusions, and where σ and τ are epimorphisms as in Lemma 4.20. Since F is f.p., τ is an isomorphism by Lemma 4.21.

By Lemma 4.21, we need only prove that σ is monic. But this is clear from the diagram since $1 \otimes f$ is monic (as $Tor_1(R^I, F/H) = 0$). $\square$

REFERENCES

[1] N. Bourbaki, *Commutative Algebra*, Hermann-Addison-Wesley, 1972.

[2] S. U. Chase, Direct products of modules, *Trans. Amer. Math. Soc*. 97 (1960), 457-473.

[3] J. Chen and N. Ding, On n-coherent rings, *Communications in Algebra* 24 (10), (1996), 3211-3216.

[4] P. M. Cohn, *Algebra*, vol. 2, by John wiley and Sons Ltd, 1977.

[5] P. M. Cohn, On the free product of associative rings I, *Math. Z.* 71 (1959), 380-398.

[6] D. L. Costa, Parameterization families of non-noetherian rings, *Communications in Algebra* 22(1994), 3997-4011.

[7] J. Dauns, Generalized pure modules, *Journal of Algebra* 242 (2001), 1-19.

[8] J. Dauns, Coherence, *Communications in Algebra* 30 (2002), 3063-3075.

[9] D. J. Fieldhouse, Pure theories, *Math. Ann.* 184 (1969), 1-18.

[10] L. Fuchs, Algebraically compact modules over Noetherian rings, *Indian J. Math*. 9 (1967), 357-374.

[11] L. Fuchs and L. Salce, *Modules Over Non-Noetherian Domains*, Mathematical Surveys and Monographs, vol. 84 (American Mathematical Society, 2000).

[12] T. W. Hungerford, *Algebra*; Holt, Rinehart and Winston, Inc, 1974.

[13] A. Kertész, System of equations over modules, *Acta sci. Math. Szeged* 18 (1957), 207-234.

[14] J. Lambek, A Module is Flat if and only if its Character Module is Injective, *Can. Math. Bull.* 7 (1964), 237-243.

[15] S. B. Lee, n-coherent rings, *Communications in Algebra* 30 (3), (2002), 1119-1126.

[16] B. H. Maddox, Absolutely pure modules, *Proc. Amer. Soc*. 18 (1967), 155-158.

[17] J. M. Maranda, On pure subgroups of abelian groups, *Arch. Math*. 13 (1960), 1-13.

[18] C. Megibben, Absolutely pure modules, *Proc. Amer. Math. Soc*. 26 (1970), 561-566.

[19] D. G. Northcott, *A First Course of Homological Algebra*, Cambridge University Press 1973.

[20] J. Rotman, *An Introduction to Homological Algebra*, Academic press: New York, 1979.

[21] B. Stenström, Coherent rings and FP-injective modules, *J. London Math. Soc.* 2 (1970), 323-329.

[22] B. Stenström, Pure submodules, *Arkiv für Math*. 7 (1967), 159-171

[23] B. Stenström, *Rings of Quotients*, Springer-Verlag New York 1965.

[24] R. B. Warfield, Purity and algebraic compactness for modules, *Pacific J. Math*. 28 (1969), 699-719.

[25] R. Wisbauer, *Foundations of Module and Ring Theory*, Gordon and Breach Science Publishers, 1991.

Vita

- Hussain Yahya Ahmed Al-Hammali
- Born in Qatif, Saudi Arabia in 1977
- Received Bachelor of Science (B. S.) degree in Mathematical Sciences from King Fahd University of Petroleum and Minerals (KFUPM), Dhahran, Saudi Arabia in 1998
- E-mail: ssuhyh@yahoo.com