STUDY OF IMPERFECT PRODUCTION PROCESSES WITH SHORTAGES

BY

WAEL IBRAHIM AL-HAJAILAN

SYSTEMS ENGINEERING

JUNE 2005

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A Thesis Presented to the
DEANSHIP OF GRADUATE STUDIES

KING FAHD UNIVERSITY OF PETROLEUM & MINERALS
DHAHRAN, SAUDI ARABIA

In Partial Fulfillment of the
Requirements for the Degree of

MASTER OF SCIENCE

In

SYSTEMS ENGINEERING

JUNE 2005

KING FAHD UNIVERSITY OF PETROLUEM & MINERALS

DHAHRAN 31261, SAUDI ARABIA

DEANSHIP OF GRADUATE STUDIES

This thesis, written by **WAEL IBRAHIM AL-HAJAILAN** under the direction of his Thesis Advisor and approved by his Thesis Committee, has been presented to and accepted by the Dean of Graduate Studies, in partial fulfilment of the requirements for the degree of **MASTER OF SCIENCE IN SYSTEMS ENGINEERING.**

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Dedicated

to

My Dear Parents,

Brothers and Sisters

Acknowledgements

All praise is for ALLAH (SWT), the most compassionate, the most merciful. May peace and blessings be upon prophet Muhammed (PBUH), his family and companions I thank almighty ALLAH (SWT) for giving me the knowledge and patience to complete this work.

I would like to acknowledge the support and facilities provided by the systems engineering department, KFUPM for the completion of this work.

Also great thanks to my sponsorship company, Saudi Aramco, for supporting me toward completing a Master of Science Degree in Systems Engineering.

I would like to thank Dr. M. A. Rahim for his usual help and support through out the course of this work. Also great appreciation to Dr. Shokri Selim for his valuable suggestions and useful discussions which made this work interesting for me. Thanks also to my thesis committee member Dr. Umar Al-Turki for his interest and cooperation.

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THESIS ABSTRACT

Name: WAEL IBRAHIM AL-HAJAILAN

Title: Study of Imperfect Production Processes with Shortages

Degree: MASTER OF SCIENCE

Major Field: SYSTEMS ENGINEERING

Date of Degree: JUNE 2005

We consider the Economic Production Quantity Model where the process goes out of control after a point within the production cycle. Shortages are allowed. The total cost function is developed which consists of set up cost, shortage cost, holding cost and cost of rework. The problem is to find the production time which minimizes the total cost. The study generalizes the work of Chung and Hou (2003), Hou (2005), Rahim and Hajailan (2006).Chung and Hou [2003] considered the case where the defective rate is a constant percentage of the production. In this work we extended their work in three directions. In the first we consider the case where the percent of defectives is proportion to the time the process is out of control to the total production cycle which is Rahim and Hajailan (2006). In the Model II when the process goes out-of-control the percentage of defectives rate increases using an exponential function. However, Model III considers the case where the process mean shifts after a random duration. The defective rate is determined by the percentage of the production outside the specification limits. In the first model we consider several distribution functions for the time to failure, namely the exponential distribution, the Weibull distribution, the Gamma distribution and the Normal distribution. In the Model II and Model III we consider only the case where the time to failure is exponentially distributed. We show that the objective function is generally nonconvex for all the cases studied regardless of the probability distribution of the time to failure. The objective function is convex for particular values of problem parameters. Sensitivity analysis is performed for the case of exponential time to failure for Rahim and Hajailan (2006) Model I, Model II, and Model III. In addition, warranty repair cost also has been incorporated for all models. The thesis is concluded by suggesting a number of recommendations for future research.

Keywords: *Deteriorating Process, Economic Production Lot Sizing, Production Cycle Length, Rate of Defectives, Convexity of Total Cost, Minimal Free Warranty.*

Master of Science Degree

King Fahd University of Peroluem and Minerals, Dhahran.

JUNE 2005

خلاصة الرسالة

الاسم : وائل بن ابراهيم الحجيلان

عنوان الدراسة : دراسة عمليات الإنتاج الغير مكتملة مع حالات العجز

التخصص : هندسة النظم

تاريخ التخرج : يونيو 2005 مـ

إننا نأخذ في اعتبارنا نموذج كمية الإنتاج الاقتصادي حيث تكون العملية خارج السيطرة عند نقطةٍ ما من دائرة الإنتاج. وتكون حالات العجز مقبولة. ويتم تطوير أداء التكلفة الإجمالية والتي تتكون من تكلفة التأسيس, تكلفة العجز, تكلفة الامتلاك (الحيازة) وتكلفة إعادة العمل. والمشكلة تكمن في إيجاد وقت الإنتاج الذي يعمل على تقليل النفقات الإجمالية. وتهدف الدراسة إلى استقراء أعمال الـ Chung and Hou (2003), والـ Hou (2005), Rahim and Hajailan (2006). وتضع الـ Chung and Hou (2003) في اعتبارها الحالة حيث يكون معدل النقص نسبة مئوية ثابتة إلى الإنتاج. وفي هذا العمل نقوم بتمديد أعمالهم في ثلاثة اتجاهات. في الأول نراعي ونأخذ في الاعتبار الحالة حيث نسبة المعيبات متناسبة مع الوقت والعملية خارج السيطرة بالنسبة لدائرة الإنتاج الكلي والتي هي Rahim and Hajailan (2006). في النموذج الثاني عندما تخرج العملية عن السيطرة, فإن النسبة المئوية لسعر المعيبات تزداد مستخدمة الدالة الأسية (Exponential Function). بينما النموذج الثالث يأخذ في الاعتبار الحالة حيث العملية تعني الانتقالات (التغييرات) بعد فترة عشوائية. ويتم تحديد القيمة المعيبة عن طريق النسبة المئوية للإنتاج خارج حدود التعيين (التخصيص). وفي النموذج الأول نعير اهتمامنا لوظائف التوزيع العديدة بالنسبة للزمن إلى القصور, وأعني التوزيع الدليلي (الأسي), توزيع الـ"Weibull" , توزيع الـ "Gamma" والتوزيع الـ "Normal". في النموذج الثاني والنموذج الثالث نأخذ في الاعتبار فقط الحالة حيث الزمن إلى القصور يتم توزيعه بطريقة دليلية (أسيّة). وقد أظهرنا بأن وظيفة الهدف هي بشكل عام غير محدبة بالنسبة لكل الحالات التي تمت دراستها بغض النظر عن توزيع الاحتمالية للزمن إلى القصور. وظيفة الهدف هي محدبة بالنسبة للقيم (النسب) المعينة لمعايير المشكلة. ويتم أداء تحليل الحساسية بالنسبة لحالة الزمن الدليلي (الأسي) للزمن إلى القصور بالنسبة لـ Rahim and Hajailan (2006) النموذج الأول والثاني والثالث. بالإضافة إلى ذلك, فقد تم أيضاً إدماج تكاليف إصلاح الضمان بالنسبة لكل النماذج. ويتم ختم الرسالة عن طريق اقتراح عدد من التوصيات للبحث المستقبلي.

الكلمات الهامة : عملية التلف (الانهيار), قياس حجم الإنتاج الاقتصادي, طول دائرة الإنتاج, معدل المعيبات, تحدب التكلفة الإجمالية , الحد الأدنى للضمان المجاني.

درجة الماجستير في العلوم

جامعة الملك فهد للبترول والمعادن

يونيو 2005 مـ

Chapter I : Problem Description

1.1 Introduction

In developing classical Economic Production Quantity (EPQ) models, it has been assumed that the product quality and production process are perfect. Indeed, product quality is not always perfect and actually depends on the state of the production process. The production process is subject to deterioration due to the occurrence of some assignable cause which may shift the process from an in-control state to an out-of-control state and produce some defective items.

The effect of an imperfect process on production run time and EPQ was initially studied by Rosenblatt and Lee (1986). In their study, the elapsed time until the process shift was assumed to be exponentially distributed. The optimal production run was found to be shorter than that of classical EPQ model. In recent years, numerous research efforts have been undertaken to extend the work of Rosenblatt and Lee (1986). Kim and Hong (1999) extended the work of Rosenblatt and Lee (1986) by assuming that elapsed time until the process shift was arbitrarily distributed. However, neither of their models took into consideration of allowable shortages. Chung and Hou (2003), however, have generalized the work of Kim and Hong (1999) by assuming that shortages were allowed.

Nevertheless, the common assumption of all the above-mentioned models was that there were a fixed percentage of defective items produced during the out-of-control period. This assumption may not be applicable in many industrial situations. Intuitively, the percentage of the defective rate should increase with an increase in the duration of the out-of-control period.

The main purpose of this thesis is to generalize the work of Chung and Hou (2003) by introducing a time-varying percentage defective rate. That is, when the process shifts to an out-of-control state, defective items are produced with variable increasing percentages depending on the duration of out-of-control period. Hou (2005) generalized the work of Rosenblatt and Lee (1986) to allow shortages and considered the restoration cost into account for a two-state continuous-time Markovian deteriorating production system. Yeh, Ho, and Tseng (2000) have studied the optimal production run length with free minimal repair warranty where the deteriorated process of the system is characterised by a two-state continuous-time Markov chain.

However, the above studied did not include that the percentage of defectives could be variable. Rahim and Hajailan (2006) have introduced the case where the percentage of defectives in a production process is time varying.

In the first model the percentage of defectives is computed by dividing the time of the process out of control to the total production run time. In the second model when the process goes out-of-control the percentage of defectives rate increases using an exponential function. In the third model we consider the case where the process mean shifts after a random duration. The defective rate is determined by the percentage of the production outside the specification limits. Convexity of the all models is studied. If for some parameters the total cost function is not convex it means that it has no optimal production run time.

Sensitivity analysis is conducted for all models to show the interaction of the model parameters values on the optimal production run time and the total cost that is incurred.

In the first model we consider several distribution functions for the time to failure, namely the exponential distribution, the Weibull distribution, the Gamma distribution and the normal distribution. In the second model we consider only the case where the time to failure is exponentially distributed.

1.2 Assumptions and Notations

Assumptions:

1. At the start of each product cycle, the production process is always in an in-control state and perfect items are produced but some items could be outside the specification limits.
2. During a production run, the production process may shift from an in-control state to an out-of-control state.
3. The elapsed time until the shift is a random variable with finite mean and variance.
4. Once the production process shifts to an out-of-control state, the shift cannot be detected until the end of the production cycle, and a variable proportion of the produced items are assumed to be defects.
5. All defective items produced are detected after the production cycle is over, and rework cost for defective items will be incurred.

6. The process is brought back to the in-control state at each setup.
7. Shortages of inventory items are allowed.
8. The demand rate is constant.
9. The production rate is greater than the demand rate.

<u>Notations:</u>

The following notations will be used.

D	=	demand rate in units per unit time,
P	=	production rate in units per unit time ($P > D$),
h	=	holding cost per unit, per unit time,
K	=	cost for setting the machine up and inspecting or resetting it to new condition before the beginning of the production cycle,
s	=	rework cost for a defective item,
x	=	an elapsed time until production process shifts,
α	=	percentage of defective items produced once the system is in the out-of-control state, it is to be taken as a function of elapsed time,
π	=	backorder cost per unit, per unit time,
$\overline{B}$	=	average backorder level,
$\bar{I}$	=	average inventory level,

I_{max}	=	maximum on-hand inventory level,
S_{max}	=	maximum shortage permitted,
T_1	=	production time when backorder is replenished,
T_2	=	production time when inventory builds up,
T_3	=	time period when there is no production or inventory depletion,
T_4	=	time period when there is no production and shortage occurs,
T	=	cycle time for each production lot, $T = T_1 + T_2 + T_3 + T_4$,
t	=	production run time in a production cycle, where $t = T_1 + T_2$ $= (D/P)T$
TC(t)	=	total cost,
TRC(T1, T2)	=	total relative cost,
*T**	=	optimal production-inventory cycle time.
λ	=	production system failure rate per unit time when an exponential probability distribution of failure is assumed,
USL	=	upper specification limit of the product quality characteristic,
LSL	=	lower specification limit of the product quality characteristic,
μ_0	=	initial process mean when process is in control,
β_τ	=	given constant used for rate of change in Model III,
μ_τ	=	mean at time τ where $\tau \geq x$,

a	=	shape parameter for Weibull and Gamma distributions,
β	=	scale parameter for Weibull and Gamma distributions,
v	=	cost of scheduled maintenance,
$f(\bullet)$	=	probability distribution function (p.d.f) denoting the transition from an in-control state to an out-of-control state,
$\phi(\bullet)$	=	standard normal probability distribution function,
$\Gamma(z)$	=	Gamma function, defined by $\Gamma(z)=\int_0^{\infty} t^{z-1}e^{-t}dt$,
$\phi(z) = \frac{2}{\sqrt{\pi}}\int_0^{z} e^{-t^2}dt$	=	error function
$M_i(t)$	=	$\int_0^{t} x^i f(x)dx$,
A(t)	=	$-2tM_1(t) + 3M_2(t)$,
$q(t)$	=	the fraction of nonconforming items,
q_1	=	the fraction of nonconforming items when process is in the in-control state,
q_2	=	the fraction of nonconforming items when process is in the out-of-control state,
$h(t)$	=	hazard rate function,
$W(t)$	=	average post-sale warranty cost,
c_r	=	repair cost per unit,
ω	=	warranty period,
k_1 *and* k_2	=	scale parameters used for hazard function,

m = shape parameter used for hazard function,

1.3 Terms and Terminology

1.3.1 Ordering Cost

Ordering costs include the cost of preparing and placing orders for replenishing inventories, the cost of handling and shipping orders, the cost of machine setups for the production run, the cost of inspecting the received orders in inventory, and all costs that do not vary with the size of the order.

1.3.2 Inventory Carrying Cost

The cost of carrying inventory can be broken down into several components: (1) the opportunity cost of money being tied up in inventory; (2) storage and space charges, representing the cost of providing storage space, as well as its cost of maintenance; (3) taxes and insurance, and the cost of physical deterioration and its prevention; (4) the cost of obsolescence due to technological change.

1.3.3 Shortage Cost

This cost is incurred if units of inventory are not available when demanded. It is the cost of lost sales, loss of goodwill, overtime payments, or customer dissatisfaction. There are two types of shortage costs: (1) one-time shortage cost per unit short, independent of the duration of the shortage; and (2) shortage cost per unit short per unit time.

1.3.4 Rework Cost

Re-work is defined as, ‘‘the process by which an item is made to conform to the original requirement’’. Re-work effects the operating costs, if we are one of the few manufacturers that never has to re-cut a component or re-make a cabinet door that was rejected, then the cost of poor quality should not be an issue. However, if we are experiencing any type of non-conformance or rework in the plant, then you cannot ignore the effect that it can have on your total annual operating expense, because rework and quality costs are just that; unnecessary additional operating costs that might as well be included in the operating budget and classified as a valid expense.

Material waste and rework are natural occurrences that happen on a daily basis, and require temporary corrective action to immediately reproduce or rework a part.

1.3.5 Warranty Cost

It is the cost related to the quality of items produced by deteriorating production system.

Under the minimal repair warranty, failures that occur within the warranty period ω result in valid warranty claims and are rectified by minimal repair at no cost to the buyers. Each minimal repair incurs a cost of c_r to the manufacturer.

1.4 Literature Review

Rosenblatt and Lee (1986) are the first who studied the effects of an imperfect production process of on the optimal production cycle time. The system is assumed to be in an in-control state at the beginning of each production cycle and may shift to an out-of-control state. The elapsed time until the shift is assumed to be exponentially distributed. The system will deteriorates during the production process and produce certain percentage of defective items, which are to be reworked or repaired at some cost. Items are sold to the customers, some costs of servicing, warranty or loss of goodwill will be incurred. The optimal production run time is found to be shorter than that of the classical economic manufacturing quantity. Lee and Rosenblatt (1987) have addressed the problem of joint determination of economic production cycle or economic manufacturing quantity EMQ and maintenance policy for a single product. It is shown that the optimal inspection schedule is equally-. The problem is solved by using an approximation to the cost function. The resulting EMQ is found to be an adjustment to the classical EMQ. Lee and Rosenblatt (1989) have studied the joint problem of production planning and maintenance schedules under the realistic assumption that the cost of process restoration is a function of the detection delay. In addition, the possibility of incurring shortages in the model is allowed. For specific restoration cost functions such as linear and exponential, an efficient solution procedure is presented to find the number of maintenance inspections in a production run, the length of the production run, the economic manufacturing quantity, and the maximum level of back orders. Cheng (1991) relaxed two major assumptions of the classical EOQ model with imperfect production process. These major assumptions are that the demand is constant and deterministic, and that the unit price (unit production cost) is independent of the order (production quantity). However, when demand is high, a company can produce more items so as to spread the fixed costs of production more widely, which will result in lower unit production costs. So, the unit cost of production is an increasing function of the demand rate. The optimal solution was obtained in a closed-form by using geometric programming (GP). Lion, Tseng, and Lin (1994) incorporated type I and type II inspection errors into the EMQ model under the imperfect production system which may seriously affect the product quality. They have derived the expected total cost when the shift of the production process follows a general distribution and the inspection interval is arbitrary. Huang and Chiu (1995) presented an imperfect production process model with two monitoring policies. Policy I represents the preventive maintenance whereas policy II represents not use it, where the cost of restoration and the proportion of defective items is a function of the detection delay. The objective is to determine the optimal production cycle time while minimizing the total cost of the imperfect production process under these two policies. They have shown that the total costs under policy I are smaller than those of policy II. So, it is necessary for the preventive maintenance procedure to be performed. Hariga and Ben-Daya (1998) extended the EPQ model to consider the general shift distributions with the imperfect process discussed by Rosenblatt and Lee (1986). They have developed distribution-based and distribution-free bounds on the optimal total cost. For the exponential distribution case, they compared the optimal solution with the Rosenblatt and Lee (1986) solution. Kim and Hong (1999) considered the EMQ model with a deteriorating production process. An optimal production run length and a minimum average cost are derived in three deteriorating processes: constant, linearly increasing, and exponentially increasing. The elapsed time until the shift is arbitrarily distributed. A numerical experiment is carried out to investigate behaviour of the proposed model and to compare the solutions with those from Rosenblatt and Lee (1986), and show that the differences in the two solutions may be significant depending on the cost and the process parameters. Ben-Daya and Hariga (2000) modelled the effects of an imperfect production process on the economic lot scheduling problem (ELSP). The mathematical model developed has taken into account the effect of imperfect quality and process restoration. Salameh and Jaber (2000) presented a modified inventory model which accounts for imperfect quality items when using the EPQ/EOQ formulae. It shows that the economic lot size quantity tends to increase as the average percentage of imperfect quality items increases. Items of imperfect quality are withdrawn from stock resulting in lower holding costs per unit per unit time and they proposed discount sales for imperfect quality items. Yeh, Hi, and Tseng (2000) studied the optimal production run length for a deteriorating production system in which the products are sold with free minimal repair warranty. The deteriorating process of the system is characterised by a two-state continuous-time Markov chain. Hayek and Salameh (2001) studied the effect of imperfect quality items on the finite production model. When production stops, defective items are assumed to be reworked at a constant rate. The percentage of imperfect quality items is considered to be a random variable with a known probability density function. The optimal operating policy that minimizes the total inventory cost per unit time is derived where shortages are allowed and back ordered. Wang and Sheu (2001) developed an EMQ model with a production process subject to random deterioration with a general discrete shift distribution, while items are being produced. Since defective items reach the marketplace the manufacturer will incur a significant cost (warranty cost), an EMQ model has been employed to consider the difference between the reworked cost before sale and warranty cost after sale. They investigated the effect of the warranty cost on optimization of the EMQ. Chung and Hou (2003) extended the work of Kim and Hong (1999) by allowing shortages. The elapsed time to shift is assumed to be arbitrarily distributed. They showed that there exists a unique optimal production run time to minimize the total relevant cost function. Chiu (2003) considered the EPQ model with the rework process of imperfect quality items under the assumption that not all of the defects are repairable. A portion of them are scrap and will not be reworked. The disposal cost for each scrap item and the repairing and holding costs for each reworked items are included in the cost analysis. The renewal reward theorem is utilized to deal with the variable cycle length, and the optimal lot size that minimizes the overall costs for the imperfect quality. The EPQ model is derived where back orders are permitted. Hou (2005) generalized the work of Rosenblatt and Lee to allow shortages and take the restoration costs into account for a two-state continuous-time Markovian deteriorating production system. When the production process is in the in-control (or out-of-control) state, $q1$(or $q2$) percent of the items produced will be nonconforming, where $q1<q2$.

1.5 Proposed Objectives

Rosenblatt and Lee (1986) have studied the effect of an imperfect production process of a single machine-single product system on the optimal production cycle time. The elapsed time till shift is assumed to be a random variable that is exponentially distributed with a known mean. Defective items will be reworked at some cost or if passed to customers, some costs of warranty or loss of goodwill. Kim and Hong (1999) have considered EMQ model with a deteriorated production process. An optimal production run and a minimum average cost are derived in three deteriorated processes constant, linearly increasing and exponentially increasing. Chung and Hou (2003) extended the problem of Kim and Hong (1999) by allowing shortages. Yeh, Ho, and Tseng (2000) have studied the optimal production run length for a deteriorated production system in which the product are sold with free minimal repair warranty. Rahim and Hajailan (2006) have extended the work of Chung and Hou (2003) by allowing the percentage of defective items to be time varying.

Chung and Hou (2003) assumed that the expected number of defects is given by:

$$ED(t) = \int_{x=0}^{t} \alpha(t-x)Pf(x)dx \qquad (1.1)$$

Where α is the percentage of defects per unit time. In this work we consider two models for determining the expected number of units which require rework.

In the first model we consider the case where α is given by α = (t-x)/t in Rahim and Hajailan (2006). This case is discussed in Chapter 2. In model II we assume that when the process is in out of control state the percentage of defectives in increasing exponentially which is discussed in chapter 3. In the Model III we assume that there are lower and upper specification limits of the quality characteristics. The percentage of defects will be determined by computing the probability of being outside those limits. Model III is discussed in Chapter 4.

For each of the above models the time to failure is random. We will examine the convexity conditions of the total cost function for these models. The optimal production run time is determined and sensitivity analysis for all the considered models is performed. In addition the free minimal repair warranty cost has been incorporated for all Models.

1.6 Thesis Organization

The remainder of this thesis is organized as follows. Chapter 2 deals with Model I what will happens when the fraction of defective items depends on the detection delay divided by the total production run time. The convexity conditions of the models are identified. Sensitivity analysis is presented to find the effect of the model parameters on the optimal production run time. Chapter 3 is discussing the case when the percentage of defectives items in out of control is increasing exponentially. Chapter 4 provides Model III issues faced when the fraction of defective items are represented by a normal probability distribution function, by calculating the fraction when the mean and standard deviation are changed, and when the process is moving from an in-control period to an out-of-control period. Examples of convexity and sensitivity analysis are also presented. In chapter 5 a summary of the work is listed, major contributions and possible directions for future research are also given. However, warranty repair cost has been incorporated for all Models where products are sold with free minimal repair warranty.

1.7 The Inventory Model

Chung and Hou (2003) presented the following production and inventory model.

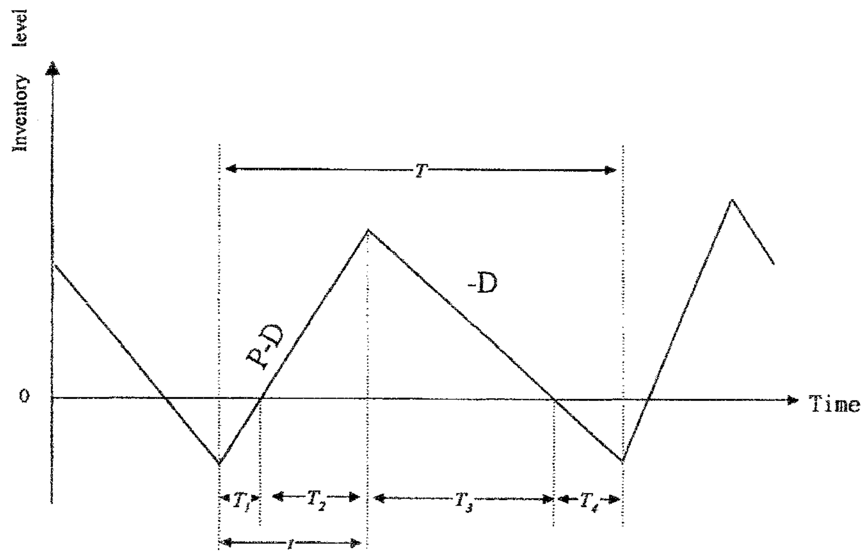

Figure 1: The production-inventory model with shortages

Figure 1 represents the production-inventory model with allowable shortages. This model can be described by four parts: production is started with constant production and demand rates when the back order is met (part 1), a period when the inventory level reaches its maximum value (part 2), inventory is consumed by a constant demand (part 3) and shortages period (part 4). The total relevant cost per unit time composed of setup costs, backorder costs, and rework costs. Based on the basic four phases of the inventory cycles, these costs are evaluated as follows:

(a) Setup Costs:

The average setup cost per unit time is : $\frac{K}{T} = \frac{KD}{Pt}$ (1.2)

(b) Holding Costs:

$$h\bar{I} = \frac{h(P-D)}{2}t - h(P-D)T_1 + \frac{h(P-D)T_1^2}{2t} \tag{1.3}$$

For details, please see Appendix 1.

(c) Back ordering costs:

$$\pi\overline{B} = \frac{\pi T_1^2 (P-D)}{2t} \tag{1.4}$$

For details, please see Appendix 2.

(d) Rework costs:

If the process becomes out-of-control after production is over then no defective parts are produced. However, if the process becomes out-of-control before the completion of the production time, then the number of defective items is proportional to the duration that the process is out-of-control, t-x, where x is the elapsed time before the process becomes out-of-control. Chung and Hou (2003) use a linear function as follows:

$$N = \begin{cases} 0 & if \quad X \geq t \\ \alpha(t-X)P & if \quad X < t \end{cases} \tag{1.5}$$

Therefore, the expected number of defective items in a production cycle is:

$$ED(t) = \int_o^t \alpha P(t-x) f(x) dx \tag{1.6}$$

(f)Total Costs:

$TRC(T_1, T_2) =$ setup costs + holding costs + back order costs + rework cost

$$\begin{aligned} &= \frac{K}{T} + h\overline{I} + \pi\overline{B} + \frac{sED(t)}{T} \\ &= \frac{KD}{P(T_1+T_2)} + (h+\pi)\frac{(P-D)}{2(T_1+T_2)}T_1^2 + \frac{h(P-D)}{2}(T_2-T_1) \\ &\quad + \frac{sD}{P(T_1+T_2)}ED(t) \end{aligned} \tag{1.7}$$

1.8 The Current Mathematical Models

Rosenblatt and Lee (1986) derived the total cost function as follows:

$$N = \begin{cases} 0 & if \quad X \geq t \\ \alpha P(t - X) & if \quad X < t \end{cases}$$, where α is constant.

$$C(t) = \left(\frac{KD}{Pt}\right) + \frac{h(P-D)t}{2} + \frac{s\alpha\mu Dt}{2},$$

which include setup cost, holding cost, and rework cost.

Chung and Hou (2003) have included the shortage cost:

$$TRC(t) = \frac{KD}{Pt} + \frac{h(P-D)t}{2}\left(\frac{\pi}{h+\pi}\right) + \frac{sD}{Pt}ED(N)$$

In the above models, the percentages of defective items in the in-control period are assumed to be zero. Hou (2005) has introduced the number of defectives as follows:

$$N = \begin{cases} q_1 Pt & if \quad X \geq t \\ q_1 PX + q_2 P(t - X) & if \quad X < t \end{cases}$$, where $q_1 < q_2$ and are constant.

So, the total cost function is:

$$TC(t) = \frac{KD}{Pt} + \frac{h(P-D)t}{2}\left(\frac{\pi}{h+\pi}\right) + \frac{rD\left(1-e^{-\lambda t}\right)}{Pt} + sDq_2 + sD(q_1 - q_2)\frac{1-e^{-\lambda t}}{\lambda t}$$

The percentage of defective items are assumed to be constant, however this is not always true for real life industry.

Chapter II : Model I

2.1 Introduction

In this chapter we will discuss the imperfect production processes when the percentage of defective units in the out-of-control period, α, is equal to the ratio of the detection delay, t-x, to the cycle production time t, i.e. $\alpha = \frac{t-x}{t}$. This model appeared in Rahim and Hajailan (2006).

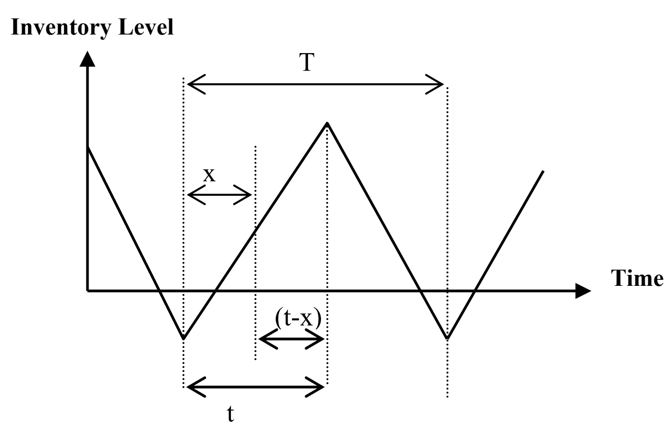

Figure 2: A typical production cycle

Figure 2 shows a typical production cycle, when t is the production time and x is the time when the process is in control.

In this case we take $ED(t) = \int_0^t \frac{(t-x)^2}{t} Pf(x)dx$ and the total cost is given by

$$\begin{aligned} \mathrm{TRC}(\mathrm{T}_1, \mathrm{T}_2) &= \frac{K}{T} + h\bar{I} + \pi\bar{B} + \frac{sED(t)}{T} \\ &= \frac{KD}{P(T_1+T_2)} + (h+\pi)\frac{(P-D)}{2(T_1+T_2)}T_1^2 + \frac{h(P-D)}{2}(T_2-T_1) \\ &+ \frac{sD}{P(T_1+T_2)}ED(t) \end{aligned}$$

2.2 Convexity Of The Total Cost Function, TRC (T_1, T_2)

In this section we examine the convexity of *TRC*. It will be shown that, in general, *TRC* is a non convex function. We accomplish this by examining its Hessian matrix at some points.

A function, g(x, y), differentiated twice, is convex if:

$$\frac{\partial^2 g}{\partial x^2} > 0, \quad \frac{\partial^2 g}{\partial y^2} > 0 \text{ and } \frac{\partial^2 g}{\partial x^2} \times \frac{\partial^2 g}{\partial y^2} \geq \left(\frac{\partial^2 g}{\partial x \partial y}\right)^2$$

The second derivatives of *TRC* are given by;

$$\frac{\partial^2 TRC}{\partial T_1^2} = \frac{2DK}{P(T_1+T_2)^3} + \frac{(P-D)(h+\pi)T_2^2}{(T_1+T_2)^3} + \frac{2Ds}{(T_1+T_2)^4}\left(-2tM_1(t) + 3M_2(t)\right) \quad (2.1)$$

$$\frac{\partial^2 TRC}{\partial T_2^2} = \frac{2DK}{P(T_1+T_2)^3} + \frac{(P-D)(h+\pi)T_1^2}{(T_1+T_2)^3} + \frac{2Ds}{(T_1+T_2)^4}\left(-2tM_1(t) + 3M_2(t)\right) \quad (2.2)$$

$$\frac{\partial^2 TRC}{\partial T_1 \partial T_2} = \frac{2DK}{P(T_1+T_2)^3} + \frac{(D-P)(h+\pi)T_1T_2}{(T_1+T_2)^3} + \frac{2Ds}{(T_1+T_2)^4}\left(-2tM_1(t) + 3M_2(t)\right) \quad (2.3)$$

where $M_i(t) = \int_0^t x^i f(x)dx$ is the i[th] moment of the density function f(x).

Note that the first two terms of (2.1), (2.2) and (2.3) are always positive. However the last term includes the moments of the density function f(t). Hence the signs of (2.1), (2.2) and (2.3) depend on the sign and magnitude of the quantity $-2tM_1(t) + 3M_2(t)$.

In the following subsections we examine the sign of $-2tM_1(t)+3M_2(t)$ for different probability density functions, f(x).

2.2.1 Case of f(x) is the Exponential Distribution

In this section we examine the sign of A(t) = $-2tM_1(t)+3M_2(t)$, and the Hessian of TRC where $f(x)=\lambda e^{-\lambda x}$. In this case

$$A(t)=\frac{e^{-t\lambda}}{\lambda^2}\left(-6-4t\lambda-t^2\lambda^2+e^{t\lambda}(6-2t\lambda)\right) \tag{2.4}$$

In the following we prove that A(t) ≤ 0 for t ≥ 0. Towards this end we show that the function $B(t;\lambda)=-6-4t\lambda-t^2\lambda^2+e^{t\lambda}(6-2t\lambda)\le 0$ for t ≥ 0 and λ > 0.

Lemma 2.1
$e^x(1-x)\le 1$ for all x and equality is satisfied at x = 0
Proof
Note that e^{-x} is a convex function. Its tangent at x = 0 is the line y = 1 – x hence $e^{-x}\ge 1-x$.

Theorem 2.2
$B(t;\lambda)=-6-4t\lambda-t^2\lambda^2+e^{t\lambda}(6-2t\lambda)$ is non-positive for t ≥ 0 and λ > 0. Strict equality holds at t = 0.
Proof
Let x = λt and $g(x) = -6-4x-x^2+2e^x(3-x)$. $d^2g/dx^2 = 2(-1+e^x(1-x))$. From Lemma 2.1, g”(x) ≤ 0. Hence g is concave. The tangent of g(x) at x = 0 is the line y = 0, hence g(x) is also non-positive.

Figure 3 below shows a plot of A(t) for λ = 0.5

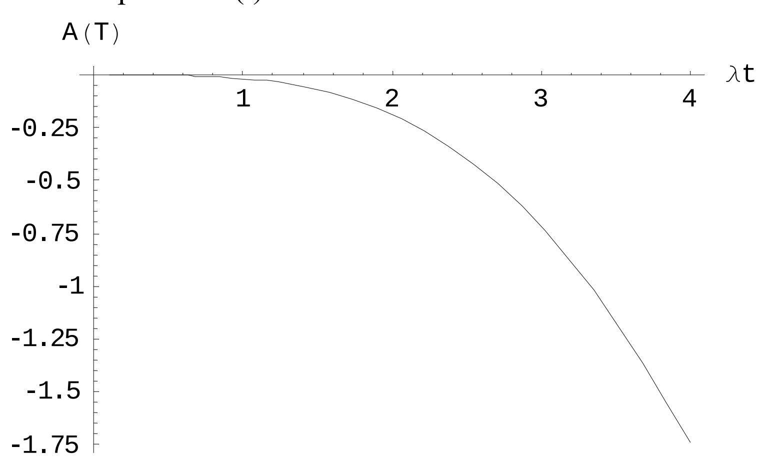

Figure 3: A(t) vs. production run time for Exponential Distribution

By examining Equations (2.1), (2.2) and (2.3) we observe that the sign of these derivatives could be negative for specific values of the parameters. In the following two cases we show that the Hessian could be made indefinite or positive definite by the proper choice of parameters.

Example 1:

We choose the following values for the parameters of the problem; K=100, D = 500, P = 1000, π= 0.3, h = 0.1, s = 2, λ=0.5. The Hessian at the point $(T_1, T_2) = (3.2, 9.6)$ and its determinant is;

$$\begin{vmatrix} 6.77426 & -4.94449 \\ -4.94449 & -1.03824 \end{vmatrix} = -31.4813$$

H is indefinite and hence the function TRC is not convex at this point.

Example 2:

The Hessain could be positive definite at some points. For example if we take K=150, D=999, P=1000, π=2.5, h=0.75, s=10, λ=0.5, and $(T_1, T_2) = (0.145237, 0.484121)$ we get the Hessian and its determinant, as given below;

$$\begin{vmatrix} 1{,}032.64 & 1{,}028.67 \\ 1{,}028.67 & 1{,}029.86 \end{vmatrix} = 5{,}316.78$$

The Hessian is positive definite and hence the function TRC is convex at this point.

We conclude that the cost function, *TRC,* under consideration is *generally* non-convex, as shown by the above two examples. The parameter values affect the function geometry drastically. By examining Equations (2.1), (2.2), and (2.3) one could make the right hand side (RHS) negative by choosing a large value for s, and very small values for K and P-D. This will result in the first two terms becoming small positive numbers and the last term a large negative number.

2.2.2 Case of f(x) is the Weibull Distribution

The Weibull Distribution has a wide industrial applications. When its shape parameter a=1 it becomes exponential. If a >1 then the distribution tends to Normal. If a <1 it has decreasing hazard rate.

The Weibull distribution is given by

$$f(x) = a\beta^{-a}x^{a-1}e^{-(x/\beta)}, \qquad (2.5)$$

where a and β are the shape and scale parameters of the distribution. If a = 1, we get the exponential distribution. We will consider two examples of a and β and show that $A(t) = -2tM_1(t) + 3M_2(t) < 0$.

Example 3:

Let a=2 and β=1,
In this case A(t) contains the Gamma and Incomplete Gamma integrals. Figure 4 shows a plot of A(t). The graph illustrates the fact that A(t) is negative for t > 1.4.

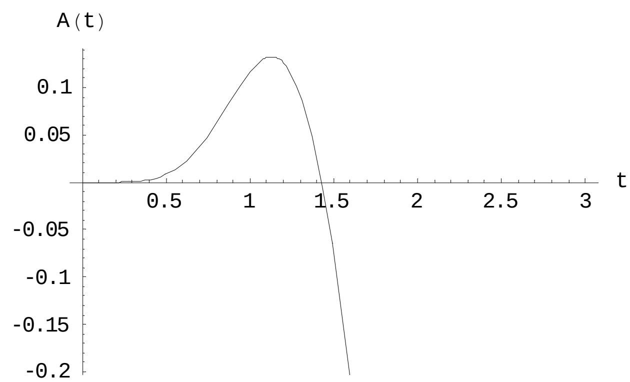

Figure 4: A(t) vs. production run time for Weibull Distribution

To examine the convexity of *TRC* we choose the following values for the parameters of the problem; K=150, D=999, P=1,000, π=2.5, h=0.75, s=15 and the compute the Hessian at $(T_1, T_2) = (2.0, 1.5)$. The Hessian and the determinant are given below;

$$\begin{vmatrix} -632.663 & -633.061 \\ -633.061 & -632.531 \end{vmatrix} = -587.632$$

Note that H is indefinite and hence TRC is non convex at this point where the time to failure is Weibull distributed.

Example 4:

Let a=2 and β=1,
Different values of the parameters are chosen the function could be convex. Here we use the following values for the parameters; K=150, D=999, P=1,000, π=2.5, h=0.75, s=5, and

use the point $(T_1, T_2) = (0.145237, 0.484121)$. The Hessian and the determinant are given below;

$$\begin{vmatrix} 2{,}204.06 & 2{,}200.09 \\ 2{,}200.09 & 2{,}201.28 \end{vmatrix} = 11{,}366$$

Note that H is positive semi definite, and the function TRC is convex at this point.

We conclude that the cost function under consideration is *generally* non-convex. As shown in the above two cases, the parameter values affect the function geometry drastically. By examining Equations (2.1), (2.2), and (2.3) one could make the RHS negative by choosing very small values for K and P-D, resulting in the first two terms being small. The third term can be made negative and large if s is large and t is also large. For the examples on hand $t > 1.4$ was sufficient to make the RHS negative.

Figure 5, 6, and 7 shows the function A(t) for different condition of a and β.

2.2.2.1 Hazard Rate

a=0.5, β=1

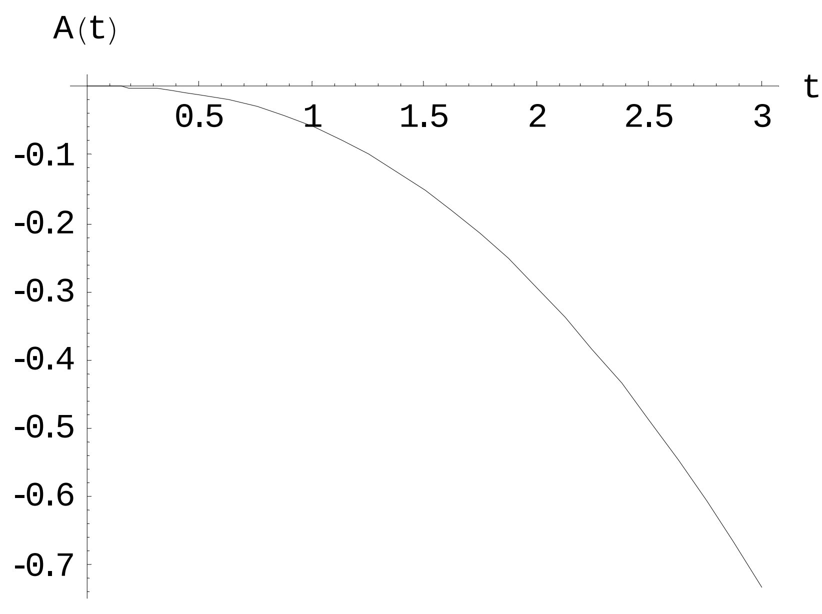

Figure 5: Graph of A(t) when a=0.5, β=1

a=1, β=1

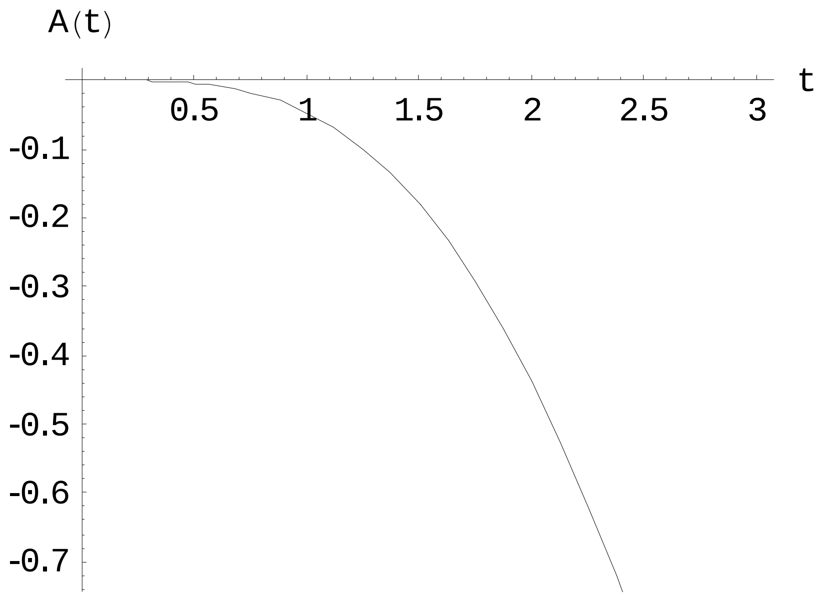

Figure 6: Graph of A(t) when a=1, β=1

a=3, β=1

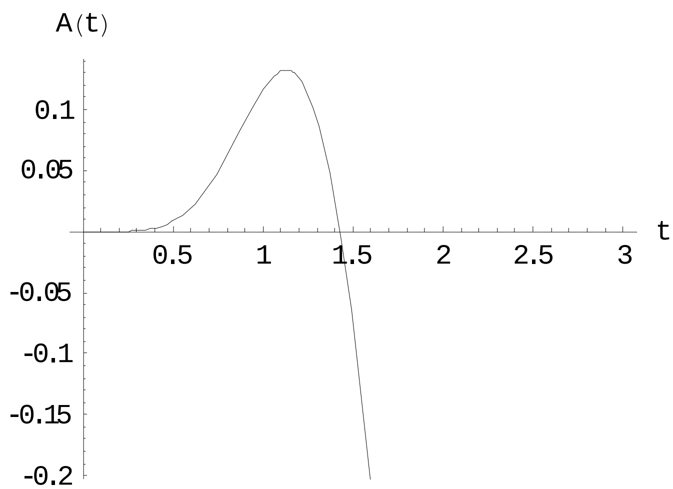

Figure 7: Graph of A(t) when a=3, β=1

2.2.3 Case of f(x) is the Gamma Distribution

The density function in this case is given by:

$$f(x) = \beta^{-a} x^{a-1} e^{-x/\beta} \quad (2.6)$$

Where a is the shape parameter and β is the scale parameter.

Consider the case where a=2 and β=1, the A(t) is given by:

$$
\begin{aligned}
A(t) &= -2tM_1(t) + 3M_2(t) \\
&= -\frac{1}{2}e^{-t}\left(72 + 12e^{t}(t-6) + 60t + 24t^2 + 6t^3 + t^4\right)
\end{aligned}
\tag{2.7}
$$

A plot of A(t) is shown in Figure 8. The graph illustrates the fact that A(t) < 0 for t > 3.6

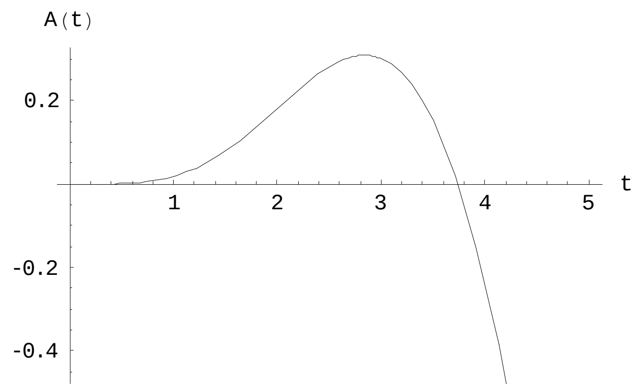

Figure 8: A(t) vs. production run time for Gamma Distribution

Example 5:

Next we consider the Hessian of TRC. We choose the following values for the parameters of the problem; K=150, D=999, P=1,000, π=2.5, h=0.75, s=15. We use the point $(T_1, T_2) = (3.5, 1.5)$. The Hessian and its determinant, in this case, are

$$\begin{vmatrix} -226.014 & -226.412 \\ -226.412 & -225.881 \end{vmatrix} = -210.029$$

Note that H is not positive semi definite, and the function TRC is not convex at this point where the time to failure is Gamma distributed.

Example 6:

If we use the same parameters as above, but take s = 2 and consider the point $(T_1, T_2) = (1, 0.5)$. We get the following Hessian and determinant.

$$\begin{vmatrix} 116.387 & 115.665 \\ 115.665 & 117.11 \end{vmatrix} = 251.651$$

Note that H is positive definite and the function TRC is convex at this point.

We conclude that the cost function under consideration is *generally* non-convex. As shown in the above two examples, the parameter values affect the function geometry drastically. By examining Equations (2.1), (2.2), and (2.3) one could make the RHS negative by choosing very small values for K and P-D, resulting in the first two terms being small. The third term can be made negative and large if s is large and t is also large. For the examples on hand t > 3.8 was sufficient to make the RHS negative.

2.2.4 Case of f(x) is the Normal Distribution

The density function of the Normal distribution is given by:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2 / 2\sigma^2} \text{ and } A(t) = -2tM_1(t) + 3M_2(t) \qquad (2.8)$$

Where μ is the mean of distribution and σ is the standard deviation.

Consider the case when: μ=10, σ=1 take a large positive value. In this case A(t) contains the error function. Figure 9 shows a plot of A(t). The graph illustrates the fact that A(t) is always negative.

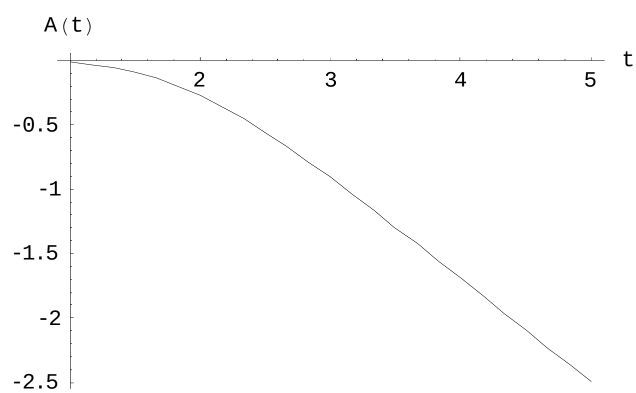

Figure 9: A(t) vs. production run time for Normal Distribution

Example7:

We chose the following values for the parameters of the problem K=100, D = 500, P = 1000, π= 0.3, h = 0.1, s = 2, λ=0.5, U=13, L=7, (T1, T2)=(3.2,9.6)

The Hessian and its determinant are,

$$\begin{vmatrix} 8.18758 & -3.53117 \\ -3.53117 & 0.375083 \end{vmatrix} = -9.39812$$

Note that H is not positive semi definite and the function TRC is not convex. By examining Equations (2.1), (2.2), and (2.3) one could make the RHS negative by choosing a large value for s, and very small values for K and P-D. This will result in the first two terms becoming small positive numbers and the last term a large negative number.

2.3 Stationary Points of TRC

In this section we derive the stationary points of function TRC given by (1.7) which is rewritten here for convenience of the reader.

$$TRC(T_1,T_2) = \frac{KD}{P(T_1+T_2)} + (h+\pi)\frac{(P-D)}{2(T_1+T_2)}T_1^2 + \frac{h(P-D)}{2}(T_2-T_1) + \frac{sD}{P(T_1+T_2)}ED(t)$$

where ED(t) is the average number of defective units. In this chapter we take $\alpha = \frac{t-x}{t}$ where $t = T_1 + T_2$. ED(t) and its partial derivatives are given by

$$ED(t) = P\int_0^t \frac{(t-x)^2}{t} f(x)dx$$
$$\frac{\partial ED(t)}{\partial T_i} = PF(T_1+T_2) - P\int_0^{T_1+T_2} \frac{x^2}{(T_1+T_2)^2} f(x)dx = ED'(t),\ i = 1 \text{ and } 2 \quad (2.9)$$

The partial derivatives of the TRC function with respect to T_1 and T_2 are shown below:

$$\frac{\partial TRC(T_1,T_2)}{\partial T_1} = -\frac{KD}{Pt^2} - \frac{(h+\pi)(P-D)T_1^2}{2t^2} + (h+\pi)(P-D)T_1/t + \frac{h(P-D)}{2} - h(P-D) - \frac{sD}{Pt^2}ED(t) + \frac{sD}{Pt}ED'(t) \quad (2.10)$$

$$\frac{\partial TRC(T_1,T_2)}{\partial T_2} = -\frac{KD}{Pt^2} - \frac{(h+\pi)(P-D)T_1^2}{2t^2} + \frac{h(P-D)}{2} - \frac{sD}{Pt^2}ED(t) + \frac{sD}{Pt}ED'(t) \quad (2.11)$$

At a stationery point both partial derivatives vanish and hence :

Note that $\frac{\partial TRC(T_1,T_2)}{\partial T_1} = \frac{\partial TRC(T_1,T_2)}{\partial T_2} + (h+\pi)(P-D)T_1/t - h(P-D)$

$(h+\pi)(P-D)T_1/t - h(P-D) = 0$

which simplifies to

$T_1^* = \frac{h}{h+\pi}t$ and $T_2^* = \frac{\pi}{h+\pi}t$

Substituting into (1.7) gives:

$$TC(t) = \frac{KD}{Pt} + \frac{h(P-D)t}{2}\left(\frac{\pi}{h+\pi}\right) + \frac{sD}{Pt}ED(t)$$

Next we consider the derivative of $TC(t)$ with respect to t which is given by:

$$\frac{dTC}{dt} = -\frac{DK}{Pt^2} + \frac{h(P-D)\pi}{2(h+\pi)} + \frac{2Ds\int_0^t (t-x)f(x)dx}{t^2} - \frac{2Ds\int_0^t (t-x)^2 f(x)dx}{t^3} \qquad (2.12)$$

The second derivative is given by:

$$\frac{d^2TC}{dt^2} = \frac{2DK}{Pt^3} + \frac{2Ds\int_0^t f(x)dx}{t^2} - \frac{8Ds\int_0^t (t-x)f(x)dx}{t^3} + \frac{6Ds\int_0^t (t-x)^2 f(x)dx}{t^4} \qquad (2.13)$$

Note that no closed form solution can be obtained from (2.12). The value of t that minimizes *TC* could be computed by a root finding algorithm of (2.12) or by a line search algorithm of *TC*.
Figure 10 below shows that a plot of function TC for the case f(x) is an exponential density function. For this example we use the following values for the parameters of the problem; K=100, D = 500, P = 1000, π= 0.3, h = 0.1, s = 2, and λ=0.5.

In conclusion, we have shown that the function TC is generally nonconvex. Equation (2.15) can be used to generate a stationary point and a plot of the function is used to verify that this point is a global minimum of the cost function.
Table 1 shows the values used in plotting the function shown in figure 10.

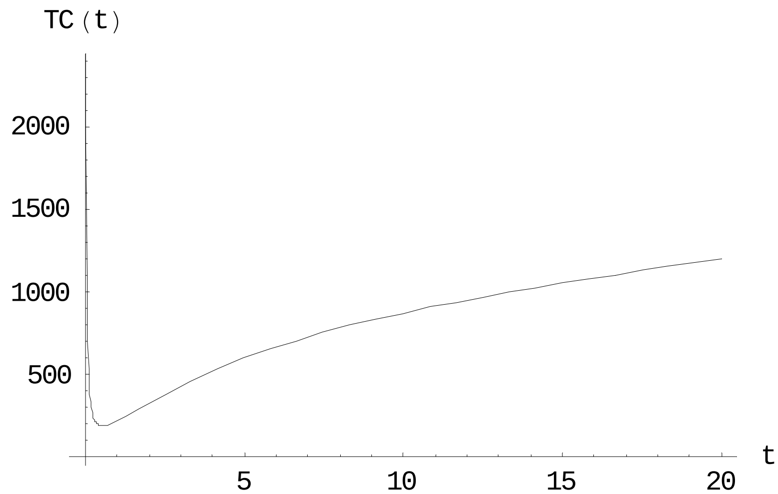

Figure 10: Total cost function versus production run time

Table 1: Table of production run time vs. total cost, TC

t	TC(t)	t	TC(t)
0.01	5001.85	10.04	819.63
0.1	518.335	10.63	846.683
0.5	186.917	11.22	872.486
1.19	236.351	11.81	897.17
1.78	176.468	12.4	920.85
2.37	246.437	12.99	943.628
2.96	313.43	13.58	965.592
3.55	375.708	14.17	986.819
4.14	433.081	14.76	1007.38
4.73	485.84	15.35	1027.33
5.32	534.418	15.94	1046.74
5.91	579.268	16.53	1065.64
6.5	620.82	17.12	1084.07
7.09	659.46	17.71	1102.09
7.68	695.533	18.3	1119.72
8.27	729.34	18.89	1136.99
8.86	761.144	19.48	1153.93
9.45	791.174	20.07	1170.57

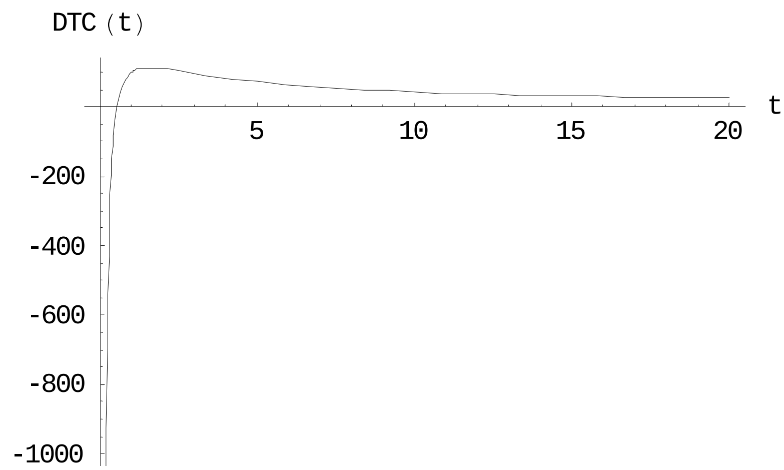

Figure 11: First derivative of TC(t) vs. production run time

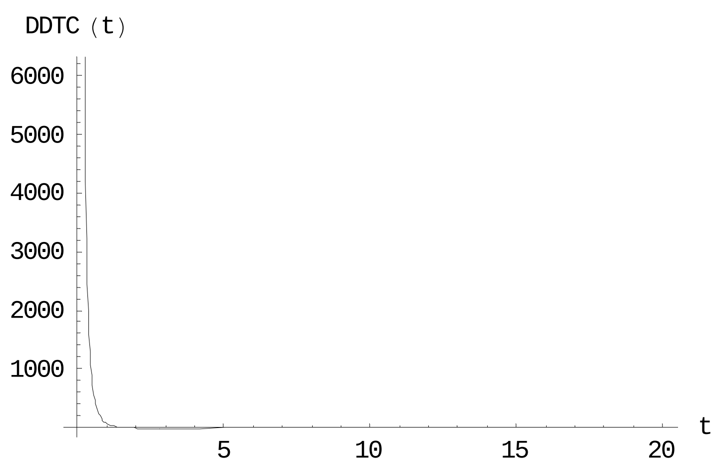

Figure 12: Second derivative of TC(t) vs. production run time

The plot of the function TC shows that it is nonconvex, while its second derivative is positive at some range close to the origin and negative otherwise.

2.4 Further Convexity Results

Equation 1.7 gives total costs as a function of T_1 and T2. Since t= T1+T2 then we can substitute t- T1 for T2. This will give a function in t and T1. Let $C(t,T_1)$ be this function.

$$C(t,T_1)=\frac{KD}{Pt}+(h+\pi)\frac{(P-D)}{2t}T_1^2+h\left[\frac{(P-D)t}{2}-(P-D)T_1\right]+\frac{sD}{Pt}E(N)$$

2.4.1 Convexity of TC(t,T_1) with respect to T_1:

For $TC(t,T_1)$ to be convex in T_1 we have to show that the Hessian is positive semi definite. Sections 2.2.2, 2.2.3, and 2.2.4 contain examples that the Hessian is indefinite, hence $TC(t,T_1)$ is not convex in T_1.

2.4.2 Existence and Uniqueness of t*:

Intermediate Value Theorem says that if f is continuous function on the closed interval [a, b], and suppose d is a real number between f(a) and f(b); then there exists c in [a, b] such that f(c) = d.

We will take the limit for the first derivative of $TC(t)$ as it goes to zero and to infinity. $\lim_{t\to 0} dTC/dt=-\infty$ and $\lim_{t\to\infty} dTC/dt=\frac{h(P-D)\pi}{2(h+\pi)}>0$, therefore $dTC/dt=0$ at some point t^*.

Let $g=t^3\,dTC/dt$, then $dg/dt=3t^2\,dTC/dt+t^3\,d^2TC/dt^2$. Substituting dTC/dt from (2.12) and d^2TC/dt^2 from (2.13) gives,

$$=-3\frac{DK}{P}+\frac{3t^2h(P-D)\pi}{2(h+\pi)}+6Ds\int_0^t(t-x)f(x)dx-\frac{6Ds}{t}\int_0^t(t-x)^2f(x)dx$$

$$+\frac{2DK}{P}+2Dst\int_0^t f(x)dx-8Ds\int_0^t(t-x)f(x)dx+\frac{6Ds\int_0^t(t-x)^2f(x)dx}{t}$$

$$=-\frac{DK}{P}+\frac{3t^2h(P-D)\pi}{2(h+\pi)}+2Dst\int_0^t f(x)dx-2Ds\int_0^t(t-x)f(x)dx$$

$$=-\frac{DK}{P}+\frac{3t^2h(P-D)\pi}{2(h+\pi)}+2Ds\int_0^t xf(x)dx \tag{2.14}$$

Note that (2.14) is negative at t=0. However, its second and third terms are strictly increasing function. Hence $\frac{dg}{dt}$ will eventually vanish for some value, say, t_0. This implies that g is decreasing for $t<t_0$ and increasing for $t>t_0$. Next we examine the behaviour of $\frac{dTC}{dt}$. We showed earlier that $\frac{dTC}{dt}=0$ at some points. Let these points t_1, t_2, …., t_r then g=0 at these points but this contradicts the fact that g is decreasing before t_0 and increasing after t_0. Therefore $\frac{dTC}{dt}=0$ at exactly one point. Hence we have shown the uniqueness of the minimum point of TC.

2.5 Free Minimal Repair Warranty

In Equation (1.7) the consideration of warranty period and warranty costs are not considered. In this section we are considering warranty.

The fraction of nonconforming items in a production run with length t denoted by $q(t)$ is given by :

$$q(t)=\frac{ED(t)}{Pt}$$

Under the free minimal repair warranty, it is well-known Hou (2005) that the failure process of a conforming (or nonconforming) item is a nonhomogenous process with intensity $h_1(t)$ (or $h_2(t)$). The expected post-sale warranty cost for a warranty period ω is:

$$W(t)=c_r\left[(1-q(t))\int_o^{\omega} h_1(\tau)d\tau+q(t)\int_o^{\omega} h_2(\tau)d\tau\right]\times\frac{Pt}{T} \qquad (2.15)$$

In this chapter we use $ED(t)=\int_0^t \frac{(t-x)^2}{t}Pf(x)dx$

Hence the fraction of nonconforming items in a production run with length t denoted by $q(t)$ is given by :

$$q(t)=\frac{ED(t)}{pt}=P\int_0^t \frac{(t-x)^2}{t^2}f(x)dx$$

Example:

Suppose $h_1(t) = k_1 t^{m_1}$ and $h_2(t) = k_2 t^{m_2}$ where $h_1(t) < h_2(t)$ for $t \geq 0$ denote the hazard rate associated with conforming and nonconforming item, respectively. Furthermore suppose that the time to failure is exponential distribution function. Assume K=150, D=1200, P=1650, π=2.5, h=.75, s=.85, λ=0.5, k_1=1, k_2=2, m_1=2, m_2=2, c_r=1.5, ω=6.

Table 2: Free minimal repair warranty for Rahim and Hajailan (2006) Model

Rahim and Hajailan (2006) Model	Without Warranty	With Warranty
Optimal production run time, t^*	0.6293	0.5562
Cost	354.109	914.295

Note that if warranty is available the optimal production time is less than the case without warranty. This will reduce the probability of being out of control and hence less nonconforming items will be produced.

2.6 Sensitivity Analysis

In this section, a sensitivity analysis of the model is conducted to study the effect of the different cost parameters on Rahim and Hajailan (2006) model. Table 2 gives the values of the parameters used to perform the analysis. The effect of each parameter is studied as well as the effect of simultaneous changes of two parameters. In addition, a comparison of the basic model versus the sequential changes of the selected parameters is made.

Table 3: Values of the parameters used in the sensitivity analysis study for Model I

Parameter	Level 1	Level 2	Level 3
D	1,200	1,350	1,550
P	1,650	1,850	2,250
K	150	220	450
h	0.75	1.2	1.95
π	2.5	3.5	5.1
λ	0.5	0.75	0.95
s	0.85	1.25	1.75

Exponential Distribution

Effect of the demand rate, D

Different values for the demand rate, *D*, are studied and presented in cases 1-3. As the demand rate increases, the optimal production run time, t^*, also increases to meet the demand.

Effect of the production rate, P

The effects of changes in the production rate, *P*, are analyzed in cases 1, 4 and 5. An increase in the production rate will decrease the production run time, but the total relevant costs increase. If there is no constraint on the production run time, current results indicate low production rates have low total costs.

Effect of the setup cost, K

Different values for the setup cost, K, are presented in cases 1, 6 and 7. By increasing the setup cost, the optimal production run time increases, so that the number of times the machines are set up for production is minimized in order to reduce the total cost of machine setups.

Effect of the holding cost, h

The effects of the holding cost, h, are presented in cases 1, 8 and 9. If the holding cost increases, the optimal production times tend to be smaller. So, if we produce smaller quantities to meet the demand, there will be smaller quantities left in the storehouse, and in turn will decrease the total holding cost.

Effect of shortage cost, π

The effects of shortage cost, π, are presented in cases 1, 10 and 11. By increasing the shortage cost, the optimal production run time is decreased. This mean the TRC will be enhanced by decreasing production run time, t, if the shortage cost is increased.

Effect of the process failure rate, λ, when the time to failure is exponentially distributed.

The results of different values for the failure rate, λ, are presented in cases 1, 12 and 13.

As the failure rate of the production system increases, the optimal production run time decreases. For this reason we will avoid having the system in the out-of-control state, which in turn will contribute to the reduction of the total relevant costs.

Effect of rework cost, s

Cases 1, 14 and 15 present different values for the rework cost, s. If we increase s, t^* tends to be smaller. So, the percentage of having defective items will be smaller.

Table 4: Effects of each parameter on the optimal production run time for Model I

Case	*Vp**	Level	*t**	TRC(*t**)	T_1	T_2
1	Basic Model	1	0.6293	354.11	0.1452	0.4841
2		2	0.7043	358.83	0.1625	0.5418
3	*D*	3	0.8222	358.51	0.1897	0.6325
4	*P*	2	0.5378	367.35	0.1241	0.4137
5	*P*	3	0.4187	385.48	0.0966	0.3221
6	*K*	2	0.7689	426.95	0.1774	0.5915
7	*K*	3	1.1236	604.06	0.2593	0.8643
8	*h*	2	0.5749	385.73	0.1865	0.3884
9	*h*	3	0.5249	420.88	0.2300	0.2949
10	π	2	0.6187	359.82	0.1092	0.5095
11	π	3	0.6097	364.82	0.0782	0.5315
12	λ	2	0.5691	396.43	0.1313	0.4378
13	λ	3	0.5343	426.25	0.1233	0.4110
14	*s*	2	0.5602	398.22	0.1293	0.4309
15	*s*	3	0.4983	447.67	0.1150	0.3833

In the following table, the effect of simultaneous changes of two parameters is also presented. For example, consider Cases 1 and 2; if we increase *h* and *D*, the optimal production run time tends to be larger. Where in case 3, when *h* and *P* increases, the optimal run time decreases. These analyses show us the interaction between paired model parameters.

Table 5: Effects of simultaneous changes of two parameters on the optimal production run time for Model I

Case	Pairs of parameters	Level of factor 1	Level of factor 2	*t**	TRC(*t**)
1	Basic Model	1	1	0.6293	354.11
2	*h D*	2	2	0.6569	382.67
3	*h P*	2	2	0.4845	406.1
4	*h K*	2	2	0.7011	465.55
5	*h π*	2	2	0.5589	396.29
6	*h λ*	2	2	0.5272	425.23
7	*h s*	2	2	0.5207	426.62
8	*π D*	2	2	0.6953	363.1
9	*π P*	2	2	0.5272	374.39
10	*π K*	2	2	0.7556	433.94
11	*π λ*	2	2	0.5611	401.61
12	*π s*	2	2	0.5526	403.32
13	*λ D*	2	2	0.6244	410.56
14	*λ P*	2	2	0.4931	404.69
15	*λ K*	2	2	0.6991	476.77
16	*λ s*	2	2	0.4967	454.23
17	*s D*	2	2	0.6112	413.22
18	*s P*	2	2	0.4876	405.78
19	*s K*	2	2	0.6849	480.03

In addition, in the following table the sensitivity analysis is comparing the basic model versus the sequential and/or simultaneous changes to the selected factors. For example, in case 2, when the demand rate *D* increases, *t** will also increase. However, in case 3, when the production rate *P,* and the demand rate *D* increase, *t** will decrease.

Table 6: Effects of sequential changes of the parameters on the optimal production run time for Model I

		Model parameters								
Case	Effects	*D*	*P*	*K*	*h*	*π*	*λ*	*s*	*t**	TRC(*t**)
1	Basic	1	1	1	1	1	1	1	0.6293	354.11
2	*D*	2	1	1	1	1	1	1	0.7043	358.83
3	*P*	2	2	1	1	1	1	1	0.5947	375.59
4	*K*	2	2	2	1	1	1	1	0.7264	452.95
5	*h*	2	2	2	2	1	1	1	0.663	493.48
6	*π*	2	2	2	2	2	1	1	0.6444	507.01
7	*λ*	2	2	2	2	2	2	1	0.5945	555.97
8	*s*	2	2	2	2	2	2	2	0.5304	624.34

Chapter III : Model II

3.1 Introduction

In this chapter we introduce a new structure for the percentage of defectives. In this model the percentage of defectives when the process in control is $q_1 \geq 0$. When the process goes out-of-control the percentage of defectives rate increases using an exponential function, $q_2(t) = u + v\left(1 - e^{-bt}\right)$. At t=0, $q_2(0) = q_1$ hence u=q_1. On the other hand as t goes to ∞ $q_2(\infty) = 1$, hence $v = 1 - q_1$. Therefore in this model $q_2(t) = q_1 + (1 - q_1)\left(1 - e^{-bt}\right)$. Note that this model generalize the models of Rosenblatt and Lee (1986), Lee and Rosenblatt (1989), Chung and Hou (2003), and Hou (2005). We call this; model II.

The number of defectives is given by:

$$N = \begin{cases} q_1 Pt & if \;\; X \geq t \\ q_1 PX + q_2(t)P(t - X) & if \;\; X < t \end{cases}$$

where x is the elapsed time until production process shifts and t is the production run time in a production cycle.

Where the expected number of defectives is given by:

$$ED(t) = \int_t^{\infty} q_1 Ptf(x)dx + \int_0^t \left(q_1 Px + q_2(t)P(t - x)\right) f(x)dx \tag{3.1}$$

3.2 Nonconvexity of the Total Cost Function, TRC (T_1, T_2)

In this section we examine the convexity. We restrict the study to the case of exponential distribution.

We show the non-convexity of TRC through an example.

Example 1:

We choose the following values for the parameters of the problem; K=100, D = 500, P = 1000, π= 0.3, h = 0.1, s = 2, λ=0.5, q1=0.15. The Hessian at the point (T_1, T_2) = (3.2, 9.6) and its determinant is;

$$\begin{vmatrix} 7.0122 & -4.70655 \\ -4.70655 & -0.800304 \end{vmatrix} = -27.7635$$

The Hessain is indefinite and hence the function TRC is not convex at this point.

In the next example we show that TRC is convex for another values for the problem parameter and a different T_1 and T_2.

Example 2:

The Hessain could be positive definite at some points. For example if we take K=150, D=999, P=1000, π=2.5, h=0.75, s=10, λ=0.5, q1=0.15 and (T_1, T_2) = (0.145237, 0.484121) we get the Hessian and its determinant, as given below;

$$\begin{vmatrix} 2093.96 & 2089.99 \\ 2089.99 & 2091.18 \end{vmatrix} = 10{,}797.4$$

The Hessian is positive definite and hence the function TRC is convex at this point.

These two examples illustrates the fact that TRC is in general nonconvex.

3.3 Free Minimal Repair Warranty

In Equation (1.7) the consideration of warranty period and warranty costs are not considered. In this section we are considering warranty.

The fraction of nonconforming items in a production run with length t denoted by $q(t)$ is given by:

$$q(t) = \frac{ED(t)}{Pt} = \frac{\int_t^{\infty} q_1 Ptf(x)dx + \int_0^t \left(q_1 Px + q_2 P(t-x)\right) f(x)dx}{Pt} \tag{3.2}$$

The warranty cost W(t) is given by (2.15).

Example:

Suppose $h_1(t) = k_1 t^{m_1}$ and $h_2(t) = k_2 t^{m_2}$ where $h_1(t) < h_2(t)$ for $t \geq 0$ denote the hazard rate associated with conforming and nonconforming item, respectively. Furthermore suppose that the time to failure is exponential distribution function. Assume K=150, D=1200, P=1650, π=2.5, h=.75, s=.85, λ=0.5, k_1=1, k_2=2, m_1=2, m_2=2, c_r=1.5, ω=6, q1=0.15

Table 7: Free minimal repair warranty for Model II

Model II	Without Warranty	With warranty
Optimal production run time, t^*	0.692	0.6358
Cost	439.69	1,048.16

Note that if warranty is available the optimal production time is less than the case without warranty. This will reduce the probability of being out of control and hence less nonconforming items will be produced.

3.4 Sensitivity Analysis

Exponential Distribution

In this section, a sensitivity analysis of the model is conducted using Exponential distribution as a time to failure to study the effect of the different cost parameters. Warranty costs have not been considered in this section. We used a mean of 10, upper specification limit and lower specification limit is 10 and 7 respectively. Also the variance is equal 1. Table 6 gives the values of the parameters used to perform the analysis. The effect of each parameter is studied as well as the effect of simultaneous changes of two parameters. In addition, a comparison of the basic model versus the sequential changes of the selected parameters is made.

Table 8: Values of the parameters used in the sensitivity analysis study for Model II

Parameter	Level 1	Level 2	Level 3
D	1,200	1,350	1,550
P	1,650	1,850	2,250
K	150	220	450
h	0.75	1.2	1.95
π	2.5	3.5	5.1
λ	0.5	0.75	0.95
s	0.85	1.25	1.75

Effect of the demand rate, D

Different values for the demand rate, *D*, are studied and presented in cases 1-3. As the demand rate increases, the optimal production run time, t^*, also increases to meet the demand.

Effect of the production rate, P

The effect of changes in the production rate, *P*, are analyzed in cases 1, 4 and 5. An increase in the production rate will decrease the production run time, but the total costs increase. If there is no constraint on the production run time, current results indicate low production rates have low total costs.

Effect of the setup cost, K

Different values for the setup cost, K, are presented in cases 1, 6 and 7. By increasing the setup cost, the optimal production run times increase, so that the number of times to setup the machines up for production is minimized in order to reduce the total setup cost.

Effect of the holding cost, h

The effects of the holding cost, h, are presented in cases 1, 8 and 9. If the holding cost increases, the optimal production times tend to be smaller. So, if we produce smaller quantities to meet the demand, there will be a smaller quanties left in the storehouse and in turn decrease the total holding cost.

Effect of the shortage cost, π

The effects of shortage costs, π, are presented in cases 1, 10 and 11. By increasing the shortage cost, the optimal production run time is decreased. This means the TRC will be enhanced by decreasing production run time t if the shortage costs are increased.

Effect of the process failure rate, λ when the time to failure is exponentially distributed.

The results of different values for the failure rate, λ, are presented in cases 1, 12 and 13. As the failure rate of the production system increases, the optimal production run time decreases. So, we will avoid having the system in the out-of-control state, which in turn will contribute to the reduction of the total relevant costs.

Effect of the rework cost, s

Cases 1, 14 and 15 present different values for the rework cost, s. If we increase s, t^* tends to be smaller. So, the percentage of having defective items will be smaller.

Table 9: Effect of each parameter on the optimal production run time for Model II

Case	*Vp**	Level	t*	TRC(t*)
1	Basic Model	1	0.6920	439.69
2	*D*	2	0.7757	451.12
3	*D*	3	0.9000	459.17
4	*P*	2	0.5930	458.21
5	*P*	3	0.4601	485.24
6	*K*	2	0.8211	506.94
7	*K*	3	1.1373	677.62
8	*h*	2	0.6298	474.40
9	*h*	3	0.5718	512.80
10	π	2	0.68	445.97
11	π	3	0.6697	451.47
12	λ	2	0.6041	473.87
13	λ	3	0.5524	500.75
14	*s*	2	0.639	528.88
15	*s*	3	0.5905	637.59

In the following table, the effect of simultaneous changes of two parameters is also presented. For example, consider Cases 1 and 2; if we increase h and D, the optimal production run times tend to be larger. Where in case 3, when h and P increase, the optimal run time decreases. These analyses show us the interaction between paired model parameters.

Table 10: Effects of simultaneous changes of two parameters on the optimal production run time for Model II

Case	Pairs of parameters	Level of factor 1	Level of factor 2	t^*	TRC(t^*)
1	Basic Model	1	1	0.6920	439.69
2	h D	2	2	0.7232	477.38
3	h P	2	2	0.5300	500.77
4	h K	2	2	0.7494	548.19
5	h π	2	2	0.6112	485.96
6	h λ	2	2	0.5617	504.51
7	h s	2	2	0.5902	561.18
8	π D	2	2	0.7658	455.83
9	π P	2	2	0.5805	465.97
10	π K	2	2	0.8072	514.40
11	π λ	2	2	0.5961	479.37
12	π s	2	2	0.6297	534.69
13	λ D	2	2	0.6575	495.63
14	λ P	2	2	0.5302	485.67
15	λ K	2	2	0.7171	550.91
16	λ s	2	2	0.5455	572.55
17	s D	2	2	0.7009	555.21
18	s P	2	2	0.5571	543.53
19	s K	2	2	0.7545	601.90

Additionally, in the following table the sensitivity analysis is comparing the basic model versus the sequential and/or simultaneous changes to the selected factors. For example, in case 2, when the demand rate D increases, t^* will also increase. However, in case 3, when the production rate P and demand rate D increase, t^* will decrease.

Table 11: Effects of sequential changes of the parameters on the optimal production run time for Model II

		Model parameters								
Case	Effects	D	P	K	h	π	λ	s	t^*	TRC(t^*)
1	Basic	1	1	1	1	1	1	1	0.6920	439.69
2	D	2	1	1	1	1	1	1	0.7757	451.12
3	P	2	2	1	1	1	1	1	0.6598	473.80
4	K	2	2	2	1	1	1	1	0.7825	544.6
5	h	2	2	2	2	1	1	1	0.7140	588.27
6	π	2	2	2	2	2	1	1	0.6935	602.83
7	λ	2	2	2	2	2	2	1	0.6212	642.21
8	s	2	2	2	2	2	2	2	0.5692	755.06

Chapter IV : Model III

4.1 Introduction

In this chapter we will discuss the imperfect production process where the rate of defective items is determined by the probability of not meeting lower and upper specifications limits. The process is in-control for a random period of time, X. During this period the process mean and variance are fixed and are given by μ_0 and σ^2. The process becomes out-of-control at time X and the mean changes linearly which variance remains constant. Let τ be the time spent while the process is out of control and β be the rate of change in the mean, then the mean at τ is given by $\mu_\tau = \mu_0 + \beta\tau$.

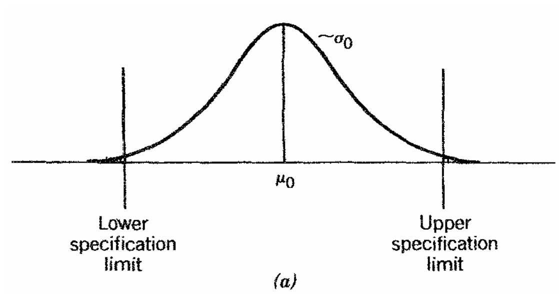

Figure 13: The production process is in control

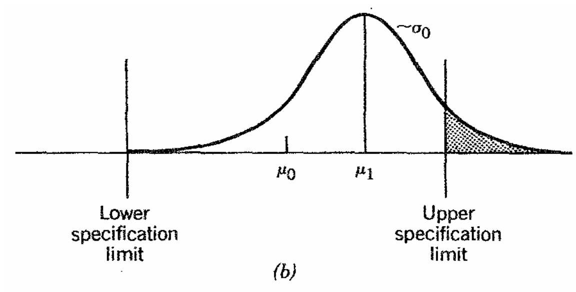

Figure 14: The production process is out of control

If the process becomes out of control at time X = x ≥ t, the number of defects is given by

$$N = Pt\left(1 - \int_{LSL}^{USL} \phi(y, \mu_0)dy\right) \tag{4.1}$$

Where y is a quality characteristic which follows a normal distribution, φ, with mean μ_0 when the process is in control. The quantity in parenthesis in eq. (4.1) represents the probability of being out side the specification limit. When the process is out of control at time X = x < t, then the number of defective parts t is given by;

$$N = P\left[x\left(1 - \int_{LSL}^{USL} \phi(y, \mu_0)dy\right) + \int_{\tau=0}^{t-x}\left(1 - \int_{LSL}^{USL} \phi(y, \mu_\tau)dy\right)d\tau\right] \tag{4.2}$$

The first term in eq. (4.2) represent the number of defectives when the process is in control for duration x and the mean is μ_0.

The bracket within the second integral gives the percentage of defectives at time τ where the mean is μ_τ. Let $\alpha_\tau = 1 - \int_{LSL}^{USL} \phi(y, \mu_\tau)d\tau$. Then $P\alpha_\tau d\tau$ represents the number of defectives in an infinitesimal interval d_τ. The second integral times P represents the number of defectives during the interval (t-x) where the process is out of control.

Hence the expected number of defective items during the production period t is given by:

$$ED(t) = Pt\left(1-\int_{LSL}^{USL}\phi(y,\mu_0)dy\right)\left(1-F(t)\right)+$$
$$P\int_{x=0}^{t}\left[x\left(1-\int_{LSL}^{USL}\phi(y,\mu_0)dy\right)+\int_{\tau=0}^{t-x}\left(1-\int_{LSL}^{USL}\phi(y,\mu_\tau)dy\right)d\tau\right]f(x)dx \qquad (4.3)$$

then ED(t) simplifies to

$$ED(t) = P\left[\alpha_0\left(t\left(1-F(t)\right)+M_1(t)\right)+\int_{x=0}^{t}f(x)\int_{\tau=0}^{t-x}\alpha_\tau d\tau dx\right] \qquad (4.4)$$

Where $M_1(t)$ is the 1st moment of the density function f(x).

For the exponential distribution, (4.3) simplifies to

$$ED(t) = P\left[\alpha_0\left(\frac{1-e^{-\lambda t}}{\lambda}\right)+\lambda\int_{x=0}^{t}e^{-\lambda x}\int_{\tau=0}^{t-x}\alpha_\tau d\tau dx\right] \qquad (4.5)$$

The total cost function TC(t) is given by (1.7). In the next section we examine the stationary points of this function.

4.2 Stationary Points of TRC

From (4.3) we have $\frac{d}{dt}ED(t) = P[\alpha_0\left(1-F(t)\right)+\int_{x=0}^{t}f(t)\alpha_{t-x}dx]$ (4.6)

To obtain the partial derivative of $ED(t)$ with respect to T_1 and T_2 note that $t = T_1 + T_2$ hence

$$\frac{\partial ED(t)}{\partial T_i} = \frac{d}{dt}ED(t)\times\frac{\partial t}{\partial T_i} = ED^{'}(t),\ \text{i} = 1 \text{ and } 2 \qquad (4.7)$$

The partial derivatives of $TRC(T_1,T_2)$ are exactly as those obtained in Chapter 2 so they are not repeated here. In Chapter 2 we showed that at a stationary point of TRC we have $T_1^* = \frac{h}{h+\pi}t$ and $T_2^* = \frac{\pi}{h+\pi}t$. Substituting in $TRC(T_1,T_2)$ we obtain:

$$TC(t) = \frac{KD}{Pt} + \frac{h(P-D)t}{2}\left(\frac{\pi}{h+\pi}\right) + \frac{sD}{Pt}ED(t)$$

If the time to failure is exponentially distributed then TC(t) simplifies to:

$$TC(t) = \frac{KD}{Pt} + \frac{h(P-D)t}{2}\left(\frac{\pi}{h+\pi}\right) + \frac{sD}{Pt}P\left[\alpha_0\left(\frac{1-e^{-\lambda t}}{\lambda}\right) + \lambda\int_{x=0}^{t} e^{-\lambda x}\int_{\tau=0}^{t-x}\alpha_\tau d\tau dx\right] \quad (4.8)$$

$$\frac{dTC}{dt} = \frac{-KD}{Pt^2} + \frac{h(P-D)\pi}{2(h+\pi)} + \frac{SD}{Pt^2}\left(tED'(t) - ED(t)\right) \quad (4.9)$$

Define $g(t) = t^2\frac{dTC}{dt}$, then

$$g(t) = -\frac{KD}{P} + \frac{h(P-D)\pi}{2(h+\pi)}t^2 + SD\left(t\frac{dED(t)}{dt} - ED(T)\right) \quad (4.10)$$

The necessary condition for t* to be optimal is g(t*)=0.

4.3 Nonconvexity of the TRC (T_1, T_2)

In this section we examine the convexity, or rather the lack of convexity of the cost function. We restrict the study to the case of exponential distribution. However, our conclusions apply to other distributions.

Exponential Distribution

The TC function is not convex. To show this we present an example. Choose the following values for the cost parameters of the problem; K=100, D=1,200, P=1,650, π=1, h=0.75, s=3. The parameter of the exponential distribution is λ=0.5. When the process becomes out of control the mean becomes $\mu_\tau=10+\tau$, where τ denotes the time since the process became out of control. The quality characteristic is normally distributed with mean 10 and variance 1. The lower and upper specification limits are 7 and 13 respectively. The plot of TC with those parameters is shown below. The graph of TC is nonconvex.

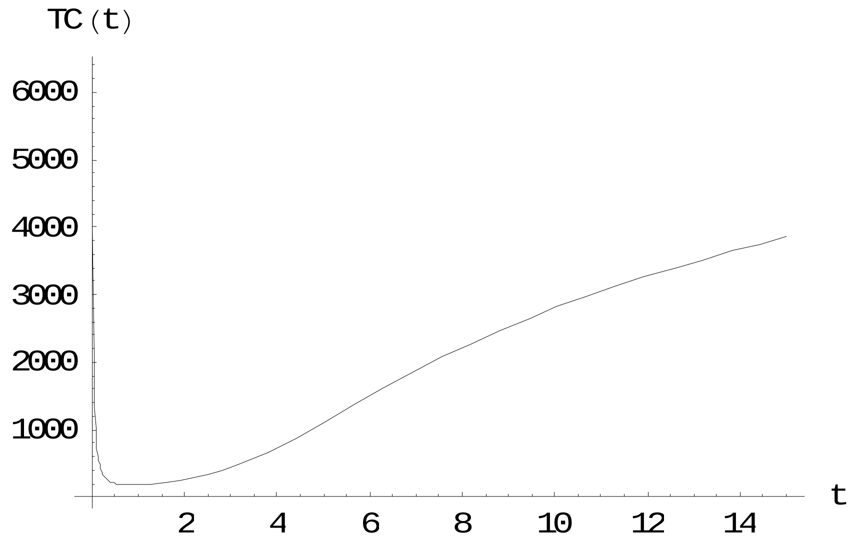

Figure 15: Graph of the Total cost function (t)

4.4 Free Minimal Repair Warranty

In Equation (1.7) the consideration of warranty period and warranty costs are not considered. In this section we are considering warranty.

The fraction of nonconforming items in a production run with length t denoted by $q(t)$ is given by :

$$q(t) = \frac{ED(t)}{pt} = \left[\alpha_0 \left(\frac{1-e^{-\lambda t}}{\lambda} \right) + \lambda \int_{x=0}^{t} e^{-\lambda x} \int_{\tau=0}^{t-x} \alpha_\tau d\tau dx \right] / t$$

The warranty cost W(t) is given by (2.15).

Example:

Let's use the time to failure is exponential distribution function.

K=150, D=1200, P=1650, π=2.5, h=.75, s=.85, λ=0.5, σ=1, k_1=1, k_2=2, m_1=2, m_2=2, c_r=1.5, ω=6, σ=1, μo=10, U=μo+3 σ, L=μo-3 σ.

Table 12: Free minimal repair warranty for Model III

Model III	Without Warranty	With Warranty
Optimal production run time, t^*	0.91107	0.9083
Cost	241.216	756.02

Note that if warranty is available the optimal production time is less than the case without warranty. This will reduce the probability of being out of control and hence less nonconforming items will be produced.

4.5 Sensitivity Analysis

Exponential Distribution

In this section, a sensitivity analysis of the model is conducted using Exponential distribution as a time to failure to study the effect of the different cost parameters. Warranty costs have not been considered in this section. We used a mean of 10, upper specification limit and lower specification limit is 10 and 7 respectively. Also the variance is equal 1. Table 10 gives the values of the parameters used to perform the analysis. The effect of each parameter is studied as well as the effect of simultaneous changes of two parameters. In addition, a comparison of the basic model versus the sequential changes of the selected parameters is made.

Table 13: Values of the parameters used in the sensitivity analysis study for Model III

Parameter	Level 1	Level 2	Level 3
D	1,200	1,350	1,550
P	1,650	1,850	2,250
K	150	220	450
h	0.75	1.2	1.95
π	2.5	3.5	5.1
λ	0.5	0.75	0.95
s	0.85	1.25	1.75

Effect of the demand rate, D

Different values for the demand rate, *D*, are studied and presented in cases 1-3. As the demand rate increases, the optimal production run time, t^*, also increases to meet the demand.

Effect of the production rate, P

The effect of changes in the production rate, *P*, are analyzed in cases 1, 4 and 5. An increase in the production rate will decrease the production run time, but the total costs increase. If there is no constraint on the production run time, current results indicate low production rates have low total costs.

Effect of the setup cost, K

Different values for the setup cost, K, are presented in cases 1, 6 and 7. By increasing the setup cost, the optimal production run times increase, so that the number of times to setup the machines up for production is minimized in order to reduce the total setup cost.

Effect of the holding cost, h

The effects of the holding cost, h, are presented in cases 1, 8 and 9. If the holding cost increases, the optimal production times tend to be smaller. So, if we produce smaller quantities to meet the demand, there will be a smaller quanties left in the storehouse and in turn decrease the total holding cost.

Effect of the shortage cost, π

The effects of shortage costs, π, are presented in cases 1, 10 and 11. By increasing the shortage cost, the optimal production run time is decreased. This means the TRC will be enhanced by decreasing production run time t if the shortage costs are increased.

Effect of the process failure rate, λ when the time to failure is exponentially distributed.

The results of different values for the failure rate, λ, are presented in cases 1, 12 and 13. As the failure rate of the production system increases, the optimal production run time decreases. So, we will avoid having the system in the out-of-control state, which in turn will contribute to the reduction of the total relevant costs.

Effect of the rework cost, s

Cases 1, 14 and 15 present different values for the rework cost, s. If we increase s, t^* tends to be smaller. So, the percentage of having defective items will be smaller.

Table 14: Effect of each parameter on the optimal production run time for Model III

Case	*Vp**	Level	*t**	TRC(*t**)
1	Basic Model	1	0.9109	241.2156
2	*D*	2	1.1699	210.4143
3	*D*	3	1.8299	139.7973
4	*P*	2	0.7199	273.1065
5	*P*	3	0.5099	314.1628
6	*K*	2	1.0999	291.8445
7	*K*	3	1.5499	417.8426
8	*h*	2	0.7699	285.1719
9	*h*	3	0.6599	330.8916
10	π	2	0.8809	249.4249
11	π	3	0.8569	256.5027
12	λ	2	0.9089	241.4167
13	λ	3	0.9069	241.5653
14	*s*	2	0.9089	242.7263
15	*s*	3	0.9049	244.612

In the following table, the effect of simultaneous changes of two parameters is also presented. For example, consider Cases 1 and 2; if we increase *h* and *D*, the optimal production run times tend to be larger. Where in case 3, when *h* and *P* increase, the optimal run time decreases. These analyses show us the interaction between paired model parameters.

Table 15: Effects of simultaneous changes of two parameters on the optimal production run time for Model III

Case	Pairs of parameters	Level of factor 1	Level of factor 2	t^*	TRC(t^*)
1	Basic Model	1	1	0.9109	241.2156
2	h D	2	2	0.9949	248.1416
3	h P	2	2	0.6069	323.1267
4	h K	2	2	0.9319	344.9480
5	h π	2	2	0.7349	299.1908
6	h λ	2	2	0.7699	285.2923
7	h s	2	2	0.7699	286.5942
8	π D	2	2	1.1319	217.4397
9	π P	2	2	0.6949	282.4558
10	π K	2	2	1.0639	301.7539
11	π λ	2	2	0.8789	249.6065
12	π s	2	2	0.8789	250.9140
13	λ D	2	2	1.1590	210.9065
14	λ P	2	2	0.7179	273.2034
15	λ K	2	2	1.0949	292.2066
16	λ s	2	2	0.9049	243.0190
17	s D	2	2	1.1589	212.4133
18	s P	2	2	0.8179	274.5035
19	s K	2	2	1.0949	293.5358

Additionally, in the following table the sensitivity analysis is comparing the basic model versus the sequential and/or simultaneous changes to the selected factors. For example, in case 2, when the demand rate D increases, t^* will also increase. However, in case 3, when the production rate P and demand rate D increase, t^* will decrease.

Table 16: Effects of sequential changes of the parameters on the optimal production run time for Model III

		Model parameters								
Case	Effects	D	P	K	h	π	λ	s	t^*	TRC(t^*)
1	Basic	1	1	1	1	1	1	1	0.9109	241.2156
2	D	2	1	1	1	1	1	1	1.1689	210.4141
3	P	2	2	1	1	1	1	1	0.8659	254.8370
4	K	2	2	2	1	1	1	1	1.0459	308.2517
5	h	2	2	2	2	1	1	1	0.8859	364.3614
6	π	2	2	2	2	2	1	1	0.8449	382.2690
7	λ	2	2	2	2	2	2	1	0.8439	382.4485
8	s	2	2	2	2	2	2	2	0.8419	384.1796

4.6 Comparison Among the Current and Proposed Models

Table 6 shows the percentages of defectives used in the literature as well as the models proposed in this thesis namely, Model II and Model III. The parameters used to obtain t^* and the optimal cost are as follows: K=32, D=200, P= 400, π=2.5, h=0.08, λ=0.1

Table 17: Models Parameters

Model	Parameters
Rosenblatt and Lee (1986)	α=.05
Lee and Rosenblatt (1989)	v=4, r=10, α=.05, a=0.15, b=1.0/.15
Chung and Hou Model (2003)	s=10, α=.05
Hou Model (2005)	s=10, q_1=.15, q_2=.65, r=200
Rahim and Hajailan (2006) Model I	s=10, β=1, alpha=2
Proposed Model II	s=10, β=1, alpha=2, q_1=.15, $q_2=q_1+(1-q_1)*(1-Exp[-\lambda t])$
Proposed Model III	s=10, σ=1, μ_o=10, $U=\mu_o+3\sigma$, $L=\mu_o-3\sigma$, β=1, ,β1=1, alpha=2, $\mu_\tau=\mu_o+\beta\tau$

Table 18: Optimal production run lengths and minimum average costs

Model	Percentages of Defects: In-Control Period	Percentages of Defects: Out-of-Control Period	Shortage	Optimal t^*	Cost
Rosenblatt and Lee (1986)	0	$\alpha + \beta t$	no	1.1581	25.5313
		$\alpha + a\left(1 - e^{-bt}\right)$		0.9	31.9
Lee and Rosenblatt (1989)	0	α	no	Linear Restoration:	
				1.43666	29.0849
				Exponential Restoration:	
				1.43666	31.0028
Chung and Hou Model (2003)	0	α	yes	1.13644	28.3616
Hou Model (2005)	q_1	q_2	yes	0.5368	370.07
Rahim and Hajailan (2006) Model I	0	$1 - \frac{x}{t}$	yes	0.4685	68.6542
Proposed Model II	q_1	$q_2 = q_1 + \left(1 - q_1\right)\left(1 - e^{-\lambda t}\right)$	yes	0.8718	331.122
Proposed Model III	$1 - \int_L^U \phi dy$	$\int_\tau \left(1 - \int_L^U \phi dy\right) d\tau$	yes	1.2998	28.4238

Chapter V : Summary and Future Research Direction

5.1 Summary

In this thesis we examine a model prepared by Rahim and Hajailan (2006). We showed that the objective function is not convex. However, a unique minimum exist. We also proposed a model which allows for defectives when the process is in control as well as out of control. When the process out of control the number of defectives items is an exponential function. This model generalizes the four well known models in the literature, [Rosenblatt and Lee (1986), Lee and Rosenblatt (1989), Chung and Hou (2003), and Hou (2005)].

In chapter 3 we introduced Model III which assumes that there are specification limit for the quality characteristics. Therefore there are defectives even when the process is in control. After random period of time the mean of quality characteristics changes linearly with time. Hence the percentage of defectives increases as the process remains in out of control. The objective function of this model is con-convex.

5.2 Possible Directions for Future Research

In this thesis we considered one type of products. A worthwhile extension is to consider several products with a constraint on space availability. One may include an inspection within the production cycle that involves cost. Based on the inspection production may be halted if the process is found to be out of control or allowed to be continue if it is in control. The problem in this case will be to find the optimal production time and the optimal inspection point during production.

Appendix 1

The average inventory is given by:

$$\bar{I} = \frac{\frac{1}{2}\left(T_2 + T_3\right)I_{\max}}{T}$$

Substituting $T_2 = \frac{I_{\max}}{P - D}$ and $T_3 = \frac{I_{\max}}{D}$ in $\bar{I}$ we get

$$\bar{I} = \frac{I_{\max}^2}{2Q(1 - D/P)}$$

$$I_{\max} = Q - S_{\max} = T \times D - S_{\max} = \left(\frac{Q}{P}\right) \times (P - D) - S_{\max}$$

$$\bar{I} = \frac{\left(Q(1 - D/P) - S_{\max}\right)^2}{2Q(1 - D/P)} = \frac{Q(1 - D/P)}{2} - S_{\max} + \frac{S_{\max}^2}{2Q(1 - D/P)},$$

$$\textit{we know that } S_{\max} = (P - D)T_1 \textit{ and } Q = Pt$$

$$\bar{I} = \frac{t(P - D)}{2} - (P - D)T_1 + \frac{(P - D)T_1^2}{2t}$$

$$\textit{The holding cost is } h\bar{I}$$

Appendix 2

Backordering cost

$$\overline{B} = \frac{\frac{1}{2}(T_1)(P - D)T_1 + \frac{1}{2}(T_4)(D)T_4}{T} = \frac{(T_1^2)(P - D) + T_4^2 D}{2T}$$

we know that $DT_4 = (P - D)T_1$, and $T = \frac{Pt}{D}$, then

$$\overline{B} = \frac{T_1^2(P - D)}{2T} + \frac{(P - D)^2 T_1^2}{2TD} = \frac{T_1^2(P - D)}{2t}$$

The Backordering cost is $= \pi\overline{B} = \frac{\pi T_1^2(P - D)}{2t}$

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